Brake orbits and heteroclinic connections for first order Mean Field Games

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Abstract

We consider first order variational MFG in the whole space, with aggregative interactions and density constraints, having stationary equilibria consisting of two disjoint compact sets of distributions with finite quadratic moments. Under general assumptions on the interaction potential, we provide a method for the construction of periodic in time solutions to the MFG, which oscillate between the two sets of static equilibria, for arbitrarily large periods. Moreover, as the period increases to infinity, we show that these periodic solutions converge, in a suitable sense, to heteroclinic connections. As a model example, we consider a MFG system where the interactions are of (aggregative) Riesz-type.

Contents

1 Introduction

Mean field games (MFG) theory describes interactions among a large numbers of indistinguishable rational individuals, in which a generic agent optimizes some functional depending both on its dynamical state and on the average collective behavior, represented by the density of the overall population. In an equilibrium regime, the optimal dynamics of the average agent is consistent with the collective evolution. Such equilibria can be described by a system of coupled PDEs, a backward Hamilton-Jacobi equation characterizing the value function of the average

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agent, and a forward continuity equation modelling the evolution of the population density, that is (in the model case of first order MFG with quadratic Hamiltonian)

$$\begin{cases} -\partial_t u + \frac{|\nabla u|^2}{2} = f(x,m), & \text{in } (0,T) \times \mathbb{R}^d \\ \partial_t m - \operatorname{div}(m\nabla u) = 0 & \text{in } (0,T) \times \mathbb{R}^d \\ m \geq 0, \ \int_{\mathbb{R}^d} m(t,x)dx = 1. \end{cases} \tag{1.1}$$

Usually the system is coupled with initial/final time conditions. This theory has been introduced in the mathematical community by Lasry and Lions in [23, 24] and since then, there has been a large development of the subject in the literature.

Here, we will focus on the widely studied class of potential (or variational [4]) MFG: these are MFG systems that can be derived as optimality conditions of suitable optimal control problems on the continuity equation, with quadratic Lagrangian and running cost f. Precisely, we assume that $f(x,m)$ is the derivative of a potential $\mathcal{W}$ defined on the space of Borel probability measures $\mathcal{P}(\mathbb{R}^d)$, that is $f(x,m) = \frac{\delta}{\delta m}\mathcal{W}(m) \in C(\mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d))$, or equivalently

$$\lim_{h \to 0^+} \frac{\mathcal{W}(m + h(m'-m)) - \mathcal{W}(m)}{h} = \int_{\mathbb{R}^d} f(x,m)d(m'-m)(x)$$

for all $m, m' \in \mathcal{P}(\mathbb{R}^d)$. In this case, the PDE system (1.1) formally appears as the first order condition for critical points of the following energy functional:

$$J_T(m,v) := \int_0^T \int_{\mathbb{R}^d} \frac{1}{2}|v(t,x)|^2 m(t,dx)dt + \int_0^T \mathcal{W}(m(t))dt, \tag{1.2}$$

to be computed among all possible evolutions of the mass distributions, that is among all couples (m,v) such that $m_t - \operatorname{div}(mv) = 0$ in the distributional sense, where $m(t) \in \mathcal{P}(\mathbb{R}^d)$ for all t, and the velocity field $v \in L^2(dt \otimes m(t,dx))$. It is well-known that when $\mathcal{W} \equiv 0$, and $m(0), m(T)$ are given, this is the so-called fluid mechanics formulation of the Monge-Kantorovich mass transfer problem introduced by Benamou and Brenier [3], which leads to the dynamic characterization of the L^2-Kantorovich-Rubinstein-Wasserstein distance d_2 between measures in $\mathcal{P}_2(\mathbb{R}^d)$ (those with finite quadratic moments in $\mathbb{R}^d$, see Definition 2.2 and [2, 32] for a general discussion). The similarities between the Benamou-Brenier formulation of optimal transport and MFG have been already explored in the study of first order MFG systems, and we refer to [20, 21, 25, 30].

We will construct in this work (constrained) critical points of J_T, rather than produce solutions to PDE systems like (1.1). We then show that these critical points $(\bar{m}, \bar{v})$ give rise to mean field Nash equilibria, in the following sense: for any admissible competitor (m,v),

$$\int_0^T \int_{\mathbb{R}^d} \frac{1}{2}|v|^2 m + \int_0^T f(\bar{m})m \geq \int_0^T \int_{\mathbb{R}^d} \frac{1}{2}|\bar{v}|^2\bar{m} + \int_0^T f(\bar{m})\bar{m}, \tag{1.3}$$

that is, $(\bar{m}, \bar{v})$ is a minimizer of a *linearized* functional. In other words, $(\bar{m}, \bar{v})$ is such that $(\bar{m}, \bar{v})$ itself is the optimum in a (infinite dimensional / McKean-Vlasov) control problem involving a quadratic Lagrangian and running cost $f(\bar{m})$; such a fixed point property is at the core of Nash equilibria in MFG. In addition, following [8], these equilibria yield solutions to PDE systems of the form (1.1). This can be shown by exploiting an optimization problem in duality with (1.3). The derivation and the analysis of first order optimality conditions in PDE form will not be carried out here; besides, our constructions of critical points of J_T does not require $\mathcal{W}$ to have a derivative with respect to m, though this assumption is crucial if one wants to make a connection with MFG systems.

Another observation which is crucial for this work is that variational MFG systems of the form (1.1) can be interpreted as Hamiltonian systems on the infinite dimensional metric space $\mathcal{P}_2(\mathbb{R}^d)$, endowed with the distance d_2. In addition, the energy $J_T(m,v)$ defined in (1.2) can

be rewritten via the Benamou-Brenier formula [3] as an energy on the space of trajectories $C([0, T], \mathcal{P}_2(\mathbb{R}^d))$, as follows:

$$J_T(m) = \int_0^T \frac{1}{2} |m'|^2(t) + \mathcal{W}(m(t)) \, dt, \tag{1.4}$$

where $|m'|(t)$ is the metric derivative of the curve with respect to the Wasserstein distance d_2, see [2]. In such a form, J_T is reminiscent of standard action functionals appearing in Hamiltonian mechanics. Let us make a short remark on the sign in front of the "potential" $\mathcal{W}$, to avoid possible confusion. In the form (1.4), J_T formally corresponds to an action functional of a mechanical system of (infinitely many) particles subject to the acceleration $\nabla_x \delta_m \mathcal{W}$. In the classical mechanics terminology, the "potential energy" would then correspond to $-\mathcal{W}$. We have adopted here the plus sign in line with the MFG viewpoint, where $\mathcal{W}$ is the "potential" (anti-derivative) of the running cost f.

We will also make use, as in the work by Benamou and Brenier, of the standard change of variables which replaces velocity by momentum, i.e. $(m, w) = (m, vm)$. The energy (1.2) then becomes, in a generalized sense,

$$J_T(m, w) = \int_0^T \int_{\mathbb{R}^d} \frac{1}{2} \left| \frac{dw}{dt \otimes m(t, dx)} \right|^2 m(t, dx) dt + \int_0^T \mathcal{W}(m) dt, \tag{1.5}$$

to be computed on the set

$$\mathcal{K} := \left\{ (m, w) \mid m \in C(\mathbb{R}, \mathcal{P}_2(\mathbb{R}^d)), \right.$$

w is a Borel d-vector measure on $\mathbb{R} \times \mathbb{R}^d$, absolutely continuous w.r.t. $dt \otimes m(t, dx)$,

$-\partial_t m + \operatorname{div}(w) = 0$ in the sense of distributions,

$$\left. \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \frac{1}{2} \left| \frac{dw}{dt \otimes m(t, dx)} \right|^2 m(t, dx) dt < \infty \text{ for all } -\infty < t_1 < t_2 < \infty \right\}. \tag{1.6}$$

The two energies (1.2) and (1.5) are equivalent, see [4]. Note that under these new variables the differential constraints become linear, that is $m_t - \operatorname{div} w = 0$, and moreover the function $(m, w) \mapsto \frac{|w|^2}{2m}$ (extended to 0 where $m = 0$) is a convex function. In the following, we are going to consider constrained minimizers of (1.5), i.e. minimizers in some suitable subset of $\mathcal{K}$.

An interesting issue in MFG is the description of the long time behavior of equilibria, that is: given some information of the system at initial and final time, say at $t = 0$ and $t = T$, such as the population distribution m and/or the final cost u, is it possible to describe m (and u) at intermediate times? A natural goal would be to characterize attractors that are approached by m as $T \to \infty$. A large part of the literature in this direction is devoted to congestion type games, that are games in which players prefer sparsely populated areas of the state space. This is typically translated into the assumption that $\mathcal{W}$ is convex, or equivalently that the interaction cost $f(x, m)$ is monotone increasing with respect to the mass distribution:

$$\int_{\mathbb{R}^d} (f(x, m) - f(x, m')) d(m - m')(x) \geq 0 \qquad \forall m, m' \in \mathcal{P}_2(\mathbb{R}^d). \tag{1.7}$$

We point out that this condition does not imply that the functional $\mathcal{W}$ is geodesically convex in $\mathcal{P}_2$ (see [2, 32]): geodesic convexity of $\mathcal{W}$ and monotonicity of f are actually unrelated conditions. We refer to [19] for a recent work on potential MFG with geodesically convex Lagrangians. Under the monotonicity assumption (1.7), one expects in general uniqueness of the equilibria, and some further regularity properties. For second order problems, the long time behavior of the PDE system is quite well understood (at least when the state space is the flat torus): in a long time horizon, solutions approach the (unique) stationary equilibrium, which is then attractive for the evolutive system. We refer to the recent papers [9, 16] and references therein for more details.

On the other hand, without the monotonicity assumption, the long time behavior is much less understood and very few is known about long time patterns. The second author obtained recently some results for viscous (second order) MFG in the flat torus with anti-monotone interactions, that is assuming that $-f(x, m)$ is monotone increasing. In particular in [14] (see also [15]), it is provided the construction, using bifurcation arguments, of an infinite number of branches of non-trivial solutions which exhibit an oscillatory (in time) behavior, and emanating from a trivial stationary solution (also for the case of two populations of players, which is non-variational in general). Finally, in [27], by using weak KAM methods in an infinite dimensional setting, it is provided an example of a second order MFG with non monotone interaction cost, settled in the periodic torus, for which solutions in the long time horizon do not converge to the stationary state (see also [7] for further results). Long time pattern formation has also been explored in MFG models arising in socioeconomics [22, 33, 34]. For first order problems, the long time behavior is even less understood (we are only aware of few results in [6], regarding the monotone case).

In this paper, we analyze long time patterns arising in some first order (potential) MFG. Differently with respect to previous works, our setting is first order (no viscosity), on the whole space and without periodicity conditions. Moreover, we consider a non-monotone case, namely $\mathcal{W}$ will be a sort of "double-well" potential. We have in mind models where players aim at aggregating, that is, they attracted towards crowded areas, see in particular Section 1.1 below. In contrast with the aggregating forces, we impose density constraints to the population density, that are particularly meaningful when one describes crowd motions for example. We indeed impose the distribution of players $m(t)$ to have a density which does not exceed some given value ρ, that is, for all t,

$$m(t) \in \mathcal{P}^r_{2,\rho}(\mathbb{R}^d) := \{ m \in \mathcal{P}_2(\mathbb{R}^d) \ : \ \exists \, 0 \leq \tilde{m} \leq \rho \ \text{a.e. on } \mathbb{R}^d \text{ s.t. } m = \tilde{m}dx \}$$

(with a slight abuse of notation, we will often identify m with its density $\tilde{m}$) and so we restrict the set $\mathcal{K}$ defined in (1.6) to

$$\mathcal{K}^\rho := \{ (m, w) \in \mathcal{K}, \ m(t) \in \mathcal{P}^r_{2,\rho}(\mathbb{R}^d), \ \forall t \in \mathbb{R} \}. \tag{1.8}$$

This constraint models an environment with finite capacity. Alternatively, it could be regarded as an infinite cost paid by players that try to cluster over saturated regions (hard congestion). We mention that first order MFG with density constraints have been studied, in the monotone case, in [8], where connections with variational models for the incompressible Euler's equations à la Brenier are also discussed (see also [25]). Another effect against concentration could be dissipation, that may appear as a viscosity term in the continuity equation for m. This setting has been considered recently in [10], where stationary solutions to second order aggregating MFG are constructed; concentration phenomena and selection type results when the dissipation term vanishes are also shown.

Throughout the paper, we assume the following general conditions on the interaction potential $\mathcal{W} : \mathcal{P}^r_{2,\rho}(\mathbb{R}^d) \to [0, \infty)$. Note that "aggregation" is not encoded explicitly in our setting. Nevertheless, if $\mathcal{W}$ has to be double-well shaped in the sense described below, then $\mathcal{W}$ has to break the convexity assumption (typically related to "competition"), and therefore it has to enforce "aggregation" to some extent (see the model in Section 1.1). First of all we assume that $\min_{\mathcal{P}^r_{2,\rho}(\mathbb{R}^d)} \mathcal{W}$ exists, and without loss of generality that $\min_{\mathcal{P}^r_{2,\rho}(\mathbb{R}^d)} \mathcal{W} = 0$. We suppose in addition that minima of $\mathcal{W}$ consists of *two disjoint compact subsets* of $\mathcal{P}_2(\mathbb{R}^d)$, that is

$$\exists \mathcal{M}^+, \mathcal{M}^- \subset\subset \mathcal{P}^r_{2,\rho}(\mathbb{R}^d) \ \text{s.t.} \ d_2(\mathcal{M}^+, \mathcal{M}^-) =: 2q_0 > 0, \ \text{and} \ \mathcal{W}(m) = 0 \Leftrightarrow m \in \mathcal{M}^\pm. \quad \textbf{(Z)}$$

(where $\mathcal{M}^\pm = \mathcal{M}^+ \cup \mathcal{M}^-$). We assume some standard lower semi-continuity (in a topology which is slightly weaker than the one of $\mathcal{P}_2$)

$$\text{for any } p < 2, \{m_n\} \subset \mathcal{P}^r_{2,\rho}(\mathbb{R}^d), \ \text{if } \lim_n d_p(m_n, m) = 0 \ \text{then} \ \liminf_n \mathcal{W}(m_n) \geq \mathcal{W}(m), \quad \textbf{(lsc)}$$

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which will be needed to construct minimizers of (1.5). Note that lower semi-continuity of the kinetic part term in J_T is standard by convexity (see Proposition 2.7). Some coercivity of $\mathcal{W}$ in $\mathcal{P}_2(\mathbb{R}^d)$ will be also needed: there exists $C_{\mathcal{W}} > 0$ such that for all $m \in \mathcal{P}_{2,\rho}^r(\mathbb{R}^d)$

$$-C_{\mathcal{W}} + C_{\mathcal{W}}^{-1} \int_{\mathbb{R}^d} |x|^2 m(x)dx \leq \mathcal{W}(m) \leq C_{\mathcal{W}} \left(1 + \int_{\mathbb{R}^d} |x|^2 m(x)dx \right). \qquad \textbf{(BDD)}$$

Note that **(BDD)** implies that $\mathcal{W}$ has compact sublevel sets in $\mathcal{P}_p(\mathbb{R}^d)$ for every $p < 2$, see Lemma 2.4 and Remark 2.5, but not necessarily for $p = 2$.

We finally assume the following continuity property in $\mathcal{P}_2(\mathbb{R}^d)$ close to the zero level-set: for any $\{m_n\} \subset \mathcal{P}_{2,\rho}^r(\mathbb{R}^d)$,

$$\text{if} \quad \lim_n \mathcal{W}(m_n) = 0, \quad \text{then} \quad \lim_n d_2(m_n, \mathcal{M}^{\pm}) = 0. \qquad \textbf{(CON)}$$

Note that if $\mathcal{W}$ is assumed to be lower semicontinuous and with compact sublevel sets in $\mathcal{P}_2(\mathbb{R}^d)$, then **(CON)** follows directly from **(Z)**.

It is clear that minima $\mathcal{M}^{\pm}$ of $\mathcal{W}$ are stationary solutions/equilibria, namely minimizers of the energy J_T. The main goal of this work is show that the MFG problem has other equilibria that exhibit peculiar patterns. First, we construct *periodic in time* critical points of J_T, that oscillate between stationary solutions (brake orbits). Then, we construct *heteroclinic connections*, that are, roughly speaking, solutions to the MFG problem which are defined for all times and approach $\mathcal{M}^-$ as $t \to -\infty$ and $\mathcal{M}^+$ at $t \to +\infty$ (see Definition 4.1). We will exploit the fact that the potential $\mathcal{W}$ in the energy (1.5) is assumed to be a double-well potential in $\mathcal{P}_{2,\rho}^r(\mathbb{R}^d)$. Written in the form (1.4), the energy can be interpreted as an action functional on the space of continuous curves with values in the metric space $\mathcal{P}_2(\mathbb{R}^d)$, and is reminiscent of classical variational problems for finite-dimensional Hamiltonian systems.

