ON HARDY AND CAFFARELLI–KOHN–NIRENBERG INEQUALITIES

By

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Abstract. We establish improved versions of the Hardy and Caffarelli–Kohn–Nirenberg inequalities by replacing the standard Dirichlet energy with some nonlocal nonconvex functionals which have been involved in estimates for the topological degree of continuous maps from a sphere into itself and characterizations of Sobolev spaces.

1 Introduction

In many branches of mathematical physics, harmonic and stochastic analysis, the classical **Hardy inequality** plays a central role. It states that, if $1 \le p < d$,

$$\Big(\frac{d-p}{p}\Big)^p \int_{\mathbb{R}^d} \frac{|u|^p}{|x|^p} dx \le \int_{\mathbb{R}^d} |\nabla u|^p dx,$$

for every $u \in C_c^1(\mathbb{R}^d)$ with optimal constant which, contrary to the Sobolev inequality, is never attained. Another class of relevant inequalities is given by the so-called **Caffarelli–Kohn–Nirenberg inequalities** [14, 15]. Precisely, let $p \ge 1$, $q \ge 1$, $\tau > 0$, $0 < a \le 1$, $\alpha,\ \beta,\ \gamma \in \mathbb{R}$ be such that

$$\begin{gathered} \frac{1}{\tau} + \frac{\gamma}{d}, \quad \frac{1}{p} + \frac{\alpha}{d}, \quad \frac{1}{q} + \frac{\beta}{d} > 0, \\ \frac{1}{\tau} + \frac{\gamma}{d} = a\Big(\frac{1}{p} + \frac{\alpha - 1}{d}\Big) + (1-a)\Big(\frac{1}{q} + \frac{\beta}{d}\Big), \end{gathered} \tag{1.1}$$

and, with $\gamma = a\sigma + (1-a)\beta$,

$$0 \le \alpha - \sigma$$

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and

$$\alpha - \sigma \leq 1 \quad \text{if } \frac{1}{\tau} + \frac{\gamma}{d} = \frac{1}{p} + \frac{\alpha - 1}{d}.$$

Then, for every $u \in C^1_c(\mathbb{R}^d)$,

$$\| |x|^\gamma u\|_{L^\tau(\mathbb{R}^d)} \leq C \| |x|^\alpha \nabla u\|^a_{L^p(\mathbb{R}^d)} \| |x|^\beta u\|^{(1-a)}_{L^q(\mathbb{R}^d)},$$

for some positive constant C independent of u. This inequality has been an object of a large amount of improvement and extensions to more general frameworks.

In the non local case, it was shown in [18, 19] that there exists $C > 0$, independent of $0 < \delta < 1$, such that

$$C \int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x|^{p\delta}} dx \leq J_\delta(u), \tag{1.2}$$

for all $u \in C^1_c(\mathbb{R}^d)$, where

$$J_\delta(u) := (1 - \delta) \iint_{\mathbb{R}^{2d}} \frac{|u(x) - u(y)|^p}{|x - y|^{d+p\delta}} dxdy.$$

In light of the results of Bourgain, Brezis, and Mironescu [3, 4] and a refinement of Davila [17], we have

$$\lim_{\delta \searrow 0} J_\delta(u) = K_{d,p} \int_{\mathbb{R}^d} |\nabla u|^p dx, \quad \text{for } u \in W^{1,p}(\mathbb{R}^d),\ K_{d,p} := \frac{1}{p} \int_{\mathbb{S}^{d-1}} |\boldsymbol{e} \cdot \sigma|^p \, d\sigma,$$

for some $\boldsymbol{e} \in \mathbb{S}^{d-1}$, $\mathbb{S}^{d-1}$ being the unit sphere in $\mathbb{R}^d$. This allows to recover the classical Hardy inequality from (1.2) by letting $\delta \searrow 0$. Various problems related to J_δ are considered in [7, 9, 10, 12, 33, 34]. The full range of Caffarelli–Kohn–Nirenberg inequalities and their variants were established in [30] (see [1] for partial results in the case $a = 1$).

For $p \geq 1$, set Ω a measurable set of $\mathbb{R}^d$, and $u \in L^1_{\text{loc}}(\Omega)$,

$$I_\delta(u, \Omega) := \underset{\{|u(x)-u(y)|>\delta\}}{\int_\Omega \int_\Omega} \frac{\delta^p}{|x - y|^{d+p}} dxdy.$$

In the case $\Omega = \mathbb{R}^d$, we simply denote $I_\delta(u, \mathbb{R}^d)$ by $I_\delta(u)$. The quantity I_δ with $p = d$ has its roots in estimates for the topological degree of continuous maps from a sphere into itself in [5, 22]. This also appears in characterizations of Sobolev spaces [6, 11, 12, 21, 24] and related contexts [8, 11, 12, 23, 25, 26, 28, 29]. It is known that (see [21, Theorem 2] and [12, Proposition 1]), for $p \geq 1$,

$$\lim_{\delta \searrow 0} I_\delta(u) = K_{d,p} \int_{\mathbb{R}^d} |\nabla u|^p \, dx, \quad \text{for } u \in C^1_c(\mathbb{R}^d) \text{ [1]} \tag{1.3}$$

[1] In the case $p > 1$, one can take $u \in W^{1,p}(\mathbb{R}^d)$ in (1.3). Nevertheless, (1.3) does not hold for $u \in W^{1,1}(\mathbb{R}^d)$ in the case $p = 1$. An example for this is due to Ponce presented in [21].

and, for $p > 1$,

$$I_\delta(u) \le C_{d,p} \int_{\mathbb{R}^d} |\nabla u|^p \, dx, \quad \text{for } u \in W^{1,p}(\mathbb{R}^d), \tag{1.4}$$

for some positive constant $C_{d,p}$ independent of u.

The aim of this paper is to get improved versions of the local Hardy and Caffarelli–Kohn–Nirenberg type inequalities and their variants which involve nonlinear nonlocal nonconvex energies $I_\delta(u)$ and its related quantities. In what follows for $R > 0$, B_R denotes the open ball of $\mathbb{R}^d$ centered at the origin of radius r. Our first main result concerning Hardy's inequality is:

Theorem 1.1 (Improved Hardy inequality)**.** *Let $d \ge 1$, $p \ge 1$, $0 < r < R$, and $u \in L^p(\mathbb{R}^d)$. We have*

(i) *if $1 \le p < d$ and $\operatorname{supp} u \subset B_R$, then*

$$\int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x|^p} \, dx \le C(I_\delta(u) + R^{d-p}\delta^p),$$

(ii) *if $p > d$ and $\operatorname{supp} u \subset \mathbb{R}^d \setminus B_r$, then*

$$\int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x|^p} \, dx \le C(I_\delta(u) + r^{d-p}\delta^p),$$

(iii) *if $p = d \ge 2$ and $\operatorname{supp} u \subset B_R$, then*

$$\int_{\mathbb{R}^d \setminus B_r} \frac{|u(x)|^d}{|x|^d \ln^d(2R/|x|)} \, dx \le C(I_\delta(u) + \ln(2R/r)\delta^d),$$

(iv) *if $p = d \ge 2$ and $\operatorname{supp} u \subset \mathbb{R}^d \setminus B_r$, then*

$$\int_{B_R} \frac{|u(x)|^d}{|x|^d \ln^d(2|x|/r)} \, dx \le C(I_\delta(u) + \ln(2R/r)\delta^d),$$

where C denotes a positive constant depending only on p and d.

In light of (1.3), by letting $\delta \to 0$, one obtains variants of (i), (ii), (iii) and (iv) of Theorem 1.1 where the RHS is replaced by $C\int_{\mathbb{R}^d} |\nabla u|^p \, dx$; see Proposition 1.1 for a more general version. By (1.3) and (1.4), Theorem 1.1 provides improvement of Hardy's inequalities in the case $p > 1$.

