THE REGULARITY PROBLEM FOR GEODESICS OF THE CONTROL DISTANCE

ROBERTO MONTI (PADOVA)

This note is based on the talk given at the "Bruno Pini Mathematical Analysis Seminar" in Bologna, on the 1st February 2018. The results are obtained with A. Pigati (ETH Zürich), D. Vittone (Padova), and with some master students of the Mathematics Department in Padova.

1. Statement of the problem

Let M be a smooth manifold and $\mathscr{D} \subset TM$ a distribution on M locally given by a system of vector-fields satisfying the Hörmander condition, $\mathscr{D} = \text{span}\{X_1, \dots, X_r\}$. The number $r = \text{rank}\mathscr{D} \geq 2$ is the rank of $\mathscr{D}$. On $\mathscr{D}$ there is a fixed positive quadratic form that measures the speed of curves with tangent in $\mathscr{D}$, the so-called horizontal curves. We can choose the one making $X_1, \dots, X_r$ orthonormal. Let d be the induced Carnot-Carathéodory distance on M. This distance is often called *control metric* by the Bologna's school, because it controls the regularity of partial differential operators built upon Hörmander vector-fields.

Proposition 1.1. If the metric space (M, d) is connected, complete and locally compact then each pair of points in M is connected by (at least) one length minimizing curve γ (a "geodesic").

The a priori regularity of a geodesic γ is the Lipschitz regularity because the existence is proved by applying Ascoli-Arzelà's theorem in the class of rectifiable curves. The main and basic open problem is the following.

Question. Is any geodesic γ of class C^1 after arc-length parameterization? Or even better of class C^∞?

The problem is difficult and interesting because of the presence of abnormal extremals, also known as singular extremals. These are the critical points of the end-point mapping, i.e., curves where the differential of the end-point mapping is not surjective.

In the analytic case, i.e., analytic manifold and vector fields, Sussmann proved that length minimizers are smooth on a dense open set of times [Sus15]. However, it is not yet clear how to show that this set has full measure.

If (nontriavial) strictly singular curves do not appear, then all geodesics are of class C^∞. This is the case of a distribution $\mathscr{D}$ that satisfies the Hörmander condition with step 2. Also, for a generic distribution of rank $r \geq 3$ there are no abnormal curves, as shown in [CJT06]. However, there are examples of singular curves that are indeed length minimizing, as first observed in [Mon94]. All such known examples are of class C^∞. In particular, the important class of *regular abnormal* extremals are always smooth and also locally length-minimizing [LS95].

The situation is in general complicated by the following example, that was discovered using the algebraic theory of [LDLMV13] and [LDLMV18].

Proposition 1.2. There is a sub-Riemannian structure $(M, \mathscr{D})$, with $M = \mathbf{R}^n$ a Carnot group, such that *for any* function $\phi \in \mathrm{Lip}([0,1])$ the curve

$$\gamma(t) = (t, \phi(t), *, \dots, *), \qquad * = \text{suitable}, \quad t \in [0,1],$$

1) satisfies the first order necessary optimality conditions of the Pontryagin Maximum Principle;
2) it satisfies the second order necessary condition of Control Theory known as Goh condition.

This means that with a differential analysis up to the second order it is not possible to answer the regularity question with full generality.

2. Existence of tangent lines at every point

We recently proved in [MPV18a] that any geodesic is differentiable at any point for a suitable infinitesimal sequence of scales. This is proved by a blow-up analysis and cut-and-adjust technique that we are going to describe in this section.

After choosing exponential coordinates of first kind at the point $x_0 \in \mathrm{spt}(\gamma)$, we can assume that $M = \mathbf{R}^n$ and that $x_0 = 0 \in \mathbf{R}^n$.

The stratification of bundles at $x_0 = 0$ induced by $\mathscr{D}$ determines suitable weights $w_i \in \mathbf{N}$ associated to the "degree" of the ith coordinate. Then we have the following dilations adapted to the stratification:

$$\delta_\lambda(x) = (\lambda^{w_1} x_1, \dots, \lambda^{w_n} x_n), \qquad \lambda > 0.$$

In each vector field X_j we can isolate the leading δ_λ-homogeneous part and define truncated vector fields X_j^∞. So we have a limit distribution $\mathscr{D}^\infty = \mathrm{span}\{X_1^\infty, \dots, X_r^\infty\}$ and a limit manifold $M^\infty = \mathbf{R}^n$. We call $(M^\infty, \mathscr{D}^\infty)$ the tangent sub-Riemannian structure to $(M, \mathscr{D})$.

