A NOTE ON EQUI-INTEGRABILITY IN DIMENSION REDUCTION PROBLEMS

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Abstract. In the framework of the asymptotic analysis of thin structures, we prove that, up to an extraction, it is possible to decompose a sequence of 'scaled gradients' $\big(\nabla_\alpha u_\varepsilon\big|\frac{1}{\varepsilon}\nabla_\beta u_\varepsilon\big)$ (where ∇_β is the gradient in the k-dimensional 'thin variable' x_β) bounded in $L^p(\Omega;\mathbb{R}^{m\times n})$ $(1<p<+\infty)$ as a sum of a sequence $\big(\nabla_\alpha v_\varepsilon\big|\frac{1}{\varepsilon}\nabla_\beta v_\varepsilon\big)$ whose p-th power is equi-integrable on Ω and a 'rest' that converges to zero in measure. In particular, for $k=1$ we recover a well-known result for thin films by Bocea and Fonseca [4].

Keywords: equi-integrability, dimension reduction.

2000 Mathematics Subject Classification: 74B20, 74K35, 74K15, 49J45.

1. Introduction

A very handy tool in the study of the asymptotic behavior of variational problems defined on Sobolev spaces is Fonseca, Müller and Pedregal's *equi-integrability Lemma* [8] (see Theorem 2.1 below; see also earlier work by Acerbi and Fusco [2] and by Kristensen [11]), which allows to substitute a sequence (w_j) with (∇w_j) bounded in L^p by a sequence (z_j) with $(|\nabla z_j|^p)$ equi-integrable, such that the two sequences are equal except on a set of vanishing measure. In this way the asymptotic behavior of integral energies of p-growth involving ∇w_j can be computed using ∇z_j and thus avoiding to consider concentration effects. This method is very helpful for example in the computation of lower bounds for Γ-limits (see, *e.g.*, [5]).

In the framework of dimensional reduction, we encounter sequences of functions (w_ε) defined on cylindrical sets with some 'thin dimension' ε; *e.g.*, in the physical three-dimensional case either *thin films* defined on some set of the type $\omega\times(0,\varepsilon)$ (see, *e.g.*, [10, 6]), or *thin wires* defined on $\varepsilon\omega\times(0,1)$ (see, *e.g.*, [1, 9]), where ω is some two-dimensional bounded open set. In order to carry on some asymptotic analysis such functions are usually rescaled to an ε-independent reference configuration Ω (see Fig. 1), so that a new sequence (u_ε) is constructed, satisfying some 'degenerate' bounds of the form

$$\int_\Omega\Big(|\nabla_\alpha u_\varepsilon|^p+\frac{1}{\varepsilon^p}|\nabla_\beta u_\varepsilon|^p\Big)dx\le C<+\infty \tag{1.1}$$

whenever the sequence of the gradients (∇w_ε) satisfied some corresponding L^p bound on the unscaled domain. Here, ∇_α represents the gradient with respect to the unscaled coordinates (denoted by x_α) and ∇_β represents the gradient with respect to the 'thin' coordinate directions (denoted by x_β). In the case described above of thin films $x_\beta=x_3$; for thin wires, $x_\beta=(x_1,x_2)$.

A theorem by Bocea and Fonseca [4] states that an analog of Fonseca, Müller and Pedregal's result still holds in this framework, and an 'equivalent sequence' v_ε

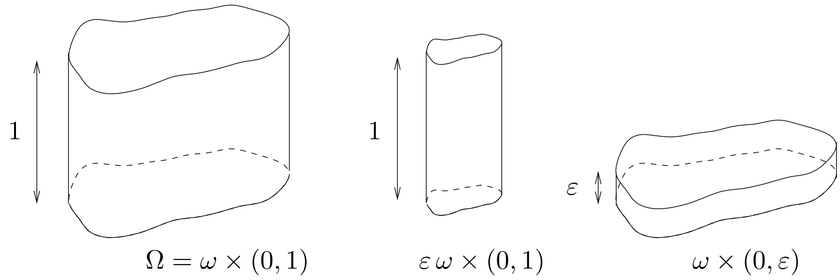

FIGURE 1. Scaled domain, a wire and a thin film.

