# $p$-hyperbolicity of ends and families of paths in metric spaces

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**Abstract**

The purpose of this note is to give an expository survey on the notions of $p$-parabolicity and $p$-hyperbolicity of metric measure spaces of locally bounded geometry. These notions are extensions of the notions of recurrence and transience to non-linear operators such as the $p$-Laplacian (with the standard Laplacian or the 2-Laplacian associated with recurrence and transience behaviors). We discuss characterizations of these notions in terms of potential theory and in terms of moduli of families of paths in the metric space.

Keywords: recurrence, $p$-hyperbolic, singular function, modulus of curve families, ends.

MSC Class: Primary: 31E05; Secondary: 43A85, 65M80

## 1 Introduction

It is now a well-known fact that Brownian motion is recurrent in $\mathbb{R}$ and $\mathbb{R}^2$ but is transient in $\mathbb{R}^n$ for $n \geq 3$. In other words, a Brownian motion, starting from a closed ball in $\mathbb{R}^n$, will almost surely return infinitely often to that ball when $n \leq 2$ but almost surely will eventually not return to the ball when $n \geq 3$. This dichotomous behavior of recurrence versus transience can be seen in more general Riemannian manifolds, leading to a classification of manifolds as parabolic (Brownian motion is recurrent, returning infinitely often to a ball) or hyperbolic (where the Brownian motion is transient). The works [29, 11] demonstrated that the recurrence or transience of the Brownian motion is intimately connected with the existence of global singular functions, also known as Green's functions. A manifold is transient if and only if it supports a non-negative singular function.

During the past twenty years the notion of first order calculus has been developed for more general non-smooth metric measure spaces where the metric space is complete and the measure is a locally doubling Radon measure supporting a local Poincaré type inequality. For such spaces, it is not clear what the Brownian motion is, but thanks to Kakutani's theorem, we know that Brownian motion on a Riemannian manifold is a probabilistic approach to harmonic functions and the Laplace-Beltrami operator on the manifold. Physics and the

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theory of Markovian process as described in [10] also back this up, with the link provided through the heat equation. Using this as a motivation, we can study recurrence or transience of a metric measure space in terms of the existence of a singular function associated with the so-called 2-harmonicity.

Indeed, the recurrence and transience properties of the space seem to be associated with a "large scale" dimension of the underlying space. To explore the effect of the geometry of a space on curves in the space, we also move away from the realm of linear operators (Laplace-Beltrami operators) to non-linear $p$-Laplace type operators. In his dissertation [16], Holopainen gave a definition of $p$-parabolicity and $p$-hyperbolicity in Riemannian manifolds and their connections to $p$-harmonic functions. In this note we will describe some of the connections between the geometry of curves in the setting of metric measure spaces, which should be thought of as a non-linear analog of recurrence versus transience, and $p$-harmonic functions in the space. Metric measure spaces that correspond to transient spaces for $p$-harmonic functions are said to be *$p$-hyperbolic* while those that are not are said to be *$p$-parabolic*.

## 2 Background notions

The context of this note is that of metric measure spaces that need not be smooth (Riemannian). Here $(X, d, \mu)$ denotes a metric measure space with the measure $\mu$ assumed to be a Radon measure such that balls have positive and finite measure. In this section we will give a brief account of the basic notions used in the study of parabolicity versus hyperbolicity of the space in terms of first order analysis. For details on these notions, we recommend [15] and the references therein.

To understand $p$-parabolicity (recurrence) and $p$-hyperbolicity (transience), we need to have a concept of a "size" of families of curves in $X$. To this end, let $\Gamma$ be a collection of curves in $X$, and we set $\mathcal{A}(\Gamma)$ to be the collection of all non-negative Borel measurable functions $\rho$ on $X$ such that for each locally rectifiable path $\gamma \in \Gamma$ we have

$$\int_\gamma \rho \, ds \geq 1.$$

Here a path is locally rectifiable if it maps an interval $I \subset \mathbb{R}$ continuously into $X$ and for each compact subinterval $J \subset I$ we have that $\gamma|_J$ has finite length. An excellent introduction to the notion of path integrals in metric setting can be found in [15, Chapter 5], [1, Chapters 4, 6] and [13, Chapter 7].

**Definition 2.1.** *Given $1 \leq p < \infty$, the* $p$-modulus *of the collection $\Gamma$ is the number*

$$\operatorname{Mod}_p(\Gamma) = \inf_{\rho \in \mathcal{A}(\Gamma)} \int_X \rho^p \, d\mu.$$

Observe that if $\Gamma$ consists only of paths that are not locally rectifiable, then by definition $\operatorname{Mod}_p(\Gamma) = 0$, whereas if $\Gamma$ includes a constant curve, then $\operatorname{Mod}_p(\Gamma) = \infty$. It is not too difficult to verify that $\operatorname{Mod}_p$ is an outer measure on the collection of all paths, and that if $\Gamma$ has even one constant path then $\operatorname{Mod}_p(\Gamma) = \infty$. It is a result of Fuglede [9] that the only sets that are $\operatorname{Mod}_p$-measurable are those of zero $p$-modulus and their complements. In this note we only consider $\operatorname{Mod}_p$ to the extent of verifying whether a family $\Gamma$ satisfies $\operatorname{Mod}_p(\Gamma) > 0$ or not. To this end, the following result of Koskela and MacManus [23] is useful.

