RANK-1 CONVEXITY IMPLIES QUASICONVEXITY ON CERTAIN HYPERSURFACES

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Abstract. We show that, if $f : \mathbb{M}^{2\times 2} \longrightarrow \mathbb{R}$ is rank-1 convex on the hyperboloid $H_D^- := \{X \in S^{2\times 2} : \det X = -D,\ X_{11} \geq c > 0\}$, $D \geq 0$, $S^{2\times 2}$ is the set of 2×2 real symmetric matrices, then f can be approximated by quasiconvex functions on $\mathbb{M}^{2\times 2}$ uniformly on compact subsets of H_D^-. Equivalently, every gradient Young measure supported on a compact subset of H_D^- is a laminate.

1. Introduction and Results

The notion of quasiconvexity was introduced by Morrey in the fundamental paper [Mo]. He proves that the variational integral

$$I(u) := \int_\Omega f(\nabla u(x))\, dx,$$

defined for sufficiently regular functions $u : \Omega \longrightarrow \mathbb{R}^m$, where Ω is a bounded open set in $\mathbb{R}^n$, $\nabla u(x)$ denotes the gradient of u at x and $f : \mathbb{M}^{m\times n} \longrightarrow \mathbb{R}$ is a continuous function, is weakly lower semicontinuous if and only if f satisfies the following so-called quasiconvexity condition: for any open bounded set $U \subset \mathbb{R}^n$,

$$\int_U (f(F + \nabla\phi) - f(F))\, dx \geq 0,\ \forall\, F \in \mathbb{M}^{m\times n},\ \forall \phi \in C_0^\infty(U).$$

There is no general procedure to verify whether a given function f is quasiconvex or not. A function $f : \mathbb{M}^{m\times n} \longrightarrow \mathbb{R}$, on the $m \times n$ real matrices is called rank-one convex if it is convex on each rank-one line, i.e., all the functions $t \mapsto f(F + ta\otimes b)$ are convex for every $F \in \mathbb{M}^{m\times n}$ and $a \in \mathbb{R}^m$, $b \in \mathbb{R}^n$. It is easy to prove that quasiconvex imply rank-one convex (see for example [Mu1]). Whether the converse is true for $m = 2$, $n \geq 2$, is major unsolved problem in the calculus of variation. In 1992, Šverák [Sv1] found a striking counterexample showing that rank-one convexity does not imply quasiconvexity for any $n \geq 2$, $m \geq 3$. Pedregal and Šverák [PS] showed that Šverák's idea of the counterexample for $m \geq 3$ cannot be used to obtain a counterexample for the 2×2 case. However, in 1999, Müller [Mu2] proved that rank-one convexity imply quasiconvexity on 2×2 diagonal matrices. Our aim

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of this article is to extend this result to the following two dimensional nonlinear hypersurface, for any $D \geq 0$, $c > 0$,

$$H_D^- := \left\{ X = (X_{ij})_{1 \leq i,j \leq 2} \in S^{2\times 2} : \det X = -D,\ X_{11} \geq c > 0 \right\},$$

where $S^{2\times 2}$ is the set of 2×2 real symmetric matrices.

The most concise statement of our result is in terms of gradient Young measures. A Young measure ν is a (weak* measurable) map from a measurable set $\Omega \subset \mathbb{R}^n$ to the space of probability measures on $\mathbb{R}^d$. The fundamental theorem for Young measures [Yo1, Yo2, BL, Ta, Ba] implies that every sequence of maps $u^{(j)} : \Omega \longrightarrow \mathbb{R}^d$ which is bounded in L^∞ contains a subsequence (not relabeled) that generates a Young measures ν in the sense that

$$\lim_{j\to\infty} \int_\Omega f(u^{(j)}(x))\phi(x)dx \ = \int_\Omega \langle \nu_x, f\rangle \phi(x) dx\,,$$

for all continuous function f and for all $\phi \in L^1(\Omega)$. Moreover ν have compact support. Here $\langle \nu_x, f\rangle := \int_{\mathbb{R}^d} f(\lambda)d\nu_x(\lambda)$. We say that ν is a $W^{1,\infty}$-gradient Young measure if Ω is open and ν is generated by a sequence of gradients $\nabla u^{(j)}$, where $(u^{(j)})$ is bounded in $W^{1,\infty}$. A Young measure is homogeneous if $x \mapsto \nu_x$ is the constant map (a.e.). Kinderlehrer and Pedregal [KP] showed that homogeneous Young measures are exactly those probability measures that satisfies Jensen's inequality for all quasiconvex functions:

$$\langle \nu, f\rangle \geq f(\langle \nu, id\rangle) \quad \forall\, f \text{ quasiconvex.}$$

A probability measure μ is called a laminate if the Jensen's inequality holds for all rank-one convex functions, see [Pe]. It is well known that the question whether rank-one convexity implies quasiconvexity can be rephrased as: Is every homogeneous gradient Young measure a laminate (see e.g., [Mu1])? Our main result is:

Theorem 1.1. *Every gradient Young measure supported on a compact subset of the hypersurface H_D^-, $D \geq 0$ is a laminate.*

This shows that rank-one convex functions on H_D^- almost admit a quasiconvex extension. More precisely the following assertion holds.

