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# Uniqueness and non-uniqueness of signed measure-valued solutions to the continuity equation

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**Abstract**

We consider the continuity equation $\partial_t \mu_t + \operatorname{div}(\boldsymbol{b}\mu_t) = 0$, where $\{\mu_t\}_{t\in\mathbb{R}}$ is a measurable family of (possibly signed) Borel measures on $\mathbb{R}^d$ and $\boldsymbol{b}\colon \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}^d$ is a bounded Borel vector field (and the equation is understood in the sense of distributions). We discuss some uniqueness and non-uniqueness results for this equation: in particular, we report on some counterexamples in which uniqueness of the flow of the vector field holds but one can construct non-trivial signed measure-valued solutions to the continuity equation with zero initial data. This is based on a joint work with N.A. Gusev [BG19].



## 1 Introduction

In this short note, we report on some uniqueness and non-uniqueness results for measure-valued solutions to the *continuity equation* in the Euclidean space. More precisely, fixed $T > 0$ and $d \in \mathbb{N}$, let $\boldsymbol{b}\colon [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ be a given bounded Borel vector field: we consider the initial value problem for the continuity equation

$$\begin{cases} \partial_t \mu_t + \operatorname{div}(\boldsymbol{b}\mu_t) = 0, \\ \mu_0 = \overline{\mu} \end{cases} \tag{PDE}$$

for finite, possibly signed, Borel measures $\{\mu_t\}_{t\in[0,T]}$ on $\mathbb{R}^d$, where the initial datum $\overline{\mu} \in \mathscr{M}(\mathbb{R}^d)$ is a given measure on $\mathbb{R}^d$. This class of measure-valued solutions arises naturally in the limit for weakly* converging subsequences of smooth solutions, and it appears in various applications including hyperbolic conservation laws, optimal transport and other areas, see e.g. [BJ98, AGS08, BPRS15].

In particular, we want to study the relationship between uniqueness of solutions to (PDE) and uniqueness to the ordinary differential equation drifted by $\boldsymbol{b}$, i.e.

$$\frac{d}{dt}\gamma(t) = \boldsymbol{b}(t,\gamma(t)), \quad t \in (0,T), \tag{ODE}$$

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where $\gamma \in C([0,T];\mathbb{R}^d)$. As usual, a solution to (PDE) is intended in the sense of distributions, while a solution to (ODE) is defined to be a continuous curve $\gamma \in C([0,T];\mathbb{R}^d)$ such that

$$\gamma(\tau) = \gamma(s) + \int_s^\tau \boldsymbol{b}(r,\gamma(r))\,dr \qquad \text{for every } (s,\tau) \subset (0,T).$$

Note that this definition is sensitive to modifications of $\boldsymbol{b}$ in a Lebesgue-negligible set, therefore we underline that $\boldsymbol{b}$ is a function defined *everywhere* and not an equivalence class.

Given a solution $\gamma \in C([0,T];\mathbb{R}^d)$ of (ODE) one readily checks that $\mu_t := \delta_{\gamma(t)}$ solves (PDE), where $\delta_p$ denotes the Dirac measure concentrated at $p$. Therefore uniqueness for (PDE) implies uniqueness for (ODE). Hence it is natural to ask whether the converse implication holds.

### 1.1 The non-negative case: Ambrosio's Superposition Principle

In the class of non-negative measure-valued solutions it turns out it is actually possible to transfer uniqueness for (ODE) to uniqueness for (PDE). This result was obtained, without regularity assumptions on the velocity field, in [AGS08] as a consequence of the so-called *superposition principle*. In order to formulate this principle, we will say that a family of Borel measures $\{\mu_t\}_{t\in[0,T]}$ is *represented by* a finite (possibly signed) Borel measure $\eta$ on $C([0,T];\mathbb{R}^d)$ if

1. $\eta$ is concentrated on $\Gamma_{\boldsymbol{b}}$;

2. $(e_t)_\sharp \eta = \mu_t$ for a.e. $t$,

where $e_t\colon C([0,T];\mathbb{R}^d) \to \mathbb{R}^d$ is the so-called *evaluation map* defined by $e_t(\gamma) := \gamma(t)$, $(e_t)_\sharp \eta$ denotes the image of $\eta$ under $e_t$, and $\Gamma_{\boldsymbol{b}}$ denotes the set of solutions of (ODE) (note the $\Gamma_{\boldsymbol{b}}$ is a Borel subset of $C([0,T];\mathbb{R}^d)$ by [Ber08, Proposition 2]).

