

Nonexistence of Minimizers for a Nonlocal Perimeter Functional with a Riesz and a Background Potential

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Abstract

We consider the nonexistence of minimizers for the energy containing a nonlocal perimeter with a general kernel K , a Riesz potential, and a background potential in \mathbb{R}^N with $N \geq 2$ under the volume constraint. We show that the energy has no minimizer for a sufficiently large mass under suitable assumptions on K . The proof is based on the partition of a minimizer and the comparison of the sum of the energy for each part with the energy for the original configuration. This strategy is based on [6] and [11].

1 Introduction

We study the minimizing problem for the following functional:

$$\mathcal{F}_{(K,A)}(E) := P_K(E) + V_1(E) - A R(E) \quad (1.1)$$

under the volume constraint $|E| = m$ where $E \subset \mathbb{R}^N$, $N \geq 2$, is measurable, and $A \geq 0$. Here $P_K(E)$ is the nonlocal perimeter with a general kernel K defined by

$$P_K(E) := \int_E \int_{E^c} K(x-y) dx dy, \quad (1.2)$$

where $E^c := \mathbb{R}^N \setminus E$ and the kernel K satisfies the following conditions:

(K1) $K(x) > 0$, and $K(-x) = K(x)$ for any $x \in \mathbb{R}^N$.

(K2) $K \in L^1(B_1^c(0))$.

(K3) there exists $\beta \geq N + s$ such that

$$\left| 1 - \frac{|h|}{|x|} \right|^\beta \leq \frac{K(x)}{K(x+h)} \leq \left| 1 + \frac{|h|}{|x|} \right|^\beta \quad (1.3)$$

for any $x, h \in \mathbb{R}^N$ with $x \neq 0$ and $0 < |h| < 1$.

Moreover, in our study, we further impose either of the following assumptions:

(K4) there exists constants $0 < s < 1$ and $\varepsilon > 2^{\frac{1}{N+s-1}} - 1 > 0$ such that, for any $0 < |x| < 1 + \varepsilon$, $K(x) \leq |x|^{-(N+s)}$ and, for any $|x| \geq 1 + \varepsilon$, $|x|K(x) \leq (1 + \varepsilon)^{-(N+s-1)}$.

(K4)' there exists constants $0 < s < 1$, $\varepsilon > 2^{\frac{1}{N+s-1}} - 1 > 0$, and $\lambda > 1$ such that $|x|^{-(N+s)} \leq K(x) \leq \lambda |x|^{-(N+s)}$ for any $0 < |x| < 1 + \varepsilon$ and $|x|^{-(N+s-1)} \leq |x|K(x) \leq (1 + \varepsilon)^{-(N+s-1)}$ for any $|x| \geq 1 + \varepsilon$.

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For instance, a kernel $K(x) := |x|^{-(N+s)}$ is one of the examples satisfying all of the above conditions on K . Indeed, we may easily show that K satisfies (K1), (K2), (K4), and (K4)'. Moreover, from the Taylor expansion of $|x|^{N+s}$, we have that K satisfies the assumption (K3). From the above conditions, we can roughly illustrate the shape of K in Figure 1 and 2.

Moreover, V_1 is the Riesz potential and is defined by

$$V_1(E) := \frac{1}{2} \int_E \int_E \frac{1}{|x-y|} dx dy, \quad (1.4)$$

for any measurable $E \subset \mathbb{R}^N$. Specifically, in the case of $N = 3$, V_1 is said to be the Coulombic repulsive potential. Finally, $R(E)$ is defined by

$$R(E) := \int_E \frac{1}{|x|} dx, \quad (1.5)$$

for any measurable $E \subset \mathbb{R}^N$.

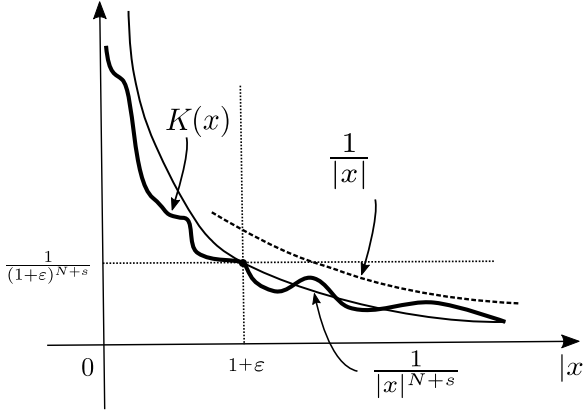


Figure 1: Graph of K with (K4)

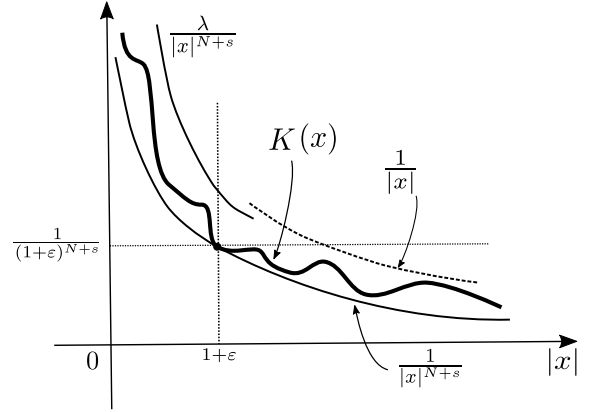


Figure 2: Graph of K with (K4)'

The study of the variational problem of (1.1) is one extension of a series of the previous works [8, 10, 12, 13], in which they treated the classical perimeter instead of the nonlocal one. Especially, when $N = 3$ and $A \equiv 0$, it is important to consider this minimizing problem in physics. It is known as the liquid drop model, introduced by Gamow [7] to model the stability of atomic nuclei and nuclear fission.

In this paper, we consider the nonexistence of the minimizers for (1.1) all over the measurable sets in \mathbb{R}^N with the volume constraint. Note that, considering the general Riesz potential V_α , instead of V_1 , which is defined by

$$V_\alpha(E) := \frac{1}{2} \int_E \int_E \frac{1}{|x-y|^\alpha} dx dy \quad (1.6)$$

for any $E \subset \mathbb{R}^N$ and some $\alpha \in (0, N)$, we are not able to show the nonexistence of minimizers so far. This is because we do not overcome the technical difficulties in estimating V_α to obtain the claim (see the proof of Theorem 2.2 for details).

If we consider each term in (1.1) separately and we choose a function $|x|^{-(N+s)}$ as the kernel K , then it is known that, by the isoperimetric inequality for the nlocal perimeter, a ball is the only minimizer for P_K all over the sets with finite volume and, by Riesz rearrangement inequality, a ball is the only maximizer for both V_α and R under the volume constraint. Thus, it is not trivial whether the minimizing problem of (1.1) has a solution under the volume constraint or

not. Moreover, we may not expect the existence of a minimizer for (1.1) if m is large. Indeed, considering the rescaled set λE , we observe

$$\mathcal{F}_{(K,A)}(\lambda E) = \lambda^{N-s} P_K(E) + \lambda^{2N-1} V_1(E) - \lambda^{N-1} A R(E). \quad (1.7)$$

Thus, when λ is large, the dominating term is $\lambda^{2N-1} V_1(E)$, however V_1 does not admit minimizers. Note that, if λ is sufficiently small, then the dominating term is $\lambda^{N-s} P_K(E)$ because of $s < 1$. Especially, in the case that $K(x) = |x|^{-(N+s)}$, it is known that P_K admits a minimizer and it is actually a ball. This implies that, if λ is small, then there may exist a minimizer for (1.1) and it should be a ball in \mathbb{R}^N .