We next discuss an improved version of Caffarelli–Kohn–Nirenberg in the case the exponent $a = 1$. The more general case is considered in Theorem 3.1 (see also Proposition 3.1). For $p \ge 1$, $\alpha \in \mathbb{R}$, and Ω a measurable subset of $\mathbb{R}^d$, set

$$I_\delta(u, \Omega, \alpha) := \int_\Omega \int_\Omega_{\{|u(x)-u(y)|>\delta\}} \frac{\delta^p |x|^{p\alpha}}{|x-y|^{d+p}} \, dx dy, \quad \text{for } u \in L^1_{\text{loc}}(\Omega).$$

If $\Omega = \mathbb{R}^d$, we simply denote $I_\delta(u, \mathbb{R}^d, \alpha)$ by $I_\delta(u, \alpha)$. We have

Theorem 1.2 (Improved Caffarelli–Kohn–Nirenberg inequality for $a = 1$). *Let $d \geq 2$, $1 < p < d$, $\tau > 0$, $0 < r < R$, and $u \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$. Assume that*

$$\frac{1}{\tau} + \frac{\gamma}{d} = \frac{1}{p} + \frac{\alpha - 1}{d} \quad \text{and} \quad 0 \leq \alpha - \gamma \leq 1.$$

We have

(i) *if $d - p + p\alpha > 0$ and $\operatorname{supp} u \subset B_R$, then*

$$\left(\int_{\mathbb{R}^d} |x|^{\gamma\tau} |u(x)|^\tau \, dx \right)^{p/\tau} \leq C(I_\delta(u, \alpha) + R^{d-p+p\alpha} \delta^p),$$

(ii) *if $d - p + p\alpha < 0$ and $\operatorname{supp} u \subset \mathbb{R}^d \setminus B_r$, then*

$$\left(\int_{\mathbb{R}^d} |x|^{\gamma\tau} |u(x)|^\tau \, dx \right)^{p/\tau} \leq C(I_\delta(u, \alpha) + r^{d-p+p\alpha} \delta^p),$$

(iii) *if $d - p + p\alpha = 0$, $\tau > 1$, and $\operatorname{supp} u \subset B_R$, then*

$$\left(\int_{\mathbb{R}^d \setminus B_r} \frac{|x|^{\gamma\tau} |u(x)|^\tau}{\ln^\tau(2R/|x|)} \, dx \right)^{p/\tau} \leq C(I_\delta(u, \alpha) + \ln(2R/r) \delta^p),$$

(iv) *if $d - p + p\alpha = 0$, $\tau > 1$, and $\operatorname{supp} u \subset \mathbb{R}^d \setminus B_r$, then*

$$\left(\int_{B_R} \frac{|x|^{\gamma\tau} |u(x)|^\tau}{\ln^\tau(2|x|/r)} \, dx \right)^{p/\tau} \leq C(I_\delta(u, \alpha) + \ln(2R/r) \delta^p).$$

Here C denotes a positive constant independent of u, r, and R.

Remark 1.1. In contrast with Theorem 1.1, in Theorem 1.2 we assume that $1 < p < d$. This assumption is required due to the use of Sobolev's inequality related to $I_\delta(u, \Omega, 0)$ (see Lemmas 3.1 and 3.2).

Remark 1.2. Using the theory of maximal functions with weights due to Muckenhoupt [20] (see also [16]), one can bound $I_\delta(u, \alpha)$ by $C \int_{\mathbb{R}^d} |x|^{p\alpha} |\nabla u|^p \, dx$ for $-1/p < \alpha < 1 - 1/p$ and get an improvement of the Caffarelli–Kohn–Nirenberg inequality for $a = 1$ via Theorem 1.2 and for $0 < a < 1$ and $0 \leq \alpha - \sigma \leq 1$ via Theorem 3.1 in Section 3. The details of this fact are given in Remark 3.3 (see also Remark 3.2 for a different approach covering a more general result).

We later prove a general version of Theorem 1.2 in Theorem 3.1, where $0 < a \leq 1$, which implies Proposition 3.1 by interpolation. As a consequence of Theorem 3.1 (see also Remark 3.2) and Proposition 3.1, we have

Proposition 1.1. *Let $p \geq 1$, $q \geq 1$, $\tau > 0$, $0 < a \leq 1$, α, β, $\gamma \in \mathbb{R}$ be such that*

$$\frac{1}{\tau} + \frac{\gamma}{d} = a\Big(\frac{1}{p} + \frac{\alpha - 1}{d}\Big) + (1 - a)\Big(\frac{1}{q} + \frac{\beta}{d}\Big),$$

and, with $\gamma = a\sigma + (1-a)\beta$,

$$0 \leq \alpha - \sigma$$

and

$$\alpha - \sigma \leq 1 \quad \text{if } \frac{1}{\tau} + \frac{\gamma}{d} = \frac{1}{p} + \frac{\alpha - 1}{d}.$$

We have, for $u \in C_c^1(\mathbb{R}^d)$,
(A1) *if $1/\tau + \gamma/d > 0$, then*

$$\Big(\int_{\mathbb{R}^d} |x|^{\gamma\tau}|u|^\tau\,dx\Big)^{1/\tau} \leq C\||x|^\alpha \nabla u\|^a_{L^p(\mathbb{R}^d)}\||x|^\beta u\|^{(1-a)}_{L^q(\mathbb{R}^d)},$$

(A2) *if $1/\tau + \gamma/d < 0$ and* $\operatorname{supp} u \subset \mathbb{R}^d \setminus \{0\}$, *then*

$$\Big(\int_{\mathbb{R}^d} |x|^{\gamma\tau}|u|^\tau\,dx\Big)^{1/\tau} \leq C\||x|^\alpha \nabla u\|^a_{L^p(\mathbb{R}^d)}\||x|^\beta u\|^{(1-a)}_{L^q(\mathbb{R}^d)}.$$

Assume in addition that $\alpha - \sigma \leq 1$ and $\tau > 1$. We have
(A3) *if $1/\tau + \gamma/d = 0$ and* $\operatorname{supp} u \subset B_R$ *for some $R > 0$, then*

$$\Big(\int_{\mathbb{R}^d} \frac{|x|^{\gamma\tau}}{\ln^\tau(2R/|x|)}|u|^\tau\,dx\Big)^{1/\tau} \leq C\||x|^\alpha \nabla u\|^a_{L^p(\mathbb{R}^d)}\||x|^\beta u\|^{(1-a)}_{L^q(\mathbb{R}^d)},$$

(A4) *if $1/\tau + \gamma/d = 0$ and* $\operatorname{supp} u \subset \mathbb{R}^d \setminus B_r$ *for some $r > 0$, then*

$$\Big(\int_{\mathbb{R}^d} \frac{|x|^{\gamma\tau}}{\ln^\tau(2|x|/r)}|u|^\tau\,dx\Big)^{1/\tau} \leq C\||x|^\alpha \nabla u\|^a_{L^p(\mathbb{R}^d)}\||x|^\beta u\|^{(1-a)}_{L^q(\mathbb{R}^d)}.$$

Here C denotes a positive constant independent of u, r, and R.

Assertion (A1) is a slight improvement of the classical Caffarelli–Kohn–Nirenberg. Indeed, in the classical setting, Assertion (A1) is established under the additional assumptions

$$1/p + \alpha/d > 0 \quad \text{and} \quad 1/q + \beta/d > 0,$$

as mentioned in (1.1) in the introduction. Assertion (A2) with $a = 1$ and $\tau = p$ was known (see, e.g., [18]). Concerning Assertion (A3) with $a = 1$, this was obtained for $d = 2$ in [13] and [2] and, for $d \geq 3$, this was established in [2]. Assertion (A4) with $a = 1$ might be known; however, we cannot find any references for it. To our knowledge, the remaining cases seem to be new.

Analogous versions in a bounded domain will be given in Section 4.

The ideas used in the proof of Theorems 1.1 and 1.2, and their general version (Theorem 3.1), are as follows. On one hand, this is based on Poincaré and Sobolev inequalities related to $I_\delta(u,\Omega)$ (see Lemmas 2.1 and 3.1). These inequalities have their roots in [25]. Using these inequalities, we derive the key estimate (see Lemma 3.2 and also Lemma 2.1) for an annulus D centered at the origin and for $\lambda > 0$,

$$\begin{aligned}(1.5)\qquad &\left(\fint_{\lambda D}\left|u - \fint_{\lambda D} u\right|^\tau dx\right)^{1/\tau} \\ &\quad \leq C(\lambda^{p-d} I_\delta(u,\lambda D) + \delta^p)^{a/p}\left(\fint_{\lambda D}\left|u - \fint_{\lambda D} u\right|^q dx\right)^{(1-a)/q},\end{aligned}$$

for some positive constant C independent of u and λ. On the other hand, decomposing $\mathbb{R}^d$ into annuli $\mathscr{A}_k$ which are defined by

$$\mathscr{A}_k := \{x \in \mathbb{R}^d : 2^k \leq |x| < 2^{k+1}\},$$

and applying (1.5) to each $\mathscr{A}_k$, we obtain

$$\left(\fint_{\mathscr{A}_k}\left|u - \fint_{\mathscr{A}_k} u\right|^\tau dx\right)^{1/\tau} \leq C(2^{-(d-p)k} I_\delta(u,\mathscr{A}_k) + \delta^p)^{a/p}\left(\fint_{\mathscr{A}_k} |u|^q\right)^{(1-a)/q},$$

A similar idea was used in [15]. Using (1.5) again in the cases (i) and (ii), we can derive an appropriate estimate for

$$2^{(\gamma\tau+d)k}\left|\fint_{\mathscr{A}_k} u\right|^\tau.$$

This is the novelty in comparison with the approach in [15]. Combining these two facts, one obtains the desired inequalities. The other cases follow similarly. A similar approach is used to establish the Caffarelli–Kohn–Nirenberg inequalities for fractional Sobolev spaces in [30].