For example, if $\gamma \in C([0,T];\mathbb{R}^d)$ solves (ODE) then $\eta := \delta_\gamma$ (as a measure on $C([0,T];\mathbb{R}^d)$) represents the solution $\mu_t := \delta_{\gamma(t)}$ of (PDE).

A straightforward computation shows that if $\{\mu_t\}_{t\in[0,T]}$ is represented by some (possibly signed) measure $\eta$ then $\mu_t$ solves (PDE). In this case we will say that $\mu_t$ is a *superposition solution* of (PDE). Clearly uniqueness for (ODE) implies uniqueness for (PDE) in the class of superposition solutions. Indeed, by uniqueness for (ODE) the continuous mapping $e_0\colon \Gamma_{\boldsymbol{b}} \to \mathbb{R}^d$ is injective, hence $e_0^{-1}$ is Borel and thus $(e_0)_\sharp \eta = \mu_0$ is equivalent to $\eta = (e_0^{-1})_\sharp \mu_0$.

Therefore, when uniqueness holds for the Cauchy problem for (ODE), uniqueness for the Cauchy problem for (PDE) holds in the class of measure-valued solutions if and only if *any* measure-valued solution of such Cauchy problem is a superposition solution. The superposition principle established in [AGS08] can now be stated by saying that any *non-negative* solution $\mu_t$ of (PDE) can be represented by some non-negative measure $\eta$ on $C([0,T];\mathbb{R}^d)$.

### 1.2 The signed setting

However the superposition principle cannot be extended to *signed* solutions, because (PDE) can have a nontrivial signed solution even when $\Gamma_{\boldsymbol{b}} = \emptyset$ (see e.g. [Gus18] for the details).

If one assumes Lipschitz bounds on the vector field $\boldsymbol{b}$ uniqueness for (PDE) within the class of signed measures can be proved by means of a duality argument, see Section 3.

Out of the Lipschitz setting, some further results are available in the literature: in [BC94] the authors considered log-Lipschitz vector fields. Later on, in the paper [AB08], the authors proved that the signed superposition principle holds provided that the vector field satisfies a quantitative two-sided diagonal Osgood condition. More precisely, in [AB08] the authors considered vector fields satisfying the following assumptions:

- it holds
$$|\langle \boldsymbol{b}(t,x) - \boldsymbol{b}(t,y), x-y\rangle| \le C(t)\|x-y\|\rho(\|x-y\|) \qquad \forall x,y \in \mathbb{R}^d, \forall t \in (0,T), \tag{1}$$
where $C \in L^1(0,T)$ and $\rho\colon [0,1) \to [0,+\infty)$ is an Osgood modulus of continuity, i.e. a continuous, non-decreasing function with $\rho(0) = 0$ and
$$\int_0^1 \frac{1}{\rho(s)}\, ds = +\infty.$$
- it holds
$$|\boldsymbol{b}(t,x)| \le D(t) \tag{2}$$
for some $D \in L^1(0,T)$ for every $t,x \in (0,T) \times \mathbb{R}^d$.

Then their main result can be stated as follows:

**Theorem 1** (Thm. 1 in [AB08])**.** *If the vector field $\boldsymbol{b}$ satisfies* (1) *and* (2)*, then there is uniqueness for* (PDE) *in the class of bounded signed measures, i.e. if $\mu_t$ is a solution of* (PDE) *such that $|\mu_t|(\mathbb{R}^d) \in L^\infty(0,T)$ then*
$$\mu_t = \boldsymbol{X}(t,\cdot)_\# \mu_0, \qquad \forall t \in (0,T),$$
*where $\boldsymbol{X}(t,\cdot)$ is the flow of $\boldsymbol{b}$, i.e. the unique map solving*
$$\begin{cases} \partial_t \boldsymbol{X}(t,x) = \boldsymbol{b}(t,\boldsymbol{X}(t,x)) & t \in [0,T], x \in \mathbb{R}^d \\ \boldsymbol{X}(0,x) = x & x \in \mathbb{R}^d. \end{cases}$$

Notice that the Osgood assumption (1) is an assumption on $\boldsymbol{b}$ which is much stronger than an implicit assumption of uniqueness for (ODE). Moreover, according to a theorem of Orlicz [Orl32] (see also [Ber08, Thm. 1]), in the space of all continuous vector fields $\boldsymbol{b}$ (equipped with the topology of the uniform convergence on compact sets) the ones for which the differential equation (ODE) has at least one non-uniqueness point is of first category: this shows that in the generic situation Lipschitz/Osgood conditions are *not* necessary for uniqueness.

