DISCRETE-TO-CONTINUUM LIMITS OF MULTI-BODY SYSTEMS WITH BULK AND SURFACE LONG-RANGE INTERACTIONS

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Abstract. We study the atomistic-to-continuum limit of a class of energy functionals for crystalline materials via Γ-convergence. We consider energy densities that may depend on interactions between all points of the lattice and we give conditions that ensure compactness and integral-representation of the continuum limit on the space of special functions of bounded variation. This abstract result is complemented by a homogenization theorem, where we provide sufficient conditions on the energy densities under which bulk- and surface contributions decouple in the limit. The results are applied to long-range and multi-body interactions in the setting of weak-membrane energies.

1. Introduction

The passage from atomistic to continuum models is of major interest in the description and understanding of many physical phenomena. Even for those atomistic systems which are driven by simple energies, the choice of the method to analyse their asymptotic behavior as the interatomic distance tends to zero is nontrivial and it may lead to different results. Compare for instance the results obtained by taking pointwise limits ([7, 8, 30]) to those obtained by variational methods (see [15, 12] for an overview).

In this paper we work within the variational framework, which amounts to analysing the asymptotic behavior of discrete sytems in terms of Γ-convergence. This has proven to be a powerful tool not only in material sciences to describe the effective behavior of atomistic systems, but also in computer vision to provide discrete approximations of given continuum energies that might be used, *e.g.*, for numerical simulations. In the context of material sciences such a discrete-to-continuum variational analysis may help to predict or better understand the macroscopic response of a material to microscopic deformations. Fixing $\varepsilon > 0$ one describes the atomistic deformation of a material occupying an open bounded domain $\Omega \subset \mathbb{R}^n$ through a map $u : Z_\varepsilon(\Omega) \to \mathbb{R}^d$, where $Z_\varepsilon(\Omega) := \Omega \cap \varepsilon\mathbb{Z}^n$ denotes the set of ε-spaced material points (or simply atoms) of the system. In the most general case one can assume such a system to be driven by an energy of the form

$$F_\varepsilon(u) = \sum_{i\in Z_\varepsilon(\Omega)} \varepsilon^n \phi_i^\varepsilon(\{u^{i+j}\}_{j\in Z_\varepsilon(\Omega-i)}). \tag{1.1}$$

Here the functions $\phi_i^\varepsilon : (\mathbb{R}^d)^{Z_\varepsilon(\Omega-i)} \to [0, +\infty)$ should be thought of as the potential energies at scale ε describing the interaction between the atom at position i and the whole configuration $\{u^j\}_{j\in Z_\varepsilon(\Omega)}$. As a consequence, energies as in (1.1) can model systems which are (at the same time) non-homogeneous, multi-body, non-local and multi-scale.

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1.1. **Aim of the paper.** In this paper we are interested in the variational description (via Γ-convergence) of the limit of F_ε as the lattice spacing ε vanishes while the density of the atoms is kept constant thanks to the scaling factor ε^n. We refer to such a coarse-graining procedure as discrete-to-continuum limit. As a matter of fact a fine description of the discrete-to-continuum limit of physical systems driven by energies as those in (1.1) turns out to be a very challenging task unless the potentials are explicitly known and take some very special form. Until now the most general result in this direction has been obtained in [17], where the authors establish a set of assumptions on the potential energies ϕ_i^ε which ensure that up to subsequences the Γ-limit of energies as in (1.1) is an integral functional defined on a Sobolev space. The aim of the present paper is the extension of such a general result to the setting of special functions of bounded variations, that is to find sufficient conditions on ϕ_i^ε under which the variational limit energy of the sequence (F_ε) is of the form

$$F(u) = \int_\Omega f(x, \nabla u)\, dx + \int_{S_u} g(x, u^+ - u^-, \nu_u)\, d\mathcal{H}^{n-1} \tag{1.2}$$

defined on those u (here we use the same notation u for both microscopic and macroscopic fields) belonging to $SBV(\Omega;\mathbb{R}^d)$. Energies of this type are usually referred to as free-discontinuity functionals and are widely used to model a number of phenomena in fracture mechanics, image reconstruction or in the theory of liquid crystals, to make only a few examples (ADD citations). The discrete-to-continuum analysis performed in the present paper thus provides a very general framework on the one hand for atomistic systems whose macroscopic behavior can be studied in the context of fracture mechanics and on the other hand for possible discrete approximations of energies used in image reconstruction, such as for instance the approximations studied in [23, 24, 16, 32].

The assumptions on the potentials ϕ_i^ε that are needed to restrict the class of possible discrete-to-continuum limits to functionals of the form (1.2) are carefully listed in Section 2. Here we limit ourselves to highlight the main ideas behind them in the case that u represents the elastic deformation field of a physical system to be studied within the theory of fracture mechanics. In this case the two energy terms in (1.2) can be interpreted as follows. The bulk integral represents the (hyper-)elastic energy stored in the system due to the contribution of bounded microscopic deformation gradients, that is of deformations with $|u^i - u^j|/\varepsilon$ of order one. The surface term represents the energy the system needs to produce the fracture S_u in Ω with opening $u^+ - u^-$. Such an energy is instead due to microscopic deformation gradients of order $1/\varepsilon$. In the simplest possible case $f(x, M) = |M|^p$ and $g = const$ the bulk and surface energies are proportional to the p-th power of the L^p norm of the macroscopic deformation gradient ∇u and to the length of the fracture, respectively.

Within this framework the assumptions on the potentials ϕ_i^ε read as follows.

(H1) invariance under translations in u: This ensures that the integrand f in (1.2) does not depend explicitly on u and g depends on u^+ and u^- only through their difference;

(H2) monotonicity in the strain: the potential energy is assumed to be a non-decreasing in the finite differences $|u^i - u^j|$ - in the simple case of pairwise interactions this translates to the fact that the elastic energy increases as the modulus of the deformation gradient increases;

(H3) weak Cauchy-Born type upper bound: only the potential energy of a microscopic affine deformation is bounded from above by the p-th power ($p > 1$) of the norm of its gradient;

(H4) lower bound that allows to deduce that the limit is defined on $SBV(\Omega)$: Keeping in mind the above interpretation of finite differences as deformation gradients, $\phi_i^\varepsilon(\{u^{i+j}\})$ is assumed to be bounded from below by $|(u^i - u^j)/\varepsilon|^p$ whenever this quantity is of order 1, and otherwise by $1/\varepsilon$;

(H5) mild non-locality: the potential energies ϕ_i^ε of different deformations that agree in a cube of side length α centred at a point i are comparable up to an error that vanishes for large α as $\varepsilon \to 0$ uniformly in i. This ensures that the Γ-limit is a local integral functional;

(H6) controlled non-convexity: the energy stored by a convex combination of two deformations is asymptotically controlled by the sum of the energies corresponding to each single deformation. This technical assumption allows us to use the abstract methods of Γ-convergence (see below) and is needed here to tame the effect of the possibly diverging number of multi-body interactions.

We take the discrete-to-continuum limit of the energies in (1.1) under this set of assumptions. To this end we regard a discrete field u as belonging to $L^1(\Omega;\mathbb{R}^d)$ by identifying it with its piecewise constant interpolation on the cells of the ε lattice. Outside this set of functions we extend F_ε to $L^1(\Omega;\mathbb{R}^d)$ by setting it equal to $+\infty$. We then define the discrete-to-continuum limit of F_ε as its Γ-limit as $\varepsilon \to 0$ with respect to the strong L^1-convergence.

1.2. **Main results, methods of proof and comparison with existing results.** In this paper we prove compactness, integral-representation and homogenization results for energies of the form (1.1). More precisely, in Theorem 3.1 we show that, up to subsequences, the discrete energies F_ε Γ-converge to a free-discontinuity functional of the type (1.2). Using this integral representation we then prove the homogenization Theorem 4.3. There we show that under additional assumptions on ϕ_i^ε which will be discussed at the end of this section the whole sequence (F_ε) Γ-converges to

$$F_{\rm hom}(u) = \int_\Omega f_{\rm hom}(\nabla u)\,dx + \int_{S_u} g_{\rm hom}(u^+ - u^-, \nu_u)\,d\mathcal{H}^{n-1}, \tag{1.3}$$

where $f_{\rm hom}$ and $g_{\rm hom}$ are some homogenized bulk and surface-energy densities, respectively.

The proof of Theorem 3.1 relies on the so-called localization method of Γ-convergence (see [28, Chapters 14–20] and also [11, Chapter 16]). Following this method we consider the energies F_ε as functions defined both on u and on the open subsets of Ω by defining for every pair (u, A) with $u : Z_\varepsilon(\Omega) \to \mathbb{R}^d$ and $A \subset \Omega$ open the localized energy $E_\varepsilon(u, A)$ according to (1.1) where now the sum is taken only over $i \in Z_\varepsilon(A)$. We then prove a general compactness result (Theorem 3.11) which ensures that for every sequence of positive numbers converging to zero there exist a subsequence (ε_j) and a functional F such that for every $A \subset \Omega$ open and with Lipschitz boundary the localized energies $F_{\varepsilon_j}(\cdot, A)$ Γ-converge to $F(\cdot, A)$. Subsequently, thanks to assumptions (H1)–(H6) we recover enough information on F both as a function in u and as a set function to write it as a free-discontinuity functional of the form (1.2) by using the general integral-representation result in [9]. Before we comment on the homogenization result below we give a short overview on the use of the localization method in the context of discrete systems.