can be constructed such that the sequence $(|\nabla_\alpha v_\varepsilon|^p + \frac{1}{\varepsilon^p}|\nabla_\beta v_\varepsilon|^p)$ is equi-integrable on Ω. In their result they deal specifically with the case of thin films; *i.e.*, when the space of the x_β is one-dimensional in the notation above. An earlier mention of the equi-integrability result in this form can be found without proof in a paper by Shu [12], where it is suggested that the same argument of [8] could be followed. This path is not pursued by Bocea and Fonseca's as it would necessitate re-proving a number of fine results for maximal functions in a periodic context; their proof instead relies on a direct argument.

This note provides an alternative proof to that of Bocea and Fonseca, that we think worth pointing out since its method could be applied to other types of problems involving thin structures and extends to a general nD-to-$(n-k)$D dimensional-reduction framework. Its argument is essentially the following: we consider the unscaled functions w_ε defined on some Ω_ε (*e.g.*, $\omega \times (0, \varepsilon)$) on which we have an L^p bound of the gradient and extend them to 2ε-periodic functions in the x_β directions. These extended functions still satisfy an L^p bound, now on each fixed Ω (*e.g.*, a cube), so that we may apply Fonseca, Müller and Pedregal's result to find z_ε with the equi-integrability property. This property is quantified by de la Vallée Poussin's Criterion, which ensures the existence of a positive Borel function φ with superlinear growth such that $\int_\Omega \varphi(|\nabla z_\varepsilon|^p)\,dx \le C < +\infty$. By this remark and a simple but careful counting argument we can choose a set differing from the original Ω_ε by a 2ε-periodic translation in the x_β directions (and hence it is not restrictive to suppose that this set is precisely Ω_ε) such that

$$\frac{1}{\varepsilon^k}\int_{\Omega_\varepsilon} \varphi(|\nabla z_\varepsilon|^p)\,dx \le C < +\infty, \tag{1.2}$$

(k denotes the dimension of the space of the x_β) and still z_ε equals w_ε except for a set with relative measure tending to zero in Ω_ε. By scaling such z_ε we conclude the proof since (1.2) exactly states the desired equi-integrability property.

Since our method does not rely on space dimensions, we state and proof our result in a general n-dimensional setting. In particular it also comprises the physical case of thin wires not covered in [4]. Thin wires are generally dealt with by more direct arguments exploiting their one-dimensional limit nature, but our general equi-integrability result may nevertheless be useful in the case of thin wires with an unprescribed heterogeneous nature, in order to obtain general compactness results as for thin films (see [6]).

2. Preliminaries

In this section we recall two results which will be the key tools in the proof of Theorem 3.1. The first one is Fonseca-Müller-Pedregal's decomposition Theorem for 'unscaled gradients' while the second is a classical equi-integrability criterion.

In what follows m, n will be two positive integers, Ω a bounded open subset of $\mathbb{R}^n$ and p a real number such that $1 < p < +\infty$.

Theorem 2.1 ([8] Lemma 1.2)**.** *Let (w_j) be a bounded sequence in $W^{1,p}(\Omega;\mathbb{R}^m)$. Then there exists a subsequence of (w_j) (not relabelled) and a sequence (z_j) in $W^{1,p}(\Omega;\mathbb{R}^m)$ such that*

$$\mathcal{L}^n(\{z_j \neq w_j\} \cup \{\nabla z_j \neq \nabla w_j\}) \to 0,$$

as $j \to +\infty$, and $(|\nabla z_j|^p)$ is equi-integrable on Ω. If Ω is Lipschitz, then each z_j can be chosen to be a Lipschitz function.