**Lemma 2.2.** *Let $\Gamma$ be a family of paths in $X$, and $1 < p < \infty$. Then $Mod_p(\Gamma) = 0$ if and only if there is a nonnegative Borel function $\rho \in L^p(X)$ such that for each $\gamma \in \Gamma$,*

$$\int_\gamma \rho \, ds = \infty.$$

**Definition 2.3.** *Given two sets $E, F \subset X$, by $\Gamma(E, F)$ we mean the collection of all curves in $X$ with one end point in $E$ and the other in $F$.*

The following definition is based on the dissertation [16].

**Definition 2.4.** *We say that $X$ is* $p$-hyperbolic *if there is a closed ball $B = \overline{B}(x_0, R) \subset X$ and a strictly monotone increasing sequence of real numbers $R_n > R$ with $\lim_n R_n = \infty$ and*

$$\lim_n Mod_p(\Gamma(\overline{B}(x_0, R), X \setminus B(x_0, R_n)) > 0.$$

*We say that $X$ is* $p$-parabolic *if it is not $p$-hyperbolic.*

There are now at least five available notions of Sobolev spaces in the metric setting: Poincaré-Sobolev, Korevaar-Schoen, Hajłasz-Sobolev, Newton-Sobolev, and Dirichlet domain spaces, see for example [15]. In this paper we will focus on the notion of Newton-Sobolev spaces as they are the closest aligned to the study of paths in a metric space, though for the case $p = 2$ one can replace this with Dirichlet forms and the corresponding Dirichlet domains whenever they are available, by considering the corresponding heat equation, see [11] for example.

Given a function $f : X \to \mathbb{R}$, we say that a non-negative Borel measurable function $g$ on $X$ is an *upper gradient* of $f$ if for each non-constant compact rectifiable curve $\gamma : [a, b] \to X$ we have

$$|f(\gamma(b)) - f(\gamma(a))| \le \int_\gamma g \, ds.$$

We say that $g$ is a *$p$-weak* upper gradient of $f$ if the collection $\Gamma$ of non-constant compact rectifiable curves for which the above inequality fails satisfies $\mathrm{Mod}_p(\Gamma) = 0$. With $D_p(f)$ denoting the collection of all $p$-weak upper gradients of $f$ that also belong to $L^p(X)$, we say that $f \in N^{1,p}(X)$ if $f \in L^p(X)$ (that is, the *function* $f$ belongs to an equivalence class in $L^p(X)$) and $D_p(f)$ is nonempty. The set $D_p(f)$ is a convex lattice subset of $L^p(X)$, and by a result in [23], it is also closed in $L^p(X)$. We set

$$\|f\|_{N^{1,p}(X)} := \|f\|_{L^p(X)} + \inf_{g \in D_p(f)} \|g\|_{L^p(X)}.$$

For $1 < p < \infty$, by the uniform convexity of $L^p(X)$ and the lattice property of $D_p(f)$ we know that there is a unique element $g_f \in D_p(f)$ with the property that for each $g \in D_p(f)$, $g_f \le g$ almost everywhere. Thus

$$\|f\|_{N^{1,p}(X)} = \|f\|_{L^p(X)} + \|g_f\|_{L^p(X)}.$$

Equipped with the norm $\| \cdot \|_{N^{1,p}(X)}$, the space $N^{1,p}(X)$ is a Banach space, see for example [26] and [15]. Classically, the measure of the set where two functions disagree determines whether the two functions belong to the same equivalence class in $L^p(X)$. In the setting of $N^{1,p}(X)$ the notion of $p$-capacity of a set plays this role, and here the value of $p$ determines what sets are of zero $p$-capacity. Given a set $E \subset X$, we set

$$\mathrm{Cap_p}(E) := \inf_f \int_X [|f|^p + g_f^p] d\mu,$$

where the infimum is over all functions $f \in N^{1,p}(X)$ such that $f \geq 1$ on $E$. A more pertinent notion related to parabolicity and hyperbolicity is that of *relative p-capacity.*

**Definition 2.5.** *Given two closed sets $E, F \subset X$ such that $E \cap F$ is empty,*

$$\mathrm{cap}_p(E, F) := \inf_f \int_X g_f^p d\mu,$$

*where the infimum is over all functions $f \in N^{1,p}(X)$ such that $f \geq 1$ on $E$ and $f \leq 0$ on $F$.*

There is a close connection between $\mathrm{cap_p}(E, F)$ and $\mathrm{Mod}_p(\Gamma(E, F))$. Indeed, if $\rho \in \mathcal{A}(\Gamma(E, F))$, then the function $u$ defined by

$$u(y) = \inf_{\gamma_y} \int_{\gamma_y} \rho ds$$

with infimum taken over all locally rectifiable curves in $X$ with one endpoint $y$ and the other end point in $E$, is measurable (see for example [20]) and satisfies $u = 0$ on $E$ and $u \geq 1$ on $F$. If then $X \setminus E$ is bounded, we would have $u \in N^{1,p}(X)$ with $\rho \in D_p(u)$, and thus we would have

$$\mathrm{cap_p}(E, F) \leq \mathrm{Mod}_p(\Gamma(E, F)).$$

Typically in this note $E$ would be $X \setminus B(x_0, R)$ for some $x_0 \in X$ and $R > 0$, and $F$ would be a compact subset of the ball $B(x_0, R)$.