There is a huge literature (see the survey [31] and references therein) on the construction of periodic or heteroclinic trajectories in Hamiltonian systems by means of variational techniques. Among periodic solutions, the so-called *brake orbits* are widely studied. These are solutions m of the Hamiltonian system in $[t_0, t_1]$ such that $|m'|(t_0) = |m'|(t_1) = 0$, which are extended to periodic curves with period $2|t_1 - t_0|$ just by reflection around t_0 or t_1. Roughly speaking, a brake orbit travels periodically between $m(t_0)$ and $m(t_1)$, and $m(t_0)$ and $m(t_1)$ are typically close to steady states. These orbits are usually found as periodic critical points of the action functional (1.4) (with Morse index 1 in the context of periodic perturbations), and not as global minimizers. To mode out their instability (and to circumvent the lack of compactness given by time translations in (1.4)) some symmetry can be added to the system. Here, we assume that there exists a reflection $\gamma : \mathbb{R}^d \to \mathbb{R}^d$, such that

$$\mathcal{W}(\gamma_{\#}m) = \mathcal{W}(m) \qquad \text{and} \qquad \mathcal{M}^+ = \gamma_{\#}\mathcal{M}^-, \qquad \textbf{(REF)}$$

that is, $\mathcal{W}$ is symmetric and vanishing on two disjoint subsets that are symmetric with respect to each other. To generate brake orbits, we then restrict to those trajectories that satisfy

$$m(-t) = \gamma_{\#}m(t), \qquad m\left(\frac{T}{4} - t\right) = m\left(\frac{T}{4} + t\right) \qquad \forall t, \qquad (1.9)$$

so that $|m'|(T/4) = |m'|(-T/4) = 0$. In this way, we construct orbits that follow back and forth the same trajectory connecting $m(-T/4)$ and $m(T/4)$ in time $T/2$, thus performing a whole cycle in time T.

Before stating our results, we recall that other extensions to the infinite dimensional setting of these kind of constructions has been considered quite recently in the literature. The existence of heteroclinic connections in the general framework of metric spaces has been provided in [29], under the assumption that the potential $\mathcal{W}$ has a finite numbers of zeros. The result is obtained by a different procedure, namely by re-parametrizing the action functional (1.4)

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to a length functional in the metric space: then an heteroclinic connection is a geodesic with respect the new length functional. Another class of infinite dimensional problems which have been considered in the literature is related to functionals $\mathcal{W}$ defined on Hilbert spaces (such as $H^1(\Omega)$, with appropriate boundary conditions) and $\mathcal{W}(u) = \|\nabla u\|^2_{L^2(\Omega)} + \int_\Omega W(x, u)dx$, where $W(x, \cdot)$ is a double well potential. In [1] (see also references therein) the authors prove the existence of brake orbits and also convergence to heteroclinic connections as the period goes to infinity by minimizing the action functional among curves with prescribed energy. Analogous results have been proved in [18], with a different approach: instead of minimizing the action functional with fixed mechanical energy, the author minimize it on a set of T-periodic maps with fixed $T > 0$. In this paper, we follow the same approach as in [18], and as far as we know, similar constructions for MFG problems have never been studied.

The first main result is about construction of brake orbits, and it is proved in Section 3. We introduce the sets of curves on which we minimize our functional

$$\mathcal{K}_T^{\rho,S} := \left\{ (m, w) \in \mathcal{K}^\rho, \ T\text{-periodic}, \ m\left(\frac{T}{4} + t\right) = m\left(\frac{T}{4} - t\right), \ m(-t) = \gamma_\# m(t), \ \forall t \in \mathbb{R} \right\}. \tag{1.10}$$

Then, we have the following result.

Theorem 1.1. *Assume* (**Z**), (**lsc**), (**BDD**), (**CON**) *and* (**REF**). *Let* $q \in (0, q_0)$, *where* q_0 *is defined in* (**Z**). *Then there exists* $\bar{T} = \bar{T}(q) > 4$ *such that, for any* $T \geq \bar{T}$, *there exists a* T-*periodic minimizer* $(m^T, w^T) \in \mathcal{K}_T^{\rho,S}$ *of the problem* $\min_{\mathcal{K}_T^{\rho,S}} J_T$. *Moreover*, (m^T, w^T) *satisfies*

$$\begin{cases} d_2(m^T(t), \mathcal{M}^+) < q & \forall t \in \left(s, \frac{T}{2} - s\right) \\ d_2(m^T(t), \mathcal{M}^-) < q & \forall t \in \left(-\frac{T}{2} + s, -s\right), \end{cases} \tag{1.11}$$

for some $0 < s = s(q)$ *(note that* $s(q)$ *does not depend on T).*

As a consequence, we show in Corollary 3.5 that there exists a brake orbit for the MFG problem in the sense of Definition 3.1, that is: a Mean Field Nash equilibrium, i.e. a minimizer as in (1.3) in the larger non-symmetric set

$$\mathcal{K}_T^\rho := \left\{ (m, w) \in \mathcal{K}^\rho, \ T\text{-periodic} \right\} \tag{1.12}$$

that exhibit (symmetric) oscillations between two nearly-steady states. Note that $\mathcal{K}_T^{\rho,S}$ contains no information on the "closeness" of m to $\mathcal{M}^\pm$. What we prove is that minimizers of J_T in $\mathcal{K}_T^{\rho,S}$ have a posteriori to be close to $\mathcal{M}^\pm$, in the sense of (1.11). Therefore, these nearly-steady states are close to be minimizers of the potential energy $\mathcal{W}$.

We stress that the transition time $2s$ between (neighborhoods of) the two steady states depends on q only, and remains bounded as $T \to \infty$. This is a key point in obtaining the second main result, which is about the construction of heteroclinic solutions and convergence of brake orbits to heteroclinics (that will proved in Section 4). Heteroclinics are equilibria (minimizers as in (1.3)), connecting different sets of steady states in an infinite time horizon: $\lim_{t\to-\infty} d_2(m(t), \mathcal{M}^-) = \lim_{t\to+\infty} d_2(m(t), \mathcal{M}^+) = 0$ (see Definition 4.1 below).

To this aim, we introduce the energy on the whole space:

$$J(m, w) = \int_{-\infty}^{+\infty} \int_{\mathbb{R}^d} \frac{1}{2} \left| \frac{dw}{dt \otimes m(t, dx)} \right|^2 m(t, dx)dt + \int_{-\infty}^{+\infty} \mathcal{W}(m(t)) \, dt,$$

and the sets of curves

$$\mathcal{K}^{\rho,S} := \left\{ (m, w) \in \mathcal{K}^\rho : \ m(-t) = \gamma_\# m(t) \ \text{for all } t, \ J(m, w) < +\infty \right\}. \tag{1.13}$$

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Note that (see Lemma 4.2), if $(m, w) \in \mathcal{K}^{\rho, S}$, then

$$\lim_{t \to \pm\infty} d_2\left(m(t), \mathcal{M}^{\pm}\right) = 0.$$

We have the following result.

Theorem 1.2. *Assume* (**Z**)*,* (**lsc**)*,* (**BDD**)*,* (**CON**) *and* (**REF**)*.*

a) *There exists a minimizer* $(m, w) \in \mathcal{K}^{\rho, S}$ *of the problem* $\min_{\mathcal{K}^{\rho, S}} J$*.*

b) *For any* $T > 0$*, let* $(m^T, w^T) \in \mathcal{K}_T^{\rho, S}$ *be a minimizer of* J_T *constructed in Theorem 1.1. Then*

$$\lim_{T \to +\infty} d_2^2\left(m^T\left(\frac{T}{4}\right), \mathcal{M}^+\right) = 0 = \lim_{T \to +\infty} d_2^2\left(m^T\left(-\frac{T}{4}\right), \mathcal{M}^-\right),$$

and up to passing to subsequences $T_n \to +\infty$*, there holds*

$$m^{T_n} \to \hat{m} \qquad \text{locally uniformly in } C(\mathbb{R}, \mathcal{P}_p(\mathbb{R}^d)) \text{ for all } p < 2$$
$$w^{T_n} \to \hat{w} \qquad \text{weakly in } L^2([-L, L] \times \mathbb{R}^d) \text{ for all } L > 0,$$

and

$$(\hat{m}, \hat{w}) \in \mathcal{K}_T^{\rho, S} \text{ is a minimizer of the problem } \min_{\mathcal{K}^{\rho, S}} J,$$

$$J(\hat{m}, \hat{w}) = \frac{1}{2} \lim_{T \to +\infty} J_T(m^T, w^T).$$

As a consequence, we obtain that minimizers obtained above are heteroclinic connections for the MFG in the sense of Definition 4.1.

We make a few final remarks in light of the two results. As we previously mentioned, the unique minimizer of $\mathcal{W}$ is an attractor of MFG equilibria under the monotonicity assumption (1.7). If one drops (1.7), the picture may then change substantially. Heteroclinics produced here connect two different minimizers of $\mathcal{W}$; hence, the state of the system can be arbitrarily close to a minimum (with respect to d_2) of $\mathcal{W}$, and converge to a different steady state as $t \to \infty$. A further study of stability of minimizers of $\mathcal{W}$ can be matter of future work.

Note again that brake orbits and heteroclinics obtained in Theorems 1.1 and 1.2 yield solutions to MFG systems of the form (1.1) in a suitable weak sense. The connection between the variational formulation and the PDE system for first order problems has been extensively studied in [8], and adaptions to our framework may require minor technical work.

We finally note that to obtain Theorems 1.1 and 1.2 it is not really necessary to work with absolutely continuous measures that are bounded in L^∞, that is on $\mathcal{P}_{2,\rho}^r(\mathbb{R}^d)$. Using the same techniques in Wasserstein spaces, it would be possible to restate them in $\mathcal{P}_2$ (reformulating accordingly all the assumptions). We present anyway our general results in $\mathcal{P}_{2,\rho}^r(\mathbb{R}^d)$ to accomodate the following "aggregation" model, that is a main motivation of this work. Without the density constraint, such a model would become probably more trivial, and surely more far from describing motions of crowds (because of likely formation of singularities). Similarly, it is not really necessary to set up everything on the whole euclidean space: analogous results could be proven on bounded domains or for periodic in space probabilities. Still, we wanted to modify the setting of previous works, to show that non-trivial long time patterns may arise in a non-periodic environment (and without the presence of viscosity).

1.1 A model problem

Finally, we present a model problem where our results apply. We consider a vatiational MFG where the potential term $\mathcal{W}$ is given by

$$\mathcal{W}(m) = \int_{\mathbb{R}^d} W(x) m(dx) - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(|x - y|) m(dx) m(dy). \tag{1.14}$$

Note that in this case $f(x,m) = \delta_m \mathcal{W} = W(x) - 2\int_{\mathbb{R}^d} K(|x-y|)m(dy)$. The first part of the energy is a potential energy, where $W : \mathbb{R}^d \to [0, +\infty)$ is a "double-well" confining function, symmetric under a reflection γ, vanishing on two disjoint balls B_1, B_2 (with $B_1 = \gamma(B_2)$), and quadratically increasing at infinity: see the assumption (5.2). The function $W(\cdot)$ models the spatial preference of the agents. The second part of the energy is an interaction energy, modeled through the interaction kernel $-K$. K is assumed to be positive definite, radially symmetric, locally integrable and increasing at zero (in an appropriate sense), see (5.5) and (5.6). In particular a model class of such interaction kernels K is given by the Riesz kernels

$$K(|x-y|) = \frac{1}{|x-y|^{d-\alpha}}, \quad \text{with } \alpha \in (0, d). \tag{1.15}$$

For instance, (1.14) can be associated to a crowd of agents that aim at minimizing their reciprocal distances (with spatial preference W). Note that the *aggregation* effect plays against the hard density constraint on m: no further aggregation is possible whenever $m = \rho$. Energies like (1.14) have been recently studied extensively, as they are directly connected to a class of self-assembly/aggregation models which have received much attention, see e.g. [13] and references therein.

It is possible to show, see Section 5, that under the previous assumptions on $W, K, \mathcal{W}$ defined in (1.14) satisfies (**BDD**), (**lsc**), (**CON**), (**REF**). Regarding the general assumption (**Z**), we are indeed able to provide a full description of minimizers of $\mathcal{W}$, and therefore our results apply. Informally speaking, the description of minimizers can be stated as follows. For precise statements, see Theorem 5.9 and Proposition 5.10.

Theorem 1.3. *Under the assumptions that W is symmetric, vanishing on two disjoint balls and with quadratic growth, and K behaves like (1.15) (i.e. (5.2), (5.3), (5.5), (5.6)), there exist minimizers of (1.14) in $\mathcal{P}_{2,\rho}^r(\mathbb{R}^d)$, and all the minimizers are given by characteristic functions (multiplied by ρ) of compact sets in $\mathbb{R}^d$.*

If, in addition, the flat zones of minima of W are sufficiently large in terms of ρ (i.e. (5.4) holds), then all the minimizers consists of two compact disjoint sets $\mathcal{M}^\pm \subset \mathcal{P}_{2,\rho}^r(\mathbb{R}^d)$, symmetric with respect to each other, whose elements are characteristic functions of balls (multiplied by ρ).

So, in the case described in Theorem 1.3, Theorems 1.1 and Theorem 1.2 apply and we may construct brake orbits and heteroclinic solutions.

Some interesting issues, in our opinion, are left open for this model problem. In particular, we know by Theorem 1.3 that stationary minimal solutions to the MFG problem are given by characteristic functions, i.e. $m = \rho\chi_E$, where E is a compact set. A natural question is whether or not minimizers (or even critical points) of the evolutive energy (1.5) enjoy these two features; that is, time-dependent equilibria have compact support and are evolving characteristic functions. In other words, given a periodic brake orbit (m^T, w^T) as in Theorem 1.1, or a minimal heteroclinic connection (m, w) as in Theorem 1.2, is it true that $m^T(t), m(t)$ are characteristic functions of a family of evolving compact sets E_t for all times ? At the moment a full answer to this question seems far to be reached.

Another natural related problem is the version of the game with a large (but finite) number of players: MFG can be interpreted (and derived) indeed as limiting models for large populations of interacting agents, where any given individual is affected by the averaged state of the other individuals. In the companion work [11], we consider the analogous variational problem involving the energy (1.2) of a finite number of interacting particles, where the density constraint appears as a bound from below on the minimal distance between particles (being in turn inversely proportional to the number of particles N). First of all, we formalize the connection between the discrete N-particles problem and the continuous MFG model by proving a Γ-convergence type result, as $N \to +\infty$, of the energies, in the same spirit of [17]. Moreover, we show that for the N-particle system, at least in the 1-dimensional case, periodic minimizers are compactly supported, and particles minimize reciprocal distances. This gives a partial answer

to our question (again, at least in dimension one), namely we provide the existence of limiting brake orbits for the continuous problem that are time-dependent characteristic functions.

Acknowledgements

Notation

We wil denote by $B(x,r) \subseteq \mathbb{R}^d$ the ball centered at x and with radius r, $B_r = B(0,r)$ and $\omega_d = |B_1|$. For any measurable set $E \subseteq \mathbb{R}^d$, we define χ_E to be the characteristic function of E.
$\mathcal{P}(\mathbb{R}^d)$, $\mathcal{P}_p(\mathbb{R}^d)$ and $\mathcal{P}_{2,\rho}^r(\mathbb{R}^d)$ are (sub)sets of Borel probability measures defined below (see Section 2). For any set $\mathcal{M} \subset \mathcal{P}_2(\mathbb{R}^d)$, $d_2(\mu, \mathcal{M}) = \inf_{m \in \mathcal{M}} d_2(\mu, m)$.

2 The Wasserstein spaces

We introduce some notions for calculus in Wasserstein spaces that will be useful in the following. For a general reference on these results we refer to [2], [32]. First, let $\mathcal{P}(\mathbb{R}^d)$ be the space of Borel probability measures on $\mathbb{R}^d$, endowed with the topology of narrow convergence, that is:

Definition 2.1. *Let $\mu_k, \mu \in \mathcal{P}(\mathbb{R}^d)$. We say that $\mu_k \to \mu$ narrowly if*

$$\lim_k \int_{\mathbb{R}^d} g(x)\mu_k(dx) = \int_{\mathbb{R}^d} g(x)\mu(dx) \qquad \forall g \in C_b(\mathbb{R}^d), \tag{2.1}$$

where $C_b(\mathbb{R}^d)$ is the space of continuous and bounded functions on $\mathbb{R}^d$.

Note that this notion of convergence is equivalent to the one of convergence in the sense of distributions (see [2, Remark 5.1.6]), where one has (2.1) for all smooth and compactly supported test functions $g \in C_0^\infty(\mathbb{R}^d)$.

Definition 2.2. *Let $p \geq 1$. The Wasserstein space of Borel probability measures with bounded p-moments is defined by*

$$\mathcal{P}_p(\mathbb{R}^d) = \left\{ \mu \in \mathcal{P}(\mathbb{R}^d) \ \Big| \ \int_{\mathbb{R}^d} |x|^p d\mu(x) < +\infty \right\}.$$

The Wasserstein space can be endowed with the p-Wasserstein distance

$$d_p^p(\mu, \nu) = \inf \left\{ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |x-y|^p d\pi(x,y) \mid \pi \in \Pi(\mu,\nu) \right\} \tag{2.2}$$

where $\Pi(\mu, \nu)$ is the set of Borel probability measures π on $\mathbb{R}^d \times \mathbb{R}^d$ such that $\pi(A \times \mathbb{R}^d) = \mu(A)$ and $\pi(\mathbb{R}^d \times A) = \nu(A)$ for any Borel set $A \subseteq \mathbb{R}^d$.

