Γ-convergence of a Family of Anisotropic Functionals with Lack of Coercivity

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Abstract

We study the Γ-convergence of a family of vectorial integral functionals, which are the sum of a vanishing anisotropic quadratic form in the gradients and a penalizing double-well potential depending only on a linear combination of the components of their argument. This particular feature arises from the study of the so called "Bidomain model" for the cardiac electric field; one of its consequence is that the L^1-norm of a minimizing sequence can be unbounded and therefore a lack of coercivity occours. We characterize the Γ-limit as a surface integral functional, whose integrand is a convex function of the normal and can be computed by solving a localized minimization problem.

1 Introduction and main results.

In a Lipschitz and bounded open set Ω of $\mathbb{R}^n$, let us consider

$$\text{a continuous family of quadratic forms } \boldsymbol{q} : \overline{\Omega} \times \mathbb{M}^{m\times n} \to \mathbb{R}, \tag{1.1a}$$

$\mathbb{M}^{m\times n}$ being the vector space of the $m \times n$-matrices, satisfying the usual uniform ellipticity condition

$$\exists\, \lambda, \Lambda > 0 : \quad \lambda|\mathfrak{a}|^2 \le \boldsymbol{q}(x, \mathfrak{a}) \le \Lambda|\mathfrak{a}|^2 \quad \forall\, x \in \overline{\Omega},\ \forall\, \mathfrak{a} \in \mathbb{M}^{m\times n}. \tag{1.1b}$$

Let $W : \mathbb{R} \to [0, +\infty]$ be a continuous double-well potential, such that

$$W(s) = 0 \quad \Leftrightarrow \quad s \in T := \{s_-, s_+\} \subset \mathbb{R}, \tag{1.2a}$$

$$0 < \liminf_{|s|\uparrow+\infty} \frac{W(s)}{|s|^{p_0}} \le \limsup_{|s|\uparrow+\infty} \frac{W(s)}{|s|^{p_0}} < +\infty \quad \text{for some } p_0 \ge 2, \tag{1.2b}$$

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and let us consider the family of integrands

$$f^\varepsilon(x,\boldsymbol{s},\mathfrak{a}) := \varepsilon\boldsymbol{q}(x,\mathfrak{a}) + \frac{1}{\varepsilon}\tilde W(\boldsymbol{s}), \quad (x,\boldsymbol{s},\mathfrak{a}) \in \Omega\times\mathbb{R}^m\times\mathbb{M}^{m\times n}, \tag{1.3a}$$

$$\tilde W(\boldsymbol{s}) := W(\ell(\boldsymbol{s})), \quad \ell(\boldsymbol{s}) := \langle \boldsymbol{\ell}, \boldsymbol{s}\rangle, \quad \boldsymbol{\ell} := (\ell_1,\dots,\ell_m) \neq 0. \tag{1.3b}$$

The aim of this paper is to investigate, in the usual framework of Γ-convergence theory, the asymptotic behaviour, as $\varepsilon\downarrow 0$, of the correponding family of integral functionals in $L^2(\Omega;\mathbb{R}^m)$

$$\begin{aligned}\mathscr{F}^\varepsilon(\boldsymbol{u}) &:= \varepsilon\int_\Omega \boldsymbol{q}(x,\nabla\boldsymbol{u}(x))\,dx + \frac{1}{\varepsilon}\int_\Omega W(\ell(\boldsymbol{u}))\,dx\\ &= \int_\Omega f^\varepsilon(x,\boldsymbol{u},\nabla\boldsymbol{u})\,dx, \quad \nabla\boldsymbol{u} := (\nabla u_1,\nabla u_2,\dots,\nabla u_m)^T,\end{aligned} \tag{1.4}$$

where, as usual, we set $\mathscr{F}^\varepsilon(\boldsymbol{u}) := +\infty$ if $\boldsymbol{u}\notin H^1(\Omega;\mathbb{R}^m)\cap L^{p_0}(\Omega;\mathbb{R}^m)$. We point out that problem (1.4) is *vectorial* and *anisotropic* with the penalizing potential only *dependent on a linear combination* of the components of $\boldsymbol{u}$.

The "Bidomain model" for the cardiac electric field. Functionals of this type naturally arise in the study of the spread of excitation in the heart using the so called *bidomain* representation of the cardiac muscle (cf. [11, 9]); this macroscopic approach follows by means of homogenization techniques applied to media consisting of interconnected cells. The bidomain model attempt to describe the averaged electric potentials and current flows inside (intra) and outside (extracellular space) the cardiac cells: in this representation,

$$\Omega\subset\mathbb{R}^3 \text{ is the physical region occupied by the heart} \tag{B_1}$$

and it coincides with the intra and extracellular domains, which are two interpenetrating and superimposed continua connected at each point by the cardiac cellular membrane. ¿From this point of view,

$$\begin{gathered}\boldsymbol{u} := (u_1,u_2) \text{ represents the couple of the intracellular}\\ \text{and (the opposite of) the extracellular potentials,}\end{gathered} \tag{B_2}$$

and the sum [1] of its components

$$\ell(\boldsymbol{u}) := u_1 + u_2 \text{ defines the transmembrane potential.} \tag{B_3}$$

Since cardiac cells form a fiberlike array, the averaged conducting structure results anisotropic and it can be described by means of

$$\begin{gathered}\text{the intra and extracellular } \textit{symmetric and positive definite}\\ \text{conductivity tensors } A_1, A_2 : \overline{\Omega}\to\mathbb{M}^{3\times3},\end{gathered} \tag{B_4}$$

and by the correponding *coercive* quadratic form

$$\boldsymbol{q}(x,\mathfrak{a}) := \sum_{i=1,2}\langle A_i(x)\mathfrak{a}_i,\mathfrak{a}_i\rangle, \quad \mathfrak{a} = (\mathfrak{a}_1,\mathfrak{a}_2)^T \in \mathbb{M}^{2\times3}. \tag{B_5}$$

[1] for convenience, we have chosen *the opposite* of the extracellular potential as basic variable u_2

In this framework, a macroscopic model governing the bioelectric activity of the cardiac tissue during the excitation phase of the heart beat is described by a *degenerate reaction-diffusion* system (cf. [10, 11, 12]) in the couple of unknowns (u_1, u_2). More precisely, in dimensionless form, this system is obtained by taking the gradient flow of the Lyapunov functional, like (1.4),

$$\begin{aligned}\mathscr{F}^\varepsilon(\boldsymbol{u}) := \varepsilon \int_\Omega \Big(\langle A_1 \nabla u_1, \nabla u_1\rangle + \langle A_2 \nabla u_2, \nabla u_2\rangle\Big)\, dx \\ + \frac{1}{\varepsilon}\int_\Omega W(u_1+u_2)\, dx,\end{aligned} \tag{B_6}$$

with respect to the positive but degenerate bilinear form in $L^2(\Omega;\mathbb{R}^2)$

$$b(\boldsymbol{v},\boldsymbol{w}) := \int_\Omega (v_1+v_2)\ (w_1+w_2)\, dx;$$

this yields the following system of variational evolution equations

$$b(\partial_t \boldsymbol{u}, \boldsymbol{v}) + \delta\mathscr{F}^\varepsilon(\boldsymbol{u},\boldsymbol{v}) = 0 \quad \forall \boldsymbol{v} \in H^1(\Omega;\mathbb{R}^2)\cap L^{p_0}(\Omega;\mathbb{R}^2), \tag{1.5}$$

where $\delta\mathscr{F}^\varepsilon(\boldsymbol{u},\cdot)$ is the Euler-Lagrange first variation of the functional $\mathscr{F}^\varepsilon$ [16, pp. 9-10].

In a simplified account of the excitability properties of the cardiac cellular membrane, the reactive term is bystable and depends, instantaneously, only on the transmembrane potential. Thus the nonlinear potential energy W depends only on $u = u_1 + u_2$, which is a fundamental feature of the model and reflects the degeneracy of the system; moreover W exhibits two minima with different depths.

The typical size of the physical coefficients correponds to $0 < \varepsilon << 1$ in (B_6) and implies slow diffusive and fast reactive media. Under these conditions, for an appropriate applied stimulus or initial condition, (1.5) exhibits solutions of traveling wave front type; therefore a thin transition layer develops around a propagating interface, *the excitation wavefront*, which separates the cardiac tissue into two different domains, namely the *excited* and the *rest* regions.

By using formal singular-perturbation arguments, the asymptotic equation of the inteface motion has been derived in [10, 11, 5]. This geometric law, also called eikonal-curvature equation, prescribes the normal velocity of the interface as a sum of two nonlinear anisotropic functions of its normal direction and its curvatures. In order to characterize the form of the anisotropic curvature term of this equation, we study the asymptotic limit, as $\varepsilon \downarrow 0$, of the stationary problem associated to $\mathscr{F}^\varepsilon$ with a potential W having wells of equal depth.

Our contribution is intended as a first step towards a rigorous mathematical theory describing and justifying the right asymptotic motion law of the cardiac excitation wavefronts. We think that understanding the limit of Lyapunov functionals associated to the family (B_6) gives insights about the correct choice of the anisotropic surface energy driving the propagation of the fronts. Morever, as we shall see in a moment, this problem seems of independent mathematical interest.

After a brief overview of the available results in the literature for singular perturbation problems of anisotropic, vectorial and coercive type, in the remaining part of this Introduction we will present the statements of the main theorems of the paper, highlighting the difficulties and the differences in dealing with our class of problems, that does not fit into the previous one due to the lack of coercivity. At the end of the Introduction we will collect a refinement of the relevant results for the cardiac bidomain model, addressing especially this survey to applicants.

Anisotropic singular perturbations: the coercive case. The Γ-convergence of the integral functionals (1.4), where f^ε has a structure like (1.3a), has been deeply studied under various assumptions on f^ε, starting from the papers [19, 18] in the isotropic scalar case; these results were generalized to anisotropic (and not necessarily of quadratic type) functions in [7, 20] (cf. also [6] for a different approach in the context of Finsler metrics), and to a very general anisotropic vectorial framework in [4].

In all these cases it is assumed that the potential $\tilde{W}$, which could also depend on the x-variable,

$$\text{has only two distinct minimum points w.r.t } \boldsymbol{s}\text{, say } \boldsymbol{s}_-, \boldsymbol{s}_+ \in \mathbb{R}^m. \tag{1.6}$$

(cf. [3, 1] for the generalization to the case of a finite or *compact* minimum set),

It is then possible to show that:

- there exists the Γ-limit $\mathscr{F}$ of $\mathscr{F}^\varepsilon$ as ε goes to 0, with respect to the strong topology of $L^p(\Omega;\mathbb{R}^m)$, $1 \le p \le p_0$;
- $\mathscr{F}(\boldsymbol{u})$ is finite if and only if $\boldsymbol{u}$ is a function of bounded variation in Ω assuming only the values $\boldsymbol{s}_-, \boldsymbol{s}_+$ a.e.;
- on these functions $\mathscr{F}$ can be represented as an integral functional on the discontinuity surface of $\boldsymbol{u}$ as

$$\mathscr{F}(\boldsymbol{u}) = \int_{S^*_{\boldsymbol{u}}} \varphi(x, \nu_{\boldsymbol{u}}(x))\, d\mathscr{H}^{n-1}(x), \tag{1.7}$$

  where[2] $S^*_{\boldsymbol{u}}$ is the common reduced boundary, in the sense of De Giorgi, of the two level set of $\boldsymbol{u}$, $\nu_{\boldsymbol{u}}$ is the approximate unit normal, and $\{\varphi(x,\cdot)\}_{x\in\Omega}$ is a suitable family of norms on $\mathbb{R}^n$.

The precise calculation of these surface energy densities is not easy in the anisotropic vectorial case: they can be characterized by considering the "minimal cost" of a transition from $\boldsymbol{s}_-$ to $\boldsymbol{s}_+$ in a cylinder across a unitary planar surface perpendicular to a direction ν.

More precisely, let us first describe what we mean by the term *cylinder*: we fix

$$\begin{gathered}\text{an hyperplane } \pi \subset \mathbb{R}^n\text{, a section } \Sigma \subset \pi,\\ \text{a vector } \eta \notin \pi\text{, and an open interval (possibly unbounded) } (a,b) \subset \mathbb{R}.\end{gathered} \tag{1.8a}$$

A general cylinder with section Σ and direction η is $\Sigma(a,b;\eta) \subset \mathbb{R}^n$ defined by

$$\Sigma(a,b;\eta) = \Big\{ x + t\eta : x \in \Sigma,\ t \in (a,b) \Big\}. \tag{1.8b}$$

[2]The precise definitions of these and other concepts are collected in the next section.

We say that a set $Q := \Sigma(a, b; \eta) \subset \mathbb{R}^n$ is an *admissible cylinder* for $\nu \in \mathbb{S}^{n-1}$ if

$$\pi \text{ is orthogonal to } \nu, \quad a < 0 < b, \tag{1.8c}$$

$$\Sigma = \Big\{ \sum_{i=1}^{n-1} r_i \eta_i : \big|r_i\big| < 1/2 \Big\} \text{ for a basis } \{\eta_1, \eta_2, \dots \eta_{n-1}\} \text{ of } \pi \tag{1.8d}$$

$$\Sigma \text{ has unitary } (n-1)\text{-Hausdorff measure.} \tag{1.8e}$$

We denote the basis of Q by

$$\Sigma_+ := \big\{ x + b\eta : x \in \Sigma \big\}, \quad \Sigma_- := \big\{ x + a\eta : x \in \Sigma \big\}$$

and we introduce the closed subspace of $H^1(Q; \mathbb{R}^2)$

$$\mathcal{A}(Q; \mathbb{R}^m) := \Big\{ \boldsymbol{u} \in H^1(Q; \mathbb{R}^m) : \boldsymbol{u}_{|\partial Q} \text{ depends only on } \langle x, \nu \rangle, \\ \boldsymbol{u}(x) = \boldsymbol{s}_\pm \text{ on } \Sigma_\pm \Big\}. \tag{1.9}$$

Then ϕ admits the following expression (cf. [4])

$$\varphi(x_0, \nu) = \inf \Big\{ \int_Q f^\varepsilon(x_0, \boldsymbol{u}, \nabla \boldsymbol{u}) : \boldsymbol{u} \in \mathcal{A}(Q; \mathbb{R}^m),\ \varepsilon > 0 \Big\}.$$

In the scalar case $m = 1$ this expression can be further simplified, since it is possible to put the minimization w.r.t. ε *inside* the integral and therefore to replace f^ε with

$$f(x, \boldsymbol{s}, \mathfrak{a}) := \inf_{\varepsilon > 0} f^\varepsilon(x, \boldsymbol{s}, \mathfrak{a}).$$

If f^ε has the structure (1.3a), where now $\mathfrak{a}$ is a vector of $\mathbb{R}^n$, $\boldsymbol{q}(x, \cdot)$ is a quadratic form in $\mathbb{R}^n$ and $\boldsymbol{s}$ is a scalar, then

$$f(x, \boldsymbol{s}, \mathfrak{a}) = 2\sqrt{\tilde{W}(\boldsymbol{s}) \boldsymbol{q}(x, \mathfrak{a})} \tag{1.10}$$

and the calculus of ϕ can be reduced to a 1-dimensional problem which can be solved by the following explicit formula

$$\phi(x_0, \nu) = 2\sqrt{\boldsymbol{q}(x_0, \nu)} \int_{s_-}^{s_+} \sqrt{\tilde{W}(s)}\, ds. \tag{1.11}$$

The non coercive case. The foundamental difference between the families of functionals considered in all these contributions and the one considered in our paper lies in the structure of the set of minimum points of the potential

$$\boldsymbol{w} \in \mathbb{R}^m \mapsto \tilde{W}(\boldsymbol{w}) = W(\ell(\boldsymbol{w})),$$
$$\text{i.e. the set } \arg\min \tilde{W} := \big\{ \boldsymbol{w} \in \mathbb{R}^m : \ell(\boldsymbol{w}) = s_\pm \big\},$$

which is the union of two parallel hyperplanes. This fact reflects that a general minimizing sequence $\boldsymbol{u}^\varepsilon$ for $\mathscr{F}^\varepsilon$ can be unbounded in $L^2(\Omega; \mathbb{R}^m)$, and *a priori* only some compactness properties for the sum $\ell(\boldsymbol{u}^\varepsilon)$ could be expected: therefore, we are dealing with an anisotropic singular perturbation problem with a *lack of coercivity.* One of the main consequences is that the study of the Γ-convergence is strongly influenced by the choice of the topology, among the L^p-ones for $1 \le p \le p_0$, which in a certain sense has to supply the lack of the control on all the components of $\boldsymbol{u}$: our first result concerns the Γ-convergence w.r.t. the *strong* topology of $L^2(\Omega;\mathbb{R}^m)$.

Theorem 1 *There exists the $\Gamma(L^2(\Omega;\mathbb{R}^m))$-limit $\mathscr{F}$ of $\mathscr{F}^\varepsilon$, as ε goes to 0; $\mathscr{F}(\boldsymbol{u})$ depends only on $u := \ell(\boldsymbol{u})$, it is finite if and only if u is a function of bounded variation in Ω assuming only the values s_-, s_+ a.e., and on this kind of function $\mathscr{F}$ can be represented as in (1.7), i.e.*

$$\mathscr{F}(\boldsymbol{u}) = \int_{S_u^*} \phi(x,\nu_u(x))\, d\mathscr{H}^{n-1}(x). \tag{1.12}$$

In this case $\{\phi(x,\cdot)\}_{x\in\Omega}$ is the family of norms on $\mathbb{R}^n$ characterized by

$$\phi(x_0,\nu) = \inf\Big\{ \int_Q f^\varepsilon(x_0,\boldsymbol{u},\nabla\boldsymbol{u}) : \boldsymbol{u} \in \mathcal{B}(Q;\mathbb{R}^m),\ \varepsilon > 0\Big\} \tag{1.13}$$

where f^ε is defined as in (1.3), Q is an admissible set for ν as in (1.8), and

$$\begin{aligned}\mathcal{B}(Q;\mathbb{R}^m) := \Big\{\boldsymbol{u} \in H^1(Q;\mathbb{R}^m) : \boldsymbol{u}_{|\partial Q} \text{ depends only on } \langle x,\nu\rangle,\\ \ell(\boldsymbol{u})(x) = s_\pm \text{ on } \Sigma_\pm\Big\}.\end{aligned} \tag{1.14}$$

It is well known that a Γ-convergence result as the previous one is equivalent to two different properties: from one hand, for every element $\boldsymbol{u} \in L^2(\Omega;\mathbb{R}^m)$ we have to be able to exhibit a family $\boldsymbol{u}^\varepsilon \in L^2(\Omega;\mathbb{R}^m)$ such that, as $\varepsilon \downarrow 0$,

$$\boldsymbol{u}^\varepsilon \to \boldsymbol{u} \quad \text{and} \quad \mathscr{F}^\varepsilon(\boldsymbol{u}^\varepsilon) \to \mathscr{F}(\boldsymbol{u}); \tag{1.15a}$$

on the other hand, a kind of "lower semicontinuity" property should also be proved, i.e.

$$\boldsymbol{u}^\varepsilon \to \boldsymbol{u} \quad \text{in } L^2(\Omega;\mathbb{R}^m) \quad \Rightarrow \quad \liminf_{\varepsilon\downarrow 0} \mathscr{F}^\varepsilon(\boldsymbol{u}^\varepsilon) \ge \mathscr{F}(\boldsymbol{u}). \tag{1.15b}$$

The stronger is the topology in (1.15a), the better is it; on the contrary, the possibility to replace the strong topology of $L^2(\Omega;\mathbb{R}^m)$ with the weak one is very important, whenever we know that a family $\boldsymbol{u}^\varepsilon$ (arising, e.g., from a suitable minimization problem) is only *bounded* in $L^2(\Omega;\mathbb{R}^m)$ and not *totally bounded.*

Of course, a result of this type is related to the Γ-convergence w.r.t. the weak topology of $L^2(\Omega;\mathbb{R}^m)$; since this topology does not satisfy the first axiom of countability, in order to avoid technical difficulties we limit ourselves to consider the Γ-convergence of $\mathscr{F}^\varepsilon$ on *bounded sets* of $L^2(\Omega;\mathbb{R}^m)$, which is sufficient for proving (1.15b) in the weak convergence case.

Theorem 2 *For every family $\boldsymbol{u}^\varepsilon \in L^2(\Omega;\mathbb{R}^m)$, $\varepsilon > 0$, we have*

$$\boldsymbol{u}^\varepsilon \rightharpoonup \boldsymbol{u} \quad \text{in } L^2(\Omega;\mathbb{R}^m) \text{ as } \varepsilon \downarrow 0 \quad \Rightarrow \quad \liminf_{\varepsilon\downarrow 0} \mathscr{F}^\varepsilon(\boldsymbol{u}^\varepsilon) \ge \mathscr{F}(\boldsymbol{u}). \tag{1.16}$$

Lower and upper bounds for ϕ. It is natural to ask if it is possible to provide a more direct formula for ϕ. We will show now two explicit lower and upper bounds for ϕ, and we will analyze them more deeply in the case of the *bidomain model*.

A first simple strategy to get an explicit *lower* bound, is to minimize the integrand in (1.4) among all the possible decompositions of ∇u into a linear combination of arbitrary $\mathbb{R}^n$-valued functions, instead of gradient vector fields.

Since

$$\ell(\boldsymbol{u}) = \langle \boldsymbol{\ell}, \boldsymbol{u} \rangle = u \quad \Rightarrow \quad \nabla \boldsymbol{u}^T \cdot \boldsymbol{\ell} = \nabla u, \tag{1.17}$$

we introduce the coercive quadratic form on $\mathbb{R}^n$

$$\underline{q}(x,\xi) := \min \Big\{ \boldsymbol{q}(x,\mathfrak{a}) : \mathfrak{a}^T \cdot \boldsymbol{\ell} = \xi \Big\}, \tag{1.18}$$

which obviously satisfies

$$\underline{q}(x,\xi) \geq \frac{\lambda}{|\boldsymbol{\ell}|^2} |\xi|^2 \quad \forall\, \xi \in \mathbb{R}^m; \tag{1.19}$$

by (1.17) it is easy to see that

$$\ell(\boldsymbol{u}) = u \quad \Rightarrow \quad \mathscr{F}^\varepsilon(\boldsymbol{u}) \geq \int_\Omega \Big(\varepsilon \underline{q}(x, \nabla u(x)) + \frac{1}{\varepsilon} W(u(x)) \Big)\, dx. \tag{1.20}$$

By applying the already quoted results on the Γ-convergence of singularly perturbed problems in the scalar case, we obtain:

Proposition 1.1 *For every $x \in \Omega$ and $\xi \in \mathbb{R}^n$,*

$$\phi(x,\xi) \geq \underline{\phi}(x,\xi) := \gamma_0 \sqrt{\underline{q}(x,\xi)}, \tag{1.21}$$

where γ_0 is defined by

$$\gamma_0 := 2 \int_{s_-}^{s_+} \sqrt{W(s)}\, ds. \tag{1.22}$$

On the other hand, an explicit upper bound for ϕ can be obtained if we limit the admissible functions to be globally dependent only on $\langle x, \nu \rangle$ in the evaluation of the infimum (1.13). This constraint reduces (1.13) to a one-dimensional problem.

When a function $\boldsymbol{u} : Q \to \mathbb{R}^m$ depends only on $\langle x, \nu \rangle$ and it satisfies $\ell(\boldsymbol{u}) = u$, it is not difficult to see that[3]

$$\nabla \boldsymbol{u} = \boldsymbol{\eta} \otimes \nabla u \quad \text{for some } \boldsymbol{\eta} \text{ with } \ell(\boldsymbol{\eta}) = 1.$$

Thus we introduce the function

$$\bar{q}(x,\xi) := \min \Big\{ \boldsymbol{q}(x, \boldsymbol{\eta} \otimes \xi) : \boldsymbol{\eta} \in \mathbb{R}^m,\ \ell(\boldsymbol{\eta}) = 1 \Big\}, \tag{1.23}$$

[3] We use the notation $\boldsymbol{\eta} \otimes \xi := \boldsymbol{\eta} \cdot \xi^T$, so that, for every $\nu \in \mathbb{R}^n$

$$(\boldsymbol{\eta} \otimes \xi)\nu = \boldsymbol{\eta} \langle \xi, \nu \rangle.$$

which is bounded from above by

$$\bar{q}(x,\xi) \leq \frac{\Lambda}{|\boldsymbol{\ell}|^2}|\xi|^2 \quad \forall\,\xi \in \mathbb{R}^m, \tag{1.24}$$

and we get

Proposition 1.2 *For every $x \in \Omega$ and $\xi \in \mathbb{R}^n$,*

$$\phi(x,\xi) \leq \bar{\phi}(x,\xi) := \gamma_0 \sqrt{\bar{q}(x,\xi)}, \tag{1.25}$$

where γ_0 is defined by (1.22).

