VARIATIONAL PROBLEMS FOR FUNCTIONALS INVOLVING THE VALUE DISTRIBUTION

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Abstract. We study variational problems involving the measure of level sets, or more precisely the push-forward of the Lebegue measure. This problem generalizes variational problems with finitely many (discrete) volume constraints. We obtain existence results for this general framework. Moreover, we show the surprising existence of asymmetric solutions to symmetric variational problems with this type of volume constraints.

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1. Introduction

Variational problems consist of a functional $\mathcal{F}$ and a class $\mathcal{A}$ of admissible functions on a set $\Omega \subset \mathbb{R}^N$ among which the functional has to be minimized (or maximized).

There are many possible constraints that can be posed on the functions in $\mathcal{A}$, e.g. regarding regularity or the average of the function on Ω.

In the last years, classes $\mathcal{A}$ which are connected with the measure of level sets of the functions have been studied in different frameworks. In particular, there is a series of works where the measure of certain level sets is prescribed. As a prototypical example we may prescribe the measure of the sets $u^{-1}(0) = \{x \in \Omega;\ u(x) = 0\}$ and $u^{-1}(1) = \{x \in \Omega;\ u(x) = 1\}$ for all functions $u \in \mathcal{A}$. Such constraints are called "volume–" or "level set constraints". Here the geometrical and topological shape of the level sets $u^{-1}(0)$ and $u^{-1}(1)$ is *a priori* completely arbitrary.

A typical variational problem with volume constraints reads as

$$\text{Minimize } \mathcal{F}(u) := \int_\Omega |\nabla u(x)|^2\, dx \tag{1.1}$$

among all functions in

$$\begin{aligned}\mathcal{A} \;:=\; \Big\{u \in H^1(\Omega,\mathbb{R});\ &|\{x \in \Omega,\ u(x) = 0\}| = \alpha,\\ &|\{x \in \Omega,\ u(x) = 1\}| = \beta\Big\},\end{aligned}$$

where $|\cdot|$ denotes the Lebesgue measure of a set, and α and β are positive constants with $\alpha+\beta<|\Omega|$.

Similar minimization problems but with only one volume constraint have been studied by various authors, see e.g. [2]. Recently problems with two or more constraints have caught attention [4, 17, 18, 19, 21, 22, 23], partially motivated by physical problems related to immiscible fluids [14] and mixtures of micromagnetic materials [1].

These problems have a very different nature than problems with only one volume constraint: In the case of one volume constraint, only additional boundary conditions or the form of the energy can induce transitions of the solution between different values. Two or more volume constraints, on the other hand, force transitions of the solution by their very nature. Ambrosio, Marcellini, Fonseca and Tartar [4] studied this class of problems for the first time and proved an existence result for the problem of two (or more) level set constraints with an energy density of the form

$$\mathcal{F}(u)=\int_\Omega f(|\nabla u|).$$

It turned out that unlike usual variational problems, lower order terms in the function f pose hard problems for the analysis and can lead, even in very easy examples, to nonexistence of optimal solutions [17, 19]. However, under certain regularity assumptions on the energy density the existence results were extended to quite general energy functionals depending on ∇u and u [19]. For the special case of one space dimension a somewhat complete analysis of existence and uniqueness has been given in [17]. These results have been partially extended to the higher dimensional setting in [18].

The original motivation for this paper was to generalize these ideas from finitely many prescribed levels to more arbitrary (possibly infinitely many). In fact, it is possible to define a notion of volume constraints on arbitrary levels, even on the whole range of a function. As an illustration imagine a transparency which casts a shadow, see Fig. 1. The density of the shadow will then be determined by the shape of the transparency. A typical problem could be to find the optimal shape of the transparency under the constraint of a prescribed shadow density. The mathematical equivalent of the shadow density is the push-forward $u^\#$ of the Lebesgue measure through u, as will be explained in Section 2. The study of $u^\#$ leads to a class of minimization problems which entail not only classical volume constrained problems, but also variational problems that have been investigated in connection with vortex dynamics [9, 10, 11] and plasma physics (compare [16] and the references therein).

The connection of our general setting to the classical volume constrained problems is explained in more details in Section 3. Additionally, we present existence results for a certain class of generalized volume constrained problems. A number of examples conclude the section.

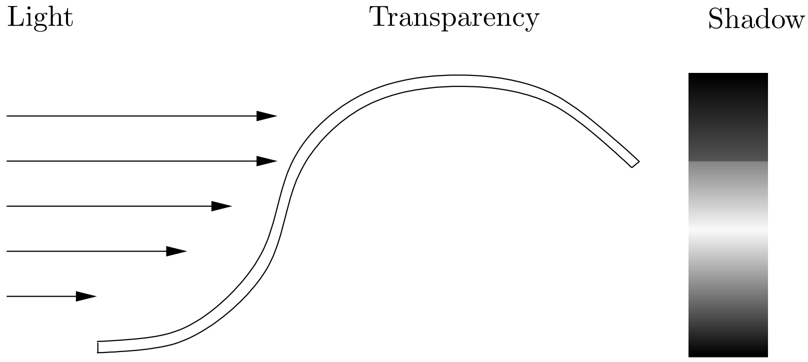

FIGURE 1. Interpretation of the push-forward $u^\#$ as the shadow of a curved transparency.

In the final Section 4, we apply methods from symmetric rearrangements to discuss the symmetry behavior of these problems and to construct some examples with interesting symmetry breaking solutions.