In particular, a natural question (raised in [AB08]) is whether uniqueness for (PDE) (in the class of signed measures) holds in the presence of a (unique) flow of homeomorphisms solving (ODE), without an explicit bound like (1) on the vector field. If one keeps the *continuity* of the vector field, then in the autonomous 1d case the answer is affirmative, see Section 4 for a sketch of proof inspired to the one presented in [BG19].

Finally, if one drops also the continuity assumption the answer is then negative: in Section 5 we show how to construct a bounded vector field $\boldsymbol{b}\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ such that for *any* $x \in \mathbb{R}$ only $\gamma(t) \equiv x$ ($\forall t \in [0,T]$) solves (ODE) but (PDE) with zero initial condition has a non-trivial measure-valued solution $\{\mu_t\}_{t\in[0,T]}$.

**Theorem 2.** *There exist a vector field $\boldsymbol{b}\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ and a measurable measure-valued map $[0,T] \ni t \mapsto \mu_t \in \mathscr{M}(\mathbb{R})$ such that*

- *is bounded and Borel (in particular it is defined everywhere);*
- *for* any $x \in \mathbb{R}$ *only $\gamma(t) \equiv x$ $\forall t \in [0,T]$ solves* (ODE)*, hence there exists a unique flow of homeomorphisms of $\boldsymbol{b}$;*
- $[t \mapsto \mu_t] \in L^1([0,T];\mathscr{M}(\mathbb{R}))$ *is a non-trivial solution of* (PDE) *with zero initial condition.*

We stress the fact that in the proof of Theorem 2, the map $t \mapsto \mu_t$ is in $L^1([0,T];\mathscr{M}(\mathbb{R}))$, i.e. $\int_{[0,T]} |\mu_t|\, dt < \infty$, but it does *not* belong to $L^\infty([0,T];\mathscr{M}(\mathbb{R}))$: in other words, the measure $\mu_t$ is *not* bounded in time on every subinterval $I \subset [0,T]$. However, as shown rigorously in [BG19], one can properly "embed" the vector field proposed in the proof of Theorem 2 in $\mathbb{R}^2$ and construct an example of non-trivial solution $[t \mapsto \mu_t] \in L^\infty([0,T];\mathscr{M}(\mathbb{R}^2))$ to (PDE) starting from $\bar{mu} = 0$. We refer the reader for this case to the paper [BG19].

## 2 Notation

In the following, we will denote by $\mathscr{B}(\mathbb{R}^d)$ the Borel $\sigma$-algebra on $\mathbb{R}^d$. Recall that a family $\{\mu_t\}_{t\in[0,T]}$ of Borel measures on $\mathbb{R}^d$ is called *a Borel family* if for any $A \in \mathscr{B}(\mathbb{R}^d)$ the map $t \mapsto \mu_t(A)$ is Borel-measurable. It is easily checked that, if $\{\mu_t\}_{t\in[0,T]}$ is a Borel family, then $\{|\mu_t|\}_{t\in[0,T]}$ is a Borel family, too. Furthermore for any bounded Borel function $g\colon [0,T]\times\mathbb{R}^d \to \mathbb{R}$ the map $t \mapsto \int_{\mathbb{R}^d} g(t,x)\,d\mu_t(x)$ is Borel.

In what follows we will write that $[t \mapsto \mu_t] \in L^1((0,T);\mathscr{M}(\mathbb{R}^d))$ if $\{\mu_t\}_{t\in[0,T]}$ is a a Borel family and

$$\int_0^T |\mu_t|(\mathbb{R}^d)\,dt < +\infty.$$

If it holds

$$\text{ess-sup}_{t\in[0,T]}|\mu_t|(\mathbb{R}^d) < +\infty,$$

then we will write $[t \mapsto \mu_t] \in L^\infty((0,T);\mathscr{M}(\mathbb{R}^d))$.

The continuity equation for measure-valued maps is understood in the sense of distributions, according to the following definition:

**Definition 1.** A family $[t \mapsto \mu_t] \in L^1((0,T);\mathscr{M}(\mathbb{R}^d))$ is called a measure-valued solution of (PDE) if for any $\varphi \in C^1_c([0,T)\times\mathbb{R}^d)$

$$\int_0^T \int_{\mathbb{R}^d} (\partial_t\varphi + \underline{\ }(t,x)\cdot\nabla_x\varphi(t,x))\,d\mu_t(x)\,dt + \int_{\mathbb{R}^d}\varphi(0,x)\,d\bar{\mu}(x) = 0. \tag{3}$$

Even though the distributional formulation of the Cauchy problem for (PDE) is well-defined for $[t \mapsto \mu_t] \in L^1((0,T);\mathscr{M}(\mathbb{R}^d))$, it is much more natural in the class $[t \mapsto \mu_t] \in L^\infty((0,T);\mathscr{M}(\mathbb{R}))$, because in this class the initial condition can be understood in the sense of traces, considering a weak* continuous representative of $[t \mapsto \mu_t]$. More precisely, we have the following Proposition (for a proof see e.g. [Bon17, Chapter 1, Prop. 1.6]).