Nonlocal functionals and the behaviour of the potentials in the "Bidomain model". Now we come back to the setting $(B_1, \dots B_6)$, trying to give a more detailed description of the behaviour of the potentials $\boldsymbol{u} := (u_1, u_2)$ in terms of the transmembrane potential $u := u_1 + u_2$. ¿From the physical point of view, the dependence of $\mathscr{F}$ only on u given by (1.12) corresponds to the fact that only the evolution of the transmembrane potential u behaves like a propagating front, whereas the properties of the potentials u_1, u_2 are much more complicated, since they are related to u by a nonlocal dependence.

This non local character can also be better understood, if we associate to $\mathscr{F}^\varepsilon$ the functionals $\mathscr{G}^\varepsilon$ depending only on u, which are obtained by minimizing $\mathscr{F}^\varepsilon$ among all the possible decomposition of u as the sum of two functions in $H^1(\Omega)$. First of all we define the quadratic functional $\mathscr{Q}$ on $H^1(\Omega)$

$$\mathscr{Q}(u) := \min\left\{\int_\Omega \boldsymbol{q}(x, \nabla\boldsymbol{u}(x))\,dx : \boldsymbol{u} \in H^1(\Omega;\mathbb{R}^2),\ u_1 + u_2 = u \text{ in } \Omega\right\} \tag{1.26}$$

and we set

$$\mathscr{G}^\varepsilon(u) := \varepsilon\mathscr{Q}(u) + \frac{1}{\varepsilon}\int_\Omega W(u(x))\,dx \quad \forall\, u \in H^1(\Omega), \tag{1.27}$$

with the usual convention to extend $\mathscr{G}^\varepsilon$ to $L^2(\Omega)$. If we denote by $\mathcal{U}(u)$ the set of all $\boldsymbol{u} \in H^1(\Omega;\mathbb{R}^2)$ which achieve the minimum in (1.26), we obviously have

$$u_1 + u_2 = u \;\Rightarrow\; \mathscr{F}^\varepsilon(\boldsymbol{u}) \geq \mathscr{G}^\varepsilon(u), \quad \text{the equality holds iff } \boldsymbol{u} \in \mathcal{U}(u). \tag{1.28}$$

In a certain sense, substituting a generic couple $\boldsymbol{u}$ with $\bar{\boldsymbol{u}} \in \mathcal{U}(u_1 + u_2)$ provides a natural renormalization of $\mathscr{F}^\varepsilon(\boldsymbol{u})$. In this way, of course, we obtain coercive functionals, and it is easy to see that their Γ-convergence is not influenced by the choice of an $L^p(\Omega)$-topology (both strong and weak) if $1 \leq p \leq p_0$,[4] since they satisfies

$$\sup_{\varepsilon>0} \mathscr{G}^\varepsilon(u^\varepsilon) < +\infty \quad \Rightarrow \quad \{u^\varepsilon\}_{\varepsilon>0} \text{ is precompact in } L^{p_0}(\Omega). \tag{1.29}$$

However new difficulties arise since $\mathscr{G}^\varepsilon$ are non local. We shall see that good a priori L^2-estimates for $\bar{\boldsymbol{u}}$ in terms of $\boldsymbol{u}$ entail a strict relationship between the Γ-limits of both families and allow a slightly better refinement of Theorems 1, 2.

Let us observe that

[4] For simplicity, in the following we will keep fixed $p = 2$ as before

$\boldsymbol{u} \in \mathcal{U}(u)$ iff $u_1 + u_2 = u$ and u_1, u_2 solve (the natural variational formulation of) the following elliptic equations

$$\begin{aligned} \mathcal{L}u_1 &= \mathcal{L}_2 u, \quad \text{in } \Omega, \quad \partial_{\bar{\boldsymbol{n}}} u_1 = \partial_{\bar{\boldsymbol{n}}_2} u \quad \text{on } \partial\Omega, \\ \mathcal{L}u_2 &= \mathcal{L}_1 u, \quad \text{in } \Omega, \quad \partial_{\bar{\boldsymbol{n}}} u_2 = \partial_{\bar{\boldsymbol{n}}_1} u \quad \text{on } \partial\Omega, \end{aligned} \tag{1.30}$$

where $\mathcal{L}, \mathcal{L}_1, \mathcal{L}_2$ are the elliptic operators ($i = 1, 2$)

$$\mathcal{L}_i : \zeta \in H^1(\Omega) \mapsto -\operatorname{div}\big(A_i(x)\nabla\zeta(x)\big), \quad \mathcal{L} := \mathcal{L}_1 + \mathcal{L}_2, \tag{1.31}$$

$\boldsymbol{n}$ is the outer unit normal to $\partial\Omega$ and $\bar{\boldsymbol{n}}, \bar{\boldsymbol{n}}_1, \bar{\boldsymbol{n}}_2$ are the associated conormal vectors on $\partial\Omega$

$$\bar{\boldsymbol{n}}_i := A_i \boldsymbol{n}, \quad \bar{\boldsymbol{n}} := \bar{\boldsymbol{n}}_1 + \bar{\boldsymbol{n}}_2.$$

It is interesting to know whether (1.28)…(1.30) hold also in the limit, as $\varepsilon \downarrow 0$. An answer to this question is given in the following result, where, according to the previous convention, q and A denote the sum of q_i and A_i respectively.

Theorem 3 *There exists the $\Gamma(L^2(\Omega))$-limit $\mathscr{G}$ of $\mathscr{G}^\varepsilon$, as ε goes to 0; $\mathscr{G}(u)$ is finite if and only if u is a function of bounded variation in Ω assuming only the two values s_-, s_+ a.e., and on this kind of function $\mathscr{G}$ can be represented as in* (1.12), *i.e.*

$$\mathscr{G}(u) = \int_{S_u^*} \phi(x, \nu_u(x))\, d\mathscr{H}^{n-1}(x), \tag{1.32}$$

where ϕ are defined by (1.13). *Finally, if*

$$\Omega \text{ is convex or of class } C^{1,1}, \quad A_1, A_2 \text{ are Lipschitz continuous} \tag{1.33}$$

and

$$u^\varepsilon \to u \in L^2(\Omega), \quad \limsup_{\varepsilon\downarrow 0} \mathscr{G}^\varepsilon(u^\varepsilon) \le \mathscr{G}(u), \tag{1.34}$$

then there exists $\boldsymbol{u}^\varepsilon \in \mathcal{U}(u^\varepsilon)$ such that $\boldsymbol{u}^\varepsilon \to \boldsymbol{u}$ in $L^2(\Omega; \mathbb{R}^2)$, and $\boldsymbol{u}$ solves the following "very weak" formulation of the elliptic problem (1.30)

$$\int_\Omega u_1\, \mathcal{L}(\zeta)\, dx = \int_\Omega u\, \mathcal{L}_2(\zeta)\, dx + \int_{\partial\Omega} u\, \partial_{\bar{\boldsymbol{n}}_2}\zeta\, d\mathscr{H}^{n-1}(x), \tag{1.35a}$$

$$\int_\Omega u_2\, \mathcal{L}(\zeta)\, dx = \int_\Omega u\, \mathcal{L}_1(\zeta)\, dx + \int_{\partial\Omega} u\, \partial_{\bar{\boldsymbol{n}}_1}\zeta\, d\mathscr{H}^{n-1}(x), \tag{1.35b}$$

for every test function $\zeta \in H^2(\Omega)$ with $\partial_{\bar{\boldsymbol{n}}}\zeta = 0$ on $\partial\Omega$.

Corollary 1.3 *Let $\boldsymbol{u}^\varepsilon \in L^2(\Omega; \mathbb{R}^2)$ such that*

$$\ell(\boldsymbol{u}^\varepsilon) \rightharpoonup u \quad \text{in } \mathscr{D}'(\Omega) \text{ as } \varepsilon \downarrow 0.$$

Then

$$\liminf_{\varepsilon\downarrow 0} \mathscr{F}^\varepsilon(\boldsymbol{u}^\varepsilon) \ge \mathscr{G}(u).$$

Remark 1.4 Let us make a few comments about these results:

- The trace on $\partial\Omega$ of u in the last integrals of (1.35a,b) is well defined in the BV-sense and it belongs to $L^\infty(\partial\Omega)$.
- The solution $\boldsymbol{u} = (u_1, u_2)$ of (1.35a,b), which is defined up to an additive constant $\boldsymbol{c} = (c, -c)$, does not belong to $H^1(\Omega;\mathbb{R}^2)$ in general; but if $\boldsymbol{u} \in H^1(\Omega;\mathbb{R}^2)$ then it is also the usual variational solution of the system (cf. (4.4) below).
- Corollary 1.3 follows by Theorem 3 and the equicoercivity of $\mathscr{G}^\varepsilon$.
- We assumed that the spatial dimension n is equal to 3 for obvious physical reasons, but this hypothesis is not relevant for the validity of the mathematical results; on the contrary, the particular dimension $m = 2$, which is related to the vectorial structure of the problem, plays an essential role in some proofs. $\square$

Thanks to these last information on the Γ-limit of $\mathscr{F}^\varepsilon$ and $\mathscr{G}^\varepsilon$, we can show that, in general, the inequality (1.21) is strict. First of all we observe that the quadratic form $\underline{q}$ defined by (1.18) is associated to the *harmonic mean* $\underline{A}$ of A_1, A_2 by the formulae (where we omit the explicit dependence on x)

$$\underline{A} := \left(A_1^{-1} + A_2^{-1}\right)^{-1} = A_1(A_1 + A_2)^{-1}A_2, \tag{1.36}$$

$$\underline{q}(\cdot, \xi) = \langle \underline{A}\xi, \xi\rangle. \tag{1.37}$$

Analogously, $\bar{q}$ defined by (1.23) can be easily calculated by the "harmonic mean" of the quadratic forms

$$\bar{q}(x,\xi) = \frac{q_1(x,\xi)q_2(x,\xi)}{q_1(x,\xi) + q_2(x,\xi)}. \tag{1.38}$$

It is obvious by the definition that $\underline{\phi} \le \bar{\phi}$, and it is not difficult to show that

$$\underline{\phi}(x,\xi) = \bar{\phi}(x,\xi) \quad \Leftrightarrow \quad A_1(x)\xi, A_2(x)\xi \ \text{ have the same direction.} \tag{1.39}$$

If this is the case, of course, we have also

$$\underline{\phi}(x,\xi) = \phi(x,\xi) = \bar{\phi}(x,\xi). \tag{1.40}$$

As we said before, in all the other cases we have a strict inequality.

Theorem 4 *If $A_1(x_0)\xi$ and $A_2(x_0)\xi$ have different directions, then*

$$\underline{\phi}(x_0,\xi) < \phi(x_0,\xi). \tag{1.41}$$

Remarks and open problems. Formal asymptotics in the Bidomain case [10], [11], [5] suggest that

$$\phi(x,\nu) = \bar{\phi}(x,\nu) = \gamma_0\sqrt{\bar{q}(x,\nu)} \tag{1.42}$$

where γ_0 and $\bar{q}$ are defined in (1.22) and (1.38) respectively; if this would be the case, (1.38) should provide an explicit and manageable formula to deal with the study, both theoretic and numeric, of the anisotropic curvature flow governed by the energy surface density ϕ (cf. [5]).

On the other hand, in some pathophysiological setting, such as regional ischemia and heales infarction, the corresponding conductivity tensors A_1, A_2 yields a not convex $\bar{\phi}$: since ϕ is always convex (cf. Theorem 1), in this case (1.42) does not hold. It would be interesting to check if the convex envelope of $\bar{\phi}$ is a good substitute for 1.42 in this case. As we already said at the beginning of this introduction, we considered only the stationary problem; the rigorous justification of the evolution law still remains to the best of our knowledge, an open problem.

Plan of the paper. We will collect in §2 some preliminary notation and results which will turn to be useful in the sequel: some of them are well known parts of deep and widely developed theories, which we recall here for the reader convenience, without claiming any completeness. We will refer to the quoted books and papers for a systematic presentation of these subjects and for more detailed references.

Section 3, devoted to the proof of Theorems 1 and 2, has been divided in some subsections in order to make more clear the logic connection of the arguments and the crucial technical points of the proof.

In the last sections we will consider in more detail the case of the "Bidomain model": Theorems 3 is considered in § 4 and the proof of the strict inequality of Theorem 4 is given in § 5.

We remark that the intrinsic length of some proofs is justified by the attempt to develop a sufficiently abstract scheme, which could be directly applied to different applications; moreover, many technical difficulties, related to the lack of coercivity, have often precluded a more direct appeal to already known results.

2 Notation and definitions

2.1 ***Preliminary notation.*** ¿From now on we fix a *bounded, Lipschitz, and connected* open set $\Omega \subset \mathbb{R}^n$. By $\boldsymbol{A}(\Omega)$ we denote the collection of the open subsets of Ω. If $\mathcal{O}_1, \mathcal{O}_2 \in \boldsymbol{A}(\Omega)$, we say that $\mathcal{O}_1 \Subset \mathcal{O}_2$ if $\overline{\mathcal{O}_1} \subset \mathcal{O}_2$.
For a given set $U \subset \mathbb{R}^n$, we denote by $\mathscr{H}^{n-1}(U)$ its $n-1$-dimensional Hausdorff measure and by $|U|$ its Lebesgue measure.
$\mathbb{S}^{n-1}$ is the set of unit vectors in $\mathbb{R}^n$. If $\mathcal{O} \subset \mathbb{R}^n$ is an open set, $w \in L^1_{loc}(\mathcal{O})$, and η is a non vanishing vector, we denote by $D_\eta w$ the distributional derivative of w along η, with coincides with the classical one $\partial_\eta w = \langle \nabla w, \eta \rangle$ if $w \in C^1(\mathcal{O})$; we also use the symbol $\partial_\eta w$ if $D_\eta w \in L^1_{loc}(\mathcal{O})$. $\partial_1, \dots, \partial_n$ and $D_1, \dots, D_n$ denote the derivatives along the directions of the canonical basis of $\mathbb{R}^n$.
We will deal with functionals depending on the continuous paramenter $\varepsilon > 0$ or on the discrete values of a suitable decreasing infinitesimal sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$; for notational convenience, we will treat both the cases in the same way, by considering a general not empty set E of real numbers such that

$$E \subset (0, +\infty), \quad \inf E = 0. \tag{2.1}$$

Expressions like $\lim\limits_{\varepsilon\downarrow 0, \varepsilon\in E}$, $\liminf\limits_{\varepsilon\downarrow 0, \varepsilon\in E}$, etc..., have an obvious meaning as limits for ε going to 0 in E. Of course, when E contains an open interval $(0, \delta)$, $\delta > 0$, we will use the usual notation $\lim_{\varepsilon\downarrow 0}$.

2.2 *Localization of functionals.* It will be useful to deal with the *localized version* of $\mathscr{F}^\varepsilon$, $\mathscr{G}^\varepsilon$ (and their Γ-limits): therefore, for every $\mathcal{O} \in \boldsymbol{A}(\Omega)$, $u \in L^2(\Omega)$, and $\boldsymbol{u} \in L^2(\Omega;\mathbb{R}^m)$ we introduce

$$\mathscr{F}^\varepsilon(\boldsymbol{u},\mathcal{O}) := \begin{cases} \int_{\mathcal{O}} f^\varepsilon(x,\boldsymbol{u},\nabla\boldsymbol{u})\,dx & \text{if } \boldsymbol{u} \in H^1(\mathcal{O};\mathbb{R}^m), \\ +\infty & \text{otherwise,} \end{cases} \tag{2.2}$$

$$\mathscr{Q}(u,\mathcal{O}) := \min\left\{ \int_{\mathcal{O}} \boldsymbol{q}(x,\nabla\boldsymbol{u}(x))\,dx : \boldsymbol{u} \in H^1(\mathcal{O};\mathbb{R}^m),\ \ell(\boldsymbol{u}) = u \text{ in } \mathcal{O} \right\}, \tag{2.3}$$

$$\mathscr{G}^\varepsilon(u,\mathcal{O}) := \varepsilon\mathscr{Q}(u,\mathcal{O}) + \frac{1}{\varepsilon}\int_{\mathcal{O}} W(u(x))\,dx, \tag{2.4}$$

where, as usual,

$$H^1(\mathcal{O};\mathbb{R}^m) := \Big\{ \boldsymbol{u} \in L^2(\mathcal{O};\mathbb{R}^m) : \partial_i\boldsymbol{u} = D_i\boldsymbol{u} \in L^2(\mathcal{O};\mathbb{R}^m), \quad i = 1,2,\dots,n \Big\}.$$

We observe that these functionals are *local, lower semicontinuous, and increasing*, in the sense of [14, Def. 1.1, 1.2], [13, Def. 15.1]: we collect here these and some other related definitions, which will turn to be useful in the sequel.

Let X be a topological space included in $L^1_{loc}(\Omega;\mathbb{R}^d)$ and let $\mathscr{J}$ be a functional defined on $X \times \boldsymbol{A}(\Omega)$ taking its values in the extended half line $[0,+\infty]$; we say that $\mathscr{J}$ is

local if for every $\mathcal{O} \in \boldsymbol{A}(\Omega)$ and $u, w \in X$

$$u_{|\mathcal{O}} = w_{|\mathcal{O}} \quad \Rightarrow \quad \mathscr{J}(u,\mathcal{O}) = \mathscr{J}(w,\mathcal{O})$$

lower semicontinuous if for every $\mathcal{O} \in \boldsymbol{A}(\Omega)$,

$$\text{the map } u \mapsto \mathscr{J}(u,\mathcal{O}) \text{ is lower semicontinuous in } X$$

increasing if for every $u \in X$ and $\mathcal{O}_1, \mathcal{O}_2 \in \boldsymbol{A}(\Omega)$

$$\mathcal{O}_1 \subset \mathcal{O}_2 \quad \Rightarrow \quad \mathscr{J}(u,\mathcal{O}_1) \le \mathscr{J}(u,\mathcal{O}_2)$$

superadditive if for every $u \in X$ and $\mathcal{O}_1, \mathcal{O}_2 \in \boldsymbol{A}(\Omega)$

$$\mathcal{O}_1 \cap \mathcal{O}_2 \neq \varnothing \quad \Rightarrow \quad \mathscr{J}(u,\mathcal{O}_1 \cup \mathcal{O}_2) \ge \mathscr{J}(u,\mathcal{O}_1) + \mathscr{J}(u,\mathcal{O}_2)$$

weakly subadditive if for every $u \in X$ and $\mathcal{O}_1, \mathcal{O}_1', \mathcal{O}_2 \in \boldsymbol{A}(\Omega)$

$$\mathcal{O}_1 \Subset \mathcal{O}_1' \quad \Rightarrow \quad \mathscr{J}(u,\mathcal{O}_1 \cup \mathcal{O}_2) \le \mathscr{J}(u,\mathcal{O}_1') + \mathscr{J}(u,\mathcal{O}_2)$$

inner regular if for every $u \in X$ and $\mathcal{O} \in \boldsymbol{A}(\Omega)$

$$\mathscr{J}(u,\mathcal{O}) = \sup_{\mathcal{O}_1 \Subset \mathcal{O}} \mathscr{J}(u,\mathcal{O}_1)$$

a measure if for every $u \in X$ the map $\mathcal{O} \mapsto \mathscr{J}(u, \mathcal{O})$ is the restriction to $\boldsymbol{A}(\Omega)$ of a regular Borel measure: in this case, its value on an arbitrary Borel set B can be recovered by

$$\mathscr{J}(u, B) := \inf \Big\{ \mathscr{J}(u, \mathcal{O}) : \mathcal{O} \in \boldsymbol{A}(\Omega),\ B \subset \mathcal{O} \Big\}.$$

An important result of [15, prop. 5.5 and Thm. 5.6] (cf. also [13, Thm. 14.23]), ensures that $\mathscr{J}$ is a measure iff $\mathscr{J}$ is increasing, superadditive, weakly subadditive and inner regular.

Finally, we recall that to every increasing functional $\mathscr{J}$ can be associated its *inner regular envelope* $\mathscr{J}^*$, which is defined, for every $u \in X$ and $\mathcal{O} \in \boldsymbol{A}(\Omega)$, by

$$\mathscr{J}^*(u, \mathcal{O}) := \sup \Big\{ \mathscr{J}(u, \mathcal{O}_1) : \mathcal{O}_1 \in \boldsymbol{A}(\Omega),\ \mathcal{O}_1 \Subset \mathcal{O} \Big\}.$$

2.3 Γ*-convergence.* (cf. [13, Def. 4.1, Prop. 8.1]) Let us fix a set E as in (2.1), a topological space X *satisfying the first axiom of countability* and a family $\{\mathscr{J}^\varepsilon\}_{\varepsilon \in E}$ of functionals defined on X with values in the extended real line $[-\infty, +\infty]$. The Γ-*lower limit* and the Γ-*upper limit* of $\mathscr{J}^\varepsilon$, as ε vanishes in E, are the (extended) real function on X defined by

$$\Gamma(X)\text{-}\liminf_{\varepsilon \downarrow 0, \varepsilon \in E} \mathscr{J}^\varepsilon(u) = \inf \Big\{ \liminf_{\varepsilon \downarrow 0, \varepsilon \in E} \mathscr{J}^\varepsilon(u^\varepsilon) : \lim_{\varepsilon \downarrow 0, \varepsilon \in E} u^\varepsilon = u \quad \text{in } X \Big\} \tag{2.5a}$$

$$\Gamma(X)\text{-}\limsup_{\varepsilon \downarrow 0, \varepsilon \in E} \mathscr{J}^\varepsilon(u) = \inf \Big\{ \limsup_{\varepsilon \downarrow 0, \varepsilon \in E} \mathscr{J}^\varepsilon(u^\varepsilon) : \lim_{\varepsilon \downarrow 0, \varepsilon \in E} u^\varepsilon = u \quad \text{in } X \Big\} \tag{2.5b}$$

$$\Gamma(X)\text{-}\liminf_{\varepsilon \downarrow 0, \varepsilon \in E} \mathscr{J}^\varepsilon(u) = \Gamma(X)\text{-}\limsup_{\varepsilon \downarrow 0, \varepsilon \in E} \mathscr{J}^\varepsilon(u), \quad \forall\, u \in X,$$

we say that $\mathscr{J}^\varepsilon$ $\Gamma(X)$-converge as $\varepsilon \downarrow 0, \varepsilon \in E$, to the common value of both functionals, denoted by $\Gamma(X)\text{-}\lim_{\varepsilon \downarrow 0, \varepsilon \in E} \mathscr{J}^\varepsilon$. When no misunderstanding is possible, we will omit the indication of X and E and we will use the symbols $\mathscr{J}'$, $\mathscr{J}''$, and $\mathscr{J}$ to denote the above Γ-limits. Since X satisfies the first axiom of countability, when

$$E \text{ is the image of a decreasing infinitesimal sequence } n \mapsto \varepsilon_n > 0, \tag{2.6}$$

the above definitions coincide with the general ones of [13, Def. 4.1]. Let us observe that (2.5a,b) imply

$$u^\varepsilon \to u \text{ in } X \quad \Rightarrow \quad \liminf_{\varepsilon \downarrow 0, \varepsilon \in E} \mathscr{J}^\varepsilon(u^\varepsilon) \geq \mathscr{J}'(u). \tag{2.7}$$

and, for every $u \in X$,

$$\exists \{u^\varepsilon\}_{\varepsilon \in E} \subset X : \quad \lim_{\varepsilon \downarrow 0, \varepsilon \in E} u^\varepsilon = u \text{ in } X, \quad \limsup_{\varepsilon \downarrow 0, \varepsilon \in E} \mathscr{J}^\varepsilon(u^\varepsilon) \leq \mathscr{J}''(u). \tag{2.8}$$

Moreover, it follows immediately that if D is a subset of E with $\inf D = 0$ (in particular, if $D = \{\varepsilon_n\}_{n \in \mathbb{N}}$ corresponds to a suitable subsequence in E) then

$$\begin{aligned} &\Gamma(X)\text{-}\liminf_{\varepsilon \downarrow 0, \varepsilon \in E} \mathscr{J}^\varepsilon \leq \Gamma(X)\text{-}\liminf_{\varepsilon \downarrow 0, \varepsilon \in D} \mathscr{J}^\varepsilon \leq \\ &\leq \Gamma(X)\text{-}\limsup_{\varepsilon \downarrow 0, \varepsilon \in D} \mathscr{J}^\varepsilon \leq \Gamma(X)\text{-}\limsup_{\varepsilon \downarrow 0, \varepsilon \in E} \mathscr{J}^\varepsilon. \end{aligned} \tag{2.9}$$

2.4 $\overline{\Gamma}$-*convergence.* If X is a topological space satisfying the first axiom of countability and $\mathscr{J}^\varepsilon : X \times \boldsymbol{A}(\Omega) \to [0,+\infty]$, $\varepsilon \in E$, is a family of local increasing functional (as considered in paragraph 2.2), for every open set $\mathcal{O}$ the functionals

$$\mathscr{J}'(\cdot,\mathcal{O}) := \Gamma(X)\text{-}\liminf_{\varepsilon\downarrow 0,\varepsilon\in E} \mathscr{J}^\varepsilon(\cdot,\mathcal{O}),$$
$$\mathscr{J}''(\cdot,\mathcal{O}) := \Gamma(X)\text{-}\limsup_{\varepsilon\downarrow 0,\varepsilon\in E} \mathscr{J}^\varepsilon(\cdot,\mathcal{O})$$

can be defined accordingly to (2.5a,b) and are increasing and lower semicontinuous; if the inner regular envelopes of $\mathscr{J}'$ and $\mathscr{J}''$ coincide, i.e. there exists an increasing, lower semicontinuous and inner regular functional $\overline{\mathscr{J}} : X \times \boldsymbol{A}(\Omega)$ such that, for every $u \in X$, $\mathcal{O}_1, \mathcal{O}_2 \in \boldsymbol{A}(\Omega)$

$$\mathcal{O}_1 \Subset \mathcal{O}_2 \quad \Rightarrow \quad \overline{\mathscr{J}}(u,\mathcal{O}_1) \le \mathscr{J}'(u,\mathcal{O}_1) \le \mathscr{J}''(u,\mathcal{O}_1) \le \overline{\mathscr{J}}(u,\mathcal{O}_2),$$

then we say that $\mathscr{J}^\varepsilon$ $\overline{\Gamma}(X)$-converge to $\overline{\mathscr{J}}$ as ε goes to 0 in E [13, Def. 16.3, Prop. 16.4].