**Definition 2.6.** *We say that the measure $\mu$ is* uniformly locally doubling *on $X$ if there is a constant $C_D \geq 1$ and a scale $0 < R_0 \leq \infty$ such that whenever $x \in X$ and $0 < r < R_0$, we have*

$$\mu(B(x, 2r)) = \mu(\{y \in X : d(x, y) < 2r\}) \leq C_D \mu(B(x, r)).$$

*We say that $(X, d, \mu)$ supports a* uniformly local $p$-Poincaré inequality *if there are constants $C > 0$, $\lambda \geq 1$ and a scale $0 < R_1 \leq \infty$ such that whenever $x \in X$, $0 < r < R_0$, and $f \in N^{1,p}(B(x, 2\lambda r))$, we have*

$$\fint_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \leq C \, r \left( \fint_{B(x,\lambda r)} g_f^p \, d\mu \right)^{1/p}.$$

*Here*

$$f_B := \fint_B f d\mu := \frac{1}{\mu(B)} \int_B f \, d\mu.$$

It is known that if $X$ is complete, $\mu$ is uniformly locally doubling, and $(X, d, \mu)$ supports a uniformly local $p$-Poincaré inequality, then for compact sets $E, F \subset X$,

$$\mathrm{cap_p}(E, F) = \mathrm{Mod}_p(\Gamma(E, F)). \tag{1}$$

A proof of this can be obtained by adapting the proof found in [21] where it was assumed that $R_0 = R_1 = \infty$. It follows immediately that $\mathrm{cap_p}(E, F) =$

$\text{cap}_\text{p}(F, E)$, even though this was not at all obvious merely from considering the definition of $\text{cap}_\text{p}(E, F)$.

**Standing assumptions:** We will assume in this note that $1 < p < \infty$, $X$ is complete, $\mu$ is uniformly locally doubling, and $(X, d, \mu)$ supports a uniformly local $p$-Poincaré inequality.

# 3 Potential theoretic characterization of $p$-hyperbolicity via $p$-singular functions

In this section we will discuss a Grigor'yan-type characterization of $p$-hyperbolicity in terms of existence of global singular functions. A $p$-singular function is a *non-negative* $p$-superharmonic function $u$ on $X$ such that there is a point $x_0 \in X$ for which $u$ is $p$-harmonic in $X \setminus \{x_0\}$, $u \in N^{1,p}(X \setminus B(x_0, r))$ for each $r > 0$, and satisfies $\lim_{y \to x_0} u(y) = \infty$. As described in [11], a manifold $X$ is transient (that is, it is 2-hyperbolic) if and only if $X$ supports a 2-singular function. In the setting of manifolds, the dissertation [16] extends this result to the non-linear setting of all $1 < p < \infty$.

Following [27], for a non-empty open set $\Omega \subset X$ and a function $u$ on $\Omega$, we say that $u$ is *$p$-harmonic* in $\Omega$ if $u \in N^{1,p}_{loc}(\Omega)$ and for each open set $V \subset \Omega$ with $\overline{V} \subset \Omega$ compact and each $v \in N^{1,p}(X)$ with $v = 0$ in $X \setminus V$ we have

$$\int_V g_u^p \, d\mu \le \int_V g_{u+v}^p \, d\mu.$$

We say that $u$ is *$p$-superharmonic* in $\Omega$ if whenever $V \subset \Omega$ with $\overline{V} \subset \Omega$ a compact set and $v \in N^{1,p}(X)$ is $p$-harmonic in a neighborhood of $\overline{V}$ with $v \le u$ on $\partial V$, we must have $v \le u$ on $V$.

**Definition 3.1.** *Let $\Omega$ be a nonempty open subset of $X$ with $X \setminus \Omega$ nonempty and $x_0 \in \Omega$. We say that a non-negative function $u$ on $X$ is a $p$-singular function on $\Omega$ with singularity at $x_0$ if*

*1 $u$ is $p$-harmonic in $\Omega \setminus \{x_0\}$,*

*2 $\lim_{\Omega \setminus \{x_0\} \ni y \to x_0} u(y) = \text{cap}_p(\{x_0\}, X \setminus \Omega)^{1/(1-p)}$,*

*3 $u \in N^{1,p}(X \setminus B(x_0, r))$ for each $r > 0$, and $u = 0$ in $X \setminus \Omega$,*

*4 and finally,*

$$\left(\frac{p-1}{p}\right)^{2(p-1)} (b-a)^{1-p} \le \text{cap}_p(\{u \ge b\}, \{u > a\}) \le p^2 (b-a)^{1-p}$$