The method was originally proposed by De Giorgi and has been successfully used in the context of homogenization of multiple integrals in the continuum setting (see [14] and references therein). It has been first adapted to study discrete-to-continuum limits in [1] in the context of pairwise-interacting discrete systems modelling nonlinear hyper-elastic materials and giving rise to continuum functionals finite on Sobolev spaces of the form $\int_\Omega f(x,\nabla u)\,dx$. After that the application of the localization method to discrete systems at a bulk scaling has been extended into several directions including stochastic lattices [4, 25], more general interaction potentials [21, 19, 17] and has also been combined with dimension-reduction techniques [3]. The most general result for discrete systems on deterministic lattices with limit energies on Sobolev spaces is by now contained in [17].

At the surface scaling the analysis of discrete systems has required the use of the abstract method for the first time in [5]. This paper derives the continuum domain wall theory in ferromagnetism from pairwise interacting Ising-type spin systems on (possibly stochastic) lattices (see also [20] for thin films). The extension of this result to more general magnetic interactions has been considered in [2]. There the authors give examples of systems not satisfying (the analog of) assumption (H5) whose discrete-to-continuum limit is a nonlocal functional (see also [10]). A first general result for discrete systems with multi-body and long-range interactions at this scaling has been obtained in [13] in the context of spin-like systems with spatially modulated phases.

We point out that in the above mentioned papers the discrete energies under consideration involve either a pure bulk or a pure surface scaling. In order to obtain a Γ-limit of the type (1.2) one needs to consider discrete energies where both scalings are present at the same time. In this case, however, it becomes more difficult to find the correct set of assumptions which makes the localization method applicable. A first result in this direction has been obtained in [32], where the author considers energies of the form (1.1) on a possibly stochastic lattice. The interaction potentials ϕ_i^ε however are independent of i and ε, have finite range and depend on finitely many particles uniformly in ε. Moreover, they depend on the configuration $\{u^j\}_j$ through the set of discrete differences $\{|u^i - u^j|\}_{i,j}$. This type of dependence is essential to decouple the contribution of bulk and surface scalings in the continuum limit, which finally allows to prove the full Γ-limit result (without extraction of a subsequence) in the case of a stationary stochastic lattice. This is done by exploiting for the first time in the discrete setting the theory of maximal functions introduced in [31] and used in [18] in the context of homogenization. This technique turns out to be useful also in the proof of the present homogenization result Theorem 4.3, which we finally describe below.

Theorem 4.3 falls into the framework of periodic homogenization and thus requires the restriction to a special class of periodic interaction-energy densities. As our interaction-energy densities at a point i may depend on the whole configuration $\{u^{i+j}\}_{j\in Z_\varepsilon(\Omega-i)}$ the meaning of periodicity needs to be clarified. A proper definition of periodicity (at least in the interior of Ω) is possible when restricting to finite-range interactions. This modelling assumption also helps to decouple the bulk and the surface scaling in the Γ-limit, which is central to characterize the homogenized integrands $f_{\rm hom}$ and $g_{\rm hom}$ in (1.3). We highlight that even under the finite-range assumption this task still requires a major effort due to the lack of a gradient structure in the interaction potentials. In fact, a crucial step in proving the homogenization result consists in establishing sufficient conditions on the potential ϕ_i^ε (without enforcing an explicit gradient structure) which make it possible to distinguish between the discretization of a macroscopic affine deformation of the form $u_M(x) = Mx$ with $M \in \mathbb{R}^{d\times n}$ and of a macroscopic jump, that is, a mapping of the form $u_\zeta(x_1,\dots,x_n) = \zeta\chi_{\{x_n>0\}}$ with $\zeta \in \mathbb{R}^d$. More in detail, to derive formulas for the homogenized integrands $f_{\rm hom}$ and $g_{\rm hom}$ in (1.3) it is essential that the potentials ϕ_i^ε reflect the different scaling properties of u_M and u_ζ when passing from the scaled lattice $\varepsilon\mathbb{Z}^n$ to the integer lattice $\mathbb{Z}^n$. Indeed, the affine function u_M satisfies $u_M(j) = \varepsilon u_M(j/\varepsilon)$ for every $j \in \varepsilon\mathbb{Z}^n$, while for the jump function u_ζ there holds $u_\zeta(j) = u_\zeta(j/\varepsilon)$ for every $j \in \varepsilon\mathbb{Z}^n$. It thus seems natural to require that for a given discrete function $u : \mathbb{Z}^n \to \mathbb{R}^d$ and $i \in \mathbb{Z}^n$ asymptotically there holds

$$\varepsilon^n\phi_{\varepsilon i}^\varepsilon(\{\varepsilon u^{j/\varepsilon}\}) \sim \varepsilon^n\psi_i^b(\{u^j\}), \qquad \varepsilon^n\phi_{\varepsilon i}^\varepsilon(\{u^{j/\varepsilon}\}) \sim \varepsilon^{n-1}\psi_i^s(\{u^j\}),$$

for some discrete bulk and surface potentials ψ_i^b, ψ_i^s. This heuristic argument is made rigorous in Section 4.1, where we carefully state the correct hypotheses on the interaction potentials and we refer the reader to this section for more details.

1.3. **Plan of the paper.** The paper is organized as follows. In Section 2 we recall some basic notation and we introduce the discrete functionals under consideration together with the precise assumptions on the potential ϕ_i^ε. Section 3 is then devoted to the proof of the integral-representation Theorem 3.1 and to the treatment of Dirichlet boundary problems. The latter allows us to obtain asymptotic minimization formulas for the integrands f and g in (1.2) (see Remark 3.13), which are a key ingredient to prove the homogenization result Theorem 4.3. This is done in Section 4, where we also state precisely the periodicity- and the separation-of-scales assumptions. We conclude the paper by giving some examples that fall into the framework of our discrete energies in Section 5.

2. Setting of the problem

Notation. Let $n \geq 1$ be a fixed integer and $\Omega \subset \mathbb{R}^n$ an open, bounded set with Lipschitz boundary. We denote by $\mathcal{A}(\Omega)$ the family of all open subsets of Ω and by $\mathcal{A}^{reg}(\Omega)$ the family of all open subsets of Ω with Lipschitz boundary.

Let $\{e_1, \dots, e_n\}$ denote the standard orthonormal basis in $\mathbb{R}^n$. If $\nu, \xi \in \mathbb{R}^n$ we use the notation $\langle \nu, \xi \rangle$ for the scalar product between ν and ξ and by $|\nu| := \sqrt{\langle \nu, \nu \rangle}$ and $|\nu|_\infty := \sup_{1 \leq k \leq n} |\langle \nu, e_k \rangle|$ we denote the euclidian norm and the supremum norm of ν, respectively. Moreover, we set $S^{n-1} := \{\nu \in \mathbb{R}^n \colon |\nu| = 1\}$ and for every $\nu \in S^{n-1}$ we denote by $\Pi_\nu := \{x \in \mathbb{R}^n \colon \langle x, \nu \rangle = 0\}$ the hyperplane orthogonal to ν and passing through the origin and $p_\nu : \mathbb{R}^n \to \Pi_\nu$ is the orthogonal projection onto Π_ν. Further, Q^ν denotes the unit cube centred at the origin and with one face orthogonal to ν, and for every $x_0 \in \mathbb{R}^n$ and $\rho > 0$ we set $Q^\nu_\rho(x_0) := x_0 + \rho Q^\nu$. If $\nu = e_k$ for some $k \in \{1, \dots, n\}$ we simply write Q and $Q_\rho(x_0)$ in place of Q^{e_k} and $Q^{e_k}_\rho(x_0)$.