We now make some comments on the magnetic Sobolev setting. If $A : \mathbb{R}^d \to \mathbb{R}^d$ is locally bounded and $u : \mathbb{R}^d \to \mathbb{C}$, we set

$$\Psi_u(x,y) := e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} u(y), \qquad x, y \in \mathbb{R}^d.$$

The following **diamagnetic inequality** holds:

$$||u(x)| - |u(y)|| \leq |\Psi_u(x,x) - \Psi_u(x,y)|, \qquad \text{for a.e. } x, y \in \mathbb{R}^d.$$

In turn, by defining

$$I_\delta^A(u,\alpha) = \int_{\mathbb{R}^d}\int_{\substack{\mathbb{R}^d \\ \{|\Psi_u(x,y)-\Psi_u(x,x)|>\delta\}}} \frac{\delta^p |x|^{p\alpha}}{|x-y|^{d+p}}\, dxdy,$$

we have, for $\alpha \in \mathbb{R}$,

$$I_\delta(|u|, \alpha) \le I_\delta^A(u, \alpha) \quad \text{for all } \delta > 0.$$

Then, the assertions of Theorems 1.1 and 1.2 keep holding with $I_\delta^A(u, 0)$ (resp., $I_\delta^A(u, \alpha)$) on the right-hand side in place of $I_\delta(u)$ (resp., $I_\delta(u, \alpha)$). For the sake of completeness, we refer the reader to [27] for some recent results about new characterizations of classical magnetic Sobolev spaces in the terms of $I_\delta^A(u, 0)$ (see [27, 32, 35] for the ones related to J_δ).

The paper is organized as follows. In Section 2 we prove Theorem 1.1. In Section 3 we prove Theorem 3.1 and Proposition 3.1 which imply Theorem 1.2 and Proposition 1.1. In Section 4 we present versions of Theorems 1.1 and 3.1 in a bounded domain Ω.

2 Improved Hardy's inequality

We first recall that a straightforward variant of [25, Theorem 1] yields the following:

Lemma 2.1. *Let $d \ge 1$, $p \ge 1$ and set*

$$D := \{x \in \mathbb{R}^d : r < |x| < R\}.$$

Then

$$\fint_D \left| u(x) - \fint_D u \right|^p dx \le C_{r,R}(I_\delta(u, D) + \delta^p), \quad \textit{for all } u \in L^p(D).$$

As a consequence, we have, for $\lambda > 0$,

$$\fint_{\lambda D} \left| u(x) - \fint_{\lambda D} u \right|^p dx \le C_{r,R}(\lambda^{p-d} I_\delta(u, \lambda D) + \delta^p), \quad \textit{for all } u \in L^p(\lambda D), \tag{2.1}$$

where $\lambda D := \{\lambda x : x \in D\}$. Here $C_{r,R}$ denotes a positive constant independent of u, δ, and λ.

The following elementary inequality will be used several times in this paper.

Lemma 2.2. *Let $\Lambda > 1$ and $\tau > 1$. There exists $C = C(\Lambda, \tau) > 0$, depending only on Λ and τ, such that, for all $1 < c < \Lambda$,*

$$(|a| + |b|)^\tau \le c|a|^\tau + \frac{C}{(c-1)^{\tau-1}}|b|^\tau, \quad \textit{for all } a, b \in \mathbb{R}. \tag{2.2}$$

Proof. Since (2.2) is clear in the case $|b| \geq |a|$ and in the case $b = 0$, by rescaling and considering $x = |a|/|b|$, it suffices to prove, for $C = C(\Lambda, \tau)$ large enough, that

$$(x+1)^\tau \leq cx^\tau + \frac{C}{(c-1)^{\tau-1}}, \quad \text{for all } x \geq 1. \tag{2.3}$$

Set

$$f(x) = (x+1)^\tau - cx^\tau - \frac{C}{(c-1)^{\tau-1}} \quad \text{for } x > 0.$$

We have

$$f'(x) = \tau(x+1)^{\tau-1} - c\tau x^{\tau-1} \quad \text{and} \quad f'(x) = 0 \text{ if and only if } x = x_0 := (c^{\frac{1}{\tau-1}} - 1)^{-1}.$$

One can check that

$$\lim_{x\to+\infty} f(x) = -\infty, \quad \lim_{x\to 1} f(x) < 0 \text{ if } C = C(\Lambda, \tau) \text{ is large enough,} \tag{2.4}$$

and

$$f(x_0) = cx_0^{\tau-1} - \frac{C}{(c-1)^{\tau-1}}. \tag{2.5}$$

If $c^{\frac{1}{\tau-1}} > 2$, then $x_0 < 1$ and (2.3) follows from (2.4). Otherwise $1 \leq s := c^{\frac{1}{\tau-1}} \leq 2$. By the mean value theorem, we have

$$s^{\tau-1} - 1 \leq (s-1) \max_{1\leq t\leq 2} (\tau-1)t^{\tau-2} \quad \text{for } 1 \leq s \leq 2.$$

We derive from (2.5) that, with $C = \Lambda[\max_{1\leq t\leq 2}(\tau-1)t^{\tau-2}]^{\tau-1}$,

$$f(x_0) < 0.$$

The conclusion now follows from (2.4). □

We are now ready to give the

Proof of Theorem 1.1. Let $m, n \in \mathbb{Z}$ be such that

$$2^{n-1} \leq R < 2^n \quad \text{and} \quad 2^m \leq r < 2^{m+1}.$$

It is clear that $n - m \geq 1$. By (2.1) of Lemma 2.1, we have, for all $k \in \mathbb{Z}$,

$$\fint_{\mathscr{A}_k} \left| u(x) - \fint_{\mathscr{A}_k} u \right|^p dx \leq C(2^{-(d-p)k} I_\delta(u, \mathscr{A}_k) + \delta^p).$$

Here and in what follows in this proof, C denotes a positive constant independent of k, u and δ. This implies

$$2^{-pk} \int_{\mathscr{A}_k} \left| u(x) - \fint_{\mathscr{A}_k} u \right|^p dx \leq C(I_\delta(u, \mathscr{A}_k) + 2^{(d-p)k}\delta^p).$$

It follows that

$$2^{-pk}\int_{\mathscr{A}_k}|u(x)|^p\,dx \le C2^{(d-p)k}\left|\fint_{\mathscr{A}_k} u\right|^p + C(I_\delta(u,\mathscr{A}_k)+2^{(d-p)k}\delta^p). \tag{2.6}$$

• **Proof of (i).** Summing (2.6) with respect to k from $-\infty$ to n, we obtain

$$\int_{\mathbb{R}^d}\frac{|u(x)|^p}{|x|^p}\,dx \le C\sum_{k=-\infty}^{n}2^{(d-p)k}\left|\fint_{\mathscr{A}_k} u\right|^p + CI_\delta(u) + C2^{(d-p)n}\delta^p, \tag{2.7}$$

since $d > p$. We also have, by (2.1), for $k \in \mathbb{Z}$,

$$\left|\fint_{\mathscr{A}_k} u - \fint_{\mathscr{A}_{k+1}} u\right| \le C(2^{-(d-p)k}I_\delta(u,\mathscr{A}_k\cup\mathscr{A}_{k+1})+\delta^p)^{1/p}.$$

This implies

$$\left|\fint_{\mathscr{A}_k} u\right| \le \left|\fint_{\mathscr{A}_{k+1}} u\right| + C(2^{-(d-p)k}I_\delta(u,\mathscr{A}_k\cup\mathscr{A}_{k+1})+\delta^p)^{1/p}.$$

Applying Lemma 2.2, we have

$$\left|\fint_{\mathscr{A}_k} u\right|^p \le \frac{2^{d-p+1}}{1+2^{d-p}}\left|\fint_{\mathscr{A}_{k+1}} u\right|^p + C(2^{-(d-p)k}I_\delta(u,\mathscr{A}_k\cup\mathscr{A}_{k+1})+\delta^p).$$

It follows that, with $c = 2/(1+2^{d-p}) < 1$,

$$2^{(d-p)k}\left|\fint_{\mathscr{A}_k} u\right|^p \le c2^{(d-p)(k+1)}\left|\fint_{\mathscr{A}_{k+1}} u\right|^p + C(I_\delta(u,\mathscr{A}_k\cup\mathscr{A}_{k+1})+2^{(d-p)k}\delta^p).$$

We derive that

$$\sum_{k=-\infty}^{n}2^{(d-p)k}\left|\fint_{\mathscr{A}_k} u\right|^p \le C\sum_{k=-\infty}^{n}I_\delta(u,\mathscr{A}_k\cup\mathscr{A}_{k+1}) + C2^{(d-p)n}\delta^p. \tag{2.8}$$

A combination of (2.7) and (2.8) yields

$$\int_{\mathbb{R}^d}\frac{|u(x)|^d}{|x|^d}\,dx \le CI_\delta(u) + C2^{(d-p)n}\delta^p.$$

The conclusion of (i) follows.