2. VALUE DISTRIBUTIONS AND OPTIMIZATION PROBLEMS

In all the paper Ω will be a bounded open subset of $\mathbb{R}^N$ with a Lipschitz boundary. We recall that, for every $u \in L^1(\Omega)$, the distribution measure $u^\#$ associated to u is the push-forward of the measure $\mathscr{L}^N \llcorner \Omega$ through u, that is

$$u^\#(E) = \mathscr{L}^N\big(\Omega \cap u^{-1}(E)\big) \tag{2.1}$$

for every Borel subset E of $\mathbb{R}$, where $\mathscr{L}^N$ denotes the N-dimensional Lebesgue measure. It turns out that $u^\#$ is a nonnegative measure on the real line, such that

$$u^\#(\mathbb{R}) = \mathscr{L}^N(\Omega). \tag{2.2}$$

We sometimes call $u^\#$ the *value distribution* of u.

Example 2.1. If $u(x) = c$ is a constant function, by (2.1) we have $u^\# = \mathscr{L}^N(\Omega) \cdot \delta_c$ where δ_c is the Dirac mass at the point $c \in \mathbb{R}$.

Analogously, if we denote the characteristic function of a set A by χ_A and consider the piecewise constant function $u = \sum_{i\in I} c_i \chi_{\Omega_i}$, we have that $u^\# = \sum_{i\in I} \mathscr{L}^N(\Omega_i) \cdot \delta_{c_i}$.

Example 2.2. In the case $N = 1$, if $u\colon (a, b) \to \mathbb{R}$ is a monotone nondecreasing function, then it is easy to see that $u^\# = (u^{-1})' \llcorner (\alpha, \beta)$ where $(u^{-1})'$ is the distributional derivative of the monotone nondecreasing function u^{-1} and $\alpha, \beta \in [-\infty, +\infty]$ are defined by $\alpha = \lim_{\varepsilon\to 0} u(a+\varepsilon)$, $\beta = \lim_{\varepsilon\to 0} u(b-\varepsilon)$.

The following result relates the convergence of a sequence of functions to the one of the corresponding value distributions.

Proposition 2.3. *Assume that $u_n \to u$ strongly in $L^1(\Omega)$. Then $u_n^\# \to u^\#$ in the weak$\star$-convergence of measures.*

Proof: By (2.2) the measures $u_n^\#$ have bounded total mass; therefore, it is enough to show that for every smooth function $\varphi : \mathbb{R} \to \mathbb{R}$ with compact support we have

$$\int_{\mathbb{R}} \varphi \, du_n^\# \to \int_{\mathbb{R}} \varphi \, du^\# \ .$$

By a change of variables we obtain

$$\begin{aligned} \lim_{n\to\infty} \int_{\mathbb{R}} \varphi \, du_n^\# &= \lim_{n\to\infty} \int_\Omega \varphi\big(u_n(x)\big)\, dx \\ &= \int_\Omega \varphi\big(u(x)\big)\, dx = \int_{\mathbb{R}} \varphi \, du^\# \ , \end{aligned} \tag{2.3}$$

where in the second equality we used the fact that $u_n \to u$ in $L^1(\Omega)$ and that φ is smooth. □

The variational problems we consider are of the form

$$\min\big\{F(u) + G(u^\#) \ : \ u \in X\big\} \tag{2.4}$$

where:

- the function space X is either a Sobolev space $W^{1,p}(\Omega)$, $W_0^{1,p}(\Omega)$ $(1 \le p \le \infty)$ or the space $BV(\Omega)$ of functions with bounded variation;
- the functional F is sequentially lower semicontinuous with respect to the weak $W^{1,p}(\Omega)$ convergence (if $p < \infty$) or the weak$\star$-convergence (in the cases $X = W^{1,\infty}(\Omega)$ or $X = BV(\Omega)$);
- the functional G is sequentially lower semicontinuous with respect to the weak$\star$-convergence on measures.

The following existence theorem is straightforward.

Theorem 2.4. *In addition to the conditions above we assume:*

(i) *F is coercive on X, that is for every $c \in \mathbb{R}$ the set $\{F(u) \le c\}$ is sequentially compact for the weak convergence (weak$\star$ if $X = W^{1,\infty}(\Omega)$ or $X = BV(\Omega)$);*

(ii) *there exists at least one function $u_0 \in X$ such that $F(u_0) + G(u_0^\#) < +\infty$.*

Then the minimum problem (2.4) admits at least one solution.

Proof: It follows by a straightforward application of the direct methods of the calculus of variations, taking into account Proposition 2.3. □

A typical choice for F is to consider integral functionals like

$$F(u) = \int_\Omega f(x, u, Du)\, dx \tag{2.5}$$

where $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, +\infty]$ is a Borel integrand such that $f(x, \cdot, \cdot)$ is lower semicontinuous and $f(x, s, \cdot)$ is convex. Then (see for instance [12, 13]) the functional in (2.5) turns out to be sequentially lower semicontinuous with respect to the weak convergence in $W^{1,p}(\Omega)$ (weak$\star$ if $p = \infty$). Some extra assumptions on the regularity of the integrand f are required in the case $X = BV(\Omega)$, as well as a refinement of the definition of the functional F, since the gradient Du is in this case a vector measure. We refer the interested reader to [5, 12] for further details.

In the case of functionals of the form (2.5) the coercivity condition (i) of Theorem 2.4 is fulfilled whenever

- $f(x, s, z) \geq \alpha |z|^p$ (with $\alpha > 0$) if $X = W^{1,p}(\Omega)$ with $1 < p < \infty$;
- $f(x, s, z) \geq \alpha |z|$ (with $\alpha > 0$) if $X = BV(\Omega)$;
- $f(x, s, z) \geq H(|z|)$ with H superlinear, that is $\lim_{t \to +\infty} H(t)/t = +\infty$, if $X = W^{1,1}(\Omega)$;
- $f(x, s, z) = +\infty$ for $|z| > \alpha$ (with $\alpha > 0$) if $X = W^{1,\infty}(\Omega)$.

