BV functions and parabolic initial boundary value problems on domains

L. Angiuli* M. Miranda Jr.† D. Pallara* F. Paronetto*

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Abstract

Given a uniformly elliptic second-order operator $\mathcal{A}$ on a (possibly unbounded) domain $\Omega \subset \mathbb{R}^N$, let $(T(t))_{t\geq 0}$ be the semigroup generated by $\mathcal{A}$ in $L^1(\Omega)$, under homogeneous co-normal boundary conditions on $\partial\Omega$. We show that the limit as $t \to 0$ of the L^1-norm of the spatial gradient $D_x T(t)u_0$ tends to the total variation of the initial datum u_0, and in particular is finite if and only if u_0 belongs to $BV(\Omega)$. This result is true also for weighted BV spaces. A further characterisation of BV functions in terms of the short-time behaviour of $(T(t))_{t\geq 0}$ is also given.

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1 Introduction

The definition of functions with bounded variation in $\mathbb{R}^N$, $N \geq 2$, has been given by E. De Giorgi in [9] using the convolution with the Gauss-Weierstrass kernel

$$g_t(x) = \frac{1}{(4\pi t)^{N/2}} e^{-\frac{|x|^2}{4t}}.$$

Given $u \in L^1(\mathbb{R}^N)$, he defined the total variation of u by

$$|Du|(\mathbb{R}^N) = \lim_{t\to 0} \int_{\mathbb{R}^N} |D(u * g_t)| dx,$$

where it is easily seen that the integral on the right hand side is a monotone function in t, and then the limit exists. Notice that g_t is the heat kernel on $\mathbb{R}^N$, and then, using the language of semigroup and setting $T(t)u = u * g_t$, formula (1.1) can be written as

$$|Du|(\mathbb{R}^N) = \lim_{t\to 0} \int_{\mathbb{R}^N} |DT(t)u| dx. \tag{1.1}$$

*Dipartimento di Matematica “Ennio De Giorgi”, Università del Salento, C.P.193, 73100, Lecce, Italy. e-mail: luciana.angiuli@unile.it, diego.pallara@unile.it, fabio.paronetto@unile.it
†Dipartimento di Matematica, Università di Ferrara, via Machiavelli, 35, 44100, Ferrara, Italy. e-mail: michele.miranda@unife.it

Moreover, the fact that $|Du|(\mathbb{R}^N)$ is finite is equivalent to saying that the distributional gradient of u is a ($\mathbb{R}^N$-valued) Radon measure, and this gives the equality

$$(1.2) \qquad |Du|(\mathbb{R}^N) = \sup\Big\{ \int_{\mathbb{R}^N} u\mathrm{div}\,\phi\; dx \ : \phi \in C_c^1(\mathbb{R}^N, \mathbb{R}^N), \|\phi\|_{L^\infty(\mathbb{R}^N)} \le 1\Big\}.$$

A formula analogous to (1.1) can also be used in more general contexts. The question then arises if the connection between semigroups and total variation is more general; recently, in [14], [6] this connection has been investigated in the setting of Riemannian manifolds with a bound on the geometry.

Formula (1.2) can be localized in a subset $\Omega \subset \mathbb{R}^N$ by

$$(1.3) \qquad |Du|(\Omega) = \sup\Big\{ \int_{\Omega} u\mathrm{div}\,\phi\; dx \ : \phi \in C_c^1(\Omega, \mathbb{R}^N), \|\phi\|_{L^\infty(\Omega)} \le 1\Big\}$$

and gives the definition of total variation of $u \in L^1(\Omega)$.

In this paper we prove that (1.1) still holds in Ω, when $(T(t))_{t\ge 0}$ is the semigroup generated by a second order uniformly elliptic operator with regular coefficients and suitable boundary conditions. Concerning the monotonicity, in $\mathbb{R}^N$ the inequality

$$\int_{\mathbb{R}^N} |DT(t)u| dx \le |Du|(\mathbb{R}^N), \quad \forall t > 0,$$

holds, whereas in the Riemannian case the following is true:

$$\int_M |DT(t)u| d\mu \le e^{kt} |Du|(M), \quad \forall t > 0$$

where $k > 0$ is a constant bounding the geometry of M. Here we consider the unifomly elliptic operator with sufficiently smooth coefficients

$$\mathcal{A}u = \sum_{i,j=1}^N D_i(A_{ij} D_j u) + \sum_{i=1}^N B_i D_i u + Cu$$

and the initial-boundary value problem

$$\begin{cases} \partial_t w - \mathcal{A}w = 0 & \text{in } (0,\infty) \times \Omega \\ w(0) = u_0 & \text{in } \Omega \\ \langle ADw, \nu \rangle = 0 & \text{in } (0,\infty) \times \partial\Omega. \end{cases} .$$

Denoting again by

$$T(t)u_0(x) = \int_\Omega p(t,x,y) u_0(y)\, dy$$

the semigroup which gives its solution, we prove that the inequality

$$\int_\Omega |DT(t)u| dx \le |Du|(\Omega) + c\|u\|_{W^{1,1}(\Omega)} t^{1/2}, \qquad t \in (0,1),$$

holds for $u \in W^{1,1}(\Omega)$, and deduce (1.1) by approximating in variation $u \in BV(\Omega)$ by $W^{1,1}$ functions, and taking the limit as $t \to 0$. We point out that equation (1.1) holds not only for classical BV functions, but also for weighted BV functions, see Theorem 5.2.

A further characterisation of BV functions can be given considering in a different way the short-time behaviour of $(T(t))_{t\geq 0}$, namely, we prove that

$$
(1.4) \qquad |Du|_A(\Omega) = \lim_{t\to 0} \frac{\sqrt{\pi}}{2\sqrt{t}} \int_\Omega \int_\Omega p(t,x,y)|u(x)-u(y)|\,dydx,
$$

where $|Du|_A$ denotes the (A_{ij})-weighted total variation of u. This characterisation is in the spirit of [13], [15] and [4], [8], where kernels depending on $|x-y|$ are considered.

The paper is organized as follows; after recalling some preliminary results in Section 2, we discuss in Section 3 the main properties of the semigroup associated with second order partial differential operator on unbounded domains. Using these results, and after giving the definitions and main properties of weighted BV functions in Section 4, we prove in Section 5 the limiting formula (1.1). Section 6 is devoted to the proof of (1.4)

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2 Notations and preliminary results

We denote by $\langle\cdot,\cdot\rangle$ the standard inner product on $\mathbb{R}^N$, by $|\cdot|$ its induced norm and by $B_\varrho(x)$ the open ball centred at x and with radius ϱ. Moreover, given a symmetric positive definite matrix P we introduce the norm

$$
(2.1) \qquad |\xi|_P^2 = |P^{1/2}\xi|^2 = \langle P\xi,\xi\rangle, \quad \forall \xi\in\mathbb{R}^N;
$$

we use the same notation even for variable matrices P. With $C_b(\Omega)$ we mean the set of continuous and bounded functions on Ω, by $C_c(\Omega)$ the set of function u with support strictly contained in Ω, that is $\operatorname{supp} u \subset\subset \Omega$ and by $C_c(\overline{\Omega})$ the set of functions with support a compact set contained in $\overline{\Omega}$ (then not necessarily zero on $\partial\Omega$). For functions $u\in C^k(\Omega)$ we define the norms

$$
\|u\|_\infty = \|u\|_{L^\infty(\Omega)}, \qquad \|\cdot\|_{k,\infty} = \sum_{|\alpha|\leq k} \|D^\alpha\cdot\|_\infty .
$$

Moreover, given a matrix Q, we define

$$
(2.2) \qquad C_Q(\Omega) = \left\{ u\in C^\infty(\Omega)\cap C^1(\overline{\Omega}); \langle QDu,\nu\rangle = 0 \text{ on } \partial\Omega \right\}.
$$

Given a subset $E\subset\mathbb{R}^N$, we denote by $|E|$ its Lebesgue measure, and by $\mathcal{H}^{N-1}(E)$ its $(N-1)$-dimensional Hausdorff measure. By $L^p(\Omega)$ $(p\geq 1)$ we denote the Lebesgue space of p-integrable function with respect to the Lebesgue measure and by $W^{k,p}(\Omega)$ the space of functions p-integrable together with their distributional derivatives up to the k-th order. We recall that if the open set Ω has regular boundary, then the trace operator is continuous from $W^{1,1}(\Omega)$ onto $L^1(\partial\Omega,\mathcal{H}^{N-1})$ (see for instance [1, Theorem 5.3.6]), that is there exists $c_\Omega > 0$ such that for every $u\in W^{1,1}(\Omega)$ the trace $v = u_{|\partial\Omega}$ of u on $\partial\Omega$ is well defined and

$$
(2.3) \qquad \|v\|_{L^1(\partial\Omega,\mathcal{H}^{N-1})} \leq c_\Omega \|u\|_{W^{1,1}(\Omega)}.
$$

In the whole paper we denote by Ω an open subset of $\mathbb{R}^N$, not necessarily bounded, and we always assume that Ω has uniformly C^2 boundary, that is, there are $\varrho, L>0$ such that for every $x\in\partial\Omega$ the set $\partial\Omega\cap B_\varrho(x)$ is the graph of a C^2 function ψ with $\|D^2\psi\|_{2,\infty}, \|D^2\psi^{-1}\|_{2,\infty}\leq L$.

We consider the second order differential operator in divergence form

$$\mathcal{A}u = \operatorname{div}(ADu) + \langle B, Du\rangle + Cu = \sum_{i,j=1}^{N} D_i(A_{ij}D_ju) + \sum_{i=1}^{N} B_iD_iu + Cu \tag{2.4}$$

and the first order operator (co-normal derivative) on $\partial\Omega$

$$\mathcal{B}u = \langle ADu, \nu\rangle = \sum_{i,j=1}^{N} A_{ij}D_ju\nu_i, \tag{2.5}$$

where ν is the outward unit normal vector to $\partial\Omega$. We say that a symmetric positive definite matrix A is λ-elliptic if there exists $\lambda \geq 1$ such that

$$\frac{1}{\lambda}|\xi|^2 \leq \langle A\xi, \xi\rangle \leq \lambda|\xi|^2, \quad \forall \xi \in \mathbb{R}^N.$$

In general, we say that A is elliptic if it is λ-elliptic for some $\lambda \geq 1$. The operator $\mathcal{A}$ is said to be elliptic or λ-elliptic if the matrix A is so. Let us state our standing hypotheses:

(H1) $\Omega \subset \mathbb{R}^N$ has uniformly C^2 boundary;

(H2) $\mathcal{A}$ is λ-elliptic for some $\lambda \geq 1$;

(H3) $A_{ij} \in C_b^1(\overline{\Omega})$, $B, C \in L^\infty(\Omega)$.

Under assumption **(H3)** it is then possible to define the finite quantity

$$M_0 = \max_{i,j}\left\{\|A_{ij}\|_{1,\infty}, \|B_i\|_\infty, \|C\|_\infty\right\}.$$

We are interested in the realization of the operator $\mathcal{A}$ with boundary condition $\mathcal{B}$ in $L^1(\Omega)$; we denote by $D(\mathcal{A})$ the domain of such a realization, which is defined as the closure, in the graph norm, of $\{u \in C^2(\overline{\Omega}) \cap L^1(\Omega) : \ \mathcal{A}u \in L^1(\Omega), \ \mathcal{B}u = 0\}$; we also have that $C_A(\Omega)$ is a core. We recall in the next section that $(\mathcal{A}, D(\mathcal{A}))$ is sectorial and generates an analytic semigroup in $L^1(\Omega)$ which we assume to be contractive, see Remark 3.3. After recalling some known properties of the semigroup, we show further estimates needed in the sequel. Finally, we use the notation $c_n = c(x, y, \ldots)$ with the meaning that the constant c_n depends upon the quantities x, y, etc.

3 Analytic semigroups in $L^1(\Omega)$ generated by elliptic operators

In this section we recall the main properties of the operator $(\mathcal{A}, D(\mathcal{A}))$ and of the semigroup generated in $L^1(\Omega)$, and derive further estimates needed in the sequel. We collect the known results in the following statement.

Theorem 3.1 *Let $\Omega, \mathcal{A}, \mathcal{B}, D(\mathcal{A})$ be as specified in Section 2. Then, $(\mathcal{A}, D(\mathcal{A}))$ is sectorial and generates an analytic semigroup of contractions $(T(t))_{t\geq 0}$ in $L^1(\Omega)$; for the kernel*

$p : (0, +\infty) \times \Omega \times \Omega \to \mathbb{R}$ *of the semigroup* $(T(t))_{t \geq 0}$ *the following estimates hold; there exist* $b, c_0 > 0$ *such that for* $|\alpha|, |\beta| < 2$, $x, y \in \Omega$ *and* $t > 0$

$$|D_x^\alpha D_y^\beta p(t, x, y)| \leq \frac{c_0}{t^{\frac{N+|\alpha|+|\beta|}{2}}} e^{-b\frac{|x-y|^2}{t}}. \tag{3.1}$$

In addition, there are constants $c_i = c(\Omega, \lambda, M_0) > 0$, $i = 1, 2, 3$ *such that, since the operator is sectorial, the following holds:*

$$\|T(t)\|_{\mathcal{L}(L^1(\Omega))} \leq c_1, \quad \sqrt{t}\|DT(t)\|_{\mathcal{L}(L^1(\Omega))} \leq c_2, \quad t\|\mathcal{A}T(t)\|_{\mathcal{L}(L^1(\Omega))} \leq c_3, \tag{3.2}$$

for $t > 0$. *Moreover,* $D(\mathcal{A})$ *is continuously embedded in* $W^{1,1}(\Omega)$, *i.e., there exists* $c_4 = c(\Omega, \lambda, M_0) > 0$ *such that* $v \in D(\mathcal{A})$ *implies* $v \in W^{1,1}(\Omega)$ *and*

$$\|v\|_{W^{1,1}(\Omega)} \leq c_4(\|v\|_{L^1(\Omega)} + \|\mathcal{A}v\|_{L^1(\Omega)}); \tag{3.3}$$

Finally,

$$\lim_{t \to 0} \|T(t)v - v\|_{W^{1,1}(\Omega)} = 0 \tag{3.4}$$

for every $v \in D(\mathcal{A})$.