For every $A \subset \mathbb{R}^n$ we write $|A|$ for the n-dimensional Lebesgue measure of A, while $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$. If $p \in [1, +\infty]$ and $d \geq 1$ is a fixed integer we use standard notation for Lebesgue spaces $L^p(\Omega; \mathbb{R}^d)$ and Sobolev spaces $W^{1,p}(\Omega; \mathbb{R}^d)$. Moreover, $SBV(\Omega; \mathbb{R}^d)$ denotes the space of $\mathbb{R}^d$-valued special functions of bounded variation in Ω (see, *e.g.*, [6] for the general theory). If $u \in SBV(\Omega; \mathbb{R}^d)$ we write ∇u for the approximate gradient of u, S_u for the approximate discontinuity set of u and ν_u is the generalised outer normal to S_u. Moreover, u^+ and u^- are the the traces of u on both sides of S_u and we set $[u] := u^+ - u^-$. We also consider the larger space $GSBV(\Omega; \mathbb{R}^d)$ defined as the space of all functions $u : \Omega \to \mathbb{R}^d$ such that $\varphi \circ u \in SBV_{\rm loc}(\Omega; \mathbb{R}^d)$ for every $\varphi \in C^1(\mathbb{R}^d; \mathbb{R}^d)$ with $\mathrm{supp}(\nabla\varphi) \subset\subset \mathbb{R}^d$. For $p \in (1, +\infty)$ it is also convenient to consider the spaces

$$SBV^p(\Omega; \mathbb{R}^d) := \{u \in SBV(\Omega; \mathbb{R}^d) \colon \nabla u \in L^p(\Omega; \mathbb{R}^{d\times n}),\ \mathcal{H}^{n-1}(S_u) < +\infty\}$$

and

$$GSBV^p(\Omega; \mathbb{R}^d) := \{u \in GSBV(\Omega; \mathbb{R}^d) \colon \nabla u \in L^p(\Omega; \mathbb{R}^{d\times n}),\ \mathcal{H}^{n-1}(S_u) < +\infty\}.$$

Notice that $GSBV^p(\Omega; \mathbb{R}^d)$ is a vector space and for every $u \in GSBV^p(\Omega; \mathbb{R}^d)$ and $\varphi \in C^1(\mathbb{R}^d; \mathbb{R}^d)$ with $\mathrm{supp}(\nabla\varphi) \subset\subset \mathbb{R}^d$ there holds $\varphi \circ u \in SBV^p(\Omega; \mathbb{R}^d) \cap L^\infty(\Omega; \mathbb{R}^d)$ (see, *e.g.*, [29, Section 2]).

For $x_0 \in \mathbb{R}^n$, $\nu \in S^{n-1}$, $\zeta \in \mathbb{R}^d$ and $M \in \mathbb{R}^{d\times n}$ we will frequently consider the jump function $u^\nu_{\zeta,x_0} : \mathbb{R}^n \to \mathbb{R}^d$ and the affine function $u_{M,x_0} : \mathbb{R}^n \to \mathbb{R}^d$ defined by setting

$$u^\nu_{\zeta,x_0}(x) := \begin{cases} \zeta & \text{if } \langle x - x_0, \nu \rangle \geq 0, \\ 0 & \text{if } \langle x - x_0, \nu \rangle < 0, \end{cases} \qquad \text{and} \qquad u_{M,x_0}(x) := M(x - x_0), \tag{2.1}$$

for every $x \in \mathbb{R}^n$.

Setting. In all that follows $\varepsilon > 0$ denotes a parameter varying in a strictly decreasing sequence of positive real numbers converging to zero. For any $\varepsilon > 0$, $u : \mathbb{R}^n \to \mathbb{R}^d$, $\xi \in \mathbb{Z}^n$ and $x \in \mathbb{R}^n$ we denote by

$$D_\varepsilon^\xi u(x) := \frac{u(x+\varepsilon\xi) - u(x)}{\varepsilon|\xi|}$$

the difference quotient of u at x in direction ξ. If $\xi = e_k$ for some $k \in \{1, \dots, n\}$ we write $D_\varepsilon^k u(x)$ in place of $D_\varepsilon^{e_k} u(x)$.

We now introduce the discrete functionals considered in this paper. To this end, for every $A \subset \mathbb{R}^n$ let $Z_\varepsilon(A) := A \cap \varepsilon\mathbb{Z}^n$ and set $\mathcal{A}_\varepsilon(\Omega;\mathbb{R}^d) := \{u : Z_\varepsilon(\Omega) \to \mathbb{R}^d\}$. It is then convenient to identify discrete functions $u \in \mathcal{A}_\varepsilon(\Omega;\mathbb{R}^d)$ with their piecewise constant counterpart belonging to $L^1(\Omega;\mathbb{R}^d)$ defined by setting

$$u(x) := u(i) =: u^i \quad \text{for every } x \in i + [0,\varepsilon)^n,\ i \in Z_\varepsilon(\Omega). \tag{2.2}$$

If (u_ε) is a sequence in $\mathcal{A}_\varepsilon(\Omega;\mathbb{R}^d)$ we say that (u_ε) converges in $L^1(\Omega;\mathbb{R}^d)$ to a function $u \in L^1(\Omega;\mathbb{R}^d)$ if the piecewise constant interpolation of (u_ε) defined as in (2.2) does so.

Finally, for every $i \in Z_\varepsilon(\Omega)$ it is convenient to consider the tranlated set $\Omega_i := \Omega - i$. We then consider functions $\phi_i^\varepsilon : (\mathbb{R}^d)^{Z_\varepsilon(\Omega_i)} \to [0,+\infty)$ and we define the discrete functionals $F_\varepsilon : L^1(\Omega;\mathbb{R}^d) \times \mathcal{A}(\Omega) \to [0,+\infty]$ as

$$F_\varepsilon(u,A) := \begin{cases} \displaystyle\sum_{i \in Z_\varepsilon(A)} \varepsilon^n \phi_i^\varepsilon(\{u^{i+j}\}_{j \in Z_\varepsilon(\Omega_i)}) & \text{if } u \in \mathcal{A}_\varepsilon(\Omega;\mathbb{R}^d), \\ +\infty & \text{otherwise in } L^1(\Omega;\mathbb{R}^d). \end{cases} \tag{2.3}$$

In the case $A = \Omega$ we omit the depedence on the set and simply write $F_\varepsilon(u)$ in place of $F_\varepsilon(u,\Omega)$. With the identification as in (2.2) and the corresponding $L^1(\Omega;\mathbb{R}^d)$-convergence we aim to describe the Γ-limit of the functionals F_ε in the strong $L^1(\Omega)$-topology under suitable conditions on the energy densities ϕ_i^ε. Namely, we assume that the functions $\phi_i^\varepsilon : (\mathbb{R}^d)^{Z_\varepsilon(\Omega_i)} \to [0,+\infty)$ satisfy the following hypotheses for every $\varepsilon > 0$ and $i \in Z_\varepsilon(\Omega)$.

(H1) (translational invariance) For all $w \in \mathbb{R}^d$ and $z : Z_\varepsilon(\Omega_i) \to \mathbb{R}^d$,

$$\phi_i^\varepsilon(\{z^j + w\}_{j \in Z_\varepsilon(\Omega_i)}) = \phi_i^\varepsilon(\{z^j\}_{j \in Z_\varepsilon(\Omega_i)});$$

(H2) (monotonicity) for all $z, w : Z_\varepsilon(\Omega_i) \to \mathbb{R}^d$ with $|z^j - z^l| \leq |w^j - w^l|$ for every $j, l \in Z_\varepsilon(\Omega_i)$ we have

$$\phi_i^\varepsilon(\{z^j\}_{j \in Z_\varepsilon(\Omega_i)}) \leq \phi_i^\varepsilon(\{w^j\}_{j \in Z_\varepsilon(\Omega_i)});$$

(H3) (upper bound for linear functions) there exist $c_1 > 0$ and $p \in (1,+\infty)$ such that for every $M \in \mathbb{R}^{d \times n}$ we have

$$\phi_i^\varepsilon(\{(Mx)^j\}_{j \in Z_\varepsilon(\Omega_i)}) \leq c_1(|M|^p + 1),$$

where by (Mx) we denote the linear function defined by $(Mx)^j := Mj$;

(H4) (lower bound) there exists $c_2 > 0$ such that

$$\phi_i^\varepsilon(\{z^j\}_{j \in Z_\varepsilon(\Omega_i)}) \geq c_2 \min\left\{\sum_{k=1}^n |D_\varepsilon^k z(0)|^p, \frac{1}{\varepsilon}\right\},$$

for all $i \in Z_\varepsilon(\Omega)$ with $i + \varepsilon e_k \in Z_\varepsilon(\Omega)$ for every $k \in \{1, \dots, n\}$ and every $z : Z_\varepsilon(\Omega_i) \to \mathbb{R}^d$.

Moreover, we require that the following is satisfied.

(H5) (mild non-locality) For every $\varepsilon > 0$, $\alpha \in \mathbb{N}$, $j \in Z_\varepsilon(\mathbb{R}^n)$ and $\xi \in \mathbb{Z}^n$ there exists $c^{j,\xi}_{\varepsilon,\alpha} \geq 0$ such that for every $i \in Z_\varepsilon(\Omega)$ and for all $z, w : Z_\varepsilon(\Omega_i) \to \mathbb{R}^d$ with $z^j = w^j$ for all $j \in Z_\varepsilon(\varepsilon\alpha Q)$ there holds

$$\phi^\varepsilon_i(\{z^j\}_{j\in Z_\varepsilon(\Omega_i)}) \leq \phi^\varepsilon_i(\{w^j\}_{j\in Z_\varepsilon(\Omega_i)}) + \sum_{j\in Z_\varepsilon(\Omega_i)} \sum_{\substack{\xi\in\mathbb{Z}^n \\ j+\varepsilon\xi\in\Omega_i}} c^{j,\xi}_{\varepsilon,\alpha} \min\left\{|D^\xi_\varepsilon z(j)|^p, \frac{1+|z(j+\varepsilon\xi)-w(j+\varepsilon\xi)|}{\varepsilon}\right\},$$

and the sequence $(c^{j,\xi}_{\varepsilon,\alpha})$ satisfies that following:

$$\limsup_{\varepsilon\to 0} \sum_{\alpha\in\mathbb{N}} \sum_{j\in Z_\varepsilon(\mathbb{R}^n)} \sum_{\xi\in\mathbb{Z}^n} c^{j,\xi}_{\varepsilon,\alpha} < +\infty \tag{2.4}$$

and for every $\eta > 0$ there exists a sequence (M^ε_η) with $\varepsilon M^\varepsilon_\eta \to 0$ as $\varepsilon \to 0$ such that

$$\limsup_{\varepsilon\to 0} \sum_{\max\left\{\alpha, \frac{1}{\varepsilon}|j|, |\xi|\right\} > M^\varepsilon_\eta} c^{j,\xi}_{\varepsilon,\alpha} < \eta; \tag{2.5}$$