In order to define the functional G, it is convenient to decompose any measure μ into an absolutely continuous part (with respect to the Lebesgue measure) with a density $\rho \in L^1$, and a singular part, which we denote by σ. The singular measure σ can be further decomposed into a Cantor part σ^c and an atomic part σ^0, so that we obtain

$$\mu = \rho \cdot dx + \sigma^c + \sigma^0 .$$

An index u to the measures $\mu, \rho, \sigma, \sigma^c, \sigma^0$ stands to denote that they are related to u via the equality $\mu_u = u^{\#}$.

The class of weakly$\star$ lower semicontinuous functionals on measures have been systematically studied by Bouchitté and Buttazzo in [6, 7, 8]; for simplicity here we limit ourselves to the ones which are invariant under translations in the x-variable. Then, in our case we have the characterization formula

$$G(\mu) = \int_{\mathbb{R}} g(\rho)\, dt + \int_{\mathbb{R}} g^\infty(\sigma^c) + \int_{\mathbb{R}} \vartheta(\sigma^0)\, d\mathscr{H}^0 \tag{2.6}$$

where:

- g is convex and lower semicontinuous;
- g^∞ is the recession function of g given by $g^\infty(z) = \lim_{s \to +\infty} g(sz)/s$;
- $\mathscr{H}^0$ is the counting measure;
- ϑ is a subadditive function satisfying the compatibility condition

$$g^\infty(z) = \lim_{s \to 0^+} \frac{\vartheta(sz)}{s} .$$

Example 2.5. Consider the functional (of *desired distribution penalization*)

$$G(\mu) = \begin{cases} \int_{\mathbb{R}} |\rho(t) - \rho_0(t)|^2\, dt & \text{if } \sigma \equiv 0 \\ +\infty & \text{otherwise.} \end{cases} \tag{2.7}$$

where ρ_0 is a given L^2 function. Then by Theorem 2.4 the minimization problem

$$\min\Big\{\int_\Omega |Du|^2\,dx \quad +\int_{\mathbb{R}} |\rho_u(t)-\rho_0(t)|^2\,dt \\ \quad : \ u\in H_0^1(\Omega),\ u^\# \ll \mathscr{L}^N\Big\}$$

admits at least one solution.

Example 2.6. Consider the Mumford-Shah like functional

$$G(\mu)=\begin{cases}\alpha\int_{\mathbb{R}}\rho^2(t)\,dt+\beta\mathscr{H}^0(\mathrm{supp}\ \sigma^0) & \text{if } \sigma^c\equiv 0\\ +\infty & \text{otherwise.}\end{cases}$$

(with $\alpha,\beta>0$) which is weakly$\star$ lower semicontinuous by the arguments above. Then by Theorem 2.4 the minimization problem

$$\min\Big\{\int_\Omega |Du|^2\,dx \quad +\alpha\int_{\mathbb{R}}\rho_u^2(t)\,dt+\beta\mathscr{H}^0(\text{atoms of } u^\#) \\ \quad : \ u\in H_0^1(\Omega),\ (u^\#)^c\equiv 0\Big\}$$

admits at least one solution. In the one-dimensional case, with $\Omega=(-1,1)$, it is easy to see that the minimum value of the problem is $\beta\wedge 6\alpha^{2/3}$, reached at $u\equiv 0$ if $\beta\le 6\alpha^{2/3}$ and at $u(x)=\alpha^{1/3}(1-|x|)$ if $\beta\ge 6\alpha^{2/3}$.

The framework above can be repeated in the case of vector valued functions $u:\Omega\to\mathbb{R}^m$. Setting again

$$u^\#(E)=\mathscr{L}^N\big(\Omega\cap u^{-1}(E)\big)$$

for every Borel subset E of $\mathbb{R}^m$, we have that $u^\#$ is a nonnegative measure on $\mathbb{R}^m$, with $u^\#(\mathbb{R}^m)=\mathscr{L}^N(\Omega)$. For instance, in the case $N=m$, if $u:\Omega\to\mathbb{R}^N$ is a regular invertible function we have that

$$u^\#=|\det Du^{-1}|\cdot\mathscr{L}^N\llcorner u(\Omega)\ .$$

By the way, this formula allows to define the Jacobian for nonregular functions v by

$$|\det Dv|:=(v^{-1})^\#,$$

compare, e.g., [20].

The previous tools, as Proposition 2.3 and Theorem 2.4, still hold and we may obtain existence results for minimum problems of the form (2.4) where now:

- F is a variational integral like
$$F(u)=\int_\Omega f(x,u,Du)\,dx$$
with $f(x,s,\cdot)$ quasiconvex and coercive (we refer to [13] for the lower semicontinuity results of integral functionals in the vector valued setting);
- G is a weakly$\star$ lower semicontinuous functional on measures.

For instance, as in Examples 2.5 and 2.6, the minimum problems

$$\min\Big\{ \int_\Omega f(Du)\,dx \quad + \textstyle\int_{\mathbb{R}^m} |\rho_u(y) - \rho_0(y)|^2\,dy \\ \qquad : \ u \in W_0^{1,p}(\Omega;\mathbb{R}^m),\ u^\# \ll \mathscr{L}^N \Big\}$$

and

$$\min\Big\{ \int_\Omega f(Du)\,dx \quad +\alpha \textstyle\int_{\mathbb{R}^m} \rho_u^2(y)\,dy + \beta\mathscr{H}^0(\text{atoms of } u^\#) \\ \qquad : \ u \in W_0^{1,p}(\Omega;\mathbb{R}^m),\ (u^\#)^c \equiv 0 \Big\}$$

both admit a solution, provided f is quasiconvex and $f(z) \geq c|z|^p$ with $c > 0$ and $p > 1$.