Proof. Estimates (3.1) are proved in [17, Theorem 5.7]; from these, estimates (3.2) and (3.3) follow. To prove (3.4), if $v \in D(\mathcal{A})$, then $T(t)\mathcal{A}v = \mathcal{A}T(t)v$ and by the strong continuity of $T(t)$ in $L^1(\Omega)$ we get

$$\begin{aligned} \|DT(t)v - Dv\|_{L^1(\Omega)} &\leq c_4 \left(\|T(t)v - v\|_{L^1(\Omega)} + \|\mathcal{A}T(t)v - \mathcal{A}v\|_{L^1(\Omega)}\right) \\ &= c_4 \left(\|T(t)v - v\|_{L^1(\Omega)} + \|T(t)\mathcal{A}v - \mathcal{A}v\|_{L^1(\Omega)}\right) \end{aligned}$$

and the statement is proved. □

Remark 3.2 **[Neumann boundary conditions]** We have stated Theorem 3.1 in the form we most frequently use, but the estimates stated are known to hold under more general assumptions. In particular, all non tangential boundary conditions are allowed. We denote by c_ν a constant which can be used in the first two inequalities in (3.2), when Neumann boundary conditions are associated with a general uniformly elliptic operator.

Remark 3.3 **[Contractivity]** We point out that the contractivity assumption on the semigroup is not restrictive for our purposes, as we may replace $\mathcal{A}$ by $\mathcal{A} - \omega$ for a suitable ω, getting a contractive semigroup $(T^\omega(t))_{t \geq 0}$ whose kernel is nothing but $p_\omega(t, x, y) = e^{\omega t} p(t, x, y)$. Since we are interested only in the behaviour of $(T(t))_{t \geq 0}$ for small t, it is not restrictive to assume contractivity from the beginning.

In order to proceed, we also need a precise L^1-estimate of the second (spatial) derivatives of $T(t)u_0$, for $u_0 \in W^{1,1}(\Omega)$. This is proved in Proposition 3.4 below. The argument used here is similar to the one used in [10, Theorem 2.4], where Ω is bounded and different boundary conditions are imposed.

Proposition 3.4 *Let* $\Omega, \mathcal{A}, \mathcal{B}$ *be as in Section 2, but assume, in addition,* $A \in W^{2,\infty}(\Omega)$ *and* $B, C \in W^{1,\infty}(\Omega)$; *then, there exists* c_5 *depending on* N, λ, Ω, $\|A\|_{2,\infty}$, $\|B\|_{1,\infty}$, $\|C\|_{1,\infty}$, c_1, c_2, c_3 c_ν *such that for every* $t \in (0, 1)$ *and* $u_0 \in W^{1,1}(\Omega)$ *we have*

$$\sqrt{t}\|D^2 T(t)u_0\|_{L^1(\Omega)} \leq c_5 \|u_0\|_{W^{1,1}(\Omega)}. \tag{3.5}$$

Proof. Set

$$M_1 = \max\{\|A\|_{2,\infty}, \|B\|_{1,\infty}, \|C\|_{1,\infty}\}.$$

From the regularity of the boundary $\partial\Omega$ we can consider a partition of unity $\{(\eta_h, U_h)\}_{h\in\mathbb{N}}$ such that supp $\eta_h \subset U_h$, $\sum_{h=0}^{\infty} \eta_h(x) = 1$ for every $x \in \overline{\Omega}$ and $0 \le \eta_h \le 1$ for every $h \in \mathbb{N}$, $\overline{U}_0 \subset \Omega$, U_h for $h \ge 1$ is a ball such that $\{U_h\}_{h\ge 1}$ is a covering of $\partial\Omega$ and $\{U_h\}_{h\in\mathbb{N}}$ is a covering of Ω with bounded overlapping, that is there is $\kappa > 0$ such that

$$\sum_{h\in\mathbb{N}} \chi_{U_h}(x) \le \kappa, \quad \forall x \in \overline{\Omega}. \tag{3.6}$$

Moreover we choose η_h in such a way $\langle A(x)D\eta_h(x), \nu(x)\rangle = 0$ for every $x \in \partial\Omega$. By the uniform C^2 regularity of $\partial\Omega$, the η_h can be chosen in such a way that

$$M := \sup_{h\in\mathbb{N}} \|\eta_h\|_{2,\infty} < +\infty\,.$$

Now consider $u_0 \in W^{1,1}(\Omega)$ and denote by $u(t) = T(t)u_0$ the solution of the problem

$$\begin{cases} \partial_t w - \mathcal{A}w = 0 & \text{in } (0,\infty)\times\Omega \\ w(0) = u_0 & \text{in } \Omega \\ \langle ADw, \nu\rangle = 0 & \text{in } (0,\infty)\times\partial\Omega. \end{cases} \tag{3.7}$$

We want to estimate the L^1–norm of $\sqrt{t}D^2u(t)$ by the $W^{1,1}$-norm of u_0. The functions $v_h(t) = u(t)\eta_h$ solve, for every $h \in \mathbb{N}$, the problem

$$\begin{cases} \partial_t w - \mathcal{A}w = \mathcal{A}_h u & \text{in } (0,\infty)\times\Omega \\ w(0) = \eta_h u_0 & \text{in } \Omega \\ \langle ADw, \nu\rangle = 0 & \text{in } (0,\infty)\times\partial\Omega \end{cases} \tag{3.8}$$

where

$$\mathcal{A}_h u = -2\langle AD\eta_h, Du\rangle - u\,\mathrm{div}(AD\eta_h) - u\,\langle B, D\eta_h\rangle\,. \tag{3.9}$$

Notice that the derivative $D_k v_h$ satisfies the equation

$$\partial_t(D_k v_h) - \mathcal{A}(D_k v_h) = \mathcal{A}_h^k u$$

where

$$\begin{aligned} \mathcal{A}_h^k u =& \mathrm{div}\left((D_k A)D(u\eta_h)\right) + \langle (D_k B), D(u\eta_h)\rangle + (D_k C)u\eta_h + D_k(\mathcal{A}_h u) \\ =& \mathrm{div}\left((D_k A)D(u\eta_h)\right) + \langle (D_k B), D(u\eta_h)\rangle + (D_k C)u\eta_h \\ &+ D_k[-2\langle AD\eta_h, Du\rangle - u\,\mathrm{div}(AD\eta_h) - u\,\langle B, D\eta_h\rangle] \end{aligned} \tag{3.10}$$

For $D_k v_h$ we consider the problem

$$\begin{cases} \partial_t w - \mathcal{A}w = \mathcal{A}_h^k u & \text{in } (0,\infty)\times\Omega \\ w(0) = D_k(\eta_h u_0) & \text{in } \Omega \\ \langle ADw, \nu\rangle = 0 & \text{in } (0,\infty)\times\partial\Omega \end{cases} \tag{3.11}$$

whose solution is

$$v_{hk(t)} = T(t)D_k(\eta_h u_0) + \int_0^t T(t-s)\mathcal{A}_h^k u(s)ds$$

Now we consider $h = 0$, i.e., we draw our attention to the inner part. Since $v_0 = \eta_0 u = 0$ in $\Omega \setminus U_0$, it turns out that $D_k v_0$ is the solution of (3.11) with $h = 0$. Then

$$D_k v_0(t) = T(t) D_k(\eta_0 u_0) + \int_0^t T(t-s)\mathcal{A}_0^k u(s) ds, \tag{3.12}$$

where $\mathcal{A}_0^k$ is the operator defined in (3.10). Then, differentiating, we obtain

$$D^2_{lk} v_0 = D_l[T(t) D_k(\eta_0 u_0)] + \int_0^t D_l[T(t-s)\mathcal{A}_0^k v(s)] ds.$$

by which, using (3.2),

$$\begin{aligned} \|D^2_{lk} v_0(t)\|_{L^1(\Omega)} &\le \|D_l T(t) D_k(\eta_0 u_0)\|_{L^1(\Omega)} + \int_0^t \|D_l T(t-s)\mathcal{A}_0^k u(s)\|_{L^1(\Omega)} ds \\ &\le \frac{c_2}{\sqrt{t}} \|\eta_0 u_0\|_{W^{1,1}(\Omega)} + \int_0^t \frac{c_2}{\sqrt{t-s}} \|\mathcal{A}_0^k u(s)\|_{L^1(\Omega)} ds. \end{aligned}$$

Finally, estimating $\|\mathcal{A}_0^k u(s)\|_{L^1(\Omega)}$ by (3.10) we get

$$\|\mathcal{A}_0^k u(s)\|_{L^1(\Omega)} \le c\, \|u(s)\|_{W^{2,1}(\Omega)}$$

where $c = c(M, M_1)$ Summing on l and k and using again (3.2), we get

$$\|D^2 v_0(t)\|_{L^1(\Omega)} \le c\, \frac{1}{\sqrt{t}} \|u_0\|_{W^{1,1}(\Omega)} + \int_0^t \frac{c}{\sqrt{t-s}} \|D^2 u(s)\|_{L^1(\Omega)} ds \tag{3.13}$$

where $c = c(M, M_1, c_1, c_2, c_3)$. We now consider $h \ge 1$, i.e., we consider a ball intersecting $\partial\Omega$. By the uniform C^2 regularity of $\partial\Omega$, we can consider coordinate functions $\psi_h : V_h \to B_1(0)$ such that $\psi_h(V_h \cap \Omega) = B_1^-(0) = \{y = (y', y_N) \in B_1(0) : y_N < 0\}$, $\psi_h(V_h \cap \partial\Omega) = \{y = (y', y_N) \in B_1(0) : y_N = 0\}$ and $d(\psi_h)_x(A(x)\nu(x)) = e_N$ for every $x \in \partial\Omega$ and that there is a constant M_ψ such that

$$\sup_{h \ge 1} \left\{ \|D^2 \psi_h\|_{2,\infty}, \|D^2 \psi_h^{-1}\|_{2,\infty} \right\} \le M_\psi.$$

Notice also that there is no loss of generality assuming that for all $h \ge 1$ the inclusion $U_h \subset\subset V_h$ holds, and that we can choose a C^2 domain E such that $\psi_h(U_h \cap \Omega) \subset E \subset B_1^-(0)$.

Using the transformation (for a generic f defined in $\Omega \cap V_h$)

$$\hat{f}(y) := f(\psi_h^{-1}(y))$$

and since v_h is the solution of (3.8), we get that for every $h \ge 1$ the function $\hat{v}_h(t, y) = \eta_h(\psi_h^{-1}(y)) v(t, \psi_h^{-1}(y))$ is the solution of the following initial-boundary value problem with homogeneous Neumann boundary conditions

$$\begin{cases} \partial_t w - \hat{\mathcal{A}} w = \hat{\mathcal{A}}_h \hat{v} & \text{in } (0, +\infty) \times E \\ w(0) = \hat{\eta}_h \hat{u}_0 & \text{in } E \\ \dfrac{\partial w}{\partial \nu} = 0 & \text{in } (0, +\infty) \times \partial E \end{cases} \tag{3.14}$$

where $\hat{\mathcal{A}}$ is the operator defined on $B_1(0)$ as follows

$$\hat{\mathcal{A}}w := \operatorname{div}(\hat{A}\, Dw) + \langle \hat{B}, Dw\rangle + \hat{C}w \tag{3.15}$$

defined by the coefficients (here for $\hat{\mathcal{A}}$ and its coefficients we omit the index h for simplify the notations and by analogy with (3.8))

$$\begin{aligned}
\hat{A}(y) :=& \big(D\psi_h\big)(\psi_h^{-1}(y)) \cdot A(\psi_h^{-1}(y)) \cdot \big(D\psi_h\big)^t(\psi_h^{-1}(y))\\
(\hat{B}(y))_l :=& \mathrm{Tr}\Big[\big(D\psi_h\big)(\psi_h^{-1}(y)) \cdot A(\psi_h^{-1}(y)) \cdot H^l(\psi_h^{-1}(y)) \cdot \big(D\psi_h^{-1}\big)^t(y)\Big]\\
&+ \mathrm{Tr}\Big[\big(D\psi_h\big)(\psi_h^{-1}(y)) \cdot G^j(y)\Big]\big(D\psi_h\big)^t_{jl}(\psi_h^{-1}(y)) - \frac{\partial}{\partial y_j}\Big[\hat{A}_{jl}(y)\Big]\\
&+ \Big[\big(D\psi_h\big)(\psi_h^{-1}(y)) \cdot B(\psi_h^{-1}(y))\Big]_l\\
\hat{C}(y) :=& C(\psi_h^{-1}(y))
\end{aligned}$$

where $H^l_{ki} = D^2_{ki}(\psi_h)_l$ and $G^j_{ki} = D_k A_{ij}(\psi_h^{-1}(y))$ and (see (3.9))

$$\hat{\mathcal{A}}_h\hat{u} = -2\big\langle A(\psi_h^{-1}(y))(D\psi_h)^t D\hat{\eta}_h, (D\psi_h)^t D\hat{u}\big\rangle - \hat{u}\big[\operatorname{div}(\hat{A}D\hat{\eta}_h) + \langle \hat{B}, D\hat{\eta}_h\rangle\big]\,.$$