Now let us briefly review the previous works concerned with our problem. In the case of the classical perimeter and the general Riesz potential V_α instead of P_K and V_1 in (1.1), Knüpfer and Muratov proved a series of the results stated in [9, 10] as follows; if $N = 2$, $\alpha \in (0, 2)$, and $A \equiv 0$, balls are the only minimizer under the volume constraint $|E| = m$ for sufficiently small $m > 0$. In addition, for sufficiently large $m > 0$, there are no minimizers. Finally, in higher dimensions, if $3 \leq N \leq 7$ and $\alpha \in (0, N - 1)$, then the balls are the only minimizers for sufficiently small $m > 0$. In addition, Bonacini and Cristofori extended some of the above results to the case $N \geq 3$ in [1]. Especially, regarding to the nonexistence of minimizers, not only Knüpfer and Muratov but also Lu and Otto showed in [13] the following result; if $N = 3$ and $A \neq 0$, then there exists a number $m_0 > 0$, which is explicitly obtained, that for any $m \geq m_0$, the minimizing problem $\min\{P(E) + V_1(E) - A R(E) \mid |E| = m\}$ has no solution. In the case of the nonlocal perimeter, Figalli et al. in [4] considered the isoperimetric problems with $K(x) = |x|^{-(N+s)}$ in the presence of the Riesz potential if $N \geq 2$. Precisely, they showed that, if m is positive number less than some number m_0 , then the problems

$$\inf \left\{ \frac{1-s}{\omega_{N-1}} P_s(E) + V_{N-\alpha}(E) \mid |E| = m \right\}, \quad P_s(E) := \int_E \int_{E^c} \frac{1}{|x-y|^{N+s}} dx dy, \quad (1.8)$$

where ω_{N-1} is the volume of the unit sphere \mathbb{S}^{N-1} , admit balls of volume m as their (unique up to translation) minimizers for any $s \in (s_0, 1)$ and $\alpha \in (\alpha_0, N)$ with some uniform lower bounds $s_0, \alpha_0 > 0$.

In this paper, we established the general results of the nonexistence of the minimizers in the case of nonlocal perimeter. Precisely, we obtain the following two results; the first one is that, if we assume that K satisfies (K1), (K2), (K3), and (K4)' and assume $A \geq 0$ and $m > 0$, then every minimizer E of $\mathcal{F}_{(K,A)}$ is bounded. The second one is that, if we assume that K satisfies (K1), (K2), and (K4) and $m > m_0(N, s, \varepsilon)$, A , then $\min\{\mathcal{F}_{(K,A)}(E) \mid |E| = m\}$ has no bounded solutions. Accordingly, if we assume that the conditions (K1), (K2), (K3), and (K4)' are valid, then we may have that $\min\{\mathcal{F}_{(K,A)}(E) \mid |E| = m\}$ has no solutions. We emphasize that our result is one complementing result shown in [4]. Indeed, if we set $A \equiv 0$ and $K(x) = |x|^{-(N+s)}$, then our result gives us the solution to the nonexistence problem for the functional $P_s(E) + V_1(E)$ under the volume constraint $|E| = m$, while the authors in [4] considered the existence of minimizers for the functional $P_s(E) + V_\alpha(E)$ where $\alpha \in (\alpha_0, N)$ for some $\alpha_0 > 0$ under the volume constraint.

Our idea for proving the boundedness of minimizers is based on [2]. First, we will show that the energy (1.1) is stable under some sufficiently small perturbation near a point in the measure-theoretic boundary of a minimizer. This stability is what we call ‘‘Almgren’s lemma’’. Secondly, if we suppose that a minimizer is not bounded, this stability and the minimality make it possible that one can derive an integral inequality of the volume of a minimizer outside of some ball, which leads to a contradiction. On the other hand, the idea for proving the main theorem, i.e., the nonexistence of minimizers is based on [8, 14, 6, 11]. First of all, we recall that, if $K(x) = |x|^{-(N+s)}$, then Knüpfer and Muratov showed in [9, 10] the nonexistence of minimizers by the following idea; first, they showed that the minimizers with large volumes must be long and thin; otherwise the Riesz potential can gain the energy. Thus, they cut the minimizers for large masses into two parts by some hyperplane. Then, they moved the two parts far away from each other to reduce

the energy. As a result, they can compare the energy of the sum of the two sets with that of the original configuration, to find that it can be represented as a differential inequality for the volumes of a cross-section of the minimizer by different hyperplanes. In our case, however, due to the last term of (1.1), even if we cut the minimizers into two parts and move them far away from each other, there is a possibility that one may decrease the total energy. Thus, the idea in [9, 10] might not work in our case. Instead, we will take the following strategy; first, we separate \mathbb{R}^N into two parts by a hyperplane which is parametrized by a directional parameter $\nu \in \mathbb{S}^{N-1}$ and a translating parameter $l \in \mathbb{R}$. Then, taking any minimizer of (1.1) and considering the intersection of the minimizer by each separated part (either of them can be an empty set), we compare the sum of the energy for each intersection with that of the original set. Integrating the resulting inequality with respect to l and then ν , we can obtain the inequality of the mass of a minimizer to find that it actually shows the upper bound of the mass.

This paper is divided into four sections. In Section 2, we state the two results, one is on the boundedness of minimizers and the other is on the nonexistence of minimizers. In Section 3, we first show the boundedness of minimizers under the assumptions (K1), (K2), (K3), and (K4)' on the kernel $K(x)$. In Section 4, we next prove the nonexistence of minimizers. Before proving the nonexistence, we first show the weak subadditivity of the infimum of the energy (1.1) with respect to the mass $m > 0$. Finally, in Section 5, we state one possible generalization of the energy and the proof of the nonexistence of minimizers for it. This is obtained by a simple modification of the proof done in Section 4.

2 Statement of results

Our main results in this paper is as follows; the first one is on the boundedness of the minimizer and the second one is on the nonexistence of the minimizers.

Lemma 2.1 *Suppose that $N \geq 2$ and K satisfies (K1), (K2), (K3), and (K4)' and let $A \geq 0$ and $m \in (0, \infty)$. Then, for every minimizer E of $\mathcal{F}_{(K,A)}$, E is bounded.*

Theorem 2.2 *Suppose that $N \geq 2$ and K satisfies (K1), (K2), and (K4) and let $A \geq 0$. Then, there exists a critical mass $m_c > 0$ given by*

$$m_c := \left(\frac{1}{2} - \frac{1}{(1+\varepsilon)^{N+s-1}} \right)^{-1} \left(\frac{\omega_{N-1}}{1-s} (1+\varepsilon)^{1-s} + A \right) \quad (2.1)$$

such that, for any $m > m_c$,

$$\min\{\mathcal{F}_{(K,A)}(E) \mid E \subset \mathbb{R}^N \text{ measurable, } |E| = m\} \quad (2.2)$$

has no bounded solutions.

In addition, if we impose the conditions (K3) and (K4)' instead of (K4) on K , then, for any $m > m_c$ where m_c is the same critical mass as (2.1), (2.2) has no solutions.

The proof is based on [6, 11], however, we need to pay more attention on the calculation of the nonlocal perimeter in order to obtain the explicit lower bound of the mass.

3 Boundedness of minimizers

In this section, we prove Lemma (2.1), i.e., the boundedness of minimizers for the functional (1.1) under some conditions on a kernel K . First of all, we start to show a generalized version of so-called Almgren's lemma (see [15] for the classical results, [5] in the case of an anisotropic perimeter, or [2] in the case of a nonlocal s-perimeter).