Proposition 2.2 (de la Vallée Poussin's Criterion)**.** *Let (w_j) be in $L^1(\Omega;\mathbb{R}^m)$; then (w_j) is equi-integrable on Ω if and only if there exists a positive Borel function $\varphi : [0,+\infty) \to [0,+\infty]$ such that*

$$\lim_{t\to+\infty} \frac{\varphi(t)}{t} = +\infty \qquad \textit{and} \qquad \sup_j \int_\Omega \varphi(|w_j|)\, dx < +\infty.$$

A proof of de la Vallée Poussin's Criterion can be found in Dellacherie-Meyer [7].

3. Statement and proof of the main result

Let k be a positive integer such that $k < n$. Given $x \in \mathbb{R}^n$, we set $x = (x_\alpha, x_\beta)$ where $x_\alpha = (x_1, \ldots, x_{n-k})$ and $x_\beta = (x_{n-k+1}, \ldots, x_n)$ is the 'thin variable'; then $\nabla_\alpha = \big(\partial_{x_1}, \ldots, \partial_{x_{n-k}}\big)$ is the gradient with respect to x_α and $\nabla_\beta = \big(\partial_{x_{n-k+1}}, \ldots, \partial_{x_n}\big)$ the gradient with respect to x_β.

Theorem 3.1. *Let $\omega_\alpha \subset \mathbb{R}^{n-k}$, $\omega_\beta \subset \mathbb{R}^k$ be open bounded sets and assume that ω_β is connected and with Lipschitz boundary. Let (ε_j) be a sequence of positive real numbers converging to zero and let (u_j) be a bounded sequence in $W^{1,p}(\omega_\alpha \times \omega_\beta;\mathbb{R}^m)$ satisfying*

$$\sup_j \int_{\omega_\alpha\times\omega_\beta} \Big(|\nabla_\alpha u_j|^p + \frac{1}{\varepsilon_j^p}|\nabla_\beta u_j|^p\Big)\, dx < +\infty. \tag{3.1}$$

Then there exists a subsequence of (u_j) (not relabelled) and a sequence (v_j) in $W^{1,p}(\omega_\alpha \times \omega_\beta;\mathbb{R}^m)$ such that

$$\mathcal{L}^n(\{v_j \neq u_j\} \cup \{\nabla v_j \neq \nabla u_j\}) \to 0, \tag{3.2}$$

as $j \to +\infty$, and $\Big(|\nabla_\alpha v_j|^p + \frac{1}{\varepsilon_j^p}|\nabla_\beta v_j|^p\Big)$ is equi-integrable on $\omega_\alpha \times \omega_\beta$. If ω_α is Lipschitz then each v_j can be chosen to be a Lipschitz function.

Proof. Let (u_j) be a bounded sequence in $W^{1,p}(\omega_\alpha\times\omega_\beta;\mathbb{R}^m)$ satisfying (3.1). Since ω_β is connected and with Lipschitz boundary, by applying a standard extension technique (see for instance Adams [3], Theorems 4.26 and 4.28, and Section 4.29 for details) we may assume to deal with a $W^{1,p}(\omega_\alpha\times Q^k;\mathbb{R}^m)$-sequence, for $Q^k \subset \mathbb{R}^k$ open cube containing ω_β, still preserving the same boundedness properties of (u_j). Moreover, up to possible scalings and translations, we can always suppose that $Q^k = (0,1)^k$.

Set $\hat{u}_j(x) := u_j(x_\alpha, \frac{x_\beta}{\varepsilon_j})$; then $(\hat{u}_j) \subset W^{1,p}(\omega_\alpha \times (0,\varepsilon_j)^k;\mathbb{R}^m)$ and by hypothesis

$$\sup_j \frac{1}{\varepsilon_j^k}\int_{\omega_\alpha\times(0,\varepsilon_j)^k} |\hat{u}_j|^p\,dx = \sup_j \int_{\omega_\alpha\times(0,1)^k} |u_j|^p\,dx < +\infty, \tag{3.3}$$

while

$$\sup_j \frac{1}{\varepsilon_j^k}\int_{\omega_\alpha\times(0,\varepsilon_j)^k} \left(|\nabla_\alpha \hat{u}_j|^p + |\nabla_\beta \hat{u}_j|^p\right)\,dx = \sup_j \int_{\omega_\alpha\times(0,1)^k} \Big(|\nabla_\alpha u_j|^p + \frac{1}{\varepsilon_j^p}|\nabla_\beta u_j|^p\Big)\,dx < +\infty, \tag{3.4}$$

and from (3.4) in particular

$$\sup_j \frac{1}{\varepsilon_j^k}\int_{\omega_\alpha\times(0,\varepsilon_j)^k} |\nabla \hat{u}_j|^p\,dx < +\infty. \tag{3.5}$$