3. Generalized volume constraints

3.1. **Smooth rearrangements and existence.** In this section we consider a special case of the problem (2.4) by setting for a given measure μ_0 and a set $S \subset \mathbb{R}$:

$$G(\mu) := \begin{cases} 0 & \text{if } \mu|_S = \mu_0|_S \\ +\infty & \text{otherwise.} \end{cases} \tag{3.1}$$

This class of problems includes standard problems with volume constraints (also called "level set constraints") as considered, e.g., in [4, 17, 19, 23]. In fact, by setting $S := \{m_1, m_2, \dots, m_n\}$ and $\mu_0 = \sum_{i=1}^n \alpha_i \delta_{m_i}$ we obtain the classical volume constraints

$$|\{x \in \Omega :\ u(x) = m_i\}| = \alpha_i, \quad \text{for } i = 1, \dots, n. \tag{3.2}$$

Obviously, problem (3.1) is in general not solvable. Take, e.g., the measure $\mu_0 = 0$, the set $\Omega = (0,1)$, $S = \mathbb{R}$ and the Sobolev space $X = H^1(\Omega)$, then there is simply no function $u \in X$ such that $u^\# \equiv 0$. A first necessary condition is therefore $\int_S \mu_0 \leq |\Omega|$ (with equality if $S = \mathbb{R}$), compare (2.2). However, this condition is not sufficient, as it can be seen from the examples in [19] and [17]. One of the difficulties is that a minimizing sequence satisfying the constraint (3.1) may have a limit which does not satisfy the constraint. Indeed, it is easy to see that this can happen whenever S is not open. In other words, we have the following proposition:

Proposition 3.1. *The functional G defined by (3.1) is weakly$\star$ lower semicontinuous if and only if S is open.*

In the classical setting of finitely many volume constraints as in (3.2), S is a finite union of points and hence not open, which leads to a lack of semicontinuity of G, and allows for non-existence of solutions to the minimization problem, compare [17, 19].

In the following, we want to consider the case where S is open. For simplicity, we assume $S = \mathbb{R}$. Similar problems have been considered, e.g., in [9, 16].

According to Theorem 2.4, it is sufficient for the existence of a minimizer to prove that there is at least one function $u_0 \in X$ such that $F(u_0)+G(u_0^{\#}) < +\infty$. In the case of G given by (3.1), this can be restated as

$$K_{\mu_0}(X) := \{u \in X;\ u^{\#} = \mu_0\} \neq \emptyset.$$

Hence we arrive at the problem of finding a "smooth rearrangement" of a given value distribution μ_0. The following theorem gives sufficient conditions for this problem which are sharp for certain spaces X:

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary and let $1 < p < +\infty$. Let μ_0 be a non-negative measure with $|\mu_0| = |\Omega|$ and absolutely continuous part ρ, and let $S := \mathrm{conv}(\mathrm{supp}\,\mu_0)$, where $\mathrm{conv}(A)$ denotes the convex envelope of A. Then we have:*

(i) *If $\rho \geq C > 0$ on S, then $K_{\mu_0}(W^{1,\infty}_{(0)}(\Omega)) \neq \emptyset$,*

(ii) *If $1/\rho \in L^{p-1}(S)$ (for $p > 1$), then $K_{\mu_0}(W^{1,p}_{(0)}(\Omega)) \neq \emptyset$,*

(iii) *If S is bounded, then $K_{\mu_0}(\mathrm{BV}(\Omega)) \neq \emptyset$.*

Here $W^{1,p}_{(0)}$ means that a zero boundary condition can be (optionally) imposed as long as $0 \in S$. The condition (iii) is sharp, (i) and (ii) are not (see examples below).

An immediate consequence of Theorem 3.2 and Theorem 2.4 is the following corollary:

Corollary 3.3. *The minimization problem (2.4) with G defined by (3.1) and F weakly lower semicontinuous with respect to the function space $X \in \{\mathrm{BV}, W^{1,p}\}$, where $1 < p \leq +\infty$, admits a solution whenever the non-negative measure μ_0 satisfies $|\mu_0| = |\Omega|$ and the corresponding condition (i), (ii) or (iii) from Theorem 3.2 holds.*

We define $\Omega_t := \{x \in \Omega;\ \mathrm{dist}(x, \partial\Omega) > t\}$ and $\mathrm{Per}(\Omega_t) := \mathscr{H}^{N-1}(\partial\Omega_t)$. The following lemma collects some results on these functions:

Lemma 3.4. *Let $\Omega \subset \mathbb{R}^N$ be a bounded set with $\partial\Omega$ Lipschitz. Let $\omega(t) := |\Omega_t|$. Then $\omega'(t)$ and $\mathrm{Per}(\Omega_t)$ are uniformly bounded for a.e. $t \geq 0$.*

Proof: The uniform bound on $\mathrm{Per}(\Omega_t)$ follows from the following result by Ambrosio, compare Theorem 3.8 in [3]:

Theorem 3.5. *Let $A \subset \mathbb{R}^N$ and $\theta, \tau > 0$ such that*

$$\mathscr{H}^{N-1}(A \cap B_\rho(x)) \geq \theta\rho^{N-1} \quad \text{for all } x \in A,\ \rho \in (0, \tau). \tag{3.3}$$

Then there exists a constant $\Gamma < \infty$ only depending on N and θ such that

$$\mathrm{ess\,sup}\left\{\mathscr{H}^{N-1}(\{x \in \mathbb{R}^N;\ d(x, A) = t\});\ 0 < t < R\right\} \leq \Gamma\left(\frac{R}{\tau}\right)^{N-1}\mathscr{H}^{N-1}(A).$$

We can apply this theorem, since a Lipschitz continuous boundary $\partial\Omega$ satisfies (3.3).