**Proposition 3** (Continuous representative)**.** *Let* $\{\mu_t\}_{t\in[0,T]}$ *be a Borel family of measures and assume* $[t \mapsto \mu_t] \in L^\infty((0,T);\mathscr{M}(\mathbb{R}^d))$*. Then there exists a* narrowly continuous *curve* $[0,T] \ni t \mapsto \tilde{\mu}_t \in \mathscr{M}(\mathbb{R}^d)$ *such that* $\mu_t = \tilde{\mu}_t$ *for a.e.* $t \in [0,T]$*.*

Finally, if $\boldsymbol{v}\colon (0,T)\times\mathbb{R}^d \to \mathbb{R}^d$ is a bounded, continuous vector field, which is Lipschitz continuous in space, uniformly in time, we denote by

$$\boldsymbol{X}\colon [0,T]\times[0,T]\times\mathbb{R}^d \to \mathbb{R}^d$$

the unique map which solves the following problem:

$$\begin{cases} \frac{\partial}{\partial t}\boldsymbol{X}(t,s,x) = \boldsymbol{v}(t,\boldsymbol{X}(t,s,x)) & t,s\in(0,T), x\in\mathbb{R}^d \\ \boldsymbol{X}(s,s,x) = x. \end{cases} \tag{4}$$

The existence and uniqueness of the map $\boldsymbol{X}$ follows from the classical Cauchy-Lipschitz theory.

*Remark* 1. In the case $\boldsymbol{v}$ is autonomous, i.e. does not depend on time, the map $\boldsymbol{Z}(t,x) := \boldsymbol{X}(t,0,x)$ satisfies the following semi-group identity:

$$\boldsymbol{Z}(t+h,x) = \boldsymbol{Z}(t,\boldsymbol{Z}(h,x))$$

for every $t,h \in \mathbb{R}$ and every $x \in \mathbb{R}^d$. In particular, differentiating the expression above w.r.t. $h$ and evaluating in $h = 0$ both members we deduce the following equality

$$\boldsymbol{v}(\boldsymbol{Z}(t,x)) = \partial_x\boldsymbol{Z}(t,x)\cdot\boldsymbol{v}(x)$$

for every $t \in \mathbb{R}$ and $x \in \mathbb{R}^d$.

## 3 Lipschitz vector fields

We propose in this section a proof of uniqueness of signed measure-valued solutions to the continuity equation drifted by a Lipschitz, autonomous vector field. The argument works in every dimension $d \geq 1$ and is well-known, see e.g. [AGS08, Prop. 8.1.7]: it is based on a duality argument (see e.g. [DPL89, Thm. II.6]). We present the proof in the autonomous case, but with minor modifications one can prove also the version for time-dependent vector fields.

**Proposition 4.** *Let* $\boldsymbol{b}\colon \mathbb{R}^d \to \mathbb{R}^d$ *be a bounded, Lipschitz continuous vector field. Let* $\{\mu_t\}_{t\in[0,T]}$ *be a bounded family of* signed *measures solving the continuity equation*

$$\partial_t \mu_t + \operatorname{div}(\boldsymbol{b}\mu_t) = 0$$

*with initial condition* $\mu_0 \leq 0$. *Then we have* $\mu_t \leq 0$ *for every* $t \in [0,T]$. *In particular, if* $\mu_0 = 0$ *then* $\mu_t \equiv 0$ *for every* $t \in [0,T]$.

**Proof.** Set $\boldsymbol{b}^\varepsilon := \boldsymbol{b} * \rho^\varepsilon$, where $\{\rho^\varepsilon\}_{\varepsilon>0}$ is a family of mollifiers in $\mathbb{R}^d$. In particular, so that $\boldsymbol{b}^\varepsilon \to \boldsymbol{b}$ uniformly on compact sets. Let $\psi \in C_c^\infty(I \times \mathbb{R}^d)$ be an arbitrary smooth, compactly supported function with $0 \leq \psi(t,x) \leq 1$ for every $t,x$ and let $w^\varepsilon$ be the solution to the backward Cauchy problem for the transport equation drifted by $\boldsymbol{b}^\varepsilon$:

$$\begin{cases} \partial_t w^\varepsilon + \boldsymbol{b}^\varepsilon \cdot \nabla w^\varepsilon = \psi, \\ w^\varepsilon(T,x) = 0. \end{cases}$$