• **Proof of (ii).** Summing (2.6) with respect to k from m to $+\infty$, we obtain

$$\int_{\mathbb{R}^d}\frac{|u(x)|^p}{|x|^p}\,dx \le C\sum_{k=m}^{+\infty}2^{(d-p)k}\left|\fint_{\mathscr{A}_k} u\right|^p + CI_\delta(u) + C2^{(d-p)m}\delta^p, \tag{2.9}$$

since $p > d$. We also have, by (2.1), for $k \in \mathbb{Z}$,

$$\left|\fint_{\mathscr{A}_k} u - \fint_{\mathscr{A}_{k+1}} u\right| \le C(2^{-(d-p)k}I_\delta(u,\mathscr{A}_k\cup\mathscr{A}_{k+1})+\delta^p)^{1/p}.$$

This implies that

$$\left| \fint_{\mathscr{A}_{k+1}} u \right| \le \left| \fint_{\mathscr{A}_k} u \right| + C(2^{-(d-p)k} I_\delta(u, \mathscr{A}_k \cup \mathscr{A}_{k+1}) + \delta^p)^{1/p}.$$

Applying Lemma 2.2, we have

$$\left| \fint_{\mathscr{A}_{k+1}} u \right|^p \le \frac{1 + 2^{d-p}}{2^{d-p+1}} \left| \fint_{\mathscr{A}_k} u \right|^p + C(2^{-(d-p)k} I_\delta(u, \mathscr{A}_k \cup \mathscr{A}_{k+1}) + \delta^p).$$

It follows that, with $c = (1 + 2^{d-p})/2 < 1$,

$$2^{(d-p)(k+1)} \left| \fint_{\mathscr{A}_{k+1}} u \right|^p \le c 2^{(d-p)k} \left| \fint_{\mathscr{A}_k} u \right|^p + C(I_\delta(u, \mathscr{A}_k \cup \mathscr{A}_{k+1}) + 2^{(d-p)k} \delta^p).$$

We derive that

$$\sum_{k=m}^{+\infty} 2^{(d-p)k} \left| \fint_{\mathscr{A}_k} u \right|^p \le C I_\delta(u) + C 2^{(d-p)m} \delta^p. \tag{2.10}$$

A combination of (2.9) and (2.10) yields

$$\int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x|^p} \, dx \le C I_\delta(u) + C 2^{(d-p)m} \delta^p.$$

The conclusion of (ii) follows.

• **Proof of (iii).** Let $\alpha > 0$. Summing (2.6) with respect to k from m to n, we obtain

$$\begin{aligned} \int_{\{2^m < |x| < 2^n\}} & \frac{|u(x)|^d}{|x|^d \ln^{\alpha+1}(2R/|x|)} \, dx \\ & \le C \sum_{k=m}^{n} \frac{1}{(n-k+1)^{\alpha+1}} \left| \fint_{\mathscr{A}_k} u \right|^d + C I_\delta(u) + C(n-m)\delta^d. \end{aligned} \tag{2.11}$$

We also have, by (2.1), for $k \in \mathbb{Z}$,

$$\left| \fint_{\mathscr{A}_k} u \right| \le \left| \fint_{\mathscr{A}_{k+1}} u \right| + C(I_\delta(u, \mathscr{A}_k \cup \mathscr{A}_{k+1})^{1/d} + \delta). \tag{2.12}$$

By applying Lemma 2.2 with

$$c = \frac{(n-k+1)^\alpha}{(n-k+1/2)^\alpha},$$

it follows from (2.12) that, for $m \le k \le n$,

$$\begin{aligned} \frac{1}{(n-k+1)^\alpha} \left| \fint_{\mathscr{A}_k} u \right|^d \le & \frac{1}{(n-k+1/2)^\alpha} \left| \fint_{\mathscr{A}_{k+1}} u \right|^d \\ & + C(n-k+1)^{d-1-\alpha} (I_\delta(u, \mathscr{A}_k \cup \mathscr{A}_{k+1}) + \delta^d). \end{aligned} \tag{2.13}$$

We have, for $m \le k \le n$,

$$\frac{1}{(n-k+1)^{\alpha}} - \frac{1}{(n-k+3/2)^{\alpha}} \sim \frac{1}{(n-k+1)^{\alpha+1}}. \tag{2.14}$$

Taking $\alpha = d-1$ and combining (2.13) and (2.14) yields

$$\sum_{k=m}^{n} \frac{1}{(n-k+1)^{d}} \left| \fint_{\mathscr{A}_k} u \right|^{d} \le C I_{\delta}(u) + C(n-m)\delta^{d}. \tag{2.15}$$

From (2.11) and (2.15), we obtain

$$\int_{\{|x|>2^m\}} \frac{|u(x)|^d}{|x|^d \ln^d(2R/|x|)}\, dx \le C I_{\delta}(u) + C(n-m)\delta^{d}.$$

This implies the conclusion of (iii).

• **Proof of (iv).** Let $\alpha > 0$. Summing (2.6) with respect to k from m to n, we obtain

$$\int_{\{2^m<|x|<2^n\}} \frac{|u(x)|^d}{|x|^d \ln^{\alpha+1}(2|x|/R)}\, dx \le C \sum_{k=m}^{n} \frac{1}{(k-m+1)^{\alpha+1}} \left| \fint_{\mathscr{A}_k} u \right|^{d} + C I_{\delta}(u) + C\delta^{d}. \tag{2.16}$$

We have, by (2.1), for $k \in \mathbb{Z}$,

$$\left| \fint_{\mathscr{A}_{k+1}} u \right| \le \left| \fint_{\mathscr{A}_k} u \right| + C\left(I_{\delta}(u, \mathscr{A}_k \cup \mathscr{A}_{k+1})^{1/d} + \delta\right). \tag{2.17}$$

By applying Lemma 2.2 with

$$c = \frac{(n-k+1)^{\alpha}}{(n-k+1/2)^{\alpha}},$$

it follows from (2.17) that, for $m \le k+1 \le n$,

$$\begin{aligned} \frac{1}{(k-m+1)^{\alpha}} \left| \fint_{\mathscr{A}_{k+1}} u \right|^{d} \le{}& \frac{1}{(k-m+1/2)^{\alpha}} \left| \fint_{\mathscr{A}_k} u \right|^{d} \\ &+ C(k-m+1)^{d-1-\alpha}\left(I_{\delta}(u, \mathscr{A}_k \cup \mathscr{A}_{k+1}) + \delta^{d}\right). \end{aligned} \tag{2.18}$$

We have, for $m \le k+1 \le n$,

$$\frac{1}{(k-m+1)^{\alpha}} - \frac{1}{(k-m+3/2)^{\alpha}} \sim \frac{1}{(k-m+1)^{\alpha+1}}. \tag{2.19}$$

Taking $\alpha = d-1$ and combining (2.18) and (2.19) yields

$$\sum_{k=m}^{n} \frac{1}{(k-m+1)^{d}} \left| \fint_{\mathscr{A}_k} u \right|^{d} \le C I_{\delta}(u) + C(n-m)\delta^{d}. \tag{2.20}$$

From (2.16) and (2.20), we obtain

$$\int_{\{2^m<|x|<2^n\}} \frac{|u(x)|^d}{|x|^d \ln^d(2|x|/R)}\, dx \le CI_\delta(u) + C(n-m)\delta^d.$$

This implies the conclusion of (iv).

The proof is complete. □

3 Improved Caffarelli–Kohn–Nirenberg inequality

In the proof of Theorem 1.2, we use the following result:

Lemma 3.1. *Let $1 < p < d$, Ω be a smooth bounded open subset of $\mathbb{R}^d$, and $v \in L^p(\Omega)$. We have*

$$\|u\|_{L^{p^*}(\Omega)} \le C_\Omega (I_\delta(u)^{1/p} + \|u\|_{L^p} + \delta),$$

where $p^* := dp/(d-p)$ denotes the Sobolev exponent of p.

Proof. For $\tau > 0$, let us set

$$\Omega_\tau := \{x \in \mathbb{R}^d : \mathrm{dist}(x, \Omega) < \tau\}.$$

Since Ω is smooth, by [12, Lemma 17], there exists $\tau > 0$ small enough and an extension U of u in Ω_τ such that

$$I_\delta(U, \Omega_\tau) \le CI_\delta(u, \Omega) \quad \text{and} \quad \|U\|_{L^p(\Omega_\tau)} \le C\|u\|_{L^p(\Omega)}, \tag{3.1}$$

for $0 < \delta < 1$. Fix such a τ. Let $\varphi \in C^1(\mathbb{R}^d)$ such that

$$\mathrm{supp}\,\varphi \subset \Omega_{2\tau/3}, \quad \varphi = 1 \text{ in } \Omega_{\tau/3}, \quad 0 \le \varphi \le 1 \text{ in } \mathbb{R}^d.$$

Define $v = \varphi U$ in $\mathbb{R}^d$. We claim that

$$I_{2\delta}(v) \le C(I_\delta(u, \Omega) + \|u\|^p_{L^p(\Omega)}). \tag{3.2}$$

Indeed, set

$$f(x, y) = \frac{\delta^p}{|x-y|^{d+p}} \mathbb{1}_{\{|v(x)-v(y)|>2\delta\}}.$$

We estimate $I_{2\delta}(v)$. We have

$$\iint_{\Omega\times\mathbb{R}^d} f(x, y)\, dxdy \le \iint_{\Omega_{\tau/3}\times\Omega_{\tau/3}} f(x, y)\, dxdy + \iint_{\substack{\Omega_\tau\times\mathbb{R}^d \\ \{|x-y|>\tau/4\}}} f(x, y)\, dxdy,$$

and, since $v = 0$ in $\Omega_\tau \setminus \Omega_{2\tau/3}$,

$$\iint_{(\mathbb{R}^d\setminus\Omega_\tau)\times\mathbb{R}^d} f(x,y)\,dxdy \le \iint_{(\mathbb{R}^d\setminus\Omega_\tau)\times(\mathbb{R}^d\setminus\Omega_\tau)} f(x,y)\,dxdy + \iint_{\substack{\Omega_\tau\times\mathbb{R}^d\\ \{|x-y|>\tau/4\}}} f(x,y)\,dxdy,$$

$$\iint_{(\Omega_\tau\setminus\Omega)\times\mathbb{R}^d} f(x,y)\,dxdy \le \iint_{(\Omega_\tau\setminus\Omega)\times(\Omega_\tau\setminus\Omega)} f(x,y)\,dxdy + \iint_{\Omega_{\tau/3}\times\Omega_{\tau/3}} f(x,y)\,dxdy + \iint_{\substack{\Omega_\tau\times\mathbb{R}^d\\ \{|x-y|>\tau/4\}}} f(x,y)\,dxdy.$$