*whenever $0 \le a < b$ such that $\{u > a\} \subset B(x_0, R_0/2)$.*

In the above definition, Condition 2 is equivalent to enforcing the condition $\lim_{\Omega \setminus \{x_0\} \ni y \to x_0} u(y) = \infty$ if $\text{cap}_p(\{x_0\}, X \setminus \Omega) = 0$ (which is the case for values of $p$ that are not larger than the dimension of the space). Thus the first three properties would be satisfied by positive scalar multiples of a $p$-singular function. The fourth condition dictates the the condensers $(\{u \ge b\}, \{u > a\})$ for $b > a$, or more specifically the value of $\text{Mod}_p(\Gamma(\{u \ge b\}, \{u \le a\})$, in terms of $(b-a)^{1-p}$. Hence this condition narrows the candidates for $p$-singular functions. Indeed, from the arguments in [16], this fourth condition guarantees uniqueness of $p$-singular functions in the context of Riemannian manifolds and other spaces where there is an Euler-Lagrange equation corresponding to the

$p$-energy minimization property. A combination of the above second and fourth conditions guarantee then that the $p$-Laplacian type operator, corresponding to the Euler-Lagrange equation, acts on the $p$-singular function to give the unit atomic measure $\delta_{x_0}$ supported at $x_0$.

From [22] we know that functions that are $p$-harmonic on an open set satisfy local Hölder continuity and (if they are non-negative) a Harnack inequality. Namely, we know that given a $p$-harmonic function $h$ on a domain $U \subset X$ and $x \in U$, there are constants $\alpha, C_h > 0$ such that if $r > 0$ with $B(x, 2r) \subset U$ and whenever $z, w \in B(x, r)$ we have $|h(z) - h(w)| \leq C_h d(z, w)^{\alpha_x}$; this is the local Hölder continuity ([22, Theorem 5.2]). Moreover, it is shown in [22, Corollary 7.3] that there is a constant $C > 0$ so that if $h$ is $p$-harmonic and non-negative on $U$ and $B(x, 2r) \subset U$, then $\sup_{B(x,r)} h \leq C \inf_{B(x,r)} h$. Using this Harnack inequality for non-negative $p$-harmonic functions in $\Omega \setminus \{x_0\}$, it is shown in [19] that if $\Omega$ is a relatively compact subset of $X$, then for each $x_0 \in \Omega$ we always have a $p$-singular function on $\Omega$ with singularity at $x_0$. Therefore the non-trivial aspect of the existence of singular functions is when $\Omega$ is unbounded.

**Definition 3.2.** *A function $u$ on $X$ is said to be a* $p$-singular function on $X$ *with singularity at $x_0 \in X$ if*

1. *$u$ is $p$-harmonic in $X \setminus \{x_0\}$ with $u > 0$ there,*
2. *there is a sequence of bounded open sets $\Omega_j \subset X$ with*
3. *$\overline{\Omega}_j \subset \Omega_{j+1}$ and $X = \bigcup_j \Omega_j$ and $r_0 > 0$ such that for $0 < r < r_0$ and $x \in X$ with $d(x, x_0) = r$,*

$$\lim_{X\setminus\{x_0\}\ni y \to x_0} u(y) \simeq \lim_j \operatorname{cap}_p(\overline{B}(x_0, r), X \setminus \Omega_j)^{1/(1-p)},$$

4. *$u \in N^{1,p}_{loc}(X \setminus \{x_0\})$,*
5. *and finally,*

$$\left(\frac{p-1}{p}\right)^{2(p-1)} (b-a)^{1-p} \leq \operatorname{cap}_p(\{u \geq b\}, \{u > a\}) \leq p^2 (b-a)^{1-p}$$

*whenever $0 \leq a < b \leq \lim_j \operatorname{cap}_p(\{x_0\}, X \setminus \Omega_j)^{1/(1-p)}$ with $b$ sufficiently large.*

Note that the notation adopted in [19] is slightly different from that here; there the relative capacity $\operatorname{cap}_p(E, F)$ is computed with respect to functions $u \in N^{1,p}(X)$ with $u = 0$ in $X \setminus F$ and $u \geq 1$ on $E$, with $E \subset F$. Hence to interpret the notation of [19] here, we should substitute the second component of $\operatorname{cap}_p(E, \Omega)$, namely $\Omega$ there, with $X \setminus \Omega$ in this current paper. In the setting of metric measure spaces, the following theorem was established in [19, Theorem 3.14].

**Theorem 3.3.** *$(X, d, \mu)$ is $p$-hyperbolic if and only if there is a point $x_0 \in X$ and a $p$-singular function on $X$ with singularity at $x_0$. If $(X, d, \mu)$ is $p$-hyperbolic, then for* every *$x_0 \in X$ there is a $p$-singular function with singularity at $x_0$.*

The idea for the proof is simple, though the details are cumbersome; we refer the interested reader to [19] for the details, and merely give a sketch of the proof now.