(H6) (controlled non-convexity) there exists $c_3 > 0$ and for every $\varepsilon > 0$, $j \in Z_\varepsilon(\mathbb{R}^n)$ and $\xi \in \mathbb{Z}^n$ there exists $c^{j,\xi}_\varepsilon \geq 0$ with

$$\limsup_{\varepsilon\to 0} \sum_{j\in Z_\varepsilon(\mathbb{R}^n)} \sum_{\xi\in\mathbb{Z}^n} c^{j,\xi}_\varepsilon < +\infty \tag{2.6}$$

such that for all $i \in Z_\varepsilon(\Omega)$, every $z, w : Z_\varepsilon(\Omega_i) \to \mathbb{R}^d$ and every cut-off $\varphi : \mathbb{R}^n \to [0,1]$ we have

$$\phi^\varepsilon_i(\{\varphi^j z^j + (1-\varphi^j)w^j\}_{j\in Z_\varepsilon(\Omega_i)}) \leq c_3 \left(\phi^\varepsilon_i(\{z^j\}_{j\in Z_\varepsilon(\Omega_i)}) + \phi^\varepsilon_i(\{w^j\}_{j\in Z_\varepsilon(\Omega_i)})\right) + R^\varepsilon_i(z,w,\varphi),$$

where

$$R^\varepsilon_i(z,w,\varphi) := \sum_{j\in Z_\varepsilon(\Omega_i)} \sum_{\substack{\xi\in\mathbb{Z}^n \\ j+\varepsilon\xi\in\Omega_i}} c^{j,\xi}_\varepsilon \left(\sup_{\substack{l\in Z_\varepsilon(\Omega_i) \\ k\in\{1,\dots,n\}}} |D^k_\varepsilon \varphi(l)|^p |z(j+\varepsilon\xi) - w(j+\varepsilon\xi)|^p \right) + \sum_{j\in Z_\varepsilon(\Omega_i)} \sum_{\substack{\xi\in\mathbb{Z}^n \\ j+\varepsilon\xi\in\Omega_i}} c^{j,\xi}_\varepsilon \left(\min\left\{|D^\xi_\varepsilon z(j)|^p, \frac{1}{\varepsilon|\xi|}\right\} + \min\left\{|D^\xi_\varepsilon w(j)|^p, \frac{1}{\varepsilon|\xi|}\right\} \right);$$

Remark 2.1. Hypotheses (H1) together with (H3) imply that for every $\varepsilon > 0$, $i \in Z_\varepsilon(\Omega)$ and for any constant function $z : Z_\varepsilon(\Omega_i) \to \mathbb{R}^d$, $z^j = w$ for all $j \in Z_\varepsilon(\Omega_i)$ we have

$$\phi^\varepsilon_i(\{z^j\}_{j\in Z_\varepsilon(\Omega_i)}) = \phi^\varepsilon_i(\{0j + w\}_{j\in \mathbb{Z}_\varepsilon(\Omega_i)}) = \phi^\varepsilon_i(\{0j\}_{j\in \mathbb{Z}_\varepsilon(\Omega_i)}) \leq c_1 + 1. \tag{2.7}$$

Notice that the condition on the decaying tail of the sequence $(c^{j,\xi}_{\varepsilon,\alpha})$ in (H5) is slightly more genereal then the corresponding conditions in [1] and [17]. In fact, therein the authors choose for every $\eta > 0$ a constant $M_\eta > 0$ uniformly in ε such that the analog of (2.5) is satisfied. Here we show that this assumption can be weakened by allowing M^ε_η to depend on ε as long as $\varepsilon M^\varepsilon_\eta \to 0$. This weaker condition makes it possible to rephrase an example considered in [10] in our framework (see Section 5.3).

Remark 2.2 (Smooth truncation). In order to apply (H2) we will need to truncate $\mathbb{R}^d$-valued functions in a suitable way. To this end, following the approach in [22] we consider $\varphi \in C_c^\infty(\mathbb{R})$ with $\varphi(t) = t$ for all $t \in \mathbb{R}$ with $|t| \leq 1$, $\varphi(t) = 0$ for all $t \geq 3$ and $\|\varphi'\|_\infty \leq 1$ and we define $\phi \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^d)$ by setting

$$\phi(\zeta) := \begin{cases} \varphi(|\zeta|)\frac{\zeta}{|\zeta|} & \text{if } \zeta \neq 0, \\ 0 & \text{if } \zeta = 0. \end{cases}$$

The function ϕ is 1-Lipschitz [22, Section 4] and for every $k \in \mathbb{N}$ the function ϕ_k defined as $\phi_k(\zeta) := k\phi(\frac{\zeta}{k})$ is also 1-Lipschitz. In particular, since $\phi_k(0) = 0$, we have

$$|\phi_k(\zeta)| \leq |\zeta| \text{ for every } \zeta \in \mathbb{R}^d. \tag{2.8}$$

For every $u : \mathbb{R}^n \to \mathbb{R}^d$ we now define the truncation $T_k u := \phi_k(u)$ and we observe that thanks to (2.8) and the 1-Lipschitzianity of ϕ_k (H2) yields

$$F_\varepsilon(T_k u, A) \leq F_\varepsilon(u, A), \tag{2.9}$$

for every $k \in \mathbb{N}$, $\varepsilon > 0$, $A \in \mathcal{A}(\Omega)$ and $u \in \mathcal{A}_\varepsilon(\Omega;\mathbb{R}^d)$. Moreover, for every $u \in GSBV^p(\Omega;\mathbb{R}^d)$ and every $k \in \mathbb{N}$ the truncation $T_k u$ belongs to $SBV^p(\Omega;\mathbb{R}^d) \cap L^\infty(\Omega;\mathbb{R}^d)$ and $\|T_k u\|_{L^\infty} \leq 3k$. Finally, if $u \in GSBV^p(\Omega;\mathbb{R}^d) \cap L^1(\Omega;\mathbb{R}^d)$ there holds (see [32, Lemma 2.1])

(i) $T_k u \to u$ a.e. and in $L^1(\Omega;\mathbb{R}^d)$ as $k \to +\infty$,

(ii) $\nabla T_k u(x) = \nabla\phi_k(u(x))\nabla u(x)$ and in particular $|\nabla T_k u(x)| \leq |\nabla u(x)|$ for a.e. $x \in \Omega$ and every $k \in \mathbb{N}$,

(iii) $S_{T_k u} \subset S_u$ and $([u], \nu_u) = ([T_k u], \nu_{T_k u})$ $\mathcal{H}^{n-1}$-a.e. on $S_{T_k u}$ up to a simultaneous change of sign of $[T_k u]$ and $\nu_{T_k u}$, and by Lipschitzianity $|(T_k u)^+ - (T_k u)^-| \leq |u^+ - u^-|$ for every $k \in \mathbb{N}$. Moreover $\lim_{k\to+\infty} \mathcal{H}^{n-1}(S_{T_k u}) = \mathcal{H}^{n-1}(S_u)$.

Remark 2.3 (Γ-liminf and Γ-limsup). In all that follows we use standard notation for the Γ-liminf and the Γ-limsup, *i.e.*, for every pair $(u, A) \in L^1(\Omega;\mathbb{R}^d) \times \mathcal{A}(\Omega)$ we set

$$F'(u, A) := \Gamma\text{-}\liminf_{\varepsilon\to0} F_\varepsilon(u, A) := \inf\{\liminf_{\varepsilon\to0} F_\varepsilon(u_\varepsilon, A) \colon u_\varepsilon \to u \text{ in } L^1(\Omega;\mathbb{R}^d)\},$$
$$F''(u, A) := \Gamma\text{-}\limsup_{\varepsilon\to0} F_\varepsilon(u, A) := \inf\{\limsup_{\varepsilon\to0} F_\varepsilon(u_\varepsilon, A) \colon u_\varepsilon \to u \text{ in } L^1(\Omega;\mathbb{R}^d)\}.$$

If $A = \Omega$ we write $F'(u)$ and $F''(u)$ in place of $F'(u, \Omega)$ and $F''(u, \Omega)$.