Lemma 3.1 Suppose that K satisfies (K1), (K2), and (K3). Let $E \subset \mathbb{R}^N$ be a measurable set with $P_K(E) + V_1(E) + R(E) < \infty$. Let $x_0 \in \mathbb{R}^N$ and $r_0 > 0$ be such that

$$|B_{r_0}(x_0) \cap E| > 0, \quad |B_{r_0}(x_0) \cap E^c| > 0. \quad (3.1)$$

Then, there exist $k_0, C > 0$ such that, for any $k \in (-k_0, k_0)$, there exists a measurable set $F_k \subset \mathbb{R}^N$ such that $P_K(F_k) + V_1(F_k) + R(F_k) < \infty$ and the following properties hold:

1. $E \Delta F_k \Subset B_{r_0}(x_0)$
2. $|F_k| - |E| = k$
3. $|\mathcal{F}_{(K,A)}(F_k) - \mathcal{F}_{(K,A)}(E)| < Ck$.

Proof. From the density assumption, we have that there exists a function $T \in C_c^1(B_{r_0}(x_0); \mathbb{R}^N)$ such that

$$M := \int_E \operatorname{div} T(x) dx > 0. \quad (3.2)$$

Otherwise, by the definition of the classical perimeter, it holds that $P(E, B_{r_0}(x_0)) = 0$. However, by the classical isoperimetric inequality, we have $|E \cap B_{r_0}(x_0)| = 0$, which contradicts the assumption of E .

For any $t \in (-1, 1)$, we define the maps $\Psi_t(x) := x + tT(x)$ for all $x \in \mathbb{R}^N$. Then, we may easily see that there exists $\delta_0 \in (0, 1)$ such that the maps Ψ_t are diffeomorphisms from \mathbb{R}^N onto itself for any $t \in (-\delta_0, \delta_0)$. Moreover, we have that $\det(\nabla \Psi_t(x)) = 1 + t \operatorname{div} T(x) + o(|t|)$ for any $t \in (-\delta_0, \delta_0)$. By the definition of Ψ_t , we also have $E \Delta \Psi_t(E) \subset \subset \mathbb{R}^N$ and thus, applying the change of variables,

$$|\Psi_t(E)| = \int_E |\det \nabla \Psi_t(x)| dx = \int_E (1 + t \operatorname{div} T(x) + o(t)) dx = |E| + tM + o(|t|), \quad (3.3)$$

for sufficiently small $t \in (-\delta_0, \delta_0)$. Therefore, there exists a constant $k_0 > 0$ such that, if we set $F_k := \Psi_{t(k)}(E)$, where $t(k) := k/M + o(|k|)$, for any $k \in (-k_0, k_0)$, F_k satisfies the first and second properties in Lemma 3.1.

Now, from (K3) and since $\Psi_t(x) \neq \Psi_t(y)$ for sufficiently small t and $x \neq y$, it holds that

$$\left(1 + |t| \frac{|T(x) - T(y)|}{|x - y|}\right)^{-\beta} \leq \frac{K(\Psi_t(x) - \Psi_t(y))}{K(x - y)} \leq \left(1 - |t| \frac{|T(x) - T(y)|}{|x - y|}\right)^{-\beta} \quad (3.4)$$

for any $x, y \in \mathbb{R}^N$ with $x \neq y$ and sufficiently small $|t| > 0$. From the compactness of the support and the regularity of T , there exists a constant $C_0 = C_0(T) > 0$ such that

$$\frac{K(x - y)}{(1 + C_0 |t|)^\beta} \leq K(x - y + t(T(x) - T(y))) \leq \frac{K(x - y)}{(1 - C_0 |t|)^\beta} \quad (3.5)$$

for any $x, y \in \mathbb{R}^N$ with $x \neq y$ and sufficiently small t . Moreover, the inequality (3.5) holds even if $x = y$ since all the sides in (3.5) take values of infinity. Thus, by applying (3.5), we may compute the difference of the energies, i.e., $|\mathcal{F}_{(K,A)}(F_k) - \mathcal{F}_{(K,A)}(E)|$ as follows; first of all, from (3.5), we may have

$$\begin{aligned} & \left| K(\Psi_t(x) - \Psi_t(y)) |\det \nabla \Psi_t(x)| |\det \nabla \Psi_t(y)| - K(x - y) \right| \\ &= \left| K(x - y + t(T(x) - T(y))) (1 + t(\operatorname{div} T(x) + \operatorname{div} T(y)) + o(|t|)) - K(x - y) \right| \\ &\leq |t|(\tilde{C}_0(T) + |o(1)|) K(x - y), \end{aligned} \quad (3.6)$$

for any $x, y \in \mathbb{R}^N$ with $x \neq y$ where $\tilde{C}_0(T) > 0$ is some constant. Hence, it holds that

$$\begin{aligned}
& |P_K(\Psi_t(E)) - P_K(E)| \\
& \leq \int_E \int_{E^c} \left| K(\Psi_t(x) - \Psi_t(y)) |\det \nabla \Psi_t(x)| |\det \nabla \Psi_t(y)| - K(x - y) \right| dx dy \\
& \leq |t| \int_E \int_{E^c} C_1(T) K(x - y) dx dy \\
& \leq C_1(T) |t| P_K(E) < \infty,
\end{aligned} \tag{3.7}$$

where $C_1(T) > 0$ is some constant. Secondly, we also have that

$$\begin{aligned}
|V_1(\Psi_t(E)) - V_1(E)| & \leq \int_E \int_E \left| \frac{|\det \nabla \Psi_t(x)| |\det \nabla \Psi_t(y)|}{|\Psi_t(x) - \Psi_t(y)|} - \frac{1}{|x - y|} \right| dx dy \\
& = \int_E \int_E \left| \frac{1 + t(\operatorname{div} T(x) + \operatorname{div} T(y)) + o(|t|)}{|x - y + t(T(x) - T(y))|} - \frac{1}{|x - y|} \right| dx dy \\
& \leq \int_E \int_E \left(\frac{C_2(T)|t|}{|x - y|} + \frac{o(|t|)}{|x - y|} \right) dx dy \\
& \leq |t| C_3(T) V_1(E) < \infty,
\end{aligned} \tag{3.8}$$

where $C_2(T)$, $C_3(T)$ are positive constants independent of t . Finally, we estimate the difference of the potentials $R(\Psi_t(E))$ and $R(E)$. Since t is sufficiently small, there exists a constant $c_0(T) > 0$ such that $|x|(1 - c_0(T)|t|) \leq |x + tT(x)| \leq |x|(1 + c_0(T)|t|)$ and we have that $(1 \pm c_0(T)|t|) = 1 \pm c_0(T)|t| + o(|t|)$. Hence, we obtain

$$\begin{aligned}
|R(\Psi_t(E)) - R(E)| & \leq \int_E \left| \frac{1 + t \operatorname{div} T(x) + o(|t|)}{|x + tT(x)|} - \frac{1}{|x|} \right| dx \leq \int_E \left(\frac{C_4(T)|t|}{|x|} + \frac{o(|t|)}{|x|} \right) dx \\
& \leq |t| C_5(T) R(E) < \infty
\end{aligned} \tag{3.9}$$

where $C_4(T)$, $C_5(T)$ are positive constants independent of t . Therefore, (3.7), (3.8), and (3.9) imply that the set F_k satisfies the third property in Lemma 3.1 and this completes the proof. \square

Before starting to prove Lemma 2.1, we need the following isoperimetric inequality of the nonlocal perimeter P_K for sets with small volumes if K satisfies the conditions (K1) and (K4)'.