This notion of convergence for increasing set functional has an important property, which holds if X has a *countable base* of open sets [13, Prop. 16.8, Thm. 16.9]: a family $\{\mathscr{J}^\varepsilon\}_{\varepsilon>0}$ $\overline{\Gamma}(X)$-converges to $\overline{\mathscr{J}}$ as $\varepsilon \downarrow 0$ iff for every infinitesimal decreasing sequence $E = \{\varepsilon_n\}_{n\in\mathbb{N}}$

$$\exists\, \mathscr{J}_E = \overline{\Gamma}(X)\text{-}\lim_{\varepsilon\downarrow 0,\varepsilon\in E} \mathscr{J}^\varepsilon \quad \Rightarrow \quad \mathscr{J}_E \equiv \overline{\mathscr{J}}; \tag{2.10}$$

in other words, in order to prove that $\{\mathscr{J}^\varepsilon\}_{\varepsilon>0}$ $\overline{\Gamma}(X)$-converges to $\overline{\mathscr{J}}$ as $\varepsilon \downarrow 0$ it is sufficient to show that the Γ-limit $\mathscr{J}_E$ on a subsequence E, whenever it exists, it is independent of E.

2.5 *Functions of bounded variation.* $BV(\Omega)$ denotes the spaces of the functions $w \in L^1(\Omega)$ of *bounded variation*, whose distributional derivatives $D_1 w, \ldots, D_n w$ are finite Borel (signed) measures, i.e.

$$\int_\Omega w(x)\,\partial_i\zeta(x)\,dx = -\int_\Omega \zeta(x)\,d\,D_i w(x), \quad \forall\,\zeta \in C^1_0(\Omega), \quad i = 1,\ldots,n.$$

We will denote by $|D_i w|$ their total variations in the sense of measures on Ω: if $\mathcal{O} \in \boldsymbol{A}(\Omega)$ we have

$$|D_i w|(\mathcal{O}) = \sup\Big\{ \int_\mathcal{O} w(x)\,\partial_i\zeta(x)\,dx : \zeta \in C^1_0(\mathcal{O}),\ |\zeta| \le 1 \Big\}.$$

The distributional gradient $Dw := (D_1 w, \ldots, D_n w)$ is therefore a Borel vector measure on Ω with

$$|Dw|(\mathcal{O}) = \sup\Big\{ \int_\mathcal{O} w(x)\ \operatorname{div}\zeta(x)\,dx : \zeta \in C^1_0(\mathcal{O};\mathbb{R}m),\ |\zeta| \le 1 \Big\} < +\infty$$

We recall that

$$w \in W^{1,1}(\Omega) \quad \Rightarrow \quad w \in BV(\Omega), \quad |Dw|(\mathcal{O}) = \int_\mathcal{O} |\nabla w(x)|\,dx.$$

We define the space $BV(\Omega;T)$ as

$$BV(\Omega;T) := \Big\{u \in BV(\Omega) : u(x) \in \{s_-, s_+\} \text{ for a.e. } x \in \Omega\Big\}.$$

If $w \in BV(\Omega;T)$, its level sets $\{x \in \Omega : w(x) = s_-\}$, $\{x \in \Omega : w(x) = s_+\}$ are both of finite perimeter; their common *reduced boundary* S^*_w is the set of point x_0 belonging to the support of $|Dw|$, such that

$$\exists \lim_{\rho \to 0^+} \frac{Dw\big(B(x_0,\rho)\big)}{|Dw|\big(B(x_0,\rho)\big)} = \nu_w(x_0), \quad \text{and} \quad |\nu_w(x_0)| = 1, \tag{2.11}$$

the Borel map $x \in S^*_w \mapsto \nu_w(x)$ is called *generalized inner normal* to $\{x \in \Omega : w(x) = s_+\}$ a Borel map, and we know that the measure $|Dw|$ is concentrated on S^*_w, i.e.

$$|Dw|(\Omega \setminus S^*_w) = 0,$$

and, for every $\zeta \in C^0_0(\Omega;\mathbb{R}^n)$,

$$\int_\Omega \langle \zeta(x), Dw\rangle := \sum_{i=1}^n \int_\Omega \zeta_i(x)\, d\, D_i(x) = \int_{S^*_w} \langle \zeta(x), \nu_w(x)\rangle\, d\mathscr{H}^{n-1}(x).$$

Slicing. Let Q be an open cylinder as in (1.8a,b) where $\pi = \pi_\nu$, and w a real function defined on Q; we consider the family of 1-dimensional sections

$$w_x : t \in (a,b) \mapsto w_x(t) := w(x + t\eta) \in \mathbb{R}.$$

It is well known that

$$w \in L^1(Q) \quad \Rightarrow \quad w_x \in L^1(a,b) \quad \text{for a.e. } x \in Q_0$$

with

$$\int_Q w(x)\, dx = |\langle \eta, \nu\rangle| \int_\Sigma \left(\int_a^b w_x(t)\, dt\right) dx. \tag{2.12}$$

If $w \in W^{1,1}_{loc}(Q)$ an analogous property holds for the integral of the partial derivative of w along η:

$$w \in W^{1,1}_{loc}(Q) \quad \Rightarrow \quad w_x \in W^{1,1}_{loc}(a,b), \quad \text{for a.e. } x \in \Sigma,$$

with

$$\int_Q \big|\partial_\eta w(x)\big|\, dx = |\langle \eta, \nu\rangle| \int_\Sigma \left(\int_a^b \big|w'_x(t)\big|\, dt\right) dx.$$

If $w \in L^1_{loc}(Q)$ and $D_\eta w$ is a finite Borel measure on Q, then the previous formula can be rewritten as

$$\begin{gathered} w_x \in BV_{loc}(a,b), \quad \text{for a.e. } x \in \Sigma, \\ \big|D_\eta w\big|(Q) = |\langle \eta, \nu\rangle| \int_\Sigma |D_t w_x|(a,b)\, dx. \end{gathered} \tag{2.13}$$

Traces. The standard *trace* operator $w \in W^{1,1}(\Omega) \mapsto w_{|\partial\Omega} \in L^1(\partial\Omega)$ can also be extended to a linear bounded operator from $BV(\Omega)$ to $L^1(\partial\Omega)$; the following related estimation will turn to be useful (the proof is an easy adaptation of [8, Claim 4,5, Thm. 1, § 5.3]):

Lemma 2.1 *Let $w_n \in W^{1,1}(\Omega)$ be a sequence such that*

$$w_n \to w \quad \text{in } L^1(\Omega), \quad \lim_{\rho\downarrow 0}\limsup_{n\uparrow\infty}\int_{\Omega_\rho}|\nabla w_n(x)|\,dx = 0, \tag{2.14}$$

where

$$\Omega_\rho := \big\{x \in \Omega : d(x,\partial\Omega) < \rho\big\}. \tag{2.15}$$

Then $w_n|_{\partial\Omega}$ converges in $L^1(\partial\Omega)$ to the trace of w on $\partial\Omega$.

If Q has the cylindrical structure of (1.8a,b), then we can combine traces and slicing: if $w \in L^1_{loc}(Q)$ and $D_\eta w$ is a bounded Borel measure, then

$$\begin{gathered} w_x(b) = w_{|\Sigma_+}(x), \quad w_x(a) = w_{|\Sigma_-}(x) \quad \text{a.e. in } \Sigma \\ |\langle\eta,\nu\rangle|\int_\Sigma \big|w_x(b) - w_x(a)\big|\,dx \le |D_\eta w|(Q). \end{gathered} \tag{2.16}$$

2.6 ***Integral representation of local functionals defined on*** $BV(\Omega;T)$**.** Let $\mathscr{J} : L^2(\Omega)\times \boldsymbol{A}(\Omega) \to [0,+\infty]$ be a *local, lower semicontinuous* functional which is a *measure* in the sense of paragraph 2.2 and which is finite on $BV(\Omega;T)$. It is interesting to know whether $\mathscr{J}$ admits an integral representation like (1.7) for a suitable function ϕ. The following result, which is a particular case of [2, Thm. 3.1], gives a very satisfactory answer to this problem; more precisely, if there exists a constant $\Lambda_0 \ge 0$ s.t.

$$\mathscr{J}(u,\mathcal{O}) \le \Lambda_0\,\mathscr{H}^{n-1}(S^*_u\cap\mathcal{O}), \quad \forall\, u \in BV(\Omega;T), \quad \forall\mathcal{O}\in\boldsymbol{A}(\Omega) \tag{2.17}$$

and if there exists a continuous modulus of continuity $\omega : [0,+\infty[\to [0,+\infty[$ such that $\omega(0) = 0$ and

$$\Big|\mathscr{J}(u,\mathcal{O}) - \mathscr{J}(v,\mathcal{O}+z)\Big| \le \omega(|z|)\mathscr{H}^{n-1}(\mathcal{O}\cap S^*_u), \tag{2.18}$$

whenever $\mathcal{O}\subset\Omega$, $z\in\mathbb{R}^n$, $|z| < \frac{1}{2}d(\mathcal{O},\partial\Omega)$ and $v(x+z) = u(x)$ in $\mathcal{O}$, then it is possible to find a uniquely determined family of norms $\phi : \Omega\times\mathbb{R}^n \to [0,+\infty[$ such that

$$\mathscr{J}(u,\mathcal{O}) = \int_{S^*_u\cap\mathcal{O}}\phi(x,\nu_u(x))\,d\mathscr{H}^{n-1}(x), \quad \forall\, u\in BV(\Omega;T), \quad \forall\,\mathcal{O}\in\boldsymbol{A}(\Omega).$$

2.7 ***A variant of Dominated Convergence Theorem.*** [8, p. 22] Let $v, w, \{v_n, w_n\}_{n=1}^{+\infty}$ be measurable real functions defined in $\mathcal{O}\subset\mathbb{R}^n$ such that

$$|v_n(x)|^p \le w_n(x) \quad \text{for a.e. } x\in\mathcal{O}, \quad \forall\, n\in\mathbb{N}, \quad 1\le p<+\infty,$$

and

$$\begin{gathered} \lim_{n\to+\infty} v_n(x) = v(x), \qquad \lim_{n\to+\infty} w_n(x) = w(x) \text{ a.e. in } \mathcal{O}, \\ \lim_{n\to+\infty}\int_{\mathcal{O}} w_n(x)\,dx = \int_{\mathcal{O}} w(x)\,dx < +\infty. \end{gathered}$$

Then

$$\lim_{n\to+\infty}\int_{\mathcal{O}}|v_n(x)-v(x)|^p\,dx = 0. \tag{2.19}$$

A simple application of this Theorem shows that

Lemma 2.2 *Let $W : \mathbb{R} \to [0, +\infty[$, $p_0 \geq 2$, as in* (1.2b)*. Then the map*

$$v \in L^{p_0}(\mathcal{O}) \mapsto W(v) \in L^1(\mathcal{O}) \quad \textit{is } \text{continuous.} \tag{2.20}$$

Moreover, if $v, \{v_n\}_{n=1}^{+\infty}$ are measurable functions such that $v_n \to v$ in $L^1(\mathcal{O})$, then

$$v_n \to v \text{ in } L^{p_0}(\mathcal{O}) \quad \Leftrightarrow \quad \lim_{n\to+\infty} \int_{\mathcal{O}} W(v_n(x))\, dx = \int_{\mathcal{O}} W(v(x))\, dx. \quad \blacksquare \tag{2.21}$$

3 Proof of Theorems 1 and 2

Let us briefly explain the sketch of the proof, which we tried to divide in several steps, in order to follow a possible general strategy to deal with this kind of singular perturbation problems.

First of all, we "embed" the original Γ-convergence problem stated in Ω into a family of similar problems depending also on the open sets $\mathcal{O} \in \boldsymbol{A}(\Omega)$: it is the standard localization technique introduced in § 2.2; for this set-depending approach we study the Γ-limits $\mathscr{F}', \mathscr{F}''$ defined by

$$\mathscr{F}'(\cdot, \mathcal{O}) := \Gamma(X)\text{-}\liminf_{\varepsilon\downarrow 0, \varepsilon\in E} \mathscr{F}^\varepsilon(\cdot, \mathcal{O}), \quad \mathscr{F}''(\cdot, \mathcal{O}) := \Gamma(X)\text{-}\limsup_{\varepsilon\downarrow 0, \varepsilon\in E} \mathscr{F}^\varepsilon(\cdot, \mathcal{O})$$

and the weaker notion of $\overline{\Gamma}$-convergence as defined in § 2.4. We already observed that showing this kind of convergence is equivalent to show that the $\overline{\Gamma}$-limit of subsequences, whenever it exists, is uniquely determined and does not depend on the particular choice of the subsequence.

So, we shall deal with such a limit $\mathscr{F}', \mathscr{F}''$, and $\overline{\mathscr{F}}(\boldsymbol{u}, \mathcal{O})$ and we will prove that

1. $\mathscr{F}'(\boldsymbol{u}, \cdot), \mathscr{F}''(\boldsymbol{u}, \cdot)$ are finite iff u belongs to $BV(\mathcal{O}; T)$: this step requires suitable lower and upper bounds for $\mathscr{F}^\varepsilon$;

2. $\overline{\mathscr{F}}(\boldsymbol{u}, \cdot)$ is a measure, in the sense of § 2.2; this property is the heart of our procedure and follows by a careful subadditivity estimate, which also plays a fundamental role in the next steps 5 and 6.

3. $\mathscr{F}'(\cdot; \mathcal{O}), \mathscr{F}''(\cdot, \mathcal{O})$ depend only on $u = \ell(\boldsymbol{u})$: here we employ a simple approximation lemma.

4. $\overline{\mathscr{F}}(\boldsymbol{u}, \cdot)$ concides with $\mathscr{F}'(\boldsymbol{u}, \cdot)$ and $\mathscr{F}''(\boldsymbol{u}, \cdot)$ on Lipschitz subsets and, in particular, on Ω.

5. $\overline{\mathscr{F}}$ can be represented by an anisotropic integral functional on the discontinuity set of u, by the general results detailed in § 2.6.

6. The surface integrand ϕ can be identified in a simple parallelepipedal geometry as the cost of a transition through a planar surface with suitable boundary conditions, and, in particular, it is independent of E.

In § 3.7 we will (quickly) repeat the same procedure to deal with the weak topology of L^2 and we will give the sketch of the proof of Theorem 2.

We fix once for all a not empty subset

$$E \subset (0, +\infty), \quad \text{with } \inf E = 0$$

as in (2.1) and we will tacitly assume that the parameter ε varies in E; we will denote by

X the Hilbert space $L^2(\Omega;\mathbb{R}^m)$ endowed with its *strong* topology.

3.1 Lower and upper bounds for $\mathscr{F}', \mathscr{F}''$; compactness.

The following results are strictly related to Propositions 1.1 and 1.2.

Lemma 3.1 *Let $\boldsymbol{u} \in X$ satisfy $\mathscr{F}'(\boldsymbol{u},\mathcal{O}) < +\infty$, with $\mathcal{O} \in \boldsymbol{A}(\Omega)$. Then the restriction to $\mathcal{O}$ of $u := \ell(\boldsymbol{u})$ belongs to $BV(\mathcal{O};T)$, and*

$$\mathscr{F}'(\boldsymbol{u},\mathcal{O}) \geq \int_{S_u^*\cap\mathcal{O}} \underline{\phi}(x,\nu_u(x))\, d\mathscr{H}^{n-1}(x) \geq \lambda_0 \mathscr{H}^{n-1}(S_u^*\cap\mathcal{O}), \tag{3.1}$$

where $\underline{\phi}$ is defined by (1.21) *and* $\lambda_0 := \gamma_0|\boldsymbol{\ell}|^{-1}\sqrt{\lambda}$.

Proof. ¿From (1.18), if $\ell(\boldsymbol{u}) = u$ we get easily

$$\mathscr{F}^\varepsilon(\boldsymbol{u},\mathcal{O}) \geq \underline{\mathscr{G}}^\varepsilon(u,\mathcal{O}) := \int_{\mathcal{O}} \Big(\varepsilon \underline{q}(x,\nabla u(x)) + \frac{1}{\varepsilon}W(u(x))\Big)\, dx; \tag{3.2}$$

then we can apply [7, Thm. 3.5, Rem. 3.6(a), and (b) pag. 299] to get

$$\Gamma(L^2(\Omega))\text{-}\lim_{\varepsilon\downarrow 0} \underline{\mathscr{G}}^\varepsilon(u,\mathcal{O}) = \begin{cases} \int_{S_u^*\cap\mathcal{O}} \underline{\phi}(x,\nu_u(x))\, d\mathscr{H}^{n-1}(x) & \text{if } u \in BV(\mathcal{O};T) \\ +\infty & \text{otherwise.}\end{cases} \tag{3.3}$$

By the obvious monotonicity property of the Γ-limits and (1.19), we conclude. ■

The following well known compactness result underlies the previous lower bound and justifies (1.29). Here the function

$$Z(s) := 2\int_0^s \sqrt{W(\tau)}\, d\tau \tag{3.4}$$

plays a foundamental role; let us remark that

$$Z \text{ is } \textit{strictly} \text{ monotone, continuous, with its inverse } Z^{-1}; \tag{3.5}$$

moreover, there exist constants $c_0, c_1 > 0$ such that

$$c_0\big(|s|^{p_0/2+1} - 1\big) \leq |Z(s)| \leq c_1\big(|s|^{p_0/2+1} + 1\big), \quad \text{with} \quad 2 \leq p_0/2 + 1 \leq p_0, \tag{3.6}$$

Lemma 3.2 *Let $\mathcal{O}$ be an open subset of Ω and $\{\boldsymbol{u}^\varepsilon\}_{\varepsilon\in E}$ be a family in X such that*

$$M := \liminf_{\varepsilon\downarrow 0, \varepsilon\in E} \mathscr{F}^\varepsilon(\boldsymbol{u}^\varepsilon,\mathcal{O}) < +\infty; \tag{3.7}$$

then there exist an infinitesimal decreasing sequence $D \subset E$, and a function $u \in BV(\mathcal{O};T)$ such that, setting $u^\varepsilon := \ell(\boldsymbol{u}^\varepsilon)$

$$\lim_{\varepsilon\downarrow 0,\varepsilon\in D} \mathscr{F}^\varepsilon(\boldsymbol{u}^\varepsilon;\mathcal{O}) = M, \qquad \lim_{\varepsilon\downarrow 0,\varepsilon\in D} u^\varepsilon = u \ \text{ in } L^{p_0}(\mathcal{O}), \tag{3.8}$$

where p_0 is the growth exponent of W defined in (1.2b). *In particular, if $\boldsymbol{u}^\varepsilon \to \boldsymbol{u}$ in X then $u^\varepsilon \to \ell(\boldsymbol{u})$ in $L^{p_0}(\mathcal{O})$.*

PROOF. Let us first observe that, by (3.2), we get

$$\mathscr{F}^\varepsilon(\boldsymbol{u}^\varepsilon, \mathcal{O}) \geq \int_{\mathcal{O}} 2\sqrt{W(u^\varepsilon(x))\underline{q}(x, \nabla u^\varepsilon(x))}\, dx \geq \int_{\mathcal{O}} \sqrt{\underline{q}(x, \nabla z^\varepsilon(x))}\, dx, \quad (3.9)$$

where $z^\varepsilon(x) := Z(u^\varepsilon(x))$. Let $D' \subset E$ be an infinitesimal decreasing sequence satisfying the first equation of (3.8); (1.19) entails

$$\sqrt{\underline{q}(x, \nabla z(x))} \geq \frac{\sqrt{\lambda}}{|\boldsymbol{\ell}|} |\nabla z(x)|,$$

and by (3.6) we deduce that

$$z^\varepsilon = Z(u^\varepsilon), \quad \varepsilon \in D', \quad \text{is bounded in } BV(\mathcal{O}). \quad (3.10)$$

By a well known argument [19], we can find another infinitesimal decreasing subsequence $D \subset D'$ such that

$$\lim_{\varepsilon \downarrow 0, \varepsilon \in D} u^\varepsilon = u, \quad \lim_{\varepsilon \downarrow 0, \varepsilon \in D} z^\varepsilon = z \quad \text{a.e. and in } L^1(\mathcal{O}).$$

Since

$$\mathscr{F}^\varepsilon(\boldsymbol{u}^\varepsilon, \mathcal{O}) \geq \frac{1}{\varepsilon} \int_{\mathcal{O}} W(u^\varepsilon(x))\, dx,$$

we also deduce

$$\lim_{\varepsilon \downarrow 0, \varepsilon \in D} \int_{\mathcal{O}} W(u^\varepsilon(x))\, dx = \int_{\mathcal{O}} W(u(x))\, dx = 0.$$

Applying Lemma 2.2 we also get the strong convergence in $L^{p_0}(\mathcal{O})$. ∎

Corollary 3.3 *Let $\mathcal{O} \in \boldsymbol{A}(\Omega)$ and let $\boldsymbol{u}^\varepsilon \in X$ with $\mathscr{F}^\varepsilon(\boldsymbol{u}^\varepsilon, \mathcal{O}) < +\infty$, $\varepsilon \in E$, satisfy*

$$u^\varepsilon := \ell(\boldsymbol{u}^\varepsilon) \to u \in L^2(\mathcal{O}), \quad \liminf_{\varepsilon \downarrow 0, \varepsilon \in E} \mathscr{F}^\varepsilon(\boldsymbol{u}^\varepsilon, \mathcal{O}) < +\infty.$$

Then $u^\varepsilon, u \in L^{p_0}(\mathcal{O})$ and

$$\lim_{\varepsilon \downarrow 0} \frac{\|u^\varepsilon - u\|_{L^{p_0}(\mathcal{O})}}{1 + \mathscr{F}^\varepsilon(\boldsymbol{u}^\varepsilon; \mathcal{O})} = 0. \quad (3.11)$$

Lemma 3.4 *Let $u \in L^2(\Omega)$ be satisfying $u_{|\mathcal{O}} \in BV(\mathcal{O}; T)$ for a* Lipschitz *open subset $\mathcal{O} \subset \Omega$. Then there exists $\boldsymbol{u} \in X$ such that $\ell(\boldsymbol{u}) = u$ and $\mathscr{F}''(\boldsymbol{u}, \mathcal{O}) < +\infty$ with*

$$\mathscr{F}''(\boldsymbol{u}, \mathcal{O}) \leq \int_{S_u^* \cap \mathcal{O}} \bar{\phi}(x, \nu_u(x))\, d\mathscr{H}^{n-1}(x) \leq \Lambda_0\, \mathscr{H}^{n-1}(S_u^* \cap \mathcal{O}), \quad (3.12)$$

where $\bar{\phi}$ is defined by (1.25) and $\Lambda_0 := \gamma_0 |\boldsymbol{\ell}|^{-1} \sqrt{\Lambda}$.