Now, as done before for $h = 0$, differentiating the equation (now $D_k = \frac{\partial}{\partial y_k}$) we obtain that $D_k\hat{v}_h$ solves

$$\partial_t(D_k\hat{v}_h) - \hat{\mathcal{A}}(D_k\hat{v}_h) = \hat{\mathcal{A}}^k_h\hat{u}$$

where $\hat{\mathcal{A}}^k_h\hat{v}$ can be obtained by taking the corresponding term in (3.10). Associated with this operator, we can consider the problem

$$\begin{cases}
\partial_t w - \hat{\mathcal{A}}w = \hat{\mathcal{A}}^k_h\hat{u} & \text{in } (0,\infty)\times E\\
w(0) = D_k\big(\hat{\eta}_h\hat{u}_0\big) & \text{in } E\\
\dfrac{\partial w}{\partial \nu} = 0 & \text{in } (0,\infty)\times\partial E
\end{cases}$$

The function $D_k\hat{v}_h$ satisfies the equation and the initial condition. Notice that if $k \neq N$ also the boundary condition is satisfied since $\hat{v}_h = 0$ in a neighbourhood of $\partial E\cap\{y\in\mathbb{R}^N \mid y_N < 0\}$, in the other part of ∂E the operator D_k is a tangential derivative and $\frac{\partial\hat{v}_h}{\partial y_N}$ is constant for $y_N = 0$. Denote by S the semigroup which gives the solution of this problem and notice that the estimates (3.2) hold for $S(t)$, see Remark 3.2. Then

$$D_k\hat{v}_h(t) = S(t)D_k\hat{v}_h(0) + \int_0^t S(t-s)\hat{\mathcal{A}}^k_h\hat{u}(s)ds. \tag{3.16}$$

Differentiating (3.16) with respect to D_j for any j, we have then proved that the following holds

$$D^2_{kj}\hat{v}_h(t) = D_jS(t)D_k\hat{v}_h(0) + \int_0^t D_jS(t-s)\hat{\mathcal{A}}^k_h\hat{u}(s)ds. \tag{3.17}$$

Thus, as for v_0, we have for $(k,j)\neq(N,N)$

$$\|D^2_{kj}\hat{v}_h(t)\|_{L^1(E)} \le \frac{c_2}{\sqrt{t}}\|\hat{\eta}_h\hat{u}_0\|_{W^{1,1}(E)} + \int_0^t \frac{c_2}{\sqrt{t-s}}\|\hat{\mathcal{A}}^k_h\hat{u}(s)\|_{L^1(E)}ds.$$

We now estimate $D^2_{NN}\hat v_h(t)$. Since

$$\begin{aligned}\hat A_{NN} D^2_{NN}\hat v_h(t) &= \hat{\mathcal A}\hat v_h(t) - \sum_{(i,j)\neq(N,N)} \hat A_{ij} D^2_{ij}\hat v_h(t) - \sum_{i,j=1}^{N} (D_i\hat A_{ij}) D_j \hat v_h(t) \\ &\quad - \sum_{i=1}^{N} \hat B_i D_i \hat v_h(t) - \hat C \hat v_h(t).\end{aligned}$$

and since $\lambda^{-1} \le \hat A_{NN} \le \lambda$, we can find a constant c (depending only on N, M_1, λ) such that

$$\begin{aligned}\|D^2_{NN}\hat v_h(t)\|_{L^1(E)} &= \Big\| \frac{1}{\hat A_{NN}} \Big(\hat{\mathcal A}\hat v_h(t) - \sum_{(i,j)\neq(N,N)} \hat A_{ij} D^2_{ij}\hat v_h(t) + \\ &\quad - \sum_{i,j=1}^{N} (D_i\hat A_{ij}) D_j\hat v_h(t) - \sum_{i=1}^{N} \hat B_i D_i\hat v_h(t) - \hat C\hat v_h(t)\Big)\Big\|_{L^1(E)} \\ &\le c\Big[\sum_{(i,j)\neq(N,N)} \|D^2_{ij}\hat v_h(t)\|_{L^1(E)} + \|\hat{\mathcal A}\hat v_h(t)\|_{L^1(E)} + \\ &\quad + \|D\hat v_h(t)\|_{L^1(E)} + \|\hat v_h(t)\|_{L^1(E)}\Big]\end{aligned}$$

Summing up, arguing as for $h = 0$, we have

$$\|D^2\hat v_h(t)\|_{L^1(E)} \le c' \frac{1}{\sqrt t} \|u_0 \circ \psi_h^{-1}\|_{W^{1,1}(E)} + \int_0^t \frac{c}{\sqrt{t-s}} \|D^2\hat u(s)\|_{L^1(E)} ds,$$

where $c' = c(M, M_1, M_\psi, N, c_\nu)$. Coming back to $\Omega \cap U_h$ we obtain

$$\|D^2 v_h(t)\|_{L^1(\Omega\cap U_h)} \le c'' \frac{1}{\sqrt t} \|u_0\|_{W^{1,1}(\Omega\cap U_h)} + \int_0^t \frac{c'}{\sqrt{t-s}} \|D^2 u(s)\|_{L^1(\Omega\cap U_h)} ds, \tag{3.18}$$

where c'' depends on M, M_1, M_ψ, N, c_ν. Now, since (3.6) holds, we have

$$\begin{aligned}\|D^2 v(t)\|_{L^1(\Omega)} &= \|D^2\big(\sum_{h=0}^{\infty} v_h(t)\big)\|_{L^1(\Omega)} = \|\sum_{h=0}^{\infty} D^2 v_h(t)\|_{L^1(\Omega)} \\ &\le \kappa\Big[c'' \frac{1}{\sqrt t}\|u_0\|_{W^{1,1}(\Omega)} + \int_0^t \frac{c''}{\sqrt{t-s}} \|D^2 u(s)\|_{L^1(\Omega)} ds\Big].\end{aligned}$$

Then we can find a constant c depending on c'', κ such that

$$\|D^2 v(t)\|_{L^1(\Omega)} \le c\Big[\frac{1}{\sqrt t}\|u_0\|_{W^{1,1}(\Omega)} + \int_0^t \frac{1}{\sqrt{t-s}} \|D^2 u(s)\|_{L^1(\Omega)} ds\Big]. \tag{3.19}$$

Now using Gronwall's generalized inequality (see for instance [12, Lemma 7.1.1]), we get

$$\|D^2 v(t)\|_{L^1(\Omega)} \le c_5 \frac{1}{\sqrt t}\|u_0\|_{W^{1,1}(\Omega)} \qquad \text{for every } t \in (0,1).$$

□

4 BV functions

Let us recall the definition and the basic properties of functions with (possibly weighted) bounded variation on Ω. For $u \in L^1(\Omega)$, given a symmetric positive definite matrix $P = (P_{ij})_{i,j=1}^N$, we can define the weighted total variation, following [3], by setting

$$|Du|_P(\Omega) = \sup \left\{ \int_\Omega u \operatorname{div}\psi dx : \psi \in C_c^1(\Omega, \mathbb{R}^N), \|P^{-1/2}\psi\|_\infty \leq 1 \right\} \tag{4.1}$$

and say that u has finite total weighted variation, $u \in BV_P(\Omega)$, if $|Du|_P(\Omega) < +\infty$. A set E is said to have finite weighted perimeter if $|D\chi_E|_P(\Omega) < +\infty$. In this case, its total variation measure is the perimeter of E and it is denoted also by $P_P(E, \Omega) = |D\chi_E|_P(\Omega)$. Notice that if P has entries $P_{ij} \in C^1(\Omega)$, then the total variation can be equivalently defined by

$$|Du|_P(\Omega) = \sup \left\{ \int_\Omega u \operatorname{div}(P^{1/2}\phi) dx : \phi \in C_c^1(\Omega, \mathbb{R}^N), \|\phi\|_\infty \leq 1 \right\}.$$

Of course, if P is the identity matrix then $|Du|_P$ reduces to (1.2) and the weighted perimeter reduces to the classical perimeter, and in this case we write $u \in BV(\Omega)$ and drop the P everywhere. The space $BV_P(\Omega)$ turns out to be a Banach space with norm

$$\|u\|_{BV_P} = \|u\|_{L^1(\Omega)} + |Du|_P(\Omega).$$

The norm topology is in some respects too strong, since for instance smooth functions are not dense with respect to it. Nevertheless, a classical weaker approximation result is given by the Anzellotti-Giaquinta theorem, see e.g. [2, Theorem 3.9]. It states that for every $u \in BV(\Omega)$ there exists a sequence of functions $(u_n)_n \subset C^\infty(\Omega)$ such that

$$\|u - u_n\|_{L^1(\Omega)} \to 0, \qquad \int_\Omega |Du_n| dx \to |Du|(\Omega);$$

such a sequence is said to converge *in variation* to u.

Let us recall a particular case of [7, Lemma 2.4], i.e., the following coarea formula:

$$|Du|_P(\Omega) = \int_{\mathbb{R}} P_P(\{u > \tau\}, \Omega) d\tau \tag{4.2}$$

which we use later.

Henceforth, we assume the following:

(H4) P is a symmetric strictly positive definite matrix with $P_{ij} \in C_b(\overline{\Omega})$.

Under the above hypothesis, the seminorms $|Du|(\Omega)$ and $|Du|_P(\Omega)$ are equivalent.

We also notice that if u is regular, then the equality

$$|Du|_P(\Omega) = \int_\Omega |Du(x)|_P dx,$$

holds, where $|Du(x)|_P$ is defined in (2.1). We notice that the weighted total variation is the supremum of the $L^1(\Omega)$ continuous functionals

$$u \mapsto \int_\Omega u \operatorname{div}\psi dx$$

and therefore it is $L^1(\Omega)$ lower semicontinuous, i.e.,

$$|Du|_P(\Omega) \leq \liminf_{n\to+\infty} |Du_n|_P(\Omega)$$

for any sequence $(u_n)_n$ with $u_n \to u$ in $L^1(\Omega)$. In particular, if $(T(t))_{t\geq 0}$ is a strongly continuous semigroup on $L^1(\Omega)$, then

$$|Du_0|_P(\Omega) \leq \liminf_{t\to 0} \int_\Omega |DT(t)u_0|_P dx. \tag{4.3}$$

The Anzellotti-Giaquinta theorem can be adapted also to the case of weighted BV functions, as is done in the following result.

Proposition 4.1 *Let Ω, $P = (P_{ij})_{i,j=1}^N$ be as above, and let $Q = (Q_{ij})_{i,j=1}^N$ be an elliptic matrix with $Q_{ij} \in C_b^1(\overline{\Omega})$. Then, for every $u \in BV_P(\Omega)$ there exists a sequence of functions $(v_n)_n \subset C_Q(\Omega)$ such that*

$$\lim_{n\to\infty} \|u - v_n\|_{L^1(\Omega)} = 0, \quad \lim_{n\to\infty} \int_\Omega |Dv_n|_P dx = |Du|_P(\Omega).$$

Proof. The proof goes as the classical one, except that we have to modify the usual approximation sequence in a neighbourhood of the boundary of Ω. The assumption on the regularity on $\partial\Omega$ is used to modify the approximating sequence to make it constant in the direction $Q\nu$. □

Remark 4.2 A particular case of Proposition 4.1 is given when $Q = A$; in this case we have that $C_A(\Omega) \subset D(\mathcal{A})$ (it is a core), and then the weighted BV functions can be approximated in variation via functions in the domain of the operator $\mathcal{A}$.

For the weighted total variation also the following continuity property under uniform convergence holds.

Proposition 4.3 *Let $P = (P_{ij})_{i,j=1}^N$ be a symmetric λ-elliptic matrix valued function and let $(P_{(n)})_{n\in\mathbb{N}}$ be a sequence of matrices valued functions uniformly convergent to P. Then, for every $u \in L^1(\Omega)$ the following holds:*

$$\lim_{n\to+\infty} |Du|_{P_{(n)}}(\Omega) = |Du|_P(\Omega). \tag{4.4}$$

Proof. We denote by $c_n = \|P^{-1/2} - P_{(n)}^{-1/2}\|_\infty$; by the uniform convergence, we have that $c_n \to 0$ as $n \to +\infty$; moreover, we may assume that the $P_{(n)}$ are $(\lambda + 1/n)$-elliptic, that is

$$\frac{1}{\lambda + 1/n}|\xi|^2 \leq |P_{(n)}^{1/2}\xi|^2 \leq (\lambda + 1/n)|\xi|^2,$$

or, simply defining $w = P_{(n)}^{1/2}\xi$,

$$\frac{1}{\sqrt{\lambda + 1/n}}|w| \leq |P_{(n)}^{-1/2}w| \leq \sqrt{\lambda + 1/n}|w|.$$

Then, if $\psi \in C_c^1(\Omega, \mathbb{R}^N)$ with $\|P_{(n)}^{-1/2}\psi\|_\infty \le 1$, we get

$$\begin{aligned} \|P^{-1/2}\psi\|_\infty &\le \|P_{(n)}^{-1/2}\psi\|_\infty + \|(P^{-1/2} - P_{(n)}^{-1/2})\psi\|_\infty \\ &\le \|P_{(n)}^{-1/2}\psi\|_\infty + c_n\|\psi\|_\infty \\ &\le \|P_{(n)}^{-1/2}\psi\|_\infty + c_n\sqrt{\lambda + 1/n}\|P_{(n)}^{-1/2}\psi\|_\infty \\ &\le 1 + c_n\sqrt{\lambda + 1/n}. \end{aligned}$$

By definition of weighted variation, we get

$$\int_\Omega u\text{div}\psi dx \le (1 + c_n\sqrt{\lambda + 1/n})|Du|_P(\Omega)$$

whence

$$|Du|_{P_{(n)}}(\Omega) \le (1 + c_n\sqrt{\lambda + 1/n})|Du|_P(\Omega).$$

With a similar computation, we also get

$$|Du|_P(\Omega) \le (1 + c_n\sqrt{\lambda})|Du|_{P_{(n)}}(\Omega),$$

and then (4.4) follows by letting $n \to +\infty$. □

5 The first characterization of BV functions

In this section we prove that for $u_0 \in L^1(\Omega)$ the equality

$$\lim_{t\to 0} \int_\Omega |DT(t)u_0|_P \, dx = |Du_0|_P(\Omega), \tag{5.1}$$

holds, where $(T(t))_{t\ge 0}$ is the semigroup generated by the operator $(\mathcal{A}, D(\mathcal{A}))$ in (2.4) and $|Du|_P(\Omega)$ is defined in (4.1).