Corollary 1.2. *Let $f : \mathbb{M}^{2\times 2} \longrightarrow \mathbb{R}$ be a function which is convex on every rank-one line contained in $H_D^- = \left\{ X = (X_{ij})_{1 \leq i,j \leq 2} \in S^{2\times 2} : \det X = -D,\ X_{11} \geq c > 0 \right\}$, $D \geq 0$. Let $K \subset H_D^-$ be compact and let $\epsilon > 0$. Then there exists a quasiconvex function $f_\epsilon : \mathbb{M}^{2\times 2} \longrightarrow \mathbb{R}$ such that, $\sup_K |f_\epsilon - f| < \epsilon$.*

Šverák [Sv3, Lemma 3] proved that a probability measure supported on connected subsets of 2×2 matrices without rank-one connections and commuting with the determinant a Dirac mass. In particular, this argument applies to gradient Young measures, since the determinant is weakly continuous. Together with Proposition 1 of [Sv2] it follows that any gradient Young measure supported on the two sheeted hyperboloid $H_D := \left\{X \in S^{2\times 2} : \det X = D\right\}$ is a Dirac mass, for $D > 0$. In contrast, if A, $B \in K \subset \mathbb{M}^{m\times n}$ differ by a matrix of rank-one then for any $\lambda \in (0,1)$, $\lambda\delta_A + (1-\lambda)\delta_B$ is a nontrivial gradient Young measure supported on the set K. One notices that the one sheeted hyperboloid $\left\{ \begin{pmatrix} z+x & y \\ y & z-x \end{pmatrix} : z^2 - x^2 - y^2 = -1 \right\}$ is made by two families of straight lines and these lines are exactly the rank-one lines. Presence of these rank-one lines is the main source of difficulties showing gradient Young measures are laminates. However our idea here is to transform the hard Jacobian constraint by means of some coordinate transformations used by Evans and Gariepy [EG], inspired by the work of Schoen and Wolfson [SW] (see [He] for the corresponding change of variables in the elliptic case) to some linear constraint and then argue by using [Mu2, Theorem 2]. We will make use of the following truncation result, which generalizes an earlier work of Zhang [Zh].

Proposition 1.3 [Mu3, Theorem 2]. *Let K be a compact, convex set in $\mathbb{M}^{m\times n}$. Suppose $u^{(j)} \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^m)$ and*

$$\int_{\mathbb{R}^n} \mathrm{dist}(\nabla u^{(j)}(x), K) dx \to 0 \,.$$

Then there exists a sequence $(v^{(j)})$ of Lipschitz functions such that

$$\|\mathrm{dist}(\nabla v^{(j)}, K)\|_\infty \longrightarrow 0, \quad \mathcal{L}^n\{u^{(j)} \neq v^{(j)}\} \to 0 \,.$$

In particular, $(\nabla u^{(j)})$ and $(\nabla v^{(j)})$ generates the same Young measure.

2. Linear constraint

The following lemma quite easily follows from Theorem 2 [Mu 2], just by rotating and reflecting of the coordinate axes but we give a proof as the idea of the proof will be used later.

Lemma 2.1. *Let Ω be a bounded domain in $\mathbb{R}^2$ and $\nu = (\nu_x)_{x\in\Omega}$ be a $W^{1,\infty}$ gradient Young measure supported on*

$$K \subset P := \left\{ X = (X_{ij})_{1\le i,j\le 2} \, : \, X_{11} + X_{22} = 0, \;\, X_{12} + X_{21} = 0 \right\} .$$

Then ν is a laminate.