PROOF. Let us fix a Lipschitz function

$$\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) : \Omega \to \mathbb{R}^m, \quad \text{such that } \ell(\boldsymbol{\alpha}) \equiv 1,$$

and for every function $v \in L^2(\Omega)$ let us define

$$\boldsymbol{v}_{\boldsymbol{\alpha}}(x) := v(x)\boldsymbol{\alpha}(x);$$

of course, $v_{|_{\mathcal{O}}} \in H^1(\mathcal{O}) \;\Rightarrow\; \boldsymbol{v}_{\boldsymbol{\alpha}}{}_{|_{\mathcal{O}}} \in H^1(\mathcal{O};\mathbb{R}^m)$

$$\nabla \boldsymbol{v}_{\boldsymbol{\alpha}} = \boldsymbol{\alpha} \otimes \nabla v + v\nabla\boldsymbol{\alpha} \in \mathbb{M}^{m\times n} \qquad \text{cf. note } ^2.$$

Cauchy-Schwarz inequality (3.34) yields, for every $\delta > 0$,

$$\begin{aligned}
\mathscr{F}^\varepsilon(\boldsymbol{v}_\alpha,\mathcal{O}) &= \int_{\mathcal{O}} \Big(\varepsilon\boldsymbol{q}(x,\nabla\boldsymbol{v}_\alpha(x)) + \frac{1}{\varepsilon}W(v(x))\Big)\,dx \\
&= \int_{\mathcal{O}} \Big(\varepsilon\boldsymbol{q}\big(x,\boldsymbol{\alpha}(x)\otimes\nabla v(x) + v(x)\nabla\boldsymbol{\alpha}(x)\big)\big) + \frac{1}{\varepsilon}W(v(x))\Big]\,dx \\
&\le (1+\delta)\mathscr{G}^\varepsilon_{\boldsymbol{\alpha}}(v,\mathcal{O}) + (1+\delta^{-1})\varepsilon\Lambda\|\nabla\boldsymbol{\alpha}\|^2_{L^\infty(\Omega;\mathbb{M}^{m\times n})}\|v\|^2_{L^2(\mathcal{O})},
\end{aligned} \tag{3.13}$$

where we defined

$$\mathscr{G}^\varepsilon_{\boldsymbol{\alpha}}(v,\mathcal{O}) := \int_{\mathcal{O}} \Big(\varepsilon\boldsymbol{q}\big(x,\boldsymbol{\alpha}(x)\otimes\nabla v(x)\big) + \frac{1}{\varepsilon}W(v(x))\Big]\,dx.$$

Now we choose $\boldsymbol{u} \in L^2(\Omega;\mathbb{R}^m)$ such that the restriction of $u := \ell(\boldsymbol{u})$ to $\mathcal{O}$ belongs to $BV(\mathcal{O};T)$; by [7, Thm. 3.5, Rem. 3.6(a), and (b) pag. 299] we can find $u^\varepsilon \in L^2(\Omega)$ such that, as $\varepsilon \downarrow 0$,

$$u^\varepsilon \to u \quad \text{in } L^2(\Omega), \quad \mathscr{G}^\varepsilon_\alpha(u^\varepsilon,\mathcal{O}) \to \int_{S^*_u\cap\mathcal{O}} \phi_\alpha(x,\nu_u(x))\,d\mathscr{H}^{n-1}(x)$$

where

$$\phi_\alpha(x,\xi) := \gamma_0\Big(\boldsymbol{q}(x,\boldsymbol{\alpha}(x)\otimes\xi)\Big)^{1/2}.$$

We observe that

$$\lim_{\varepsilon\downarrow 0} \boldsymbol{u}^\varepsilon_{\boldsymbol{\alpha}} = u\,\boldsymbol{\alpha} \quad \text{in } L^2(\Omega;\mathbb{R}^m) \quad \text{and} \quad \ell(u\,\boldsymbol{\alpha}) = u = \ell(\boldsymbol{u}) \quad \text{a.e. in } \mathcal{O}$$

so that, since $\|u^\varepsilon\|_{L^2(\Omega)}$ are uniformly bounded, by (3.13) we get

$$\mathscr{F}''(u\,\boldsymbol{\alpha},\mathcal{O}) \le \limsup_{\varepsilon\downarrow 0} \mathscr{G}^\varepsilon_{\boldsymbol{\alpha}}(u^\varepsilon,\mathcal{O}) \le (1+\delta)\int_{S^*_u\cap\mathcal{O}} \phi_\alpha(x,\nu_u(x))\,d\mathscr{H}^{n-1}(x),$$

and finally

$$\mathscr{F}''(u\,\boldsymbol{\alpha},\mathcal{O}) \le \int_{S^*_u\cap\mathcal{O}} \phi_\alpha(x,\nu_u(x))\,d\mathscr{H}^{n-1}(x).$$

In this formula, we are free to choose $\boldsymbol{\alpha}$ among the $\mathbb{R}^m$-valued Lipschitz real functions of Ω, with the constraint $\ell(\boldsymbol{\alpha}) = 1$; the uniform estimate

$$\begin{aligned}
\big|\phi_{\boldsymbol{\alpha}_1}(x,\xi) - \phi_{\boldsymbol{\alpha}_2}(x,\xi)\big| \le& \sqrt{\boldsymbol{q}(x,(\boldsymbol{\alpha}_1(x)-\boldsymbol{\alpha}_2(x))\otimes\xi)} \\
\le& \sqrt{\Lambda}\big|\boldsymbol{\alpha}_1(x)-\boldsymbol{\alpha}_2(x)\big|, \quad \forall\, x\in\Omega,\ \xi\in\mathbb{S}^{n-1}
\end{aligned}$$

the X-lower semicontinuity of $\mathscr{F}''$ and a standard approximation argument ensure that we can also choose every $\boldsymbol{\alpha} \in C^0(\overline{\Omega}; \mathbb{R}^m)$; finally, an application of Lusin's Theorem to Ω endowed with the bounded Borel measure

$$\mu(B) := \mathscr{H}^{n-1}(B \cap S_u^* \cap \mathcal{O}) + |B|$$

allows the choice of an arbitrary bounded Borel function $\boldsymbol{\alpha} : \Omega \to \mathbb{R}^m$. We choose $\boldsymbol{u} := u\,\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}$ such that

$$\boldsymbol{q}(x, \boldsymbol{\alpha} \otimes \nu_u(x)) = \bar{q}(x, \nu_u(x)) \quad \forall\, x \in S_u^* \cap \mathcal{O}. \quad \blacksquare$$

3.2 Additivity properties.

In this section we want to study some natural properties of the functionals $\mathcal{O} \in \boldsymbol{A}(\Omega) \mapsto \mathscr{F}'(\boldsymbol{u}, \mathcal{O}), \mathscr{F}''(\boldsymbol{u}, \mathcal{O})$, as defined in 3.13 for a fixed $\boldsymbol{u} \in X$. The first one is very simple, and follows by the general theory of [13, Prop. 16.12].

Lemma 3.5 *For every $\boldsymbol{u} \in X$ the map $\mathcal{O} \mapsto \mathscr{F}'(\boldsymbol{u}, \mathcal{O})$ is* superadditve, *i.e.* $\forall\, \mathcal{O}_1, \mathcal{O}_2 \in \boldsymbol{A}(\Omega)$

$$\mathcal{O}_1 \cap \mathcal{O}_2 = \varnothing \quad \Rightarrow \quad \mathscr{F}'(\boldsymbol{u}, \mathcal{O}_1 \cup \mathcal{O}_2) \geq \mathscr{F}'(\boldsymbol{u}, \mathcal{O}_1) + \mathscr{F}'(\boldsymbol{u}, \mathcal{O}_2). \tag{3.14}$$

The *subadditivity* of $\mathscr{F}''$ is much more delicate to prove, even in the weak form introduced in paragraph 2.2. The main calculations, which are a careful modification of the techniques introduced by [14, Appendix] and [4, Lemma 3.2, Step 2], are collected in the following lemma.

Lemma 3.6 *Let $\mathcal{O}_1, \mathcal{O}_1', \mathcal{O}_2$ be open subsets of Ω with $\mathcal{O}_1 \Subset \mathcal{O}_1'$, and suppose we are given two families $\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon \in L^2(\Omega; \mathbb{R}^m)$, $\varepsilon \in E$, such that*

$$\int_{\mathcal{O}_1'} |\boldsymbol{u}^\varepsilon(x)|^2\, dx + \int_{\mathcal{O}_2} |\boldsymbol{v}^\varepsilon(x)|^2\, dx \leq C \tag{3.15}$$

$$\begin{aligned} u^\varepsilon &:= \ell(\boldsymbol{u}^\varepsilon) \to u \quad \textit{in } L^2(\mathcal{O}_1'), \\ v^\varepsilon &:= \ell(\boldsymbol{v}^\varepsilon) \to v \quad \textit{in } L^2(\mathcal{O}_2) \end{aligned} \qquad u \equiv v \ \textit{in } \mathcal{O}_1' \cap \mathcal{O}_2. \tag{3.16}$$

Then there exist $\boldsymbol{w}^\varepsilon \in L^2(\Omega; \mathbb{R}^m)$ which are a pointwise convex combination of $\boldsymbol{u}^\varepsilon$ and $\boldsymbol{v}^\varepsilon$ such that

$$\begin{aligned} \boldsymbol{w}^\varepsilon &\equiv \boldsymbol{u}^\varepsilon \quad \textit{in } \mathcal{O}_1, \\ \boldsymbol{w}^\varepsilon &\equiv \boldsymbol{v}^\varepsilon \quad \textit{in } \mathcal{O}_2 \setminus \mathcal{O}_1', \end{aligned} \qquad \lim_{\varepsilon \downarrow 0} \ell(\boldsymbol{w}^\varepsilon) = u \quad \textit{in } L^2(\mathcal{O}_1' \cap \mathcal{O}_2), \tag{3.17}$$

and

$$\limsup_{\varepsilon \downarrow 0} \mathscr{F}^\varepsilon(\boldsymbol{w}^\varepsilon, \mathcal{O}_1 \cup \mathcal{O}_2) \leq \limsup_{\varepsilon \downarrow 0} \Big(\mathscr{F}^\varepsilon(\boldsymbol{u}^\varepsilon, \mathcal{O}_1') + \mathscr{F}^\varepsilon(\boldsymbol{v}^\varepsilon, \mathcal{O}_2) \Big) \tag{3.18}$$

$$\liminf_{\varepsilon \downarrow 0} \mathscr{F}^\varepsilon(\boldsymbol{w}^\varepsilon, \mathcal{O}_1 \cup \mathcal{O}_2) \leq \liminf_{\varepsilon \downarrow 0} \Big(\mathscr{F}^\varepsilon(\boldsymbol{u}^\varepsilon, \mathcal{O}_1') + \mathscr{F}^\varepsilon(\boldsymbol{v}^\varepsilon, \mathcal{O}_2) \Big). \tag{3.19}$$

Moreover, if

$$\begin{aligned} \boldsymbol{u}^\varepsilon &\rightharpoonup \boldsymbol{u} \quad \textit{in } L^2(\mathcal{O}_1'; \mathbb{R}^m), \\ \boldsymbol{v}^\varepsilon &\rightharpoonup \boldsymbol{v} \quad \textit{in } L^2(\mathcal{O}_2; \mathbb{R}^m), \end{aligned} \qquad \boldsymbol{u} \equiv \boldsymbol{v} \quad \textit{in } \mathcal{O}_1' \cap \mathcal{O}_2, \tag{3.20}$$

then we can also ask that

$$\boldsymbol{w}^\varepsilon \rightharpoonup \boldsymbol{w} \in L^2(\Omega; \mathbb{R}^m), \quad \boldsymbol{w} = \boldsymbol{u} \quad \textit{in } \mathcal{O}_1', \quad \boldsymbol{w} = \boldsymbol{v} \quad \textit{in } \mathcal{O}_2. \tag{3.21}$$

Remark 3.7 If the convergences in (3.20) are *strong*, i.e.

$$\begin{aligned} \boldsymbol{u}^\varepsilon \to \boldsymbol{u} \quad \text{in } L^2(\mathcal{O}_1'; \mathbb{R}^m),\\ \boldsymbol{v}^\varepsilon \to \boldsymbol{v} \quad \text{in } L^2(\mathcal{O}_2; \mathbb{R}^m), \end{aligned} \qquad \boldsymbol{u} \equiv \boldsymbol{v} \quad \text{in } \mathcal{O}_1' \cap \mathcal{O}_2, \tag{3.22}$$

then it is easy to see that also (3.21) holds in the stronger form

$$\boldsymbol{w}^\varepsilon \to \boldsymbol{w} \in L^2(\Omega; \mathbb{R}^m), \quad \boldsymbol{w} = \boldsymbol{u} \quad \text{in } \mathcal{O}_1', \quad \boldsymbol{w} = \boldsymbol{v} \quad \text{in } \mathcal{O}_2, \tag{3.23}$$

since, by construction, $\boldsymbol{w}^\varepsilon$ is a pointwise convex combination of $\boldsymbol{u}^\varepsilon$ and $\boldsymbol{v}^\varepsilon$. □

Proof. First of all, we observe that the interesting case is $\mathcal{O}_1 \cap \mathcal{O}_2 \neq \varnothing$; we denote by $g^\varepsilon, h^\varepsilon$ the sums

$$g^\varepsilon := \mathscr{F}^\varepsilon(\boldsymbol{u}^\varepsilon, \mathcal{O}_1') + \mathscr{F}^\varepsilon(\boldsymbol{v}^\varepsilon, \mathcal{O}_2),$$

$$h^\varepsilon := \|u^\varepsilon - u\|_{L^{p_0}(\mathcal{O}_1')} + \|v^\varepsilon - v\|_{L^{p_0}(\mathcal{O}_2)}$$

and we know by Corollary 3.3 that

$$\sigma^\varepsilon := \frac{h^\varepsilon}{1 + g^\varepsilon} \quad \text{vanishes as } \varepsilon \downarrow 0. \tag{3.24}$$

Moreover, it is not restrictive to assume that $\liminf_{\varepsilon\downarrow 0} g^\varepsilon < +\infty$ and $g^\varepsilon < +\infty$ for every $\varepsilon \in E$;[5] in particular, we can suppose that $\boldsymbol{u}^\varepsilon \in H^1(\mathcal{O}_1'; \mathbb{R}^m)$ and $\boldsymbol{v}^\varepsilon \in H^1(\mathcal{O}_2; \mathbb{R}^m)$. We know that

$$\int_{\mathcal{O}_1' \cup \mathcal{O}_2} W(w(x))\, dx = 0, \quad \text{where} \quad w \equiv u \text{ in } \mathcal{O}_1', \quad w \equiv v \text{ in } \mathcal{O}_2;$$

so that, introducing the modulus of continuity

$$\rho(s) := \sup\Big\{ \int_{\mathcal{O}_1' \cap \mathcal{O}_2} W(z(x))\, dx : \|z - w\|_{L^{p_0}(\mathcal{O}_1' \cap \mathcal{O}_2)} \le s \Big\}$$

by the result quoted in paragraph 2.5 we know that $\lim_{s\downarrow 0} \rho(s) = 0$.

Let $\mathcal{Q}$ be a fixed Lipschitz open set, such that

$$\mathcal{O}_1 \subset \mathcal{Q} \Subset \mathcal{O}_1',$$

and let us fix $\varepsilon \in E$ and a positive integer k; as in [14, p. 216], let

$$d := \tfrac{1}{2} d(\mathcal{Q}, \partial\mathcal{O}_1') > 0$$

be the half-distance between $\mathcal{Q}$ and the boundary of $\mathcal{O}_1'$, let us choose other k open sets

$$\mathcal{Q} = \mathcal{Q}_0 \Subset \mathcal{Q}_1 \Subset \ldots \Subset \mathcal{Q}_k \Subset \mathcal{O}_1'$$

[5] otherwise we set

$$\boldsymbol{w}^\varepsilon \equiv \boldsymbol{u}^\varepsilon \quad \text{in } \mathcal{O}_1, \quad \boldsymbol{w}^\varepsilon \equiv \boldsymbol{v}^\varepsilon \quad \text{in } \Omega \setminus \mathcal{O}_1, \quad \text{if } g^\varepsilon = +\infty$$

and we restrict our attention to $E' := \{\varepsilon \in E : g^\varepsilon < +\infty\}$

separated each other by a distance of width $r := d/k$, and k Lipschitz real functions

$$\alpha_1, \dots, \alpha_k : \Omega \to [0,1],$$

in the following way:

$$\mathcal{Q}_j := \Big\{ x \in \Omega : d(x, \partial \mathcal{Q}) < jr \Big\},$$
$$\alpha_j|_{\mathcal{Q}_{j-1}} \equiv 1, \quad \alpha_j|_{\Omega \setminus \mathcal{Q}_j} \equiv 0, \quad \alpha_j|_{\mathcal{Q}_j \setminus \mathcal{Q}_{j-1}}(x) = j - d(x, \mathcal{Q})/r$$

in particular

$$\|\nabla \alpha_j\|_{L^\infty(\Omega)} \le r^{-1} = k/d, \quad \int_\Omega |\nabla \alpha_j(x)|\,dx \le r^{-1} |\mathcal{Q}_{j+1} \setminus \mathcal{Q}_j| \le C_{\mathcal{O}_1} \quad (3.25)$$

where $C_{\mathcal{O}_1}$ is a constant depending on the Lipschitz character of $\mathcal{O}_1$. We define

$$\boldsymbol{w}_j(x) := \alpha_j(x) \boldsymbol{u}^\varepsilon(x) + (1 - \alpha_j(x)) \boldsymbol{v}^\varepsilon(x), \quad j = 1, \dots, k,$$

and we want to estimate $\mathscr{F}^\varepsilon(\boldsymbol{w}_j, \mathcal{O}_1 \cup \mathcal{O}_2)$. We consider the partition covering $\mathcal{O}_1 \cup \mathcal{O}_2$ given by

$$\mathcal{Q}_j, \ \mathcal{O}_2 \cap (\mathcal{Q}_{j+1} \setminus \mathcal{Q}_j), \ \mathcal{O}_2 \setminus \mathcal{Q}_{j+1}$$

and we observe that $\boldsymbol{w}_j \in H^1(\mathcal{O}_1' \cap \mathcal{O}_2)$,

$$\boldsymbol{w}_j \equiv \boldsymbol{u}^\varepsilon \quad \text{on } \mathcal{Q}_j \supset \mathcal{O}_1, \quad \boldsymbol{w}_j \equiv \boldsymbol{v}^\varepsilon \quad \text{on } \mathcal{O}_2 \setminus \mathcal{Q}_{j+1} = \mathcal{O}_2 \setminus \mathcal{Q}_{j+1}$$

and

$$\nabla \boldsymbol{w}_j = \alpha_j \nabla \boldsymbol{u}^\varepsilon + (1 - \alpha_j) \nabla \boldsymbol{v}^\varepsilon + \nabla \alpha_j (\boldsymbol{u}^\varepsilon - \boldsymbol{v}^\varepsilon), \quad \text{in } \mathcal{O}_2 \cap (\mathcal{Q}_{j+1} \setminus \mathcal{Q}_j).$$

By the convexity of $\boldsymbol{q}$ and the Cauchy inequality (3.34), for every $\delta > 0$ and a.e. $x \in \mathcal{O}_1' \cap \mathcal{O}_2$ we get

$$\begin{aligned} \boldsymbol{q}(x, \nabla \boldsymbol{w}_j) \le (1+\delta) \Big[\alpha_j \boldsymbol{q}(x, \nabla \boldsymbol{u}^\varepsilon) + (1-\alpha_j) \boldsymbol{q}(x, \nabla \boldsymbol{v}^\varepsilon) \Big] \\ + \Lambda (1 + \delta^{-1}) \frac{k^2}{d^2} |\boldsymbol{u}^\varepsilon - \boldsymbol{v}^\varepsilon|^2 \end{aligned}$$

and therefore

$$\begin{aligned} \int_{\mathcal{O}_1 \cup \mathcal{O}_2} \boldsymbol{q}(x, \nabla \boldsymbol{w}_j)\,dx \le (1+\delta) \left(\int_{\mathcal{O}_1'} \boldsymbol{q}(x, \nabla \boldsymbol{u}^\varepsilon)\,dx + \int_{\mathcal{O}_2} \boldsymbol{q}(x, \nabla \boldsymbol{v}^\varepsilon)\,dx \right) \quad (3.26) \\ + \Lambda \tfrac{(1+\delta^{-1})}{d^2} k^2 \int_{\mathcal{O}_2 \cap (\mathcal{Q}_{j+1} \setminus \mathcal{Q}_j)} |\boldsymbol{u}^\varepsilon - \boldsymbol{v}^\varepsilon|^2\,dx. \end{aligned}$$

On the other hand, we easily get

$$\int_{\mathcal{O}_1 \cup \mathcal{O}_2} \tilde{W}(\boldsymbol{w}_j)\,dx \le \int_{\mathcal{O}_1'} \tilde{W}(\boldsymbol{u}^\varepsilon)\,dx + \int_{\mathcal{O}_2} \tilde{W}(\boldsymbol{v}^\varepsilon)\,dx + \int_{\mathcal{O}_2 \cap (\mathcal{Q}_{j+1} \setminus \mathcal{Q}_j)} \tilde{W}(\boldsymbol{w}_j)\,dx$$

Summing up these contributions and calling $c_0 := \frac{\Lambda(1+\delta^{-1})}{d^2}$, we get

$$\begin{aligned}\mathscr{F}^\varepsilon(\boldsymbol{w}_j, \mathcal{O}_1 \cup \mathcal{O}_2) &\le (1+\delta)\Big(\mathscr{F}^\varepsilon(\boldsymbol{u}^\varepsilon, \mathcal{O}_1') + \mathscr{F}^\varepsilon(\boldsymbol{v}^\varepsilon, \mathcal{O}_2)\Big) \\ &\quad + \int_{\mathcal{O}_2 \cap (\mathcal{Q}_{j+1} \setminus \mathcal{Q}_j)} \Big(c_0\, \varepsilon k^2 |\boldsymbol{u}^\varepsilon - \boldsymbol{v}^\varepsilon|^2 + \frac{1}{\varepsilon}\tilde{W}(\boldsymbol{w}_j)\Big)\, dx.\end{aligned}$$

Since j can be arbitrarily chosen between 1 and k, we deduce that there exists $j^* \in \{1, \dots, k\}$ such that

$$\begin{aligned}\mathscr{F}^\varepsilon(\boldsymbol{w}_{j^*}, \mathcal{O}_1 \cup \mathcal{O}_2) &\le (1+\delta)\Big(\mathscr{F}^\varepsilon(\boldsymbol{u}^\varepsilon, \mathcal{O}_1') + \mathscr{F}^\varepsilon(\boldsymbol{v}^\varepsilon, \mathcal{O}_2)\Big) \\ &\quad + \frac{1}{k}\sum_{j=1}^{k} \int_{\mathcal{O}_2 \cap (\mathcal{Q}_{j+1} \setminus \mathcal{Q}_j)} \Big(c_0\, \varepsilon k^2 |\boldsymbol{u}^\varepsilon - \boldsymbol{v}^\varepsilon|^2 + \frac{1}{\varepsilon}\tilde{W}(\boldsymbol{w}_j)\Big)\, dx \\ &\le (1+\delta) g^\varepsilon + \int_{\mathcal{O}_2 \cap \mathcal{O}_1'} \Big(c_0\, \varepsilon k |\boldsymbol{u}^\varepsilon - \boldsymbol{v}^\varepsilon|^2 + \frac{1}{\varepsilon k}\tilde{W}(\boldsymbol{w}^*)\Big)\, dx \\ &\le (1+\delta) g^\varepsilon + c_1\, \varepsilon k + \frac{1}{\varepsilon k}\int_{\mathcal{O}_2 \cap \mathcal{O}_1'} \tilde{W}(\boldsymbol{w}^*)\, dx \\ &\le (1+\delta) g^\varepsilon + c_1\, \varepsilon k + \frac{1}{\varepsilon k}\rho\big(\|w^* - u\|_{L^p(\mathcal{O}_2 \cap \mathcal{O}_1')}\big)\end{aligned}$$

where we defined $c_1 := 2 c_0 C$ and $\boldsymbol{w}^*$ as

$$\boldsymbol{w}^* \equiv \boldsymbol{w}_j \quad \text{on} \quad \mathcal{Q}_{j+1} \setminus \mathcal{Q}, \quad w^* = \ell(\boldsymbol{w}^*).$$

We observe that $w^*(x)$ is always a convex combination of u^ε and v^ε, therefore

$$\|w^* - w\|_{L^p(\mathcal{O}_2 \cap \mathcal{O}_1')} \le h^\varepsilon$$

Choosing k as the integer part of $\varepsilon^{-1}\sqrt{\rho(\omega^\varepsilon) + \varepsilon}$ and $\boldsymbol{w}^\varepsilon := \boldsymbol{w}_{j^*}$, we get (cf. (3.24))

$$\begin{aligned}\mathscr{F}^\varepsilon(\boldsymbol{w}^\varepsilon, \mathcal{O}_1 \cup \mathcal{O}_2) &\le (1+\delta) g^\varepsilon + 2c_1\sqrt{\rho(h^\varepsilon) + \varepsilon} \\ &\le (1+\delta) g^\varepsilon + 2c_1\sqrt{\rho(\sigma^\varepsilon g^\varepsilon) + \varepsilon}.\end{aligned}$$

Since $\sigma^\varepsilon \to 0$ as $\varepsilon \downarrow 0$, we deduce that

$$\begin{aligned}\limsup_{\varepsilon \downarrow 0} \mathscr{F}^\varepsilon(\boldsymbol{w}^\varepsilon, \mathcal{O}_1 \cup \mathcal{O}_2) &\le (1+\delta) \limsup_{\varepsilon \downarrow 0} g^\varepsilon \\ \liminf_{\varepsilon \downarrow 0} \mathscr{F}^\varepsilon(\boldsymbol{w}^\varepsilon, \mathcal{O}_1 \cup \mathcal{O}_2) &\le (1+\delta) \liminf_{\varepsilon \downarrow 0} g^\varepsilon\end{aligned}$$

Being $\delta > 0$ arbitrary, we obtain (3.18).