Theorem 3.5 together with the fact that $\mathrm{Per}(\Omega_t) = 0$ for all $t > \mathrm{diam}(\Omega)/2$ proves the uniform bound on $\mathrm{Per}(\Omega_t)$.

Now using the Coarea Formula for the characteristic function of Ω_t we deduce

$$\omega(t) = \int_t^\infty \mathrm{Per}(\Omega_\tau)\, d\tau.$$

With this we can compute

$$\omega'(t) = \lim_{h\to 0} \frac{\omega(t+h) - \omega(t)}{h} = \lim_{h\to 0} \frac{1}{h} \int_t^{t+h} \mathrm{Per}(\Omega_\tau)\, d\tau.$$

Using the boundedness of $\mathrm{Per}(\Omega_t)$ a.e., the right hand side is bounded a.e. Hence a limit exists and is bounded a.e. $\square$

We remark that the use of Theorem 3.5 for the proof of Lemma 3.4 can be replaced at least in the two-dimensional case by a much easier result using some simple geometrical observations:

Lemma 3.6. *Let $\Omega \subset \mathbb{R}^2$ be a bounded set with $\partial\Omega$ Lipschitz. Then*

$$\mathrm{Per}(\Omega_t) \le \mathrm{Per}(\Omega) - 2\pi t(1 - h(\Omega)),$$

where $h(\Omega)$ denotes the number of holes in Ω, i.e. the number of bounded connected components of $\mathbb{R}^2 \setminus \Omega$. (This number is finite since Ω is bounded and $\partial\Omega$ is Lipschitz.)

Proof: By polygonal approximation of $\partial\Omega$ from inside and by the weak semi-lower-continuity of the perimeter we can reduce the problem to the case where Ω is a polygon with k sides.

Let us assume for simplicity that Ω is simply connected. Let E_i denote the edges of Ω and α_i the angle between E_i and E_{i+1} for $i = 1, \dots, k$ (where $E_{k+1} := E_1$). Then an easy geometric construction shows that

$$\begin{aligned} \mathrm{Per}(\Omega_t) &\le \sum_{i=1}^{N} \big(\mathscr{H}^1(E_i) + (\alpha_i - \pi)t\big) \\ &= \mathrm{Per}(\Omega) - 2\pi t, \end{aligned}$$

since the sum of all angles in a k-sided polygon is $(k-2)\pi$.

The proof can be easily modified to the general case where Ω is not simply connected by an additional approximation of the holes by polygons. $\square$

Proof of Theorem 3.2:
We only need to construct a function u in the class X, satisfying the constraint $u^\# = \mu_0$.

In case (iii), since in one-dimension BV functions are bounded, the boundedness of S is a necessary condition. In cases (i) and (ii), we have $\rho \in L^1$ and $1/\rho \in L^\varepsilon$ for some $\varepsilon > 0$ and a short computation using Jensen's Inequality shows that therefore the support of ρ must be bounded. Hence S is bounded as well.

For all three cases, we apply the following construction:

Define $\sigma := \mu_0 - \rho\, dt$ and $R' := \rho$ (R exists, since ρ is integrable). Using standard mollifiers ψ_ε one can set $\rho_\varepsilon := \rho + \sigma * \psi_\varepsilon$ where ρ_ε is absolutely continuous and $\lim_{\varepsilon\to 0} \rho_\varepsilon = \rho$. We remark that if ρ nonnegative, then also the (partially) mollified measures ρ_ε are nonnegative. Similar as above we define $R'_\varepsilon := \rho_\varepsilon$.

Then we can make the following ansatz, using $t(x) := \operatorname{dist}(x, \partial\Omega)$:

$$u_\varepsilon(x) := g_\varepsilon(t(x)),$$

where we set

$$g_\varepsilon(t) := R_\varepsilon^{-1}(\omega(t)).$$

The function R_ε is invertible, since by assumption $1/\rho \in L^{p-1}(S)$ and hence $|\{\rho = 0\}| = 0$ and $|\{\rho_\varepsilon = 0\}| = 0$.

Computing the gradient of u_ε we get

$$\begin{aligned} |\nabla u_\varepsilon(x)| = |g'_\varepsilon(t(x))| &= \left|\frac{1}{\rho_\varepsilon(R_\varepsilon^{-1}(\omega(t(x)))}\right| \omega'(t) \\ &\leq \left|\frac{1}{\rho(R_\varepsilon^{-1}(\omega(t(x)))}\right| \omega'(t). \end{aligned} \tag{3.4}$$

In the last step we used that σ is nonnegative and hence $\rho_\varepsilon = \rho + \sigma * \psi_\varepsilon \geq \rho$.

Case (i):
If $\rho \geq C > 0$, we see from (3.4) that $|\nabla u_\varepsilon(x)|$ is bounded uniformly, independently of ε. Thus we can take the limit in $W^{1,\infty}$ (for a subsequence) and obtain a limit function u which is Lipschitz continuous. Together with $u_\varepsilon^\# = \mu_\varepsilon$ and Proposition 2.3 we derive $u^\# = \mu_0$. Hence case (i) is proved.
Cases (ii)-(iii):

We compute the $W^{1,p}$-seminorm (for $p \in [1, \infty)$), introducing geometric constants $c_i > 0$, depending only on the space dimension and on the shape of Ω:

$$\begin{aligned} \int_\Omega |\nabla u_\varepsilon(x)|^p\, dx &= \int_\Omega \frac{1}{|\rho_\varepsilon(R_\varepsilon^{-1}(\omega(t(x)))|^p} |\omega'(t)|^p\, dx \\ &\leq \int_\Omega \frac{1}{|\rho_\varepsilon(R_\varepsilon^{-1}(\omega(t(x)))|^p} c_1\, dx \\ &\leq \int_0^T \frac{c_2}{|\rho(R_\varepsilon^{-1}(\omega(t))|^p}\, dt, \end{aligned}$$

where T is the largest distance of a point in Ω from $\partial\Omega$.