Note that $\mathcal{P}_q(\mathbb{R}^d) \subset \mathcal{P}_p(\mathbb{R}^d)$ for $p < q$, and by Jensen inequality, $d_p(\mu, \nu) \leq d_q(\mu, \nu)$ for $p < q$. We then recall the following results about narrow convergence and convergence in Wasserstein spaces.

Lemma 2.3. *Let $\mu_k, \mu \in \mathcal{P}(\mathbb{R}^d)$ such that μ_k converges to μ narrowly.*

(i) *Let $g : \mathbb{R}^d \to [0, +\infty]$ be lower semicontinuous. Then*

$$\liminf_k \int_{\mathbb{R}^d} g(x) d\mu_k(x) \geq \int_{\mathbb{R}^d} g(x) d\mu(x).$$

(ii) *Let $g : \mathbb{R}^d \to [0, +\infty)$, continuous and μ_k-integrable, be such that*

$$\limsup_k \int_{\mathbb{R}^d} g(x) d\mu_k(x) \leq \int_{\mathbb{R}^d} g(x) d\mu(x) < \infty.$$

Then, g is uniformly integrable with respect to μ_k, that is

$$\lim_{R \to +\infty} \sup_k \int_{\{x \,|\, g(x) \geq R\}} g(x) d\mu_k(x) = 0.$$

Proof. We refer to [2, Lemma 5.1.7]. □

Lemma 2.4. *$\mathcal{P}_p(\mathbb{R}^d)$ endowed with the p-Wasserstein distance is a separable complete metric space. A set $\mathcal{M} \subset \mathcal{P}_p(\mathbb{R}^d)$ is relatively compact if and only if it has uniformly integrable p-moments, that is*

$$\lim_{R \to +\infty} \sup_{\mu \in \mathcal{M}} \int_{\mathbb{R}^d \setminus B(0,R)} |x|^p d\mu(x) = 0.$$

Let now $\mu_k, \mu \in \mathcal{P}_p(\mathbb{R}^d)$ for some $p \geq 1$. Then the statements below are equivalent:

(i) *$d_p(\mu_k, \mu) \to 0$*

(ii) *μ_k converges to μ narrowly and μ_k have uniformly integrable p-moments.*

Finally, for any $\nu \in \mathcal{P}_p(\mathbb{R}^d)$, the map $\mu \to d_p(\mu, \nu)$ is lower semicontinuous with respect to narrow convergence.

Proof. We refer to [2, Prop. 7.1.5]. Note that if $\mathcal{M}$ has uniformly integrable p-moments then it is tight, i.e. for all $\varepsilon > 0$ there exists $K_\varepsilon \subseteq \mathbb{R}^d$ compact for which $\sup_{\mu \in \mathcal{M}} \int_{\mathbb{R}^d \setminus K_\varepsilon} d\mu(x) \leq \varepsilon$.

The lower semicontinuity of the Wasserstein distance is proved in [2, Proposition 7.1.3]. □

Remark 2.5. Note that, if for some $q > p$,

$$\sup_{\mu \in \mathcal{M}} \int_{\mathbb{R}^d} |x|^q d\mu(x) < +\infty$$

then $\mathcal{M}$ has uniformly integrable p-moments.

Finally, we introduce a subspace of regular measures as follows.

Definition 2.6. *We define $\mathcal{P}^r_{2,\rho}(\mathbb{R}^d)$ to be the set of measures belonging to $\mathcal{P}_2(\mathbb{R}^d)$ and having density in $L^\infty(\mathbb{R}^d)$, with $L^\infty(\mathbb{R}^d)$-norm bounded by $\rho > 0$:*

$$\mathcal{P}^r_{2,\rho}(\mathbb{R}^d) = \{m \in \mathcal{P}_2(\mathbb{R}^d) \,:\, \exists\, 0 \leq \tilde{m} \leq \rho \text{ a.e. on } \mathbb{R}^d \text{ s.t. } m = \tilde{m} dx\}.$$

Since we will work with measures with density in L^∞, we recall here the notion (or characterization) of weak-* convergence in L^∞: for $\mu_k, \mu \in L^\infty(\mathbb{R}^d)$, μ_k is said to converge to μ weak-* in L^∞ if

$$\lim_k \int_{\mathbb{R}^d} g(x) \mu_k(dx) = \int_{\mathbb{R}^d} g(x) \mu(dx) \qquad \forall g \in L^1(\mathbb{R}^d).$$

We now make a few considerations on the kinetic part of the energy in (1.5), that is on the functional

$$(m, w) \mapsto \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \frac{1}{2} \left| \frac{dw}{dt \otimes m(t, dx)} \right|^2 m(t, dx) dt,$$

which can be defined in general for couples (m, w), $m \in C(\mathbb{R}, \mathcal{P}_1(\mathbb{R}^d))$ and w a Borel d-vector measure on $\mathbb{R} \times \mathbb{R}^d$, absolutely continuous w.r.t. $dt \otimes m(t, dx)$. These properties are indeed part of the definition of admissible couples $(m, w) \in \mathcal{K}$. Throughout the paper, $m(t)$ will further satisfy the L^∞ constaint $m(t) \in \mathcal{P}_{2,\rho}^r(\mathbb{R}^d)$. We immediately note that if $(m, w) \in \mathcal{K}^\rho$, then w has a density which is in $L_{\text{loc}}^2(\mathbb{R}, L^2(\mathbb{R}^d))$, that is

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^d} |w|^2 dx dt = \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \left| \frac{w(t, x)}{m(t, x)} \right|^2 m^2(t, x) \, dx dt \leq \rho \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \left| \frac{w(t, x)}{m(t, x)} \right|^2 m(t, x) \, dx dt.$$

Moreover, by Hölder inequality and recalling that $m(t) \in \mathcal{P}_2(\mathbb{R}^d)$, we have

$$\left(\int_{t_1}^{t_2} \int_{\mathbb{R}^d} |x| |w(t, x)| dx dt \right)^2 \leq \left(\int_{t_1}^{t_2} \int_{\mathbb{R}^d} |x|^2 m(t, x) dx dt \right) \left(\int_{t_1}^{t_2} \int_{\mathbb{R}^d} \left| \frac{w(t, x)}{m(t, x)} \right|^2 m(t, x) \, dx dt \right) < \infty.$$

We now state a lower semi-continuity result (which could be stated for weaker convergence in the variables m, w, but it will be used below in the present form).

Proposition 2.7. *Suppose that $m_n \to m$ in $C(\mathbb{R}, \mathcal{P}_p(\mathbb{R}^d))$ for some $p \geq 1$, $w_n \rightharpoonup w$ (weakly) in $L^2((-\infty, \infty) \times \mathbb{R}^d)$, and $m_n(t), m(t)$ are absolutely continuous with respect to the Lebesgue measure for all t, n. Then,*

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^d} \left| \frac{w(t, x)}{m(t, x)} \right|^2 m(t, x) dx dt \leq \liminf_{n \to \infty} \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \left| \frac{w_n(t, x)}{m_n(t, x)} \right|^2 m_n(t, x) dx dt.$$

Proof. See [32, Proposition 5.18]. $\qquad\square$

Finally, we recall the following uniform continuity property of elements belonging to $\mathcal{K}$, that will be useful in the sequel.

Proposition 2.8. *Let $(m, w) \in \mathcal{K}$, as defined in (1.6). Then*

$$\left(d_2(m(t), m(s)) \right)^2 \leq (t - s) \int_s^t \int_{\mathbb{R}^d} \left| \frac{dw}{dt \otimes m(\tau, dx)} \right|^2 m(\tau, dx) d\tau, \qquad \forall t > s. \tag{2.3}$$

Proof. This can be proved using Hölder inequality and [2, Thm 8.3.1]. $\qquad\square$

3 Brake periodic solutions

Let us first define brake orbits. For the sake of clarity, let us summarize first our constraint sets, from larger to smaller:

$\mathcal{K}$: the set of admissible flows of mass-momentum (m, w), def. in (1.5) $\supset$

$\mathcal{K}^\rho$: flows such that $m(t)$ is bounded in $L^\infty(\mathbb{R}^d)$, def. in (1.6) $\supset$

$\mathcal{K}_T^\rho$: T-periodic in time flows, def. in (1.12) $\supset$

$\mathcal{K}_T^{\rho,S}$: T-periodic and symmetric flows, def. in (1.10).

Definition 3.1. *Let $(\bar{m}, \bar{w}) \in \mathcal{K}^\rho$. We say that $(\bar{m}, \bar{w})$ is a **brake orbit** for the MFG if $(\bar{m}, \bar{w}) \in \mathcal{K}_T^{\rho,S}$ and*

$$\int_0^T \int_{\mathbb{R}^d} \left| \frac{\bar{w}(t, x)}{\bar{m}(t, x)} \right|^2 \bar{m}(t, x) + f(x, \bar{m}(t)) \bar{m}(t, x) dx dt \leq$$

$$\int_0^T \int_{\mathbb{R}^d} \left| \frac{w(t, x)}{m(t, x)} \right|^2 m(t, x) + f(x, \bar{m}(t)) m(t, x) dx dt \qquad \forall (m, w) \in \mathcal{K}_T^\rho.$$

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In other words, a brake orbit is a periodic flow enjoying symmetries, which is an Nash equilibrium in a suitable Mean Field sense, i.e. a minimizer of an infinite-dimensional control problem among *non-symmetric* competitors.

We now turn to the proof of Theorem 1.1. Denote by $Q = (-\infty, \infty) \times \mathbb{R}^d$ and, for any $T > 0$, $C_T((-\infty, \infty), \mathcal{P}_2(\mathbb{R}^d))$ be the subset of T-periodic elements of $C((-\infty, \infty), \mathcal{P}_2(\mathbb{R}^d))$. We provide a preliminary result on the existence of constant speed geodesics in $\mathcal{K}^\rho$ based on displacement convexity introduced by McCann [28].

Lemma 3.2. *Let* $t_1 < t_2$, *and* $m_1, m_2 \in \mathcal{P}_{2,\rho}^r(\mathbb{R}^d)$. *Then, there exists a couple* $(m, w) \in \mathcal{K}^\rho$, *as defined in* (1.8), *such that*

$$m(t_1) = m_1, \qquad m(t_2) = m_2, \qquad \int_{\tau_1}^{\tau_2} \int_{\mathbb{R}^d} \left| \frac{w(t,x)}{m(t,x)} \right|^2 m(t,x)\, dx dt = \frac{d_2^2(m(\tau_1), m(\tau_2))}{\tau_2 - \tau_1}.$$

for all $t_1 \leq \tau_1 \leq \tau_2 \leq t_2$.

Proof. Let $\hat{m}$ be the unique constant speed geodesic $\hat{m} \in AC([0,1], \mathcal{P}_2(\mathbb{R}^d))$ (see [2, Section 7.2]) connecting m_1 and m_2 (i.e. $\hat{m}(0) = m_1$, $\hat{m}(1) = m_2$), which satisfies for all $0 \leq s_1 \leq s_2 \leq 1$

$$d_2(\hat{m}(s_1), \hat{m}(s_2)) = (s_2 - s_1) d_2(m_1, m_2),$$

that is,

$$|\hat{m}'|(s) = d_2(m_1, m_2) \qquad \text{for all } s \in (0,1).$$

The functional $m \mapsto \|m\|_{L^q(\mathbb{R}^d)}$ is *geodesically convex* in $\mathcal{P}_2(\mathbb{R}^d)$ for every $q > 1$ (see [32, Proposition 7.29]), namely $\hat{m}(s)$ is in $L^q(\mathbb{R}^d)$ for every $q > 1$ and every s, and it satisfies

$$\|\hat{m}(s)\|_{L^q(\mathbb{R}^d)} \leq \max\{\|m_1\|_{L^q(\mathbb{R}^d)}, \|m_2\|_{L^q(\mathbb{R}^d)}\} \leq \max\{\|m_1\|_{L^\infty(\mathbb{R}^d)}^{1-1/q}, \|m_2\|_{L^\infty(\mathbb{R}^d)}^{1-1/q}\} \leq \rho^{1-1/q}.$$

So, by Chebycheff inequality, for all $c > 0$ and all $q > 1$ we have $|\{x \in \mathbb{R}^d : \hat{m}(s) > c\}|^{1/q} \leq c^{-1}\rho^{1-1/q}$, and therefore $\hat{m}(s) \in L^\infty(\mathbb{R}^d)$ with $\|\hat{m}(s)\|_{L^\infty(\mathbb{R}^d)} \leq \rho$.

Being $\hat{m}(t)$ a constant speed geodesic connecting m_1 and m_2 ,

$$d_2^2(\hat{m}(s_1), \hat{m}(s_2)) = (s_2 - s_1)^2 \big(d_2(m_1, m_2)\big)^2 = (s_2 - s_1) \int_{s_1}^{s_2} |\hat{m}'|^2(s)\, ds \qquad \text{for all } 0 \leq s_1 \leq s_2 \leq 1.$$

$$(3.1)$$

By [2, Thm 8.3.1] we get for a.e. $s \in (0,1)$ the existence of a vector field $\hat{v}(s) \in L^2(\hat{m}(s); \mathbb{R}^d)$ such that $-\partial_t \hat{m} + \text{div}(\hat{v}\hat{m}) = 0$ is satisfied in the distributional sense, and for a.e. s

$$|\hat{m}'|(s) = \left(\int_{\mathbb{R}^d} |\hat{v}(s,x)|^2 \hat{m}(s,x) dx \right)^{1/2}.$$

Hence, substituting $|\hat{m}'|(s)$ into (3.1) and setting $\hat{w} = \hat{v}\hat{m}$ (on the set $\{m > 0\}$, and identically zero elsewhere), we obtain

$$d_2^2(\hat{m}(s_1), \hat{m}(s_2)) = (s_2 - s_1) \int_{s_1}^{s_2} \int_{\mathbb{R}^d} \left| \frac{\hat{w}(s,x)}{\hat{m}(s,x)} \right|^2 \hat{m}(s,x)\, dx ds.$$

We then have that the couple $(\hat{m}, \hat{w})$ belongs to $\mathcal{K}$. To obtain the required couple (m, w) it is enough to perform a linear change of variables, i.e.

$$m(t,x) := \hat{m}\left(\frac{t - t_1}{t_2 - t_1}, x \right), \quad w(t,x) := \frac{1}{t_2 - t_1} \hat{w}\left(\frac{t - t_1}{t_2 - t_1}, x \right).$$

Finally we extend $m(t,x)$ to all $t \in \mathbb{R}$ by setting $m(t,x) = m_1(x)$, $w(t,x) = 0$ for $t < t_1$, and $m(t,x) = m_2(x)$, $w(t,x) = 0$ for $t > t_2$. $\qquad \square$

Now we need a technical lemma about positivity properties of the functional $\mathcal{W}$ outside $\mathcal{M}^{\pm}$.

Lemma 3.3. *For any $q > 0$, we have*

$$\inf\left\{\mathcal{W}(m) \mid m \in \mathcal{P}_{2,\rho}^r(\mathbb{R}^d),\ d_2(m, \mathcal{M}^{\pm}) \geq q\right\} =: \delta(q, \mathcal{W}) > 0. \tag{3.2}$$

Proof. Assume by contradiction that there exists $q > 0$ for which $\delta = 0$. We consider $m_n \in \mathcal{P}_{2,\rho}^r(\mathbb{R}^d)$ such that $q \leq d_2(m_n, \mathcal{M}^{\pm})$ and $0 \leq \mathcal{W}(m_n) \leq 1/n$. By the lower bound in the assumption (**BDD**) and Lemma 2.4, we have that the sublevel set $\mathcal{W}(m) \leq 1$ is compact in $\mathcal{P}_p(\mathbb{R}^d)$, for any $p < 2$, so by (**lsc**) and (**Z**), we conclude that

$$\lim_n \mathcal{W}(m_n) = 0.$$

Hence, by the continuity property (**CON**) we get that $\lim_n d_2(m_n, \mathcal{M}^{\pm}) = 0$, which is in contradiction with the fact that $d_2(m_n, \mathcal{M}^{\pm}) \geq q$. $\qquad\square$

The next lemma 3.4 is crucial, and roughly states the following: suppose that $(m, w) \in \mathcal{K}^{\rho}$ is a (bounded energy) competitor, and that $m(t)$ is sufficiently close to $\mathcal{M}^+$ (resp. $\mathcal{M}^-$) at some times t_1, t_2. Then, if it does not remain close to $\mathcal{M}^+$ (resp. $\mathcal{M}^-$) in the whole time interval $[t_1, t_2]$, it is possible to modify it to decrease its energy. The lemma is based on a cut argument, which has been already used in the analysis of periodic orbits and heteroclinic connections for Hamiltonian systems, see e.g. [18, Lemma 2.1].