It remains to show that (3.20) entails (3.21). By the first two properties of (3.17), it is sufficient to prove the weak convergence of $\boldsymbol{w}^\varepsilon$ in $L^2(\mathcal{O}_1' \cap \mathcal{O}_2; \mathbb{R}^m)$. We will apply the result of the next lemma, where we define α^ε as the function α_{j^*} corresponding to the index j^* introduced in the previous calculations . ■

Lemma 3.8 *Let $\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon$, $\varepsilon \in E$, be two families weakly converging to the same limit $\boldsymbol{u} = \boldsymbol{v}$ in $L^2(\mathcal{O};\mathbb{R}^m)$, and let $\alpha^\varepsilon : \mathcal{O} \to [0,1]$ be a family of Lipschitz functions such that*

$$\int_{\mathcal{O}} |\nabla \alpha^\varepsilon(x)|\, dx \le A \quad \forall \varepsilon \in E. \tag{3.27}$$

Then, setting

$$\boldsymbol{w}^\varepsilon = \alpha^\varepsilon \boldsymbol{u}^\varepsilon + (1-\alpha^\varepsilon)\boldsymbol{v}^\varepsilon, \tag{3.28}$$

$\boldsymbol{w}^\varepsilon$ weakly converges to $\boldsymbol{w} = \boldsymbol{u} = \boldsymbol{v}$ in $L^2(\mathcal{O};\mathbb{R}^m)$.

Proof. ¿From (3.28) we get

$$\boldsymbol{w}^\varepsilon - \boldsymbol{w} = \alpha^\varepsilon(\boldsymbol{u}^\varepsilon - \boldsymbol{u}) + (1-\alpha^\varepsilon)(\boldsymbol{v}^\varepsilon - \boldsymbol{v});$$

since $\boldsymbol{u}^\varepsilon - \boldsymbol{u} \rightharpoonup 0$ and $\boldsymbol{v}^\varepsilon - \boldsymbol{v} \rightharpoonup 0$ in $L^2(\mathcal{O};\mathbb{R}^m)$, our thesis follows if we show that for every test function $\zeta \in L^2(\mathcal{O})$ the family

$$\Big\{\alpha\zeta : \alpha : \mathcal{O} \to [0,1], |\nabla\alpha|(\mathcal{O}) \le A\Big\} \quad \text{is compact in } L^2(\mathcal{O}).$$

But this property is an easy consequence of the fact that every sequence of functions with values in $[0,1]$ and uniformly bounded total variation posseses an a.e. convergent subsequence. ■

Let us write some important consequences of Lemma 3.6 (cf. § 2.2).

Corollary 3.9 *The local functional $\mathscr{F}''$ is weakly subadditive.*

Proof. For a fixed $\boldsymbol{u} \in X$, choose $\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon \in X$ converging to $\boldsymbol{u}$ in X; applying Lemma 3.6 we can find $\boldsymbol{w}^\varepsilon \in X$ converging to $\boldsymbol{u}$ such that

$$\limsup_{\varepsilon\downarrow 0} \mathscr{F}^\varepsilon(\boldsymbol{w}^\varepsilon, \mathcal{O}_1 \cup \mathcal{O}_2) \le \limsup_{\varepsilon\downarrow 0} \Big(\mathscr{F}^\varepsilon(\boldsymbol{u}^\varepsilon, \mathcal{O}_1') + \mathscr{F}^\varepsilon(\boldsymbol{v}^\varepsilon, \mathcal{O}_2)\Big) \tag{3.29}$$

$$\le \limsup_{\varepsilon\downarrow 0} \mathscr{F}^\varepsilon(\boldsymbol{u}^\varepsilon, \mathcal{O}_1') + \limsup_{\varepsilon\downarrow 0} \mathscr{F}^\varepsilon(\boldsymbol{v}^\varepsilon, \mathcal{O}_2). \tag{3.30}$$

It follows immediately that

$$\mathscr{F}''(\boldsymbol{u}, \mathcal{O}_1 \cup \mathcal{O}_2) \le \mathscr{F}''(\boldsymbol{u}, \mathcal{O}_1') + \mathscr{F}''(\boldsymbol{u}, \mathcal{O}_2). \quad ■$$

Combining Corollary 3.9 and Lemma 3.5 we prove the third step described at the beginning of this section.

Corollary 3.10 *Let us assume that there exists the $\overline{\Gamma}(X)$-limit (as defined in* § 2.4*)*

$$\overline{\mathscr{F}} = \overline{\Gamma}(X)\text{-}\lim_{\varepsilon\downarrow 0} \mathscr{F}^\varepsilon.$$

Then $\overline{\mathscr{F}}$ is a measure, in the sense of § 2.2.

3.3 The Γ-limits depend only on $u = \ell(\boldsymbol{u})$.

Now we want to show that the Γ-limits $\mathscr{F}', \mathscr{F}''$ (and therefore $\overline{\mathscr{F}}$) depend only on $\ell(\boldsymbol{u})$. This fact follows by the following simple approximation lemma.

Lemma 3.11 *Let $\boldsymbol{w} \in X$ be satisfying $\ell(\boldsymbol{w}) = 0$ a.e. in the open set $\mathcal{O} \subset \Omega$; then there exists a family $\boldsymbol{w}^\varepsilon \in X$, $\varepsilon \in E$, such that*

$$\ell(\boldsymbol{w}^\varepsilon) = 0 \ \textit{in}\ \mathcal{O}, \quad \lim_{\varepsilon \downarrow 0} \Big(\varepsilon \int_{\mathcal{O}} \boldsymbol{q}(x, \nabla \boldsymbol{w}^\varepsilon(x))\, dx + \int_\Omega |\boldsymbol{w}^\varepsilon - \boldsymbol{w}|^2\, dx \Big) = 0. \quad (3.31)$$

Proof. Of course, we simply have to define $\boldsymbol{w}^\varepsilon$ in $\mathcal{O}$, since we can set $\boldsymbol{w}^\varepsilon \equiv \boldsymbol{w}$ in $\Omega \setminus \mathcal{O}$. Then we can invoke the density of $H^1(\mathcal{O}; \mathbb{R}^m)$ in $L^2(\mathcal{O}; \mathbb{R}^m)$ to find $\tilde{\boldsymbol{w}}^\varepsilon \in H^1(\mathcal{O}; \mathbb{R}^m)$ satisfying (3.31). Taking $\boldsymbol{w}^\varepsilon := P(\tilde{\boldsymbol{w}}^\varepsilon)$, $P : \mathbb{R}^m \to \mathbb{R}^m$ being the orthogonal projection on the kernel of ℓ, we conclude. ∎

Proposition 3.12 *Let $\mathcal{O} \in \boldsymbol{A}(\Omega)$ and $\boldsymbol{u}, \boldsymbol{v}$ be two functions in X satisfying*

$$u = \ell(\boldsymbol{u}) = \ell(\boldsymbol{v}) = v \ \textit{a.e. in}\ \mathcal{O}.$$

Then we have

$$\mathscr{F}'(\boldsymbol{u}, \mathcal{O}) = \mathscr{F}'(\boldsymbol{v}, \mathcal{O}), \quad \mathscr{F}''(\boldsymbol{u}, \mathcal{O}) = \mathscr{F}''(\boldsymbol{v}, \mathcal{O}). \quad (3.32)$$

Proof. Let $\{\boldsymbol{u}^\varepsilon\}_{\varepsilon \in E}$ be a family such that

$$\lim_{\varepsilon \downarrow 0} \boldsymbol{u}^\varepsilon = \boldsymbol{u} \quad \text{in } X; \quad (3.33)$$

by applying the previous lemma to the function $\boldsymbol{w} := \boldsymbol{v} - \boldsymbol{u}$ we find $\boldsymbol{w}^\varepsilon \in X \cap H^1(\mathcal{O}; \mathbb{R}^m)$, $\varepsilon \in E$, such that $\ell(\boldsymbol{w}^\varepsilon) = 0$ in $\mathcal{O}$ and (3.31) holds.

Setting $\boldsymbol{v}^\varepsilon := \boldsymbol{u}^\varepsilon + \boldsymbol{w}^\varepsilon$, we get $\ell(\boldsymbol{v}^\varepsilon) = \ell(\boldsymbol{u}^\varepsilon)$, $\lim_{\varepsilon \in E} \boldsymbol{v}^\varepsilon = \boldsymbol{v}$ in X; in order to bound $\boldsymbol{q}(x, \nabla \boldsymbol{w}^\varepsilon)$, let us recall that Cauchy-Schwarz inequality yields for every $x \in \Omega$

$$\boldsymbol{q}(x, \mathfrak{a} + \mathfrak{b}) \le (1 + \delta)\boldsymbol{q}(x, \mathfrak{a}) + (1 + \delta^{-1})\boldsymbol{q}(x, \mathfrak{b}), \quad \forall\, \mathfrak{a}, \mathfrak{b} \in \mathbb{M}^{m \times n},\ \forall \delta > 0. \quad (3.34)$$

In particular, we have (the inequality is trivial if $\boldsymbol{u}^\varepsilon|_{\mathcal{O}} \notin H^1(\mathcal{O}; \mathbb{R}^m)$)

$$\mathscr{F}^\varepsilon(\boldsymbol{v}^\varepsilon; \mathcal{O}) \le (1 + \delta)\mathscr{F}^\varepsilon(\boldsymbol{u}^\varepsilon; \mathcal{O}) + (1 + \delta^{-1})\varepsilon \int_{\mathcal{O}} \boldsymbol{q}(x, \nabla \boldsymbol{w}^\varepsilon)\, dx,$$

which implies, by passing to the limit as $\varepsilon \downarrow 0$ and taking δ arbitrary small,

$$\liminf_{\varepsilon \downarrow 0} \mathscr{F}^\varepsilon(\boldsymbol{v}^\varepsilon) \le \liminf_{\varepsilon \downarrow 0} \mathscr{F}^\varepsilon(\boldsymbol{u}^\varepsilon), \quad \limsup_{\varepsilon \downarrow 0} \mathscr{F}^\varepsilon(\boldsymbol{v}^\varepsilon) \le \limsup_{\varepsilon \downarrow 0} \mathscr{F}^\varepsilon(\boldsymbol{u}^\varepsilon).$$

Recalling definitions (2.5a), (2.5b), we get

$$\mathscr{F}'(\boldsymbol{v}; \mathcal{O}) \le \mathscr{F}'(\boldsymbol{u}; \mathcal{O}), \quad \mathscr{F}''(\boldsymbol{v}; \mathcal{O}) \le \mathscr{F}''(\boldsymbol{u}; \mathcal{O}).$$

Changing the role of $\boldsymbol{u}$ and $\boldsymbol{v}$ we conclude. ∎

By this result, the following definition is well posed:

Definition 3.13 *For every $u \in L^2(\Omega)$, $\mathcal{O} \in \boldsymbol{A}(\Omega)$, we set*

$$\mathscr{G}'(u,\mathcal{O}) := \mathscr{F}'(\boldsymbol{u},\mathcal{O}), \qquad \mathscr{G}''(u,\mathcal{O}) := \mathscr{F}''(\boldsymbol{u},\mathcal{O})$$

whenever $u = \ell(\boldsymbol{u})$ a.e. in $\mathcal{O}$. Moreover, if $\mathscr{F}^\varepsilon$ $\overline{\Gamma}(X)$-converges to $\overline{\mathscr{F}}$ as $\varepsilon \downarrow 0$, then we set

$$\overline{\mathscr{G}}(u,\mathcal{O}) := \overline{\mathscr{F}}(\boldsymbol{u},\mathcal{O}), \quad \forall\, u \in L^2(\Omega),\ \mathcal{O} \in \boldsymbol{A}(\Omega). \tag{3.35}$$

We summarize some of the properties of $\mathscr{G}', \mathscr{G}''$ we just proved in the following lemma (cf. paragraph 2.2):

Lemma 3.14 *The functionals $\mathscr{G}', \mathscr{G}'' : L^2(\Omega) \times \boldsymbol{A}(\Omega) \to [0,+\infty]$ are local, increasing, and lower semicontinuous; $\mathscr{G}'$ is superadditive, $\mathscr{G}''$ is weakly subadditive, and $\overline{\mathscr{G}}$ (when it exists) is a measure; morever, they satisfy for every Lipschitz open set $\mathcal{O} \in \boldsymbol{A}(\Omega)$*

$$\mathscr{G}'(u,\mathcal{O}) < +\infty \quad \Leftrightarrow \quad \mathscr{G}''(u,\mathcal{O}) < +\infty \quad \Leftrightarrow \quad u_{|_\mathcal{O}} \in BV(\mathcal{O};T) \tag{3.36}$$

and, for the same constants λ_0, Λ_0 defined above,

$$\lambda_0\, \mathscr{H}^{n-1}(S_u^* \cap \mathcal{O}) \le \mathscr{G}'(u,\mathcal{O}) \le \mathscr{G}''(u,\mathcal{O}) \le \Lambda_0\, \mathscr{H}^{n-1}(S_u^* \cap \mathcal{O}). \tag{3.37}$$

Finally, (3.36) *and* (3.37) *hold also for $\overline{\mathscr{G}}$ and* every *open set.*

3.4 Lipschitz subsets.

Let us start with a simple modification of Corollary 3.9.

Corollary 3.15 *Let $\boldsymbol{u} \in X$ and $\mathcal{O}_1, \mathcal{O}_1', \mathcal{O}_2 \in \boldsymbol{A}(\Omega)$ with $\mathcal{O}_1 \Subset \mathcal{O}_1'$. Then*

$$\mathscr{F}'(\boldsymbol{u},\mathcal{O}_1 \cup \mathcal{O}_2) \le \mathscr{F}'(\boldsymbol{u},\mathcal{O}_1') + \mathscr{F}''(\boldsymbol{u},\mathcal{O}_2) \tag{3.38a}$$

and

$$\mathscr{F}'(\boldsymbol{u},\mathcal{O}_1 \cup \mathcal{O}_2) \le \mathscr{F}''(\boldsymbol{u},\mathcal{O}_1') + \mathscr{F}'(\boldsymbol{u},\mathcal{O}_2). \tag{3.38b}$$

Proof. (3.38a) follows as the previous corollary, by substituting (3.29) with

$$\liminf_{\varepsilon\downarrow 0} \mathscr{F}^\varepsilon(\boldsymbol{w}^\varepsilon,\mathcal{O}_1 \cup \mathcal{O}_2) \le \liminf_{\varepsilon\downarrow 0} \Big(\mathscr{F}^\varepsilon(\boldsymbol{u}^\varepsilon,\mathcal{O}_1') + \mathscr{F}^\varepsilon(\boldsymbol{v}^\varepsilon,\mathcal{O}_2)\Big) \tag{3.39}$$

$$\le \liminf_{\varepsilon\downarrow 0} \mathscr{F}^\varepsilon(\boldsymbol{u}^\varepsilon,\mathcal{O}_1') + \limsup_{\varepsilon\downarrow 0} \mathscr{F}^\varepsilon(\boldsymbol{v}^\varepsilon,\mathcal{O}_2). \tag{3.40}$$

Analogously, (3.38b) follows by the other estimate

$$\liminf_{\varepsilon\downarrow 0} \mathscr{F}^\varepsilon(\boldsymbol{w}^\varepsilon,\mathcal{O}_1 \cup \mathcal{O}_2) \le \liminf_{\varepsilon\downarrow 0} \Big(\mathscr{F}^\varepsilon(\boldsymbol{u}^\varepsilon,\mathcal{O}_1') + \mathscr{F}^\varepsilon(\boldsymbol{v}^\varepsilon,\mathcal{O}_2)\Big) \tag{3.41}$$

$$\le \limsup_{\varepsilon\downarrow 0} \mathscr{F}^\varepsilon(\boldsymbol{u}^\varepsilon,\mathcal{O}_1') + \liminf_{\varepsilon\downarrow 0} \mathscr{F}^\varepsilon(\boldsymbol{v}^\varepsilon,\mathcal{O}_2). \quad \blacksquare \tag{3.42}$$

Corollary 3.16 *If $\mathcal{O} \in \boldsymbol{A}(\Omega)$ is Lipschitz then $\mathscr{F}'(\cdot,\mathcal{O})$ and $\mathscr{F}''(\cdot,\mathcal{O})$ coincide with their inner regular envelopes (cf. § 2.3); in particular, if there exists the $\overline{\Gamma}(X)$-limit of $\mathscr{F}^\varepsilon$, then $\mathscr{F}'(\cdot,\mathcal{O}) = \mathscr{F}''(\cdot,\mathcal{O})$.*

PROOF. Let us first consider the case of the superior Γ limit $\mathscr{F}''$: we fix $\mathcal{O} \in \boldsymbol{A}(\Omega)$ and we choose $\mathcal{O}_1 \Subset \mathcal{O}_2 \Subset \mathcal{O}$, $\mathcal{O}_1$ Lipschitz. By the weak subadditivity and Lemma 3.4, we get, for every $\boldsymbol{u} \in X$, with $u_{|\mathcal{O}} \in BV(\mathcal{O};T)$ for $u := \ell(\boldsymbol{u})$

$$\begin{aligned}\mathscr{F}''(\boldsymbol{u},\mathcal{O}) &\leq \mathscr{F}''(\boldsymbol{u},\mathcal{O}_2) + \mathscr{F}''(\boldsymbol{u},\mathcal{O}\setminus\bar{\mathcal{O}}_1) \leq \\ &\leq \mathscr{F}''(\boldsymbol{u},\mathcal{O}_2) + \Lambda_0\mathscr{H}^{n-1}(S_u^*\cap(\mathcal{O}\setminus\mathcal{O}_1)).\end{aligned}$$

Therefore,

$$\mathscr{F}''(\boldsymbol{u},\mathcal{O}) \leq \sup_{\mathcal{O}_2\Subset\mathcal{O}} \mathscr{F}''(\boldsymbol{u},\mathcal{O}_2) + c''\mathscr{H}^{n-1}(S_u^*\cap(\mathcal{O}\setminus\mathcal{O}_1));$$

Being $\mathcal{O}_1 \Subset \mathcal{O}$ an arbitrary Lipschitz open set, we conclude. The proof of the corresponding result for $\mathscr{F}'$ follows by the same argument and applying (3.38a). ■

3.5 Representation results.

In this section we apply the additivity result of Corollary 3.10 and the representation results of [2] quoted in § 2.6 to obtain formula (1.12) of Theorem 1 *assuming the existence of the* $\overline{\Gamma}(X)$*-limit* $\overline{\mathscr{F}}$. As before, we set $X := L^2(\Omega;\mathbb{R}^m)$.

First of all we prove a continuous dependence result under small perturbations of the open set which is related to (2.18). Let us first recall that, by the continuity assumptions on $\boldsymbol{q}$, there exists a continuous function $\omega_{\boldsymbol{q}} : [0,+\infty[\to [0,+\infty[$ such that $\omega_{\boldsymbol{q}}(0) = 0$ and

$$\Big|\boldsymbol{q}(x,\mathfrak{a}) - \boldsymbol{q}(y,\mathfrak{a})\Big| \leq \omega_{\boldsymbol{q}}(|x-y|)\boldsymbol{q}(x,\mathfrak{a}), \quad \forall\, x,y\in\Omega, \quad \mathfrak{a}\in\mathbb{M}^{m\times n}.$$

Lemma 3.17 *For every $\mathcal{O} \Subset \Omega$ there exists a continuous function $\omega : [0,+\infty[\to [0,+\infty[$ such that $\omega(0) = 0$ and*

$$\Big|\mathscr{G}'(u,\mathcal{O}_1) - \mathscr{G}'(v,\mathcal{O}_1+z)\Big| \leq \omega(|z|)\mathscr{H}^{n-1}(\mathcal{O}_1\cap S_u^*),$$

whenever $\mathcal{O}_1 \subset \mathcal{O}$, $z \in \mathbb{R}^n$, $|z| < \frac{1}{2}d(\mathcal{O},\partial\Omega)$ and $v(x+z) = u(x)$ in $\mathcal{O}_1$. The same property holds also for $\mathscr{G}''$.

PROOF. Let us observe that for every $\boldsymbol{w} \in H^1(\mathcal{O};\mathbb{R}^m) \cap L^{p_0}(\mathcal{O};\mathbb{R}^m)$, $\boldsymbol{w}_z(x) := \boldsymbol{w}(x-z)$, we have

$$\begin{aligned}\Big|\mathscr{F}^\varepsilon(\boldsymbol{w},\mathcal{O}) - \mathscr{F}^\varepsilon(\boldsymbol{w}_z,\mathcal{O}+z)\Big| &= \varepsilon\Big|\int_{\mathcal{O}}\big(\boldsymbol{q}(x,\nabla\boldsymbol{w}(x)) - \boldsymbol{q}(x+z,\nabla\boldsymbol{w}(x))\big)\,dx\Big| \\ &\leq \omega(|z|)\int_{\mathcal{O}}\varepsilon\boldsymbol{q}(x,\nabla\boldsymbol{w}(x))\,dx \leq \omega(|z|)\mathscr{F}^\varepsilon(\boldsymbol{w},\mathcal{O}).\end{aligned}$$

Now we choose $u \in BV(\Omega;T)$, $\delta > 0$, and $\boldsymbol{u}^\varepsilon \in X$, $\varepsilon \in E$, such that

$$\boldsymbol{u}^\varepsilon \to \boldsymbol{u}, \quad \text{in } X, \quad \ell(\boldsymbol{u}) = u, \quad \liminf_{\varepsilon\in D}\mathscr{F}^\varepsilon(\boldsymbol{u}^\varepsilon,\mathcal{O}) \leq \mathscr{G}'(u,\mathcal{O}) + \delta.$$

Choosing $z \in \mathbb{R}^n$ with $|z| < \frac{1}{2}d(\mathcal{O},\partial\Omega)$, $\boldsymbol{v}^\varepsilon(x) = \boldsymbol{u}^\varepsilon(x-z)$, we deduce

$$\begin{aligned}\mathscr{G}'(v,\mathcal{O}+z) - \mathscr{G}'(u,\mathcal{O}) &\leq \omega(|z|)\liminf_{\varepsilon\in D}\mathscr{F}^\varepsilon(\boldsymbol{u}^\varepsilon,\mathcal{O}) + \delta \\ &\leq \omega(|z|)(\mathscr{G}'(u,\mathcal{O})+\delta) + \delta \leq \\ &\leq \delta + \omega(|z|)(\delta + \Lambda_0\mathscr{H}^{n-1}(S_u^*\cap\mathcal{O})).\end{aligned}$$

Being $\delta > 0$ arbitrary, and changing the role of u and v, we complete the proof for the lower Γ-limits; an analogous argument covers the case of $\mathscr{G}''$. ■

Corollary 3.18 *Let assume that there exists the $\overline{\Gamma}(X)$ limit $\overline{\mathscr{F}}$ of $\mathscr{F}^\varepsilon$ as $\varepsilon \downarrow 0$ in E, and let us set, as usual, $\overline{\mathscr{G}}(u,\cdot) = \overline{\mathscr{F}}(\boldsymbol{u},\cdot)$ whenever $u = \ell(\boldsymbol{u})$. Then there exists a continuous family of norms $\phi(x,\cdot)$, $x \in \Omega$, such that for every $u \in BV(\Omega;T)$ and $\mathcal{O} \in \boldsymbol{A}(\Omega)$*

$$\overline{\mathscr{G}}(u,\mathcal{O}) = \int_{S_u^*\cap\mathcal{O}} \phi(x,\nu_u(x))\, d\mathscr{H}^{n-1}(x). \tag{3.43}$$

Proof. We have simply to apply the result quoted in § 2.6 to $\overline{\mathscr{G}} : L^2(\Omega)\times\boldsymbol{A}(\Omega)$, which is local, lower semicontinuous, a measure in the sense of § 2.2, and satisfies (2.17) by Lemma 3.14; in order to check also (2.18), we simply observe that

$$\overline{\mathscr{G}}(u,\mathcal{O}) = \sup_{\mathcal{O}_1\Subset\mathcal{O}} \mathscr{G}'(u,\mathcal{O}_1)$$

and we apply Corollary 3.17. ■

3.6 Characterization of ϕ.

The last step in the proof of Theorem 1 is to show that ϕ is *independent* of E: if this is the case, we will obtain the $\overline{\Gamma}$-convergence of the whole family $\mathscr{F}^\varepsilon$, $\varepsilon > 0$. We will prove this uniqueness by showing (1.13).

Admissible boundary conditions. First of all, we address the questions posed by the following definition.