Proof. Let $(u^{(j)})$ be a bounded sequence in $W^{1,\infty}(\Omega, \mathbb{R}^2)$ and $(\nabla u^{(j)})$ generates the Young measure ν. Therefore, $\mathrm{dist}(\nabla u^{(j)}, K) \longrightarrow 0$ in $L^p(\Omega)$ for all $p < \infty$ and hence $u^{(j)}_{1,1} + u^{(j)}_{2,2} \longrightarrow 0$ and $u^{(j)}_{1,2} + u^{(j)}_{2,1} \longrightarrow 0$ in $L^p(\Omega)$ for all $p < \infty$. Let $\nabla u^{(j)} = \begin{pmatrix} u^{(j)}_{1,1} & u^{(j)}_{1,2} \\ u^{(j)}_{2,1} & u^{(j)}_{2,2} \end{pmatrix}$, $u^{(j)}_{\alpha,\beta}(x) := \frac{\partial}{\partial x_\beta} u^{(j)}_\alpha(x), 1 \le \alpha\,, \beta \le 2$ and $u^{(j)} \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}(\Omega, \mathbb{R}^2)$. Then the centre of mass satisfies $\overline{\nu}_x := \langle \nu_x, id\rangle = \nabla u(x)$ for *a.e.*, x in Ω. Now consider, $T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \in SO(2)$ and $S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \in SO(2) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Define $v^{(j)} : T(\Omega) \longrightarrow \mathbb{R}^2$ by $v^{(j)}(Tx) := Su^{(j)}(x)$. Then $\nabla v^{(j)}(Tx) = S\nabla u^{(j)}(x)T^{-1}$ and it is easy to see that the non-diagonal terms in the gradient matrix $\nabla v^{(j)}$ converges to zero strongly in $L^p(T(\Omega))$ for all $p < \infty$. Assume $v^{(j)} \overset{*}{\rightharpoonup} v$ in $W^{1,\infty}(T(\Omega), \mathbb{R}^2)$. Let $\mu = (\mu_y)_{y\in T(\Omega)}$ be the Young measure generated by the sequence $(\nabla v^{(j)})$. The centre of mass satisfies $\overline{\mu}_y = \nabla v(y)$ and μ is supported on the 2×2 diagonal matrices. Hence by Theorem 2 [Mu2], μ is a laminate. Now we need to show that ν is also a laminate. Let $f : \mathbb{M}^{2\times 2} \longrightarrow \mathbb{R}$ be a rank-one convex function. Then the function $g : \mathbb{M}^{2\times 2} \longrightarrow \mathbb{R}$ defined by $g(X) := f(SXT)$, is also rank-one convex. By the fundamental theorem of Young measure [Ba], and by passege to a subsequence, for any $U \subset\subset \Omega$ we obtain,

$$
\begin{aligned}
\int_{T(U)} g(\langle \mu_y, id\rangle)\, dy &\le \int_{T(U)} \langle \mu_y, g\rangle\, dy \\
&= \lim_{j\to\infty} \int_{T(U)} g(\nabla v^{(j)}(y))\, dy \\
&= \lim_{j\to\infty} \int_{U} g(\nabla v^{(j)}(Tx))\, dx \\
&= \lim_{j\to\infty} \int_{U} g(S\nabla u^{(j)}(x)T^{-1})\, dx \\
&= \lim_{j\to\infty} \int_{U} f(\nabla u^{(j)}(x))\, dx \\
&= \int_{U} \langle \nu_x, f\rangle\, dx\,.
\end{aligned}
$$

By change of variables and by the definition of g, we have $\int_{T(U)} g(\nabla v(y))\, dy = \int_U f(\nabla u(x))\, dx$ and the proof is finished. □

Lemma 2.2. *Any gradient Young measure supported on*

$$P_c := \left\{ X = (X_{ij})_{1\le i,j\le 2} \;:\; X_{11} + X_{22} = c,\;\; X_{12} + X_{21} = 0 \right\}$$

$c \neq 0$ *is a laminate.*

Proof. This follows from the change of variables $u(x) \mapsto u(x) + (0, -cx_2)$. □

3. Proof of theorem 1.2

Case I: $D > 0$.

Without loss of generality, we can assume that $D = 1$, that the Young measure $\nu = (\nu_x)_{x\in\Omega}$ is homogeneous and that $\Omega = (0,1)^2$. Let $(\nabla u^{(j)}) \subset W^{1,\infty}(\Omega, \mathbb{R}^{2\times 2})$, generate the Young measure ν, $u^{(j)} \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}(\Omega, \mathbb{R}^2)$ and $\operatorname{supp}\nu = K \subset H_D^-$. Since K is compact, $K \subset \widetilde{K} := B_R \cap \left\{X \in S^{2\times 2} : X_{11} \ge c > 0\right\}$ for some $R > 0$, where $B_R := \left\{X \in \mathbb{M}^{2\times 2} : |X| \le R\right\}$. Since $\widetilde{K}$ is a compact, convex set and $\operatorname{dist}(\nabla u^{(j)}, \widetilde{K}) \longrightarrow 0$ in $L^p(\Omega)$ for all $p < \infty$, by Proposition 1.3, there exists a sequence $(v^{(j)})$, with uniformly bounded Lipschitz constant such that $(\nabla v^{(j)})$ generates the same measure ν and $\|\operatorname{dist}(\nabla v^{(j)}, \widetilde{K})\|_\infty \longrightarrow 0$ as $j \to \infty$. Hence we can assume that our original generating sequence $(u^{(j)})$, satisfies $u^{(j)}_{1,1} \ge c/2$ and $|\nabla u^{(j)}| \le 2R$. By Ascoli-Arzela Theorem $u^{(j)} \longrightarrow u$ uniformly on Ω. Since ν is supported on H_D^-, it is easy to see that $\det(\nabla u^{(j)}(x)) + 1$ and $u^{(j)}_{1,2} - u^{(j)}_{2,1}$, converge to zero strongly in $L^p(\Omega)$ for all $p < \infty$. Now our idea is to obtain a new sequence of uniformly bounded Lipschitz functions on some suitable domain which generates a new Young measure μ, supported on the set P defined in Lemma 2.1. Then by Lemma 2.1, such a measure μ will be a laminate and finally we will argue in similar way as in the proof of Lemma 2.1 to show that the original measure ν is a laminate. This will be obtained through the following steps.