Lemma 3.2 *Suppose that K satisfies (K1) and (K4)'. Then, there exists a constant $C = C(N, s, \lambda) > 0$ such that, for any measurable set $F \subset \mathbb{R}^N$ with $0 < |F| \leq \omega_N(1 + \varepsilon)^N$, we have*

$$C |F|^{\frac{N-s}{N}} \leq P_K(F). \tag{3.10}$$

Proof. If $P_K(F) = \infty$, then the lemma is proved. Thus, we can assume that $P_K(F)$ is finite. Setting $\rho := \omega_N^{-1/N} |F|^{1/N} \in (0, 1 + \varepsilon]$, we may compute the nonlocal perimeter $P_K(F)$ as follows:

$$\begin{aligned}
\int_F \int_{F^c} K(x - y) dx dy & = \int_F \int_{F^c \cap B_\rho(y)} K(x - y) dx dy + \int_F \int_{F^c \cap (B_\rho(y))^c} K(x - y) dx dy \\
& \geq \frac{1}{\rho^{N+s}} \int_F |F^c \cap B_\rho(y)| dy + \int_F \int_{F^c \cap (B_\rho(y))^c} K(x - y) dx dy.
\end{aligned} \tag{3.11}$$

Here, by the assumption of F , we have $|F| = \omega_N \rho^N = |B_\rho(y)|$ for any y . Thus, it holds that

$$\begin{aligned}
|A^c \cap B_\rho(y)| & = |B_\rho(y)| - |A \cap B_\rho(y)| \\
& = |A| - |A \cap B_\rho(y)| = |A \cap (B_\rho(y))^c|
\end{aligned} \tag{3.12}$$

for any $y \in F$. Thus, from (3.11) and (3.12) and recalling the assumption of K and the fact that $\lambda > 1$ and $\rho < 1 + \varepsilon$, we obtain the following inequality:

$$\begin{aligned}
\int_F \int_{F^c} K(x-y) dx dy &\geq \frac{1}{\rho^{N+s}} \int_F |F \cap (B_\rho(y))^c| dy + \int_F \int_{F^c \cap (B_\rho(y))^c} K(x-y) dx dy \\
&\geq \left(\frac{1+\varepsilon}{\rho} \right)^{-N-s+1} \int_F \int_{F \cap (B_{1+\varepsilon}(y))^c} \frac{(1+\varepsilon)^{-(N+s-1)}}{|x-y|} dx dy \\
&\quad + \frac{1}{\lambda} \int_F \int_{F \cap (B_\rho^c \cap B_{1+\varepsilon}(y))} \frac{\lambda}{\rho^{N+s}} dx dy \\
&\quad + \int_F \int_{F^c \cap (B_\rho(y))^c} K(x-y) dx dy \\
&\geq \frac{1}{\lambda} \int_F \int_{F \cap (B_\rho(y))^c} K(x-y) dx dy + \int_F \int_{F^c \cap (B_\rho(y))^c} K(x-y) dx dy \\
&\geq \frac{1}{\lambda} \int_F \int_{(B_\rho(y))^c} K(x-y) dx dy \\
&\geq \frac{1}{\lambda} \int_F \int_{(B_\rho(y))^c} \frac{1}{|x-y|^{N+s}} dx dy. \tag{3.13}
\end{aligned}$$

Moreover, by applying the change of variables, we can further calculate the last term in (3.13) in the following manner:

$$\begin{aligned}
\int_F \int_{(B_\rho(y))^c} \frac{1}{|x-y|^{N+s}} dx dy &= |F| \omega_{N-1} \int_\rho^\infty \frac{1}{r^{1+s}} dr \\
&= |F| \frac{\omega_{N-1}}{s} \rho^{-s} = \frac{\omega_{N-1} \omega_N^{\frac{s}{N}}}{s} |F|^{\frac{N-s}{N}}. \tag{3.14}
\end{aligned}$$

Therefore, from (3.13) and (3.14) we obtain

$$P_K(F) \geq \frac{\omega_{N-1} \omega_N^{\frac{s}{N}}}{s \lambda} |F|^{\frac{N-s}{N}} \tag{3.15}$$

which is a required inequality. \square

Now we are prepared to prove Lemma 2.1 by applying the above two claims. The proof is based on the strategy shown in [2] for instance.

Proposition 3.3 *Suppose that K satisfies (K1), (K2), (K3), and (K4)'. Every solution E of*

$$\min\{\mathcal{F}_{(K,A)}(E) \mid E \subset \mathbb{R}^N \text{ measurable, } |E| = m\} \tag{3.16}$$

is bounded.

Proof. Let E be a minimizer of $\min_{|E|} \mathcal{F}_{(K,A)}(E)$. For any $r > 0$, we define $f(r) := |E \setminus B_r(0)|$. Then, by the continuity of measure and $|E| = m$, f is a non-increasing function and converges to zero as $r \rightarrow \infty$. Moreover, the coarea formula implies

$$f'(r) = -\mathcal{H}^{N-1}(E \cap \partial B_r(0)). \tag{3.17}$$

We now show that there exists $R > 0$ such that $f(r) = 0$ for all $r > R$. This implies the boundedness of minimizers because $\mathcal{F}_{(K,A)}(E)$ coincides with $\mathcal{F}_{(K,A)}(E')$ if $E \setminus E'$ is a set of Lebesgue measure zero and thus we can identify E with E' . Suppose by contradiction that $f(r) > 0$ for any $r > 0$. Without loss of generality, we can assume that $|E \cap B_1(0)| > 0$ and $|CE \cap B_1(0)| > 0$. Choosing k_0 as in Lemma 3.1, we may fix $R_0 > 0$ such that $f(r) < k_0$ for any $r \geq R_0$. Then, by Lemma 3.1, for any $r \geq R_0$, there exists $F_r \subset \mathbb{R}^N$ such that the followings are true:

1. $E \Delta F_r \subset \subset B_1(0) \subset B_r(0)$.
2. $|F_r| = |E| + f(r)$.
3. $|\mathcal{F}_{(K,A)}(E) - \mathcal{F}_{(K,A)}(F_r)| < C f(r)$ for some constant $C > 0$ independent of $r > 0$.

Now, letting $G_r := F_r \cap B_r(0)$ and recalling the first and second properties of Lemma 3.1, we have that $|G_r| = |E|$. Here, in the same way as in [3] for instance, one can prove the equality for a nonlocal perimeter

$$P_K(U) + P_K(W) = P_K(U \cup W) + 2 \int_U \int_W K(x - y) dx dy. \quad (3.18)$$

for any measurable sets $U, W \subset \mathbb{R}^N$ such that $|U \cap W| = 0$. Moreover, we also have the equality for a potential functional

$$V_1(U \cup W) = V_1(U) + V_1(W) + \int_U \int_W \frac{1}{|x - y|} dx dy. \quad (3.19)$$

for any measurable $U, W \subset \mathbb{R}^N$ such that $|U \cap W| = 0$. Therefore, by the minimality of E and using (3.18) and (3.19), we obtain

$$\begin{aligned} \mathcal{F}_{(K,A)}(E) &\leq \mathcal{F}_{(K,A)}(G_r) = P_K(G_r) + V_1(G_r) - A R(G_r) \\ &\leq P_K(F_r) - P_K(F_r \setminus B_r(0)) + V_1(F_r) - V_1(F_r \setminus B_r(0)) \\ &\quad - A R(F_r) + A R(F_r \setminus B_r(0)) + 2 \int_{F_r \setminus B_r} \int_{F_r \cap B_r} K(x - y) dx dy \\ &\leq \mathcal{F}_{(K,A)}(E) + C f(r) - P_K(F_r \setminus B_r(0)) \\ &\quad + A \int_{F_r \setminus B_r} \frac{1}{|x|} dx + 2 \int_{F_r \setminus B_r} \int_{F_r \cap B_r} K(x - y) dx dy. \end{aligned} \quad (3.20)$$