Definition 3.19 *Let us fix $\mathcal{O} \in \boldsymbol{A}(\Omega)$ and $u \in L^2(\Omega)\cap BV(\mathcal{O};T)$, and let us consider a subset H of $H^1(\mathcal{O};\mathbb{R}^m)$ such that*

$$H + H^1_0(\mathcal{O};\mathbb{R}^m) \subset H. \tag{3.44}$$

*We say that H is X-*admissible *for u if for every family $\boldsymbol{u}^\varepsilon$, $\varepsilon \in E$, converging to $\boldsymbol{u}$ in X, $\ell(\boldsymbol{u}) = u$ in $\mathcal{O}$, it is possible to find $\boldsymbol{w}^\varepsilon \in X$ converging to $\boldsymbol{w}$ such that $\ell(\boldsymbol{w}) = u$ in $\mathcal{O}$ and*

$$\boldsymbol{w}^\varepsilon|_{\mathcal{O}} \in H, \quad \limsup_{\varepsilon\downarrow 0} \mathscr{F}^\varepsilon(\boldsymbol{w}^\varepsilon,\mathcal{O}) \le \limsup_{\varepsilon\downarrow 0} \mathscr{F}^\varepsilon(\boldsymbol{u}^\varepsilon,\mathcal{O}).$$

Remark 3.20 The previous property can be also reformulated in terms of the indicator function $\mathcal{I}_H$ of H, which is defined by

$$\mathcal{I}_H(\boldsymbol{u}) = \begin{cases} 0 & \text{if } \boldsymbol{u} \in H, \\ +\infty & \text{otherwise.} \end{cases}$$

Then H is admissible for u if the $\Gamma(X)$-limits as $\varepsilon \downarrow 0$ in E of $\mathscr{F}^\varepsilon(\cdot,\mathcal{O})$ and $\mathscr{F}^\varepsilon(\cdot,\mathcal{O}) + \mathcal{I}_H$ coincide at every $\boldsymbol{u} \in X$ with $\ell(\boldsymbol{u}) = u$ in $\mathcal{O}$. □

The next result, which is an abstract formulation of some ideas hidden in [4, Lemma 3.2], provides a useful tool when we want to "change the boundary conditions" of a local minimizing sequence.

Proposition 3.21 *Let $\mathcal{O}$ be an open subset of Ω, let $\boldsymbol{u} \in X$ be a function such that the restriction of $u = \ell(\boldsymbol{u})$ to $\mathcal{O}$ belongs to $BV(\mathcal{O};T)$, and let $H \subset H^1(\mathcal{O};\mathbb{R}^m)$ satisfy* (3.44). *If there exists a family $\boldsymbol{v}^\varepsilon \in H$, $\varepsilon \in E$, converging to $\boldsymbol{v}$ in $L^2(\Omega;\mathbb{R}^m)$ such that*

$$\ell(\boldsymbol{v}) = u \ \textit{in}\ \mathcal{O}, \quad \inf_{\mathcal{Q} \Subset \mathcal{O}} \Big\{ \limsup_{\varepsilon\downarrow 0} \mathscr{F}^\varepsilon(\boldsymbol{v}^\varepsilon, \mathcal{O}\setminus\mathcal{Q}) \Big\} = 0, \tag{3.45}$$

then H is admissible for $\boldsymbol{u}$.

Proof. Let us fix $\delta > 0$, an open set $\mathcal{Q}_\delta, \mathcal{Q}'_\delta$ such that $\mathcal{Q}_\delta \Subset \mathcal{Q}'_\delta \Subset \mathcal{O}$ and a family $\boldsymbol{v}^\varepsilon$ as in the statement of the proposition satisfying

$$\limsup_{\varepsilon\downarrow 0} \mathscr{F}^\varepsilon(\boldsymbol{v}^\varepsilon, \mathcal{O}\setminus\mathcal{Q}) \le \delta, \quad \sup_{x\in\mathcal{Q}_\delta} d(x,\partial\mathcal{O}) \le \delta.$$

We consider a family $\boldsymbol{u}^\varepsilon$, $\varepsilon \in E$, as in definition 3.19 (we first consider the case of the strong convergence) and we apply lemma 3.6 with

$$\mathcal{O}_1 := \mathcal{Q}_\delta, \quad \mathcal{O}'_1 := \mathcal{Q}'_\delta, \quad \mathcal{O}_2 := \mathcal{O}\setminus\mathcal{Q}_\delta.$$

We find a family $\boldsymbol{w}^{\varepsilon,\delta}$ such that

$$\lim_{\varepsilon\downarrow 0} \boldsymbol{w}^{\varepsilon,\delta} = \boldsymbol{u} \text{ in } X, \ \textit{uniformly w.r.t. } \delta,$$

$\boldsymbol{w}^{\varepsilon,\delta} - \boldsymbol{v}^\varepsilon \in H^1_0(\mathcal{O};\mathbb{R}^m)$ so that $\boldsymbol{w}^{\varepsilon,\delta}{}_{|\mathcal{O}} \in H$, and

$$\begin{aligned}\limsup_{\varepsilon\downarrow 0} \mathscr{F}^\varepsilon(\boldsymbol{w}^{\varepsilon,\delta},\mathcal{O}) &\le \limsup_{\varepsilon\downarrow 0} \mathscr{F}^\varepsilon(\boldsymbol{u}^\varepsilon,\mathcal{Q}'_\delta) + \limsup_{\varepsilon\downarrow 0} \mathscr{F}^\varepsilon(\boldsymbol{v}^\varepsilon,\mathcal{O}\setminus\mathcal{Q}_\delta)\\ &\le \limsup_{\varepsilon\downarrow 0} \mathscr{F}^\varepsilon(\boldsymbol{u}^\varepsilon,\mathcal{O}) + \delta.\end{aligned}$$

Defining now $\boldsymbol{w}^\varepsilon := \boldsymbol{w}^{\varepsilon,\varepsilon}$ we conclude the proof. ■

Characterization of ϕ when $\boldsymbol{q}$ is independent of x. As we shall see at the end of this section, it is not difficult to reduce the problem to the case of a single quadratic form $\boldsymbol{q}$, independent of x; therefore, for the time being we assume that

$$\boldsymbol{q}(x,\mathfrak{a}) \equiv \boldsymbol{q}(\mathfrak{a}), \quad \text{constant w.r.t. } x \in \Omega. \tag{3.46}$$

Lemma 3.22 *Let ϕ be the function defined in Corollary 3.18, $\nu \in \mathbb{S}^{n-1}$, and Q be an admissible set for ν as in* (1.8). *Then*

$$\phi(\nu) \ge \inf\Big\{ \int_Q f^\varepsilon(\boldsymbol{u},\nabla\boldsymbol{u}) : \boldsymbol{u} \in \mathcal{B}(Q;\mathbb{R}^m),\ \varepsilon > 0 \Big\}. \tag{3.47}$$

Proof. It is not restrictive to assume $Q \subset \Omega$; we introduce the functions

$$j_\varepsilon(t) := \begin{cases} -1 & \text{if } t < -\varepsilon \\ t/\varepsilon & \text{if } -\varepsilon \le t \le \varepsilon \\ 1 & \text{if } t > \varepsilon, \end{cases} \quad j_0(t) := \mathrm{sign}(t), \tag{3.48}$$

and we define

$$u(x) := \tfrac{1}{2}(s_- + s_+) + \tfrac{1}{2}(s_+ - s_-)j(\langle x, \nu\rangle). \tag{3.49}$$

By (3.43) we easily have

$$\overline{\mathscr{G}}(u, Q) = \mathscr{G}'(u, Q) = \mathscr{G}''(u, Q) = \phi(\nu). \tag{3.50}$$

The thesis follows if we show that $\mathcal{B}(Q;\mathbb{R}^m)$ is admissible for u, as Definition 3.19 and Corollary 3.16 yield

$$\inf\Big\{\limsup_{\varepsilon\downarrow 0}\mathscr{F}^\varepsilon(\boldsymbol{u}^\varepsilon, Q) : \boldsymbol{u}^\varepsilon \to \boldsymbol{u} \text{ in } X,\ \boldsymbol{u}^\varepsilon \in \mathcal{B}(Q;\mathbb{R}^m)\Big\} = \mathscr{G}''(u, Q) = \phi(\nu).$$

This relation obviously implies (3.47).

In order to prove the admissibility of $\mathcal{B}(Q;\mathbb{R}^m)$, we apply Proposition 3.21: we choose two points $\boldsymbol{s}_\pm \in \mathbb{R}^m$ such that $\ell(\boldsymbol{s}_\pm) = s_\pm$ and we set

$$\boldsymbol{v}^\varepsilon(x) := \tfrac{1}{2}(\boldsymbol{s}_- + \boldsymbol{s}_+) + \tfrac{1}{2}(\boldsymbol{s}_+ - \boldsymbol{s}_-)j_\varepsilon(\langle x, \nu\rangle)$$

which obviously belong to $\mathcal{B}(Q;\mathbb{R}^m)$; choosing, e.g., $Q_\rho := \rho Q$, $\rho < 1$, a simple calculation shows that

$$\mathscr{F}^\varepsilon(\boldsymbol{v}^\varepsilon, Q\setminus Q_\rho) \le \mathscr{H}^{n-1}(Q\setminus Q_\rho)\Big(\Lambda|\boldsymbol{s}_+ - \boldsymbol{s}_-|^2 + \max\big(W(s_-), W(s_+)\big)\Big).$$

Letting $\rho\uparrow 1$, we get (3.45). ∎

Now we prove the opposite inequality: this conclude the proof of Theorem 1 in the constant coefficient case.

Lemma 3.23 *Let ϕ, ν, Q as in the previous lemma, let $\boldsymbol{u} \in L^2(\Omega;\mathbb{R}^m)$ be a function of $\mathcal{B}(Q;\mathbb{R}^m)$, and $\varepsilon_0 > 0$. Then*

$$\phi(\nu) \le \int_Q f^{\varepsilon_0}(\boldsymbol{u}(x), \nabla\boldsymbol{u}(x))\,dx. \tag{3.51}$$

Proof. Let $\boldsymbol{s}_-, \boldsymbol{s}_+$ be the (constant) traces of $\boldsymbol{u}$ on the basis $Q^\pm$, which by definition of $\mathcal{B}(Q;\mathbb{R}^n)$ satisfy $\ell(\boldsymbol{s}_\pm) = s_\pm$ and for every $r \in \mathbb{R}$ let us denote by r^* the unique element of $[-1/2, 1/2[$ such that $r - r^*$ is an integer.

Denoting a point $x \in \mathbb{R}^n$ with its coordinates $r_1, \dots, r_n$ with respect to the basis $\eta_1, \dots, \eta_{n-1}, \eta$ of $\mathbb{R}^n$ introduced in (1.8), we extend $\boldsymbol{u}$ to the whole $\mathbb{R}^n$ by setting

$$\boldsymbol{u}(r_1, \dots, r_n) := \begin{cases} \boldsymbol{s}_- & \text{if } r_n \le a, \\ \boldsymbol{u}(r_1^*, r_2^*, \dots, r_{n-1}^*, r_n) & \text{if } a < r_n < b, \\ \boldsymbol{s}_+ & \text{if } r_n > b. \end{cases}$$

Therefore, $\boldsymbol{u}$ is the unique $H^1_{loc}(\mathbb{R}^n;\mathbb{R}^m)$ extension which is periodic along the directions $\eta_1, \dots, \eta_{n-1}$ and constant outside the strip $a < r_n < b$.

We fix an odd integer $k = 2h + 1$, $h \in \mathbb{N}$, and we define

$$\boldsymbol{u}^k := \boldsymbol{u}(kx) \quad \text{which belongs to} \quad \mathcal{B}(Q;\mathbb{R}^m), \text{ too.}$$

It is easy to see that

$$\begin{aligned}\int_Q \boldsymbol{q}(\nabla \boldsymbol{u}^k(x))\,dx &= \int_Q k^2 \boldsymbol{q}(\nabla \boldsymbol{u}(kx))\,dx = \\ &= \int_{kQ} k^2 \boldsymbol{q}(\nabla \boldsymbol{u}(y)) k^{-n}\,dy = \\ &= k^{n-1} \int_Q k^{2-n} \boldsymbol{q}(\nabla \boldsymbol{u}(y))\,dy = \\ &= k \int_Q \boldsymbol{q}(\nabla \boldsymbol{u}(x))\,dx\end{aligned}$$

and

$$\begin{aligned}\int_Q W(u^k(x))\,dx &= \int_Q W(u(kx))\,dx = \\ &= \int_{kQ} W(u(y)) k^{-n}\,dy = \\ &= k^{n-1} \int_Q W(u(y)) k^{-n}\,dy = \\ &= k^{-1} \int_Q W(u(y))\,dy.\end{aligned}$$

Therefore,

$$\mathscr{F}^\varepsilon(\boldsymbol{u}^k, Q) = \mathscr{F}^{\varepsilon k}(\boldsymbol{u}, Q).$$

Choosing now $k := k(\varepsilon)$ as the (odd) integer part of $\varepsilon_0/\varepsilon$, $\varepsilon \in E$, it is obvious that

$$\begin{gathered}\mathscr{F}^\varepsilon(\boldsymbol{u}^{k(\varepsilon)}, Q) = \mathscr{F}^{\varepsilon k(\varepsilon)}(\boldsymbol{u}, Q) \to \mathscr{F}^{\varepsilon_0}(\boldsymbol{u}, Q), \\ \boldsymbol{u}^{k(\varepsilon)} \to \boldsymbol{u} := \tfrac{1}{2}(\boldsymbol{s}_- + \boldsymbol{s}_+) + \tfrac{1}{2}(\boldsymbol{s}_+ - \boldsymbol{s}_-) j(\langle x, \nu \rangle)\end{gathered} \tag{3.52}$$

in $L^2(Q;\mathbb{R}^m)$ as $\varepsilon \downarrow 0$, j being defined in (3.48). On the other hand, we have as in (3.50),

$$\phi(\nu) = \mathscr{F}'(\boldsymbol{u}, Q) \le \liminf_{\varepsilon \downarrow 0} \mathscr{F}^\varepsilon(\boldsymbol{u}^{k_\varepsilon}, Q). \quad \blacksquare$$

Corollary 3.24 *Let us suppose that $\boldsymbol{q}(\cdot, \mathfrak{a}) \equiv \boldsymbol{q}(\mathfrak{a})$ is constant; then the family $\mathscr{F}^\varepsilon(\boldsymbol{u}, \cdot)$, $\varepsilon > 0$, $\overline{\Gamma}(X)$-converges to the functional $\mathscr{G}(\ell(\boldsymbol{u}), \cdot)$, where*

$$\mathscr{G}(u, \mathcal{O}) := \int_{S_u^* \cap \mathcal{O}} \phi(\nu_u(x))\, d\mathscr{H}^{n-1}(x),$$

where ϕ is defined as in Theorem 1. Moreover, if $\mathcal{O}$ is Lipschitz, we have

$$\mathscr{F}'(\boldsymbol{u}, \mathcal{O}) = \mathscr{F}''(\boldsymbol{u}, \mathcal{O}) = \mathscr{G}(\ell(\boldsymbol{u}), \mathcal{O}).$$

Characterization of ϕ in the general case. We conclude now the proof of an analogous result in the non-constant coefficient case, by adapting the same arguments of Lemma 3.17.

Lemma 3.25 *For every $x_0 \in \Omega$, let us denote by $\mathscr{F}^\varepsilon_{x_0}$*

$$\mathscr{F}^\varepsilon_{x_0}(\boldsymbol{u}, \mathcal{O}) := \int_{\mathcal{O}} f^\varepsilon(x_0, \boldsymbol{u}(x), \nabla \boldsymbol{u}(x))\, dx,$$

and by $\mathscr{F}_{x_0}$ its $\overline{\Gamma}$-limit. For every Lipshitz open set $\mathcal{O}$ and $\boldsymbol{w} \in L^2(\Omega; \mathbb{R}^m)$ we have

$$\Big|\mathscr{F}'(\boldsymbol{w}, \mathcal{O}) - \mathscr{F}_{x_0}(\boldsymbol{w}, \mathcal{O})\Big| \le \vartheta(\mathcal{O}) \mathscr{H}^{n-1}(S^*_w \cap \mathcal{O}), \tag{3.53}$$

where $\vartheta(\mathcal{O}) := \Lambda_0 \sup_{x \in \mathcal{O}} \omega(|x - x_0|)$. The same bound holds also for the superior Γ-limit $\mathscr{F}''$.

Proof. As in Lemma 3.17 we get for every $\boldsymbol{w} \in H^1(\mathcal{O}; \mathbb{R}^m) \cap L^{p_0}(\mathcal{O}; \mathbb{R}^m)$

$$\begin{aligned}\Big|\mathscr{F}^\varepsilon(\boldsymbol{w}, \mathcal{O}) - \mathscr{F}^\varepsilon_{x_0}(\boldsymbol{w}, \mathcal{O})\Big| &= \varepsilon \Big| \int_{\mathcal{O}} \big(\boldsymbol{q}(x, \nabla \boldsymbol{w}(x)) - \boldsymbol{q}(x_0, \nabla \boldsymbol{w}(x))\big)\, dx \Big| \\ &\le \vartheta(\mathcal{O}) \int_{\mathcal{O}} \varepsilon \boldsymbol{q}(x_0, \nabla \boldsymbol{w}(x))\, dx \le \vartheta(\mathcal{O}) \mathscr{F}^\varepsilon_{x_0}(\boldsymbol{w}, \mathcal{O}).\end{aligned}$$

The proof follows then as in that lemma. ■

Corollary 3.26 *The family $\mathscr{F}^\varepsilon(\boldsymbol{u}, \cdot)$ $\overline{\Gamma}(X)$-converges to the functional $\mathscr{G}(\ell(\boldsymbol{u}), \cdot)$, where*

$$\mathscr{G}(u, \mathcal{O}) := \int_{S^*_u \cap \mathcal{O}} \phi(x, \nu_u(x))\, d\mathscr{H}^{n-1}(x),$$

where ϕ is defined as in Theorem 1. Moreover, if $\mathcal{O}$ is Lipschitz, we have

$$\mathscr{F}'(\boldsymbol{u}, \mathcal{O}) = \mathscr{F}''(\boldsymbol{u}, \mathcal{O}) = \mathscr{G}(\ell(\boldsymbol{u}), \mathcal{O}).$$

Proof. Let us assume that there exists the $\overline{\Gamma}$-limit as $\varepsilon \downarrow 0$ in E; by the representation result of corollary 3.18, choosing

$$\boldsymbol{u} := \tfrac{1}{2}(\boldsymbol{s}_- + \boldsymbol{s}_+) + \tfrac{1}{2}(\boldsymbol{s}_+ - \boldsymbol{s}_-) j(\langle x - x_0, \nu \rangle)$$

we easily have

$$\phi(x_0, \nu) = \lim_{\rho \downarrow 0} \frac{\overline{\mathscr{F}}(\boldsymbol{u}, \rho Q)}{\rho^{n-1}}.$$

Combining this asymptotic behaviour with the estimates of the previous lemma, we conclude. ■

3.7 Γ-convergence in the weak topology.

The aim of this section is to prove Theorem 2, by adapting the arguments of the previous paragraphs. Therefore, we fix a constant $C > 0$, and we consider the topological space

$$X_w := \Big\{ \boldsymbol{u} \in L^2(\Omega; \mathbb{R}^m) : \|\boldsymbol{u}\|_{L^2(\Omega; \mathbb{R}^m)} \le C \Big\} \tag{3.54a}$$

endowed with the coarsest topology which makes continuous the map

$$\boldsymbol{u} \in X_w \to \ell(\boldsymbol{u}) \in L^2(\Omega) \text{ with its weak topology.} \tag{3.54b}$$

In other words, as $\varepsilon \downarrow 0$

$$\boldsymbol{u}_\varepsilon \to \boldsymbol{u} \text{ in } X_w \quad \Leftrightarrow \quad u_\varepsilon := \ell(\boldsymbol{u}_\varepsilon) \rightharpoonup \ell(\boldsymbol{u}) = u \text{ in } L^2(\Omega).$$

It is not difficult to see that X_w is not an Hausdorff space but nevertheless it satisfies the second countability axiom, i.e. it has a countable base of open sets: in fact, a set $Y \subset X_w$ is open iff $\ell(Y)$ is open in the weak topology of the ball of radius $C|\boldsymbol{\ell}|$ of $L^2(\Omega)$. Therefore, the definitions and the properties presented in § 2.2, ... ,2.4 make sense also for this choice of X_w, and we can consider the family of local functionals $\mathscr{F}, \mathscr{F}^\varepsilon : X_w \times \boldsymbol{A}(\Omega) \to [0, +\infty]$, $\varepsilon > 0$, in this different setting.

Let us observe that a function $u \in L^2(\Omega)$ whose norm is bounded by $C \min_{i=1,\dots,m} |\ell_i|^{-1}$ surely belongs to the image of X_w throgh ℓ; in particular, we are dealing in the limit with functions $u \in BV(\Omega; T)$ which satisfy

$$\|u\|_{L^\infty(\Omega)} \leq s_{max} := \max\big(|s_-|, |s_+|\big), \quad \|u\|_{L^2(\Omega)} \leq s_{max}|\Omega|^{1/2},$$

so that $BV(\Omega; T) \subset \ell(X_w)$ whenever C is sufficiently large.

In order to prove Theorem 2 we have simply to check that the various statements of § 3 also hold with X replaced by X_w. We limit ourselves to notice the relevant differences.

- § 3.1: all these results do not deal with convergence, so they hold also in the new setting; in Lemma 3.4 we have only to specify that

  $$\|u\|_{L^\infty(\Omega)} \leq C|\Omega|^{-1/2} \min_{i=1,\dots,m} |\ell_i|^{-1}.$$

- *Lemma* 3.6 is already in the form we need; observe that if $\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon \in X_w$ then also $\boldsymbol{w}^\varepsilon$ belongs to X_w.

- *Corollary* 3.9 also holds, since if $\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon \to \boldsymbol{u}$ in X_w then (3.15) is obviously satified and (3.16) can be deduced by the convergence in X_w and the compactness provided by Lemma 3.2.

- § 3.3 is not necessary in this context, since the topology on X_w forces $\mathscr{F}'$ and $\mathscr{F}''$ to depend only on $\ell(\boldsymbol{u})$, since they have to be l.s.c. in X_w.

- § 3.4 *and* 3.5 holds without any change.

- *Definition* 3.19 makes sense also for X_w and *Proposition* 3.21 is still true: we only have to observe that if we know only the weak convergence of $\ell(\boldsymbol{u}^\varepsilon)$ to $\ell(\boldsymbol{u})$ then the family $\boldsymbol{w}^\varepsilon$ defined in the proof of 3.21 converges to $\boldsymbol{u}$ in X_w: in fact it is uniformly bounded by the constant C in $L^2(\mathcal{O}; \mathbb{R}^m)$ and, by construction,

  $$\boldsymbol{w}^\varepsilon \equiv \boldsymbol{u}^\varepsilon \quad \text{on every set } \mathcal{Q} \text{ such that} \quad \inf_{x \in \mathcal{Q}} d(x, \mathcal{O}) > \varepsilon.$$

  Choosing test functions compactly supported in $\mathcal{O}$, we easily deduce the weak convergence of $\ell(\boldsymbol{w}^\varepsilon)$ to $\ell(\boldsymbol{u})$.

- *Lemma* 3.22, 3.23, and their x-dependent version hold without any change.

This concludes the proof in the weak convergence case. ■

4 Γ-convergence of the non-local functionals $\mathscr{G}^\varepsilon$ for the "Bidomain" formulation.

¿From now on we will assume that $\mathscr{F}^\varepsilon$ have the particular "Bidomain structure" outlined in $(B_1, \dots, B_6)$.

Dual a priori estimates for smooth coefficients. We first consider the case of two matrices A_1, A_2 with *Lipschitz coefficients*, i.e.

$$\exists\, L \geq 0: \quad |A_i(x) - A_i(y)| \leq L|x-y| \quad \forall\, x, y \in \overline{\Omega}, \quad i = 1, 2. \tag{4.1}$$

At the end of this section we will show how the results can be extended to continuous coefficients.

Let us first consider the minimum problem (1.26) defining the functional $\mathscr{Q}(\cdot, \mathcal{O})$, $\mathcal{O}$ being at least a Lipschitz and connected open subset of Ω; we want to define a canonical choice among its minimizing solutions.