Applying the method of characteristics and Duhamel's principle we have the representation formula

$$w^\varepsilon(t,x) = -\int_t^T \psi(s, \boldsymbol{X}^\varepsilon(s,t,x))\, ds$$

where $\boldsymbol{X}^\varepsilon$ denotes the flow of the vector field $\boldsymbol{b}^\varepsilon$. Notice that the gradient of $w^\varepsilon$ is uniformly bounded w.r.t. $\varepsilon$: indeed,

$$\|\nabla w^\varepsilon\|_\infty \leq \int_0^T \|\nabla \psi\|_\infty e^{Ls}\, ds = C(\psi,T,L) < \infty$$

where $L$ is the Lipschitz constant of $\boldsymbol{b}$. Notice furthermore that, being $0 \leq \psi(t,x) \leq 1$ it holds

$$-T \leq w^\varepsilon \leq 0.$$

We now use $w^\varepsilon$ as a test function in the continuity equation getting

$$\int_{\mathbb{R}^d}\int_0^T (\partial_t w^\varepsilon(t,x) + \boldsymbol{b}(x)\cdot \nabla w^\varepsilon(t,x))\, d\mu_t(dx)\, dt = \int_{\mathbb{R}^d} w^\varepsilon(T,x) d\mu_T(x) - \int_{\mathbb{R}^d} w^\varepsilon(0,x)\, d\mu_0(x).$$

Being $\mu_0 \leq 0$ and $w^\varepsilon \leq 0$, we have

$$\int_{\mathbb{R}^d} w^\varepsilon(0,x)\, d\mu_0(x) \geq 0.$$

Furthermore, $w^\varepsilon(T,x) \equiv 0$. Thus we have

$$\begin{aligned} 0 &\geq \int_{\mathbb{R}^d}\int_0^T (\partial_t w^\varepsilon(s,x) + \boldsymbol{b}(x)\cdot\nabla w^\varepsilon(s,x))\, d\mu_s(x)\, ds \\ &= \int_{\mathbb{R}^d}\int_0^T \psi(s,x)\, d\mu_s(x)\, ds + \iint_{(0,T)\times\mathbb{R}^d} \langle \boldsymbol{b}(x) - \boldsymbol{b}^\varepsilon(x), \nabla w^\varepsilon(s,x)\rangle\, d\mu_s(x)\, ds. \end{aligned}$$

The second term goes to 0 because $\boldsymbol{b}^\varepsilon \to \boldsymbol{b}$ uniformly on compact sets and $(\nabla w^\varepsilon)_\varepsilon$ is equi-uniformly bounded (in particular, it is bounded in $L^1(\mu_s)$). Thus it remains

$$0 \geq \int_{\mathbb{R}^d} \int_0^T \psi(s,x)\, d\mu_s(x)\, ds,$$

and this concludes the proof, being $\psi$ arbitrary. Possibly changing sign with $\mu_t \mapsto -\mu_t$ we also conclude that, if $\mu_0 = 0$, then $\mu_t \equiv 0$ for every $t$. □

## 4 Continuous vector fields

In proof of Proposition 4, a crucial role is played by Lipschitz condition on $\boldsymbol{b}$: indeed, without explicit Lipschitz bounds on $\boldsymbol{b}$ we cannot find uniform bounds on $\|\nabla w^\varepsilon\|_\infty$. Thus, a different strategy of proof has to be found, if one wants to leave the Lipschitz setting.

As a first step, one could relax the Lipschitz assumption to a general continuity condition on $\boldsymbol{b}$. An important step in this direction was taken in the paper [AB08]: as already mentioned in the Introduction, in [AB08] the authors considered vector fields satisfying the diagonal Osgood condition (1) and the boundedness assumption (2). For such fields, uniqueness of solutions to (ODE) is well-known and Ambrosio-Bernard showed that uniqueness of signed measure-valued solutions to (PDE) holds true (see Theorem 1).

It was noticed in [BG19, Prop. 5.1] that vector fields enjoying (1) and (2) are indeed (equivalent a.e. to a vector field which is) continuous. Thus, one is finally led to ask if uniqueness for (PDE) (in the class of signed measures) holds for merely continuous vector fields (without explicit assumptions on the modulus of continuity like (1)), in the presence of a unique flow of diffeomorphisms for (ODE). More rigorously, one would like to consider the following class of vector fields $\boldsymbol{b}\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$:

(A1) $\boldsymbol{b}(t,\cdot)$ is continuous for every $t \in [0,T]$;

(A2) $\boldsymbol{b}$ is uniformly bounded;

(A3) for every $(t,x) \in [0,T] \times \mathbb{R}^d$ there exists a unique $\gamma = \gamma_{(t,x)} \in \Gamma_{\boldsymbol{b}}$ solving the ODE and $\gamma(t) = x$. We will denote by $\boldsymbol{X} = \boldsymbol{X}(t,x) = \boldsymbol{X}_t(x)$ this map.