Note that we also used the third property of Lemma 3.1 in the last inequality of (3.20). Since $E \setminus B_r(0) = F_r \setminus B_r(0)$, it holds that

$$\int_{F_r \setminus B_r} \frac{1}{|x|} dx = \int_{E \setminus B_r} \frac{1}{|x|} dx \leq \frac{1}{r} |E \setminus B_r| = \frac{f(r)}{r}. \quad (3.21)$$

By the assumption of K and using the coarea formula, we have that

$$\begin{aligned} \int_{F_r \setminus B_r} \int_{F_r \cap B_r} K(x - y) dx dy &= \int_{E \setminus B_r} \int_{F_r \cap B_r \cap B_{1+\varepsilon}(y)} K(x - y) dx dy \\ &\quad + \int_{E \setminus B_r} \int_{F_r \cap B_r \cap B_{1+\varepsilon}^c(y)} K(x - y) dx dy \\ &\leq \int_{E \setminus B_r} \int_{B_r \cap B_{1+\varepsilon}(y)} K(x - y) dx dy \\ &\quad + \int_{E \setminus B_r} \int_{F_r \cap B_{1+\varepsilon}^c(y)} K(x - y) dx dy \\ &\leq \lambda \int_{E \setminus B_r} \int_{B_{|y|-r}^c(y)} \frac{1}{|x - y|^{N+s}} dx dy \\ &\quad + \frac{1}{(1 + \varepsilon)^{N+s}} \int_{E \setminus B_r} |F_r| dy \\ &\leq \frac{\lambda \omega_{N-1}}{s} \int_{E \setminus B_r} \frac{1}{(|y| - r)^s} dy + \frac{1}{(1 + \varepsilon)^{N+s}} f(r) |E| \end{aligned}$$

$$\begin{aligned}
&\leq \frac{\lambda\omega_{N-1}}{s} \int_r^\infty \frac{1}{(t-r)^s} \mathcal{H}^{N-1}(E \cap \partial B_t(0)) dt + \frac{m}{(1+\varepsilon)^{N+s}} f(r) \\
&= -\frac{\lambda\omega_{N-1}}{s} \int_r^\infty \frac{1}{(t-r)^s} f'(t) dt + \frac{m}{(1+\varepsilon)^{N+s}} f(r). \tag{3.22}
\end{aligned}$$

(We may compute this integral in a more generalized way under as few assumptions as possible. First we assume that K is a continuous function on \mathbb{R}^N into $(0, +\infty)$ and satisfies the following conditions:

$\widetilde{(K1)}$ $K(x) > 0$ and $K(-x) = K(x)$ for any $x \in \mathbb{R}^N$.

$\widetilde{(K2)}$ $K \in L^1(B_1^c(0))$.

$\widetilde{(K3)}$ $\Phi_K(1) < +\infty$.

$\widetilde{(K4)}$ Assumptions on K which ensure that the isoperimetric inequality for P_K holds are satisfied. (\Rightarrow for example,)

Here we define $\phi_K(r)$ by $\phi_K(r) := \int_{B_r^c(0)} K(x) dx$ (it can happen that $\phi_K(0) = \int_{\mathbb{R}^N} K(x) dx = +\infty$) and $\Phi_K(r)$ by the primitive of ϕ_K , i.e., $\Phi_K(r) := \int_0^r \phi_K(t) dt$. Remark that, from $\widetilde{(K2)}$, we may have the inequality that $\|K\|_{L^1(B_r^c(0))} \leq \|K\|_{L^1(B_1^c(0))} + [\max_{x \in \partial B_1(0)} K(x)] |B_1 \setminus B_r(0)|$ for any $0 < r < 1$, and thus $K \in L^1(B_r^c(0))$ for each $r > 0$. Moreover, from $\widetilde{(K3)}$, we may also derive $\Phi_K(r) < +\infty$ for each $r > 0$. Indeed, since $\phi_K \geq 0$, Φ_K is a nondecreasing function and thus we have that, for any $r \geq 1$,

$$\Phi_K(r) \leq \int_0^1 \phi_K(t) dt + \phi_K(1)(r-1) = \Phi_K(1) + \phi_K(1)(r-1) < +\infty.$$

Note that $\phi_K(r)$ is finite for every $r > 0$ from $\widetilde{(K2)}$.

We now compute the integral $\int_{F_r \setminus B_r} \int_{F_r \cap B_r} K(x-y) dx dy$ as follows; by the coarea formula and the assumptions on K , we have

$$\begin{aligned}
\int_{F_r \setminus B_r} \int_{F_r \cap B_r} K(x-y) dx dy &\leq \int_{E \setminus B_r(0)} \int_{B_r(0)} K(x-y) dx dy \\
&\leq \int_{E \setminus B_r(0)} \int_{B_{|y|-r}^c(y)} K(x-y) dx dy \\
&= \int_{E \setminus B_r(0)} \int_{B_{|y|-r}^c(0)} K(x) dx dy \\
&= \int_{E \setminus B_r(0)} \phi_K(|y|-r) dy \\
&= \int_r^\infty \phi_K(t-r) \mathcal{H}^{N-1}(E \cap \partial B_t(0)) dt \\
&= - \int_r^\infty \phi_K(t-r) f'(t) dt \tag{3.23}
\end{aligned}$$

for a.e. $r > 0$. Then, by combining (3.20) and (3.23), we obtain

$$P_K(E \setminus B_r(0)) \leq C_7 f(r) - \int_r^\infty \phi_K(t-r) f'(t) dt \tag{3.24}$$

for a.e. $r > 0$.

\Rightarrow By taking several steps and from (3.24), we may have

$$C_7' f(r)^{\frac{N-s}{N}} \leq - \int_r^{+\infty} \phi_K(t-r) f'(t) dt \tag{3.25}$$

for a.e. $r > 0$ and some $C'_7 > 0$.)

Substituting (3.22) in (3.20), we obtain

$$P_K(E \setminus B_r(0)) \leq C_7 f(r) - \frac{\lambda \omega_{N-1}}{s} \int_r^\infty \frac{1}{(t-r)^s} f'(t) dt \quad (3.26)$$

for some constant $C_7 > 0$ independent of $r > 0$. For sufficiently large $r > 0$, we can assume that $|E \setminus B_r(0)| < \omega_N(1 + \varepsilon)^N$ and thus, by the isoperimetric inequality stated in Lemma 3.2, it holds that

$$C_8 |E \setminus B_r(0)|^{\frac{N-s}{N}} \leq P_K(E \setminus B_r(0)), \quad (3.27)$$

where C_8 is a positive constant independent of r . Thus, combining (3.26) with (3.27), we have

$$C_8 f(r)^{\frac{N-s}{N}} \leq C f(r) - \frac{\lambda \omega_{N-1}}{s} \int_r^\infty \frac{1}{(t-r)^s} f'(t) dt. \quad (3.28)$$

Recalling the fact that $f(r) \rightarrow 0$ as $r \rightarrow \infty$, we can choose $R_1 > R_0$ such that

$$C_9 f(r) \leq \frac{C_9}{2} f(r)^{\frac{N-s}{N}} \quad (3.29)$$

for any $r \geq R_1$. Therefore, for all $r \geq R_1$, we obtain

$$\frac{s C_9}{\lambda \omega_{N-1}} f(r)^{\frac{N-s}{N}} \leq - \int_r^\infty \frac{1}{(t-r)^s} f'(t) dt. \quad (3.30)$$

We integrate over (R, ∞) where $R \geq R_1$ and then we exchange the order of integration to obtain

$$\begin{aligned} \frac{s C_9}{\lambda_1 \omega_{N-1}} \int_R^\infty f(r)^{\frac{N-s}{N}} dr &\leq - \int_R^\infty \int_r^\infty \frac{1}{(t-r)^s} f'(t) dt dr \\ &= - \int_R^\infty \int_R^t \frac{1}{(t-r)^s} f'(t) dr dt = - \frac{1}{1-s} \int_R^\infty f'(t) (t-R)^{1-s} dt. \end{aligned} \quad (3.31)$$

(Integrating the both sides over $r \in (R, +\infty)$ and exchanging the order of integration, we have

$$\begin{aligned} C'_9 \int_R^\infty f(r)^{\frac{N-s}{N}} dr &\leq - \int_R^\infty \int_r^\infty \phi_K(t-r) f'(t) dt dr \\ &= - \int_R^\infty \int_R^t \phi_K(t-r) dr f'(t) dt \\ &= - \int_R^\infty \int_0^{t-R} \phi_K(r) dr f'(t) dt \\ &= - \int_R^\infty \Phi_K(t-R) f'(t) dt \end{aligned} \quad (3.32)$$

for all $R > 1$ and some constant $C'_9 > 0$.)