Lemma 4.1 *For every $u \in H^1(\mathcal{O})$ there exists a unique element $\bar{\boldsymbol{u}} := \bar{\boldsymbol{u}}(u, \mathcal{O}) \in H^1(\mathcal{O}; \mathbb{R}^2)$ such that*

$$\bar{u}_1 + \bar{u}_2 = u, \quad \mathscr{Q}(u, \mathcal{O}) = \int_{\mathcal{O}} q(x, \nabla \bar{\boldsymbol{u}}(x))\, dx, \ \textit{and} \tag{4.2}$$

$$\int_{\mathcal{O}} \bar{u}_1\, dx = \int_{\mathcal{O}} \bar{u}_2\, dx. \tag{4.3}$$

It satisfies the equations (here $i' := 2, 1$ if respectively $i = 1, 2$)

$$\int_{\mathcal{O}} \langle A \nabla \bar{u}_i, \nabla w \rangle\, dx = \int_{\mathcal{O}} \langle A_{i'} \nabla u, \nabla w \rangle\, dx \quad \forall\, w \in H^1(\mathcal{O}). \tag{4.4}$$

Moreover, if

$$\mathcal{O} \ \textit{is convex or of class}\ C^{1,1} \tag{4.5}$$

there exists a constant $C > 0$ (depending only on $\mathcal{O}, \lambda, \Lambda, L$) such that

$$\|\bar{\boldsymbol{u}}\|_{L^2(\mathcal{O};\mathbb{R}^2)} \leq C\Big(\|u\|_{L^2(\mathcal{O})} + \|u_{|_{\partial\mathcal{O}}}\|_* \Big), \tag{4.6}$$

where, for every $g \in L^2(\partial\mathcal{O})$

$$\|g\|_* := \sup\Big\{ \int_{\partial\mathcal{O}} g\, \partial_{\bar{\boldsymbol{n}}} \zeta\, d\mathscr{H}^{n-1} : \zeta \in H^2(\mathcal{O}),\ \|\zeta\|_{H^2(\mathcal{O})} \leq 1 \Big\}. \tag{4.7}$$

Proof. The existence of $\bar{\boldsymbol{u}}(u)$[6] is a straightforward application in $H^1(\mathcal{O}; \mathbb{R}^2)$ of the Poincaré inequality and of the Lions-Stampacchia Theorem on the closed affine subspace K_u defined by

$$K_u := \Big\{ \boldsymbol{w} \in H^1(\mathcal{O}; \mathbb{R}^2) : w_1 + w_2 = u,\ \int_{\mathcal{O}} w_1\, dx = \int_{\mathcal{O}} w_2\, dx \Big\};$$

[6] which does not require (4.1) but only (1.1b)

in fact, $\bar{\boldsymbol{u}}(u)$ is characterized by

$$\bar{\boldsymbol{u}} \in K_u, \quad \sum_{i=1,2} \int_{\mathcal{O}} \langle A_i \nabla \bar{u}_i, \nabla w_i \rangle \, dx = 0 \quad \forall \boldsymbol{w} \in K_0.$$

Since $\boldsymbol{w} \in K_0 \;\Leftrightarrow\; w_1 = -w_2, \quad \int_{\mathcal{O}_k} w_1 \, dx = 0$, we get

$$\int_{\mathcal{O}} \langle A_1 \nabla \bar{u}_1, \nabla w \rangle \, dx = \int_{\mathcal{O}} \langle A_2 \nabla \bar{u}_2, \nabla w \rangle \, dx, \quad \forall w \in H^1(\mathcal{O}).$$

Finally, by adding $\int_{\mathcal{O}} \langle A_2 \nabla \bar{u}_1, \nabla w \rangle \, dx$ to both the members, we obtain (4.4) for $i = 1$; the other equation can be derived analogously.

In order to prove the $L^2(\mathcal{O})$-estimate when $\mathcal{O}$ satisfies (4.5), we use a standard duality argument; we set

$$\bar{u}_i^* := \bar{u}_i - \tfrac{1}{|\mathcal{O}|} \int_{\mathcal{O}} \bar{u}_i(x) \, dx$$

and we observe that the L^2-estimate of $\bar{\boldsymbol{u}}$ follows from (4.2),(4.3) if we are able to control the L^2-norms of $\bar{u}_i^*$.

So, let us consider the unique variational solution $\bar{w} \in H^1(\mathcal{O})$, with $\int_{\mathcal{O}} w(x) \, dx = 0$ (we assumed that $\mathcal{O}$ is connected), of the problem

$$\int_{\mathcal{O}} A(x) \nabla \bar{w}(x) \, \nabla \zeta(x) \, dx = \int_{\mathcal{O}} \bar{u}_i^*(x) \zeta(x) \, dx, \quad \forall \zeta \in H^1(\Omega). \tag{4.8}$$

Since $\mathcal{O}$ is convex or $C^{1,1}$ and (4.1) holds, then it is well known that $\bar{w} \in H^2(\mathcal{O})$ and (see e.g. [17], Theorems 2.2.2.3 and 3.2.1.2).

$$\|w\|_{H^2(\mathcal{O})} \leq C \|\bar{u}_i^*\|_{L^2(\Omega)}. \tag{4.9}$$

Inserting (4.8) into (4.4) and choosing $\zeta := \bar{u}_i$, we get after an integration by part

$$\int_{\mathcal{O}} |\bar{u}_i^*|^2 \, dx = \int_{\mathcal{O}} \langle A \nabla \bar{w} \; \nabla \bar{u}_i \rangle \, dx = \int_{\mathcal{O}} \langle A_{i'} \nabla u, \nabla \bar{w} \rangle \, dx =$$
$$- \int_{\mathcal{O}} u \mathcal{L}_{i'} \bar{w} \, dx + \int_{\partial \mathcal{O}} u \, \partial_{\bar{\boldsymbol{n}}_{i'}} \bar{w} \, d\mathscr{H}^{n-1} \leq C \Big(\|u\|_{L^2(\mathcal{O})} + \|u_{|\partial \mathcal{O}}\|_* \Big) \|\bar{w}\|_{H^2(\Omega)}.$$

By (4.9) we conclude. ■

Γ-*limit of* $\mathscr{G}^\varepsilon$. The previous estimates in the case (4.1) are useful to derive compactness properties for $\mathscr{G}^\varepsilon$, defined by (1.27).

Lemma 4.2 *Let $\mathcal{O} \in \boldsymbol{A}(\Omega)$ be Lipschitz, let $\{u^\varepsilon\}_{\varepsilon > 0}$ be a family in $L^2(\Omega)$ such that*

$$M := \liminf_{\varepsilon \to 0^+} \mathscr{G}^\varepsilon(u^\varepsilon, \mathcal{O}) < +\infty; \tag{4.10}$$

and let $\bar{\boldsymbol{u}}^\varepsilon := \bar{\boldsymbol{u}}(u^\varepsilon, \mathcal{O})$; then there exist an infinitesimal decreasing sequence $D \subset E$ and a function $u \in BV(\mathcal{O}; T)$ such that

$$\lim_{\varepsilon \downarrow 0, \varepsilon \in D} \mathscr{G}^\varepsilon(u^\varepsilon, \mathcal{O}) = M, \quad \lim_{\varepsilon \downarrow 0, \varepsilon \in D} u^\varepsilon = u \; \text{in} \; L^{p_0}(\mathcal{O}), \tag{4.11}$$

where p_0 is the growth exponent of W defined in (1.2b). *Moreover, if* (4.5) *holds, we can choose D in such a way that*

$$\lim_{\varepsilon\downarrow 0,\varepsilon\in D} u^\varepsilon|_{\partial\mathcal{O}} = g \text{ weakly in } L^2(\partial\mathcal{O}) \text{ and } \lim_{\varepsilon\downarrow 0,\varepsilon\in D} \bar{\boldsymbol{u}}^\varepsilon = \bar{\boldsymbol{u}} \text{ in } L^2(\mathcal{O};\mathbb{R}^2) \tag{4.12}$$

where $\bar{\boldsymbol{u}}$ solves ($i = 1, 2$, $i' := 2, 1$ respectively)

$$\int_{\mathcal{O}} \bar{u}_i\, \mathcal{L}(\zeta)\, dx = \int_{\mathcal{O}} u\, \mathcal{L}_{i'}(\zeta)\, dx + \int_{\partial\mathcal{O}} g\, \partial_{\bar{\boldsymbol{n}}_{i'}}\zeta\, d\mathscr{H}^{n-1}(x), \tag{4.13}$$

$$\text{for every test function } \zeta \in H^2(\mathcal{O}) \text{ with } \partial_{\bar{\boldsymbol{n}}}\zeta = 0 \text{ on } \partial\mathcal{O}.$$

PROOF. The first part of the Lemma follows directly by Lemma 3.2 since $\mathscr{G}^\varepsilon(u^\varepsilon) = \mathscr{F}^\varepsilon(\bar{\boldsymbol{u}}^\varepsilon)$; in order to show (4.12), we recall that, by the trace estimates for $W^{1,1}$-functions, the trace of $z^\varepsilon = Z(u^\varepsilon)$, $\varepsilon \in D$, (cf. (3.4),(3.6)) on $\partial\mathcal{O}$ are bounded in $L^1(\partial\mathcal{O})$; therefore,

$$u^\varepsilon|_{\partial\mathcal{O}} \text{ are bounded in } L^2(\partial\mathcal{O}) \text{ if } \varepsilon \in D.$$

Up to a possible extraction of another subsequence, we deduce also the first relation of (4.12). It remains to show the second one, as (4.13) follows from (4.4) by a simple integration by parts.

Since u^ε strongly converges to u in $L^2(\mathcal{O})$ as $\varepsilon\downarrow 0, \varepsilon \in E$, the apriori estimate (4.6) and the next lemma yield the strong convergence of $\bar{\boldsymbol{u}}^\varepsilon$. ∎

Lemma 4.3 *Let us assume that $g^\varepsilon \rightharpoonup g$ in $L^2(\partial\mathcal{O})$ as $\varepsilon \downarrow 0, \varepsilon \in D$; then*

$$\lim_{\varepsilon\downarrow 0,\varepsilon\in D} \|g^\varepsilon - g\|_* = 0 \tag{4.14}$$

PROOF. Here the only difficulty arises when $\mathcal{O}$ is not $C^{1,1}$, so we can suppose it is a convex set. Now we fix $\delta \in (0, 1/2)$ and we observe that, by the standard trace results, the map

$$g \in L^2(\partial\mathcal{O}) \mapsto L_g \in (H^{2-\delta}(\mathcal{O}))', \quad \langle L_g, \zeta\rangle := \int_{\partial\mathcal{O}} g\, \partial_{\bar{\boldsymbol{n}}_i}\zeta\, d\mathscr{H}^{n-1}$$

is bounded and satisfies

$$\|g\|_* = \|L_g\|_{(H^2(\mathcal{O}))'}. \tag{4.15}$$

On the other hand, $H^2(\Omega)$ is densely and compactly embedded into $H^{2-\delta}(\mathcal{O})$, so that the dual of $H^{2-\delta}(\mathcal{O})$ is compactly imbedded into the dual of $H^2(\mathcal{O})$. Since L_{g^ε} weakly converge to L_g in the first space, they also strongly converge in the second one; by (4.15), they converge w.r.t. the norm $\|\cdot\|_*$. ∎

Corollary 4.4 *If $\mathcal{O} \in \boldsymbol{A}(\Omega)$ is Lipschitz, then*

$$u^\varepsilon \to u \quad \text{in } \mathscr{D}'(\mathcal{O}) \quad \Rightarrow \quad \liminf_{\varepsilon\downarrow 0} \mathscr{G}^\varepsilon(u^\varepsilon, \mathcal{O}) \geq \mathscr{G}(u, \mathcal{O}).$$

PROOF. We can assume $\mathscr{G}(u, \mathcal{O}) < +\infty$; by the previous compactness result, we can assume that $u^\varepsilon \to u$ in $L^2(\mathcal{O})$ and, if $\mathcal{O}$ is also of class $C^{1,1}$, we can find $\bar{\boldsymbol{u}}^\varepsilon \in L^2(\mathcal{O};\mathbb{R}^2)$ such that

$$\mathscr{G}^\varepsilon(u^\varepsilon, \mathcal{O}) = \mathscr{F}^\varepsilon(\bar{\boldsymbol{u}}^\varepsilon, \mathcal{O}), \quad \bar{\boldsymbol{u}}^\varepsilon \to \bar{\boldsymbol{u}} \quad \text{in } L^2(\Omega;\mathbb{R}^2), \quad \ell(\bar{\boldsymbol{u}}) = u.$$

Applying Theorem 1, we get

$$\liminf_{\varepsilon\downarrow 0} \mathscr{F}^\varepsilon(\bar{\boldsymbol{u}}^\varepsilon, \mathcal{O}) \geq \mathscr{F}(\bar{\boldsymbol{u}}, \mathcal{O}) = \mathscr{G}(u, \mathcal{O}).$$

If $\mathcal{O}$ is only Lipschitz, we apply the previous result to all the open subsets $\mathcal{O}_1 \Subset \mathcal{O}$ of class $C^{1,1}$, and we find

$$\liminf_{\varepsilon\downarrow 0} \mathscr{G}^\varepsilon(u^\varepsilon, \mathcal{O}) \geq \liminf_{\varepsilon\downarrow 0} \mathscr{G}^\varepsilon(u^\varepsilon, \mathcal{O}_1) \geq \mathscr{G}(u, \mathcal{O}_1)$$

Taking the supremum w.r.t. $\mathcal{O}_1$, we conclude. ■

The proof of the first part of Theorem 3 and its Corollary quickly follows: by the previous corollary, we know that for every Lipschitz open set $\mathcal{O}$

$$\mathscr{G}'(u, \mathcal{O}) \geq \mathscr{G}(u, \mathcal{O}) = \mathscr{F}(\boldsymbol{u}, \mathcal{O}) \quad \forall\, \boldsymbol{u} : \ell(\boldsymbol{u}) = u;$$

on the other hand, being

$$\mathscr{G}^\varepsilon(u, \mathcal{O}) \leq \mathscr{F}^\varepsilon(\boldsymbol{u}, \mathcal{O}) \quad \text{whenever } u = \ell(\boldsymbol{u}),$$

it is easy to see that

$$\mathscr{G}''(u, \mathcal{O}) \leq \inf\Big\{\mathscr{F}''(\boldsymbol{u}, \mathcal{O}) : \ell(\boldsymbol{u}) = u\Big\} = \mathscr{F}(\boldsymbol{u}, \mathcal{O}) = \mathscr{G}'(u, \mathcal{O}).$$

Strong convergence of the traces and the potentials for optimal sequences. In order to conclude the proof of the second part of Theorem 3 we need the following two Lemmata.

Lemma 4.5 *Let $\boldsymbol{u}^\varepsilon$ be a bounded family in $L^2(\Omega; \mathbb{R}^2)$ satisfying*

$$\ell(\boldsymbol{u}^\varepsilon) \to u \text{ in } L^2(\Omega), \quad \limsup_{\varepsilon\downarrow 0} \mathscr{F}^\varepsilon(\boldsymbol{u}^\varepsilon, \Omega) \leq \mathscr{G}(u, \Omega).$$

Then for every Lipschitz open set $\mathcal{O} \Subset \Omega$ we have

$$\limsup_{\varepsilon\downarrow 0} \mathscr{F}^\varepsilon(\boldsymbol{u}^\varepsilon, \mathcal{O}) \leq \inf\Big\{\mathscr{G}(u, \mathcal{O}') : \mathcal{O} \Subset \mathcal{O}'\Big\}, \tag{4.16a}$$

$$\limsup_{\varepsilon\downarrow 0} \mathscr{F}^\varepsilon(\boldsymbol{u}^\varepsilon, \Omega \setminus \overline{\mathcal{O}}) \leq \inf\Big\{\mathscr{G}(u, \Omega \setminus \overline{\mathcal{O}''}) : \mathcal{O}'' \Subset \mathcal{O}\Big\}. \tag{4.16b}$$

In particular, recalling the definition of Ω_ρ given by (2.15), we have

$$\lim_{\rho\downarrow 0} \limsup_{\varepsilon\downarrow 0} \mathscr{F}^\varepsilon(\boldsymbol{u}^\varepsilon, \Omega_\rho) = 0. \tag{4.17}$$

Proof. Let us only check (4.16b), since the first formula follows by the same arguments.

Let us choose $\mathcal{O}_1, \mathcal{O}_2 \in \boldsymbol{A}(\Omega)$ such that

$$\mathcal{O}_1 \Subset \mathcal{O}_2 \Subset \mathcal{O}.$$

We easily have

$$\begin{aligned}
\limsup_{\varepsilon\downarrow 0} \mathscr{F}^\varepsilon(\boldsymbol{u}^\varepsilon, \Omega \setminus \overline{\mathcal{O}}) + \mathscr{G}(u, \mathcal{O}_1) &\leq \limsup_{\varepsilon\downarrow 0} \mathscr{F}^\varepsilon(\boldsymbol{u}^\varepsilon, \Omega \setminus \overline{\mathcal{O}}) + \liminf_{\varepsilon\downarrow 0} \mathscr{F}^\varepsilon(\boldsymbol{u}^\varepsilon, \mathcal{O}_1) \\
&\leq \limsup_{\varepsilon\downarrow 0} \mathscr{F}^\varepsilon(\boldsymbol{u}^\varepsilon, (\Omega \setminus \mathcal{O}) \cup \mathcal{O}_1) \\
&\leq \limsup_{\varepsilon\downarrow 0} \mathscr{F}^\varepsilon(\boldsymbol{u}^\varepsilon, \Omega) \\
&\leq \mathscr{G}(u, \Omega).
\end{aligned}$$

¿From these inequalities we get

$$\limsup_{\varepsilon\downarrow 0} \mathscr{F}^\varepsilon(\boldsymbol{u}^\varepsilon, \Omega\setminus\overline{\mathcal{O}}) \le \mathscr{G}(u,\Omega) - \mathscr{G}(u,\mathcal{O}_1) \le \mathscr{G}(u,\Omega\setminus\overline{\mathcal{O}_2}).$$

Being $\mathcal{O}_2 \Subset \mathcal{O}$ arbitary, we obtain (4.16b).

In order to prove (4.17), we call

$$\Omega'_\rho := \Big\{x\in\Omega : d(x,\partial\Omega) > \rho\Big\}, \quad \Omega_\rho = \Omega\setminus\overline{\Omega}'_\rho$$

and we observe that if ρ,σ are sufficiently small,

$$\rho<\sigma \quad \Rightarrow \quad \Omega'_\sigma \Subset \Omega'_\rho;$$

by the previous formula

$$\limsup_{\varepsilon\downarrow 0} \mathscr{F}^\varepsilon(\boldsymbol{u}^\varepsilon, \Omega_\rho) \le \mathscr{G}(u,\Omega_\sigma), \quad \forall\,\sigma>\rho>0$$

and finally

$$\lim_{\rho\downarrow 0}\limsup_{\varepsilon\downarrow 0} \mathscr{F}^\varepsilon(\boldsymbol{u}^\varepsilon, \Omega_\rho) \le \lim_{\sigma\downarrow 0}\mathscr{G}(u,\Omega_\sigma) = 0 \quad \blacksquare$$

Lemma 4.6 *Let Ω be of class $C^{1,1}$ (or convex) and let u^ε be a family in $L^2(\Omega)$ satisfying* (1.34). *Then the traces of u^ε on $\partial\Omega$ strongly converge to the trace of u in $L^2(\partial\Omega)$.*

Proof. We already know that the traces of u^ε are uniformly bounded in $L^2(\partial\Omega)$. We introduce the auxiliary functions $z^\varepsilon := Z(u^\varepsilon)$ (cf. (3.4)) and we observe that, for every $\rho>0$

$$\int_{\Omega_\rho} |\nabla z^\varepsilon|\,dx \le \frac{|\ell|}{\sqrt{\lambda}}\mathscr{F}^\varepsilon(\bar{\boldsymbol{u}}^\varepsilon,\Omega_\rho),$$

where $\bar{\boldsymbol{u}}^\varepsilon := \bar{\boldsymbol{u}}^\varepsilon(u^\varepsilon,\Omega)$ are defined from u^ε as in Lemma 4.1. By applying the previous lemma, we get

$$\lim_{\rho\downarrow 0}\limsup_{\varepsilon\downarrow 0}\int_{\Omega_\rho} |\nabla z^\varepsilon|\,dx \le \frac{|\ell|}{\sqrt{\lambda}}\lim_{\rho\downarrow 0}\limsup_{\varepsilon\downarrow 0}\mathscr{F}^\varepsilon(\bar{\boldsymbol{u}}^\varepsilon,\Omega_\rho) = 0;$$

Lemma 2.1 therefore entails

$$z^\varepsilon{}_{|\partial\Omega} \to z_{|\partial\Omega} \quad \text{in } L^1(\partial\Omega).$$

By (3.5) and (3.6), we deduce that the inverse function Z^{-1} of Z can be bounded by

$$|Z^{-1}(s)| \le c_2\sqrt{1+|s|}, \quad \text{for some constant } c_2>0.$$

Therefore

$$u_\varepsilon{}_{|\partial\Omega} = Z^{-1}(z^\varepsilon)_{|\partial\Omega} \to Z^{-1}(z)_{|\partial\Omega} \quad \text{in } L^2(\partial\Omega)$$

and it is obvious that

$$Z^{-1}(z)_{|\partial\Omega} = u_{|\partial\Omega}. \quad \blacksquare$$

The final part of Theorem 3 follows now by another application of Lemma 4.2.

4.8 Γ-*convergence of $\mathscr{G}^\varepsilon$ in the case of continuous coefficients.* When the matrix-valued functions $A_1, A_2 : \overline{\Omega} \to \mathbb{M}^{3\times 3}$ are only continuous, the dual *a priori* estimate (4.6) is no longer true, in general, and the weak formulation (1.35a,b) does not make sense. However, we can prove the Γ-convergence of $\mathscr{G}^\varepsilon$ to $\mathscr{G}$ by a simple approximation result.

First of all, we define the distance between two continuous and positive definite quadratic forms $\boldsymbol{q}_1, \boldsymbol{q}_2$ satisfying (1.1a,b) by

$$d(\boldsymbol{q}_1, \boldsymbol{q}_2) := \sup_{x\in\Omega, \mathfrak{a}\in\mathbb{M}^{m\times n}\setminus\{0\}} \frac{\left|\boldsymbol{q}_1(x,\mathfrak{a}) - \boldsymbol{q}_2(x,\mathfrak{a})\right|}{|\mathfrak{a}|^2}$$

The basic fact is contained in the following Lemma.

Lemma 4.7 *For $i = 1,2$ let $\boldsymbol{q}_i$ be satisfying $(1.1a,b)$ and let $\mathscr{F}_i^\varepsilon, \mathscr{F}_i$ be the corresponding functionals defined by* (1.4) *and Theorem 1. Then for every $\boldsymbol{v} \in L^2(\Omega;\mathbb{R}^2)$ we have*

$$\left|\mathscr{F}_1^\varepsilon(\boldsymbol{v}) - \mathscr{F}_2^\varepsilon(\boldsymbol{v})\right| \le \frac{d(\boldsymbol{q}_1,\boldsymbol{q}_2)}{\lambda} \min\{\mathscr{F}_1^\varepsilon(\boldsymbol{v}), \mathscr{F}_2^\varepsilon(\boldsymbol{v})\}, \tag{4.18}$$

$$\left|\mathscr{F}_1(\boldsymbol{v}) - \mathscr{F}_2(\boldsymbol{v})\right| \le \frac{d(\boldsymbol{q}_1,\boldsymbol{q}_2)}{\lambda} \max\{\mathscr{F}_1(\boldsymbol{v}), \mathscr{F}_2(\boldsymbol{v})\}. \tag{4.19}$$

Proof. The first inequality follows easily by

$$\begin{aligned}
\left|\mathscr{F}_1^\varepsilon(\boldsymbol{v}) - \mathscr{F}_2^\varepsilon(\boldsymbol{v})\right| &\le \varepsilon \left| \int_\Omega \big(\boldsymbol{q}_1(x,\nabla\boldsymbol{v}(x)) - \boldsymbol{q}_2(x,\nabla\boldsymbol{v}(x))\big)\,dx \right| \\
&\le \varepsilon d(\boldsymbol{q}_1,\boldsymbol{q}_2) \int_\Omega |\nabla\boldsymbol{v}(x)|^2\,dx \\
&\le \frac{d(\boldsymbol{q}_1,\boldsymbol{q}_2)}{\lambda} \min\Big\{\mathscr{F}_1^\varepsilon(\boldsymbol{v}), \mathscr{F}_2^\varepsilon(\boldsymbol{v})\Big\}.
\end{aligned}$$

In order to check the other one, we take a family $\boldsymbol{v}^\varepsilon \in L^2(\Omega;\mathbb{R}^2)$ such that

$$\boldsymbol{v}^\varepsilon \to \boldsymbol{v} \quad \text{in } L^2(\Omega;\mathbb{R}^2), \quad \lim_{\varepsilon\downarrow 0} \mathscr{F}_1^\varepsilon(\boldsymbol{v}^\varepsilon) = \mathscr{F}_1(\boldsymbol{v}).$$

We obtain

$$\begin{aligned}
\mathscr{F}_2(\boldsymbol{v}) - \mathscr{F}_1(\boldsymbol{v}) &\le \liminf_{\varepsilon\downarrow 0} \Big(\mathscr{F}_2^\varepsilon(\boldsymbol{v}^\varepsilon) - \mathscr{F}_1^\varepsilon(\boldsymbol{v}^\varepsilon)\Big) \\
&\le \frac{d(\boldsymbol{q}_1,\boldsymbol{q}_2)}{\lambda} \liminf_{\varepsilon\downarrow 0}\, \min\Big\{\mathscr{F}_1^\varepsilon(\boldsymbol{v}^\varepsilon), \mathscr{F}_2^\varepsilon(\boldsymbol{v}^\varepsilon)\Big\} \\
&\le \frac{d(\boldsymbol{q}_1,\boldsymbol{q}_2)}{\lambda} \mathscr{F}_1(\boldsymbol{v}).
\end{aligned}$$

Changing the role of $\mathscr{F}_1$ and $\mathscr{F}_2$ we conclude. ∎

We can now complete the last step of the proof of Theorem 3 for a continuous family of quadratic forms.