Using the transformation $s := \omega(t)$ (remember that ω is a decreasing function and ω' is bounded) and $\xi := R_\varepsilon^{-1}(s)$, and defining $a := R_\varepsilon^{-1}(0)$ and $b := R_\varepsilon^{-1}(T)$ we obtain:

$$\int_\Omega |\nabla u_\varepsilon(x)|^p\, dx \leq c_3 \int_a^b \frac{1}{|\rho(\xi)|^{p-1}}\, d\xi. \tag{3.5}$$

Since the function R is bounded by $\mu_0(\mathbb{R}) = |\Omega| < \infty$, the $W^{1,p}$-seminorm is uniformly bounded. The L^p-norm is obviously finite, since u_ε is bounded. Thus there exists a subsequence of u_ε converging to a function u in $W^{1,p}$. Together with $u_\varepsilon^\# = \mu_\varepsilon$ and Proposition 2.3 we obtain $u^\# = \mu_0$.

In case (iii) we define

$$\tilde{\mu}_\varepsilon := (1-\varepsilon)\mu_0 + \varepsilon \frac{\mathscr{L}^1 \llcorner S}{|S|},$$

where $\mathscr{L}^1 \llcorner S$ denotes the Lebesgue measure restricted to the set S.

Since S is by assumption bounded, each $\tilde{\mu}_\varepsilon$ satisfies the condition of case (i), i.e. $\tilde{\mu}_\varepsilon \geq C > 0$. Hence we can construct a Lipschitz continuous function $\tilde{u}_\varepsilon$ with $\tilde{u}_\varepsilon^\# = \tilde{\mu}_\varepsilon$ and obtain an estimate for $|D\tilde{u}_\varepsilon|$ corresponding to (3.5) for the special case $p = 1$, i.e.

$$\int_\Omega |\nabla \tilde{u}_\varepsilon(x)|\, dx \quad \leq \quad c_3(b-a). \tag{3.6}$$

Since this estimate is independent of ε, and on the other hand the sequence $\tilde{u}_\varepsilon$ converges in $L^1(\Omega)$ to a function u, a standard result on BV-functions gives that $u \in BV(\Omega)$. Moreover, by Proposition 2.3 we obtain $u^\# = \mu_0$. Thus we have proved case (iii). □

3.2. **Examples.** We consider first some elementary situations:

Example 3.7. *Let $\mu_0 = \alpha\delta_a + \beta\delta_b$ and $|\Omega| = \alpha + \beta$.*

According to Theorem 3.2, we can find a function $u \in \mathrm{BV}(\Omega)$ with $u^\# = \mu_0$. Indeed, any BV-function which takes the value a on a set of measure α and b on a set of measure β satisfies this constraint. It is easy to see that if $a \neq b$ no $W^{1,p}$-function u satisfies the equality $u^\# = \mu_0$.

This example occurs for instance in shape optimization problems when we search for the optimal distribution of two materials a and b of given amounts α and β in a set Ω. Here the function u satisfying the above constraint corresponds to such a distribution.

Example 3.8. *Let $\mu_0 = \chi_{[0,1]}\, dy$ and $\Omega = (0,1)$.*

Again by Theorem 3.2, we can find a function $u \in W^{1,\infty}(\Omega)$ with $u^\# = \mu_0$. In fact, we can simply choose $u(x) := x$. However, we may also choose $u(x) := 2x \wedge 4 - 2x$ and other highly oscillating functions (see Fig. 2). We could even choose a function u which has a jump, take, e.g.,

$$u(x) := \begin{cases} 2x & \text{for } x \leq 1/2, \\ 2x - 2 & \text{for } x > 1/2. \end{cases}$$

The control on $u^\#$ gives therefore no more regularity than the obvious L^∞.

This example shows that a naive attempt to a characterization of Sobolev functions via their value distribution fails. However, we could try to impose

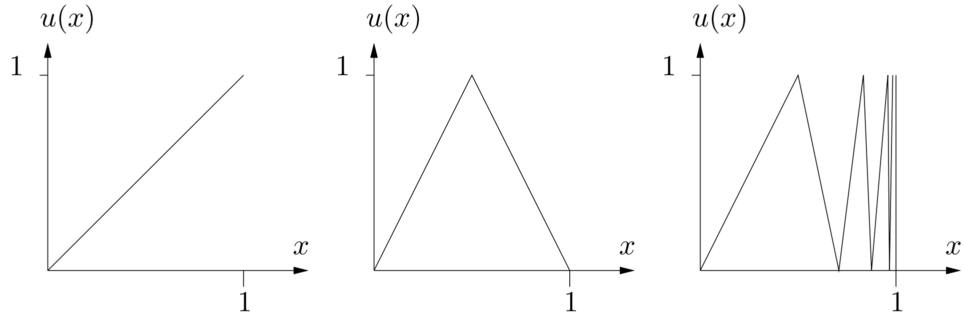

FIGURE 2. Functions satisfying the constraint of Example 3.8.