The functional F' is superadditive as a set function [28, Proposition 16.12] and both the functionals F' and F'' are increasing as set functions [28, Proposition 6.7] and $L^1(\Omega;\mathbb{R}^d)$-lower semicontinuous in u [28, Proposition 6.8]. Moreover, from (2.9) it is easy to deduce that $F'(T_k u, A) \leq F'(u, A)$ and $F''(T_k u, A) \leq F''(u, A)$ for every $(u, A) \in L^1(\Omega;\mathbb{R}^d) \times \mathcal{A}(\Omega)$ and $k \in \mathbb{N}$. Hence the $L^1(\Omega;\mathbb{R}^d)$-lower semicontinuity together with (i) in Remark 2.2 ensure that

$$\begin{aligned} \lim_{k\to+\infty} F'(T_k u, A) &= F'(u, A), \\ \lim_{k\to+\infty} F''(T_k u, A) &= F''(u, A). \end{aligned} \tag{2.10}$$

Finally, we also consider the inner regular envelopes of F' and F'' defined as

$$\begin{aligned} F'_-(u, A) &:= \sup\{F'(u, A') \colon A' \in \mathcal{A}(\Omega),\ A' \subset\subset A\}, \\ F''_-(u, A) &:= \sup\{F''(u, A') \colon A' \in \mathcal{A}(\Omega),\ A' \subset\subset A\}, \end{aligned} \tag{2.11}$$

respectively. Then F'_- and F''_- are inner regular by definition, increasing and $L^1(\Omega;\mathbb{R}^d)$-lower semicontinuous [28, Remark 15.10].

3. Compactness and integral representation

In this section we state and prove the first main result of the paper, which is the following integral-representation result for the Γ-limit of the functionals F_ε.

Theorem 3.1 (Integral representation). *Let F_ε be as in (2.3) and suppose that $\phi_i^\varepsilon : (\mathbb{R}^d)^{Z_\varepsilon(\Omega_i)} \to [0,+\infty)$ satisfy (H1)-(H6). For every sequence of positive numbers converging to 0 there exists a subsequence (ε_j) such that (F_{ε_j}) Γ-converges to a functional $F : L^1(\Omega;\mathbb{R}^d) \to [0,+\infty]$ of the form*

$$F(u) = \begin{cases} \displaystyle\int_\Omega f(x,\nabla u)\,dx + \int_{S_u} g(x,[u],\nu_u)\,d\mathcal{H}^{n-1} & \text{if } u \in GSBV^p(\Omega;\mathbb{R}^d)\cap L^1(\Omega;\mathbb{R}^d), \\ +\infty & \text{otherwise in } L^1(\Omega;\mathbb{R}^d). \end{cases} \tag{3.1}$$

Here, for every $x_0 \in \mathbb{R}^n$, $\nu \in S^{n-1}$, $\zeta \in \mathbb{R}^d$ and $M \in \mathbb{R}^{d\times n}$ the integrands are given by the formulas

$$f(x_0,M) = \limsup_{\rho\to 0} \frac{1}{\rho^n}\mathbf{m}(u_{M,x_0}, Q_\rho^\nu(x_0)), \qquad g(x_0,\zeta,\nu) = \limsup_{\rho\to 0}\frac{1}{\rho^{n-1}}\mathbf{m}(u^\nu_{\zeta,x_0}, Q^\nu_\rho(x_0)), \tag{3.2}$$

where $u_{M,x_0}, u^\nu_{\zeta,x_0}$ are given by (2.1) and for every $\bar u \in SBV^p(\Omega;\mathbb{R}^d)$ and every $A \in \mathcal{A}^{reg}(\Omega)$ we have set

$$\mathbf{m}(\bar u, A) := \inf\{F(u,A) \colon u \in SBV^p(A;\mathbb{R}^d),\ u = \bar u \text{ in a neighborhood of } \partial A\}. \tag{3.3}$$

In particular, $g(x,t,\nu) = g(x,-t,-\nu)$ for every $(x,t,\nu) \in \Omega\times\mathbb{R}^d\times S^{n-1}$. Moreover, for every $A \in \mathcal{A}^{reg}(\Omega)$ and every $u \in GSBV^p(\Omega;\mathbb{R}^d)$ there holds

$$\Gamma\text{-}\lim_{j\to+\infty} F_{\varepsilon_j}(u,A) = \int_A f(x,\nabla u)\,dx + \int_{S_u\cap A} g(x,[u],\nu_u)\,d\mathcal{H}^{n-1}. \tag{3.4}$$

We will prove Theorem 3.1 gathering Propositions 3.2 3.3, 3.5, 3.7 and 3.9 below which together with the general compactness result Theorem 3.11 ensure that the Γ-limit F exists up to subsequences and that a suitable perturbation of F satisfies all hypotheses of [9, Theorem 1]. As a first step we show that $F''(\cdot,A)$ is local for every $A \in \mathcal{A}^{reg}(\Omega)$.

Proposition 3.2 (Locality). *Let $\phi_i^\varepsilon : (\mathbb{R}^d)^{Z_\varepsilon(\Omega_i)} \to [0,+\infty)$ satisfy hypotheses (H1)-(H6). Then for any $A \in \mathcal{A}^{reg}(\Omega)$ and $u,v \in GSBV^p(\Omega;\mathbb{R}^d)\cap L^1(\Omega;\mathbb{R}^d)$ with $u = v$ a.e. in A we have*

$$F''(u,A) = F''(v,A).$$

Proof. Let A,u,v be as in the statement. Thanks to (2.10) it suffices to consider the case $u,v \in SBV^p(\Omega;\mathbb{R}^d)\cap L^\infty(\Omega;\mathbb{R}^d)$. Let us show that $F''(u,A) \le F''(v,A)$. To this end, choose $u_\varepsilon, v_\varepsilon \in \mathcal{A}_\varepsilon(\Omega;\mathbb{R}^d)$ converging in $L^1(\Omega;\mathbb{R}^d)$ to u,v, respectively and satisfying

$$\lim_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon,A) = F''(u,A), \qquad \lim_{\varepsilon\to 0} F_\varepsilon(v_\varepsilon,A) = F''(v,A). \tag{3.5}$$

Up to considering the truncated functions $T_{\|u\|_{L^\infty}}u_\varepsilon$, $T_{\|v\|_{L^\infty}}v_\varepsilon$ we can assume that $\|u_\varepsilon\|_{L^\infty} \le 3\|u\|_{L^\infty}$, $\|v_\varepsilon\|_{L^\infty} \le 3\|u\|_{L^\infty}$.

For fixed $\eta > 0$ and every $\varepsilon > 0$ let $M^\varepsilon_\eta > 0$ be given by (2.5) and define $w_\varepsilon \in \mathcal{A}_\varepsilon(\Omega;\mathbb{R}^d)$ by setting

$$w^i_\varepsilon := \begin{cases} v^i_\varepsilon & \text{if } \operatorname{dist}_\infty(i,A) \le \varepsilon M^\varepsilon_\eta, \\ u^i_\varepsilon & \text{otherwise in } Z_\varepsilon(\Omega). \end{cases}$$

Since the sequences $(u_\varepsilon),(v_\varepsilon)$ are bounded in $L^\infty(\Omega;\mathbb{R}^d)$ uniformly in ε and $u=v$ a.e. in A we have

$$\|w_\varepsilon-u\|_{L^1(\Omega)}\le\|v_\varepsilon-v\|_{L^1(A)}+\|u_\varepsilon-u\|_{L^1(\Omega\setminus A)}+c\varepsilon^n\#\{i\in Z_\varepsilon(\Omega)\colon\,\mathrm{dist}(i,\partial A)<\varepsilon M_\eta^\varepsilon\}.$$

Moreover, since ∂A is Lipschitz it admits an upper Minkowsky content, hence

$$(\varepsilon M_\eta^\varepsilon)^{n-1}\#\{i\in Z_\varepsilon(\Omega)\colon\,\mathrm{dist}(i,\partial A)<\varepsilon M_\eta^\varepsilon\}\le c\mathcal{H}^{n-1}(\partial A)+o_{\varepsilon M_\eta^\varepsilon}(1).$$

Thus, the assumption on M_η^ε ensures that $w_\varepsilon\to u$ in $L^1(\Omega;\mathbb{R}^d)$, which implies that

$$F''(u,A)\le\limsup_{\varepsilon\to0}F_\varepsilon(w_\varepsilon,A). \tag{3.6}$$

We now come to estimate $F_\varepsilon(w_\varepsilon,A)$. For every $i\in Z_\varepsilon(A)$ we set

$$\alpha_\varepsilon(i):=\sup\{\alpha\in\mathbb{N}\colon\,w_\varepsilon^j=v_\varepsilon^j\text{ for every }j\in Z_\varepsilon(i+\varepsilon\alpha Q)\},$$

so that condition (H5) yields

$$\begin{aligned}F_\varepsilon(w_\varepsilon,A)\le&\sum_{i\in Z_\varepsilon(A)}\varepsilon^n\phi_i^\varepsilon(\{v_\varepsilon^{i+j}\}_{j\in Z_\varepsilon(\Omega_i)})\\&+\sum_{i\in Z_\varepsilon(A)}\varepsilon^n\sum_{j\in Z_\varepsilon(\Omega_i)}\sum_{\substack{\xi\in\mathbb{Z}^n\\ j+\varepsilon\xi\in\Omega_i}}c^{j,\xi}_{\varepsilon,\alpha_\varepsilon(i)}\min\left\{|D_\varepsilon^\xi w_\varepsilon^{i+j}|^p,\frac{1+|w_\varepsilon^{i+j+\xi}-v_\varepsilon^{i+j+\xi}|}{\varepsilon}\right\}.\end{aligned} \tag{3.7}$$