Lemma 3.4. *Let $0 < t_1 < t_2 < T$. Let $(m, w) \in \mathcal{K}^{\rho}$ and assume that (m, w) satisfies*

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^d} \left|\frac{w(t,x)}{m(t,x)}\right|^2 m(t,x)\, dx\, dt \leq C'.$$

*Let now $q \in (0, q_0]$, where q_0 is as in (**Z**). Then, there exists $q' = q'(q, C', \mathcal{W})$ such that, if the following conditions are fulfilled for some $\bar{m}^+ \in \mathcal{M}^+$ and $t^* \in (t_1, t_2)$:*

- $d_2(m(t_1), \bar{m}^+) \leq q'$, $\quad d_2(m(t_2), \bar{m}^+) \leq q'$,
- $d_2(m(t^*), \mathcal{M}^+) > q$,

then there exists $(\mu, v) \in \mathcal{K}^{\rho}$ with the following properties:

- $(\mu(t), v(t)) = (m(t), w(t)) \quad$ *for all $t \in \mathbb{R} \setminus (t_1, t_2)$,*
- $d_2(\mu(t), \mathcal{M}^+) < q \quad$ *for all $t \in (t_1, t_2)$,*
- $\displaystyle\int_{t_1}^{t_2} \int_{\mathbb{R}^d} \frac{|v(t,x)|^2}{\mu(t,x)}\, dx\, dt + \int_{t_1}^{t_2} \mathcal{W}(\mu(t))\, dt < \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \frac{|w(t,x)|^2}{m(t,x)}\, dx\, dt + \int_{t_1}^{t_2} \mathcal{W}(m(t))\, dt.$

Proof. For any $0 < q' < q/2$ set

$$\tau_1 = \max\{t > t_1 : d_2(m(s), \mathcal{M}^+) \leq q, \quad \text{for all} \quad s \leq t\},$$
$$\tau_1' = \max\{t < \tau_1 : d_2(m(t), \mathcal{M}^+) \leq q'\}.$$

It holds $t_1 \leq \tau_1' < \tau_1 < t^* < t_2$, and $q' \leq d_2(m(t), \mathcal{M}^+) \leq q$ for all $t \in [\tau_1', \tau_1]$. Note that, by (2.3) and the triangle inequality,

$$(C'(\tau_1 - t_1))^{1/2} \geq d_2(m(\tau_1), m(t_1)) \geq d_2(m(\tau_1), \bar{m}^+) - d_2(m(t_1), \bar{m}^+)$$
$$\geq d_2(m(\tau_1), \mathcal{M}^+) - d_2(m(t_1), \bar{m}^+) \geq q - q' > \frac{q}{2},$$
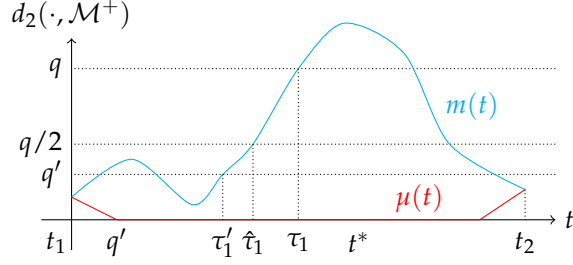
Figure 1: The construction of the alternative competitor (μ, v).

hence

$$t_2 > \tau_1 > t_1 + \frac{q^2}{4C'}.$$

We construct (μ, v) as follows. Choose $0 < q' < \frac{1}{2} \min\left\{ \frac{q^2}{4C'}, q \right\}$. By means of Lemma 3.2, there are two couples (m_1, w_1) and (m_2, w_2) belonging to $\mathcal{K}^\rho$ which connect $m(t_1)$ to $\bar{m}^+$ at time $t_1 + q'$ and $\bar{m}^+$ at time $t_2 - q'$ to $m(t_2)$, respectively. Set then

$$\mu(t) := \begin{cases} m_1(t) & t \in [t_1, t_1 + q'], \\ \bar{m}^+ & t \in [t_1 + q', t_2 - q'], \\ m_2(t) & t \in [t_2 - q', t_2], \\ m(t) & \text{otherwise} \end{cases} \qquad v(t) := \begin{cases} w_1(t) & t \in [t_1, t_1 + q'], \\ 0 & t \in [t_1 + q', t_2 - q'], \\ w_2(t) & t \in [t_2 - q', t_2], \\ w(t) & \text{otherwise} \end{cases}$$

The constraint $(\mu, v) \in \mathcal{K}^\rho$ is easily verified. Note that for $t \in [t_1, t_1 + q']$, since $\mu(t)$ is a constant speed geodesic connecting $\mu(t_1) = m(t_1)$ and $\mu(t_1 + q') = \bar{m}^+$,

$$d_2(\mu(t), \mathcal{M}^+) \le d_2(\mu(t), \bar{m}^+) \le d_2(\mu(t_1), \bar{m}^+) = d_2(m(t_1), \bar{m}^+) \le q'.$$

The same inequalities holds on $t \in [t_2 - q', t_2]$, hence

$$d_2(\mu(t), \mathcal{M}^+) \le q' \le q \qquad \text{for all } t \in (t_1, t_2). \tag{3.3}$$

Since $\mathcal{M}^+$ is compact,

$$\int_{\mathbb{R}^d} |x|^2 \mu(t, x) dx = d_2(\mu(t), \delta_0) \le d_2(\mu(t), \mathcal{M}^+) + d_2(\mathcal{M}^+, \delta_0) \le \hat{c} + q,$$

for some $\hat{c} > 0$.

Therefore, by Lemma 3.2 and the growth assumption on $\mathcal{W}$ given by (**BDD**), we get

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^d} \frac{|v|^2}{\mu} dx dt + \int_{t_1}^{t_2} \mathcal{W}(\mu(t)) dt = \tag{3.4}$$

$$= \int_{t_1}^{t_1 + q'} \int_{\mathbb{R}^d} \frac{|v_1|^2}{m_1} dx dt + \int_{t_1}^{t_1 + q'} \mathcal{W}(m_1(t)) dt + \int_{t_2 - q'}^{t_2} \int_{\mathbb{R}^d} \frac{|v_2|^2}{m_2} dx dt + \int_{t_2 - q'}^{t_2} \mathcal{W}(m_2(t)) dt$$

$$= \frac{d_2^2(m(t_1), \bar{m}^+)}{q'} + \int_{t_1}^{t_1 + q'} \mathcal{W}(m_1(t)) dt + \frac{d_2^2(m(t_2), \bar{m}^+)}{q'} + \int_{t_2 - q'}^{t_2} \mathcal{W}(m_2(t)) dt$$

$$\le 2q' + 2q' C_{\mathcal{W}}(1 + \hat{c} + q).$$

We now introduce a further intermediate time $\hat{\tau}_1 := \max\{t < \tau_1 : d_2(m(t), \mathcal{M}^+) \le q/2\}$. It holds $\tau_1' < \hat{\tau}_1 < \tau_1$, and $q/2 \le d_2(m(t), \mathcal{M}^+) \le q$ for all $t \in [\hat{\tau}_1, \tau_1]$.

By the triangular inequality and the compactness of $\mathcal{M}^\pm$, recalling the definition of q_0, we get $d_2(m(t), \mathcal{M}^-) \ge 2q_0 - d_2(m(t), \mathcal{M}^+) \ge 2q_0 - q \ge q$ for all $t \in [\hat{\tau}_1, \tau_1]$. Therefore by Lemma

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3.3, we get that there exists $\delta = \delta(q/2, \mathcal{W})$ such that $\mathcal{W}(m(t)) \geq \delta > 0$ for all $t \in [\hat{\tau}_1, \tau_1]$. Recall that $(m, w) \in \mathcal{K}$, so [2, Theorem 8.3.1] guarantees that $\int_{\mathbb{R}^d} \frac{|w(t)|^2}{m(t)} dx \geq (|m'|(t))^2$ for a.e. t. Hence, by Young's inequality and the triangle inequality,

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^d} \frac{|w|^2}{m} \, dx dt + \int_{t_1}^{t_2} \mathcal{W}(m(t)) \, dt \geq \int_{\hat{\tau}_1}^{\tau_1} \int_{\mathbb{R}^d} \frac{|w|^2}{m} \, dx dt + \int_{\hat{\tau}_1}^{\tau_1} \mathcal{W}(m(t)) dt$$

$$\geq \sqrt{2} \int_{\hat{\tau}_1}^{\tau_1} \left(\int_{\mathbb{R}^d} \frac{|w|^2}{m} \, dx \right)^{1/2} \sqrt{\mathcal{W}(m(t))} \, dt \geq \sqrt{2\delta} \int_{\hat{\tau}_1}^{\tau_1} |m'|(t) \, dt = \sqrt{2\delta} \, d_2(m(\hat{\tau}_1), m(\tau_1))$$

$$\geq \sqrt{2\delta} \big(d_2(m(\tau_1), \mathcal{M}^+) - d_2(m(\hat{\tau}_1), \mathcal{M}^+) \big) = \frac{q}{2} \sqrt{2\delta}.$$

Combining this inequality with (3.4) we complete the proof of the lemma, decreasing eventually q' so that $2q' + 2q' C_{\mathcal{W}}(1 + \hat{c} + q) < \frac{q}{2}\sqrt{2\delta}$. □

We are now ready to construct T-periodic minimizers of J_T. We restrict the class $\mathcal{K}$ to flows of probability measures that are T-periodic and enjoy additional symmetries, so we introduce the set $\mathcal{K}_T^{\rho, S}$ as defined in (1.10). We observe that the second symmetry constraint $m(-t) = \gamma m(t)$ rules out orbits which remain for all time in $\mathcal{M}^+$ or in $\mathcal{M}^-$. The first symmetry constraint $m(T/4 + t) = m(T/4 - t)$ is due to the fact that we are looking for brake periodic orbits, which oscillate twice in a period between $\mathcal{M}^+$ and $\mathcal{M}^-$. Note that we are using the notation $\gamma m(t) = \gamma_\# m(t)$; since $m(t)$ has a density, this means that $(\gamma m)(t, x) = m(-t, \gamma(x))$ a.e..

We provide now the proof of the first main result, that is Theorem 1.1.

Proof of Theorem 1.1. Step 1: Energy bounds. Choose any $m_0 \in \mathcal{P}_{2,\rho}^r(\mathbb{R}^d)$ with compact support such that $m_0 = \gamma m_0$. Observe that by (**REF**) we have $\gamma \mathcal{M}^+ = \mathcal{M}^-$. Since d_2 is preserved by the transformation γ, we can define

$$h := d_2(m_0, \mathcal{M}^+) = d_2(m_0, \mathcal{M}^-).$$

Let $\bar{m}_+ \in \mathcal{M}^+$, such that $d_2(m_0, \mathcal{M}^+) = d_2(m_0, \bar{m}_+)$. So $d_2(m_0, \mathcal{M}^-) = d_2(m_0, \gamma \bar{m}_+)$. By Lemma 3.2, there exists a couple $(m, w) \in \mathcal{K}^\rho$ that connects m_0 at time $t = 0$ to $\bar{m}_+$ at time $t = 1$. Let $T > 4$, and for $t \in [0, T/2]$,

$$\tilde{m}(t) := \begin{cases} m(t) & t \in [0, 1], \\ \bar{m}_+ & t \in [1, T/2 - 1], \\ m(T/2 - t) & t \in [T/2 - 1, T/2] \end{cases} \qquad \tilde{w}(t, x) := \begin{cases} w(t, x) & t \in [0, 1], \\ 0 & t \in [1, T/2 - 1], \\ -w(T/2 - t, x) & t \in [T/2 - 1, T/2]. \end{cases}$$

Observe that $d_2(\tilde{m}(t), \mathcal{M}^+) \leq d_2(\tilde{m}(t), \bar{m}_+) \leq h$ for all $t \in [0, T/2]$. On the interval $[-T/2, 0]$, $(\tilde{m}, \tilde{w})$ can be extended symmetrically:

$$(\tilde{m}(t), \tilde{w}(t)) := (\gamma \tilde{m}(-t), -\gamma \tilde{w}(-t)),$$

Finally, $(\tilde{m}, \tilde{w})$ can be extended periodically over the whole time interval, so $(\tilde{m}, \tilde{w}) \in \mathcal{K}_T^{\rho, S}$.

Moreover we compute, recalling Lemma 3.2, and the growth condition (**BDD**) on $\mathcal{W}$,

$$0 \leq J_T(\tilde{m}, \tilde{w}) = 4 \int_0^1 \int_{\mathbb{R}^d} \left| \frac{\tilde{w}(t, x)}{\tilde{m}(t, x)} \right|^2 \tilde{m}(t, x) \, dx dt + 4 \int_0^1 \mathcal{W}(\tilde{m}(t)) \, dt$$

$$\leq 4d^2 + 4C_{\mathcal{W}}(1 + d_2^2(\tilde{m}(t), \delta_0)) \leq 4d^2 + 4C_{\mathcal{W}}(1 + (h + d_2(\mathcal{M}^+, \delta_0))^2) =: C'. \quad (3.5)$$

Note that $C' > 0$ does not depend on T. We may then suppose that along any minimizing sequence (m_n, w_n),

$$\int_0^T \int_{\mathbb{R}^d} \left| \frac{w_n(t, x)}{m_n(t, x)} \right|^2 m_n(t, x) \, dx dt \leq J_T(m_n, w_n) \leq C'. \quad (3.6)$$

Step 2: Minimizing sequences can be chosen to be close to $\mathcal{M}^{\pm}$. Pick any minimizing sequence $(m_n, w_n) \in \mathcal{K}_T^{\rho, S}$ of J_T. Fix now $n \in \mathbb{N}$. Let $q \in (0, q_0]$, and $0 < q' < q$ be as in Lemma 3.4 (with C' as in (3.6)).

Note that the triangle inequality, the invariance of d_2 under γ, $m_n(0) = \gamma m_n(0)$, $\mathcal{M}^+ = \gamma \mathcal{M}^-$ imply that $d_2(\mathcal{M}^+, m_n(0)) = d_2(\mathcal{M}^-, m_n(0))$ and then

$$2q' < 2q_0 = d_2(\mathcal{M}^+, \mathcal{M}^-) \leq 2d_2(\mathcal{M}^+, m_n(0)) = 2d_2(\mathcal{M}^-, m_n(0)).$$

Let $\delta(q') = \inf_{m \in \mathcal{P}_{2,\rho}^r(\mathbb{R}^d), d_2(m, \mathcal{M}^{\pm}) \geq q'} \mathcal{W}_0(m) > 0$, as in Lemma 3.3. Let $0 \leq s \leq T$. Note that if $d_2(m(t), \mathcal{M}^{\pm}) \geq q'$ for all $t \in [0, s]$, then this implies

$$s\delta(q') \leq \int_0^s \mathcal{W}(m_n(t)) \, dt \leq J_T(m_n, w_n) \leq C'.$$

Hence, for $T > \bar{s} := C'(\delta(q'))^{-1}$, by continuity of $m_n(t)$, since $d_2(m_n(0), \mathcal{M}^{\pm}) > q'$, there exists $s \in (0, \bar{s})$ such that

$$d_2(m_n(t), \mathcal{M}^{\pm}) > q' \quad \text{for all } t \in [0, s) \quad \text{and} \quad d_2(m_n(s), \mathcal{M}^{\pm}) = q'.$$

Let $\bar{m} \in \mathcal{M}^+ \cup \mathcal{M}^-$ such that $d_2(m_n(s), \bar{m}) = q'$. We may assume without loss of generality that $\bar{m} \in \mathcal{M}^+$ (the proof is completely analogous if $\bar{m} \in \mathcal{M}^-$).

So $d_2(m_n(s), \mathcal{M}^+) = d_2(m_n(s), \bar{m}) = q'$. Note that by symmetry of $m_n(t)$ we also have $d_2(m_n(T/2 - s), \mathcal{M}^+) = d_2(m_n(T/2 - s), \bar{m}) = q'$. Hence, if $d_2(m_n(t), \mathcal{M}^+) > q$ for some $t \in (s, T/2 - s)$, by Lemma 3.4 it is possible to modify (m_n, w_n) in $(s, T/2 - s)$ to construct a competitor (μ_n, v_n) with $J_T(\mu_n, v_n) < J_T(m_n, w_n)$. Therefore, we can further restrict the minimization process to competitors $(m, w) \in \mathcal{K}_T$ that satisfy for some s

$$\begin{cases} d_2(m(t), \mathcal{M}^+) < q & \forall t \in \left(s, \frac{T}{2} - s\right) \\ d_2(m(t), \mathcal{M}^-) < q & \forall t \in \left(s - \frac{T}{2}, -s\right). \end{cases} \tag{3.7}$$

Note that $0 < s \leq \bar{s} = C'(\delta(q'))^{-1}$ and that $T > C'(\delta(q'))^{-1} \to +\infty$ as $q \to 0$.