Notice that, by the result obtained in Section 4, equality (5.1) holds for $u_0 \in D(\mathcal{A})$, see (3.3), and moreover by semicontinuity inequality (4.3) always holds.

In this section we require more regularity on the coefficients A_{ij}, and replace hypothesis **(H3)** with the following:

(H3)$'$ $A_{ij} \in W^{2,\infty}(\Omega)$, $B, C \in L^\infty(\Omega)$.

We need the following result, which gives a localized version of (5.1). Here we state the proposition for an operator $\mathcal{A}$ such as that defined in (2.4), even if we need the result only for operators with B and C null, i.e. for operators of the type $\mathcal{A}u = \text{div}(ADu)$, in the proof of Theorem 5.2.

Proposition 5.1 *Let $v \in D(\mathcal{A})$, where $\mathcal{A}$ is as in (2.4), with coefficients $A_{ij} \in W^{2,\infty}(\Omega)$, $B_i, C \in W^{1,\infty}(\Omega)$. Let $P = (P_{ij})_{i,j=1}^N$ be a non-negative λ-elliptic matrix with $P_{ij} \in W^{1,\infty}(\Omega)$ and $P_{ij} = A_{ij}$ on $\partial\Omega$. Then for every $\eta \in C_b^1(\overline{\Omega})$, η non-negative, there exists a constant*

$$c_6 = c(M_1, N, \|P\|_{1,\infty}, \|\eta\|_{1,\infty}, \lambda)$$

such that

$$\int_\Omega \eta|DT(t)v|_P \, dx \le \int_\Omega \eta|Dv|_P \, dx + c_6\sqrt{t}\|v\|_{W^{1,1}(\Omega)} \tag{5.2}$$

holds for every $t \in (0, 1)$.

Proof. For $v \in D(\mathcal{A})$ and $\eta \in C_b^1(\overline{\Omega})$, $\eta \geq 0$, we define the function $F_\eta : (0,1) \to \mathbb{R}$ by

$$F_\eta(t) = \int_\Omega \eta |DT(t)v|_P \, dx.$$

This function is differentiable since $T(t)v$ is regular for every $t > 0$ and the equality

$$\partial_t |DT(t)v|_P = \frac{1}{|DT(t)v|_P} \langle PDT(t)v, D\mathcal{A}T(t)v \rangle$$

holds for a.e. $x \in \Omega$. Moreover, $T(t)v \in D(\mathcal{A})$ for every $t > 0$ and then

$$\mathcal{A}T(t)v = T\Big(\frac{t}{2}\Big)\mathcal{A}T\Big(\frac{t}{2}\Big)v;$$

this implies also that $\mathcal{A}T(t)v \in D(\mathcal{A})$. Then, thanks to (3.3) and from the fact that

$$\frac{|\langle PDT(t)v, D\mathcal{A}T(t)v \rangle|}{|DT(t)u_0|_P} \leq |D\mathcal{A}T(t)v|_P,$$

we can differentiate under the integral sign. Denoting by $u(t,x)$ the solution $(T(t)v)(x)$, we obtain

$$\begin{aligned}
F'_\eta(t) &= \frac{d}{dt}\int_\Omega \eta |Du|_P \, dx = \int_\Omega \frac{\eta}{|Du|_P} \langle PDu, D\mathcal{A}u \rangle \, dx \\
&= \sum_{i,j,h,k=1}^N \int_\Omega \eta \frac{P_{ij} D_j u D_i (D_h (A_{hk} D_k u))}{|Du|_P} \, dx \\
&\quad + \sum_{i,j,h=1}^N \int_\Omega \eta \frac{P_{ij} D_j u D_i (B_h D_h u)}{|Du|_P} \, dx + \sum_{i,j=1}^N \int_\Omega \eta \frac{P_{ij} D_j u D_i (Cu)}{|Du|_P} \, dx \\
(I_1) &= \sum_{i,j,h,k=1}^N \int_\Omega \eta \frac{P_{ij} D_j u (D^2_{ih} A_{hk} D_k u + D_h A_{hk} D^2_{ik} u + D_i A_{hk} D^2_{hk} u)}{|Du|_P} \, dx \\
(I_2) &\quad + \sum_{i,j,h,k=1}^N \int_\Omega \eta \frac{1}{|Du|_P} P_{ij} D_j u \, A_{hk} D^3_{ihk} u \, dx \\
(I_3) &\quad + \sum_{i,j,h,k=1}^N \int_\Omega \eta \frac{1}{|Du|_P} P_{ij} D_j u \Big(D_i B_h D_h u + B_h D^2_{ih} u \Big) \, dx \\
(I_4) &\quad + \sum_{i,j,h,k=1}^N \int_\Omega \eta \frac{1}{|Du|_P} P_{ij} D_j u \Big(D_i C \, u + C D_i u \Big) \, dx \, .
\end{aligned}$$

Notice that there is a constant $c_8 = c(N, M_1, \|\eta\|_\infty, \|P\|_\infty)$ such that

$$|I_1| + |I_3| + |I_4| \leq c_8 \|u\|_{W^{2,1}(\Omega)} \, .$$

It remains to estimate I_2; integrating by parts with respect to x_k, we have that

$$
\begin{aligned}
& \sum_{i,j,h,k=1}^{N} \int_\Omega \frac{\eta}{|Du|_P} P_{ij} D_j u\, A_{hk} D^3_{ihk} u\, dx \\
(II_1) \quad =\ & \frac{1}{2} \sum_{i,j,h,k,l,m=1}^{N} \int_\Omega \frac{\eta}{|Du|_P^3} P_{ij} D_j u\, A_{hk} D^2_{ih} u D_k P_{lm} D_m u D_l u\, dx \\
(II_2) \quad & + \sum_{i,j,h,k,l,m=1}^{N} \int_\Omega \frac{\eta}{|Du|_P^3} P_{ij} D_j u\, A_{hk} D^2_{ih} u\, P_{lm} D_m u D^2_{kl} u\, dx \\
(II_3) \quad & - \sum_{i,j,h,k=1}^{N} \int_\Omega \frac{\eta}{|Du|_P} \Big(D_k P_{ij} D_j u\, A_{hk} + P_{ij} D_j u D_k A_{hk} \Big) D^2_{ih} u\, dx \\
(II_4) \quad & - \sum_{i,j,h,k=1}^{N} \int_\Omega \frac{\eta}{|Du|_P} P_{ij} D^2_{kj} u\, A_{hk} D^2_{ih} u\, dx \\
(II_5) \quad & - \sum_{i,j,h,k=1}^{N} \int_\Omega \frac{1}{|Du|_P} P_{ij} D_j u\, A_{hk} D^2_{ih} u\, D_k \eta\, dx \\
(II_6) \quad & + \sum_{i,j,h,k=1}^{N} \int_{\partial\Omega} \frac{\eta}{|Du|_P} P_{ij} D_j u\, A_{hk} D^2_{ih} u\, \nu_k\, d\mathcal{H}^{N-1}
\end{aligned}
$$

This implies the existence of a constant $c_9 = c(M_1, \|P\|_{1,\infty}, \|\eta\|_{1,\infty})$, such that

$$
|II_1| + |II_3| + |II_5| \le c_9 \int_\Omega |D^2 u|\, dx .
$$

Notice that for II_2 we have

$$
\begin{aligned}
\sum_{i,j,k,l,m=1}^{N} P_{ij} D_j u A_{hk} D^2_{ih} u\, P_{lm} D_m u D^2_{kl} u &= \big\langle D^2 u\, A\, D^2 u\, P Du, P Du \big\rangle \\
&= \Big\langle P^{1/2} D^2 u\, A\, D^2 u\, P^{1/2} (P^{1/2} Du), P^{1/2} Du \Big\rangle,
\end{aligned}
$$

and for II_4 we can write

$$
\begin{aligned}
\sum_{i,j,h,k=1}^{N} P_{ij} D^2_{kj} u\, A_{hk} D^2_{ih} u &= \sum_{i,j,h,k=1}^{N} P^{1/2}_{im} P^{1/2}_{mj} D^2_{kj} u\, A_{hk} D^2_{ih} u \\
&= \mathrm{Tr}\, \Big(P^{1/2} D^2 u\, A\, D^2 u\, P^{1/2} \Big),
\end{aligned}
$$

where Tr denotes the trace of a matrix. Then

$$
\begin{aligned}
II_2 + II_4 &= \int_\Omega \frac{1}{|Du|_P} \bigg(\Big\langle P^{1/2} D^2 u\, A\, D^2 u\, P^{1/2} \frac{P^{1/2} Du}{|Du|_P}, \frac{P^{1/2} Du}{|Du|_P} \Big\rangle \\
&\quad - \mathrm{Tr}\, \Big(P^{1/2} D^2 u\, A\, D^2 u\, P^{1/2} \Big) \bigg)\, \eta\, dx \le 0
\end{aligned}
\tag{5.3}
$$

since $P^{1/2} D^2 u\, A\, D^2 u\, P^{1/2}$ is positive definite because

$$\Big\langle (P^{1/2} D^2 u\, A\, D^2 u\, P^{1/2})\xi, \xi \Big\rangle = \Big\langle A^{1/2} D^2 u\, P^{1/2}\xi, A^{1/2} D^2 u\, P^{1/2}\xi \Big\rangle.$$

Finally, for the term II_6, we notice that

$$\begin{aligned}
&\sum_{i,j,h,k=1}^{N} P_{ij} D_j u\, A_{hk} D^2_{ih} u\, \nu_k = \sum_{i=1}^{N} \Big(\sum_{h,k=1}^{N} A_{hk} D^2_{ih} u\, \nu_k \sum_{j=1}^{N} P_{ij} D_j u \Big) \\
(5.4) \qquad &= \sum_{i=1}^{N} \sum_{h,k=1}^{N} \Big(D_i \Big(A_{hk} D_h u\, \nu_k \Big) - D_h u D_i (A_{hk} \nu_k) \Big) \sum_{j=1}^{N} P_{ij} D_j u \\
&= \langle D \langle A D u, \nu \rangle, P D u \rangle - \langle D(A\nu) D u, P D u \rangle \\
&= -\langle D(A\nu) D u, P D u \rangle
\end{aligned}$$

since $P \equiv A$ on $\partial\Omega$. Observe that the regularity of the boundary and the ellipticity of A_{ij} imply that there exists a constant $c_{10} = c(L, M_1)$ such that $|D(A\nu)| \le c_{10}$. As a consequence, we obtain that

$$\begin{aligned}
&\Big| \sum_{i,j,h,k=1}^{N} \int_{\partial\Omega} \frac{1}{|Du|_P} \eta\, P_{ij} D_j u\, A_{hk} D^2_{ih} u\, \nu_k \, d\mathcal{H}^{N-1} \Big| \\
= \; & \Big| \int_{\partial\Omega} \frac{1}{|Du|_P} \eta \, \langle D(A\nu) Du, PDu \rangle \, d\mathcal{H}^{N-1} \Big| \\
\le \; & c_{10} \int_{\partial\Omega} \eta |Du|_P d\mathcal{H}^{N-1} \\
\le \; & c_{10} \|\eta\|_\infty \sqrt{\lambda} \int_{\partial\Omega} |Du| d\mathcal{H}^{N-1} \\
\le \; & c_{11} \int_{\Omega} \big[|Du| + |D^2 u| \big] \, dx \,,
\end{aligned}$$

where $c_{11} = c(M_1, L, \lambda, \|\eta\|_\infty, c_\Omega)$, where c_Ω is introduced in (2.3).

Taking now into account that $u(t,x)$ satisfies (3.2) and (3.5), we have proved there is a constant c depending only on the dimension N, on the geometry of Ω, on the norms $\|P\|_{1,\infty}$, $\|\eta\|_{1,\infty}$ and on the constant M_1, c_1 such that for every $t > 0$ the inequality

$$F'_\eta(t) = \frac{d}{dt} \int_\Omega \eta |Du|_P \, dx \le c \left(\|u_0\|_{L^1(\Omega)} + \frac{1}{\sqrt{t}} \|u_0\|_{W^{1,1}(\Omega)} \right) \le c \frac{1}{\sqrt{t}} \|u_0\|_{W^{1,1}(\Omega)}.$$

holds. Then, by integration (5.2) follows. □

Theorem 5.2 *Assume* **(H1), (H2)**, $(\mathbf{H3})'$, **(H4)**, *and let $(T(t))_{t\ge 0}$ be the semigroup generated by $(\mathcal{A}, D(\mathcal{A}))$ in $L^1(\Omega)$. Then, for every $u_0 \in L^1(\Omega)$, the equality*

$$\lim_{t \to 0} \int_\Omega |DT(t)u_0(x)|_P \, dx = |Du_0|_P(\Omega)$$

holds. In particular, u_0 belongs to $BV(\Omega)$ if and only if the above limit is finite.