Let $\gamma : [-1,1] \to \mathbf{R}^n$ be a $\mathscr{D}$-horizontal curve with $\gamma(0) = 0$.

Definition 2.3. The tangent cone $\mathrm{Tan}(\gamma;0)$ is the set of all $\mathscr{D}^\infty$-horizontal curves $\kappa : \mathbf{R} \to M^\infty$ such that there is an infinitesimal sequence $\eta_i \downarrow 0$ satisfying

$$\lim_{i\to\infty} \delta_{1/\eta_i}\gamma(\eta_i t) = \kappa(t), \qquad t \in \mathbf{R},$$

with locally uniform convergence.

Theorem 2.4. Let γ be a length-minimizing curve in $(M, \mathscr{D})$. Then $\mathrm{Tan}(\gamma;x)$ contains a line, for any interior point $x \in \mathrm{spt}(\gamma)$.

This theorem is proved in [MPV18a]. We give a sketch of the proof in the next section. In the recent work [HLD18], the authors prove the same result with a slightly different approach and also obtain other results using a blow-down technique.

Proving that $\mathrm{Tan}(\gamma;x) = \{1 \text{ line}\}$, i.e., that the tangent cone consists precisely of one line, would mean that γ is differentiable at x. We are not yet able to do this. And this is not yet the C^1 regularity. The higher regularity seems even more difficult. Some $C^{1,\alpha}$-regularity results for length minimizers are obtained in [Mon14].

As a corollary of Theorem 2.4, we obtain the following result originally proved by Leonardi-Monti [LM08] with some restrictions and by Le Donne-Hakavuori [HLD16] in full generality. The technical step of reducing to problem from a general manifold to a Carnot group is detailed in [MPV18b].

Theorem 2.5. Let γ be length-minimizing in $(M, \mathscr{D})$. Then γ does not have corner-like singularities.

In fact, if x is a corner point then $\mathrm{Tan}(\gamma;x)$ consists of one curve that is a corner (and not a line). A nice application of this theorem has recently appeared in [BCJ+18].

Theorem 2.6. In rank $r = 2$ and step $s \le 4$ length minimizing curves are of class C^1.

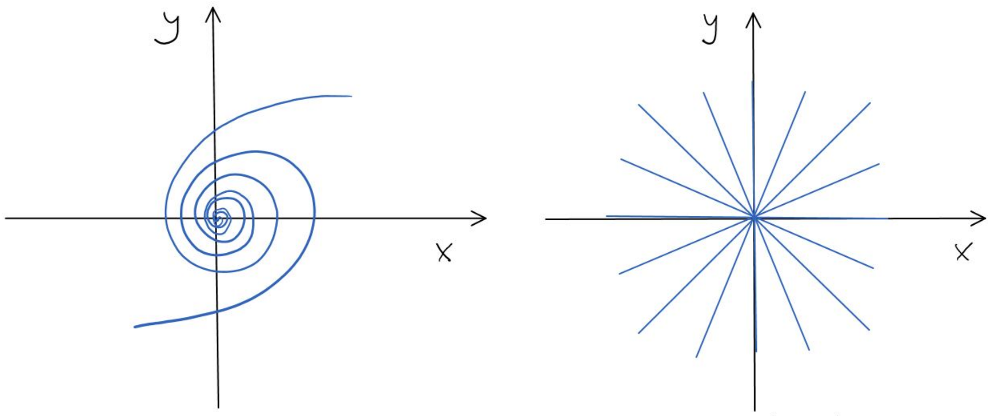
After a careful analysis of the conditions for an abnormal extremal, the authors show that it can have only isolated singularities of corner-type, which are not compatible with minimality.

Example 2.7. A limit of Theorem 2.4 is shown by the following example. Consider the planar double-logarithmic spiral

$$\gamma(t) = te^{i\log(-\log|t|)}, \qquad 0 < |t| \le 1/2$$

with $\gamma(0) = 0$.

In this case we have $\mathrm{Tan}(\gamma;0) = \{\text{all lines through } 0\}$. The spiral γ has finite length and moreover it may appear as (part of the coordinates of) an extremal curve in some sub-Riemannian manifold. The information given by Theorem 2.4 is empty and the techniques used in its proof do not seem sufficient to show that this spiral cannot be length-minimizing.