Step 3: Existence of a minimizer. By the growth condition (**BDD**), we get that there exists $t_n \in [0, T]$ such that $m_n(t_n)$ are uniformly bounded in $\mathcal{P}_2(\mathbb{R}^d)$ with respect to n. Moreover by (2.3)

$$d_2^2(m_n(t), m_n(s)) \leq C'|t - s| \tag{3.8}$$

for all $t, s \in [0, T]$. This implies that (m_n) is uniformly continuous as a sequence of $\mathcal{P}_2(\mathbb{R}^d)$-valued periodic functions, and

$$\sup_n \sup_{t \in [0,T]} \int_{\mathbb{R}^d} |x|^2 m_n(x, t) dx < \infty. \tag{3.9}$$

Therefore, by Ascoli-Arzelà theorem and Lemma 2.4, (m_n) has a subsequence (still denoted by (m_n)) which converges in $C(\mathbb{R}, \mathcal{P}_p(\mathbb{R}^d))$ for all $p < 2$ to some $m^T \in C_T(\mathbb{R}, \mathcal{P}_p(\mathbb{R}^d))$. Due to the lower semicontinuity (**lsc**), and the growth assumption (**BDD**) of $\mathcal{W}$, we get that $m \in C_T(\mathbb{R}, \mathcal{P}_2(\mathbb{R}^d))$. Note that by convergence in $C(\mathbb{R}, \mathcal{P}_p(\mathbb{R}^d))$ symmetry properties pass to the limit. Moreover also (3.7) passes to the limit, due to lower semicontinuity of d_2 with respect to narrow convergence, see Lemma 2.4.

Finally, (m_n) is bounded in $L^\infty(Q)$, so we can extract a further subsequence that converges $L^\infty(Q)$-weak-* to m^T, and $0 \leq m^T(t, x) \leq \rho$ a.e.

Regarding (w_n), we have

$$\int_0^T \int_{\mathbb{R}^d} |w_n(t, x)|^2 \, dx dt \leq \rho \int_0^T \int_{\mathbb{R}^d} \frac{|w_n(t, x)|^2}{m_n(t, x)} \, dx dt, \tag{3.10}$$

hence w_n converges weakly (up to a subsequence) in $L^2(Q)$ to some w^T.
It is easy to check that $-\partial_t m^T + \text{div}(w^T) = 0$ in the distributional sense.
So we are just left to check that (m^T, w^T) minimizes J_T. We use the lower semicontinuity of the kinetic part of the energy recalled in Proposition 2.7, and for the potential part, we use the lower semicontinuity (**lsc**) of $\mathcal{W}$ and Fatou lemma. $\square$

We now observe that $(m^T, w^T) \in \mathcal{K}_T^{\rho,S}$ obtained in Theorem 1.1 is a global minimizer of the control problem in Definition 3.1. Note that, following [8], this property can be used as a starting point to derive first order optimality conditions, that are of the form (1.1). We mention that additional "pressure" terms and an ergodic constant may appear in the Hamilton-Jacobi equation, due to density constraints and T-periodicity. In any case, the key observation here is that no further multipliers related to $m(T/4 + t) = m(T/4 - t)$, $m(-t) = \gamma_\# m(t)$ are going to appear in the optimality conditions (by the symmetry assumptions on $\mathcal{W}$).

Corollary 3.5. $(m^T, w^T) \in \mathcal{K}_T^{\rho,S}$ obtained in Theorem 1.1 is a brake orbit in the sense of Definition 3.1

Proof. Having denoted by f the derivative $f(x, m) = \frac{\delta}{\delta m} \mathcal{W}(m) \in C(\mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d))$, given any minimizer $(\bar{m}, \bar{w})$ of J_T in $\mathcal{K}_T^{\rho,S}$, it is possible to show by convexity of $(m, w) \mapsto \frac{|w|^2}{m}$ and arguing as in [5] that for all $(m, w) \in \mathcal{K}_T^{\rho,S}$,

$$\int_0^T \int_{\mathbb{R}^d} \left| \frac{\bar{w}(t,x)}{\bar{m}(t,x)} \right|^2 \bar{m}(t,x) + f(x, \bar{m}(t))\bar{m}(t,x)dxdt \leq$$

$$\int_0^T \int_{\mathbb{R}^d} \left| \frac{w(t,x)}{m(t,x)} \right|^2 m(t,x) + f(x, \bar{m}(t))m(t,x)dxdt. \quad (3.11)$$

Hence, we just need to show that the minimization property (3.11) can be extended to the more general class of non-symmetric competitors $(m, w) \in \mathcal{K}_T^{\rho} \supset \mathcal{K}_T^{\rho,S}$.

We detail only the fact that the symmetry condition $m(-t) = \gamma_\# m(t)$ can be dropped (arguing analogously, it can be shown the symmetry constraint around $T/4$ can be also dropped). Indeed, for $(m, w) \in \mathcal{K}_T^{\rho}$ satisfying $m(T/4 + t) = m(T/4 - t)$ only, let

$$\tilde{m}(t) = \frac{1}{2}m(t) + \frac{1}{2}\gamma m(-t), \qquad \tilde{w}(t) = \frac{1}{2}w(t) + \frac{1}{2}\gamma w(-t).$$

Note that $\mathcal{W}(m) = \mathcal{W}(\gamma_\# m)$ yields $f(x, m) = f(\gamma(x), \gamma m)$ (recall that $f = \delta_m \mathcal{W}$), and therefore, since $\bar{m}(t,x) = \bar{m}(-t, \gamma(x))$ via a change of variables and convexity,

$$\int_0^T \int_{\mathbb{R}^d} \frac{|w(t,x)|^2}{m(t,x)} + f(x, \bar{m}(t))m(t,x)dxdt = \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} \frac{|w(t,x)|^2}{m(t,x)} + f(x, \bar{m}(t))m(t,x)dxdt +$$

$$+ \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} \frac{|w(-t, \gamma(x))|^2}{m(-t, \gamma(x))} + f(\gamma(x), \bar{m}(-t))m(-t, \gamma(x))dxdt$$

$$\geq \int_0^T \int_{\mathbb{R}^d} \frac{|\tilde{w}(t,x)|^2}{\tilde{m}(t,x)} + f(x, \bar{m}(t))\tilde{m}(t,x)dxdt.$$

Then, since we have that $(\tilde{m}, \tilde{w}) \in \mathcal{K}_T^{\rho,S}$,

$$\int_0^T \int_{\mathbb{R}^d} \frac{|w(t,x)|^2}{m(t,x)} + f(x, \bar{m}(t))m(t,x)dxdt \geq \int_0^T \int_{\mathbb{R}^d} \frac{|\tilde{w}(t,x)|^2}{\tilde{m}(t,x)} + f(x, \bar{m}(t))\tilde{m}(t,x)dxdt.$$

$\square$

4 Heteroclinic connections

In this section we provide the proof of the second main result, that is Theorem 1.2.

We introduce our definition of heteroclinic connection. First of all we recall the definition of the energy on the whole space:

$$J(m,w) = \int_{-\infty}^{+\infty} \int_{\mathbb{R}^d} \frac{1}{2} \left| \frac{dw}{dt \otimes m(t,dx)} \right|^2 m(t,dx)dt + \int_{-\infty}^{+\infty} \mathcal{W}(m(t))\,dt. \qquad (4.1)$$

Recall that $f(x,m) = \frac{\delta}{\delta m}\mathcal{W}(m)$, and couples $(m,w) \in \mathcal{K}^\rho$ are admissible flows that are absolutely continuous with respect to the Lebsegue measure $dt \otimes dx$.

Definition 4.1. *Let $(\bar{m},\bar{w}) \in \mathcal{K}^\rho$. We say that $(\bar{m},\bar{w})$ is a* **heteroclinic connection** *for the MFG if $\lim_{t\to-\infty} d_2(m(t),\mathcal{M}^-) = 0 = \lim_{t\to+\infty} d_2(m(t),\mathcal{M}^+)$, and $(\bar{m},\bar{w})$ satisfies*

$$\int_{-\infty}^{+\infty} \int_{\mathbb{R}^d} \left| \frac{\bar{w}(t,x)}{\bar{m}(t,x)} \right|^2 \bar{m}(t,x) + f(x,\bar{m}(t))\bar{m}(t,x)dxdt \leq$$

$$\int_{-\infty}^{+\infty} \int_{\mathbb{R}^d} \left| \frac{w(t,x)}{m(t,x)} \right|^2 m(t,x) + f(x,\bar{m}(t))m(t,x)dxdt \qquad \forall (m,w) \in \mathcal{K}^\rho. \quad (4.2)$$

We start observing that if (m,w) has bounded energy, then m should approach at $\pm\infty$ the stationary sets $\mathcal{M}^\pm$.

Lemma 4.2. *Let $(m,w) \in \mathcal{K}^\rho$, and suppose $J(m,w) < +\infty$. Then*

$$either \quad \lim_{t\to+\infty} d_2\left(m(t),\mathcal{M}^+\right) = 0 \quad or \quad \lim_{t\to+\infty} d_2\left(m(t),\mathcal{M}^-\right) = 0$$

and analogously for $t \to -\infty$. In particular if $(m,w) \in \mathcal{K}^{\rho,S}$ then either

$$\lim_{t\to+\infty} d_2\left(m(t),\mathcal{M}^+\right) = 0 \quad and \quad \lim_{t\to-\infty} d_2\left(m(t),\mathcal{M}^-\right) = 0$$

or

$$\lim_{t\to+\infty} d_2\left(m(t),\mathcal{M}^-\right) = 0 \quad and \quad \lim_{t\to-\infty} d_2\left(m(t),\mathcal{M}^+\right) = 0.$$

Proof. Setting $C = J(m,w)$, observe that by (2.3)

$$d_2^2(m(t),m(s)) \leq C|t-s| \qquad \forall t,s \in \mathbb{R}. \qquad (4.3)$$

So, due to the uniform continuity given by (4.3) and to the fact that $J(m,w) = C < +\infty$, it is not possible that along some subsequence $t_n \to +\infty$, there holds that $d_2(m(t_n),\mathcal{M}^\pm) \geq r$ for some $r > 0$ independent of n. Indeed, using (4.3), we get that there exists $\eta = \eta(r) > 0$ independent of n such that $d_2(m(t),\mathcal{M}^\pm) \geq r/2 > 0$ for all $t \in [t_n - \eta, t_n + \eta]$. By Lemma 3.3, $\inf_{\{m \in \mathcal{P}_{2,\rho}^r(\mathbb{R}^d),\, d_2(m,\mathcal{M}^\pm) \geq r/2\}} \mathcal{W}(m) = \delta(r/2) > 0$ and then we would get a contradiction with the fact that $J(m,w) = C < +\infty$ since

$$C \geq \sum_n \int_{t_n-\eta}^{t_n+\eta} \mathcal{W}(m)dt \geq \sum_n \int_{t_n-\eta}^{t_n+\eta} \delta(r/2)dt \to +\infty.$$

Therefore, using this observation, first of all we deduce that for any sequence $t_n \to +\infty$, either $\lim_n d_2(m(t_n),\mathcal{M}^+) = 0$ or $\lim_n d_2(m(t_n),\mathcal{M}^-) = 0$. Indeed, if it were not the case, then there would exist $\varepsilon > 0$ and a subsequence t_n such that both $d_2(m(t_n),\mathcal{M}^+) \geq 2\varepsilon$ and $d_2(m(t_n),\mathcal{M}^-) \geq 2\varepsilon > 0$, contradicting the previous assertion.

Assume now that there are sequences $t_n, s_n \to +\infty$ for which $\lim_n d_2(m(t_n),\mathcal{M}^+) = 0$ and $\lim_n d_2(m(s_n),\mathcal{M}^-) = 0$. We may assume that $s_n \leq t_n - 1 \leq t_n$ for all n.

Fix $\delta \in (0,1)$ such that $2\delta < q_0 = \frac{1}{2}d_2(\mathcal{M}^+, \mathcal{M}^-)$ and let n_0 such that for all $n \geq n_0$

$$d_2(m(t_n), \mathcal{M}^+) \leq \delta < \frac{q_0}{2} \quad \text{and} \quad d_2(m(s_n), \mathcal{M}^-) \leq \delta < \frac{q_0}{2}.$$

Note that by triangular inequality

$$d_2(m(s_n), \mathcal{M}^+) \geq d_2(\mathcal{M}^+, \mathcal{M}^-) - d_2(m(s_n), \mathcal{M}^-) \geq 2q_0 - \delta \geq \frac{3}{2}q_0.$$

The function $t \in (s_n, t_n) \to m(t)$ is a continuous function with value in $\mathcal{P}_2(\mathbb{R}^d)$. Therefore there exists $\bar{t}_n \in (s_n, t_n)$ such that
$$d_2(m(\bar{t}_n), \mathcal{M}^+) = q_0.$$
Again by triangular inequality we get that

$$d_2(m(\bar{t}_n), \mathcal{M}^-) \geq d_2(\mathcal{M}^+, \mathcal{M}^-) - d_2(m(\bar{t}_n), \mathcal{M}^+) \geq 2q_0 - q_0 = q_0.$$

And this, again, would contradict the boundedness of the energy.

Therefore, we get that either for all $t_n \to +\infty$, $\lim_n d_2(m(t_n), \mathcal{M}^+) = 0$, or for all $t_n \to +\infty$, $\lim_n d_2(m(t_n), \mathcal{M}^-) = 0$. This implies in particular the conclusion, for $t \to +\infty$. Proceeding analogously we get the statement for $t \to -\infty$. Finally, if $(m, w) \in \mathcal{K}^{\rho,S}$ we conclude by recalling that $m(-t) = \gamma_\# m(t)$ and that $\mathcal{M}^+ = \gamma_\# \mathcal{M}^-$. $\qquad\square$

We provide now the existence of a solution to the constrained minimization problem

$$J(\bar{m}, \bar{w}) = \min_{(m,w) \in \mathcal{K}^{\rho,S}} J(m, w). \tag{4.4}$$

Note that in $\mathcal{K}^{\rho,S}$ a symmetry constraint with respect to $t = 0$ appears. Arguing as in Corollary 3.5, one can show that any constrained minimizer is an heteroclinic connection, in the sense of Definition 4.1. In other words, it has the minimality property (4.2) in the broader class of competitors $\mathcal{K}^\rho$.

Proof of Theorem 1.2.
Proof of item a). We use similar arguments to those in the proof of Theorem 1.1.

Step 1: energy bounds. First of all we show that $\mathcal{K}^{\rho,S} \neq \emptyset$. Choose $m_0 \in \mathcal{P}_{2,\rho}^r(\mathbb{R}^d)$ with compact support such that $m_0 = \gamma m_0$ and let

$$d := d_2(m_0, \mathcal{M}^+) = d_2(m_0, \mathcal{M}^-).$$

Let $\bar{m}_+ \in \mathcal{M}^+$ such that $d_2(m_0, \bar{m}_+) = d$. By Lemma 3.2, there exists a couple $(m, w) \in \mathcal{K}^\rho$ that connects m_0 at time $t = 0$ to $\bar{m}_+$ at time $t = 1$.

$$\tilde{m}(t) := \begin{cases} m(t) & t \in [0,1], \\ \bar{m}_+ & t \in [1, +\infty), \end{cases} \qquad \tilde{w}(t, x) := \begin{cases} w(t, x) & t \in [0,1], \\ 0 & t \in [1, +\infty). \end{cases}$$

Observe that $d_2(\tilde{m}(t), \mathcal{M}^+) \leq d$ for all $t \in [0,1]$. We extend $(\tilde{m}, \tilde{w})$ on $(-\infty, 0)$ symmetrically:

$$(\tilde{m}(t), \tilde{w}(t)) := (\gamma\tilde{m}(-t), -\gamma\tilde{w}(-t)),$$

Note that

$$J(\tilde{m}, \tilde{w}) = 2\int_0^1 \int_{\mathbb{R}^d} \left| \frac{\tilde{w}(t,x)}{\tilde{m}(t,x)} \right|^2 \tilde{m}(t,x)\,dx dt + 2\int_0^1 \mathcal{W}(\tilde{m}(t))\,dt \leq \frac{C'}{2},$$

where C' is defined in (3.5).