PROOF. We start first assuming that $P_{ij} \in C_b^2(\overline{\Omega})$ and considering the operator $\hat{\mathcal{A}}u = \mathrm{div}(ADu)$, i.e., $B_i = C = 0$, $i = 1, \dots N$, and the generated semigroup, denoted $\hat{T}$. Thanks to (4.3), we have only to prove that

$$\limsup_{t\to 0} \int_\Omega |D\hat{T}(t)u_0(x)|_P\, dx \le |Du_0|_P(\Omega), \tag{5.5}$$

which is trivially satified if $u_0 \in L^1(\Omega) \setminus BV(\Omega)$. We then consider $u_0 \in BV(\Omega)$. Fix $\varepsilon > 0$ and consider two open neighbourhoods $U \subset V$ of $\partial\Omega$ with disjoint boundaries such that, if we take $S' = \Omega \cap U$ and $S = \Omega \cap \overline{V}$, we get

$$|Du_0|_P(S) < \varepsilon. \tag{5.6}$$

Let then $\eta \in C^2(\Omega)$ be a function such that

$$0 \le \eta \le 1, \quad \eta \equiv 1 \text{ on } S', \quad \eta \equiv 0 \text{ on } \Omega \setminus S$$

and define the matrix

$$P_A = \eta^2 A + (1 - \eta^2)P.$$

By Proposition 4.1 there exists a sequence

$$\begin{aligned}(u_n)_n &\subset \left\{v \in C_c^\infty(\overline{\Omega}) : \langle ADv, \nu\rangle = 0 \text{ on } \partial\Omega\right\}\\ &= \left\{v \in C_c^\infty(\overline{\Omega}) : \langle P_A Dv, \nu\rangle = 0 \text{ on } \partial\Omega\right\}\\ &\subset D(\mathcal{A})\end{aligned}$$

such that $u_n \to u_0$ in $L^1(\Omega)$ and

$$\lim_{n\to+\infty} \int_\Omega |Du_n|_P\, dx = |Du_0|_P(\Omega).$$

Notice that since P and A are λ-elliptic we get

$$\int_\Omega |Du_n| dx \le \sqrt{\lambda} \int_\Omega |Du_n|_P dx$$

and then there exists $M > 0$ such that

$$\|u_n\|_{W^{1,1}(\Omega)} \le M. \tag{5.7}$$

Since $\Omega \setminus S$ is an open set, by lower semicontinuity we have

$$|Du_0|_P(\Omega \setminus S) \le \liminf_{n\to+\infty} \int_{\Omega\setminus S} |Du_n|_P dx$$

and also

$$\int_S |Du_n|_P dx = \int_\Omega |Du_n|_P dx - \int_{\Omega\setminus S} |Du_n|_P dx$$

whence

$$\begin{aligned}\limsup_{n\to+\infty} \int_S |Du_n|_P dx &\le \lim_{n\to+\infty} \int_\Omega |Du_n|_P dx - \liminf_{n\to+\infty} \int_{\Omega\setminus S} |Du_n|_P dx\\ &\le |Du_0|_P(\Omega) - |Du_0|_P(\Omega \setminus S) = |Du_0|_P(S).\end{aligned}$$

This proves that

$$\limsup_{n\to+\infty}\int_S |Du_n|_P dx \le |Du_0|_P(S); \tag{5.8}$$

by the λ-ellipticity of A and P, the following holds:

$$\begin{aligned}\limsup_{n\to+\infty}\int_S |Du_n|_A dx &= \limsup_{n\to+\infty}\int_S \langle ADu_n, Du_n\rangle^{1/2} dx\\ &= \limsup_{n\to+\infty}\int_S \left\langle AP^{-1}PDu_n, Du_n\right\rangle^{1/2} dx\\ &\le \lambda \limsup_{n\to+\infty}\int_S |Du_n|_P dx,\end{aligned}$$

whence by (5.8) and (5.6)

$$\limsup_{n\to+\infty}\int_S |Du_n|_A dx \le \lambda\varepsilon. \tag{5.9}$$

We also notice that

$$\begin{aligned}|\xi|_P^2 &= \langle P\xi,\xi\rangle = \langle P_A\xi,\xi\rangle + \langle (P-P_A)\xi,\xi\rangle\\ &= \langle P_A\xi,\xi\rangle + \eta^2\langle (P-A)\xi,\xi\rangle\\ &= |\xi|_{P_A}^2 + \eta^2\langle (P-A)\xi,\xi\rangle\end{aligned}$$

and, since P, A and A^{-1} are λ-elliptic,

$$|\langle (P-P_A)\xi,\xi\rangle| \le 2\lambda|\xi|^2 \le 2\lambda^2\langle A\xi,\xi\rangle, \quad \forall \xi\in\mathbb{R}^N.$$

We have then obtained that $|\xi|_P \le |\xi|_{P_A} + \lambda\sqrt{2}\eta|\xi|_A$ and as a consequence

$$\int_\Omega |D\hat{T}(t)u_n|_P dx \le \int_\Omega |D\hat{T}(t)u_n|_{P_A} dx + \lambda\sqrt{2}\int_\Omega \eta|D\hat{T}(t)u_n|_A dx.$$

We can apply Proposition 5.1 to both terms in the right hand side in order to obtain, using (5.7), that

$$\int_\Omega |D\hat{T}(t)u_n|_P dx \le \int_\Omega |Du_n|_{P_A} dx + \lambda\sqrt{2}\int_\Omega \eta|Du_n|_A dx + (1+\lambda\sqrt{2})c_9 M\sqrt{t}.$$

By definition of P_A, we have that

$$|\xi|_{P_A}^2 = \eta^2|\xi|_A^2 + (1-\eta^2)|\xi|_P^2, \quad \forall \xi\in\mathbb{R}^N,$$

and then

$$\begin{aligned}\int_\Omega |Du_n|_{P_A} dx &\le \int_\Omega \eta|Du_n|_A dx + \int_\Omega \sqrt{1-\eta^2}|Du_n|_P dx\\ &\le \int_S |Du_n|_A dx + \int_\Omega |Du_n|_P dx.\end{aligned}$$

We have then obtained the following estimate

$$\int_\Omega |D\hat{T}(t)u_n|_P dx \leq \int_\Omega |Du_n|_P dx + (1+\lambda\sqrt{2}) \int_S |Du_n|_A dx + (1+\lambda\sqrt{2})c_9 M\sqrt{t}.$$

Using (5.9) and the fact that $\hat{T}(t)u_n \to \hat{T}(t)u_0$ in $L^1(\Omega)$ as $n \to +\infty$, we get

$$\begin{aligned}\int_\Omega |D\hat{T}(t)u_0|_P dx &\leq \liminf_{n\to+\infty} \int_\Omega |D\hat{T}(t)u_n|_P dx \leq \limsup_{n\to+\infty} \int_\Omega |D\hat{T}(t)u_n|_P dx \\ &\leq |Du_0|_P(\Omega) + \lambda(1+\lambda\sqrt{2})\varepsilon + (1+\lambda\sqrt{2})c_9 M\sqrt{t}\end{aligned}$$

and the result for P regular then follows by letting $t \to 0$, since ε is arbitrary. The case with $P_{ij} \in C_b(\overline{\Omega})$ is a consequence of the approximation result given in Proposition 4.3.

Finally, we consider non zero coefficients B_i and C and $\mathcal{A}u = \operatorname{div}(ADu) + \langle B, Du\rangle + Cu$ with $B_i, C \in L^\infty(\Omega)$, $i = 1,\dots N$. Notice that the boundary operators associated with $\mathcal{A}$ and $\hat{\mathcal{A}}$ as in (2.5) coincide, and then the set $C_A(\Omega)$ defined in (2.2) is a core both for $(\mathcal{A}, D(\mathcal{A}))$ and $(\hat{\mathcal{A}}, D(\hat{\mathcal{A}}))$. We denote by $(T(t))_{t\geq 0}$ the semigroup generated by $\mathcal{A}, D(\mathcal{A}))$. Notice that if we define $\hat{u}(t) := \hat{T}(t)u_0$ and $u = T(t)u_0$ the function $w := \hat{u} - u$ is the solution of the problem

$$\begin{cases} \partial_t w - \mathcal{A}w = \mathcal{E}\hat{u} := -\langle B, D\hat{u}\rangle - C\hat{u} & \text{in } (0,\infty)\times\Omega \\ w(0) = 0 & \text{in } \Omega \\ \langle ADw, \nu\rangle = 0 & \text{in } (0,\infty)\times\partial\Omega\,. \end{cases}$$

Thus, since $w(t) = \int_0^t T(t-s)\mathcal{E}\hat{u}(s)ds$, we get

$$Dw(t) = D(\hat{u} - u)(t) = \int_0^t DT(t-s)\mathcal{E}\hat{u}(s)ds$$

and then using (3.2)

$$\begin{aligned}\|D\hat{u}(t) - Du(t)\|_{L^1(\Omega)} &\leq c_2\|\mathcal{E}\hat{u}\|_{L^1(\Omega)} \int_0^t \frac{1}{\sqrt{t-s}} ds \\ &\leq 2c_2\sqrt{t}\left(\|B\|_\infty \|D\hat{u}(t)\|_{L^1(\Omega)} + \|C\|_\infty \|\hat{u}(t)\|_{L^1(\Omega)}\right).\end{aligned}$$

Since $\|\hat{u}(t)\|_{L^1(\Omega)} \to \|u_0\|_{L^1(\Omega)}$ and by what seen before $\limsup_{t\to 0}\|D\hat{u}(t)\|_{L^1(\Omega)}$ is bounded we can conclude that $\lim_{t\to 0}\|D\hat{u}(t) - Du(t)\|_{L^1(\Omega)} = 0$ and consequently, for $v \in C_A(\Omega)$, it follows

$$\begin{aligned}&\limsup_{t\to 0}\int_\Omega |DT(t)v|_P\, dx \leq \\ &\qquad\leq \limsup_{t\to 0}\int_\Omega |DT(t)v|_P\, dx + \lim_{t\to 0}\int_\Omega |D\hat{T}(t)v - DT(t)v|_P\, dx \\ &\qquad\leq \int_\Omega |Dv|_P\, dx.\end{aligned}$$

The thesis then follows from the density of $C_A(\Omega)$ in $BV_P(\Omega)$ (see Proposition 4.1); given $u_0 \in BV_P(\Omega)$, we take a sequence $(u_n) \subset C_A(\Omega)$ approximating u_0 in P-variation. Then

$$\begin{aligned}
|Du_0|_P(\Omega) &\le \liminf_{t\to 0} \int_\Omega |DT(t)u_0|_P dx \le \liminf_{t\to 0} \liminf_{n\to+\infty} \int_\Omega |DT(t)u_n|_P dx \\
&\le \limsup_{t\to 0} \limsup_{n\to+\infty} \Big(\int_\Omega |Du_n|_P dx + c_6\sqrt{t}\|u_n\|_{W^{1,1}} \Big) \\
&\le \limsup_{t\to 0} \Big(|Du_0|_P(\Omega) + c_6 M\sqrt{t} \Big) = |Du_0|_P(\Omega).
\end{aligned}$$

□

We end with the discussion of the simplest application of Theorem 5.2.

Example 5.3 Of course, the simplest case is with $A = P = I$, $B = C = 0$, i.e., $(T(t))_{t\ge 0}$ is the heat semigroup generated by the Neumann Laplacian and the total variation is the classical (nonweighted) one. In this case, it is easily seen that $F(t) = \|DT(t)u_0\|_{L^1(\Omega)}$ is decreasing (as is the case if $\Omega = \mathbb{R}^N$), *provided that* Ω *is convex.* In fact, our computations significantly simplify and go as follows, where as in the proof of Theorem 5.2 we set $u(x,t) = (T(t)u_0)(x)$:

$$\begin{aligned}
F'(t) &= \int_\Omega \partial_t |Du|\, dx = \int_\Omega \frac{1}{|Du|}\langle Du, D\partial_t u\rangle\, dx = \int_\Omega \frac{1}{|Du|} \sum_{i,k} D_i u D_i D^2_{kk} u\, dx \\
&= \int_{\partial\Omega} \frac{1}{|Du|} \sum_{i,k} D_i u D^2_{ik} u \nu_k \, d\mathcal{H}^{N-1} - \int_\Omega \sum_{i,k} D_k \frac{D_i u}{|Du|} D^2_{ik} u\, dx \\
&= -\int_{\partial\Omega} \frac{1}{|Du|}\langle D\nu Du, Du\rangle\, d\mathcal{H}^{N-1} - \int_\Omega \left[\Big| D^2 u \frac{Du}{|Du|}\Big|^2 - \mathrm{Tr}\,(D^2u)^2 \right] dx \le 0
\end{aligned}$$

where we have taken into account (5.3), (5.4) and the fact that if Ω is convex then all the curvatures (i.e., the eigenvalues of the matrix $D\nu$) are non-negative.

Notice that it has been proved in [11, Theorem 2.16] that there is a (nonconvex) Ω such that $F'(0) > 0$.

6 A second characterization of BV functions

In this section we give a characterization of BV functions using in a different way the semigroup generated by the operator $\mathcal{A}$, in the spirit of [15]; more precisely, we prove that for every $u \in L^1(\Omega)$ equality (1.4) holds. The right hand side there significantly simplifies if $u = \chi_E$ is the characteristic function of a measurable set $E \subset \mathbb{R}^N$ of finite perimeter, and reads

$$\lim_{t\to 0} \sqrt{\frac{\pi}{t}} \int_{E^c\cap\Omega} T(t)\chi_E dx = \int_{\mathcal{F}E\cap\Omega} |A^{1/2}(x)\nu_E(x)| d\mathcal{H}^{N-1}(x), \tag{6.1}$$

where $\mathcal{F}E$ is the reduced boundary of E. Indeed, we first prove (6.1), and then, using it in connection with the coarea formula, we deduce (1.4). Let us recall some of the main notions on sets of finite perimeter; for a detailed description see for instance [2]. We denote by $\mathcal{F}E$ the reduced boundary of E, defined as the set

$$\mathcal{F}E = \left\{ x \in \mathrm{supp}\,|D\chi_E| : \exists \lim_{\varrho\to 0} \frac{D\chi_E(B_\varrho(x))}{|D\chi_E|(B_\varrho(x))} = \nu_E(x), \text{ and } |\nu_E(x)| = 1 \right\}.$$

Moreover, we denote by E^α the set of points of $\mathbb{R}^N$ with density α at x, that is

$$E^\alpha = \left\{ x \in \mathbb{R}^N : \exists \lim_{\varrho \to 0} \frac{|E \cap B_\varrho(x)|}{|B_\varrho(x)|} = \alpha \right\};$$

the essential boundary is then defined as $\partial^* E = \mathbb{R}^N \setminus (E^0 \cup E^1)$. The main properties of sets of finite perimeter we need are that $\mathcal{F}E \subset E^{1/2}$ and that $\mathcal{H}^{N-1}(\partial^* E \setminus \mathcal{F}E) = 0$.