$\gamma(t) = te^{\mathrm{i}\log(-\log|t|)}$ $\quad Tan(\gamma;0)$

3. Sketch of the proof of Theorem 2.4

The goal is to find a line in the tangent cone. The first two step in the argument are the following:

Step 1. If $\bar\gamma \in \mathrm{Tan}(\gamma;0)$ and $\bar{\bar\gamma} \in \mathrm{Tan}(\bar\gamma;0)$ then $\bar{\bar\gamma} \in \mathrm{Tan}(\gamma;0)$.

Step 2. If γ is length minimizing and $\bar\gamma \in \mathrm{Tan}(\gamma;0))$ then $\bar\gamma$ is length minimizing in the limit structure.

For these reasons, we can without loss of generality assume that $M = G = \mathbf{R}^n$ is a nilpotent stratified Lie group (Carnot group). The Lie algebra of G has the stratification

$$\mathfrak{g} = V_1 \oplus V_2 \oplus \ldots \oplus V_s, \qquad s := \text{step},$$

where $\mathscr{D} = V_1$ is the first layer, i.e., the generating layer. On V_1 there is a fixed left-invariant scalar product.

Let $\gamma : [-1,1] \to G = \mathbf{R}^n$ be a horizontal curve parameterized by arc length and with $\gamma(0) = 0$.

Definition 3.8. The excess of γ on the interval $[-\eta,\eta]$ is

$$\mathrm{Exc}(\gamma;[-\eta,\eta]) = \inf_{v\in V_1,|v|=1} \left(\frac{1}{2\eta}\int_{-\eta}^{\eta} \langle\dot\gamma(t), v\rangle^2 dt\right)^{1/2}.$$

Here are some elementary considerations:

1) If $\mathrm{Exc}(\gamma;[-\eta,\eta]) = 0$ then $\dot\gamma$ is contained in a proper subspace of V_1, and thus γ is contained in a proper subgroup of G.
2) When $r = \dim(V_1) = 2$ and $\mathrm{Exc}(\gamma;[-\eta,\eta]) = 0$ then γ is a line.
3) The previous statements also hold in the infinitesimal version. For instance, if $\mathrm{Exc}(\gamma;[-\eta,\eta]) \to 0$ and the blow-up of γ is converging to some curve, then this curve is contained in a proper subgroup of G.

For these reasons, Theorem 2.4 is a consequence of the following new claim, by an iteration argument.

Theorem 3.9. Let γ be length minimizing. Then there exists an infinitesimal sequence $\eta_i \to 0$ such that

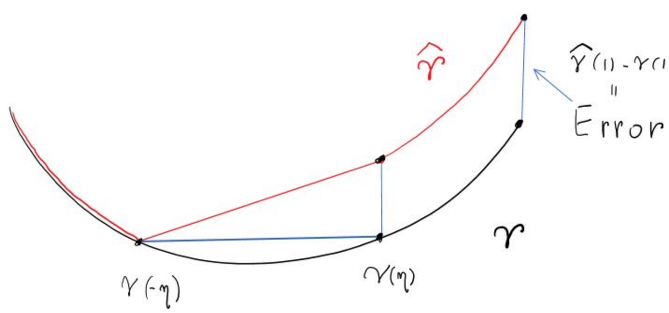
$$\lim_{i\to\infty} \mathrm{Exc}(\gamma; [-\eta_i, \eta_i]) = 0.$$

The proof goes by contradiction. Assume there exists an $\varepsilon > 0$ such that for all $\eta > 0$ we have:

$$\mathrm{E} := \mathrm{Exc}(\gamma; [-\eta, \eta]) \geq \varepsilon. \tag{$*$}$$

Condition $(*)$ has two consequences.

1) Let $\widehat{\gamma}$ be the curve obtained from γ replacing $\gamma|_{[-\eta,\eta]}$ with a line segment and then lifting to a horizontal curve. Then we have $\mathrm{length}(\gamma) - \mathrm{length}(\widehat{\gamma}) \geq \eta \mathrm{E}^2 \geq \eta\varepsilon^2$. However, there is a final error $\widehat{\gamma}(1) - \gamma(1) \neq 0$.