Moreover, recalling $f(r) \rightarrow 0$ as $r \rightarrow \infty$, we have that

$$\begin{aligned} - \int_R^\infty f'(t) (t-R)^{1-s} dt &= - \int_R^{R+1} f'(t) (t-R)^{1-s} dt - \int_{R+1}^\infty f'(t) (t-R)^{1-s} dt \\ &\leq f(R) - f(R+1) - \int_{R+1}^\infty f'(t) (t-R)^{1-s} dt \\ &\leq f(R) + \int_{R+1}^\infty f'(t) (1 - (t-R)^{1-s}) dt \\ &\leq f(R) + (1-s) \int_{R+1}^\infty f(t) (t-R)^{-s} dt \leq f(R) + \int_R^\infty f(t) dt \end{aligned} \quad (3.33)$$

for any $R > 1$.

(We may further compute as follows:

$$\begin{aligned}
-\int_R^\infty \Phi_K(t-R) f'(t) dt &= -\int_R^{R+1} \Phi_K(t-R) f'(t) dt - \int_{R+1}^{+\infty} \Phi_K(t-R) f'(t) dt \\
&\leq \Phi_K(1)(f(R) - f(R+1)) - \int_{R+1}^{+\infty} \Phi_K(t-R) f'(t) dt \\
&= \Phi_K(1)f(R) + \int_{R+1}^{+\infty} f'(t) (\Phi_K(1) - \Phi_K(t-R)) dt \\
&\leq \Phi_K(1)f(R) + [f(t) (\Phi_K(1) - \Phi_K(t-R))]_{t=R+1}^{+\infty} \\
&\quad + \int_{R+1}^{+\infty} f(t) \phi_K(t-R) dt \\
&\leq \Phi_K(1)f(R) + \phi_K(1) \int_R^{+\infty} f(t) dt
\end{aligned} \tag{3.34}$$

for any $R > 1$. Therefore, by combining (3.34) with (3.25) and (3.32), we have

$$C'_9 \int_R^{+\infty} f(r)^{\frac{N-s}{N}} dr \leq \Phi_K(1)f(R) + \phi_K(1) \int_R^{+\infty} f(r) dr. \tag{3.35}$$

for any large $R > 1$. Recall that $\phi_K(1) = \|K\|_{L^1(B_1^c(0))} < +\infty$.

\Rightarrow It holds that

$$\frac{C'_{10}}{\Phi_K(1)} \int_R^{+\infty} f(r)^{\frac{N-s}{N}} dr \leq f(R) \tag{3.36}$$

for sufficiently large $R > 1$ and some $C'_{10} > 0$. Note that $+\infty > \Phi_K(1) = \int_0^1 \int_{B_r^c(0)} K(x) dx dr > 0$ since $K > 0$ in \mathbb{R}^N and $(\widetilde{K3})$ holds.)

Thus, substituting (3.33) with (3.31), we obtain

$$\frac{s C_9}{\lambda \omega_{N-1}} \int_R^\infty f(r)^{\frac{N-s}{N}} dr \leq f(R) + \int_R^\infty f(r) dr. \tag{3.37}$$

Since $f(r)$ is small for sufficiently large $R > R_1$, we may assume

$$2 \int_R^\infty f(r) dr \leq \frac{s C_9}{\lambda \omega_{N-1}} \int_R^\infty f(r)^{\frac{N-s}{N}} dr \tag{3.38}$$

for any $R > R_2$ and some $R_2 > R_1$. Therefore, we conclude that, for any $R > R_2$,

$$C_{10} \int_R^\infty f(r)^{\frac{N-s}{N}} dr \leq f(R), \tag{3.39}$$

where $C_{10} = C_{10}(N, s, \lambda) := \frac{s C_9}{2 \lambda \omega_{N-1}} > 0$.

Let $R > R_2$ be fixed such that $w_0 = |E \setminus B_R(0)| > 0$ is sufficiently small. For any $k \in \mathbb{Z}$ with $k \geq 0$, we set $\alpha := \frac{N-s}{N}$, $R_k := R + 1 - 2^{-k}$, and $w_k := f(R_k)$. Then, from (3.39), we have that $R_k \rightarrow R_\infty := R + 1$ as $k \rightarrow \infty$ and

$$C_{10} 2^{-(k+1)} w_{k+1}^\alpha \leq w_k \tag{3.40}$$

for any k . Then, by iterating this estimate and recalling that w_0 can be chosen sufficiently small, we obtain that $w_k \rightarrow 0$ as $k \rightarrow \infty$. However, by the assumption, we also have that $\lim_{k \rightarrow \infty} w_k = f(R+1) = |E \setminus B_{R+1}(0)| > 0$, which is a contradiction. \square

4 Nonexistence of minimizers for $\mathcal{F}_{(K,A)}$

In this section, we show the proof of Theorem 2.2. First of all, we define the quantity

$$F_{(K,A)}[m] := \inf\{\mathcal{F}_{(K,A)}(E) \mid |E| = m, E : \text{bounded}\} \quad (4.1)$$

for a given number $m \in (0, \infty)$. Because of the last term of the functional (1.1), we cannot expect the subadditivity of the functional $\mathcal{F}_{(K,A)}$. Moreover, if we decompose the Riesz potential $V_1(E)$ into two parts $V_1(E_1)$ and $V_1(E_2)$ where $E = E_1 \cup E_2$, then we usually gain the energy while the nonlocal perimeter P_K is not the case. This also implies the invalidity of the subadditivity. However, since we can move the two bounded sets far away from each other to decrease the extra potential energy arising from the decomposition, we may have a weak version of the subadditivity in terms of the quantity (4.1) as follows:

Lemma 4.1 *Suppose that K satisfies (K1) and (K2). Let m_1 and m_2 be a positive number. Then, it holds that*

$$F_{(K,A)}[m_1 + m_2] \leq F_{(K,A)}[m_1] + F_{(K,0)}[m_2]. \quad (4.2)$$

Proof of Lemma 4.1. The proof can be done in a similar way with in [13], and thus, we basically follow their strategy. Let $\varepsilon > 0$ be an arbitrary number. Then, by the definition of (4.1), there exist bounded subsets $E_1, E_2 \subset \mathbb{R}^N$ with the volume constraints $|E_1| = m_1$ and $|E_2| = m_2$ such that

$$\mathcal{F}_{(K,A)}(E_1) + \mathcal{F}_{(K,0)}(E_2) \leq F_{(K,A)}[m_1] + F_{(K,0)}[m_2] + \varepsilon. \quad (4.3)$$