Step 1. Change of variables:

As in [EG] consider the maps $T^{(j)},\ T:\Omega\longrightarrow\mathbb{R}^2$, defined by $T^{(j)}(x_1,x_2):=(u_1^{(j)}(x),x_2)$ and $T(x_1,x_2):=(u_1(x),x_2)$, respectively. Since $u_1^{(j)}(\cdot,t)$ and $u_1(\cdot,t)$ are strictly monotonically increasing on $(0,1)$ for each $0<t<1$, the maps $T^{(j)}:\Omega\longrightarrow T^{(j)}(\Omega)$ and $T:\Omega\longrightarrow T(\Omega)$ are bi-Lipschitz, where

$$T^{(j)}(\Omega)=\left\{(y_1,y_2)\,:\,u_1^{(j)}(0,y_2)<y_1<u_1^{(j)}(1,y_2),\,0<y_2<1\right\}$$

and

$$T(\Omega)=\{(y_1,y_2)\,:\,u_1(0,y_2)<y_1<u_1(1,y_2),\,0<y_2<1\}\,.$$

Hence there exist Lipschitz maps $g^{(j)}:T^{(j)}(\Omega)\longrightarrow\mathbb{R}^2$ and $g:T(\Omega)\longrightarrow\mathbb{R}^2$, such that

$$x_1=g_1^{(j)}(u_1^{(j)}(x),x_2)\quad,u_2^{(j)}(x)=g_2^{(j)}(u_1^{(j)}(x),x_2) \tag{3.1}$$

and

$$x_1=g_1(u_1(x),x_2)\quad,u_2(x)=g_2(u_1(x),x_2)\,. \tag{3.2}$$

From the definition of $T^{(j)}$, T and differentiating (3.1), with respect to x_1, x_2 we obtain, for a.e. x

$$\nabla T^{(j)}(x)=\begin{pmatrix}u_{1,1}^{(j)}(x) & u_{1,2}^{(j)}(x)\\ 0 & 1\end{pmatrix}\,,$$

$$\nabla T(x)=\begin{pmatrix}u_{1,1}(x) & u_{1,2}(x)\\ 0 & 1\end{pmatrix}$$

and

$$\begin{cases}1=g_{1,1}^{(j)}(T^{(j)}(x))u_{1,1}^{(j)}(x)\\ 0=g_{1,1}^{(j)}(T^{(j)}(x))u_{1,2}^{(j)}(x)+g_{1,2}^{(j)}(T^{(j)}(x))\\ u_{2,1}^{(j)}(x)=g_{2,1}^{(j)}(T^{(j)}(x))u_{1,1}^{(j)}(x)\\ u_{2,2}^{(j)}(x)=g_{2,1}^{(j)}(T^{(j)}(x))u_{1,2}^{(j)}(x)+g_{2,2}^{(j)}(T^{(j)}(x))\,.\end{cases} \tag{3.3}$$

From (3.3), we have

$$\nabla g^{(j)}(T^{(j)}(x))=\frac{1}{u_{1,1}^{(j)}(x)}\begin{pmatrix}1 & -u_{1,2}^{(j)}(x)\\ u_{2,1}^{(j)}(x) & \det\nabla u^{(j)}(x)\end{pmatrix} \tag{3.4}$$

and similarly from (3.2), we obtain

$$\nabla g(T(x)) = \frac{1}{u_{1,1}(x)} \begin{pmatrix} 1 & -u_{1,2}(x) \\ u_{2,1}(x) & \det\nabla u(x) \end{pmatrix}, \tag{3.5}$$

for $a.e.$ x in Ω. Now observe that

$$\begin{aligned} \nabla(g^{(j)} \circ T^{(j)})(x) &= \nabla g^{(j)}(T^{(j)}(x))\, \nabla T^{(j)}(x) \\ &= \begin{pmatrix} 1 & 0 \\ u^{(j)}_{2,1}(x) & u^{(j)}_{2,2}(x) \end{pmatrix} \end{aligned}$$

and hence from (3.5), we conclude that $\nabla(g^{(j)} \circ T^{(j)}) \overset{*}{\rightharpoonup} \nabla(g \circ T)$ in $L^\infty(\Omega, \mathbb{M}^{2\times 2})$. From (3.1) and (3.2) it follows that $g^{(j)} \circ T^{(j)} \overset{*}{\rightharpoonup} g \circ T$ in $W^{1,\infty}(\Omega, \mathbb{R}^2)$.