*Sketch.* Suppose first that $X$ is $p$-hyperbolic; then there is some $x_0 \in X$, $R > 0$, and a strictly monotone increasing sequence of positive real numbers $R_n$, $n \in \mathbb{N}$, with $R_1 > R$, such that

$$\lim_n \operatorname{Mod}_p(\Gamma(\overline{B}(x_0, R), X \setminus B(x_0, R_n)) > 0.$$

Since each curve in $\Gamma(\overline{B}(x_0, R), X \setminus B(x_0, R_{n+1})$ has a subcurve that belongs to the family $\Gamma(\overline{B}(x_0, R), X \setminus B(x_0, R_n)$, it follows that

$$\operatorname{Mod}_p(\Gamma(\overline{B}(x_0, R), X \setminus B(x_0, R_{n+1})) \leq \operatorname{Mod}_p(\Gamma(\overline{B}(x_0, R), X \setminus B(x_0, R_n)),$$

and so the above limit is well-defined. Then by (1) we know that

$$0 < \lim_n \operatorname{cap_p}(X \setminus B(x_0, R_n), \overline{B}(x_0, R)) \leq \operatorname{cap_p}(X \setminus B(x_0, R_1), \overline{B}(x_0, R)) < \infty.$$

For each $n$ let $u_n$ be a $p$-singular function in $B(x_0, R_n)$ with singularity at $x_0$; Thanks to the uniformly local version of Harnack's inequality and the definition of $p$-singular functions, for each $n \in \mathbb{N}$ the sequence $u_m$, $m \geq n$, is locally uniformly bounded in $B(x_0, R_n) \setminus \{x_0\}$. A stability result for $p$-harmonic functions (see [28]) then gives us a subsequence of $u_m$, and a function $u_\infty$, such that $u_m$ converges locally uniformly in $X \setminus \{x_0\}$ to $u_\infty$ with $u_\infty$ a $p$-harmonic function on $X \setminus \{x_0\}$. A direct argument would show that $u_\infty$ is a $p$-singular function on $X$ with singularity at $x_0$.

Now suppose that $X$ supports a $p$-singular function $u$ with singularity at some $x_0 \in X$. Then for sufficiently small $r > 0$ and a nested sequence of open sets $\Omega_j$ with $X = \bigcup_j \Omega_j$ such that

$$u \simeq \lim_j \operatorname{cap_p}(X \setminus \Omega_j, \overline{B}(x_0, r))^{1/(1-p)}$$

on the sphere $S(x_0, r) = \{y \in X \,:\, d(x_0, y) = r\}$. Thus

$$\lim_j \operatorname{cap_p}(X \setminus \Omega_j, \overline{B}(x_0, r)) > 0.$$

By passing to a subsequence if necessary, we may assume that $R_j := \operatorname{dist}(x_0, X \setminus \Omega_j)$ is a strictly monotone increasing sequence; as $X = \bigcup_j \Omega_j$, it follows that $\lim_j R_j = \infty$, and so

$$\operatorname{cap_p}(X \setminus B(x_0, R_j), \overline{B}(x_0, r)) \geq \operatorname{cap_p}(X \setminus \Omega_j, \overline{B}(x_0, r)).$$

Hence we now have

$$\lim_j \operatorname{cap_p}(X \setminus B(x_0, R_j), \overline{B}(x_0, r)) > 0,$$

that is, $X$ is $p$-hyperbolic. □ □

Note that here we require the singular functions to be non-negative. Reverting back to the setting of Euclidean spaces $\mathbb{R}^n$, we know that $\mathbb{R}^n$ supports $p$-singular functions for $1 < p < n$, but does not support a $p$-singular function for $p = n$; in the case of $p = n$ we have Green's functions, which are functions $u$ that are $p$-harmonic in $\mathbb{R}^n \setminus \{x_0\}$, $\lim_{y \to x_0} u(y) = \infty$, and $\Delta_n u = \delta_{x_0}$; however, in this case $u$ is not non-negative, and indeed we have that $\lim_{y \to \infty} u(y) = -\infty$. For more on singular functions and $p$-parabolicity, see for example [2, 3, 11, 16, 17].

# 4 $p$-hyperbolicity and $p$-modulus of a family of curves connecting a ball to $\infty$

In the setting of manifolds and with $p = 2$, we know from [11] that a manifold $M$ is 2-parabolic if and only if the (Brownian) probability measure of the collection of all Brownian paths $\gamma$ in $M$ that eventually never return to a given ball in $M$ is zero; that is, if $B$ is a ball in $M$ and $\Gamma$ is the collection of all Brownian paths $\gamma : [0, \infty) \to M$ such that $\gamma(t) = x_0$ and $\gamma(t) \notin B$ for all $t \geq t_B \in [0, \infty)$, then $\mathbb{P}(\Gamma) = 0$. In the non-linear setting of $p \neq 2$, and even when $p = 2$ but in the setting of metric measure spaces where the upper gradient structure does not come from an inner product structure on the space, the connection to Brownian motion is more tenuous. However, there is a connection between $p$-parabolicity and $p$-modulus of families of curves connecting $B$ to $\infty$; the focus of this section is to explore this idea further.