For every $t > 0$ and $x_0 \in \Omega$, we set

$$\Omega^{t,x_0} = \frac{\Omega - x_0}{\sqrt{t}} = \left\{ y \in \mathbb{R}^N : x_0 + \sqrt{t} y \in \Omega \right\}$$

and, given $f : \Omega \to \mathbb{R}$,

$$f^{t,x_0}(y) = f(x_0 + \sqrt{t} y);$$

with this notation, we define the operator $\mathcal{A}^{t,x_0}$ on Ω^{t,x_0} by

$$\begin{aligned}
\mathcal{A}^{t,x_0}(y)v(y) &= \operatorname{div}(A^{t,x_0}(y)Dv(y)) + \sqrt{t}\langle B^{t,x_0}(y), Dv(y)\rangle + tC^{t,x_0}(y)v(y) \\
&= \sum_{h,k=1}^N A_{hk}(x_0 + \sqrt{t}y) \frac{\partial^2 v}{\partial y^h \partial y^k}(y) \\
&\quad + \sqrt{t} \sum_{k=1}^N \left(\sum_{h=1}^N D_h A_{hk}(x_0 + \sqrt{t}y) \right) \frac{\partial v}{\partial y^k}(y) \\
&\quad + \sqrt{t} \sum_{h=1}^N B_h(x_0 + \sqrt{t}y) \frac{\partial v}{\partial y^h}(y) + tC(x_0 + \sqrt{t}y)v(y),
\end{aligned}$$

and the operator $\mathcal{A}^x$ on $\mathbb{R}^N$ by

$$\mathcal{A}^x v(y) = \sum_{h,k=1}^N A_{hk}(x) \frac{\partial^2 v}{\partial y^h \partial y^k}(y).$$

By setting $x = x_0 + \sqrt{t}y$, it is easily seen that $\mathcal{A}^{t,x_0}(y) = t\mathcal{A}(x)$. We have the following Lemma.

Lemma 6.1 *Setting $u(s,x) = T(s)u_0(x)$, we can define the function $v : (0,+\infty) \times \Omega^{t,x_0} \to \mathbb{R}$ by $v(s,y) = u(ts, x_0 + \sqrt{t}y)$; then v is the solution of the problem*

$$\left\{ \begin{array}{ll} \partial_s w = \mathcal{A}^{t,x_0}(y)w & \text{in } (0,+\infty) \times \Omega^{t,x_0} \\ w(0,y) = u_0^{t,x_0}(y) & \text{in } \Omega^{t,x_0} \\ \langle A^{t,x_0} Dw, \nu \rangle = 0 & \text{in } (0,+\infty) \times \partial\Omega^{t,x_0}. \end{array} \right. \tag{6.2}$$

Proof. By definition, we have $v(0,y) = u(0, x_0 + \sqrt{t}y) = u_0(x_0 + \sqrt{t}y) = u_0^{t,x_0}(y)$. Moreover, if we set $x = x_0 + \sqrt{t}y$, we have that $\partial/\partial y^h = \sqrt{t}\partial/\partial x^h$ and also that the unit outward normal to $\partial\Omega^{t,x_0}$ at y coincides with the unit outward normal to $\partial\Omega$ at x; therefore,

$$\left\langle A^{t,x_0}(y) D_y v(s,y), \nu(y) \right\rangle = \sqrt{t} \langle A(x) D_x u(ts,x), \nu(x) \rangle = 0,$$

In the same way, we have

$$\begin{aligned}
\partial_s v(s,y) &= tu'(ts, x_0 + \sqrt{t}y) = tu'(ts,x) \\
&= t\mathcal{A}(x)u(ts,x) = \mathcal{A}^{t,x_0}(y)v(s,y),
\end{aligned}$$

where u' denotes the derivative of u with respect to its first variable, and this concludes the proof. □

We also denote by $(T^{t,x_0}(s))_{s\geq 0}$ the semigroup associated with problem (6.2) and by $p^{t,x_0}(s,y,z)$ its kernel. We also denote by $(T^{x_0}(s))_{s\geq 0}$ the semigroup associated with the problem

$$\begin{cases} \partial_s w(s,y) = \mathcal{A}^{x_0}(y)w(s,y) & \text{in } (0,+\infty)\times\mathbb{R}^N \\ w(0,y) = w_0(y) & \text{in } \mathbb{R}^N \end{cases}$$

and by $p^{x_0}(s,y,z)$ its kernel.

Lemma 6.2 *For the kernels the following holds*

$$p(s,x,y) = t^{-N/2}p^{t,x_0}\Big(\frac{s}{t},\frac{x-x_0}{\sqrt{t}},\frac{y-x_0}{\sqrt{t}}\Big). \tag{6.3}$$

Proof. The proof of Lemma 6.1 gives that $v(s,y) = T^{t,x_0}(s)u_0^{t,x_0}(y) = T(ts)u_0(x_0+\sqrt{t}y)$; using the kernels, we get that

$$\begin{aligned} \int_\Omega p(s,x,y)u_0(y)dy &= T(s)u_0(x) = T^{t,x_0}\left(\frac{s}{t}\right)u_0^{t,x_0}\left(\frac{x-x_0}{\sqrt{t}}\right) \\ &= \int_{\Omega^{t,x_0}} p^{t,x_0}\left(\frac{s}{t},\frac{x-x_0}{\sqrt{t}},z\right)u_0(x_0+\sqrt{t}z)dz \\ &= t^{-N/2}\int_\Omega p^{t,x_0}\left(\frac{s}{t},\frac{x-x_0}{\sqrt{t}},\frac{y-x_0}{\sqrt{t}}\right)u_0(y)dy. \end{aligned}$$

The arbitrarity of u_0 gives the thesis. □

We have the following result.

Proposition 6.3 *For every $f\in L^1(\mathbb{R}^N)$, let $u^{t,x}(s,\xi)$ be the solution of the problem*

$$\begin{cases} \partial_s w(s,\xi) = A^{t,x}(\xi)w(s,\xi) & \text{in } (0,+\infty)\times\Omega^{t,x} \\ \big\langle A(x+\sqrt{t}\xi)Dw(s,\xi),\nu_{\Omega^{t,x}}(\xi)\big\rangle = 0 & \text{in } (0,+\infty)\times\partial\Omega^{t,x} \\ w(0,\xi) = f(\xi) & \text{in } \Omega^{t,x} \end{cases}$$

and let $u^x(s,\xi)$ be the solution of the problem

$$\begin{cases} \partial_s w(s,\xi) = A^{x}(\xi)w(s,\xi) & \text{in } (0,+\infty)\times\mathbb{R}^N \\ w(0,\xi) = f(\xi) & \text{in } \mathbb{R}^N \end{cases}.$$

Then for every $s>0$ we have that $u^{t,x}(s,\cdot)$ converges to $u^x(s,\cdot)$ in $L^1_{\rm loc}(\mathbb{R}^N)$ as $t\to 0$.

Proof. We start by taking $f\in C_c(\mathbb{R}^N)$ and denote by $u^{t,x}(s,\xi)$ the solution of the problem

$$\begin{cases} \partial_s w(s,\xi) = A^{x}(\xi)w(s,\xi) & \text{in } (0,+\infty)\times\Omega^{t,x} \\ \langle A^{t,x}(\xi)D_\xi w(s,\xi),\nu(\xi)\rangle = 0 & \text{in } (0,+\infty)\times\partial\Omega^{t,x} \\ w(0,\xi) = f(\xi) & \text{in } \Omega^{t,x}. \end{cases} \tag{6.4}$$

Since $u^{t,x}$ is a classical solution, for every regular function $\varphi:[0,s_0]\times\mathbb{R}^N\to\mathbb{R}$ with $\varphi(s_0,\cdot)=0$, the following holds:

$$\begin{aligned} -\int_{\Omega^{t,x}} f(\xi)\varphi(0,\xi)d\xi &= \int_0^{s_0}\int_{\Omega^{t,x}}\Big\{u^{t,x}(s,\xi)\left(\partial_s\varphi(s,\xi)+tC^{t,x}(\xi)\right) \\ &\quad +\frac{\partial u^{t,x}(s,\xi)}{\partial\xi^k}\Big[-A^{t,x}_{hk}(\xi)\frac{\partial\varphi(s,\xi)}{\partial\xi^h}+\sqrt{t}\varphi(s,\xi)B^{t,x}_k(\xi)\Big]\Big\}d\xi ds. \end{aligned} \tag{6.5}$$

Moreover, notice that $tC^{t,x} \to 0$, $A^{t,x}_{hk} \to A_{hk}(x)$, $\sqrt{t}B^{t,x}_k \to 0$ uniformly on compact sets as $t \to 0$.

As an auxiliary tool, let us use the L^2 theory, see e.g. [17, Section 5.4], recalling that there is $M > 0$ such that

$$\|u^{t,x}(s)\|_{L^2(\Omega^{t,x})} \le M\|f\|_{L^2(\Omega^{t,x})} \le M\|f\|_{L^2(\mathbb{R}^N)}, \tag{6.6}$$

$$\|Du^{t,x}(s)\|_{L^2(\Omega^{t,x})} \le \frac{M}{\sqrt{s}}\|f\|_{L^2(\Omega^{t,x})} \le \frac{M}{\sqrt{s}}\|f\|_{L^2(\mathbb{R}^N)}, \tag{6.7}$$

and

$$\|D^2u^{t,x}(s)\|_{L^2(\Omega^{t,x})} \le \frac{M}{s}\|f\|_{L^2(\Omega^{t,x})} \le \frac{M}{s}\|f\|_{L^2(\mathbb{R}^N)}. \tag{6.8}$$

These conditions imply that for every bounded open set $K \subset \mathbb{R}^N$, $s > 0$ fixed and t_0 small, the family $(u^{t,x}(s,\cdot))_{0<t<t_0}$ is bounded in $W^{2,2}(K)$, and then, up to subsequences, it is strongly convergent in $W^{1,2}(K)$ and also in $W^{1,1}(K)$.

We can now fix a dense countable set $D \subset [0, s_0]$ in such a way that $u^{t_h,x}(s,\cdot)$ converges to some $g(s,\cdot)$ in $W^{1,1}(K)$ for every $s \in D$ and some sequence $t_h \to 0$. By Theorem 3.1 we get that

$$\begin{aligned}
\|u^{t,x}(s_2,\cdot) - u^{t,x}(s_1,\cdot)\|_{L^1(\Omega^{t,x})} &= \Big\| \int_{s_1}^{s_2} \partial_s u^{t,x}(s,\cdot)ds \Big\|_{L^1(\Omega^{t,x})} \\
&\le \int_{s_1}^{s_2} \|A^{t,x}u^{t,x}(s,\cdot)\|_{L^1(\Omega^{t,x})}ds \\
&\le c_2\|f\|_{L^1(\Omega^{t,x})} \int_{s_1}^{s_2} \frac{1}{s}ds \le c_2\|f\|_{L^1(\mathbb{R}^N)} \log\frac{s_2}{s_1},
\end{aligned}$$

that is, the function $s \mapsto u^{t,x}(s,\cdot)$ is continuous from $(0, s_0)$ to $L^1(\Omega^{t,x})$; in particular, if we consider $s_1, s_2 \in D$, then the inequality

$$\begin{aligned}
\|g(s_2,\cdot) - g(s_1,\cdot)\|_{L^1(K)} \le &\|g(s_2,\cdot) - u^{t_h,x}(s_2,\cdot)\|_{L^1(K)} + \|u^{t_h,x}(s_2,\cdot) - u^{t_h,x}(s_1,\cdot)\|_{L^1(K)} \\
&+ \|u^{t_h,x}(s_1,\cdot) - g(s_1,\cdot)\|_{L^1(K)}
\end{aligned}$$

holds and the convergence of $u^{t,x}$ on D shows that we can extend g to a continuous map from $(0, s_0)$ to $L^1_{\text{loc}}(\mathbb{R}^N)$; we also notice that by (3.2) we deduce also that $g(s,\cdot) \in W^{1,1}(K)$ for every $s \in (0, s_0)$. By continuity, and by the convergence of $u^{t_h,x}(s,\cdot)$ on D we deduce that $u^{t_h,x}(s,\cdot) \to g(s,\cdot)$ in $L^1_{\text{loc}}(\mathbb{R}^N)$ for every $s \in (0, s_0)$. In addition, conditions (3.2) allow us to apply the dominated convergence theorem, and then, taking the limit in (6.5), we get

$$-\int_K f(\xi)\varphi(0,\xi)d\xi = \int_0^{s_0}\int_K \Big(g(s,\xi)\partial_s\varphi(s,\xi) - \langle A(x)D_\xi\varphi(s,\xi), D_\xi g(s,\xi)\rangle\Big)d\xi ds$$

for all φ as above, and then (see e.g. [16, Prop. 2.1, Ch. III]) $g(s,\cdot)$ is the solution of the problem

$$\begin{cases} \partial_s w(s,\xi) = A_{hk}(x)\frac{\partial^2 w}{\partial\xi^h\partial\xi^k}(s,\xi) & \text{in } (0,s_0) \times \mathbb{R}^N \\ w(0,\xi) = f(\xi) & \text{in } \mathbb{R}^N \end{cases}$$

for every $f \in C_c(\mathbb{R}^N)$. Then, it follows that

$$g(s,\xi) = u^x(s,\xi) = \int_{\mathbb{R}^N} p^x(s,\xi,z) f(z) dz,$$

where using the Fourier transform the kernel p^x is given by

$$p^x(s,\xi,z) = \frac{1}{(4\pi s)^{N/2} |\det A^{1/2}(x)|} \exp\left(-\frac{\langle A^{-1}(x)(\xi - z), (\xi - z)\rangle}{4s} \right). \tag{6.9}$$

From the density of C_c in L^1 we conclude. □

The following statement is an immediate consequence of Proposition 6.3.