We observe that by construction $\alpha_\varepsilon(i)>M_\eta^\varepsilon$ for every $i\in Z_\varepsilon(A)$. Estimating the minimum in (3.7) with $(1+|w_\varepsilon^{i+j+\xi}-v_\varepsilon^{i+j+\xi}|)/\varepsilon$ and using the uniform bound on $\|v_\varepsilon\|_{L^\infty}$ and $\|w_\varepsilon\|_{L^\infty}$ thus gives

$$F_\varepsilon(w_\varepsilon,A)\le F_\varepsilon(v_\varepsilon,A)+(1+3\|u\|_{L^\infty}+3\|v\|_{L^\infty})\sum_{\alpha>M_\eta^\varepsilon}\sum_{j\in Z_\varepsilon(\mathbb{R}^n)}\sum_{\xi\in\mathbb{Z}^n}c^{j,\xi}_{\varepsilon,\alpha}\varepsilon^{n-1}\#\{i\in Z_\varepsilon(A)\colon\,\alpha_\varepsilon(i)=\alpha\}.$$

Moreover, the Lipschitz regularity of A yields

$$\varepsilon^{n-1}\#\{i\in Z_\varepsilon(A)\colon\,\alpha_\varepsilon(i)=\alpha\}\le c\mathcal{H}^{n-1}(\partial A)+o_\varepsilon(1),$$

which in view of the choice of M_η^ε and (2.5) gives

$$\limsup_{\varepsilon\to0}F_\varepsilon(w_\varepsilon,A)\le\limsup_{\varepsilon\to+\infty}F_\varepsilon(v_\varepsilon,A)+c\eta.$$

Gathering (3.5) and (3.6) we thus obtain

$$F''(u,A)\le F''(v,A)+c\eta,$$

and the desired inequality follows by the arbitrariness of $\eta>0$. □

As a next step towards the proof of Theorem 3.1 the following two propositions show that F' and F'' are finite only on $GSBV^p(\Omega;\mathbb{R}^d)\cap L^1(\Omega;\mathbb{R}^d)$ and satisfy suitable growth conditions.

Proposition 3.3 (Lower bound)**.** *Let F_ε be given by (2.3) and suppose that the functions $\phi_i^\varepsilon:(\mathbb{R}^d)^{Z_\varepsilon(\Omega_i)}\to[0,+\infty)$ satisfy (H4). Let $A\in\mathcal{A}^{reg}(\Omega)$ and $u\in L^1(\Omega;\mathbb{R}^d)$ with $F'(u,A)<+\infty$. Then $u\in GSBV^p(A;\mathbb{R}^d)$ and*

$$F'(u,A)\ge c\left(\int_A|\nabla u|^p\,dx+\mathcal{H}^{n-1}(S_u\cap A)\right) \tag{3.8}$$

for some $c>0$ independent of u and A.

Proof. Let $(u,A)\in L^1(\Omega;\mathbb{R}^d)\times\mathcal{A}^{reg}(\Omega)$ be as in the statement and let $(u_\varepsilon)\subset\mathcal{A}_\varepsilon(\Omega;\mathbb{R}^d)$ be a sequence converging to u in $L^1(\Omega;\mathbb{R}^d)$ and satisfying $\sup_\varepsilon F_\varepsilon(u_\varepsilon,A)<+\infty$. In view of (H4) we have

$$F_\varepsilon(u_\varepsilon,A)\geq c_2\sum_{i\in Z_\varepsilon(A)}\varepsilon^n\min\left\{\sum_{k=1}^n|D^k_\varepsilon u_\varepsilon(i)|^p,\frac{1}{\varepsilon}\right\},\tag{3.9}$$

hence [32, Lemma 3.3] applied to $\mathcal{L}=\mathbb{Z}^n$ and $f(p)=\min\{\|p\|_1,\frac{1}{\varepsilon}\}$ together with the uniform bound on $F_\varepsilon(u_\varepsilon,A)$ yield $u\in GSBV^p(A;\mathbb{R}^d)$. Moreover, from [32, Lemma 3.3] and (3.9) we also deduce

$$\liminf_{\varepsilon\to0}F_\varepsilon(u_\varepsilon,A)\geq c\left(\int_A|\nabla u|^p\,dx+\mathcal{H}^{n-1}(S_u\cap A)\right)$$

for some $c>0$ independent of u and A, hence (3.8) follows. □

In order to prove an upper bound for $F''(u)$ we need to restrict to a suitable dense class of functions. To this end, it is convenient to introduce the following definition of a regular triangulation.

Definition 3.4. Let $A\subset\mathbb{R}^n$ be open, bounded and with Lipschitz boundary. We say that a family $(U_l)_{l=1,\dots,N}$ of pairwise disjoint open n-simplices $U_1,\dots,U_N$ is a regular triangulation of A if $A\subset\bigcup_{l=1}^N\overline{U}_l$ and if for any $(l,l')\in\{1,\dots,N\}^2$ the intersection $S_{l,l'}:=\overline{U}_l\cap\overline{U}_{l'}$ is either the emptyset or an $(n-k)$-dimensional simplex for some $k\in\{1,\dots,n\}$. The $(n-1)$-dimensional simplices $S_{l,l'}$ are called the faces of the triangulation and by $\theta\in(0,\pi)$ we denote the minimal angle between two faces of such a triangulation.

Proposition 3.5 (Upper bound)**.** *Let $A\in\mathcal{A}^{reg}(\Omega)$ and $u\in GSBV^p(A;\mathbb{R}^d)\cap L^1(\Omega;\mathbb{R}^d)$ and suppose that the functions ϕ_i^ε satisfy (H1)-(H6). Then*

$$F''(u,A)\leq c\left(\int_A\left(|\nabla u|^p+1\right)\,dx+\int_{S_u\cap A}(1+|u^+(y)-u^-(y)|)\,d\mathcal{H}^{n-1}(y)\right)\tag{3.10}$$

for some $c>0$ independent of u and A.

Proof. Let $\tilde\Omega\subset\mathbb{R}^n$ be any open bounded set with Lipschitz boundary such that $\Omega\subset\subset\tilde\Omega$.

Step 1: As a preliminary step we prove the existence of some constant $c>0$ such that for any $u\in SBV^p(\tilde\Omega;\mathbb{R}^d)\cap L^\infty(\tilde\Omega;\mathbb{R}^d)$ and any $A\in\mathcal{A}^{reg}(\Omega)$ there holds

$$F''(u,A)\leq c\left(\int_A\left(|\nabla u|^p+1\right)\,dx+\int_{S_u\cap\overline{A}}(1+|u^+(y)-u^-(y)|)\,d\mathcal{H}^{n-1}(y)\right).\tag{3.11}$$

We first prove (3.11) for A polyhedral set.

Thanks to [27, Theorem 3.1] (see also [26, Theorem 3.9]), employing a standard density argument it suffices to prove (3.11) for $u\in SBV^p(\tilde\Omega;\mathbb{R}^d)\cap L^\infty(\tilde\Omega;\mathbb{R}^d)$ such that S_u is essentially closed (i.e., $\mathcal{H}^{n-1}(\overline{S_u}\setminus S_u)=0$), $\overline{S_u}$ is the intersection of Ω with a finite union of $(n-1)$-dimensional simplices and $u\in W^{1,\infty}(\tilde\Omega\setminus\overline{S_u};\mathbb{R}^d)$. Moreover, since $u\in W^{1,\infty}(\tilde\Omega\setminus\overline{S_u};\mathbb{R}^d)$, arguing again by density we may assume that u is piecewise affine on $\tilde\Omega\setminus\overline{S_u}$.

More precisely, we may assume that there exist a regular triangulation $(U_l)_{l=1,\dots,N}$ of $\tilde\Omega$ and $M_1,\dots,M_N\in\mathbb{R}^{d\times n}$, $b_1,\dots,b_N\in\mathbb{R}^d$ such that u satisfies the following.