Since E_1, E_2 are bounded, we can find a sufficiently large number $d = d(\varepsilon) > 0$ such that $\text{dist}(E_1, (E_2 + d e_1)) \geq d/2$. Then, from (3.18) and (3.19), we may calculate as follows:

$$\begin{aligned} \mathcal{F}_{(K,A)}(E_1 \cup (E_2 + d e_1)) &= P_K(E_1 \cup (E_2 + d e_1)) + V_1(E_1 \cup (E_2 + d e_1)) - A R(E_1 \cup (E_2 + d e_1)) \\ &\leq P_K(E_1) + P_K(E_2 + d e_1) + V_1(E_1) + V_1(E_2 + d e_1) \\ &\quad + \int_{E_1} \int_{E_2 + d e_1} \frac{1}{|x - y|} dx dy - A R(E_1) \\ &\leq \mathcal{F}_{(K,A)}(E_1) + \mathcal{F}_{(K,0)}(E_2) + \frac{2m_1 m_2}{d}. \end{aligned} \quad (4.4)$$

Note that P_K and V_1 is invariant under translations and $|x - y| \geq d/2$ for any $x \in E_1$ and $y \in E_2 + d e_1$. Hence, by the definition of (4.1), we obtain

$$F_{(K,A)}[m_1 + m_2] \leq F_{(K,A)}[m_1] + F_{(K,0)}[m_2] + \varepsilon + \frac{2m_1 m_2}{d}. \quad (4.5)$$

Letting $d \rightarrow \infty$, and then $\varepsilon \rightarrow 0$, we conclude that the lemma holds. \square

Proof of Theorem 2.2. First of all, we prove the former claim of the main theorem. To do this, we assume that K satisfies (K1), (K2), and (K4). We suppose that there exists a bounded minimizer $E \subset \mathbb{R}^N$ with $|E| = m$ of (4.1) for given m . Then we will show that m actually satisfies the opposite inequality to (2.1). In order to divide \mathbb{R}^N into two parts, we define the hyperplane $H_{\nu,l}$ by $H_{\nu,l} := \{x \in \mathbb{R}^N \mid x \cdot \nu = l\}$ for any parameters $\nu \in \mathbb{S}^{N-1}$ and $l \in \mathbb{R}$. Moreover, we set

$$H_{\nu,l}^+ := \{x \in \mathbb{R}^N \mid x \cdot \nu \geq l\}, \quad H_{\nu,l}^- := \mathbb{R}^N \setminus H_{\nu,l}^+. \quad (4.6)$$

and

$$E_{\nu,l}^+ := E \cap H_{\nu,l}^+, \quad E_{\nu,l}^- := E \cap H_{\nu,l}^- \quad (4.7)$$

for any set $E \subset \mathbb{R}^N$ for any $\nu \in \mathbb{S}^{N-1}$ and $l \in \mathbb{R}$. Next, we want to compare the sum of the energies for $E_{\nu,l}^+$ and $E_{\nu,l}^-$ with the energy for E . To do this, we apply the Lemma 4.1 and use the minimality of E and then we have

$$\mathcal{F}_{(K,A)}(E) = F_{(K,A)}[m] \leq F_{(K,A)}[|E_{\nu,l}^+|] + F_{(K,0)}[|E_{\nu,l}^-|] \leq \mathcal{F}_{(K,A)}(E_{\nu,l}^+) + \mathcal{F}_{(K,0)}(E_{\nu,l}^-), \quad (4.8)$$

Thus, it can be rewritten as

$$P_K(E) + V_1(E) - A R(E) \leq P_K(E_{\nu,l}^+) + V_1(E_{\nu,l}^+) - A R(E_{\nu,l}^+) + P_K(E_{\nu,l}^-) + V_1(E_{\nu,l}^-). \quad (4.9)$$

Therefore, from (3.18) and (3.19), we obtain

$$\int_{E_{\nu,l}^+} \int_{E_{\nu,l}^-} \frac{1}{|x-y|} dx dy \leq 2 \int_{E_{\nu,l}^+} \int_{E_{\nu,l}^-} K(x-y) dx dy + A \int_{E_{\nu,l}^-} \frac{1}{|x|} dx. \quad (4.10)$$

By the layer cake formula and Fubini's theorem, we may obtain the following integration result:

$$\begin{aligned} \int_{E_{\nu,0}^-} \frac{-x \cdot \nu}{|x|} dx &= \int_{E_{\nu,0}^-} \int_{-\infty}^0 \frac{\chi_{\{x \cdot \nu > 0\}}(l)}{|x|} dl dx \\ &= \int_{-\infty}^0 \int_{E_{\nu,0}^-} \frac{\chi_{\{x \cdot \nu < l\}}(x)}{|x|} dx dl = \int_{-\infty}^0 \int_{E_{\nu,l}^-} \frac{1}{|x|} dx dl. \end{aligned} \quad (4.11)$$

Integrating the inequality (4.10) with respect to l from $-\infty$ to 0 and substituting (4.11) for (4.10), we have

$$\int_{-\infty}^0 \int_{E_{\nu,l}^+} \int_{E_{\nu,l}^-} \frac{1}{|x-y|} dx dy dl \leq 2 \int_{-\infty}^0 \int_{E_{\nu,l}^+} \int_{E_{\nu,l}^-} K(x-y) dx dy dl + A \int_{E_{\nu,0}^-} \frac{|x \cdot \nu|}{|x|} dx. \quad (4.12)$$

By interchanging the role of $E_{\nu,l}^+$ and $E_{\nu,l}^-$ in the above calculations, we obtain

$$\begin{aligned} \int_0^{+\infty} \int_{E_{\nu,l}^+} \int_{E_{\nu,l}^-} \frac{1}{|x-y|} dx dy dl \\ \leq 2 \int_0^{+\infty} \int_{E_{\nu,l}^+} \int_{E_{\nu,l}^-} K(x-y) dx dy dl + A \int_{E_{\nu,0}^+} \frac{|x \cdot \nu|}{|x|} dx. \end{aligned} \quad (4.13)$$

Thus, summing up (4.12) and (4.13), we have that

$$\int_{-\infty}^{+\infty} \int_{E_{\nu,l}^+} \int_{E_{\nu,l}^-} \frac{1}{|x-y|} dx dy dl \leq 2 \int_{-\infty}^{+\infty} \int_{E_{\nu,l}^+} \int_{E_{\nu,l}^-} K(x-y) dx dy dl + A \int_E \frac{|x \cdot \nu|}{|x|} dx \quad (4.14)$$

Now, using the layer cake formula and Fubini's theorem again, we have that

$$\begin{aligned} \int_{-\infty}^{+\infty} \int_{E_{\nu,l}^+} \int_{E_{\nu,l}^-} \frac{1}{|x-y|} dx dy dl &= \int_E \int_E \int_{-\infty}^{+\infty} \frac{\chi_{\{x \cdot \nu < l\}}(x) \chi_{\{y \cdot \nu \geq l\}}(y)}{|x-y|} dl dx dy \\ &= \int_E \int_E \int_{-\infty}^{+\infty} \frac{\chi_{\{x \cdot \nu < l < y \cdot \nu\}}(l)}{|x-y|} dl dx dy \\ &= \int_E \int_E \frac{((y-x) \cdot \nu)_+}{|x-y|} dx dy. \end{aligned} \quad (4.15)$$

For any fixed $x \in \mathbb{R}^N$, by the spherical polar coordinates with x located on the x_N -axis, we obtain

$$\int_{\mathbb{S}^{N-1}} (x \cdot \nu)_+ d\mathcal{H}^{N-1}(\nu) = \int_{\mathbb{S}^{N-2}} \int_0^{\frac{\pi}{2}} |x| \cos \theta d\theta d\mathcal{H}^{N-2} = \omega_{N-2} |x|. \quad (4.16)$$