Corollary 6.4 *For every $s > 0$, $\xi \in \mathbb{R}^N$, the measures $d\mu^{t,x} = p^{t,x}(s,\xi,\cdot) d\mathcal{L}^N \llcorner \Omega^{t,x}$ are weakly* convergent to the measure $d\mu^x = p^x(s,\xi,\cdot) d\mathcal{L}^N$ as $t \to 0$, that is, for every $\varphi \in C_c(\mathbb{R}^N)$ the following equality holds*

$$\lim_{t \to 0} \int_{\Omega^{t,x}} \varphi(z) p^{t,x}(s,\xi,z) dz = \int_{\mathbb{R}^N} \varphi(z) p^x(s,\xi,z) dz.$$

Henceforth, given the function $p(s,\xi,z)$, we shall denote by $D_1 p(s,\xi,z)$ the gradient with respect to the first spatial variables ξ and by $D_2 p(s,\xi,z)$ the gradient with respect to the second spatial variables z.

Proposition 6.5 *For every $s > 0$ and every $\xi \in \mathbb{R}^N$, the equality*

$$\lim_{t \to 0} \int_{\Omega^{t,x}} \langle D_2 p^{t,x}(s,\xi,z), \varphi(z)\rangle dz = \int_{\mathbb{R}^N} \langle D_2 p^x(s,\xi,z), \varphi(z)\rangle dz \tag{6.10}$$

holds for every $\varphi \in C_c(\mathbb{R}^N, \mathbb{R}^N)$.

Proof. We start by considering $\varphi \in C_c^1(\mathbb{R}^N, \mathbb{R}^N)$; we choose $t_0 > 0$ in such a way that $\operatorname{supp} \varphi \subset \Omega^{t,x}$ for all $t \leq t_0$; then

$$\int_{\Omega^{t,x}} \langle D_2 p^{t,x}(s,\xi,z), \varphi(z)\rangle dz = -\int_{\Omega^{t,x}} p^{t,x}(s,\xi,z) \operatorname{div}\varphi(z) dz$$

and then, by Corollary 6.4

$$\begin{aligned}
\lim_{t \to 0} \int_{\Omega^{t,x}} \langle D_2 p^{t,x}(s,\xi,z), \varphi(z)\rangle dz &= \lim_{t \to 0} -\int_{\Omega^{t,x}} p^{t,x}(s,\xi,z) \operatorname{div}\varphi(z) dz \\
&= -\int_{\mathbb{R}^N} p^x(s,\xi,z) \operatorname{div}\varphi(z) dz \\
&= \int_{\mathbb{R}^N} \langle D_2 p^x(s,\xi,z), \varphi(z)\rangle dz.
\end{aligned}$$

For an arbitrary $\varphi \in C_c(\mathbb{R}^N, \mathbb{R}^N)$ we use an approximation procedure; we select $\varphi_\varepsilon \in C_c^1(\mathbb{R}^N, \mathbb{R}^N)$ such that $\|\varphi - \varphi_\varepsilon\|_\infty \leq \varepsilon$ and then

$$\begin{aligned}
\int_{\Omega^{t,x}} \langle D_2 p^{t,x}(s,\xi,z), \varphi(z)\rangle dz &= \int_{\Omega^{t,x}} \langle D_2 p^{t,x}(s,\xi,z), \varphi_\varepsilon(z)\rangle dz \\
&\quad + \int_{\Omega^{t,x}} \langle D_2 p^{t,x}(s,\xi,z), (\varphi(z) - \varphi_\varepsilon(z))\rangle dz
\end{aligned}$$

Taking into account that $p^{t,x}(s,\xi,z) = t^{N/2}p(ts, x+\sqrt{t}\xi, x+\sqrt{t}z)$ and also that

$$\begin{aligned} D_2 p^{t,x}(s,\xi,z) &= D_z t^{N/2} p(ts, x+\sqrt{t}\xi, x+\sqrt{t}z) \\ &= t^{(N+1)/2} D_2 p(ts, x+\sqrt{t}\xi, x+\sqrt{t}z) \end{aligned}$$

by (3.1) we obtain

$$\begin{aligned} &\Big| \int_{\Omega^{t,x}} \langle D_2 p^{t,x}(s,\xi,z), (\varphi(z) - \varphi_\varepsilon(z))\rangle dz \Big| \\ &\qquad \le t^{(N+1)/2} \|\varphi - \varphi_\varepsilon\|_\infty \int_{\Omega^{t,x}} |D_2 p(ts, x+\sqrt{t}\xi, x+\sqrt{t}z)| dz \\ &\qquad \le C\varepsilon \end{aligned}$$

with C independent of t. Of course, the inequality

$$\Big| \int_{\mathbb{R}^N} \langle D_2 p^x(s,\xi,z), (\varphi(z) - \varphi_\varepsilon(z))\rangle dz \Big| \le C\varepsilon$$

holds as well, and then

$$\begin{aligned} &\lim_{t\to 0} \Big| \int_{\Omega^{t,x}} \langle D_2 p^{t,x}(s,\xi,z), \varphi(z)\rangle dz - \int_{\mathbb{R}^N} \langle D_2 p^x(s,\xi,z), \varphi(z)\rangle dz \Big| \\ &\le \lim_{t\to 0} \Big| \int_{\Omega^{t,x}} \langle D_2 p^{t,x}(s,\xi,z), (\varphi(z) - \varphi_\varepsilon(z))\rangle dz \Big| \\ &\quad + \lim_{t\to 0} \Big| \int_{\Omega^{t,x}} \langle D_2 p^{t,x}(s,\xi,z), \varphi_\varepsilon(z)\rangle dz - \int_{\mathbb{R}^N} \langle D_2 p^x(s,\xi,z), \varphi_\varepsilon(z) dz\rangle \Big| \\ &\quad + \lim_{t\to 0} \Big| \int_{\mathbb{R}^N} \langle D_2 p^x(s,\xi,z), (\varphi(z) - \varphi_\varepsilon(z))\rangle dz \Big| \le C\varepsilon \end{aligned}$$

and the thesis follows from the arbitrariness of ε. □

The main step in the proof of (1.4) is the following result, where an asymptotic formula relating two sets of finite perimeter is shown. In the statement, we assume that E has finite measure in order to give a meaning to the left hand side in (6.11). But, notice that, since E has finite peimeter in Ω, then by the relative isoperimetric inequality in Ω

$$\min\{|E\cap\Omega|, |\Omega\setminus E|\} \le cP(E,\Omega)^{N/N-1},$$

either $|E\cap\Omega|$ or $|\Omega\setminus E|$ is finite. Therefore, if $|E\cap\Omega|$ is infinite, then $|\Omega\setminus E|$ is finite and (6.11) applies with $\Omega\setminus E$ in place of E.

Theorem 6.6 *Assume* **(H1), (H2), (H3)***, and let* $(T(t))_{t\ge 0}$ *be the semigroup generated by* $(\mathcal{A}, D(\mathcal{A}))$ *in* $L^1(\Omega)$*; then, if* $E, F \subset \mathbb{R}^N$ *are sets of finite perimeter in* Ω*, the following holds*

$$\lim_{t\to 0} \sqrt{\frac{\pi}{t}} \int_{\Omega\cap F} (\chi_E(x) - T(t)\chi_E(x)) dx = \int_{\Omega\cap\mathcal{F}F\cap\mathcal{F}E} \langle A(x)\nu_E(x), \nu_F(x)\rangle d\mathcal{H}^{N-1}(x). \tag{6.11}$$

Proof. We have

$$
\begin{aligned}
(6.12)\qquad \int_{\Omega\cap F}(T(t)\chi_E(x)-\chi_E(x))dx &= \int_{\Omega\cap F}\int_0^t \frac{d}{ds}T(s)\chi_E(x)dsdx\\
&= \int_0^t\int_{\Omega\cap F}\mathcal{A}T(s)\chi_E(x)dxds\\
(6.13)\qquad\qquad &= \int_0^t\Big(\int_{\Omega\cap F}\mathrm{div}_x(A(x)D_xT(s)\chi_E(x))dx\\
&\quad+\int_{\Omega\cap F}\langle B(x),D_xT(s)\chi_E(x)\rangle dx\\
&\quad+\int_{\Omega\cap F}C(x)T(s)\chi_E(x)dx\Big)ds.
\end{aligned}
$$

For the last term we have that

$$\left|\int_{\Omega\cap F}C(x)T(s)\chi_E(x)dx\right|\leq \|C\|_\infty\min\{|\Omega\cap E|,,|\Omega\cap F|\}$$

and then

$$\lim_{t\to 0}\frac{1}{\sqrt{t}}\int_0^t\int_{\Omega\cap F}C(x)T(s)\chi_E(x)dxds=0.$$

For the second term in (6.12), we notice that

$$\left|\int_{\Omega\cap F}\int_{\Omega\cap E}\langle B(x),D_xp(s,x,y)\rangle dydx\right|\leq\|B\|_\infty\min\{|\Omega\cap E|,|\Omega\cap F|\}\int_\Omega|D_xp(s,x,y)|dx$$

and using Gaussian estimates (3.1) we get

$$\int_\Omega|D_xp(s,x,y)|dx\leq\frac{c}{\sqrt{s}}$$

for some constant c depending only on the operator $\mathcal{A}$ and the dimension N. We introduce now the kernel $p_*(s,x,y)$ of the semigroup generated by the adjoint operator $\mathcal{A}^*$ of $\mathcal{A}$; by the symmetry of the matrix A, the second order part of $\mathcal{A}^*$ is the same as $\mathcal{A}$. In this way we have that $p(s,x,y)=p_*(s,y,x)$ (see for instance [17, Theorem 5.6]) and since

$$
\begin{aligned}
\frac{\partial}{\partial x^i}p(s,x,y) &= \lim_{h\to 0}\frac{p(s,x+he_i,y)-p(s,x,y)}{h}=\lim_{h\to 0}\frac{p_*(s,y,x+he_i)-p_*(s,y,x)}{h}\\
&= s^{-N/2}\lim_{h\to 0}\frac{p_*^{s,x}(1,\frac{y-x}{\sqrt{s}},\frac{he_i}{\sqrt{s}})-p_*^{s,x}(1,\frac{y-x}{\sqrt{s}},0)}{h}\\
&= s^{-(N+1)/2}D_2^ip_*^{s,x}\Big(1,\frac{y-x}{\sqrt{s}},0\Big)
\end{aligned}
$$

where D_2^i denotes the i-th component of the gradient with respect to the second variables.

Then $D_x p(s,x,y) = s^{-(N+1)/2} D_2 p_*^{s,x}(1, \frac{y-x}{\sqrt{s}}, 0)$; hence

$$\begin{aligned}
\int_{\Omega\cap F} \langle B, D_x T(s)\chi_E\rangle dx &= \int_{\Omega\cap F} dx \int_{\Omega\cap E} \langle B(x), D_x p(s,x,y)\rangle dy \\
&= s^{-(N+1)/2} \int_{\Omega\cap F} dx \int_{\Omega\cap E} \left\langle B(x), D_2 p_*^{s,x}\left(1, \frac{y-x}{\sqrt{s}}, 0\right)\right\rangle dy \\
&= \frac{1}{\sqrt{s}} \int_{\Omega\cap F} dx \int_{\Omega^{s,x}\cap E^{s,x}} \langle B(x), D_2 p_*^{s,x}(1,z,0)\rangle dz \\
&= \frac{1}{\sqrt{s}} \int_{\Omega\cap F} dx \int_{\mathbb{R}^N} \langle B(x), D_2 p_*^{s,x}(1,z,0)\rangle d\mu^{s,x}(z).
\end{aligned}$$

where we have denoted by $\mu^{s,x}$ the measure

$$\mu^{s,x} = \mathcal{L}^N \llcorner (\Omega^{s,x} \cap E^{s,x}).$$

These measures verify the following properties:

1. $\mu^{s,x} \rightharpoonup^* 0$ if $x \in E^0$:
2. $\mu^{s,x} \rightharpoonup^* \mathcal{L}^N$ if $x \in E^1$;
3. $\mu^{s,x} \rightharpoonup^* \mathcal{L}^N \llcorner H_{\nu_E(x)}$ for $x \in \mathcal{F}E$, where $H_{\nu_E(x)} = \{z \in \mathbb{R}^N : \langle z, \nu_E(x)\rangle \le 0\}$.