We extend $\hat{u}_j$ to $\omega_\alpha \times (-\varepsilon_j,\varepsilon_j)^k$ by reflection in the k variables $x_{n-k+1},\dots,x_n$ by defining

$$\tilde{u}_j(x) := \hat{u}_j(x_\alpha, |x_{n-k+1}|,\dots,|x_n|) \quad \text{in } \omega_\alpha \times (-\varepsilon_j,\varepsilon_j)^k.$$

Note that $(\tilde{u}_j) \subset W^{1,p}(\omega_\alpha \times (-\varepsilon_j,\varepsilon_j)^k;\mathbb{R}^m)$ and $\tilde{u}_j(x_\alpha,\cdot)$ has the same trace on the opposite faces of $(-\varepsilon_j,\varepsilon_j)^k$ for a.e. $x_\alpha \in \omega_\alpha$. Thus $\tilde{u}_j$ can be extended by $(-\varepsilon_j,\varepsilon_j)^k$-periodicity in x_β, to the whole $\omega_\alpha \times \mathbb{R}^k$ obtaining the $W^{1,p}_{\rm loc}(\omega_\alpha \times \mathbb{R}^k;\mathbb{R}^m)$-sequence defined as follows

$$\bar{u}_j(x) := \tilde{u}_j(x_\alpha, x_\beta - 2\varepsilon_j\, i) \quad \text{in} \quad \omega_\alpha \times (2\varepsilon_j\, i + (-\varepsilon_j,\varepsilon_j)^k), \text{ for } i=(i_1,\dots,i_k)\in\mathbb{Z}^k.$$

We want to prove that $(\bar{u}_j)$ is bounded in $W^{1,p}(\omega_\alpha \times (0,1)^k;\mathbb{R}^m)$. By the periodicity and symmetry properties of $\bar{u}_j$, denoting by $[t]$ the integer part of $t\in\mathbb{R}$, we have

$$\begin{aligned}
\int_{\omega_\alpha\times(0,1)^k} |\bar{u}_j|^p\,dx &\le \sum_{i_1,\dots,i_k=0}^{[1/2\varepsilon_j]+1} \int_{\omega_\alpha\times(2\varepsilon_j\, i + (-\varepsilon_j,\varepsilon_j)^k)} |\bar{u}_j|^p\,dx \\
&= \sum_{i_1,\dots,i_k} \int_{\omega_\alpha\times(-\varepsilon_j,\varepsilon_j)^k} |\tilde{u}_j|^p\,dx = 2^k \sum_{i_1,\dots,i_k} \int_{\omega_\alpha\times(0,\varepsilon_j)^k} |\hat{u}_j|^p\,dx \\
&= 2^k \left(\left[\frac{1}{2\varepsilon_j}\right]+2\right)^k \int_{\omega_\alpha\times(0,\varepsilon_j)^k} |\hat{u}_j|^p\,dx \\
&\le \frac{2^k}{\varepsilon_j^k}\int_{\omega_\alpha\times(0,\varepsilon_j)^k} |\hat{u}_j|^p\,dx
\end{aligned} \tag{3.6}$$

for j sufficiently large.