a local condition in the following way: For all open subsets $\omega \subset \Omega$ assume that $\mu := (u|_\omega)^\#$ satisfies the condition (i), i.e. ρ, the absolutely continuous part of μ, fulfills the estimate $\rho \geq C > 0$ on $S := \mathrm{conv}(\mathrm{supp}\,\mu)$. This excludes examples with jumps, since in a small neighborhood of a jump the set S has a "gap", i.e. $\rho = 0$ on a non-zero subset of S. However, the following example shows, that such a "local" classification fails as well:

Example 3.9. *Let $u : (-1,1) \to \mathbb{R}$ be given by*

$$u(x) := \begin{cases} 2^n(x + 2^{1-n}) & \text{for } -2^{1-n} < x \leq -2^{-n},\ n \text{ odd}, \\ 1 - 2^n(x + 2^{1-n}) & \text{for } -2^{1-n} < x \leq -2^{-n},\ n \text{ even}, \\ 0 & \text{for } x = 0 \end{cases}$$

and $u(x) := u(-x)$ for $0 < x < 1$, see Fig. 3.

Then in any neighborhood ω of the "irregular" point $x_0 = 0$ the corresponding value distribution $\mu = \rho\, dx + \sigma$ satisfies $\rho \geq C > 0$ on $S := \mathrm{conv}(\mathrm{supp}\,\mu)$ for some constant $C = C(\omega)$.

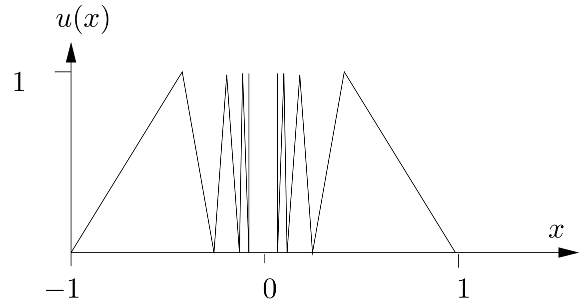

FIGURE 3. The non-continuous function of Example 3.9.

We conclude with a two-dimensional example on a disk:

Example 3.10. *Let $\Omega := B(0,1) \subset \mathbb{R}^2$, $\mu = c(1-y)^\alpha \chi_{[0,1]}$ where the constant c is chosen such that $|\mu| = |\Omega|$.*

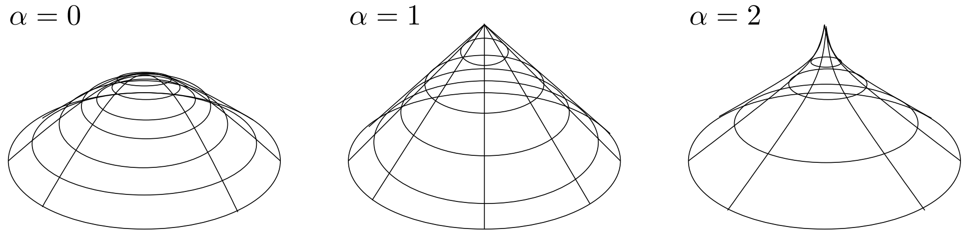

FIGURE 4. Functions u with $u^{\#} = c(1-y)^{\alpha}\chi_{[0,1]}$ for different values of α.

Using the computation of the proof of Theorem 3.2, we get $u(x) = R^{-1}(\omega(1-|x|)$. With $\omega(1-|x|) = c_1(1-|x|)^n$ and $R(t) = c\alpha^{-1}(1-t)^{\alpha+1}+c_4$, we finally compute that $u(|x|)$ is proportional to $|x|^2\alpha + 1$ (modulo a constant, needed to adjust the boundary value). Hence we get (compare Fig. 4),

- for $\alpha = 0$, the solution u is of order $O(|x|^2)$ near $x = 0$;
- for $\alpha = 1$, the solution u is of order $O(|x|)$ near $x = 0$;
- for $\alpha = 2$, the solution u is of order $O(|x|^{2/3})$ near $x = 0$.

This means in particular that for $\alpha = 1$ we have Lipschitz continuity, although the condition (i) of Theorem 3.2 is violated. The potential regularity problem disappears, since the critically small value of the measure $u^{\#}$ at $y = 1$ is mapped to a point. If it were mapped to a line (for instance, if we imposed the boundary condition $u = 1$), the gradient of u had to grow too fast to allow for Lipschitz regularity.

In the next section we will use this observation for the construction of a symmetry breaking solution.

4. Symmetry breaking solutions

It is interesting to study the symmetry of solutions in symmetric domains. As an immediate consequence of standard results on symmetric rearrangements we first obtain the following theorem.

Theorem 4.1. *Let Ω be a unit ball and assume that there exists a minimizer $u \in W_0^{1,2}(\Omega)$ to a functional $E(u) := \int_\Omega f(|\nabla u|, u)\, dx + G(u^{\#})$ where f is strictly increasing and convex in the first variable and G is defined as in (3.1). Then there exists a minimizer which is* radially symmetric.

PROOF: This follows immediately by applying the Schwarz rearrangement (see [15]), since the push-forward of a function does not change when the function is rearranged. □

It is clear that this result strongly depends on the symmetry of the domain Ω. However, it is interesting to see that its generalization to arbitrary radially symmetric domains may fail. In fact, we can construct an example where a radially symmetric problem admits only asymmetric solutions. (As

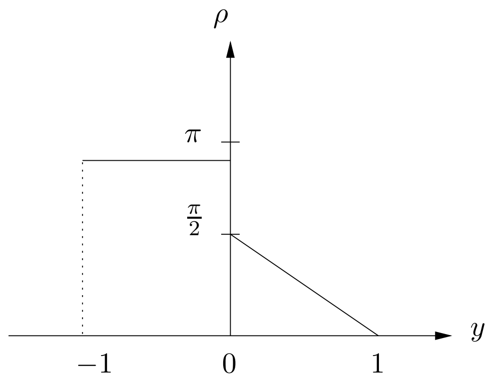

FIGURE 5. The auxiliary function ρ.

we will show later, even on the ball such examples exists when we omit the zero boundary condition.)