Step 2: limit of minimizing sequences. We consider now a minimizing sequence $(m_n, w_n) \in \mathcal{K}^{\rho,S}$ such that $J(m_n, w_n) \leq \frac{C'}{2}$. By the growth condition **(BDD)** on $\mathcal{W}$, since $\int_0^1 \mathcal{W}(m_n)dt < C'/2$,

there exists $t_n \in [0,1]$ such that $m_n(t_n)$ is uniformly bounded in $\mathcal{P}_2(\mathbb{R}^d)$. By (2.3), we get that $(m_n) \subset C(\mathbb{R}, \mathcal{P}_2(\mathbb{R}^d))$ is equicontinuous and then $m_n(t)$ is uniformly bounded in $\mathcal{P}_2(\mathbb{R}^d)$ for all t. By Ascoli-Arzelà theorem and Lemma 2.3, up to extracting a subsequence and to a diagonalization procedure, we get that m_n converges uniformly in $C([-L, L], \mathcal{P}_p(\mathbb{R}^d))$ for all $p < 2$ and all $L > 0$, to some $\bar{m} \in C(\mathbb{R}, \mathcal{P}_p(\mathbb{R}^d))$. Again by lower semicontinuity (**lsc**), and the growth condition (**BDD**), there holds that $\bar{m} \in C(\mathbb{R}, \mathcal{P}_2(\mathbb{R}^d))$. Moreover $\bar{m}(-t) = \gamma \bar{m}(t)$ since symmetry properties pass to the limit, and we can extract a further subsequence that converges also in $L^\infty([-L, L] \times \mathbb{R}^d)$-weak-* to $\bar{m}$, so $0 \le \bar{m}(x, t) \le \rho$ a.e.. Finally, reasoning as in (3.10), we get that w_n converges weakly (up to the extraction of a subsequence and a diagonalization procedure) in $L^2([-L, L] \times \mathbb{R}^d)$ to some $\bar{w}$ for every $L > 0$. In particular we get that $-\partial_t \bar{m} + \operatorname{div}(\bar{w}) = 0$ in distributional sense in $(-\infty, +\infty) \times \mathbb{R}^d$.

Step 3: finite energy. We fix $L > 0$. By the lower semicontinuity properties and Fatou lemma, we get that for every $L > 0$,

$$
0 \le \int_{-L}^{L} \int_{\mathbb{R}^d} \left| \frac{\bar{w}(t, x)}{\bar{m}(t, x)} \right|^2 \bar{m}(t, x) \, dx dt + \int_{-L}^{L} \mathcal{W}(\bar{m}(t)) \, dt \le \liminf_n J(m_n, w_n) =: \eta \le \frac{C'}{2}
$$

and so again by Fatou lemma

$$
0 \le J(\bar{m}, \bar{w}) \le \eta = \inf_{(\mu, v) \in \mathcal{K}^{\rho, S}} J(\mu, v) \le \frac{C'}{2}. \tag{4.5}
$$

This implies that $(\bar{m}, \bar{w}) \in \mathcal{K}^{\rho, S}$ and moreover that $(\bar{m}, \bar{w})$ is a minimizer.

Proof of item b).

Step 4: limit of $m^T(\pm T/4)$ as $T \to +\infty$. First of all, by Theorem 1.1 observe that for all $\varepsilon > 0$ small and $T > \max(4s, \bar{T})$ (where $s = s(\varepsilon)$ and $\bar{T} = \bar{T}(\varepsilon)$ according to the notation of Theorem 1.1, in which $q = \varepsilon$), there exists a minimizer (m^T, w^T) such that

$$
0 \le d_2^2 \left(m^T(t), \mathcal{M}^+ \right) \le \varepsilon \qquad \forall t \in \left(s, \frac{T}{2} - s \right).
$$

Since $\frac{T}{4} \in \left(s, \frac{T}{2} - s \right)$, sending $T \to +\infty$, we get that $\lim_{T \to +\infty} d_2^2 \left(m^T(T/4), \mathcal{M}^+ \right) = 0$ and, by the symmetry conditions, $\lim_{T \to +\infty} d_2^2 \left(m^T(-T/4), \mathcal{M}^- \right) = 0$.

Step 5: equicontinuity of m^T and passage to the limit. Let fix $q \in (0, q_0)$ and let $(m^T, w^T) \in \mathcal{K}_T^{\rho, S}$ be a minimizer of J_T constructed in Theorem 1.1 with $T > \bar{T}(q)(\ge 4)$. First of all observe that by (3.6) there exists C' independent of T such that $0 \le J_T(m^T, w^T) \le C'$, and so in particular by (2.3), we get that $(m^T(\cdot))_T \subset C(\mathbb{R}, \mathcal{P}_2(\mathbb{R}^d))$ is equicontinuous. By the growth condition (**BDD**), since $\int_0^1 \mathcal{W}(m^T) dt \le C'$, there exists $t(T) \in [0, 1]$ such that $m^T(t(T))$ is bounded in $\mathcal{P}_2(\mathbb{R}^d)$, uniformly with respect to T.

By (2.3) and triangular inequality we conclude that for all $t \in [0, T]$, $m^T(t)$ is bounded in $\mathcal{P}_2(\mathbb{R}^d)$, uniformly with respect to T. By Ascoli-Arzelà theorem and Lemma 2.3, we get that up to extracting a subsequence $T_n \to +\infty$ and using a diagonalization procedure, we get that m^{T_n} converges uniformly in $C([-L, L], \mathcal{P}_p(\mathbb{R}^d))$ for all $p < 2$ and all $L > 0$, to some $m \in C(\mathbb{R}, \mathcal{P}_p(\mathbb{R}^d))$, which a posteriori, due to (**lsc**) and (**BDD**), satisfies $m \in C(\mathbb{R}, \mathcal{P}_2(\mathbb{R}^d))$. Moreover $m(-t) = \gamma m(t)$ since symmetry properties pass to the limit, and we can extract a further subsequence that converges also in $L^\infty([-L, L] \times \mathbb{R}^d)$-weak-* to m, and $0 \le m(x, t) \le \rho$ a.e.. Finally, reasoning as in (3.10), we get that w_n converges weakly (up to the extraction of a subsequence and a diagonalization procedure) in $L^2([-L, L] \times \mathbb{R}^d)$ to some w for every $L > 0$. In particular we get that $-\partial_t m + \operatorname{div}(w) = 0$ in distributional sense in $(-\infty, +\infty) \times \mathbb{R}^d$.

It is immediate to check that the same argument applies to every limit point of (m^T, w^T).

Step 6: finite energy of (m, w). Let (m, w) the limit of (m^{T_n}, w^{T_n}) as obtained in the previous step. Fix now $L > 0$ and let n_0 such that $T_n \ge 4L$ for all $n \ge n_0$. By the lower semicontinuity

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properties and Fatou lemma, we get that for every $L > 0$, we get that

$$0 \leq \int_{-L}^{L} \int_{\mathbb{R}^d} \left| \frac{w(t,x)}{m(t,x)} \right|^2 m(t,x) \, dxdt + \int_{-L}^{L} \mathcal{W}(m(t)) \, dt$$

$$\leq \liminf_n \int_{-L}^{L} \int_{\mathbb{R}^d} \left| \frac{w^{T_n}(t,x)}{m^{T_n}(t,x)} \right|^2 m^{T_n}(t,x) \, dxdt + \int_{-L}^{L} \liminf_n \mathcal{W}(m^{T_n}(t)) \, dt$$

$$\leq \liminf_n \int_{-L}^{L} \int_{\mathbb{R}^d} \left| \frac{w^{T_n}(t,x)}{m^{T_n}(t,x)} \right|^2 m^{T_n}(t,x) \, dxdt + \liminf_n \int_{-L}^{L} \mathcal{W}(m^{T_n}(t)) \, dt$$

$$\leq \liminf_n \left[\int_{-T_n/4}^{T_n/4} \int_{\mathbb{R}^d} \left| \frac{w^{T_n}(t,x)}{m^{T_n}(t,x)} \right|^2 m^{T_n}(t,x) \, dxdt + \int_{-T_n/4}^{T_n/4} \mathcal{W}(m^{T_n}(t)) \, dt \right]$$

$$= \frac{1}{2} \liminf_n J_{T_n}(m^{T_n}, w^{T_n}) \leq \frac{1}{2} C'$$

and so by Fatou lemma

$$0 \leq J(m,w) \leq \frac{1}{2} C'. \tag{4.6}$$

This, along with the properties proved in Step 5, implies that $(m,w) \in \mathcal{K}^{\rho,S}$.

Step 7: (m,w) is a solution of (4.4). Since $J_{T_n}(m^{T_n}, w^{T_n})$ is equibounded, up to passing to a further subsequence we may assume that $\lim_n J_{T_n}(m^{T_n}, w^{T_n}) = e > 0$. Arguing as above it is immediate to check that

$$e = \lim_n J_{T_n}(m^{T_n}, w^{T_n}) \geq 2J(m,w).$$

We claim that

$$e \leq 2J(\bar{m}, \bar{w})$$

where $(\bar{m}, \bar{w})$ is a minimizer constructed in item a). If the claim is true, then we have that $2J(m,w) \leq e \leq 2J(\bar{m}, \bar{w})$ which implies immediately that (m,w) is a minimizer and moreover that $e = 2J(\bar{m}, \bar{w})$.

Assume by contradiction that for some $\delta > 0$, there holds

$$e > 2J(\bar{m}, \bar{w}) + \delta.$$

Let us fix T_n and consider $(\bar{m}, \bar{w})$ restricted to $\left[-\frac{T_n}{4}, \frac{T_n}{4} \right]$. Extend them to $\left[-\frac{T_n}{4}, \frac{3T_n}{4} \right]$ by putting $\tilde{m}_n \left(t + \frac{T_n}{4} \right) := \bar{m} \left(\frac{T_n}{4} - t \right)$, $\tilde{w}_n \left(t + \frac{T_n}{4} \right) := -\bar{w} \left(\frac{T_n}{4} - t \right)$ for $t \in \left[0, \frac{T_n}{2} \right]$ and then extend them periodically in $\mathbb{R}$. It is easy to check that $\gamma \tilde{m}_n(t) = \tilde{m}_n(-t)$. So $(\tilde{m}_n, \tilde{w}_n) \in \mathcal{K}_{T_n}$ and therefore

$$J_{T_n}(m^{T_n}, w^{T_n}) \leq J_{T_n}(\tilde{m}_n, \tilde{w}_n) = 2 \int_{-\frac{T_n}{4}}^{\frac{T_n}{4}} \int_{\mathbb{R}^d} \left| \frac{\bar{w}(t,x)}{\bar{m}(t,x)} \right|^2 \bar{m}(t,x) \, dx + \mathcal{W}(\bar{m}(t)) dt \leq 2J(\bar{m}, \bar{w}) < e - \delta.$$

Taking n sufficiently large, this gives a contradiction with the fact that $e = \lim_n J_{T_n}(m^{T_n}, w^{T_n})$.

$\square$

5 A model problem

In this section we describe a model to which results of the previous sections apply. We define on $\mathcal{P}_{2,\rho}^r(\mathbb{R}^d)$ (for any fixed $\rho > 0$) the following potential energy

$$\mathcal{W}(m) = \int_{\mathbb{R}^d} W(x) m(dx) - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(|x-y|) m(dx) m(dy). \tag{5.1}$$

First of all we describe our main assumptions on K and W and then we check all the conditions that are needed in Theorems 1.1, 1.2. Note that, as we will see below, $\mathcal{W}(m)$ has minimizers on

$\mathcal{P}^r_{2,\rho}(\mathbb{R}^d)$, but $\min_{\mathcal{P}^r_{2,\rho}(\mathbb{R}^d)} \mathcal{W} < 0$. Therefore, to apply Theorems 1.1, 1.2 one just needs to add to $\mathcal{W}$ the renormalization constant $\min_{\mathcal{P}^r_{2,\rho}(\mathbb{R}^d)} \mathcal{W}$, that is to consider

$$\mathcal{W}_0(m) = \mathcal{W}(m) - \min_{\mathcal{P}^r_{2,\rho}(\mathbb{R}^d)} \mathcal{W}.$$

5.1 Standing assumptions on W and K

We start describing the assumptions on the local energy $\int_{\mathbb{R}^d} W(x)dm(x)$. Let $W : \mathbb{R}^d \to [0,+\infty)$ be a confining double-well potential such that

- $W \in C(\mathbb{R}^d)$ is non-negative,
- $\exists C > 0$, such that $C^{-1}|x|^2 - C \le W(x) \le C|x|^2 + C$
- $\exists a^+, a^- \in \mathbb{R}^d, \tilde{r} > 0$ such that $B(a^+, \tilde{r}) \cap B(a^-, \tilde{r}) = \varnothing$
  $\quad and \quad W(x) = 0 \Leftrightarrow x \in B(a^+, \tilde{r}) \cup B(a^-, \tilde{r}).$

(5.2)

Note that we require W to have two disjoint flat regions $B(a^\pm, \tilde{r})$. Moreover, we assume that W is invariant under a reflection $\gamma : \mathbb{R}^d \to \mathbb{R}^d$, that is

$$W(x) = W(\gamma(x)) \qquad x \in \mathbb{R}^d. \tag{5.3}$$

In particular this implies that $\gamma(a^+) = a^-$ and $|a^\pm| - \tilde{r} > 0$. Finally, we assume that the plateaus of W are sufficiently large with respect to the density constraint ρ, in the following sense:

$$\rho \ge \tilde{\rho} := \frac{1}{\omega_d \tilde{r}^d}, \tag{5.4}$$

where $\tilde{r}$ is defined in (5.2), and ω_d is the Lebesgue measure of B_1. See Figure 2 for an example of W satisfying our assumptions.

We describe now the assumptions on the interaction energy $- \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(|x-y|)m(dx)m(dy)$ and some basic properties. We consider a radially symmetric interaction kernel $K(|x|)$, where $K : [0,+\infty) \to [0,\infty)$ is a function such that

- $r \mapsto r^{d-1}K(r) \in L^1_{loc}([0,+\infty), [0,+\infty))$,
- K is nonincreasing,
- $\lim_{r \to 0} K(r) - K(t+r) > 0$ for every t,
- $\lim_{r \to +\infty} K(r) = 0$.

(5.5)

Moreover, we assume that K is positive definite, which means that

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x)f(y)K(|x-y|)dxdy \ge 0 \quad \text{for all } f \in L^1(\mathbb{R}^d) \tag{5.6}$$

and $\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x)f(y)K(|x-y|)dxdy = 0$ if and only if $f = 0$. Note that positive definiteness is equivalent to assume that the Fourier transform of K is a positive function.

We define the energy interaction functional for $f \in L^1(\mathbb{R}^d)$

$$\mathcal{I}(f) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x)f(y)K(|x-y|)dxdy = \int_{\mathbb{R}^d} f(x)V_f(x)dx \tag{5.7}$$

where V_f is the interaction potential

$$V_f(x) = f * K(x) = \int_{\mathbb{R}^d} f(y)K(|x-y|)dy. \tag{5.8}$$

Remark 5.1. If K is positive definite, then the map $f \mapsto V_f = K * f$ is monotone increasing in the sense that

$$\int_{\mathbb{R}^d} (V_{f_1} - V_{f_2})(f_1 - f_2) dx \geq 0 \qquad \forall f_1, f_2 \in L^1(\mathbb{R}^d).$$

We recall a well known result on the interaction potential.

Lemma 5.2. *Assume* (5.5). *Let* $f \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. *Then* $V_f \in C(\mathbb{R}^d)$ *and* $\lim_{|x| \to +\infty} V_f(x) = 0$.

Proof. Let $\eta : [0, +\infty) \to [0, 1]$ be a smooth function such that $\eta = 0$ in $[0, 1]$ and $\eta = 1$ in $[2, +\infty)$ and define $K_\varepsilon(|x|) = K(|x|)\eta(|x|/\varepsilon)$ for $\varepsilon > 0$. Then $V_f^\varepsilon = K_\varepsilon * f \in C(\mathbb{R}^d)$ since $|V_f^\varepsilon(x+h) - V_f^\varepsilon(x)| \leq \|K_\varepsilon\|_\infty \|f(\cdot + h) - f(\cdot)\|_1 \to 0$ as $|h| \to 0$. Moreover $\lim_{|x| \to +\infty} V_f^\varepsilon(x) = 0$: indeed for $R > 2$ and $\varepsilon \leq 1$, recalling the assumptions on K,

$$
\begin{aligned}
|V_f^\varepsilon(x)| &\leq \int_{B(0,R)} K_\varepsilon(|y|)|f(x-y)|dy + \int_{\mathbb{R}^d \setminus B(0,R)} K_\varepsilon(|y|)|f(x-y)dy| \\
&\leq K(\varepsilon) \int_{B(x,R)} |f(y)|dy + K(R)\|f\|_1 \to 0 \qquad \text{as } |x| \to +\infty \text{ and } R \to +\infty.
\end{aligned}
$$

We conclude observing that V_f^ε converges uniformly to V_f as $\varepsilon \to 0$ since

$$|V_f^\varepsilon(x) - V_f(x)| \leq \int_{B(0,2\varepsilon)} K(|y|)|f(x-y)|dy \leq \|f\|_\infty \int_{B(0,2\varepsilon)} K(|y|)dy.$$

$\square$

We recall the Riesz rearrangement inequality (see [26]). For $f \in L^1(\mathbb{R}^d)$ such that $f \geq 0$, we define the f^* is the spherical rearrangement of f, that is

$$f^*(x) = \int_0^{+\infty} \chi_{\{y \, |f(y)>t\}^*}(x)dt \quad \text{where } \{y \, |f(y) > t\}^* = B(0,r), \text{ with } \omega_n r^n = |\{y \, |f(y) > t\}|.$$

The Riesz rearrangement inequality states that

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x)f(y)K(|x-y|)dxdy \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f^*(x)f^*(y)K^*(|x-y|)dxdy.$$

Note that $K^*(|x|) = K(|x|)$ since K is radially symmetric and nonincreasing. So we conclude that for every $f \in L^1(\mathbb{R}^d)$ such that $f \geq 0$,

$$\mathcal{I}(f) \leq \mathcal{I}(f^*). \tag{5.9}$$

We recall a well known result, see [26].