From Definition 2.4, a metric measure space $X$ is $p$-hyperbolic if there is a ball $B = B(x_0, R_0)$ and a positive number $\tau > 0$ such that whenever $R > R_0$, the $p$-modulus of the collection of all paths connecting $B$ to $X \setminus B(x_0, R)$ is at least $\tau$. Let $\Gamma(R)$ denote this collection of paths. Set $\Gamma := \bigcap_{R>0} \Gamma(R)$. Then $\Gamma$ consists of all paths that have one end point in $B$ and leave each bounded subset of $X$. Moreover, for $R_0 < R < T$ we have $\Gamma(T) \subset \Gamma(R)$, and so the family $(\Gamma(R))_{R>R_0}$ is a decreasing sequence of families of paths. However, in general it is not true that if $\Gamma_n$, $n \in \mathbb{N}$, is a decreasing sequence of families of curves, then $\lim_n \text{Mod}_p(\Gamma_n) = \text{Mod}_p(\bigcap_n \Gamma_n)$. However, we will see in this section that we can still conclude that $\text{Mod}_p(\Gamma) > 0$. As far as I know, this fact is not proven in currently existing literature on analysis on metric spaces, we provide a complete proof of this here. Note that this result is new even in the Euclidean setting. We first need the following lemma.

**Lemma 4.1.** *There is a non-negative Borel measurable function $h \in L^p(X)$ such that for each $x_0 \in X$ and $R > 0$,*

$$\inf_{B(x_0,R)} h := \beta_R > 0.$$

*Proof.* We fix $x_0 \in X$ and $R_0 > 0$, and set

$$h := \sum_{k \in \mathbb{N}} \frac{1}{2^k \, \mu(\overline{B}(x_0, (k+2)R_0) \setminus B(x_0, kR_0))^{1/p}} \chi_{B(x_0,(k+2)R_0) \setminus \overline{B}(x_0,kR_0)}.$$

Then $h$ is lower semicontinuous, and satisfies the desired requirements. □ □

Now we are ready to prove the main result of this section.

**Theorem 4.2.** *Let $B$ be a ball in $X$ and let $\Gamma$ be the collection of all paths $\gamma : [0, \infty) \to X$ such that $\gamma(0) \in B$ and for each $R > 0$ there is some $t_{\gamma,R} > 0$ such that $\gamma(t) \notin B(x_0, R)$ whenever $t > t_{\gamma,R}$. Then $X$ is $p$-hyperbolic if and only if $Mod_p(\Gamma) > 0$.*

*Proof.* Suppose first that $X$ is $p$-hyperbolic. Then

$$\lim_{R \to \infty} \text{Mod}_p(\Gamma(R)) =: \tau > 0. \tag{2}$$

Suppose that with $\Gamma = \bigcap_{R>R_0} \Gamma(R)$ satisfies $\text{Mod}_p(\Gamma) = 0$. Then we know from Fuglede's theorem (see the discussion following Definition 2.1) that there is a non-negative Borel function $\rho \in L^p(X)$ such that $\int_\gamma \rho \, ds = \infty$ for each locally rectifiable path $\gamma \in \Gamma$. An application of the Vitali-Carathéodory theorem allows us to assume that $\rho$ is also lower semicontinuous. Moreover, by replacing $\rho$ with $\max\{\rho, h\}$ with $h$ as in Lemma 4.1, we may also assume that for each $R > 0$,

$$\inf_{B(x_0,R)} \rho := \beta_R > 0.$$

Scaling $\rho$ by a positive constant if necessary, we can also assume that

$$\int_X \rho^p \, d\mu \leq \tau/2.$$

Then by (2) and by the fact that $R \mapsto \mathrm{Mod}_p(\Gamma(R))$ is monotone *decreasing*, we know that $\rho \notin \mathcal{A}(\Gamma(R))$ for each $R > R_0$. Thus, for each positive integer $n \geq 2$, there is a rectifiable curve $\gamma_n \in \Gamma(nR_0)$ such that $\int_{\gamma_n} \rho ds < 1$.

For each positive integer $n \geq 2$ and each positive integer $k \geq n$, we now have that

$$\ell(\gamma_k \cap B(x_0, nR_0)) \leq \frac{1}{\beta_{nR_0}} \int_{\gamma_n} \rho \, ds < \frac{1}{\beta_{nR_0}} < \infty.$$

It follows that the sequence $\gamma_n$, $n \in \mathbb{N}$, of paths in $X$ (using arc-length parametrization) is locally equicontinuous and locally equibounded. Given that the metric space $X$ is complete and doubling, it follows that $X$ is proper (that is, closed and bounded subsets of $X$ are compact, see for example [13]). Therefore we can invoke the Arzelà-Ascoli theorem and a Cantor diagonalization argument to obtain a subsequence of paths, denoted $\gamma_{n_j}$, $j \in \mathbb{N}$, and and a locally rectifiable path $\gamma$ with one end point in $B = B(x_0, R_0)$, such that $\gamma_{n_j} \to \gamma$ locally uniformly in $[0, \infty)$. Recall that $\rho$ is lower semicontinuous. Hence an adaptation of the argument found in [13, Page 13–14], we have

$$\int_\gamma \rho \, ds \leq \liminf_k \int_{\gamma_k} \rho \, ds \leq 1.$$

On the other hand, as $\gamma_n \in \Gamma(nR_0)$, it follows that $\gamma \in \Gamma(kR_0)$ for each positive integer $k \geq 2$; whence we have that $\gamma \in \Gamma$. This violates our choice of $\rho$ as a function in $L^p(X)$ such that for each $\widetilde{\gamma} \in \Gamma$ we have $\int_{\widetilde{\gamma}} \rho \, ds = \infty$. We can therefore conclude that we must have $\mathrm{Mod}_p(\Gamma) > 0$ as desired.