Step 2. Domain selection:

Define, $v^{(j)}_\alpha(t) := u^{(j)}_1(\alpha, t)$ and $v_\alpha(t) := u_1(\alpha, t)$ for $\alpha = 0, 1$ on $(0,1)$. Since $u^{(j)}_{1,1}(x) \geq c/2$ on Ω, it follows that $v^{(j)}_1(t) - v^{(j)}_0(t) \geq c/2 > 0$ on $(0,1)$ and from the uniform convergence of $(u^{(j)})$ we have, $\inf_{t\in(0,1)} (v_1(t) - v_0(t)) \geq c/2$. Choose $0 < \epsilon < \frac{1}{4} \inf_{t\in(0,1)} (v_1(t) - v_0(t))$. Then for sufficiently large j_0,

$$V_\epsilon := \{(y_1, y_2) \,:\, v_0(y_2) + \epsilon < y_1 < v_1(y_2) - \epsilon,\ 0 < y_2 < 1\} \subset \bigcap T^{(j)}(\Omega)\,, \tag{3.6}$$

and trivially $V_\epsilon \subset T(\Omega)$. Define $f^{(j)} := g^{(j)}\big|_{V_\epsilon}$. We need to prove that the sequence $(f^{(j)})$ is uniformly Lipschitz on V_ϵ. Observe that for $y \in V_\epsilon$ there exists $x^{(j)} \in \Omega$ such that $y = T^{(j)}(x^{(j)})$, so $\nabla f^{(j)}(y) = \nabla g^{(j)}(T^{(j)}(x^{(j)})) = \nabla(g^{(j)} \circ T^{(j)})(x^{(j)}) \left(\nabla T^{(j)}(x^{(j)})\right)^{-1}$. Hence from Step 1 and from the fact that $u^{(j)}_{1,1} \geq c/2$, it follows that $\|f^{(j)}\|_{W^{1,\infty}(V_\epsilon, \mathbb{R}^2)} \leq M$, for some $M > 0$. Suppose $f^{(j)} \overset{*}{\rightharpoonup} f$ in $W^{1,\infty}(V_\epsilon, \mathbb{R}^2)$. We prove that $f = g$ on the smaller domain $\widetilde{V}_\epsilon := \left\{(y_1, y_2) \,:\, v_0(y_2) + \frac{3}{2}\epsilon < y_1 < v_1(y_2) - \frac{3}{2}\epsilon,\ 0 < y_2 < 1\right\} \subset V_\epsilon$. Let $y = Tx \in \widetilde{V}_\epsilon \subset T(\Omega)$ for some $x \in \Omega$, then by the definition of $\widetilde{V}_\epsilon$, $T^{(j)}(x) \in V_\epsilon$. Since $f^{(j)}$ is uniformly Lipschitz on V_ϵ and $T^{(j)} \longrightarrow T$ on Ω, we get

$$\lim_{j\to\infty} \left(f^{(j)} \circ T^{(j)}(x) - f^{(j)} \circ T(x)\right) = 0\,. \tag{3.7}$$

From Step 1 and (3.7) we obtain, $f(T(x)) = \lim_{j\to\infty} f^{(j)}(Tx) = \lim_{j\to\infty} g^{(j)}(Tx) = \lim_{j\to\infty} \left[g^{(j)}(T^{(j)}(x) + (g^{(j)}(Tx) - g^{(j)}(T^{(j)}(x))\right] = g(T(x))$ and hence $f = g$ on $\widetilde{V}_\epsilon$.