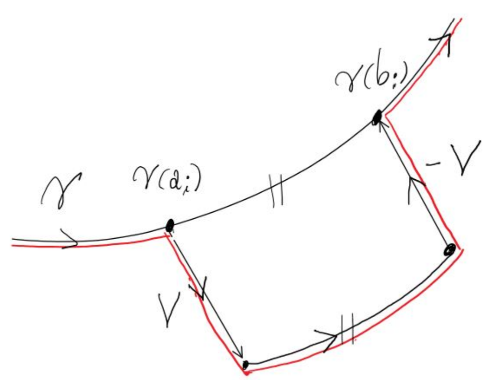


2) There exist r subintervals $[a_1, b_1], \dots, [a_r, b_r] \subseteq [-\eta, \eta]$, with $a_i < b_i \leq a_{i+1}$, such that

$$|\det(\widetilde{\gamma}(b_1) - \widetilde{\gamma}(a_1), \dots, \widetilde{\gamma}(b_r) - \widetilde{\gamma}(a_r))| \geq c(\varepsilon)\eta^r, \tag{$**$}$$

where $\widetilde{\gamma} \in \mathbf{R}^r$ are the "horizontal coordinates".

On each $[a_i, b_i]$ and for suitable $V_i \in \mathfrak{g}$ we construct a "correction device" as in the picture below:



If the displacement $V = V_i$ is short then the device is "cheap", i.e., the error can be corrected adding a small amount of length.

Proposition 3.10. There are "devices" correcting the final error $\widehat{\gamma}(1) - \gamma(1)$ adding an amount of length $o(\eta)$.

This proposition ends the proof of Theorem 3.9, because we were able to construct a curve shorter than γ and joining the same initial and final points. Condition $(**)$ controls the constants in solving a certain linear system.

4. Height-estimate and Lipschitz graphs along X_1

By Theorem 3.9 we know that, at some suitable infinitesimal scale, the excess is vanishing. For this reason it is important to understand the consequences of the small-excess regime. In this section we describe some results in this direction proved in the master thesis [Zac18].

Let $G = \mathbf{R}^n$ be a *free* Carnot group with vector fields

$$X_1 = \frac{\partial}{\partial x_1} \quad \text{and} \quad X_j = \frac{\partial}{\partial x_j} + \sum_{k=j+1}^{n} p_{jk}(x) \frac{\partial}{\partial x_k}, \qquad j = 2, \dots, r,$$

where r is the rank and p_{jk} are suitable polynomials, namely the polynomial given by the Hall basis theorem.

Consider a horizontal curve $\gamma : [-1, 1] \to \mathbf{R}^n$ with $\gamma(0) = 0$.

Definition 4.11. The directional excess of γ along X_1 at scale $\varrho > 0$ is

$$E(\gamma; 0; \varrho; X_1) = \fint_{\mathrm{spt}(\gamma) \cap B_\varrho(0)} |\dot{\gamma} - X_1|^2 d\mathscr{H}^1,$$

where $\mathscr{H}^1$ is the natural length measure and $B_\varrho(0)$ is a ball in the sub-Riemannian distance.

In Geometric Measure Theory, minimal surfaces satisfy the so-called "Height-estimate". In our setting we have the following result:

Theorem 4.12. Let γ be a length-minimizer parameterized by arc-length with $\gamma(0) = 0$. There exist integers α_i, β_i such that:

1) $\alpha_i + \beta_i = w_i$ =weight of the ith coordinate of $\mathbf{R}^n$.

2) For $0 < |t| \le \varrho$ and $i \ge 2$

$$\Big(\frac{|\gamma_i(t)|}{|t|^{\alpha_i}} \Big)^{\frac{1}{\beta_i + 1}} \le 2\varrho \sqrt{E(\gamma; 0; \varrho; X_1)}.$$

This means that in the small-excess regime, the curve is contained in a thin tube around X_1. The proof is by induction on the coordinates x_i. The numbers α_i and β_i are determined by the polynomials p_{jk} given by the Hall basis theorem.

Theorem 4.13. Let $\gamma : [-1,1] \to \mathbf{R}^n$ be a length minimizer parameterized by arc-length, with $\gamma(0) = 0$. For any $\varepsilon > 0$ there exist a set $I \subset [-1/4, 1/4]$ and a curve $\bar{\gamma} : I \to \mathbf{R}^n$ such that:

i) $\mathrm{spt}(\bar{\gamma}) \subset \mathrm{spt}(\gamma)$;

ii) $\bar{\gamma}_1(t) = t$ for $t \in I$, i.e., $\bar{\gamma}$ is a graph along X_1;

iii) for $i \geq 2$ and $s, t \in I$

$$|(\bar{\gamma}(s)^{-1} \cdot \bar{\gamma}(t))_i|^{1/w_i} \leq \varepsilon |t - s|$$

iv) $\mathscr{H}^1\big(B_{1/4} \cap \mathrm{spt}(\gamma) \setminus \mathrm{spt}(\bar{\gamma})\big) \leq C_\varepsilon E(\gamma; 0; 1; X_1)$.