Since $\nu \in \mathbb{S}^{N-1}$ is any element, we have, by using Fubini's theorem again and (4.16), that

$$\begin{aligned} \int_{\mathbb{S}^{N-1}} \int_E \int_E \frac{((y-x) \cdot \nu)_+}{|x-y|} dx dy d\mathcal{H}^{N-1}(\nu) &= \int_E \int_E \int_{\mathbb{S}^{N-1}} \frac{((y-x) \cdot \nu)_+}{|x-y|} d\mathcal{H}^{N-1}(\nu) dx dy \\ &= \omega_{N-2} |E|_N^2. \end{aligned} \quad (4.17)$$

Moreover, by Fubini's theorem and (4.16), we also obtain

$$\int_{\mathbb{S}^{N-1}} \int_E \frac{|x \cdot \nu|}{|x|} dx d\mathcal{H}^{N-1}(\nu) = 2\omega_{N-2} |E|. \quad (4.18)$$

Now we consider the third term in (4.10). By the three assumptions on K , and applying the calculation in (4.15) and Fubini's theorem, we may also compute as follows:

$$\begin{aligned} & \int_{\mathbb{S}^{N-1}} \int_{-\infty}^{+\infty} \int_{E_{\nu,l}^+} \int_{E_{\nu,l}^-} K(x-y) dx dy dl d\mathcal{H}^{N-1}(\nu) \\ &= \int_E \int_E \int_{\mathbb{S}^{N-1}} ((y-x) \cdot \nu)_+ K(x-y) d\mathcal{H}^{N-1}(\nu) dx dy \\ &= \omega_{N-2} \int_E \int_E |x-y| K(x-y) dx dy. \\ &\leq \omega_{N-2} \int_E \int_{B_{1+\varepsilon}(y)} |x-y| K(x-y) dx dy + \omega_{N-2} \int_E \int_{E \cap B_{1+\varepsilon}^c(y)} |x-y| K(x-y) dx dy \\ &\leq \omega_{N-2} \int_E \int_{B_{1+\varepsilon}(y)} \frac{1}{|x-y|^{N+s-1}} dx dy + \omega_{N-2} \int_E \int_E \frac{1}{(1+\varepsilon)^{N+s-1}} dx dy \\ &= \omega_{N-2} \int_{\mathbb{S}^{N-1}} \int_0^{1+\varepsilon} \frac{1}{r^s} dr d\mathcal{H}^{N-1} + \frac{\omega_{N-2}}{(1+\varepsilon)^{N+s-1}} |E|_N^2 \\ &= \frac{\omega_{N-2} \omega_{N-1}}{1-s} (1+\varepsilon)^{1-s} |E| + \frac{\omega_{N-2}}{(1+\varepsilon)^{N+s-1}} |E|_N^2. \end{aligned} \quad (4.19)$$

Therefore, integrating the both sides in (4.14) with respect to ν in \mathbb{S}^{N-1} and substituting (4.17), (4.18), and (4.19), we obtain

$$\omega_{N-2} |E|_N^2 \leq \frac{2\omega_{N-2} \omega_{N-1}}{1-s} (1+\varepsilon)^{1-s} |E| + \frac{2\omega_{N-2}}{(1+\varepsilon)^{N+s-1}} |E|_N^2 + 2A\omega_{N-2} |E|. \quad (4.20)$$

Hence, recalling $|E| = m$, we conclude that

$$\left(\frac{1}{2} - \frac{1}{(1+\varepsilon)^{N+s-1}} \right) m \leq \frac{\omega_{N-1} (1+\varepsilon)^{1-s}}{1-s} + A. \quad (4.21)$$

Note that, by the definition of ε , it holds $(1+\varepsilon)^{N+s-1} > 2$.

The second claim of the main theorem is readily obtained in the following manner; if we assume that K satisfies (K1), (K2), (K3), and (K4)', then, by Lemma 2.1, every minimizer of the functional $\mathcal{F}_{(K,A)}$ under the volume constraint is actually bounded. Then, applying the argument shown in the above, we may derive the same result. \square

5 Generalization of results

In this appendix, we slightly modify the energy (1.1) and consider the nonexistence of its minimizers. The modification energy of (1.1) denoted by $\mathcal{F}_{(K,A,\gamma)}$ is defined as follows:

$$\mathcal{F}_{(K,A,\gamma)}(E) := P_K(E) + V_1(E) - A \int_E \frac{1}{|x|^\gamma} dx, \quad (5.1)$$

for any measurable $E \subset \mathbb{R}^N$ with a volume constraint, where $\gamma \in [0, N+1)$. Note that, if $\gamma = 1$, then we may obtain the same results as discussed in the above. If $\gamma \neq 1$, then we may also obtain the similar results stated in our main theorem, although the explicit value of the critical mass cannot be obtained. Indeed, if $\gamma \in [0, N+1)$, then, by the symmetric rearrangement inequality, it holds that

$$\int_E \frac{1}{|x|^\gamma} dx \leq \int_{E^*} \frac{1}{|x|^\gamma} dx \quad (5.2)$$

where E^* is the symmetric rearrangement of any measurable set E with a finite volume. Thus, setting $r_E > 0$ as a radius of the open ball E^* with $\omega_N (r_E)^N = |E| = m$ and taking $\gamma < N + 1$, (4.14), (4.16), and (4.18) into consideration, we may compute as follows:

$$\begin{aligned} \int_{\mathbb{S}^{N-1}} \int_{E^*} \frac{|x \cdot \nu|}{|x|^\gamma} dx d\mathcal{H}^{N-1}(\nu) &= 2\omega_{N-2} \int_{E^*} \frac{1}{|x|^{\gamma-1}} dx \\ &= 2\omega_{N-2} \int_{\mathbb{S}^{N-1}} d\mathcal{H}^{N-1} \int_0^{r_E} \frac{1}{r^{\gamma-1}} r^{N-1} dr \\ &= \frac{2\omega_{N-2}\omega_{N-1}}{(N+1-\gamma)\omega_N^{1-\frac{\gamma-1}{N}}} m^{1-\frac{\gamma-1}{N}}. \end{aligned} \quad (5.3)$$

Thus, following the argument of the proof of Theorem 2.2 as above and using (5.2) and (5.3), we have that

$$\left(\frac{1}{2} - \frac{1}{(1+\varepsilon)^{N+1-s}} \right) m^2 \leq \frac{\omega_{N-1}}{1-s} (1+\varepsilon)^{1-s} m + A \frac{\omega_{N-1}}{N+1-\gamma} \omega_N^{-1+\frac{\gamma-1}{N}} m^{1-\frac{\gamma-1}{N}}. \quad (5.4)$$

Here we set $p := \frac{\gamma-1}{N} \in [-\frac{1}{N}, 1)$ and define constants $C_1, C_2, C_3 > 0$ as

$$C_1 := \left(\frac{1}{2} - \frac{1}{(1+\varepsilon)^{N+1-s}} \right), \quad C_2 := \frac{\omega_{N-1}}{1-s} (1+\varepsilon)^{1-s}, \quad C_3 := \frac{\omega_{N-1}}{N+1-\gamma} \omega_N^{-1+\frac{\gamma-1}{N}}. \quad (5.5)$$

Now, considering a profile of the function $\phi(x) := C_1 x^{1+p} - C_2 x^p - A C_3$ for any $x > 0$, we may obtain that there exists a unique number $m_p > 0$ such that $\phi(m_p) = 0$ and $\phi(m) > 0$ for any $m > m_p$. Therefore, imposing the same assumptions as in Theorem 2.2 for instance, we have that there exists a critical mass $m_p > 0$ such that, for any $m > m_p$, the problem $\min\{\mathcal{F}_{(K,A,\gamma)}(E) \mid |E| = m\}$ has no solutions.

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