It is clear that, by (3.1),

$$\iint_{\Omega_{\tau/3}\times\Omega_{\tau/3}} f(x,y)\,dx\,dy \le CI_\delta(u,\Omega), \tag{3.3}$$

by the fact that $\varphi = 0$ in $\mathbb{R}^d \setminus \Omega_\tau$,

$$\iint_{(\mathbb{R}^d\setminus\Omega_\tau)\times(\mathbb{R}^d\setminus\Omega_\tau)} f(x,y)\,dxdy = 0, \tag{3.4}$$

and, by a straightforward computation,

$$\iint_{\substack{\Omega_\tau\times\mathbb{R}^d\\ \{|x-y|>\tau/4\}}} f(x,y)\,dx\,dy \le C\delta^p. \tag{3.5}$$

We have, for $x, y \in \mathbb{R}^d$,

$$v(x) - v(y) = \varphi(x)\big(U(x) - U(y)\big) + U(y)\big(\varphi(x) - \varphi(y)\big).$$

It follows that if $|v(x) - v(y)| > 2\delta$, then either

$$|U(x) - U(y)| \ge |\varphi(x)(U(x) - U(y))| > \delta$$

or

$$C|U(y)||x - y| \ge |U(y)(\varphi(x) - \varphi(y))| > \delta.$$

We thus derive that

$$\iint_{(\Omega_\tau\setminus\Omega)\times(\Omega_\tau\setminus\Omega)} f(x,y)\,dx\,dy \le \int_{(\Omega_\tau\setminus\Omega)}\int_{\substack{(\Omega_\tau\setminus\Omega)\\ \{|U(x)-U(y)|>\delta\}}} \frac{\delta^p}{|x-y|^{d+p}}\,dx\,dy + \int_{(\Omega_\tau\setminus\Omega)}\int_{\substack{(\Omega_\tau\setminus\Omega)\\ \{|x-y|>C\delta/|U(y)|\}}} \frac{\delta^p}{|x-y|^{d+p}}\,dx\,dy. \tag{3.6}$$

A straightforward computation yields

$$\int_{\substack{(\Omega_\tau\setminus\Omega)\\ \{|x-y|>C\delta/|U(y)|\}}}\int_{(\Omega_\tau\setminus\Omega)} \frac{\delta^p}{|x-y|^{d+p}}\,dx\,dy \le \int_{\Omega_\tau} dy \int_{\{|x-y|>C\delta/|U(y)|\}} \frac{\delta^p}{|x-y|^{d+p}}\,dx$$
$$= C\int_{\Omega_\tau} |U(y)|^p\,dy.$$

Using (3.1), we deduce from (3.6) that

$$\iint_{(\Omega_\tau\setminus\Omega)\times(\Omega_\tau\setminus\Omega)} f(x,y)\,dx\,dy \le CI_\delta(u,\Omega) + C\|u\|^p_{L^p(\Omega)}. \tag{3.7}$$

A combination of (3.3), (3.4), (3.5), and (3.7) yields claim (3.2). By applying [25, Theorem 3] and using the fact $\operatorname{supp} v \subset \Omega_\tau$, we have

$$\|v\|_{L^{p^*}(\mathbb{R}^d)} \le CI_{2\delta}(v)^{1/p} + C\delta. \tag{3.8}$$

The conclusion now follows from claim (3.2). □

Remark 3.1. The assumption $p > 1$ is required in (3.8).

As a consequence of Lemmas 2.1 and 3.1, we obtain

Corollary 3.1. *Let $d \ge 2$, $1 < p < d$, $0 < r < R$, and $\lambda > 0$, and set*

$$\lambda D := \{\lambda x \in \mathbb{R}^d : r < |x| < R\}.$$

We have, for $1 \le q \le p^*$,

$$\left(\fint_{\lambda D}\left|u(x) - \fint_{\lambda D} u\right|^q dx\right)^{1/q} \le C_{r,R}(\lambda^{p-d}I_\delta(u,\lambda D) + \delta^p)^{1/p}, \quad \text{for } u \in L^p(\lambda D),$$

where $C_{r,R}$ denotes a positive constant independent of u, δ, and λ.

Here is an application of Corollary 3.1 which plays a crucial role in the proof of Theorem 3.1 below.

Lemma 3.2. *Let $d \ge 1$, $1 < p < d$, $q \ge 1$, $\tau > 0$, and $0 \le a \le 1$ be such that*

$$\frac{1}{\tau} \ge a\Big(\frac{1}{p} - \frac{1}{d}\Big) + \frac{1-a}{q}.$$

Let $0 < r < R$, and $\lambda > 0$ and set

$$\lambda D := \{\lambda x \in \mathbb{R}^d : r < |x| < R\}.$$

Then, for $u \in L^1(\lambda D)$,

$$\left(\fint_{\lambda D}\left|u - \fint_{\lambda D} u\right|^\tau dx\right)^{1/\tau} \le C(\lambda^{p-d}I_\delta(u,\lambda D) + \delta^p)^{a/p}\left(\fint_{\lambda D}\left|u - \fint_{\lambda D} u\right|^q dx\right)^{(1-a)/q}$$

for some positive constant C independent of u, λ, and δ.

Proof. Let $\tau, \sigma, t > 0$, be such that

$$\frac{1}{\tau} \geq \frac{a}{\sigma} + \frac{1-a}{t}.$$

We have, by a standard interpolation inequality, that

$$\left(\fint_{\lambda D} \left|u - \fint_{\lambda D} u\right|^{\tau} dx\right)^{1/\tau} \leq \left(\fint_{\lambda D} \left|u - \fint_{\lambda D} u\right|^{\sigma} dx\right)^{a/\sigma} \left(\fint_{\lambda D} \left|u - \fint_{\lambda D} u\right|^{t} dx\right)^{(1-a)/t}.$$

Applying this inequality with $\sigma = p^*$ and $t = q$ and using Corollary 3.1, one obtains the conclusion. □

We also have, (see [31, Theorem on page 125 and the following remarks])

Lemma 3.3 (Nirenberg's interpolation inequality)**.** *Let $d \geq 1$, $p \geq 1$, $q \geq 1$, $\tau > 0$, and $0 \leq a \leq 1$ be such that*

$$\frac{1}{\tau} \geq a\Big(\frac{1}{p} - \frac{1}{d}\Big) + \frac{1-a}{q}.$$

Let $0 < r < R$, and let $\lambda > 0$ and set

$$\lambda D := \{\lambda x \in \mathbb{R}^d : r < |x| < R\}.$$

Then, for $u \in L^1(\lambda D)$,

$$\left(\fint_{\lambda D} \left|u - \fint_{\lambda D} u\right|^{\tau} dx\right)^{1/\tau} \leq C\|\nabla u\|^{a}_{L^p(\lambda D)} C\|u\|^{1-a}_{L^q(\lambda D)},$$

for some positive constant C independent of u, λ, and δ.

We prove the following more general version of Theorem 1.2:

Theorem 3.1. *Let $p \geq 1$, $q \geq 1$, $\tau > 0$, $0 < a \leq 1$, α, β, $\gamma \in \mathbb{R}$ be such that*

$$\frac{1}{\tau} + \frac{\gamma}{d} = a\Big(\frac{1}{p} + \frac{\alpha - 1}{d}\Big) + (1-a)\Big(\frac{1}{q} + \frac{\beta}{d}\Big), \tag{3.9}$$

and, with $\gamma = a\sigma + (1-a)\beta$,

$$0 \leq \alpha - \sigma \leq 1.$$

Set, for $k \in \mathbb{Z}$,

$$I_\delta(k, u) := \begin{cases} I_\delta(u, \mathscr{A}_k \cup \mathscr{A}_{k+1}, \alpha) + 2^{k(\alpha p + d - p)}\delta^p & \text{if } 1 < p < d, \\ \||x|^\alpha \nabla u\|^p_{L^p(\mathscr{A}_k \cup \mathscr{A}_{k+1})} & \text{otherwise.} \end{cases} \tag{3.10}$$

We have, for $u \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$ and $m, n \in \mathbb{Z}$ with $m < n$,