Step 3. Transformed Young measure:

Let $\mu = (\mu_y)_{y\in V_\epsilon}$ be the Young measure generated by the sequence $(\nabla f^{(j)})$, obtained in Step 2. Suppose E is the support of the measure μ. Now observe that for any $p < \infty$

$$
\begin{aligned}
\lim_{j\to\infty}\int_{V_\epsilon}\left|f_{1,1}^{(j)} + f_{2,2}^{(j)}\right|^p dy &= \lim_{j\to\infty}\int_{V_\epsilon}\left|g_{1,1}^{(j)} + g_{2,2}^{(j)}\right|^p dy \\
&\le \lim_{j\to\infty}\int_{T^{(j)}(\Omega)}\left|g_{1,1}^{(j)} + g_{2,2}^{(j)}\right|^p dy \\
&= \lim_{j\to\infty}\int_{\Omega}\left|g_{1,1}^{(j)}(T^{(j)}(x)) + g_{2,2}^{(j)}(T^{(j)}(x))\right|^p u_{1,1}^{(j)}(x)dx \\
&\le M \lim_{j\to\infty}\int_{\Omega}\left|\det\nabla u^{(j)}(x) + 1\right|^p dx \\
&= 0\,,
\end{aligned}
$$

and similarly we can show that $\lim_{j\to\infty}\int_{V_\epsilon}\left|f_{1,2}^{(j)} + f_{2,1}^{(j)}\right|^p dy = 0$. Thus the support E of μ is contained in $P := \Big\{X = (X_{ij})_{1\le i,j\le 2} \ : \ X_{11} + X_{22} = 0 \ \ X_{12} + X_{21} = 0\Big\}$ and hence by Lemma 2.1 μ is a laminate.

Step 4. Conclusion of the proof:

Define, $\mathbb{M}_+^{2\times 2} := \{X = (X_{ij})_{1\le i,j\le 2} \in \mathbb{M}^{2\times 2} \ : X_{11} > 0\}$ and consider the map $\Phi \ : \mathbb{M}_+^{2\times 2} \longrightarrow \mathbb{M}_+^{2\times 2}$ by

$$
\Phi(X) := \frac{1}{X_{11}}\begin{pmatrix} 1 & -X_{12} \\ X_{21} & \det X \end{pmatrix}. \tag{3.8}
$$

From the definition of the map Φ, it follows that $\Phi = \Phi^{-1}$ and by using the formula $\det(A - B) = \det(A) - \mathrm{Cof}(A) : B + \det(B)$ for 2×2 matrices A, B one obtains, $\det(\Phi(X) - \Phi(Y)) = -\dfrac{1}{X_{11}Y_{11}}\det(X - Y)$, for any matrices $X, Y \in \mathbb{M}_+^{2\times 2}$. Hence $\mathrm{rank}(X - Y) = 1$ if and only if $\mathrm{rank}(\Phi(X) - \Phi(Y)) = 1$. Since $\det \ : \mathbb{M}^{2\times 2} \longrightarrow \mathbb{R}$, is linear along any rank-one direction, by direct computation it follows that

$$
\Phi(\lambda X + (1-\lambda) Y) = \frac{\lambda X_{11}}{\lambda X_{11} + (1-\lambda) Y_{11}}\Phi(X) + \frac{(1-\lambda) Y_{11}}{\lambda X_{11} + (1-\lambda) Y_{11}}\Phi(Y)\,,
$$

for any $X, Y \in \mathbb{M}_+^{2\times 2}$, $\mathrm{rank}(X - Y) = 1$ and $0 \le \lambda \le 1$. Let $h : \mathbb{M}^{2\times 2} \longrightarrow \mathbb{R}$ be a rank-one convex function and define $\widetilde{h} : \mathbb{M}_+^{2\times 2} \longrightarrow \mathbb{R}$ by

$$
\widetilde{h}(X) := X_{11} h(\Phi(X))\ , \ \text{for} \ X \in \mathbb{M}_+^{2\times 2}\,.
$$

Now we show that $\widetilde{h}(X)$ is rank-one convex on $\mathbb{M}_+^{2\times 2}$. Let $X, Y \in \mathbb{M}_+^{2\times 2}$, $\det(X - Y) = 0$ and $\widetilde{\lambda} := \dfrac{\lambda X_{11}}{\lambda X_{11} + (1-\lambda) Y_{11}}$. Then (3.8) and the rank-one convexity of h, imply that

$$
\begin{aligned}
\widetilde{h}(\lambda X + (1-\lambda) Y) &= (\lambda X_{11} + (1-\lambda) Y_{11})\, h\left(\widetilde{\lambda}\,\Phi(X) + (1-\widetilde{\lambda})\,\Phi(Y)\right) \\
&\le \lambda X_{11}\, h(\Phi(X)) + (1-\lambda) Y_{11}\, h(\Phi(Y)) \\
&= \lambda\,\widetilde{h}(X) + (1-\lambda)\,\widetilde{h}(Y)\,.
\end{aligned}
$$