Clearly, this uniqueness assumption (A3) is strictly weaker than any Lipschitz type condition. For instance, one can take $\boldsymbol{b}(x) = 1 + f(x)$, where $f$ is the standard Cantor function on $(0,1)$ (extended to $\mathbb{R}$ with the values at 0 and 1). It is also strictly weaker than Osgood condition: for instance, one can easily check that the one-dimensional vector field $\boldsymbol{b}(x) = \sqrt[3]{x}$ satisfies (A1)-(A2 locally)-(A3) but not an Osgood condition.

In the case $d = 1$ it is possible to show that, for autonomous vector fields, the assumptions (A1), (A2), (A3) are sufficient to get uniqueness of signed measure-valued solutions to PDE.

**Proposition 5** ( [BG19, Prop. 5.2]). *Suppose that* $\boldsymbol{b}\colon \mathbb{R} \to \mathbb{R}$ *satisfies* (A1), (A2), (A3). *Then for any* $\bar{\mu} \in \mathscr{M}(\mathbb{R})$ *the Cauchy problem for* (PDE) *with the initial condition* $\mu_t|_{t=0} = \bar{\mu}$ *has a unique solution* $[t \mapsto \mu_t] \in L^1(0,T;\mathscr{M}(\mathbb{R}))$.

For a rigorous proof of Proposition 5 we refer the reader to [BG19]: we propose here a small variant of the original argument, assuming for simplicity that everything is compactly supported.

**Proof.** The problem is linear, thus it is enough to verify the uniqueness with initial datum $\bar{\mu} = 0$. For let $[t \mapsto \mu_t] \in L^1(0,T;\mathscr{M}(\mathbb{R}))$ be a signed measure-valued solution to PDE. By an easy approximation argument, it can be proved that there exists a Lebesgue negligible set $N \subset (0,T)$ such that for all $\tau \in (0,T) \setminus N$ for any $\Phi \in C^1_c([0,\tau] \times \mathbb{R})$ it holds that

$$\int_{\mathbb{R}} \Phi(\tau,x)\, d\mu_\tau(x) - \underbrace{\int_{\mathbb{R}} \Phi(0,x) d\bar{\mu}(x)}_{=0} = \int_0^\tau \int_{\mathbb{R}} [\partial_t \Phi(t,x) + \boldsymbol{b} \cdot \partial_x \Phi(t,x)]\, d\mu_t(x)\, dt. \tag{5}$$

The basic idea, somehow reminiscent of the duality method exploited in the proof of Proposition 4, is to use in (5) test functions of the form

$$\varphi(t,x) := \omega(\boldsymbol{X}(T-t,x)) \tag{6}$$

where $\omega \in C^\infty_c(\mathbb{R})$ is an arbitrary smooth function and $\boldsymbol{X}$ is the flow of $\boldsymbol{b}$. Were $\varphi$ sufficiently regular, it would be immediate to check (by chain-rule) that $\varphi$ satisfies the transport equation $\partial_t \varphi + \boldsymbol{b} \partial_x \varphi = 0$ (pointwise) with the final condition $\varphi(\tau,x) = \omega(x)$. Plugging $\varphi$ into (5) we would then get

$$\int_{\mathbb{R}} \omega(x)\, d\mu_\tau(x) = 0$$

and, by arbitrariness of $\omega$, the desired conclusion $\mu_\tau = 0$.

Thus the only missing point in this approach is the regularity of $\varphi$, which translates into the regularity of the flow map $\boldsymbol{X}$. A first, direct consequence we can derive from (A3) is the strict monotonicity of the map $x \mapsto \boldsymbol{X}(t,x)$, for a.e. $t \in I$: indeed, if it were not monotone, we would have by continuity an intersection between two different trajectories of the vector field, violating (A3). In particular, being monotone, the map $x \mapsto \boldsymbol{X}(t,x)$ is also of bounded variation and hence we can define $\sigma_t := D\boldsymbol{X}_t$, i.e. the measure on $\mathbb{R}$ given by the (spatial) derivative of $\boldsymbol{X}_t$.