Gathering (3.6) and (3.3) we deduce

$$\sup_j \int_{\omega_\alpha\times(0,1)^k} |\bar{u}_j|^p\,dx < +\infty;$$

an analogous argument combined with (3.5) yields

$$\sup_j \int_{\omega_\alpha\times(0,1)^k} |\nabla \bar{u}_j|^p\,dx < +\infty.$$

By these estimates $(\bar{u}_j)$ fulfills the hypothesis of Theorem 2.1, which ensures (up to an extraction) the existence of a sequence $(z_j) \subset W^{1,p}(\omega_\alpha \times (0,1)^k;\mathbb{R}^m)$ satisfying

$$\mathcal{L}^n((\{z_j \ne \bar{u}_j\} \cup \{\nabla z_j \ne \nabla \bar{u}_j\}) \cap (\omega_\alpha \times (0,1)^k)) \to 0, \quad \text{as } j \to 0$$

and such that $(|\nabla z_j|^p)$ (or equivalently $(|\nabla_\alpha z_j|^p + |\nabla_\beta z_j|^p)$) is equi-integrable on $\omega_\alpha \times (0,1)^k$. As a consequence, in view of Proposition 2.2, there exists a positive Borel function $\varphi : [0,+\infty) \to [0,+\infty]$ such that

$$\lim_{t\to+\infty} \frac{\varphi(t)}{t} = +\infty \quad \text{and} \quad \sup_j \int_{\omega_\alpha\times(0,1)^k} \varphi(|\nabla_\alpha z_j|^p + |\nabla_\beta z_j|^p)\,dx < +\infty.$$

Hence, $(0,[1/\varepsilon_j]\varepsilon_j)^k \subset (0,1)^k$ and the nonnegative character of φ yield

$$\int_{\omega_\alpha\times(0,[1/\varepsilon_j]\varepsilon_j)^k} \varphi(|\nabla_\alpha z_j|^p + |\nabla_\beta z_j|^p)\,dx \le \int_{\omega_\alpha\times(0,1)^k} \varphi(|\nabla_\alpha z_j|^p + |\nabla_\beta z_j|^p)\,dx \tag{3.7}$$

while the monotonicity of the Lebesgue measure implies

$$\begin{aligned}&\mathcal{L}^n((\{z_j \neq \bar{u}_j\} \cup \{\nabla z_j \neq \nabla \bar{u}_j\}) \cap (\omega_\alpha \times (0,[1/\varepsilon_j]\varepsilon_j)^k)) \\ &\qquad\qquad \le \mathcal{L}^n((\{z_j \neq \bar{u}_j\} \cup \{\nabla z_j \neq \nabla \bar{u}_j\}) \cap (\omega_\alpha \times (0,1)^k)).\end{aligned} \tag{3.8}$$

To shorten notation, set

$$\begin{aligned} M_j &:= \int_{\omega_\alpha\times(0,1)^k} \varphi\left(|\nabla_\alpha z_j|^p + |\nabla_\beta z_j|^p\right)\,dx, \\ m_j &:= \mathcal{L}^n((\{z_j \neq \bar{u}_j\} \cup \{\nabla z_j \neq \nabla \bar{u}_j\}) \cap (\omega_\alpha \times (0,1)^k)) \end{aligned} \tag{3.9}$$

and recall that

$$\text{(i)} \quad \sup_j M_j < +\infty, \qquad \text{(ii)} \quad m_j \to 0. \tag{3.10}$$

From (3.9) and $(0,[1/\varepsilon_j]\varepsilon_j)^k = \bigcup_{i_1,\dots,i_k=0}^{[1/\varepsilon_j]-1} (\varepsilon_j\, i + (0,\varepsilon_j)^k)$, (3.7)-(3.8) can be rewritten respectively as

$$\sum_{i_1,\dots,i_k=0}^{[1/\varepsilon_j]-1} \int_{\omega_\alpha\times(\varepsilon_j i+(0,\varepsilon_j)^k)} \varphi\left(|\nabla_\alpha z_j|^p + |\nabla_\beta z_j|^p\right)dx \le M_j, \tag{3.11}$$

and

$$\sum_{i_1,\dots,i_k=0}^{[1/\varepsilon_j]-1} \mathcal{L}^n((\{z_j \neq \bar{u}_j\} \cup \{\nabla z_j \neq \nabla \bar{u}_j\}) \cap (\omega_\alpha \times (\varepsilon_j\, i + (0,\varepsilon_j)^k))) \le m_j. \tag{3.12}$$