(i) *if* $1/\tau + \gamma/d > 0$ *and* $\operatorname{supp} u \subset B_{2^n}$, *then*

$$\left(\int_{\mathbb{R}^d \setminus B_{2^m}} |x|^{\gamma\tau} |u|^\tau \, dx \right)^{1/\tau} \le C \left(\sum_{k=m-1}^{n} I_\delta(k, u) \right)^{a/p} \| |x|^\beta u \|_{L^q(\mathbb{R}^d)}^{(1-a)},$$

(ii) *if* $1/\tau + \gamma/d < 0$ *and* $\operatorname{supp} u \subset \mathbb{R}^d \setminus B_{2^m}$, *then*

$$\left(\int_{B_{2^n}} |x|^{\gamma\tau} |u|^\tau \, dx \right)^{1/\tau} \le C \left(\sum_{k=m-1}^{n} I_\delta(k, u) \right)^{a/p} \| |x|^\beta u \|_{L^q(\mathbb{R}^d)}^{(1-a)},$$

(iii) *if* $1/\tau + \gamma/d = 0$, $\tau > 1$, *and* $\operatorname{supp} u \subset B_{2^n}$, *then*

$$\left(\int_{\mathbb{R}^d \setminus B_{2^m}} \frac{|x|^{\gamma\tau}}{\ln^\tau (2^{n+1}/|x|)} |u|^\tau \, dx \right)^{1/\tau} \le C \left(\sum_{k=m-1}^{n} I_\delta(k, u) \right)^{a/p} \| |x|^\beta u \|_{L^q(\mathbb{R}^d)}^{(1-a)},$$

(iv) *if* $1/\tau + \gamma/d = 0$, $\tau > 1$, *and* $\operatorname{supp} u \subset \mathbb{R}^d \setminus B_{2^m}$, *then*

$$\left(\int_{B_{2^n}} \frac{|x|^{\gamma\tau}}{\ln^\tau (2^{n+1}/|x|)} |u|^\tau \, dx \right)^{1/\tau} \le C \left(\sum_{k=m-1}^{n} I_\delta(k, u) \right)^{a/p} \| |x|^\beta u \|_{L^q(\mathbb{R}^d)}^{(1-a)}.$$

Here C denotes a positive constant independent of u, δ, k, n, and m.

Proof. We only present the proof in the case $1 < p < d$. The proof for the other case follows similarly, however instead of using Lemma 3.2, one applies Lemma 3.3. We now assume that $1 < p < d$. Since $\alpha - \sigma \ge 0$, by Lemma 3.2, we have

$$\left(\fint_{\mathcal{A}_k} \left| u - \fint_{\mathcal{A}_k} u \right|^\tau dx \right)^{1/\tau} \le C (2^{-(d-p)k} I_\delta(u, \mathcal{A}_k) + \delta^p)^{a/p} \left(\fint_{\mathcal{A}_k} |u|^q \right)^{(1-a)/q}. \tag{3.11}$$

Using (3.9), we derive from (3.11) that

$$\begin{aligned} \int_{\mathcal{A}_k} |x|^{\gamma\tau} |u|^\tau \, dx \le & C 2^{(\gamma\tau + d)k} \left| \fint_{\mathcal{A}_k} u \right|^\tau \\ & + C (I_\delta(u, \mathcal{A}_k, \alpha) + 2^{k(\alpha p + d - p)} \delta^p)^{a\tau/p} \| |x|^\beta u \|_{L^q(\mathcal{A}_k)}^{(1-a)\tau}. \end{aligned} \tag{3.12}$$

• **Proof of (i).** Summing (3.12) with respect to k from m to n, we obtain

$$\begin{aligned} \int_{\{|x| > 2^m\}} |x|^{\gamma\tau} |u|^\tau \, dx \le & C \sum_{k=m}^{n} 2^{(\gamma\tau + d)k} \left| \fint_{\mathcal{A}_k} u \right|^\tau \\ & + C \sum_{k=m}^{n} (I_\delta(u, \mathcal{A}_k, \alpha) + 2^{k(\alpha p + d - p)} \delta^p)^{a\tau/p} \| |x|^\beta u \|_{L^q(\mathcal{A}_k)}^{(1-a)\tau}. \end{aligned} \tag{3.13}$$

By Lemma 3.2, we have

$$\left| \fint_{\mathscr{A}_k} u \right| \le \left| \fint_{\mathscr{A}_{k+1}} u \right| + C(2^{-(d-p)k} I_\delta(u, \mathscr{A}_k \cup \mathscr{A}_{k+1}) + \delta^p)^{a/p} \left(\fint_{\mathscr{A}_k \cup \mathscr{A}_{k+1}} |u|^q \right)^{\frac{1-a}{q}}.$$

Applying Lemma 2.2, we derive that

$$\begin{aligned} \left| \fint_{\mathscr{A}_k} u \right|^\tau \le & \frac{2^{\gamma\tau+d+1}}{1+2^{\gamma\tau+d}} \left| \fint_{\mathscr{A}_{k+1}} u \right|^\tau \\ & + C(2^{-(d-p)k} I_\delta(u, \mathscr{A}_k \cup \mathscr{A}_{k+1}) + \delta^p)^{a\tau/p} \left(\fint_{\mathscr{A}_k \cup \mathscr{A}_{k+1}} |u|^q \right)^{\frac{(1-a)\tau}{q}}. \end{aligned}$$

It follows that, with $c = 2/(1+2^{\gamma\tau+d}) < 1$,

$$\begin{aligned} 2^{(\gamma\tau+d)k} \left| \fint_{\mathscr{A}_k} u \right|^\tau \le & c 2^{(\gamma\tau+d)(k+1)} \left| \fint_{\mathscr{A}_{k+1}} u \right|^\tau \\ & + C(I_\delta(u, \mathscr{A}_k \cup \mathscr{A}_{k+1}, \alpha) + 2^{k(\alpha p+d-p)}\delta^p)^{a\tau/p} \| |x|^\beta u \|^{(1-a)\tau}_{L^q(\mathscr{A}_k \cup \mathscr{A}_{k+1})}. \end{aligned}$$

This yields

$$\begin{aligned} & \sum_{k=m}^n 2^{(\gamma\tau+d)k} \left| \fint_{\mathscr{A}_k} u \right|^\tau \\ & \le C \sum_{k=m}^n (I_\delta(u, \mathscr{A}_k \cup \mathscr{A}_{k+1}, \alpha) + 2^{k(\alpha p+d-p)}\delta^p)^{a\tau/p} \| |x|^\beta u \|^{(1-a)\tau}_{L^q(\mathscr{A}_k \cup \mathscr{A}_{k+1})}. \end{aligned} \tag{3.14}$$

Combining (3.13) and (3.14) yields

$$\begin{aligned} & \int_{\{|x|>2^m\}} |x|^{\gamma\tau} |u|^\tau \, dx \\ & \le C \sum_{k=m-1}^n (I_\delta(u, \mathscr{A}_k \cup \mathscr{A}_{k+1}, \alpha) + 2^{k(\alpha p+d-p)}\delta^p)^{a\tau/p} \| |x|^\beta u \|^{(1-a)\tau}_{L^q(\mathscr{A}_k \cup \mathscr{A}_{k+1})}. \end{aligned} \tag{3.15}$$

Applying the inequality, for $s \ge 0$, $t \ge 0$ with $s+t \ge 1$, and for $x_k \ge 0$ and $y_k \ge 0$,

$$\sum_{k=m}^n x_k^s y_k^t \le C_{s,t} \left(\sum_{k=m}^n x_k \right)^s \left(\sum_{k=m}^n y_k \right)^t,$$

to $s = a\tau/p$ and $t = (1-a)\tau/q$, we obtain from (3.15) that

$$\int_{\{|x|>2^m\}} |x|^{\gamma\tau} |u|^\tau \, dx \le C \left(\sum_{k=m}^n I_\delta(k, u) \right)^{a\tau/p} \| |x|^\beta u \|^{(1-a)\tau}_{L^q(\mathbb{R}^d)} \tag{3.16}$$

since $a/p + (1-a)/q \ge 1/\tau$ thanks to the fact $\alpha - \sigma - 1 \le 0$.

• **Proof of (ii).** The proof is in the spirit of the proof of (ii) of Theorem 1.1. The details are left to the reader.