- (i) $u(x)=\sum_{l=1}^N\chi_{U_l\cap\tilde\Omega}(x)(M_lx+b_l)$ for any $x\in\tilde\Omega\cap\bigcup_{l=1}^NU_l$;
- (ii) $\overline{S_u}=\tilde\Omega\cap\bigcup_{k=1}^KS_{l_k,l'_k}$, where $(S_{l_k,l'_k})_{k=1,\dots,K}$ is a collection of faces of the triangulation;
- (iii) for any face $S_{l,l'}$ with $(l,l')\neq(l_k,l'_k)$ for every $k\in\{1,\dots,K\}$ we have

$$u(x)=M_lx+b_l=M_{l'}x+b_{l'}\quad\text{for every }x\in S_{l,l'}.$$

Since A is a polyhedral set, up to refining the triangulation and renumbering the simplices we may also assume that

$$\overline{A} = \bigcup_{l=1}^{L} \overline{U}_l$$

for some $L < N$. Finally, we can assume that $\bigcup_{l,l'} S_{l,l'} \cap \varepsilon\mathbb{Z}^n = \emptyset$, since otherwise we may consider the shifted lattice $\varepsilon\mathbb{Z}^n + \xi_\varepsilon$ for a suitable sequence $\xi_\varepsilon \to 0$. We then define a sequence $(u_\varepsilon) \subset \mathcal{A}_\varepsilon(\tilde{\Omega};\mathbb{R}^d)$ by setting

$$u_\varepsilon^i := u(i) \quad \text{for every } i \in Z_\varepsilon(\tilde{\Omega})$$

and we notice that $u_\varepsilon \to u \in L^1(\Omega;\mathbb{R}^d)$. Moreover, we write

$$F_\varepsilon(u_\varepsilon, A) = \sum_{l=1}^{L} F_\varepsilon(u_\varepsilon, U_l), \tag{3.12}$$

and we estimate $F_\varepsilon(u_\varepsilon, U_l)$ for every $l \in \{1,\dots,L\}$. To this end, for $l \in \{1,\dots,L\}$ fixed and for $i \in Z_\varepsilon(U_l)$ set

$$\alpha_\varepsilon^l(i) := \sup\{\alpha \in \mathbb{N}\colon u_\varepsilon^j = M_l j + b_l \text{ for every } j \in i + \varepsilon\alpha Q\}.$$

Thanks to (H1) and (H5) we deduce

$$\begin{aligned}
F_\varepsilon(u_\varepsilon, U_l) &\leq \sum_{i\in Z_\varepsilon(U_l)} \varepsilon^n \phi_i^\varepsilon(\{(M_l x)(i+j)\}_{j\in Z_\varepsilon(\Omega_i)}) \\
&+ \sum_{i\in Z_\varepsilon(U_l)} \varepsilon^n \sum_{j\in Z_\varepsilon(\Omega_i)} \sum_{\substack{\xi\in\mathbb{Z}^n \\ j+\varepsilon\xi\in\Omega_i}} c^{j,\xi}_{\varepsilon,\alpha_\varepsilon^l(i)} \min\left\{|D_\varepsilon^\xi u(i+j)|^p, \frac{1+|u(i+j+\varepsilon\xi)-(M_l x+b_l)(i+j+\varepsilon\xi)|}{\varepsilon}\right\} \\
&=: I^l_{\varepsilon,1} + I^l_{\varepsilon,2}.
\end{aligned} \tag{3.13}$$

Moreover, (H3) gives

$$I^l_{\varepsilon,1} \leq c_1 \sum_{i\in Z_\varepsilon(U_l)} \varepsilon^n(|M|^p+1) = c_1\int_{U_l}(|\nabla u|^p+1)\,dx + o(1), \tag{3.14}$$

so that it remains to estimate $I^l_{\varepsilon,2}$. To do so, we need to introduce some notation. In what follows for $\varepsilon > 0$, $i \in Z_\varepsilon(U_l)$, $j \in Z_\varepsilon(\Omega_i)$ and $\xi \in \mathbb{Z}^n$ we use the abbreviation

$$\mathbf{m}^{j,\xi}_{\varepsilon,l}u(i) := \min\left\{|D_\varepsilon^\xi u(i+j)|^p, \frac{1+|u(i+j+\varepsilon\xi)-(M_l x+b_l)(i+j+\varepsilon\xi)|}{\varepsilon}\right\}.$$

Further, by

$$\mathcal{N}(l) := \big\{l' \in \{1,\dots,N\}\colon S_{l,l'} \text{ is an } (n-1)\text{-dimensional simplex}\big\}$$

we denote the set of all indices which label the "neighbouring" simplices of U_l. Moreover, for $\eta > 0$ fixed and every $\varepsilon > 0$ we choose $M_\eta > 0$ such that

$$\limsup_{\varepsilon\to 0} \sum_{\max\left\{\alpha,\frac{1}{\varepsilon}|j|,|\xi|\right\}>M_\eta^\varepsilon} c^{j,\xi}_{\varepsilon,\alpha} < \eta,$$

and we find $m_\varepsilon \in \mathbb{N}$ such that $\varepsilon m_\varepsilon \to 0$ and $m_\varepsilon > \frac{4M_\eta\cos\theta}{\sin\theta}$, where $\theta \in (0,\pi)$ is as in Definition 3.4. Finally, for any $l' \in \mathcal{N}(l)$ set

$$\mathcal{I}_\varepsilon^{l'} := \{i \in Z_\varepsilon(U_l)\colon \operatorname{dist}_\infty(i, U_{l'}) \leq \varepsilon m_\varepsilon\}$$

and

$$\mathcal{J}_\varepsilon^{l'} := Z_\varepsilon(U_l) \setminus \mathcal{I}_\varepsilon^{l'}.$$

Setting $U_l^\varepsilon := \{x \in U_l \colon \operatorname{dist}_\infty(x, \mathbb{R}^n \setminus U_l) > \varepsilon\}$ we get

$$\bigcap_{l' \in \mathcal{N}(l)} \mathcal{J}_\varepsilon^{l'} = Z_\varepsilon(U_l^\varepsilon).$$

For $l' \in \mathcal{N}(l)$ we also set

$$\mathcal{L}_\varepsilon^{l'} := \bigcap_{\substack{l'' \in \mathcal{N}(l) \\ l'' \neq l'}} \mathcal{J}_\varepsilon^{l''}$$

and we rewrite $I_{\varepsilon,2}^l$ as

$$\begin{aligned} I_{\varepsilon,2}^l = & \sum_{i \in Z_\varepsilon(U_l^\varepsilon)} \varepsilon^n \sum_{j \in Z_\varepsilon(\Omega_i)} \sum_{\substack{\xi \in \mathbb{Z}^n \\ j+\varepsilon\xi \in \Omega_i}} c_{\varepsilon, \alpha_\varepsilon^l(i)}^{j,\xi} \mathbf{m}_{\varepsilon,l}^{j,\xi} u(i) \\ & + \sum_{\substack{l', l'' \in \mathcal{N}(l) \\ l' \neq l''}} \sum_{i \in \mathcal{I}_{\varepsilon,m}^{l'} \cap \mathcal{I}_{\varepsilon,m}^{l''}} \varepsilon^n \sum_{j \in Z_\varepsilon(\Omega_i)} \sum_{\substack{\xi \in \mathbb{Z}^n \\ j+\varepsilon\xi \in \Omega_i}} c_{\varepsilon, \alpha_\varepsilon^l(i)}^{j,\xi} \mathbf{m}_{\varepsilon,l}^{j,\xi} u(i) \\ & + \sum_{l' \in \mathcal{N}(l)} \sum_{i \in \mathcal{I}_\varepsilon^{l'} \cap \mathcal{L}_\varepsilon^{l'}} \varepsilon^n \sum_{j \in Z_\varepsilon(\Omega_i)} \sum_{\substack{\xi \in \mathbb{Z}^n \\ j+\varepsilon\xi \in \Omega_i}} c_{\varepsilon, \alpha_\varepsilon^l(i)}^{j,\xi} \mathbf{m}_{\varepsilon,l}^{j,\xi} u(i). \end{aligned} \tag{3.15}$$

In order to estimate the first term in (3.15) we notice that

$$\varepsilon^{n-1} \#\{i \in Z_\varepsilon(U_l^\varepsilon) \colon \alpha_\varepsilon^l(i) = \alpha\} \leq c\mathcal{H}^{n-1}(\partial U_l) + o_\varepsilon(1)$$

for every $\alpha \in \mathbb{N}$. Moreover, for every $i \in Z_\varepsilon(U_l^\varepsilon)$ we have $\alpha_\varepsilon^l(i) \geq 2m_\varepsilon$. Thus, the estimate $\mathbf{m}_{\varepsilon,l}^{j,\xi} u(i) \leq (2\|u\|_{L^\infty} + 1)\varepsilon^{-1}$ yields

$$\begin{aligned} & \sum_{i \in Z_\varepsilon(U_l^\varepsilon)} \varepsilon^n \sum_{j \in Z_\varepsilon(\Omega_i)} \sum_{\substack{\xi \in \mathbb{Z}^n \\ j+\varepsilon\xi \in \Omega_i}} c_{\varepsilon, \alpha_\varepsilon^l(i)}^{j,\xi} \mathbf{m}_{\varepsilon,l}^{j,\xi} u(i) \\ & \leq (1 + 2\|u\|_{L^\infty}) \sum_{\alpha \geq 2m} \sum_{j \in Z_\varepsilon(\mathbb{R}^n)} \sum_{\xi \in \mathbb{Z}^n} c_{\varepsilon,\alpha}^{j,\xi} \varepsilon^{n-1} \#\{i \in Z_\varepsilon(U_l^\varepsilon) \colon \alpha_\varepsilon^l(i) = \alpha\} \\ & \leq c(u) \sum_{\max\{\alpha, \frac{1}{\varepsilon}|j|, |\xi|\} > M_\eta^\varepsilon} c_{\varepsilon,\alpha}^{j,\xi}. \end{aligned} \tag{3.16}$$