For fixed j, we now consider only those cubes $\varepsilon_j\, i + (0,\varepsilon_j)^k$ with $i = 2h$ for h in $\mathcal{I}_j := \{h \in \mathbb{Z}^k : \ 0 \le h_1, \dots, h_k \le \frac{1}{2}([1/\varepsilon_j] - 1)\}$. Note that for $h \in \mathcal{I}_j$, $\bar{u}_j|_{\omega_\alpha\times 2\varepsilon_j h+(0,\varepsilon_j)^k}$ coincide with the $2\varepsilon_j h$-translation of $\hat{u}_j$ in the x_β variable.

By (3.11) and (3.12) we have that in particular

$$\sum_{h\in\mathcal{I}_j} \int_{\omega_\alpha\times(2\varepsilon_j h+(0,\varepsilon_j)^k)} \varphi\left(|\nabla_\alpha z_j|^p + |\nabla_\beta z_j|^p\right)dx \le M_j \tag{3.13}$$

$$\sum_{h\in\mathcal{I}_j} \mathcal{L}^n((\{z_j \neq \bar{u}_j\} \cup \{\nabla z_j \neq \nabla \bar{u}_j\}) \cap (\omega_\alpha \times (2\varepsilon_j\, h + (0,\varepsilon_j)^k))) \le m_j. \tag{3.14}$$

Then from (3.13), for at least half of the indices $h \in \mathcal{I}_j$ (*i.e.*, for $[1/2\,\#(\mathcal{I}_j)]$ indices) we must have

$$\int_{\omega_\alpha\times(2\varepsilon_j h+(0,\varepsilon_j)^k)} \varphi\left(|\nabla_\alpha z_j|^p + |\nabla_\beta z_j|^p\right)dx \le (\#(\mathcal{I}_j) - [1/2\,\#(\mathcal{I}_j)] + 1)^{-1} M_j. \tag{3.15}$$

In fact, let otherwise $\mathcal{I}_j^{'} := \{h \in \mathcal{I}_j : \text{ (3.15) does not hold}\}$ be such that

$$\#(\mathcal{I}_j^{'}) \geq \#(\mathcal{I}_j) - [1/2\,\#(\mathcal{I}_j)] + 1 \tag{3.16}$$

then

$$\begin{aligned} &\sum_{h\in\mathcal{I}_j} \int_{\omega_\alpha\times(2\varepsilon_j h+(0,\varepsilon_j)^k)} \varphi\left(|\nabla_\alpha z_j|^p + |\nabla_\beta z_j|^p\right) dx \\ \geq\ &\sum_{h\in\mathcal{I}_j^{'}} \int_{\omega_\alpha\times(2\varepsilon_j h+(0,\varepsilon_j)^k)} \varphi\left(|\nabla_\alpha z_j|^p + |\nabla_\beta z_j|^p\right) dx \\ >\ &\#(\mathcal{I}_j^{'})(\#(\mathcal{I}_j) - [1/2\,\#(\mathcal{I}_j)] + 1)^{-1} M_j \end{aligned}$$

and combining it with (3.16), by (3.13) we find a contradiction.

Since $\#(\mathcal{I}_j) = (\,[\frac{1}{2}\,(\,[1/\varepsilon_j] - 1)] + 1)^k$ it can be easily checked that, for j large enough

$$\#(\mathcal{I}_j) - [1/2\,\#(\mathcal{I}_j)] + 1 > \frac{1}{2^{2k+1}\varepsilon_j^k};$$

therefore from (3.15) we get that for at least $[1/2\,\#(\mathcal{I}_j)]$ indices $h \in \mathcal{I}_j$

$$\int_{\omega_\alpha\times(2\varepsilon_j h+(0,\varepsilon_j)^k)} \varphi\left(|\nabla_\alpha z_j|^p + |\nabla_\beta z_j|^p\right) < 2^{2k+1}\varepsilon_j^k M_j, \tag{3.17}$$