It is well known that rank-one convex functions are locally Lipschitz, see e.g. [Da, p.157]. Since $\|\nabla f^{(j)}\|_\infty \le R$, $\|\widetilde{h}\|_{L^\infty(B_R)} \le M$, where $B_R = \{X \in \mathbb{M}^{2\times 2} \,:\, |X| \le R\}$. Recall the definition, $\widetilde{V}_\epsilon = \{(y_1,y_2) \,:\, v_0(y_2) + \frac{3}{2}\epsilon < y_1 < v_1(y_2) - \frac{3}{2}\epsilon,\ 0 < y_2 < 1\}$ and $T(\Omega) = \{(y_1,y_2) \,:\, u_1(0,y_2) < y_1 < u_1(1,y_2),\ 0 < y_2 < 1\}$. It follows that $\mathcal{L}^2\left(T(\Omega)\setminus \widetilde{V}_\epsilon\right) \longrightarrow 0$ as $\epsilon \to 0$. Since, μ is a laminate and the generating sequence satisfies $\nabla f^{(j)}(y) \in \mathbb{M}^{2\times 2}_+$ *a.e.* $y \in V_\epsilon$, we have for *a.e.* $y \in V_\epsilon$

$$\widetilde{h}(\nabla f(y)) = \widetilde{h}(\langle \mu_y, id\rangle) \le \langle \mu_y, \widetilde{h}\rangle\ . \tag{3.9}$$

Hence for any $0 < \epsilon < \frac{1}{4}\inf_{t\in(0,1)}\left(v_1(t) - v_0(t)\right)$, we have

$$\begin{aligned}
\int_{\widetilde{V}_\epsilon} \langle \mu_y, \widetilde{h}\rangle\, dy &= \lim_{j\to\infty} \int_{\widetilde{V}_\epsilon} \widetilde{h}(\nabla f^{(j)}(y))\, dy \\
&= \lim_{j\to\infty} \int_{\widetilde{V}_\epsilon} \widetilde{h}(\nabla g^{(j)}(y))\, dy \\
&= \lim_{j\to\infty} \left[\int_{T^{(j)}(\Omega)} \widetilde{h}(\nabla g^{(j)}(y))\, dy \;-\; \int_{T^{(j)}(\Omega)\setminus\widetilde{V}_\epsilon} \widetilde{h}(\nabla g^{(j)}(y))\, dy\right] \\
&\le \lim_{j\to\infty} \left[\int_{T^{(j)}(\Omega)} \widetilde{h}(\nabla g^{(j)}(y))\, dy \;+\; M\,\mathcal{L}^2\left(T^{(j)}(\Omega)\setminus \widetilde{V}_\epsilon\right)\right] \\
&= \lim_{j\to\infty} \left[\int_{\Omega} \widetilde{h}(\nabla g^{(j)}(T^{(j)}(x)))\, u^{(j)}_{1,1}\, dx \;+\; M\,\mathcal{L}^2\left(T^{(j)}(\Omega)\setminus \widetilde{V}_\epsilon\right)\right] \\
&= \lim_{j\to\infty} \left[\int_{\Omega} \widetilde{h}(\Phi(\nabla u^{(j)}(x)))\, u^{(j)}_{1,1}\, dx \;+\; M\,\mathcal{L}^2\left(T^{(j)}(\Omega)\setminus \widetilde{V}_\epsilon\right)\right] \\
&= \lim_{j\to\infty} \left[\int_{\Omega} h(\nabla u^{(j)}(x))\, dx \;+\; M\,\mathcal{L}^2\left(T^{(j)}(\Omega)\setminus \widetilde{V}_\epsilon\right)\right] \\
&= \int_\Omega \langle \nu, h\rangle\, dx + M\,\mathcal{L}^2\left(T(\Omega)\setminus \widetilde{V}_\epsilon\right) \\
&= \langle \nu, h\rangle \;+ M\,\mathcal{L}^2\left(T(\Omega)\setminus \widetilde{V}_\epsilon\right)
\end{aligned} \tag{3.10}$$

Therefore from (3.9) and (3.10), for sufficiently small ϵ,

$$\begin{aligned}
\int_{\widetilde{V}_\epsilon} \widetilde{h}(\nabla g(y))\, dy &= \int_{\widetilde{V}_\epsilon} \widetilde{h}(\nabla f(y))\, dy \\
&\le \langle \nu, h\rangle \;+ M\,\mathcal{L}^2\left(T(\Omega)\setminus \widetilde{V}_\epsilon\right)\ ,
\end{aligned}$$

and hence by passing the limit $\epsilon \to 0$, we obtain

$$\int_{T(\Omega)} \widetilde{h}(\nabla g(y))\, dy \le \langle \nu, h\rangle\,. \tag{3.11}$$

On the other hand by change of variables, the definition of $\widetilde{h}$ and Φ, and by using $\nabla g(T(x)) = \Phi(\nabla u(x))$, we obtain

$$\int_{T(\Omega)} \widetilde{h}(\nabla g(y))\, dy \;=\; \int_\Omega h(\nabla f(x))\, dx \;=\; h(\langle \nu, id\rangle)\,. \tag{3.12}$$

Hence Theorem 1.1 follows from (3.11) and (3.12).