Lemma 4.8 *Let $\boldsymbol{q}$ be satisfying $(1.1a,b)$ and let $\mathscr{G}^\varepsilon$ be defined as in* (1.27)*; if*

$$u^\varepsilon \rightharpoonup u \quad \text{in } L^2(\Omega)$$

then

$$\liminf_{\varepsilon\downarrow 0} \mathscr{G}^\varepsilon(u^\varepsilon) \ge \mathscr{G}(u).$$

PROOF. It is not restrictive to assume that

$$\limsup_{\varepsilon\downarrow 0} \mathscr{G}^\varepsilon(u^\varepsilon) = M < +\infty;$$

let $\bar{\boldsymbol{u}}^\varepsilon = \bar{\boldsymbol{u}}(u^\varepsilon, \Omega)$ be defined as in Lemma 4.1 so that

$$\mathscr{G}^\varepsilon(u^\varepsilon) = \mathscr{F}^\varepsilon(\bar{\boldsymbol{u}}^\varepsilon)$$

and let $\boldsymbol{q}_* : \overline{\Omega} \times \mathbb{M}^{2\times 3} \to \mathbb{R}$ be another family of quadratic forms with *Lipschitz* coefficients satisfying (1.1a,b) and (4.1); we denote by $\mathscr{F}_*^\varepsilon, \mathscr{G}_*^\varepsilon$ the corresponding functionals defined by (1.4),(1.27) respectively, and by $\mathscr{F}_*, \mathscr{G}_*$ their Γ-limits as in Theorems 1,2,3; we also take

$$\bar{\boldsymbol{u}}_* \in L^2(\Omega;\mathbb{R}^2) \quad \text{such that } \ell(\boldsymbol{u}_*) = u, \quad \mathscr{G}_*(u) = \mathscr{F}_*(\bar{\boldsymbol{u}}_*).$$

Applying the previous Lemma we get

$$\begin{aligned}
\mathscr{G}^\varepsilon(u^\varepsilon) &= \mathscr{G}^\varepsilon(u^\varepsilon) - \mathscr{G}_*^\varepsilon(u^\varepsilon) + \mathscr{G}_*^\varepsilon(u^\varepsilon)\\
&\geq \mathscr{F}^\varepsilon(\bar{\boldsymbol{u}}^\varepsilon) - \mathscr{F}_*^\varepsilon(\bar{\boldsymbol{u}}^\varepsilon) + \mathscr{G}_*^\varepsilon(u^\varepsilon)\\
&\geq -d(\boldsymbol{q},\boldsymbol{q}_*)\mathscr{F}^\varepsilon(\bar{\boldsymbol{u}}^\varepsilon) + \mathscr{G}_*^\varepsilon(u^\varepsilon)\\
&= -d(\boldsymbol{q},\boldsymbol{q}_*)\mathscr{G}^\varepsilon(u^\varepsilon) + \mathscr{G}_*^\varepsilon(u^\varepsilon).
\end{aligned}$$

and passing to the limit as $\varepsilon \downarrow 0$ we get

$$\begin{aligned}
\liminf_{\varepsilon\downarrow 0} \mathscr{G}^\varepsilon(u^\varepsilon) &\geq \mathscr{G}_*(u) - Md(\boldsymbol{q},\boldsymbol{q}_*)\\
&= \mathscr{F}(\bar{\boldsymbol{u}}_*) + \Big(\mathscr{G}_*(u) - \mathscr{F}(\bar{\boldsymbol{u}}_*)\Big) - Md(\boldsymbol{q},\boldsymbol{q}_*)\\
&= \mathscr{G}(u) + \Big(\mathscr{F}_*(\bar{\boldsymbol{u}}_*) - \mathscr{F}(\bar{\boldsymbol{u}}_*)\Big) - Md(\boldsymbol{q},\boldsymbol{q}_*)\\
&\geq \mathscr{G}(u) - d(\boldsymbol{q},\boldsymbol{q}_*)\max\big\{\mathscr{F}_*(\bar{\boldsymbol{u}}_*), \mathscr{F}(\bar{\boldsymbol{u}}_*)\big\} - Md(\boldsymbol{q},\boldsymbol{q}_*)\\
&\geq \mathscr{G}(u)\Big(1 - Cd(\boldsymbol{q},\boldsymbol{q}_*)\Big)
\end{aligned}$$

where C is a positive constant depending on $\mathscr{G}(u), \lambda, \Lambda$. Since by standard approximation results we can find a Lipschitz continuous family of quadratic forms $\boldsymbol{q}_*$ arbitrarily close to $\boldsymbol{q}$ w.r.t. the distance d, we conclude. ■

5 A strict estimate from below.

This last section contains the proof of Theorem 4 for the "bidomain model".

As we have seen in § 3, it is not restrictive to assume A_1, A_2 independent of the space variable x. In the next lemma, we show a simple but important algebraic relation between $\boldsymbol{q}$ and $\underline{q}$. Let us recall that $\underline{A}$ is the symmetric and positive definite matrix

$$\underline{A} = A_1 A^{-1} A_2 = A_2 A^{-1} A_1 = \left(A_1^{-1} + A_2^{-1}\right)^{-1}.$$

Lemma 5.1 *Let A_1, A_2, $A := A_1 + A_2$ be symmetric and definite positive matrices, let $\boldsymbol{q}, q$ and $\underline{q}$ be the quadratic forms*

$$\boldsymbol{q}(\boldsymbol{\xi}) := \langle A_1\xi_1, \xi_1\rangle + \langle A_2\xi_2,\xi_2\rangle \quad \forall \boldsymbol{\xi} = (\xi_1,\xi_2) \in \mathbb{R}^n \times \mathbb{R}^n$$

$$q(\xi) = \langle A\xi, \xi\rangle, \quad \underline{q}(\xi) := \langle \underline{A}\xi, \xi\rangle \quad \forall\, \xi \in \mathbb{R}^n.$$

If $R_i := A^{-1}A_i$, then for every choice of $\boldsymbol{\xi} \in \mathbb{R}^n \times \mathbb{R}^n$, $\xi := \xi_1 + \xi_2$, we have

$$\boldsymbol{q}(\boldsymbol{\xi}) = \underline{q}(\xi) + q(R_1\xi_1 - R_2\xi_2). \tag{5.1}$$

Proof. We simply have

$$\begin{aligned}\boldsymbol{q}(\boldsymbol{\xi}) - \underline{q}(\xi) &= \langle A_1\xi_1, \xi_1\rangle + \langle A_2\xi_2, \xi_2\rangle - \langle \underline{A}(\xi_1+\xi_2), \xi_1+\xi_2\rangle \\ &= \langle (A_1 - \underline{A})\xi_1, \xi_1\rangle + \langle (A_2 - \underline{A})\xi_2, \xi_2\rangle - 2\langle \underline{A}\xi_1, \xi_2\rangle\end{aligned}$$

We observe that

$$\begin{aligned}A_1 - \underline{A} &= A_1 - A_1A^{-1}A_2 = A_1A^{-1}A - A_1A^{-1}A_2 \\ &= A_1A^{-1}(A - A_2) = A_1A^{-1}A_1,\end{aligned}$$

and analogously

$$A_2 - \underline{A} = A_2A^{-1}A_2.$$

Substituting into the previous equation we get

$$\boldsymbol{q}(\boldsymbol{\xi}) - \underline{q}(\xi) = \langle A_1A^{-1}A_1\xi_1, \xi_1\rangle + \langle A_2A^{-1}A_2\xi_2, \xi_2\rangle - 2\langle \underline{A}\xi_1, \xi_2\rangle.$$

Now we observe that

$$\begin{aligned}\langle A_1A^{-1}A_1\xi_1, \xi_1\rangle &= \langle (A_1A^{-1})A(A^{-1}A_1)\xi_1, \xi_1\rangle \\ &= \langle A(A^{-1}A_1)\xi_1, (A^{-1}A_1)\xi_1\rangle = q(R_1\xi_1),\end{aligned}$$

and

$$\langle A_2A^{-1}A_2\xi_2, \xi_2\rangle = q(R_2\xi_2),$$

$$\langle \underline{A}\xi_1, \xi_2\rangle = \langle \underline{A}\xi_2, \xi_1\rangle = \langle (A_1A^{-1})A(A^{-1}A_2)\xi_2, \xi_1\rangle = \langle AR_2\xi_2, R_1\xi_1\rangle$$

Combining these three relations we get the thesis. ■

Let us fix now a direction $\nu \in \mathbb{S}^{n-1}$ and an admissible cylinder $Q = \Sigma(a, b; \eta)$ w.r.t. ν, as in definition (1.8a,…,e); we will also introduce the corresponding infinitely stretched cylinders

$$Q^\infty := \Sigma(-\infty, \infty; \eta), \quad Q^\infty_+ := \Sigma(0, \infty; \eta), \quad Q^\infty_- := \Sigma(-\infty, 0; \eta). \tag{5.2}$$

Since we are dealing with unbounded domains, we modify a little bit the definition of $\mathcal{B}(Q; \mathbb{R}^2)$ as

$$\begin{aligned}\mathcal{C}(Q^\infty; \mathbb{R}^2) := \Big\{ \boldsymbol{u} \in H^1_{loc}(Q^\infty; \mathbb{R}^2) : \int_{Q^\infty} |\nabla \boldsymbol{u}(x)|^2\, dx < +\infty, \\ \exists\, C > 0 : u(x \pm t\eta) \equiv s_\pm \text{ if } x \in \Sigma, t \geq C \Big\}.\end{aligned} \tag{5.3}$$

Lemma 5.2 *Let $\phi : \mathbb{S}^{n-1} \to \mathbb{R}$ be the function defined as in* (1.13)*; then there exists a family $\boldsymbol{u}^k \in \mathcal{C}(Q^\infty; \mathbb{R}^2)$, $u^k = u_1^k + u_2^k$, such that*

$$\phi(\nu) \geq \limsup_{k \to +\infty} \int_{Q^\infty} \Big(\boldsymbol{q}(\nabla \boldsymbol{v}^k(x)) + W(v^k(x)) \Big)\, dx, \tag{5.4}$$

and, for $u^k := Z(u^k)$ (cf. (3.4)*)*

$$\int_{Q_-^\infty} \sqrt{\underline{q}(\nabla z^k)}\, dx = \int_{Q_+^\infty} \sqrt{\underline{q}(\nabla z^k)}\, dx \quad \forall\, k \in \mathbb{N}. \tag{5.5}$$

Proof. ¿From Theorem 1 we know that there exists a family of functions $\boldsymbol{v}^\varepsilon \in \mathcal{B}(Q; \mathbb{R}^2)$, $\varepsilon > 0$, such that

$$\phi(\nu) = \lim_{\varepsilon \downarrow 0} \varepsilon \int_Q \boldsymbol{q}(\boldsymbol{v}^\varepsilon(x))\, dx + \frac{1}{\varepsilon} \int_Q W(v^\varepsilon(x))\, dx.$$

Let $\boldsymbol{s}_\pm^\varepsilon$ be the (constant) traces of $\boldsymbol{v}^\varepsilon$ on the basis $\Sigma_\pm$ of the cylinder Q; by definition, the sum of their coordinates is exatly $s_\pm$. We extend $\boldsymbol{v}^\varepsilon$ to the whole Q^∞ by setting

$$\boldsymbol{v}^\varepsilon(x) = \begin{cases} \boldsymbol{s}_-^\varepsilon & \text{if } x \in Q_-^\infty \setminus Q, \\ \boldsymbol{s}_+^\varepsilon & \text{if } x \in Q_+^\infty \setminus Q. \end{cases}$$

It is easy to see that $\boldsymbol{v}^\varepsilon \in \mathcal{C}(Q^\infty; \mathbb{R}^2)$; we set

$$\boldsymbol{w}^\varepsilon(y) := \boldsymbol{v}^\varepsilon(\varepsilon y), \quad \forall\, y \in \varepsilon^{-1} Q^\infty,$$

so that $\nabla \boldsymbol{w}^\varepsilon(x) = \varepsilon \nabla \boldsymbol{v}^\varepsilon(\varepsilon x)$. We easily find

$$\begin{aligned} &\varepsilon \int_{Q^\infty} \boldsymbol{q}(\nabla \boldsymbol{v}^\varepsilon(x))\, dx + \frac{1}{\varepsilon} \int_{Q^\infty} W(v^\varepsilon(x))\, dx = \\ &\varepsilon^{n+1} \int_{\varepsilon^{-1} Q^\infty} \boldsymbol{q}(\nabla \boldsymbol{v}^\varepsilon(\varepsilon y))\, dy + \varepsilon^{n-1} \int_{\varepsilon^{-1} Q^\infty} W(v^\varepsilon(\varepsilon y))\, dy = \\ &\varepsilon^{n-1} \Big(\int_{\varepsilon^{-1} Q^\infty} \boldsymbol{q}(\nabla \boldsymbol{w}^\varepsilon(y))\, dy + \int_{\varepsilon^{-1} Q^\infty} W(w^\varepsilon(y))\, dy \Big). \end{aligned}$$

Now we choose $\varepsilon := (2k+1)^{-1}$, $k \in \mathbb{N}$, and we observe that $\varepsilon^{-1} Q^\infty = (2k+1) Q^\infty$ is the disjoint union (up to a negligible set) of

$$(2k+1) Q^\infty = \bigcup_{\max |j_m| \leq k} \Big(Q^\infty + \sum_{m=1}^{n-1} j_m \eta_m \Big).$$

Therefore there exist integers $j_1^*, j_2^*, \ldots, j_{n-1}^*$ between $-k$ and k and a cylinder

$$Q^* := Q^\infty + \sum_{m=1}^{n-1} j_m^* \eta_m,$$

such that

$$\begin{aligned} &\varepsilon^{n-1} \Big(\int_{\varepsilon^{-1} Q^\infty} \boldsymbol{q}(\nabla \boldsymbol{w}^\varepsilon(y))\, dy + \int_{\varepsilon^{-1} Q^\infty} W(w^\varepsilon(y))\, dy \Big) \\ &\geq \Big(\int_{Q^*} \boldsymbol{q}(\nabla \boldsymbol{w}^\varepsilon(y))\, dy + \int_{Q^*} W(w^\varepsilon(y))\, dy \Big). \end{aligned}$$

Finally we set

$$\tilde{\boldsymbol{u}}^k(x) := \boldsymbol{w}^\varepsilon(x - \sum_{m=1}^{n-1} j_m^* \eta_m)$$

which satisfy (5.4), i.e.

$$\phi(\nu) \geq \limsup_{k\to+\infty} \int_{Q^\infty} \Big(\boldsymbol{q}(\nabla\tilde{\boldsymbol{u}}^k(x)) + W(\tilde{u}^k(x))\Big)\, dx.$$

Now we have only to modify $\tilde{\boldsymbol{u}}^k$ in order to satisfy (5.5), too. Let us remark that, by (3.9), $\tilde{z}^k := Z(\tilde{u}^k) \in W^{1,1}_{loc}(Q^\infty)$ and

$$\int_{Q^\infty} \sqrt{\underline{q}(\nabla\tilde{z}^k)}\, dx \leq \int_{Q^\infty} \Big(\boldsymbol{q}(\nabla\tilde{\boldsymbol{u}}^k(x)) + W(\tilde{u}^k(x))\Big)\, dx.$$

We introduce the function

$$\beta^k(s) := \int_{-\infty}^{s} \Big(\int_\Sigma \sqrt{\underline{q}(\nabla\tilde{z}^k(x+s\eta))}\, d\mathscr{H}^{n-1}(x)\Big)\, ds$$

which is continuous, monotone and satisfies

$$\beta^k(-\infty) = 0, \quad \beta^k(+\infty) = |\langle\eta,\nu\rangle| \int_{Q^\infty} \sqrt{\underline{q}(\nabla\tilde{z}^k)}\, dx;$$

it is obvious that there exists real numbers s^k such that $\beta^k(s^k) = \frac{\beta^k(+\infty)}{2}$; choosing $\boldsymbol{u}^k(x) := \tilde{\boldsymbol{u}}^k(x + s^k\eta)$ we conclude. ■

Corollary 5.3 *Let $\phi : \mathbb{S}^{n-1} \to \mathbb{R}$ be the function defined as in* (1.13)*, $\nu \in \mathbb{S}^{n-1}$ and Q^∞ be an admissible infinite cylinder for ν. Then*

$$\phi(\nu) \geq I(\nu) := \inf\Big\{\int_{Q^\infty} \boldsymbol{q}(\nabla\boldsymbol{u}(x)) + W(u(x))\, dx : \boldsymbol{u} \in \mathcal{C}(Q^\infty;\mathbb{R}^2)\Big\}.$$

In order to conclude the proof of Theorem 4, we want to show that if $\phi(\nu) = \underline{\phi}(\nu)$, then ν is a common eigenvector of A_1, A_2. Let us denote by $\|\cdot\|_{\underline{q}}$ the norm associated to $\underline{q}$,

$$\|\xi\|_{\underline{q}} := \sqrt{\underline{q}(\xi)},$$

let us choose η as

$$\eta := \frac{\underline{A}\nu}{\underline{q}(\nu)}, \quad \text{so that } \langle\eta,\nu\rangle = 1, \tag{5.6}$$

and let us consider $\underline{q}$-projection along ν

$$\Pi\xi := \frac{\langle\underline{A}\xi,\nu\rangle}{\underline{q}(\nu)}\nu = \langle\xi,\eta\rangle\,\nu, \quad \Pi'\xi := \xi - \Pi\xi. \tag{5.7}$$

Of course we have

$$\underline{q}(\xi) = \underline{q}(\Pi'\xi) + \underline{q}(\Pi\xi) = \underline{q}(\Pi'\xi) + \frac{|\langle\underline{A}\xi,\nu\rangle|^2}{\underline{q}(\nu)} = \underline{q}(\Pi'\xi) + |\langle\xi,\eta\rangle|^2\,\underline{q}(\nu).$$

Lemma 5.4 *Let us choose $\eta := \underline{A}\nu$ as in (5.6) and let us given $\boldsymbol{u} \in \mathcal{C}(Q^\infty;\mathbb{R}^2)$ with $z := Z(u)$; then*

$$\begin{aligned}\underline{\phi}(\nu) &\le \int_{Q^\infty} \|\Pi(\nabla z)\|_{\underline{q}}\,dx \le \Big(\int_{Q^\infty} \|\Pi(\nabla u)\|_{\underline{q}}^2 + W(u)\Big)\,dx\\ &\le \int_{Q^\infty} \boldsymbol{q}(\nabla \boldsymbol{u}(x)) + W(u(x))\,dx;\end{aligned}$$

in particular, $\underline{\phi}(\nu) \le I(\nu)$.

Proof. We easily have

$$\begin{aligned}\int_{Q^\infty} \Big(\underline{q}(\Pi(\nabla u)) + W(u)\Big)\,dx &\ge \int_{Q^\infty} 2\sqrt{\underline{q}(\Pi(\nabla u))W(u)}\,dx\\ &= \int_{Q^\infty} \|\Pi(\nabla u)\|_{\underline{q}}\,\big|Z'(u)\big|\,dx\\ &= \int_{Q^\infty} \|\Pi(\nabla z)\|_{\underline{q}}\,dx.\end{aligned}$$

Now we observe that (5.7) yields

$$\|\Pi(\nabla z)\|_{\underline{q}} = |\langle \nabla z, \eta\rangle|\,\|\nu\|_{\underline{q}} = |\partial_\eta z|\,\|\nu\|_{\underline{q}}$$

so that by (2.16) and the definition of $\underline{\phi}$ in (1.21)

$$\begin{aligned}\int_{Q^\infty} \|\Pi(\nabla z)\|_{\underline{q}}\,dx &= \|\nu\|_{\underline{q}} \int_{Q^\infty} |\partial_\eta z|\,dx\\ &\ge \|\nu\|_{\underline{q}} \int_{\Sigma} \Big(Z(s_+) - Z(s_-)\Big)\,d\mathscr{H}^{n-1}(x)\\ &= 2\sqrt{\underline{q}(\nu)} \int_{s_-}^{s_+} \sqrt{W(s)}\,ds = \underline{\phi}(\nu). \quad \blacksquare\end{aligned}$$

Corollary 5.5 *Let $\boldsymbol{u}^k \in \mathcal{C}(Q^\infty;\mathbb{R}^2)$, with $z^k := Z(u^k)$, be a family satisfying (5.4) and (5.5) as in Lemma 5.2; if $\phi(\nu) = \underline{\phi}(\nu)$, then we have*

$$\lim_{k\uparrow+\infty} \int_{Q^\infty} \underline{q}\big(\Pi'(\nabla u^k)\big) + q\big(R_1(\nabla u_1^k) - R_2(\nabla u_2^k)\big)\,dx = 0 \tag{5.8}$$

and

$$\begin{aligned}\lim_{k\uparrow+\infty} \int_{Q_-^\infty} \|\nabla z^k\|_{\underline{q}}\,dx &= \lim_{k\uparrow+\infty} \int_{Q_+^\infty} \|\nabla z^k\|_{\underline{q}}\,dx\\ &= \tfrac{1}{2} \lim_{k\uparrow+\infty} \int_{Q^\infty} \|\nabla z^k\|_{\underline{q}}\,dx = \tfrac{1}{2}\underline{\phi}(\nu) = \tfrac{1}{2} I(\nu) \quad \blacksquare\end{aligned} \tag{5.9}$$

Proposition 5.6 *Let us assume that $\underline{\phi}(\nu) = \phi(\nu)$; then ν is a common eigenvector of A_1 and A_2.*

Proof. Let us denote by Σ_t the t-section of Q^∞:

$$\Sigma_t := \Sigma + t\eta;$$

let us observe that (5.5), (2.16), and the definition of $\mathcal{C}(Q^\infty;\mathbb{R}^2)$ entail for every $t>0$

$$\begin{aligned}\|\nu\|_{\underline{q}}\int_{\Sigma_t}\Big|z^k(x)-Z(s_+)\Big|\,d\mathscr{H}^{n-1}(x) &\le \|\nu\|_{\underline{q}}\int_{Q^\infty_+}\big|\partial_\eta z^k\big|\,dx\\ &\le \int_{Q^\infty_+}\|\nabla z^k\|_{\underline{q}}\,dx\\ &\le \tfrac12\int_{Q^\infty}\Big(\boldsymbol{q}(\nabla\boldsymbol{u}^k)+W(u^k)\Big)\,dx;\end{aligned}$$

analogously, for $t<0$

$$\|\nu\|_{\underline{q}}\int_{\Sigma_t}\Big|z^k(x)-Z(s_-)\Big|\,d\mathscr{H}^{n-1}(x)\le \tfrac12\int_{Q^\infty}\Big(\boldsymbol{q}(\nabla\boldsymbol{u}^k)+W(u^k)\Big)\,dx.$$

By standard compactness results, we can extract a suitable subsequence (still denoted by $\boldsymbol{u}^k$) such that

$$\boldsymbol{u}^k\to\boldsymbol{u},\quad z^k\to z=Z(u)\quad\text{in } L^1_{loc}(Q^\infty)\text{ and a.e.},\quad\text{with } \boldsymbol{u},\,z\in W^{1,1}_{loc}(Q^\infty).$$

By lower semicontinuity, recalling $R_i=A^{-1}A_i$, we get

$$\Pi'(\nabla v)=0,\quad A_1\nabla v_1=A_2\nabla v_2. \tag{5.10}$$

By the first relation of (5.10) we deduce the existence of a function $f\in W^{1,1}_{loc}(\mathbb{R})$ such that

$$u(x)=f(\langle x,\nu\rangle),\quad z=Z(f(\langle x,\nu\rangle))\quad\text{a.e. in } Q^\infty.$$

On Σ_t we have

$$z(y)=z(x+t\eta)=Z(f(t\langle\eta,\nu\rangle))=Z(f(t))$$

so that

$$\|\nu\|_{\underline{q}}|Z(f(t))-Z(s_+)|\le\tfrac12\underline{\phi}(\nu)=\tfrac12\|\nu\|_{\underline{q}}(Z(s_+)-Z(s_-))$$

and in particular

$$|Z(f(t))-Z(s_+)|\le\tfrac12(Z(s_+)-Z(s_-))\quad\forall t>0.$$

Analogously,

$$|Z(f(t))-Z(s_-)|\le\tfrac12(Z(s_+)-Z(s_-))\quad\forall t<0.$$

If $f(t)\equiv f_0$ would be a constant function, then

$$Z(f_0)=\tfrac12(Z(s_+)+Z(s_-)),\quad\text{and therefore } s_-<f_0<s_+$$

but this is not possible, since

$$\int_{Q^\infty}W(u(x))\,dx=\int_{-\infty}^{+\infty}W(f(t))\,dt<+\infty.$$

Thus f is not constant; from the second relation of (5.10) we get

$$\nabla v_1=R_2\nabla v=f'(\langle x,\nu\rangle)R_2\nu,\quad \nabla v_2=R_1\nabla v=f'(\langle x,\nu\rangle)R_1\nu.$$

Since f is not constant, we deduce that $R_2\nu$ and $R_1\nu$ are parallel to ν, i.e.

$$A_1\nu,\ A_2\nu\text{ are parallel to } A^{-1}\nu\quad\blacksquare$$

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