• **Proof of (iii).** Fix $\xi > 0$. Summing (3.12) with respect to k from m to n, we obtain

$$\begin{aligned}\int_{\{|x|>2^m\}} \frac{1}{\ln^{1+\xi}(\tau/|x|)} |x|^{\gamma\tau} |u|^\tau \, dx \\ \le C \sum_{k=m}^{n} \frac{1}{(n-k+1)^{1+\xi}} \left| \fint_{\mathscr{A}_k} u \right|^\tau \\ + C \sum_{k=m}^{n} (I_\delta(u, \mathscr{A}_k, \alpha) + 2^{k(\alpha p + d - p)} \delta^p)^{a\tau/p} \| |x|^\beta u \|_{L^q(\mathscr{A}_k)}^{(1-a)\tau}.\end{aligned} \tag{3.17}$$

By Lemma 3.2, we have

$$\left| \fint_{\mathscr{A}_k} u \right| \le \left| \fint_{\mathscr{A}_{k+1}} u \right| + C (2^{-(d-p)k} I_\delta(u, \mathscr{A}_k \cup \mathscr{A}_{k+1}) + \delta^p)^{a/p} \left(\fint_{\mathscr{A}_k \cup \mathscr{A}_{k+1}} |u|^q \right)^{\frac{1-a}{q}}.$$

Applying Lemma 2.2 with

$$c = \frac{(n-k+1)^\xi}{(n-k+1/2)^\xi},$$

we deduce that

$$\begin{aligned}\frac{1}{(n-k+1)^\xi} \left| \fint_{\mathscr{A}_k} u \right|^\tau \le \frac{1}{(n-k+1/2)^\xi} \left| \fint_{\mathscr{A}_{k+1}} u \right|^\tau \\ + C (n-k+1)^{\tau - 1 - \xi} (2^{-(d-p)k} I_\delta(u, \mathscr{A}_k \cup \mathscr{A}_{k+1}) + \delta^p)^{a\tau/p} \\ \times \left(\fint_{\mathscr{A}_k \cup \mathscr{A}_{k+1}} |u|^q \right)^{\frac{(1-a)\tau}{q}}.\end{aligned} \tag{3.18}$$

Recall that, for $k \le n$ and $\xi > 0$,

$$\frac{1}{(n-k+1)^\xi} - \frac{1}{(n-k+3/2)^\xi} \sim \frac{1}{(n-k+1)^{\xi+1}}. \tag{3.19}$$

Taking $\xi = \tau - 1$, we derive from (3.18) and (3.19) that

$$\sum_{k=m}^{n} 2^{(\gamma\tau + d)k} \frac{1}{(n-k+1)^\tau} \left| \fint_{\mathscr{A}_k} u \right|^\tau \le \sum_{k=m}^{n} C (I_\delta(k, u))^{a\tau/p} \| |x|^\beta u \|_{L^q(\mathscr{A}_k \cup \mathscr{A}_{k+1})}^{(1-a)\tau}. \tag{3.20}$$

Combining (3.17) and (3.20), as in (3.16), we obtain

$$\int_{\{|x|>2^m\}} \frac{|x|^{\gamma\tau}}{\ln^\tau(2^{n+1}/|x|)} |u|^\tau \, dx \le C \left(\sum_{k=m}^{n} I_\delta(k, u) \right)^{a\tau/p} \| |x|^\beta u \|_{L^q(\mathbb{R}^d)}^{(1-a)\tau}.$$

• **Proof of (iv).** The proof is in the spirit of the proof of (iv) of Theorem 1.1. The details are left to the reader.

The proof is complete. □

Remark 3.2. For $p > 1$, we have (see [21, Theorem 4])

$$I_\delta(k, u) \le C \int_{\mathscr{A}_k \cup \mathscr{A}_{k+1}} |x|^{p\alpha} |\nabla u|^p \, dx \quad \text{for } k \in \mathbb{Z},$$

for some positive constant C independent of k and u. This implies

$$\Big(\sum_{k=m-1}^{n} I_\delta(k, u) \Big)^{1/p} \le C \| |x|^\alpha \nabla u \|_{L^p(\mathbb{R}^d)}.$$

From Theorem 3.1, one obtains an improvement of Caffarelli–Kohn–Nirenberg's inequality for the case $0 \le \alpha - \sigma \le 1$ and for $1 < p < d$.

Using Theorem 3.1, we can derive

Proposition 3.1. *Let $p \ge 1$, $q \ge 1$, $\tau > 0$, $0 < a < 1$, $\alpha, \beta, \gamma \in \mathbb{R}$ be such that*

$$\frac{1}{\tau} + \frac{\gamma}{d} = a\Big(\frac{1}{p} + \frac{\alpha - 1}{d}\Big) + (1 - a)\Big(\frac{1}{q} + \frac{\beta}{d}\Big),$$

and, with $\gamma = a\sigma + (1 - a)\beta$,

$$\alpha - \sigma > 1 \quad \textit{and} \quad \frac{1}{\tau} + \frac{\gamma}{d} \ne \frac{1}{p} + \frac{\alpha - 1}{d}.$$

We have, for $u \in C^1_c(\mathbb{R}^d)$,

(i) *if $1/\tau + \gamma/d > 0$, then*

$$\Big(\int_{\mathbb{R}^d} |x|^{\gamma\tau} |u|^\tau \, dx \Big)^{1/\tau} \le C \| |x|^\alpha \nabla u \|^a_{L^p(\mathbb{R}^d)} \| |x|^\beta u \|^{(1-a)}_{L^q(\mathbb{R}^d)},$$

(ii) *if $1/\tau + \gamma/d < 0$ and* $\operatorname{supp} u \subset \mathbb{R}^d \setminus \{0\}$, *then*

$$\Big(\int_{\mathbb{R}^d} |x|^{\gamma\tau} |u|^\tau \, dx \Big)^{1/\tau} \le C \| |x|^\alpha \nabla u \|^a_{L^p(\mathbb{R}^d)} \| |x|^\beta u \|^{(1-a)}_{L^q(\mathbb{R}^d)},$$

for some positive constant C independent of u.

Proof. The proof is in the spirit of the approach in [15] (see also [30]). Since

$$\frac{1}{p} + \frac{\alpha - 1}{d} \neq \frac{1}{q} + \frac{\beta}{d},$$

by scaling, one might assume that

$$\| |x|^{\alpha} \nabla u \|_{L^p(\mathbb{R}^d)} = 1 \quad \text{and} \quad \| |x|^{\beta} u \|_{L^q(\mathbb{R}^d)} = 1.$$

Let $0 < a_2 < 1$ be such that

$$|a_2 - a| \text{ is small enough,} \tag{3.21}$$

and set

$$\frac{1}{\tau_2} = \frac{a_2}{p} + \frac{1 - a_2}{q} \quad \text{and} \quad \gamma_2 = a_2(\alpha - 1) + (1 - a_2)\beta.$$

We have

$$\frac{1}{\tau_2} + \frac{\gamma_2}{d} = a_2\Big(\frac{1}{p} + \frac{\alpha - 1}{d}\Big) + (1 - a_2)\Big(\frac{1}{q} + \frac{\beta}{d}\Big). \tag{3.22}$$

Recall that

$$\frac{1}{\tau} + \frac{\gamma}{d} = a\Big(\frac{1}{p} + \frac{\alpha - 1}{d}\Big) + (1 - a)\Big(\frac{1}{q} + \frac{\beta}{d}\Big). \tag{3.23}$$

Since $a > 0$ and $\alpha - \sigma > 1$, it follows from (3.21) that

$$\frac{1}{\tau} - \frac{1}{\tau_2} = (a - a_2)\Big(\frac{1}{p} - \frac{1}{q}\Big) + \frac{a}{d}(\alpha - \sigma - 1) > 0. \tag{3.24}$$

We first choose a_2 such that

$$a_2 < a \quad \text{if } \frac{1}{p} + \frac{\alpha - 1}{d} < \frac{1}{q} + \frac{\beta}{d}, \tag{3.25}$$

$$a < a_2 \quad \text{if } \frac{1}{p} + \frac{\alpha - 1}{d} > \frac{1}{q} + \frac{\beta}{d}. \tag{3.26}$$

Using (3.21), (3.25) and (3.26), we derive from (3.22), and (3.23) that

$$\frac{1}{\tau} + \frac{\gamma}{d} < \frac{1}{\tau_2} + \frac{\gamma_2}{d} \quad \text{and} \quad \Big(\frac{1}{\tau} + \frac{\gamma}{d}\Big)\Big(\frac{1}{\tau_2} + \frac{\gamma_2}{d}\Big) > 0. \tag{3.27}$$

It follows from (3.24), (3.27), and Hölder's inequality that

$$\| |x|^{\gamma} u \|_{L^{\tau}(\mathbb{R}^d \setminus B_1)} \le C \| |x|^{\gamma_2} u \|_{L^{\tau_2}(\mathbb{R}^d)}.$$

Applying Theorem 3.1 (see also Remark 3.2), we have

$$\| |x|^{\gamma_2} u \|_{L^{\tau_2}(\mathbb{R}^d)} \le C \| |x|^{\alpha} \nabla u \|^{a_2}_{L^p(\mathbb{R}^d)} \| |x|^{\beta} u \|^{(1-a_2)}_{L^q(\mathbb{R}^d)} \le C,$$

which yields

$$\| |x|^{\gamma} u\|_{L^{\tau}(\mathbb{R}^d \setminus B_1)} \le C. \tag{3.28}$$

We next choose a_2 such that

$$a < a_2 \quad \text{if } \frac{1}{p} + \frac{\alpha - 1}{d} < \frac{1}{q} + \frac{\beta}{d}, \tag{3.29}$$

$$a_2 < a \quad \text{if } \frac{1}{p} + \frac{\alpha - 1}{d} > \frac{1}{q} + \frac{\beta}{d}. \tag{3.30}$$

Using (3.21), (3.29) and (3.30), we derive from (3.22) and (3.23) that

$$\frac{1}{\tau_2} + \frac{\gamma_2}{d} < \frac{1}{\tau} + \frac{\gamma}{d} \quad \text{and} \quad \Big(\frac{1}{\tau} + \frac{\gamma}{d}\Big)\Big(\frac{1}{\tau_2} + \frac{\gamma_2}{d}\Big) > 0. \tag{3.31}$$