Above, the dot $\cdot$ is the group law. Statement iii) means that $\bar{\gamma}$ is a "Lipschitz-graph" along X_1 for the Carnot-Carathéodory metric, with Lipschitz constant ε.

The proof of Theorem 4.13 goes as follows. For fixed $\eta > 0$ consider the set

$$\Gamma = \big\{x \in \mathrm{spt}(\gamma) \cap B_{1/4} : E(\gamma; x, r; X_1) \leq \eta \text{ for } 0 < r \leq 1/2\big\}.$$

Take points $x, y \in \Gamma$ and let $\lambda = d(x, y)$. The curve $\gamma_\lambda = \delta_{1/\lambda}(y^{-1} \cdot \gamma)$ is still length-minimizing and $0 \in \mathrm{spt}(\gamma_\lambda)$. We apply the height-estimate to the point $z = \delta_{1/\lambda}(y^{-1} \cdot x) \in \mathrm{spt}(\gamma_\lambda)$. Choosing $\eta > 0$ small compared to ε we discover that the points x and y are on an ε-Lipschitz graph.

In the regularity theory of minimal surfaces, the Lipschitz-graph approximation is the first step towards the "harmonic approximation". This approximation permits to "transfer" the regularity of harmonic functions to minimal surfaces. In the case of a sub-Riemannian length minimizing curve, it is not clear what regularity could provide a similar "harmonic approximation".

References

[BCJ+18] Davide Barilari, Yacine Chitour, Frédéric Jean, Dario Prandi, and Mario Sigalotti, *On the regularity of abnormal minimizers for rank 2 sub-Riemannian structures*, preprint (2018).

[CJT06] Yacine Chitour, Frédéric Jean, and Emmanuel Trélat, *Genericity results for singular curves*, J. Differential Geom. **73** (2006), no. 1, 45–73.

[HLD16] Eero Hakavuori and Enrico Le Donne, *Non-minimality of corners in subriemannian geometry*, Invent. Math. **206** (2016), no. 3, 693–704.

[HLD18] ———, *Blowups and blowdowns of geodesics in Carnot groups*, preprint (2018).

[LDLMV13] Enrico Le Donne, Gian Paolo Leonardi, Roberto Monti, and Davide Vittone, *Extremal curves in nilpotent Lie groups*, Geom. Funct. Anal. **23** (2013), no. 4, 1371–1401.

[LDLMV18] ———, *Extremal polynomials in stratified groups*, Comm. An. Geom., in press (2018).

[LM08] Gian Paolo Leonardi and Roberto Monti, *End-point equations and regularity of sub-Riemannian geodesics*, Geom. Funct. Anal. **18** (2008), no. 2, 552–582.

[LS95] Wensheng Liu and Héctor J. Sussman, *Shortest paths for sub-Riemannian metrics on rank-two distributions*, Mem. Amer. Math. Soc. **118** (1995), no. 564, x+104.

[Mon94] Richard Montgomery, *Abnormal minimizers*, SIAM J. Control Optim. **32** (1994), no. 6, 1605–1620.

[Mon14] Roberto Monti, *Regularity results for sub-Riemannian geodesics*, Calc. Var. Partial Differential Equations **49** (2014), no. 1-2, 549–582.

[MPV18a] Roberto Monti, Alessandro Pigati, and Davide Vittone, *Existence of tangent lines to Carnot-Carathéodory geodesics*, Calc. Var. Partial Differential Equations **57** (2018), no. 3, Art. 75, 18.

[MPV18b] ______, *On tangent cones to length minimizers in Carnot–Carathéodory spaces*, SIAM J. Control Optim. **56** (2018), no. 5, 3351–3369.

[Sus15] Héctor J. Sussmann, *A regularity theorem for minimizers of real-analytic subriemannian metrics*, Proceedings of the 53rd IEEE Conference on Decision and Control (2015), 4801–4806.

[Zac18] Michele Zaccaron, *Height estimate and Lipschitz-graph approximation of length minimizers*, Master Thesis, Dipartimento di Matematica, Università di Padova (2018), http://www.math.unipd.it/∼monti/tesi/Zaccaron2.pdf.