Finally, if $\mathrm{Mod}_p(\Gamma) > 0$, then for each $R > R_0$ we must have

$$\mathrm{Mod}_p(\Gamma(R)) \geq \mathrm{Mod}_p(\Gamma) > 0,$$

and therefore $X$ is $p$-hyperbolic. This concludes the proof of the theorem. □ □

Note that the outer measure $\mathrm{Mod}_p$, on the family of all paths in $X$, sees only locally rectifiable paths. Hence $p$-hyperbolicity of the metric measure space $X$ (or a Riemannian manifold $M$) tells us that there is a plenitude of locally rectifiable curves $\gamma$ in $X$ beginning from a given ball $B$ and eventually leaving every bounded subset of $X$. The key here is that these curves are locally rectifiable. In the event that $p = 2$ and we are in the setting of Riemannian manifolds $M$, this perspective is dual to the perspective of Brownian paths which are almost surely not even locally rectifiable (though they are almost surely locally Hölder continuous). It would be interesting to know whether there is an object analogous to Brownian motion for the non-linear setting of $p \neq 2$ that sees locally non-recitifable paths. One possible process associated with the $p$-Laplacian, called tug-of-war with noise in [25], might shed some light on this, but this direction of study has so far not focused on properties of paths associated with the tug-of-war with noise process. The paper [24] gives a nice introduction to the tug-of-war process, and the regularity theory associated with the rug-of-war with noise is explored in [4].

# 5 $p$-parabolicity and a Liouville-type theorem

The classical Liouville theorem states that there is no non-constant bounded complex-analytic function on the entire complex plane. A version of this theorem states that there is no non-constant positive harmonic function on the Euclidean space $\mathbb{R}^n$. In the non-smooth setting, if $\mu$ is globally doubling and supports a global $p$-Poincaré inequality, then by the results in [22] we know that non-negative $p$-harmonic functions satisfy a Harnack inequality, and hence there are no non-constant positive $p$-harmonic functions on such metric measure spaces. The situation is different when considering metric measure spaces equipped with a measure that is locally doubling and supports a local $p$-Poincaré inequality. The hyperbolic spaces $\mathbb{H}^n$ are examples of such spaces, as are infinite trees with bounded degree that are not homeomorphic to $\mathbb{R}$. As we know, $\mathbb{H}^n$ does support a non-constant positive harmonic function. It was shown in [6] that if the measure is globally doubling and supports a global $p$-Poincaré inequality, and in addition the metric space is *annular quasiconvex*, then there are no global non-constant $p$-harmonic functions (whether non-negative or not) with finite energy. Here a metric space $X$ is annular quasiconvex if there is a constant $C \geq 1$ such that whenever $x_0 \in X$ and $r > 0$, and whenever $x, y \in B(x_0, r)\setminus B(x_0, r/2)$, there is a rectifiable path $\gamma$ in $B(x_0, Cr)\setminus B(x_0, r/C)$ with end points $x$ and $y$, and with length $\ell(\gamma) \leq Cd(x, y)$. This version of Liouville theorem (finite energy Liouville theorem) is not equivalent to the standard Liouville theorem described above. In this section we discuss the effect of $p$-hyperbolicity on the existence of global non-constant positive/finite energy $p$-harmonic functions.

Note that when $1 < p < n$, the Euclidean space $\mathbb{R}^n$ is $p$-hyperbolic, but does not have a non-constant positive $p$-harmonic function nor a non-constant finite energy $p$-harmonic function; here we say that a function $u$ on a metric space $X$ has finite energy if it has an upper gradient $g_u \in L^p(X)$. Hence $p$-hyperbolicity of a space does not guarantee existence of non-constant global $p$-harmonic functions. The results of [6] indicate that we need the space to fail to be annular quasiconvex, and strongly so. The following notion of ends of a metric space is a direct analog of the theory of ends of Riemannian manifolds as described in [2].

**Definition 5.1.** *A sequence of connected sets $\{E_k\}_k$ is said to be an end (or end at infinity) of $X$ if there is a sequence of balls $B_k \subset X$ with $B_k \subset B_{k+1}$ such that $E_k$ is a component of $X \setminus B_k$ and $E_{k+1} \subset E_k$ for each positive integer $k$. We say that an end $\{E_k\}$ is a $p$-hyperbolic end if*