It follows from (3.24), (3.31), and Hölder's inequality that

$$\| |x|^{\gamma} u\|_{L^{\tau}(B_1)} \le C \| |x|^{\gamma_2} u\|_{L^{\tau_2}(\mathbb{R}^d)}.$$

Applying Theorem 3.1 (see also Remark 3.2), we have

$$\| |x|^{\gamma_2} u\|_{L^{\tau_2}(\mathbb{R}^d)} \le C \| |x|^{\alpha} \nabla u\|_{L^p(\mathbb{R}^d)}^{a_2} \| |x|^{\beta} u\|_{L^q(\mathbb{R}^d)}^{(1-a_2)} \le C,$$

which yields

$$\| |x|^{\gamma} u\|_{L^{\tau}(\mathbb{R}^d \setminus B_1)} \le C. \tag{3.32}$$

The conclusion now follows from (3.28) and (3.32). □

Remark 3.3. Using the approach in the proof of [21, Theorem 2], one can prove that, for $p > 1$,

$$I_{\delta}(u, \alpha) \le C \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} |x|^{p\alpha} |\mathcal{M}(\sigma, \nabla u)(x)|^p \, d\sigma \, dx, \tag{3.33}$$

where

$$\mathcal{M}(\sigma, \nabla u)(x) := \sup_{r>0} \frac{1}{r} \int_0^r |\nabla u(x + s\sigma) \cdot \sigma| \, ds.$$

We claim that, for $-1/p < \alpha < 1 - 1/p$, we have

$$\int_{\mathbb{R}^d} |x|^{p\alpha} |\mathcal{M}(\sigma, \nabla u)(x)|^p d\sigma dx \le C \int_{\mathbb{R}^d} |x|^{p\alpha} |\nabla u(x) \cdot \sigma|^p dx, \quad \text{for all } \sigma \in \mathbb{S}^{d-1}, \tag{3.34}$$

for some positive constant C independent of σ and u. Then, combining (3.33) and (3.34) yields

$$I_{\delta}(u, \alpha) \le C \int_{\mathbb{R}^d} |x|^{p\alpha} |\nabla u|^p \, dx, \tag{3.35}$$

as mentioned in Remark 1.2. For simplicity, we assume that

$$\sigma = e_d = (0, \dots, 0, 1) \in \mathbb{R}^d$$

and prove (3.34). We have, for any bounded interval (a, b) and for any $x' \in \mathbb{R}^{d-1}$,

$$\fint_a^b (|x'| + |s|)^{p\alpha} \, ds \left(\fint_a^b (|x'| + |s|)^{-p\alpha/(p-1)} \, ds \right)^{p-1} \le C, \tag{3.36}$$

for some positive constant C independent of (a, b) and x' since $-1/p < \alpha < 1-1/p$. Applying the theory of maximal functions with weights due to Muckenhoupt [20, Corollary 4] (see also [16, Theorem 1]), which holds whenever the weight satisfies (3.36), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} |x|^{p\alpha} |\mathcal{M}(e_d, \nabla u)(x)|^p \, dx \\
&\qquad \le C \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} (|x'| + |x_d|)^{p\alpha} |\mathcal{M}(e_d, \nabla u)(x', x_d)|^p \, dx_d \, dx' \\
&\qquad \le C \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} (|x'| + |x_d|)^{p\alpha} |\partial_{x_d} u(x', x_d)|^p \, dx_d \, dx' \\
&\qquad \le C \int_{\mathbb{R}^d} |x|^{p\alpha} |\nabla u|^p \, dx.
\end{aligned}$$

The claim (3.34) is proved.

4 Results in bounded domains

In this section, we present some results in the spirit of Theorems 1.1 and 3.1 for a smooth bounded domain Ω. As a consequence of Theorem 1.1 and the extension argument in the proof of Lemma 3.1, we obtain

Proposition 4.1. *Let $d \ge 1$, $1 \le p \le d$, $\Omega \Subset B_R$ a smooth open subset of $\mathbb{R}^d$, and $u \in L^p(\Omega)$. We have*

(i) *if $1 \le p < d$, then*

$$\int_\Omega \frac{|u(x)|^p}{|x|^p} \, dx \le C_\Omega (I_\delta(u, \Omega) + \|u\|^p_{L^p(\Omega)} + \delta^p),$$

(ii) *if $p > d$ and* $\operatorname{supp} u \subset \bar{\Omega} \setminus B_r$, *then*

$$\int_\Omega \frac{|u(x)|^p}{|x|^p} \, dx \le C_\Omega (I_\delta(u, \Omega) + \|u\|^p_{L^p(\Omega)} + r^{d-p}\delta^p),$$

(iii) *if $p = d \ge 2$, then*

$$\int_{\Omega \setminus B_r} \frac{|u(x)|^d}{|x|^d \ln^d(2R/|x|)} \, dx \le C_\Omega (I_\delta(u, \Omega) + \|u\|^p_{L^p(\Omega)} + \ln(2R/r)\delta^d),$$

(iv) *if* $p = d \geq 2$ *and* $\operatorname{supp} u \subset \Omega \setminus B_r$*, then*

$$\int_{\Omega \cap B_R} \frac{|u(x)|^d}{|x|^d \ln^d(2|x|/r)}\, dx \leq C_\Omega(I_\delta(u, \Omega) + \|u\|_{L^p(\Omega)}^p + \ln(2R/r)\delta^d).$$

Here C_Ω *denotes a positive constant depending only on* p *and* Ω*.*

Using Theorem 1.2, we derive

Proposition 4.2. *Let* $d \geq 2$*,* $1 < p < d$*,* $q \geq 1$*,* $\tau > 0$*,* $0 < a \leq 1$*,* $\alpha, \beta, \gamma \in \mathbb{R}$*,* $0 \in \Omega \subset B_R$ *a smooth bounded open subset of* $\mathbb{R}^d$*, and* $u \in L^p(\Omega)$ *be such that*

$$\frac{1}{\tau} + \frac{\gamma}{d} = a\Big(\frac{1}{p} + \frac{\alpha - 1}{d}\Big) + (1 - a)\Big(\frac{1}{q} + \frac{\beta}{d}\Big),$$

and, with $\gamma = a\sigma + (1 - a)\beta$*,*

$$0 \leq \alpha - \sigma \leq 1.$$

We have

(i) *if* $1/\tau + \gamma/d > 0$*, then*

$$\Big(\int_\Omega |x|^{\gamma\tau}|u|^\tau\, dx\Big)^{1/\tau} \leq C(I_\delta(u, \Omega, \alpha) + \|u\|_{L^p(\Omega)}^p + \delta^p)^{a/p}\||x|^\beta u\|_{L^q(\Omega)}^{(1-a)},$$

(ii) *if* $1/\tau + \gamma/d < 0$ *and* $\operatorname{supp} u \subset \Omega \setminus \{0\}$*, then*

$$\Big(\int_\Omega |x|^{\gamma\tau}|u|^\tau\, dx\Big)^{1/\tau} \leq C(I_\delta(u, \Omega, \alpha) + \|u\|_{L^p(\Omega)}^p + \delta^p)^{a/p}\||x|^\beta u\|_{L^q(\Omega)}^{(1-a)},$$

(iii) *if* $1/\tau + \gamma/d = 0$ *and* $\tau > 1$*, then*

$$\Big(\int_{\Omega \setminus B_r} \frac{|x|^{\gamma\tau}}{\ln^\tau(2R/|x|)}|u|^\tau\, dx\Big)^{1/\tau} \leq C(I_\delta(u, \Omega, \alpha) + \|u\|_{L^p(\Omega)}^p + \delta^p \ln(2R/r))^{a/p}\||x|^\beta u\|_{L^q(\Omega)}^{(1-a)},$$

(iv) *if* $1/\tau + \gamma/d = 0$*,* $\tau > 1$*, and* $\operatorname{supp} u \subset \Omega \setminus B_r$*, then*

$$\Big(\int_\Omega \frac{|x|^{\gamma\tau}}{\ln^\tau(2|x|/r)}|u|^\tau\, dx\Big)^{1/\tau} \leq C(I_\delta(u, \Omega, \alpha) + \|u\|_{L^p(\Omega)}^p + \delta^p \ln(2R/r))^{a/p}\||x|^\beta u\|_{L^q(\Omega)}^{(1-a)}.$$

Here C *denotes a positive constant independent of* u *and* δ*.*

Proof. Let v be the extension of u in $\mathbb{R}^d$ as in the proof of Lemma 3.1. As in the proof of Lemma 3.1, we have, since $0 \in \Omega$,

$$I_{2\delta}(v, \alpha) \leq C\Big(I_\delta(u, \Omega, \alpha) + \|u\|_{L^p(\Omega)}\Big).$$

We also have, since $0 \in \Omega$,

$$\||x|^{\beta} v\|_{L^q(\mathbb{R}^d)} \le C\||x|^{\beta} u\|_{L^q(\Omega)}.$$

The conclusion now follows from Theorem 3.1. $\square$

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