Case II: $D = 0$.

In this case we follow the same steps as for $D > 0$, In Step 1, the equation (3.5) becomes

$$\nabla g(T(x)) = \frac{1}{u_{1,1}(x)} \begin{pmatrix} 1 & -u_{1,2}(x) \\ u_{2,1}(x) & 0 \end{pmatrix}$$

and Step 2 remains unchanged. The only difference to be noticed in Step 3 is $\int_{V_\epsilon} |f_{1,1}^{(j)} + f_{2,2}^{(j)} - 1|^p \to 0$, instead of $\int_{V_\epsilon} |f_{1,1}^{(j)} + f_{2,2}^{(j)}|^p \to 0$. This shows that the Young measure μ, generated by the sequence $(\nabla f^{(j)})$ is supported on $P_1 = \left\{ X = (X_{ij})_{1 \le i,j \le 2} \,:\, X_{11} + X_{22} = 1,\ X_{12} + X_{21} = 0 \right\}$ and hence by Lemma 2.2, μ is a laminate. By step 4, it again follows that the original measure is laminate. □

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References

[Ba] J. M. Ball, *A Version of the fundamental Theorem for Young measures*, PDEs and continuum models of phase transitions, Lecture Notes in Phys. **344** (1988), Springer, Berlin., 207–215.

[BL] H. Berliocchi and J. M. Lasry, *Intégrandes normales et mesures paramétrées en calcul des variations*, Bull. Soc. Math. France **101** (1973), 129-184.

[Da] B. Dacorogna, *Direct methods in the calculus of variations*, Springer, 1989.

[EG] L. C. Evans and R. F. Gariepy, *On the Partial regularity of energy-minimizing, area-preserving maps*, Calc. Var. Partial Differential Equations **9** (1999), 357–372.

[He] E. Heinz, *Über die Lösungen der Minimalflchengleichung*, Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. Math.-Phys.-Chem. Abt. **IIa, 51** (1952), 51-56.

[KP] D. Kinderlehrer and P. Pedregal, *Characterization of Young measures generated by gradients*, Arch. Rat. Mech. Anal. **115** (1991), 329-365.

[Mo] C. B. Morrey, *Quasi-convexity and the lower semicontinuity of multiple integrals*, Pacific J. Math. **2** (1952), 25–53.

[Mu1] S. Müller, *Variational models for microstructure and phase transitions*, Calculus of variations and geometric evolution problems, Lecture Notes in Math., 1713, (1999), Springer, Berlin, 85-210.

[Mu2] S. Müller, *Rank-one convexity implies quasiconvexity on diagonal matrices*, Internat. Math. Res. Notices **20** (1999), 1087–1095.

[Mu3] S. Müller, *A sharp version of Zhang's Theorem on truncating sequences of gradients*, Trans. Amer. Math. Soc. **351** (1999), 4585–4597.

[Pe] P. Pedregal, *Laminates and Microstructure*, European J. Appl. Math. **4** (1993), 121-149.

[PS] P. Pedregal and V. Šverák, *A note on quasiconvexity and rank-one convexity for* 2×2 *matrices*, J. Convex Anal. **5** (1998), 107-117.

[SW] R. Schoen and J. Wolfson, *Minimizing volume among Lagrangian submanifolds,*, Differential equations: La Pietra 1996 (Florence), 181–199, Proc. Sympos. Pure Math. **65**, Amer. Math. Soc., Providence, RI, 1999.

[Sv1] V. Šverák, *Rank-one convexity does not imply quasiconvexity*, Proc. Roy. Soc. Edinburgh Sec. A **120** (1992), 185–189.

[Sv2] V. Šverák, *New examples of quasiconvex functions*, Arch. Rat. Mech. Anal. **119** (1992), 293–300.

[Sv3] V. Šverák, *On Tartar's conjecture*, Ann. Inst. Henri Poincaré **10** (1993), 405–412.

[Ta] L. Tartar, *Compensated compactness and applications to partial differenetial equations*, Nonlinear Analysis and Mechanics, Heriot-Watt Symposium, Res. notes in Math **39** (1979), Pitman, Boston, 136-212.

[Yo1] L. C. Young, *Generalized curves and the existence of an attained absolute minimum in the calculus of variations*, Comptes Rendues de la Société des Sciences et des Letters de Varsovie, classe III **30** (1937), 191-234.

[Yo2] L. C. Young, *Lectures on the calculus of variations and optimal control theory*, Saunders, Philadelphia, 1969.

[Zh] K. Zhang, *A constraction of quasiconvex functions with linear growth at infinity*, Ann. Scuola Norm. Sup. Pisa Cl.Sci. (4) **19** (1992), 313-326.

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