Lemma 5.3. *Assume* (5.5). *Let* $r_\rho = (1/\rho\omega_d)^{1/d}$. *There holds*

$$\sup_{m \in \mathcal{P}_{2,\rho}^r(\mathbb{R}^d)} \mathcal{I}(m) = \mathcal{I}(\rho\chi_{B_{r_\rho}}) > 0.$$

Proof. It follows from the Riesz rearrangement inequality that maximizers are radially symmetric and nonincreasing. The fact that maximizers of $\mathcal{I}(f)$ in $\mathcal{P}_{2,\rho}^r(\mathbb{R}^d)$ are characteristic functions it can be proven looking at the second variation of the functional (see [26], and see also the following Proposition 5.8 for a similar argument). $\square$

Remark 5.4. Note that, due to the fact that $W(x) \geq 0$ and to Lemma 5.3, we get that for all $m \in \mathcal{P}_{2,\rho}^r(\mathbb{R}^d)$ there holds

$$-\mathcal{I}(\rho\chi_{B_{r_\rho}}) \leq \mathcal{W}(m) \leq \int_{\mathbb{R}^d} W(x)m(x).$$

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5.2 Assumptions (REF), (BDD)

We check that $\mathcal{W}$ defined in (5.1), under the standing assumptions in Section 5.1, satisfies the growth condition (**BDD**), and the reflection invariance (**REF**).

Proposition 5.5. *Under the assumptions* (5.2), (5.3), (5.5), (5.6), *the functional* $\mathcal{W}$ *in* (5.1) *satisfies* (**REF**), (**BDD**).

Since $\mathcal{W}$ and $\mathcal{W}_0$ differ by a constant, the same conclusion holds for $\mathcal{W}_0$. Moreover, note that $\frac{\delta}{\delta m}\mathcal{W}(m) = W(x) - \int_{\mathbb{R}^d} K(|x-y|)m(dy)$. Hence, as a direct consequence of the positivity of K assumed in (5.6), $\mathcal{W}$ (and $\mathcal{W}_0$) is "aggregating", namely it satisfies $\int_{\mathbb{R}^d}(f(x,m) - f(x,m'))d(m - m')(x) \leq 0$ for all m, m'.

Proof. We observe that by (5.3) and the symmetry properties of K,

$$\mathcal{W}(\gamma(m)) = \int_{\mathbb{R}^d} W(x)\gamma m(dx) - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(|x-y|)\gamma m(dx)\gamma m(dy)$$

$$= \int_{\mathbb{R}^d} W(\gamma x)m(dx) - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(|\gamma x - \gamma y|)m(dx)m(dy) = \mathcal{W}(m)$$

which is (**REF**). Finally, by (5.2) and Lemma 5.3 we get that for all $m \in \mathcal{P}_{2,\rho}^r(\mathbb{R}^d)$, there holds

$$C^{-1}\int_{\mathbb{R}^d} |x|^2 m(dx) - C - \mathcal{I}(\rho\chi_{B_{r_\rho}}) \leq \mathcal{W}(m) \leq C\int_{\mathbb{R}^d} |x|^2 m(dx) + C,$$

which is (**BDD**). $\qquad\square$

5.3 Lower semicontinuity properties of $\mathcal{W}$: assumption (lsc)

We provide continuity and semicontinuity properties of $\mathcal{W}$ (and $\mathcal{W}_0$). Let us first check that $\mathcal{W}$ is lower semicontinuous with respect narrow convergence, which implies (**lsc**).

Proposition 5.6. *Assume that K satisfies* (5.5) *and let $\mathcal{I}$ be as in* (5.7). *Let $m_k, m \in \mathcal{P}_{2,\rho}^r(\mathbb{R}^d)$ and suppose that m_k converges to m narrowly. Then,* $\lim_k \mathcal{I}(m_k) = \mathcal{I}(m)$. *In particular, the functional $\mathcal{W}$ satisfies* (**lsc**).

Proof. Once $\lim_k \mathcal{I}(m_k) = \mathcal{I}(m)$ is established, the lower semicontinuity (**lsc**) follows by applying Lemma 2.3 (and i) $\Rightarrow$ ii) of Lemma 2.4) to the term $\int W(x)m_k(dx)$. We sketch briefly the proof of $\lim_k \mathcal{I}(m_k) = \mathcal{I}(m)$. Similar arguments have been used in [13, Lemma 3.3].

By Cauchy-Schwartz inequality, recalling the positive definiteness of K, we get (note that we are identifying m_k, m with their densities)

$$\left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} m_k(x)m(y)K(|x-y|)dxdy\right| \leq \mathcal{I}(m_k)^{1/2}\mathcal{I}(m)^{1/2}$$

from which we deduce

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (m_k(x) - m(x))(m_k(y) - m(y))K(|x-y|)dxdy \geq \left((\mathcal{I}(m_k)^{1/2} - (\mathcal{I}(m)^{1/2})^2\right).$$

We fix $R > 0$ and we write, recalling the conditions on K,

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (m_k(x) - m(x))(m_k(y) - m(y))K(|x-y|)dxdy$$

$$\leq \left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (m_k(x) - m(x))(m_k(y) - m(y))K(|x-y|)\chi_{|x-y|\leq R}dxdy\right| + 4K(R). \quad (5.10)$$

We define

$$F_k(x) = \int_{\mathbb{R}^d} m_k(y)K(|x-y|)\chi_{|x-y|\leq R}dy, \quad \text{and} \quad F(x) = \int_{\mathbb{R}^d} m(y)K(|x-y|)\chi_{|x-y|\leq R}dy$$

and we observe that $F_k, F \in L^1(\mathbb{R}^d)$. Then

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (m_k(x) - m(x))(m_k(y) - m(y))K(|x-y|)\chi_{|x-y|\leq R}dxdy$$

$$= -\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (m_k(x) - m(x))F(x)dx + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} m_k(x)(F_k(x) - F(x))dx. \quad (5.11)$$

Observe that since $m_k \to m$ narrowly and $m_k, m \leq \rho$, then $m_k \to m$ weak* in L^∞, due to density of continuous functions in L^1. Therefore

$$\lim_k \int_{\mathbb{R}^d} (m_k(x) - m(x))F(x)dx = 0. \quad (5.12)$$

Moreover $\lim_k F_k(x) = F(x)$ for a.e. x and

$$\|F_k\|_1 = \|K(|x|)\chi_{|x|\leq R}\|_1 \to \|F\|_1 = \|K(|x|)\chi_{|x|\leq R}\|_1.$$

Therefore by Fatou lemma, $F_k \to F$ in $L^1(\mathbb{R}^d)$, which implies, recalling that $m_k \leq \rho$ that

$$\lim_k \int_{\mathbb{R}^d} (F_k(x) - F(x))m_k(x)dx = 0. \quad (5.13)$$

So, using (5.12), (5.13) in (5.10) and recalling that $K(R) \to 0$ as $R \to +\infty$, we get the conclusion. $\qquad\square$

We observe the following fact about uniformly integrability of narrowly convergent sequences of measures.

Lemma 5.7. *Let W be as in (5.2) and $\mu_k, \mu \in \mathcal{P}_2(\mathbb{R}^d)$ such that $\mu_k \to \mu$ narrowly. Then, $\lim_k d_2(\mu_k, \mu) = 0$ if and only if $\lim_k \int_{\mathbb{R}^d} W(x)\mu_k(dx) = \int_{\mathbb{R}^d} W(x)\mu(dx)$.*

Proof. We observe that, by Lemma 2.4, $\lim_k d_2(\mu_k, \mu) = 0$ is equivalent to the fact that μ_k has uniformly integrable 2-moments, that is

$$\lim_{R \to +\infty} \sup_k \int_{\mathbb{R}^d \setminus B(0,R)} |x|^2 d\mu_k(x) = 0. \quad (5.14)$$

Let $R > 0$, sufficiently large, such that $RC - C^2 > 1$ and $RC^{-1} > 3$, where C is the constant appearing in (5.2). We denote $A_R := \{W(x) \geq R\}$. Then by (5.2) we get $\mathbb{R}^d \setminus B(0, \sqrt{RC^{-1} + 1}) \subseteq A_R \subseteq \mathbb{R}^d \setminus B(0, \sqrt{RC - C^2})$. Then, recalling (5.2), we get

$$\frac{C}{2} \sup_k \int_{\mathbb{R}^d \setminus B(0,\sqrt{RC^{-1}+1})} |x|^2 d\mu_k(x) \leq \sup_k \int_{A_R} W(x)d\mu_k(x)$$

$$\leq (C^{-1} + C) \sup_k \int_{\mathbb{R}^d \setminus B(0,\sqrt{RC-C^2})} |x|^2 d\mu_k(x).$$

Sending $R \to +\infty$, we get that μ_k has uniformly integrable 2-moments, that is (5.14) holds, if and only if W is uniformly integrable with respect to μ_k, that is

$$\lim_{R \to +\infty} \sup_k \int_{\{W(x) \geq R\}} W(x)d\mu_k(x) = 0. \quad (5.15)$$

Now observe that if $\mu_k \to \mu$ narrowly, then $\lim_k \int_{\{W(x) \leq R\}} W(x)d\mu_k(x) = \int_{W(x) \leq R} d\mu(x)$ for all $R > 0$. Therefore if $\mu_k \to \mu$ narrowly, then $\lim_k \int_{\mathbb{R}^d} W(x)d\mu_k(x) = \int_{\mathbb{R}^d} d\mu(x)$ if and only if (5.15) holds. This gives the conclusion. $\qquad\square$

5.4 Minimizers for the stationary problem: assumptions (Z) and (CON)

We start proving existence and qualitative properties of minimizers of $\mathcal{W}(m)$ in the set $\mathcal{P}_{2,\rho}^r(\mathbb{R}^d)$. In particular we will show that under assumptions (5.2) and (5.5) the minimizers are characteristic functions. Moreover we show that assumption (5.4) assures that the minimizers are characteristic functions of balls. Then, in Proposition 5.10 we show that (Z) is satisfied by $\mathcal{W}_0$ and in Proposition 5.11, we prove that $\mathcal{W}_0$ satisfies assumption (CON).

The first result is about qualitative properties of minimizers, by looking at the first and second variation of the functional.

Proposition 5.8. *Assume (5.2) and (5.5). Let $m \in \mathcal{P}_{2,\rho}^r(\mathbb{R}^d)$ be a minimizer of the functional $\mathcal{W}$. Then there exists a bounded, measurable set $E \subseteq \mathbb{R}^d$ such that $m = \rho \chi_E$.*

Proof. The proof is based on analogous arguments as in [12, Prop. 5.4, Thm 5.7]. Let $m \in \mathcal{P}_{2,\rho}^r(\mathbb{R}^d)$ be a minimizer of the functional $\mathcal{W}$, and let define the sets

$$S = \{x \mid m(x) = \rho\}, \qquad N = \{x \mid m(x) = 0\},$$

as subsets of the set of density points of m.

We start showing that m has bounded support. We compute the first variation of the functional $\mathcal{W}$ as in [15, Lemma 5.3]. Pick any $\psi, \phi \in L^1(\mathbb{R}^d, [0, \rho])$ such that $\int_{\mathbb{R}^d} \phi(x)dx = \int_{\mathbb{R}^d} \psi(x)dx$, $\psi = 0$ a.e. in S and $\phi = 0$ a.e. in N. Then $\int_{\mathbb{R}^d} m + \lambda(\psi - \phi)dx = 1$ for every λ and moreover for $\lambda \in (0, 1)$ we get that $0 \leq m(x) + \lambda(\psi(x) - \phi(x)) \leq \rho$ for a.e. $x \in N \cup S$.

Now we consider two sequences $\psi_\varepsilon \to \psi$, $\phi_\varepsilon \to \phi$ in L^1 such that $\int_{\mathbb{R}^d} \phi_\varepsilon(x)dx = \int_{\mathbb{R}^d} \psi_\varepsilon(x)dx = \int_{\mathbb{R}^d} \phi(x)dx$ and such that $\phi_\varepsilon = 0$ a.e. on the set $\{m(x) \leq \varepsilon\}$ and $\psi_\varepsilon = 0$ a.e. on the set $\{m(x) \geq 1 - \varepsilon\}$. Choosing λ sufficiently small (depending on ε) we get that $m + \lambda(\psi_\varepsilon - \phi_\varepsilon) \in \mathcal{P}_{2,\rho}^r(\mathbb{R}^d)$. Using the fact that $\mathcal{W}(m) \leq \mathcal{W}(m + \lambda(\psi_\varepsilon - \phi_\varepsilon))$, we get, sending $\lambda \to 0^+$,

$$\int_{\mathbb{R}^d} (-2V_m(x) + W(x))(\psi_\varepsilon(x) - \phi_\varepsilon(x))dx \geq 0$$

where V_m is the potential of m defined in (5.8). Sending $\varepsilon \to 0$ we conclude that

$$\int_{\mathbb{R}^d} (-2V_m(x) + W(x))(\psi(x) - \phi(x))dx \geq 0. \tag{5.16}$$

If we choose ψ, ϕ such that $\psi = \phi = 0$ in $N \cup S$, then we can exchange the role of ϕ, ψ in (5.16) and obtain

$$\int_{\mathbb{R}^d \setminus (S \cup N)} (-2V_m(x) + W(x))(\psi(x) - \phi(x))dx = 0$$

for all $\phi - \psi \in L^1(\mathbb{R}^d)$, such that $\phi - \psi = 0$ in $S \cup N$, with $\int_{\mathbb{R}^d} (\phi - \psi)dx = 0$. This implies by the fundamental lemma of calculus of variations, that there exists a constant c such that

$$-2V_m(x) + W(x) = c \qquad x \in \mathbb{R}^d \setminus (N \cup S). \tag{5.17}$$

Using this fact in (5.16) we get, taking $\psi = 0$ in $S \cup N$, and observing that $\int_{\mathbb{R}^d \setminus (S \cup N)} \psi dx = \int_{\mathbb{R}^d} \phi dx$,

$$0 \leq \int_S (2V_m(x) - W(x))\phi(x)dx + c \int_{\mathbb{R}^d \setminus S \cup N} (\psi - \phi)dx = \int_S (2V_m(x) - W(x) + c)\phi(x)dx$$

an analogously, taking $\phi = 0$ in $S \cup N$,

$$0 \leq \int_N (-2V_m(x) + W(x))\psi(x)dx + c \int_{\mathbb{R}^d \setminus S \cup N} (\psi - \phi)dx = \int_N (-2V_m(x) + W(x) - c)\psi dx.$$

This implies that

$$\begin{cases} -2V_m(x) + W(x) \geq c & x \in N \\ -2V_m(x) + W(x) \leq c & x \in S, \end{cases} \tag{5.18}$$

Recalling that W is coercive (see assumption (5.2)) and V_m vanishes at infinity, see Lemma 5.2, we conclude from (5.17), (5.18) that both S and $\mathbb{R}^d \setminus (N \cup S)$ have to be bounded. This implies that the support of m, which is given by $\mathbb{R}^d \setminus N$, is bounded.

Now we show that $m(x) \in \{0, \rho\}$ for a.e. x. We compute the second variation of the functional as in [12, Lemma 5.5]. We consider ϕ, ψ as above, and we assume that $\psi = \phi = 0$ in $N \cup S$. Now we consider two sequences $\psi_\varepsilon \to \psi$, $\phi_\varepsilon \to \phi$ in L^1 such that $\int_{\mathbb{R}^d} \phi_\varepsilon(x) dx = \int_{\mathbb{R}^d} \psi_\varepsilon(x) dx = \int_{\mathbb{R}^d} \phi(x) dx$ and such that $\phi_\varepsilon = 0$ and $\psi_\varepsilon = 0$ a.e. on the set $\{m(x) \leq \varepsilon\} \cup \{m(x) \geq 1 - \varepsilon\}$. Now, for λ sufficiently small, depending on ε, $m + \lambda(\psi_\varepsilon - \phi_\varepsilon) \in \mathcal{P}^r_{2,\rho}(\mathbb{R}^d)$. By minimality of m, recalling that $W(x) - 2V_m(x)$ is constant on $\mathbb{R}^d \setminus (N \cup S)$, that $\psi_\varepsilon - \phi_\varepsilon = 0$ in $N \cup S$ and that $\int_{\mathbb{R}^d}(\psi_\varepsilon(x) - \phi_\varepsilon(x)) dx = 0$, we get

$$-\lambda^2 \int_{\mathbb{R}^d} K(|x - y|)(\psi_\varepsilon(x) - \phi_\varepsilon(x))(\psi_\varepsilon(y) - \phi_\varepsilon(y)) dx dy \geq 0.$$

Sending $\varepsilon \to 0$ we get

$$\int_{\mathbb{R}^d \setminus (N \cup S)} \int_{\mathbb{R}^d \setminus (N \cup S)} (\psi(x) - \phi(x))(\psi(y) - \phi(y)) K(|x - y|) dx dy \leq 0. \tag{5.19}$$

Assume now by contradiction that there are two Lebesgue points x, y of m such that $0 < m(x), m(y) < \rho$. Let $d = |x - y|$. Since x, y are density points for m, it is possible to find, for $0 < \varepsilon < d/4$ sufficiently small, two sets with positive measure $A_\varepsilon(x)$, $A_\varepsilon(y)$ such that

- $A_\varepsilon(x) \subseteq B(x, \varepsilon) \cap \mathbb{R}^d \setminus (N \cup S)$,
- $A_\varepsilon(y) \subseteq B(y, \varepsilon) \cap \mathbb{R}^d \setminus (N \cup S)$,
- $d(A_\varepsilon(x), A_\varepsilon(y)) \geq d/2$,
- $|A_\varepsilon(x)| = |A_\varepsilon(y)| > 0$.