Letting now $\boldsymbol{b}^\varepsilon$ be a smooth approximation of $\boldsymbol{b}$, e.g. mollification, we have for every $\varepsilon > 0$

$$\boldsymbol{b}^\varepsilon(\boldsymbol{X}^\varepsilon(t,x)) = \partial_x \boldsymbol{X}^\varepsilon(t,x) \cdot \boldsymbol{b}^\varepsilon(x),$$

in view of Remark 1. Observe that $\boldsymbol{b}^\varepsilon \to \boldsymbol{b}$ uniformly on compact sets; furthermore, for fixed $t$, the flows $\{\boldsymbol{X}^\varepsilon(t,\cdot)\}_{\varepsilon>0}$ are pre-compact in $C^0$ by Ascoli-Arzelà (they are equi-Lipschitz). In particular, it holds

$$\boldsymbol{X}^\varepsilon(t,x) \to \boldsymbol{X}(t,x)$$

pointwise for every $x$ (for every fixed $t$). This together with the uniform convergence of $\boldsymbol{b}^\varepsilon$ implies

$$\boldsymbol{b}^\varepsilon(\boldsymbol{X}^\varepsilon(t,x)) \to \boldsymbol{b}(\boldsymbol{X}(t,x))$$

pointwise everywhere (and hence also in $L^1$ by dominated convergence) which means

$$\partial_x \boldsymbol{X}^\varepsilon(t,x) \cdot \boldsymbol{b}^\varepsilon(x) \to (\boldsymbol{b} \circ \boldsymbol{X})(t,x).$$

On the other hand, we have

$$\partial_x \boldsymbol{X}^\varepsilon(t,\cdot) \rightharpoonup \sigma_t$$

weakly in the sense of measures and, again by uniform convergence of $\boldsymbol{b}^\varepsilon$ to $\boldsymbol{b}$ we have

$$\partial_x \boldsymbol{X}^\varepsilon(t,x) \cdot \boldsymbol{b}^\varepsilon(x) \rightharpoonup \boldsymbol{b}\sigma_t$$

as measures. In particular, we arrive at the following identity, in the sense of measures on $\mathbb{R}$,

$$\boldsymbol{b}\sigma_t = (\boldsymbol{b} \circ \boldsymbol{X}_t)\mathscr{L}^1$$

being $\mathscr{L}^1$ the Lebesgue measure. In particular, let us consider the open set $\{\boldsymbol{b}\neq 0\}$ and let $(\alpha,\beta)$ one of its connected components. Then

$$\sigma_t \llcorner (\alpha,\beta) \ll \mathscr{L}^1 \tag{7}$$

with density

$$\frac{d\sigma_t\llcorner(\alpha,\beta)}{d\mathscr{L}^1} = \frac{(\boldsymbol{b}\circ \boldsymbol{X}_t)}{\boldsymbol{b}} \in C^0((\alpha,\beta))$$

so that

$$\boldsymbol{X}(t,\cdot)\in C^1((\alpha,\beta))$$

and can now be used as test function: indeed, choosing $\omega\in C_c^\infty(\alpha,\beta)$ and defining $\varphi$ as in (6) we have $\varphi\in C_c^1([0,\tau]\times(\alpha,\beta))$. Using it as a test function as described above, we obtain that the solution $(\mu_t)\llcorner_{(\alpha,\beta)}=0$ for a.e. $t$: hence $\mu_t$ vanishes on every connected component of the set $\{\boldsymbol{b}\neq 0\}$. We have thus proved that $\mu_t$ is concentrated on $\{\boldsymbol{b}=0\}$ and then it solves (PDE) with $\boldsymbol{b}\equiv 0$. Hence $\mu_t=0$ globally for a.e. $t\in(0,T)$ and this concludes the proof. □

## 5 The general case: counterexamples

Having established a uniqueness result for continuous vector fields, one can still wonder whether the continuity assumption is needed. In this section we summarize the proof of the following result:

**Theorem 6** ( [BG19, Thm. 3.1])**.** *There exist $T>0$, a bounded Borel $\boldsymbol{b}\colon[0,T]\times\mathbb{R}\to\mathbb{R}$ and $[t\mapsto\mu_t]\in L^1((0,T);\mathscr{M}(\mathbb{R}))$ satisfying the following conditions:*

*(i) $\boldsymbol{b}$ satisfies* (A2) *and* (A3)*. Furthermore, the characteristics of $\boldsymbol{b}$ are constant, i.e. $\gamma\in\Gamma_{\boldsymbol{b}}$ if and only if there exists $x\in\mathbb{R}$ such that $\gamma(t)=x$ for all $t\in[0,T]$;*

*(ii) $\{\mu_t\}_{t\in[0,T]}$ is not identically zero and solves* (PDE) *with zero initial condition.*

The construction of the vector field is based on the following result, which is well known in measure theory. We denote by $|A|$ the Lebesgue measure of $A\subset\mathbb{R}$.