for any sufficiently large j. Moreover, in view of (3.14) we can again use an averaging procedure to find among those $[1/2\,\#(\mathcal{I}_j)]$ indices h satisfying (3.17), an index such that

$$\begin{aligned} \mathcal{L}^n((\{z_j \neq \bar{u}_j\} \cup \{\nabla z_j \neq \nabla \bar{u}_j\}) \cap (\omega_\alpha \times (2\varepsilon_j\, h + (0,\varepsilon_j)^k))) \\ \leq [1/2\,\#(\mathcal{I}_j)]^{-1} m_j \leq 2^{3k+1}\varepsilon_j^k\, m_j, \end{aligned} \tag{3.18}$$

for j large enough.

Finally, we have selected an index in $\mathcal{I}_j$ for which both (3.17) and (3.18) (definitively) hold true. Let us call this index $h^\star$. Then by the $(-\varepsilon_j, \varepsilon_j)^k$-periodicity of $\bar{u}_j$ in the x_β variable, up to at most k translations in the $x_{n-k+1}, \ldots, x_n$-directions, we can always suppose that $h^\star = (0, \ldots, 0)$.

Abusing notation we denote by z_j the restriction of z_j to $\omega_\alpha \times (0,\varepsilon_j)^k$; we show that our (v_j) can be obtained from (z_j) just by unscaling. In fact, having set

$$v_j(x) := z_j(x_\alpha, \varepsilon_j x_\beta),$$

then $(v_j) \subset W^{1,p}(\omega_\alpha \times (0,1)^k; \mathbb{R}^m)$ and by (3.17) with $h = h^\star = (0, \ldots, 0)$ we have that

$$\begin{aligned} &\int_{\omega_\alpha\times(0,1)^k} \varphi\Big(|\nabla_\alpha v_j|^p + \frac{1}{\varepsilon_j^p}|\nabla_\beta v_j|^p\Big)\, dx \\ &\qquad = \frac{1}{\varepsilon_j^k} \int_{\omega_\alpha\times(0,\varepsilon_j)^k} \varphi(|\nabla_\alpha z_j|^p + |\nabla_\beta z_j|^p)\, dx < 2^{2k+1}\, M_j. \end{aligned}$$

Thus, by virtue of (3.10)(i), again applying de la Vallée Poussin's Criterion we get that $\big(|\nabla_\alpha v_j|^p + \frac{1}{\varepsilon_j^p}|\nabla_\beta v_j|^p\big)$ is equi-integrable on $\omega_\alpha \times (0,1)^k$. Moreover by (3.18) we deduce

$$\mathcal{L}^n(\{v_j \neq u_j\} \cup \{\nabla v_j \neq \nabla u_j\}) \\ = \frac{1}{\varepsilon_j^k} \mathcal{L}^n((\{z_j \neq \bar{u}_j\} \cup \{\nabla z_j \neq \nabla \bar{u}_j\}) \cap (\omega_\alpha \times (0, \varepsilon_j)^k)) \leq 2^{3k+1} m_j$$

and by (3.10)(ii) we find (3.2). Clearly these two conditions can be restricted to $\omega_\alpha \times \omega_\beta$ if such was the domain of the starting sequence.

Finally, note that if ω_α is Lipschitz, by appealing to Theorem 2.1 we can choose any z_j to be a Lipschitz function, then for every $x, y \in \omega_\alpha \times (0,1)^k$

$$|v_j(x) - v_j(y)| = |z_j(x_\alpha, \varepsilon_j x_\beta) - z_j(y_\alpha, \varepsilon_j y_\beta)| \leq \mathrm{Lip}_{z_j} |x - y|,$$

thus v_j is still a Lipschitz function and $\mathrm{Lip}_{v_j} \leq \mathrm{Lip}_{z_j}$. □

Acknowledgements. Our attention on this problem was drawn by discussions with I. Fonseca. This work has been carried out within the Marie Curie Research Training Network MULTIMAT (MRTN-CT-2004-505226).

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