A "symmetry breaking" variational problem can be constructed by taking in the plane $\mathbb{R}^2$ the annulus $\Omega = \{1 < |x| < 2\}$ and the function (compare Fig. 5)

$$\rho(t) := \begin{cases} c & \text{if } -1 \le t \le 0, \\ k(1-t) & \text{if } 0 < t \le 1, \end{cases}$$

with $c = 11\pi/12$ and $k = \pi/2$. If B is the ball centered at $(3/2, 3/2)$ and with radius $1/2$ (compare Fig. 6), we can easily check that

$$\int_{-1}^{0} \rho(t)\,dt = |\Omega \setminus B|, \qquad \int_{0}^{1} \rho(t)\,dt = |B|.$$

Theorem 4.2. *The minimization problem*

$$\min\Big\{ \int_\Omega |\nabla u|^2\,dx \;:\; u \in H_0^1(\Omega),\ u^{\#} = \rho\cdot dt \Big\} \tag{4.1}$$

admits a solution, and every solution is asymmetric.

PROOF: We first prove that K_μ is nonempty. To this aim we define

$$u_1(r) = 1 - \sqrt{2 - 4r^2} \qquad r \in [0, 1/2]$$

inside the ball B, where r stands for the corresponding radial coordinate. In particular, $u_1 \in H_0^1(B)$. By using Theorem 3.2 we then find a function u_2 such that:

$$u_2 \in H_0^1(\Omega \setminus B), \qquad u_2^{\#} = \rho \cdot dt \llcorner \mathbb{R}^-.$$

We finally glue u_1 to u_2 and we obtain the function

$$u(x) := \begin{cases} u_1(x) & \text{if } x \in B, \\ u_2(x) & \text{if } x \in \Omega \setminus B, \end{cases}$$

which satisfies:

$$u \in H_0^1(\Omega), \qquad u^{\#} = \rho \cdot dt.$$

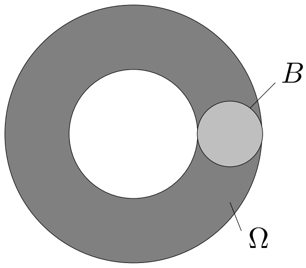

FIGURE 6. Construction of a symmetry breaking solution.

The existence of a solution to problem (4.1) is therefore guaranteed by Theorem 2.4.

It remains to prove the nonexistence of a radially symmetric solution. Let us assume that such a solution u exists. Since the decreasing rearrangement does not increase the Dirichlet integral $\int_\Omega |\nabla u|^2\,dx$ and does not change the push-forward measure $u^\#$, we may also assume that u coincides with its decreasing rearrangement. By the coarea formula we then obtain

$$\begin{aligned}\int_\Omega |\nabla u|^2\,dx &= \int_{-1}^{1}\Big(\int_{\{u=t\}} |\nabla u|\,d\mathscr{H}^1\Big)\,dt\\ &= \int_{-1}^{1}\frac{|2\pi u^{-1}(t)|^2}{\rho(t)}\,dt \geq 4\pi^2\int_{-1}^{1}\frac{1}{\rho(t)}\,dt.\end{aligned}$$

This is clearly a contradiction, since $1/\rho(t)$ is not summable on $(-1,1)$ as it is easily checked. □

Heuristically, this symmetry breaking can be explained in the following way: A function satisfying the constraint has to have very small level sets close to $y=1$. This is either possible if the set $u^{-1}(1)$ is a point (and hence the surrounding level sets can shrink making $u^\#$ small), or if the function becomes very steep (which is also making $u^\#$ small). In the latter case, the function has a singularity and hence is not admissible for the variational problem. However, the first case is excluded by the geometry of Ω when we allow only radially symmetric functions.

We can use this idea to construct a similar example on the ball which shows that the possibility for a "symmetry breaking solution" is not restricted to topological complicated domains, but is a natural property of our variational problem:

Theorem 4.3. *Let Ω be the open unit ball in $\mathbb{R}^2$ and $\rho(t) := \pi - \pi|t|$ for $t \in [-1,+1]$. Then the minimization problem*

$$\min\Big\{\int_\Omega |\nabla u|^2\,dx \ :\ u \in H^1(\Omega),\ u^\# = \rho\cdot dt\Big\} \tag{4.2}$$

admits a solution, and every solution is asymmetric.

PROOF: Again, we first construct an admissible function. This time we cut out two small disjoint balls in Ω on which we define u explicitly as above with values in $(-1, -1+\varepsilon)$ and $(1-\varepsilon, 1)$ for suitable $\varepsilon > 0$. Then the problem on the remaining set can be solved by Corollary 3.3 (with only minor modifications due to the slightly different boundary condition).

To prove that there cannot be a radially symmetric solutions, we observe that $u(0)$ can only take the value of one of the "critical" points -1 and $+1$. The other one has to be stretched out along a circle which leads to an infinite Sobolev norm as in the example above. $\square$

The above examples also work for non-constrained situations as the following simple observation shows:

Remark 4.4. *Minimize* $\int_\Omega |\nabla u|^2 + \lambda |u^\# - \rho|^2 \, dx$ *without constraints on* $u^\#$. *Then for* λ *sufficiently large, the symmetry breaking examples above still hold, since* $u^\#$ *will be forced to be sufficiently close to* ρ.

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