**Lemma 7.** *There exists a non-empty Borel set $P\subsetneq\mathbb{R}$ with the following property: for any non-empty bounded open interval $I\subset\mathbb{R}$ it holds that $|I\cap P|>0$ and $|I\cap(\mathbb{R}\setminus P)|>0$.*

Given the set $P$ constructed in Lemma 7 we now set $N:=\mathbb{R}\setminus P$ and

$$f(\tau):=2+\int_0^\tau(\mathbf{1}_P(r)-\mathbf{1}_N(r))\,dr \qquad\text{and}\qquad F(\tau):=(f(\tau),\tau) \tag{8}$$

where $\tau\in[0,1]$. Since the derivative of $f$ is equal to $\mathbf{1}_P-\mathbf{1}_N$ a.e., for convenience we denote $f':=\mathbf{1}_P-\mathbf{1}_N$. Observe that the function $f'$ is defined everywhere and takes values in $\{\pm1\}$ and it is a Borel representative of the derivative of the function $f$ defined in (8).

We now set $T:=4$ and define

$$\boldsymbol{b}(t,x):=\mathbf{1}_{F[0,1]}(t,x)\cdot\frac{1}{f'(x)} \qquad\text{and}\qquad \tilde{\mu}_t:=\sum_{x\in f^{-1}(t)}\operatorname{sign}(f'(x))\delta_x. \tag{9}$$

By definition $\boldsymbol{b}$ is Borel and bounded. Moreover by the area formula $\{\tilde{\mu}_t\}_{t\in[0,T]}$ is a measurable family of Borel measures. A simple computation allows to show the following Lemma:

**Lemma 8.** *For $\boldsymbol{b}$ and $\tilde{\mu}_t$ defined above*

$$\partial_t\tilde{\mu}_t+\operatorname{div}(\boldsymbol{b}\tilde{\mu}_t)=-\delta_{F(1)}+\delta_{F(0)} \qquad \textit{in} \quad \mathscr{D}'((0,T)\times\mathbb{R}).$$

To get rid of the defect $-\delta_{F(1)} + \delta_{F(0)}$ we simply add to $\tilde{\mu}_t$ solutions concentrated on constant in time trajectories (since $\boldsymbol{b}$ is 0 outside $F([0,1])$). More precisely, one readily checks that

$$\mu_t := \tilde{\mu}_t + \mathbf{1}_{[f(1),+\infty)}(t)\,\delta_1 - \mathbf{1}_{[f(0),+\infty)}(t)\,\delta_0$$

solves (PDE).

To conclude the proof of Theorem 6, it remains to study the integral curves of $\boldsymbol{b}$. This issue is addressed in the following Lemma:

**Lemma 9.** *For any* $(t,x)$ *there exists a unique characteristic of* $\boldsymbol{b}$ *passing through* $x$.

Therefore we have constructed a vector field $\boldsymbol{b}$ for which the characteristics are unique, but there exists a nontrivial signed solution of the CE. Using a minor modification of the present construction one can construct a similar example of $(\mu_t, \boldsymbol{b})$ having compact support in spacetime.

*Remark* 2. We remark that the crucial fact used in the proof of Lemma 9 is the fact that if $f$ is nowhere monotone. Were $f$ monotone on some interval $I$ then uniqueness would fail for the Cauchy problem for (ODE) with $\boldsymbol{b}$ constructed in the proof of Theorem 6: indeed, without loss of generality suppose that $f$ is strictly increasing on $I$. Then for any $x \in I$ there exist at least two (actually, infinitely many) integral curves $\gamma \in \Gamma_{\boldsymbol{b}}$ such that $\gamma(0) = x$. Indeed, clearly $\gamma(t) := x$ ($\forall t \in [0,T]$) belongs to $\Gamma_{\boldsymbol{b}}$. On the other hand, for any $y \in I$ such that $y > x$ one can define $\gamma$ by

$$\gamma(t) := \begin{cases} x, & t < f(x); \\ f^{-1}(t), & f(x) \le t < f(y); \\ y, & t \ge f(y). \end{cases}$$

Then one readily checks that $\gamma \in \Gamma_{\boldsymbol{b}}$, since for a.e. $t \in (f(x), f(y))$ it holds that

$$\gamma'(t) = \frac{1}{f'(f^{-1}(t))} = \frac{1}{f'(\gamma(t))} = \boldsymbol{b}(t, \gamma(t)).$$

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