These facts imply that, for $x \in E^0$,

$$\int_{\mathbb{R}^N} \langle B(x), D_2 p_*^{s,x}(1,z,0)\rangle d\mu^{s,x}(z) \to 0;$$

moreover, for $x \in E^1$

$$\int_{\mathbb{R}^N} \langle B(x), D_2 p_*^{s,x}(1,z,0)\rangle d\mu^{s,x}(z) \to B(x) \cdot \int_{\mathbb{R}^N} D_2 p_*^{x}(1,z,0) dz = 0.$$

Taking into account that $|F \setminus (E^0 \cup E^1)| = 0$ and the dominated convergence Theorem, we have then obtained that

$$\lim_{t\to 0} \frac{1}{\sqrt{t}} \int_0^t \int_{\Omega\cap F} \int_{\Omega\cap E} \langle B(x), D_x p(s,x,y)\rangle dy dx ds = 0$$

It remains to study the first term of (6.12); with an integration by parts and recalling that $\mathcal{H}^{N-1}(\partial^* E \setminus \mathcal{F}E) = 0$, we get

$$\begin{aligned}
&\int_{\Omega\cap F} \operatorname{div}(A D_x T(s)\chi_E(x)) dx = \int_{\Omega\cap\mathcal{F}F} \langle D_x T(s)\chi_E(x), A(x)\nu_F(x)\rangle d\mathcal{H}^{N-1}(x) \\
&\quad = \int_{\Omega\cap\mathcal{F}F} \int_{\Omega\cap E} s^{-(N+1)/2} \left\langle D_2 p_*^{s,x}\left(1, \frac{y-x}{\sqrt{s}}, 0\right), A(x)\nu_F(x)\right\rangle dy d\mathcal{H}^{N-1}(x) \\
&\quad = -\frac{1}{\sqrt{s}} \int_{\Omega\cap\mathcal{F}F} \int_{\Omega^{s,x}\cap E^{s,x}} \langle D_2 p_*^{s,x}(1,z,0), A(x)\nu_F(x)\rangle dz d\mathcal{H}^{N-1}(x) \\
&\quad = -\frac{1}{\sqrt{s}} \int_{\Omega\cap\mathcal{F}F} \int_{\mathbb{R}^N} \langle D_2 p_*^{s,x}(1,z,0), A(x)\nu_F(x)\rangle d\mu^{s,x}(z) d\mathcal{H}^{N-1}(x).
\end{aligned}$$

With the same argument previously used, we can deduce that for $x \in E^0 \cup E^1$, the limit of the above integral as $t \to 0$ vanishes; then we can only consider points $x \in \mathcal{F}F \cap \mathcal{F}E$; in this case we obtain that

$$\int_{\mathbb{R}^N} \langle D_2 p_*^{s,x}(1,z,0), A(x)\nu_F(x)\rangle d\mu^{s,x}(z) \longrightarrow \int_{H_{\nu_E(x)}} \langle D_2 p_*^x(1,z,0), A(x)\nu_F(x)\rangle dz.$$

Taking into account (6.9) and the symmetry of A, we get that

$$D_2 p_*^x(1,z,0) = -\frac{1}{2(4\pi)^{N/2}|\det A^{1/2}(x)|} \exp\left(-\left\langle A^{-1}(x)z, z\right\rangle/4\right) A^{-1}(x)z,$$

and then, since for $x \in \mathcal{F}F \cap \mathcal{F}E$ we have $\nu_F(x) = \langle \nu_E(x), \nu_F(x)\rangle \nu_E(x)$

$$\begin{aligned}
&\int_{H_{\nu_E(x)}} \langle D_2 p_*^x(1,z,0), A(x)\nu_F(x)\rangle dz = \\
&\qquad = -\frac{\langle \nu_E(x), \nu_F(x)\rangle}{2(4\pi)^{N/2}|\det A^{1/2}(x)|} \int_{H_{\nu_E(x)}} \exp\left(-\left\langle A^{-1}(x)z, z\right\rangle/4\right) \langle z, \nu_E(x)\rangle dz \\
&\qquad = -\frac{\langle \nu_E(x), \nu_F(x)\rangle}{\pi^{N/2}} \int_{H_{A^{1/2}(x)\nu_E(x)}} e^{-|z|^2} \left\langle z, A^{1/2}(x)\nu_E(x)\right\rangle dz.
\end{aligned}$$

For the computation of this last integral, we consider an orthonormal basis $\{e_1, \ldots, e_N\}$ of $\mathbb{R}^N$ with

$$e_N = \frac{A^{1/2}(x)\nu_E(x)}{|A^{1/2}(x)\nu_E(x)|};$$

we then obtain

$$\begin{aligned}
\int_{H_{A^{1/2}(x)\nu_E(x)}} \left\langle z, A^{1/2}(x)\nu_E(x)\right\rangle e^{-|z|^2} dz &= |A^{1/2}(x)\nu_E(x)| \int_{H_{A^{1/2}(x)\nu_E(x)}} z_N e^{-|z|^2} dz \\
&= \pi^{(N-1)/2} |A^{1/2}(x)\nu_E(x)| \int_{-\infty}^0 z_N e^{-z_N^2} dz_N \\
&= -\frac{\pi^{(N-1)/2}}{2} |A^{1/2}(x)\nu_E(x)|.
\end{aligned}$$

At the end, we have obtained that

$$\lim_{t\to 0} \sqrt{\frac{\pi}{t}} \int_{\Omega\cap F} (T(t)\chi_E - \chi_E) dx = -\int_{\Omega\cap\mathcal{F}F\cap\mathcal{F}E} \langle \nu_E, \nu_F\rangle |A^{1/2}\nu_E| d\mathcal{H}^{N-1}.$$

□

Specializing the above result for $F = E^c$ we get the following

Corollary 6.7 *Assume* **(H1), (H2), (H3)**, *and let $(T(t))_{t\geq 0}$ be the semigroup generated by $(\mathcal{A}, D(\mathcal{A}))$ in $L^1(\Omega)$; then, if $E \subset \mathbb{R}^N$ is a set with finite perimeter in Ω, the following equality holds:*

$$\lim_{t\to 0} \sqrt{\frac{\pi}{t}} \int_{\Omega\cap E^c} T(t)\chi_E dx = \int_{\Omega\cap\mathcal{F}E} |A^{1/2}(x)\nu_E(x)| d\mathcal{H}^{N-1}(x). \tag{6.14}$$

Remark 6.8 Using an argument similar to the one used in [15, Theorem 3.4], it is possible to prove that if E is a set with finite measure such that

$$\liminf_{t\to 0}\frac{1}{\sqrt{t}}\int_{E^c\cap\Omega} T(t)\chi_E(x)dx < +\infty,$$

then E has finite perimeter in Ω, that is $\chi_E \in BV(\Omega)$. In fact, denoting by

$$|D_\nu \chi_E|(z) = \liminf_{t\to 0}\frac{|E\Delta(E-\sqrt{t}z)|}{t},$$

we get

$$\begin{aligned}\int_\Omega\int_{\mathbb{R}^N}|z|p^x(1,0,z)|D_{\frac{z}{|z|}}\chi_E|(z)dzdx &\le \liminf_{t\to 0}\int_\Omega\int_{\Omega^{t,x}}|z|p^{t,x}(1,0,z)\frac{|E\Delta(E-\sqrt{t}z)|}{\sqrt{t}z}dzdx\\ &= \liminf_{t\to 0}\frac{1}{\sqrt{t}}\int_{\Omega\times\Omega}\chi_E(y)\chi_{E^c}(x)p(t,x,y)dxdy < +\infty,\end{aligned}$$

and this implies that almost every directional derivative of χ_E is a finite measure, that is $|D\chi_E|(\Omega) < +\infty$.

We are now in a position to prove the main result of this section, namely, the announced characterization of BV functions (1.4). The strategy is the same as for $\mathbb{R}^N$, see [15], and is based on (4.2).

Theorem 6.9 *Assume* **(H1), (H2), (H3)***, and let* $(T(t))_{t\ge 0}$ *be the semigroup generated by* $(\mathcal{A}, D(\mathcal{A}))$ *in* $L^1(\Omega)$ *and let* $u \in L^1(\Omega)$*; then* $u \in BV(\Omega)$ *if and only if*

$$\liminf_{t\to 0}\frac{1}{\sqrt{t}}\int_{\Omega\times\Omega}|u(x)-u(y)|p(t,x,y)dxdy < +\infty;$$

moreover, in this case the following equality holds

$$|Du|_A(\Omega) = \lim_{t\to 0}\frac{\sqrt{\pi}}{2\sqrt{t}}\int_{\Omega\times\Omega}|u(x)-u(y)|p(t,x,y)dxdy. \tag{6.15}$$

Proof. We start by considering $u \in L^1(\Omega)$; for $\tau \in \mathbb{R}$ we denote by $E_\tau = \{u > \tau\}$ and, since the semigroup is positive and contractive, we obtain that

$$\begin{aligned} 0 &\le \int_{\mathbb{R}}\liminf_{t\to 0}\frac{1}{\sqrt{t}}\int_{E_\tau^c\cap\Omega}T(t)\chi_{E_\tau}dxd\tau\\ &\le \liminf_{t\to 0}\frac{1}{\sqrt{t}}\int_{\mathbb{R}}\int_{E_\tau^c\cap\Omega}T(t)\chi_{E_\tau}dxd\tau\\ &\le \liminf_{t\to 0}\frac{1}{\sqrt{t}}\int_{\Omega\times\Omega}\int_{\mathbb{R}}|\chi_{E_\tau}(x)-\chi_{E_\tau}(y)|p(t,x,y)dxdyd\tau\\ &= \liminf_{t\to 0}\frac{1}{\sqrt{t}}\int_{\Omega\times\Omega}|u(x)-u(y)|p(t,x,y)dxdy < +\infty\end{aligned}$$

and then, thanks to Remark 6.8, almost every level E_τ has finite perimeter and equation (6.14) holds. Then, using coarea formula (4.2), we get

$$\begin{aligned}|Du|_A(\Omega) &= \int_{\mathbb{R}}P_A(E_\tau,\Omega)d\tau = \int_{\mathbb{R}}\lim_{t\to 0}\sqrt{\frac{\pi}{t}}\int_{E_\tau^c\cap\Omega}T(t)\chi_{E_\tau}dxd\tau\\ &\le \liminf_{t\to 0}\sqrt{\frac{\pi}{t}}\int_{\Omega\times\Omega}|u(x)-u(y)|p(t,x,y)dxdy < +\infty\end{aligned}$$

that is $u \in BV_A(\Omega)$. The other implication follows from (6.15). To prove (6.15), we define the function

$$g_t(\tau) = \sqrt{\frac{\pi}{t}} \int_{E_\tau^c \cap \Omega} T(t)\chi_{E_\tau}(x)dx.$$

For this function we have the following estimate

$$\begin{aligned}
|g_t(\tau)| =& \sqrt{\frac{\pi}{t}} \Big| \int_0^t \int_{E_\tau^c \cap \Omega} \mathcal{A}T(s)\chi_{E_\tau} dxds \Big| \\
=& \sqrt{\frac{\pi}{t}} \Big| \int_0^t \Big(\int_{\mathcal{F}E_\tau \cap \Omega} \langle ADT(s)\chi_{E_\tau}, \nu_{E_\tau} \rangle d\mathcal{H}^{N-1} + \int_{E_\tau^c \cap \Omega} \langle B, DT(s)\chi_{E_\tau} \rangle dx \\
&+ \int_{E_\tau^c \cap \Omega} CT(s)\chi_{E_\tau})dx \Big) ds \Big| \\
\leq& \sqrt{\frac{\pi}{t}} \int_0^t \Big(\|A\|_\infty \int_{\mathcal{F}E_\tau} |DT(s)\chi_{E_\tau}| d\mathcal{H}^{N-1} \\
&+ \|B\|_\infty \int_{E_\tau^c \cap \Omega} \int_{E_\tau \cap \Omega} |D_x p(s,x,y)| dxdy \\
&+ \|C\|_\infty \int_{E_\tau^c \cap \Omega} \int_{E_\tau \cap \Omega} |p(s,x,y)| dxdy \Big) ds \\
\leq& cM_0(P(E_\tau, \Omega) + \min\{|E_\tau \cap \Omega|, |E_\tau^c \cap \Omega|\}) = h(\tau)
\end{aligned}$$

where the last inequality follows from the estimates (3.1) on the kernel $p(s,x,y)$. We have that $h \in L^1(\mathbb{R})$ since

$$\int_{\mathbb{R}} P(E_\tau, \Omega) d\tau = |Du|(\Omega)$$

and, denoted by $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$,

$$\begin{aligned}
\int_{\mathbb{R}} \min\{|E_\tau \cap \Omega|, |E_\tau^c \cap \Omega|\} d\tau &\leq \int_0^{+\infty} |E_\tau \cap \Omega| d\tau + \int_{-\infty}^0 |E_\tau^c \cap \Omega| d\tau \\
&= \int_0^{+\infty} \int_\Omega \chi_{E_\tau} dxd\tau + \int_{-\infty}^0 \int_\Omega \chi_{E_\tau^c} dxd\tau \\
&= \int_\Omega \int_0^{+\infty} \chi_{\{u>\tau\}} d\tau dx + \int_\Omega \int_0^{+\infty} \chi_{\{-u \geq \tau\}} d\tau dx \\
&= \int_\Omega u^+ dx + \int_\Omega u^- dx = \int_\Omega |u| dx.
\end{aligned}$$

Then we can apply Corollary 6.7 and Lebesgue dominated convergence to the functions g_t in order to obtain

$$\begin{aligned}
|Du|_A(\Omega) &= \int_{\mathbb{R}} P_A(E_\tau, \Omega) d\tau = \int_{\mathbb{R}} \lim_{t \to 0} \sqrt{\frac{\pi}{t}} \int_{E_\tau^c \cap \Omega} T(t)\chi_{E_\tau} dx \\
&= \lim_{t \to 0} \sqrt{\frac{\pi}{t}} \int_{\mathbb{R}} \int_{\Omega \times \Omega} (\chi_{E_\tau}(y) - \chi_{E_\tau}(y)\chi_{E_\tau}(x)) p(t,x,y) dxdyd\tau \\
&= \lim_{t \to 0} \sqrt{\frac{\pi}{t}} \int_{\Omega \times \Omega} (u(y) - \min\{u(y), u(x)\}) p(t,x,y) dxdy
\end{aligned}$$

since $\chi_{E_\tau}(y)\chi_{E_\tau}(x) \neq 0$ iff $\tau < \min\{u(x), u(y)\}$; noticing that

$$\min\{u(y), u(x)\} = \frac{1}{2}(u(x) + u(y) - |u(x) - u(y)|),$$

the assertion follows. □

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