We recall that $B(x, \varepsilon)$ is the ball centered at x of radius ε, and analogously $B(y, \varepsilon)$. Observe that if either $t, z \in A_\varepsilon(x)$ or $t, z \in A_\varepsilon(y) \subseteq B(y, \varepsilon)$, then $|t - z| \leq 2\varepsilon$, and if $t \in A_\varepsilon(x)$, $z \in A_\varepsilon(y)$, then $|t - z| \geq d(A_\varepsilon(x), A_\varepsilon(y)) \geq d/2$.

We choose $\psi = \chi_{A_\varepsilon(x)}$ and $\phi = \chi_{A_\varepsilon(y)}$, and substituting in (5.19) we find, recalling that K is decreasing, that

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\psi(x) - \phi(x))(\psi(y) - \phi(y)) K(|x - y|) dx dy \\ &= \int_{A_\varepsilon(x)} \int_{A_\varepsilon(x)} K(|t - z|) dt dz + \int_{A_\varepsilon(y)} \int_{A_\varepsilon(y)} K(|t - z|) dt dz - 2 \int_{A_\varepsilon(x)} \int_{A_\varepsilon(y)} K(|t - z|) dt dz \\ &\geq K(2\varepsilon)|A_\varepsilon(x)|^2 + K(2\varepsilon)|A_\varepsilon(y)|^2 - 2K(d/2)|A_\varepsilon(x)||A_\varepsilon(y)| \\ &= 2|A_\varepsilon(x)|^2(K(2\varepsilon) - K(d/2)) > 0, \end{aligned}$$

which gives a contradiction. $\qquad \square$

We provide now the existence and characterization of minimizers to $\mathcal{W}$.

Theorem 5.9. *Under the assumptions* (5.5), (5.2), *the problem*

$$\min_{m \in \mathcal{P}^r_{2,\rho}(\mathbb{R}^d)} \mathcal{W}(m) \tag{5.20}$$

admits at least one solution. Each solution is given by $\rho\chi_E$ for some measurable set E such that $|E| = \rho^{-1}$. Moreover if $\rho\chi_E$ is a minimizer then also $\rho\chi_{\gamma E}$ is a minimizer.

Proof. The result is an application of the direct method in calculus of variations. By Remark 5.4, $\inf_{\mathcal{P}^r_{2,\rho}(\mathbb{R}^d)} \mathcal{W} \geq -\mathcal{I}(\rho\chi_{B_{r_\rho}})$. Let m_n be a minimizing sequence. By (**BDD**) and Lemma 2.4,, up to a subsequence, there exists m such that $m \in \mathcal{P}_p(\mathbb{R}^d)$ for every $p < 2$ such that m_n converges to m narrowly and also weak-* in L^∞. Again by the growth condition (**BDD**) and the lower semicontinuity property (**lsc**), $m \in \mathcal{P}_2(\mathbb{R}^d)$. Moreover $\|m\|_\infty \leq \liminf_n \|m_n\|_\infty \leq \rho$, and so $m \in \mathcal{P}^r_{2,\rho}(\mathbb{R}^d)$ and, again by Proposition 5.6, $\liminf_n \mathcal{W}(m_n) \geq \mathcal{W}(m)$, which implies that m is a minimizer. Finally by Proposition 5.8, $m = \rho\chi_E$ for some bounded measurable set E. The fact that $\rho\chi_{\gamma E}$ is still a minimizers comes from (5.3). $\qquad\square$

Proposition 5.10. *Assume* (5.2), (5.3) *and* (5.4). *Then* $\min_{\mathcal{P}^r_{2,\rho}(\mathbb{R}^d)} \mathcal{W} = -\mathcal{I}(\rho\chi_{B_{r_\rho}})$ *for* $r_\rho = (\omega_d\rho)^{-1/d}$, *and all the minimizers of* (5.1) *are given by* $\mathcal{M}^+ \cup \mathcal{M}^-$, *where* $\mathcal{M}^- = \gamma\mathcal{M}^+$ *and*

$$\mathcal{M}^+ = \{\rho\chi_E, \text{ where } E = B(x', (\omega_d\rho)^{-1/d}) \subseteq B(a^+, \tilde{r}) \text{ for some } x' \in \mathbb{R}^d\}.$$

If $\rho = \tilde{\rho}$ ($= (\omega_d\tilde{r}^d)^{-1}$), *then* $\mathcal{M}^+ = \{\rho\chi_{B(a^+,\tilde{r})}\}$.
   Moreover, $\mathcal{M}^+$ *and* $\mathcal{M}^-$ *are compact subsets of* $\mathcal{P}_2(\mathbb{R}^d)$ *and* $d_2(\mathcal{M}^+, \mathcal{M}^-) > 0$.

Proof. Let $\rho\chi_E$ be a minimizer. We get, recalling Remark 5.4,

$$-\mathcal{I}(\rho\chi_{B(0,r_\rho)}) \leq \rho \int_E W(x)dx - \mathcal{I}(\rho\chi_E) \leq \rho \int_{B(a^+,r_\rho)} W(x)dx - \mathcal{I}(\rho\chi_{B(0,r_\rho)}).$$

Note that under assumption (5.4), $\omega_d r_\rho^d \leq \omega_d\tilde{r}^d = |B(a^+,\tilde{r})|$, then $\mathcal{W}(\rho\chi_{B(a^\pm,r_\rho)}) = -\mathcal{I}(\rho\chi_{B(0,r_\rho)})$, and so $\rho\chi_{B(a^\pm,r_\rho)}$ are minimizers. Moreover, due to Lemma 5.3, $\int_E W(x)dx = 0$ for every E such that $\rho\chi_E$ is a minimizer. If $r_\rho < \tilde{r}$, there are infinitely many minimizers, which are given by all possible balls $B(x',r_\rho) \subseteq B(a^\pm,\tilde{r})$, whereas if $r_\rho = \tilde{r}$, the only minimizers are $B(a^\pm,\tilde{r})$.
   The compactness of $\mathcal{M}^\pm$ is straightforward. To evaluate $d_2(\mathcal{M}^+, \mathcal{M}^-)$, we just notice that for any $\bar{m}^+ \in \mathcal{M}^+$ and $\bar{m}^- \in \mathcal{M}^-$, $\bar{m}^\pm = \rho\chi_{E^\pm}$, where $E^+ = B(x',r_\rho) \subseteq B(a^+,r)$ and $E^- = B(y',r_\rho) \subseteq B(a^-,\tilde{r})$. Therefore, for all $\pi \in \Pi(\bar{m}^+dx, \bar{m}^-dx)$,

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |x-y|^2 d\pi(x,y) = \int_{B(a^+,\tilde{r})} \int_{B(a^-,\tilde{r})} |x-y|^2 d\pi(x,y) \geq [2(|a^+| - \tilde{r})]^2 > 0,$$

since $B(a^\pm,\tilde{r})$ are symmetric and disjoint. Hence,

$$d_2(\mathcal{M}^+, \mathcal{M}^-) \geq d_2(\bar{m}^+, \bar{m}^-) \geq [2(|a^+| - \tilde{r})]^2 > 0.$$

$\qquad\square$

Therefore,
$$\mathcal{W}_0(m) = \mathcal{W}(m) + \mathcal{I}(\rho\chi_{B_{r_\rho}}).$$

It remains to prove the property (**CON**) for $\mathcal{W}_0$.

Proposition 5.11. *Assume that* $m_k \in \mathcal{P}^r_{2,\rho}(\mathbb{R}^d)$ *is such that* $\lim_k \mathcal{W}(m_k) = -\mathcal{I}(\rho\chi_{B_{r_\rho}})$, *with the notation of Lemma 5.3. Then up to passing to a subsequence there exists* $m \in \mathcal{P}^r_{2,\rho}(\mathbb{R}^d)$ *such that* $\mathcal{W}(m) = -\mathcal{I}(\rho\chi_{B_{r_\rho}})$ *and* $\lim_k d_2(m_k, m) = 0$.

Proof. By the growth condition (**BDD**), we get that m_k are uniformly bounded in $\mathcal{P}_2(\mathbb{R}^d)$. So, by Lemma 2.3, up to passing to a subsequence we get that there exists $m \in \mathcal{P}_p(\mathbb{R}^d)$ such that $m_k \to m$ in $\mathcal{P}_p(\mathbb{R}^d)$ for all $p < 2$, and a posteriori by the growth condition (5.2), and the lower semicontinuity in Proposition 5.6, $m \in \mathcal{P}_2(\mathbb{R}^d)$. Again by the lower semicontinuity in Proposition 5.6 we have that $\mathcal{W}(m) \leq -\mathcal{I}(\rho\chi_{B_{r_\rho}})$, and so, recalling Remark 5.4, $\mathcal{W}(m) = -\mathcal{I}(\rho\chi_{B_{r_\rho}})$. Since, by Proposition 5.6, $\lim_k \mathcal{I}(m_k) = \mathcal{I}(m)$ this implies in particular that $\lim_n \int_{\mathbb{R}^d} W(x)m_k(dx) = \int_{\mathbb{R}^d} W(x)m(dx)$, so by Lemma 5.7, $\lim_k d_2(m_k, m) = 0$. $\qquad\square$
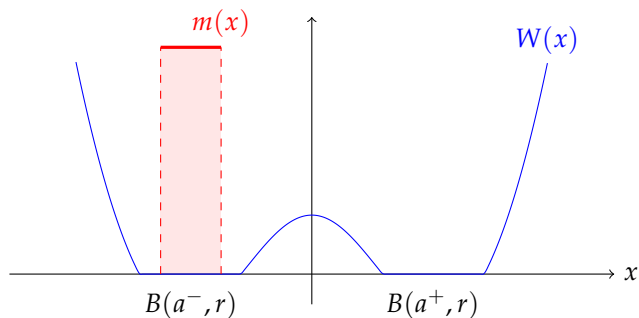
Figure 2: An example of symmetric potential W and a (compactly supported) minimizer m of $\mathcal{W}$.

The proof of Theorem 1.3 then follows by Theorem 5.9 and Proposition 5.10. Moreover, we obtain the following conclusion:

Corollary 5.12. *Under the assumptions* (5.2), (5.3), (5.4), (5.5), (5.6), $\mathcal{W}_0$ *satisfies* (**Z**), (**lsc**), (**BDD**), (**CON**) *and* (**REF**).

References

[1] F. Alessio and P. Montecchiari. Brake orbit solutions for semilinear elliptic systems with asymmetric double well potential. *J. Fixed Point Theory Appl.*, 19(1):691–717, 2017.

[2] L. Ambrosio, N. Gigli, and G. Savaré. *Gradient flows in metric spaces and in the space of probability measures*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, second edition, 2008.

[3] J.-D. Benamou and Y. Brenier. A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem. *Numer. Math.*, 84(3):375–393, 2000.

[4] J.-D. Benamou, G. Carlier, and F. Santambrogio. Variational mean field games. *Active particles. Vol. 1. Advances in theory, models, and applications*, pages 141–171, 2017.

[5] A. Briani and P. Cardaliaguet. Stable solutions in potential mean field game systems. *NoDEA Nonlinear Differential Equations Appl.*, 25(1):25:1, 2018.

[6] P. Cardaliaguet and P. J. Graber. Mean field games systems of first order. *ESAIM Control Optim. Calc. Var.*, 21(3):690–722, 2015.

[7] P. Cardaliaguet and M. Masoero. Weak KAM theory for potential MFG. *Journal of Differential Equations*, 2019.

[8] P. Cardaliaguet, A. R. Mészáros, and F. Santambrogio. First order mean field games with density constraints: pressure equals price. *SIAM J. Control Optim.*, 54(5):2672–2709, 2016.

[9] P. Cardaliaguet and A. Porretta. Long time behavior of the master equation in mean field game theory. *Anal. PDE*, 12(6):1397–1453, 2019.

[10] A. Cesaroni and M. Cirant. Concentration of ground states in stationary mean-field games systems. *Anal. PDE*, 12(3):737–787, 2019.

[11] A. Cesaroni and M. Cirant. 1-dimensional multi-agent optimal control with aggregation and distance constraints: qualitative properties and mean-field limit. *Nonlinearity, to appear*, 2020. arXiv preprint,https://arxiv.org/abs/2003.01980.

[12] A. Cesaroni and M. Novaga. The isoperimetric problem for nonlocal perimeters. *Discrete Contin. Dyn. Syst. Ser. S*, 11(3):425–440, 2018.

[13] R. Choksi, R. C. Fetecau, and I. Topaloglu. On minimizers of interaction functionals with competing attractive and repulsive potentials. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 32(6):1283–1305, 2015.

[14] M. Cirant. On the existence of oscillating solutions in non-monotone mean-field games. *J. Differential Equations*, 266(12):8067–8093, 2019.

[15] M. Cirant and L. Nurbekyan. The variational structure and time-periodic solutions for mean-field games systems. *Minimax Theory Appl.*, 3(2):227–260, 2018.

[16] M. Cirant and A. Porretta. Long time behaviour and turnpike solutions in mildly non-monotone mean field games. submitted, 2020.

[17] M. Fornasier, S. Lisini, C. Orrieri, and G. Savarè. Mean-field optimal control as Γ-limit of finite agent controls. *European Journal of Applied Mathematics*, 30(6):1153–1186., 2019.

[18] G. Fusco, G. F. Gronchi, and M. Novaga. Existence of periodic orbits near heteroclinic connections. *Minimax Theory Appl.*, 4(1):113–149, 2019.

[19] W. Gangbo and A. R. Mészáros. Global well-posedness of master equations for deterministic displacement convex potential mean field games. *https://arxiv.org/abs/2004.01660*, 2020.

[20] D. A. Gomes and T. Seneci. Displacement convexity for first-order mean-field games. *Minimax Theory Appl.*, 3(2):261–284, 2018.

[21] P. J. Graber, A. R. Mészáros, F. J. Silva, and D. Tonon. The planning problem in mean field games as regularized mass transport. *Calc. Var. Partial Differential Equations*, 58(3):Art. 115, 28, 2019.

[22] M.-O. Hongler. Mean-field games and swarms dynamics in Gaussian and non-Gaussian environments. *J. Dyn. Games*, 7(1):1–20, 2020.

[23] J.-M. Lasry and P.-L. Lions. Jeux à champ moyen. I. Le cas stationnaire. *C. R. Math. Acad. Sci. Paris*, 343(9):619–625, 2006.

[24] J.-M. Lasry and P.-L. Lions. Jeux à champ moyen. II. Horizon fini et contrôle optimal. *C. R. Math. Acad. Sci. Paris*, 343(10):679–684, 2006.

[25] H. Lavenant and F. Santambrogio. Optimal density evolution with congestion: L^∞ bounds via flow interchange techniques and applications to variational mean field games. *Comm. Partial Differential Equations*, 43(12):1761–1802, 2018.

[26] E. H. Lieb and M. Loss. *Analysis*, volume 14 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1997.

[27] M. Masoero. On the long time convergence of potential MFG. *NoDEA Nonlinear Differential Equations Appl.*, 26(2):Art. 15, 45, 2019.

[28] R. J. McCann. A convexity principle for interacting gases. *Adv. Math.*, 128(1):153–179, 1997.

[29] A. Monteil and F. Santambrogio. Metric methods for heteroclinic connections in infinite-dimensional spaces. *Indiana Univ. Math. J.*, 69(4):1445–1503, 2020.

[30] C. Orrieri, A. Porretta, and G. Savaré. A variational approach to the mean field planning problem. *J. Funct. Anal.*, 277(6):1868–1957, 2019.

[31] P. H. Rabinowitz. Periodic solutions of Hamiltonian systems: a survey. *SIAM J. Math. Anal.*, 13(3):343–352, 1982.

[32] F. Santambrogio. *Optimal transport for applied mathematicians*, volume 87 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser/Springer, Cham, 2015. Calculus of variations, PDEs, and modeling.

[33] D. Ullmo, I. Swiecicki, and T. Gobron. Quadratic mean field games. *Phys. Rep.*, 799:1–35, 2019.

[34] H. Yin, P. G. Mehta, S. P. Meyn, and U. V. Shanbhag. Bifurcation analysis of a heterogeneous mean-field oscillator game model. In *Proceedings of the 50th IEEE Conference on Decision and Control and European Control Conference, CDC-ECC 2011*, pages 3895–3900, 2011.

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