$$\liminf_{k\to\infty} Mod_p(\Gamma(\overline{B}_1, E_k)) > 0.$$

*We say that an end is $p$-parabolic if it is not $p$-hyperbolic.*

It is possible for a metric measure space to be $p$-hyperbolic but have only $p$-parabolic ends. Indeed, if $X$ is a $K$-regular tree (that is, each vertex has exactly $K$ number of edges attached to it) with $K \geq 3$, with the edges of unit length and equipped with the Lebesgue measure $\mathcal{L}^1$, then the measure on $X$ is uniformly locally doubling and supports a uniformly local 1-Poincaré inequality, see for example [5]. Observe that each end of $X$ corresponds to a geodesic ray starting from a vertex in $X$. Fix such an end, and we list the vertices that make up the corresponding geodesic ray by $x_k$, $k \in \mathbb{N}$. We fix $B = B(x_1, 1)$. The function $\rho_k$ given by setting $\rho_k = 0$ on all the edges except on the edges $[x_2, x_3], \cdots, [x_{k-1}, x_k]$, where it is set to take on the value of $1/(k-1)$. Then $\rho_k \in \mathcal{A}(\Gamma(B, X_k))$ with $X_k$ the connected component of $X \setminus x_k$ containing $x_{k+1}$. Therefore

$$\mathrm{Mod}_p(\Gamma(B, X_k)) \le \int_X \rho_k^p \, d\mu = \frac{1}{(k-1)^p} k,$$

and so

$$\lim_{k \to \infty} \mathrm{Mod}_p(\Gamma(B, X_k)) = 0.$$

Therefore the end is a $p$-parabolic end of $X$. However, $X$ itself is $p$-hyperbolic for each $p > 1$. This is a consequence of the following result from [7, Theorem 1.2] together with the fact that there is a non-constant $p$-harmonic function on $X$ with finite energy (see [6]). Indeed, fixing a base vertex $v_0$, we set $u = 0$ at $v_0$. We will define the value of $u$ at each vertex, with the understanding that a linear interpolation will extend the function to the edges that make up $X$. For ease of computation, we will focus on $p = 2$ and $K = 3$. Then with $v_{1,1}$, $v_{1,2}$ and $v_{1,3}$ denoting the three vertices that are neighbors of $v_0$, we set $u(v_{1,1}) = 0$, $u(v_{1,2}) = 1/2$, and $u(v_{1,3}) = -1/2$. On the connected component of $X \setminus \{v_0\}$ containing $v_{1,1}$ we set $u = 0$. We can then extend $u$ to vertices in the connected component of $X \setminus \{v_0\}$ containing $v_{1,2}$ by setting $u(w) = \sum_{j=1}^k 2^{-j}$ where $w$ is a vertex in this component that is a distance $k$ from $v_{1,2}$. We set $u(w) = -\sum_{j=1}^k 2^{-j}$ where $w$ is a vertex in the component of $X \setminus \{v_0\}$ containing $v_{1,3}$, with $k$ the distance between $w$ and $v_{1,3}$. A direct computation shows that $u$ is 2-harmonic in $X$ with finite energy $\sum_{j=1}^\infty 2^{-kp} 2^k$ with $p = 2$.

**Theorem 5.2.** *Suppose that in addition to being uniformly locally doubling and supporting a uniformly local $p$-Poincaré inequality, we have that $X$ is unbounded and proper. Then*

- *if $X$ has a non-constant $p$-harmonic function with finite energy, then $X$ is $p$-hyperbolic.*
- *If $X$ has at least two $p$-hyperbolic ends, then it has a non-constant bounded $p$-harmonic function with finite energy.*

Observe that when $n > 1$, the hyperbolic space $\mathbb{H}^n$ has only one end, and this end is indeed $p$-hyperbolic when $p < n$; the Euclidean space $\mathbb{R}^n$ also has only one end, and this end is $p$-hyperbolic when $p < n$. On the other hand, $\mathbb{R}^n$ supports no non-constant positive $p$-harmonic functions while $\mathbb{H}^n$ does.

Unlike the property of supporting a $p$-singular function, the property of supporting a non-constant positive or finite energy $p$-harmonic function does not characterize $p$-hyperbolic spaces; however, the above discussion shows that there is a connection between the existence of non-constant positive/finite energy $p$-harmonic functions and $p$-hyperbolicity. A deeper understanding of the structures of $p$-hyperbolic ends and $p$-parabolic ends of $X$ might lead to a characterization of the property of supporting a non-constant positive or finite energy $p$-harmonic functions, and this field of enquiry is still under development. For other partial characterizations of $p$-hyperbolicity using volume growth conditions see [17] (Riemannian manifold setting) and [18] (metric setting). It was shown in [17, Proposition 1.7] that if $X$ is a non-compact complete Riemannian manifold and

$$\int_1^\infty \left(\frac{1}{\mu(B(x_0,t))}\right)^{1/(p-1)} dt = \infty,$$

then it is $p$-parabolic. Moreover, it is shown in [17, Corollary 2.29] that if there is a constant $C > 0$ and a point $x_0 \in X$ such that each sequence $x_k \in X$ with $2 < d(x_k, x_0) \to \infty$ as $k \to \infty$ can be connected to $x_0$ by geodesics $\gamma_k$ with the property that

$$\int_1^{d(x_k,x_0)} \left(\frac{1}{\mu(B(\gamma_k(t), t/8))}\right)^{1/(p-1)} dt \leq C,$$

then $X$ is $p$-hyperbolic. Versions of these results in the metric setting can be found in [18], where large-scale dimension conditions are given to guarantee $p$-parabolicity and $p$-hyperbolicity of the space.

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