To bound the second term in (3.15) we observe that for every $l', l'' \in \mathcal{N}(l)$ with $l' \neq l''$ and every $\alpha \in \mathbb{N}$ we have

$$\varepsilon^{n-1} \#\{i \in \mathcal{I}_\varepsilon^{l'} \cap \mathcal{I}_\varepsilon^{l''} \colon \alpha_\varepsilon^l(i) = \alpha\} \leq \varepsilon m_\varepsilon c \left(\mathcal{H}^{n-1}(S_{l,l'}) + \mathcal{H}^{n-1}(S_{l,l''}) + o_\varepsilon(1)\right).$$

Hence, as in (3.16) we obtain

$$\begin{aligned} & \sum_{\substack{l', l'' \in \mathcal{N}(l) \\ l' \neq l''}} \sum_{i \in \mathcal{I}_{\varepsilon,m}^{l'} \cap \mathcal{I}_{\varepsilon,m}^{l''}} \varepsilon^n \sum_{j \in Z_\varepsilon(\Omega_i)} \sum_{\substack{\xi \in \mathbb{Z}^n \\ j+\varepsilon\xi \in \Omega_i}} c_{\varepsilon, \alpha_\varepsilon^l(i)}^{j,\xi} \mathbf{m}_{\varepsilon,l}^{j,\xi} u(i) \\ & \leq (1 + 2\|u\|_{L^\infty}) \sum_{\substack{l', l'' \in \mathcal{N}(l) \\ l' \neq l''}} \sum_{\alpha \in \mathbb{N}} \sum_{j \in Z_\varepsilon(\mathbb{R}^n)} \sum_{\xi \in \mathbb{Z}^n} c_{\varepsilon,\alpha}^{j,\xi} \varepsilon^{n-1} \#\{i \in \mathcal{I}_{\varepsilon,m}^{l'} \cap \mathcal{I}_{\varepsilon,m}^{l''} \colon \alpha_\varepsilon^l(i) = \alpha\} \end{aligned}$$

$$\leq c(u,n)\varepsilon m\sum_{\alpha\in\mathbb{N}}\sum_{j\in Z_\varepsilon(\mathbb{R}^n)}\sum_{\xi\in\mathbb{Z}^n}c^{j,\xi}_{\varepsilon,\alpha}\ \to 0 \text{ as } \varepsilon\to 0. \tag{3.17}$$

Finally, the last term in (3.15) can be estimated as follows. If $j\in\varepsilon\mathbb{Z}^n$ and $\xi\in\mathbb{Z}^n$ are such that $\max\{\frac{1}{\varepsilon}|j|,|\xi|\}\geq\frac{m_\varepsilon\sin\theta}{4\cos\theta}$ then the choice of m_ε allows us to deduce that

$$\begin{aligned}
&\sum_{\substack{l',l''\in\mathcal{N}(l)\\ l'\neq l''}}\sum_{i\in\mathcal{I}^{l'}_\varepsilon\cap\mathcal{L}^{l'}_\varepsilon}\varepsilon^n\sum_{\max\{\frac{1}{\varepsilon}|j|,|\xi|\}\geq\frac{m_\varepsilon\sin\theta}{4\cos\theta}}c^{j,\xi}_{\varepsilon,\alpha^l_\varepsilon(i)}\mathbf{m}^{j,\xi}_{\varepsilon,l}u(i)\\
&\leq(1+2\|u\|_{L^\infty})\sum_{\substack{l',l''\in\mathcal{N}(l)\\ l'\neq l''}}\sum_{\alpha\in\mathbb{N}}\sum_{\max\{\frac{1}{\varepsilon}|j|,|\xi|\}>M^\varepsilon_\eta}c^{j,\xi}_{\varepsilon,\alpha}\varepsilon^{n-1}\#\big\{i\in\mathcal{I}^{l'}_\varepsilon\cap\mathcal{L}^{l'}_\varepsilon\colon\alpha^l_\varepsilon(i)=\alpha\big\}\\
&\leq c(u,n)\sum_{\max\{\alpha,\frac{1}{\varepsilon}|j|,|\xi|\}>M^\varepsilon_\eta}c^{j,\xi}_{\varepsilon,\alpha},
\end{aligned}\tag{3.18}$$

where in the last step we have used that

$$\varepsilon^{n-1}\#\{i\in\mathcal{I}^{l'}_\varepsilon\cap\mathcal{L}^{l'}_\varepsilon\colon\alpha^l_\varepsilon(i)=\alpha\}\leq c\mathcal{H}^{n-1}(S_{l,l'})+o_\varepsilon(1).$$

Otherwise, for every $l'\in\mathcal{N}(l)$, $i\in\mathcal{I}^{l'}_\varepsilon\cap\mathcal{L}^{l'}_\varepsilon$ and $j\in\mathbb{Z}_\varepsilon(\Omega_i)$, $\xi\in\mathbb{Z}^n$ with $\max\{\frac{1}{\varepsilon}|j|,|\xi|\}<\frac{m_\varepsilon\sin\theta}{4\cos\theta}$ we have $[i+j,i+j+\varepsilon\xi]\subset U_l\cup U_{l'}$. We now distinguish between the case where $S_{l,l'}$ does not belong to $\overline{S_u}$ (*i.e.*, $(l,l')\neq(l_k,l'_k)$ for every $k\in\{1,\dots,K\}$) and the case where $(l,l')=(l_k,l'_k)$ for some $k\in\{1,\dots,K\}$.

In the first case we have $u\in W^{1,\infty}(U_l\cup U_{l'};\mathbb{R}^d)$, hence the inclusion $[i+j,i+j+\varepsilon\xi]\subset U_l\cup U_{l'}$ together with Jensen's inequality yield

$$\begin{aligned}
\mathbf{m}^{j,\xi}_{\varepsilon,l}u(i)&\leq|D^\xi_\varepsilon u(i+j)|^p=\frac{1}{|\xi|^p}\left|\int_0^1\nabla u(i+j+\varepsilon t\xi)\xi\,dt\right|^p\\
&\leq\frac{1}{|\xi|^p}\int_0^1|\nabla u(i+j+\varepsilon t\xi)|^p|\xi|^p\,dt\leq\|\nabla u\|_{L^\infty(U_l\cup U_{l'};\mathbb{R}^d)},
\end{aligned}$$

so that

$$\begin{aligned}
&\sum_{i\in\mathcal{I}^{l'}_\varepsilon\cap\mathcal{L}^{l'}_\varepsilon}\sum_{\substack{j\in Z_\varepsilon(\Omega_i)\\ |j|<\frac{\varepsilon m_\varepsilon\sin\theta}{4\cos\theta}}}\sum_{\substack{\xi\in\mathbb{Z}^n\\ |\xi|<\frac{m_\varepsilon\sin\theta}{4\cos\theta}}}c^{j,\xi}_{\varepsilon,\alpha^l_\varepsilon(i)}\varepsilon^n\mathbf{m}^{j,\xi}_{\varepsilon,l}u(i)\\
&\leq\sum_{\alpha\in\mathbb{N}}\sum_{j\in Z_\varepsilon(\mathbb{R}^n)}\sum_{\xi\in\mathbb{Z}^n}\|\nabla u\|_{L^\infty(U_l\cup U_{l'};\mathbb{R}^d)}c^{j,\xi}_{\varepsilon,\alpha}\varepsilon^n\#\big\{i\in\mathcal{I}^{l'}_\varepsilon\cap\mathcal{L}^{l'}_\varepsilon\colon\alpha^l_\varepsilon(i)=\alpha\big\}\\
&\leq\varepsilon c(u)\sum_{\alpha\in\mathbb{N}}\sum_{j\in Z_\varepsilon(\mathbb{R}^n)}\sum_{\xi\in\mathbb{Z}^n}c^{j,\xi}_{\varepsilon,\alpha}\ \to 0 \text{ as } \varepsilon\to 0.
\end{aligned}\tag{3.19}$$

Suppose finally that $S_{l,l'}=S_{l_k,l'_k}$ for some $k\in\{1,\dots,K\}$. Then we may estimate $\varepsilon^n\mathbf{m}^{j,\xi}_{\varepsilon,l}u(i)$ as follows,

$$\begin{aligned}
\varepsilon^n\mathbf{m}^{j,\xi}_{\varepsilon,l_k}u(i)&\leq\varepsilon^{n-1}\big(1+|(M_{l'_k}x+b_{l'_k})(i+j+\varepsilon\xi)-(M_{l_k}x+b_{l_k})(i+j+\varepsilon\xi)|\big)\\
&\leq\varepsilon^{n-1}\big(1+|(M_{l'_k}x+b_{l'_k})(p_{\nu_k}(i)+\mathrm{dist}(i,\Pi_{\nu_k})+j+\varepsilon\xi)\\
&\qquad-(M_{l_k}x+b_{l_k})(p_{\nu_k}(i)+\mathrm{dist}(i,\Pi_{\nu_k})+j+\varepsilon\xi)|\big)\\
&\leq\varepsilon^{n-1}\Big(1+|M_{l'_k}p_{\nu_k}(i)+b_{l'_k}-(M_{l_k}p_{\nu_k}(i)+b_{l_k})|+|M_{l'_k}-M_{l_k}|\Big(\sqrt{n}+\frac{\sin\theta}{4\cos\theta}\Big)\varepsilon m_\varepsilon\Big)
\end{aligned}$$

$$\leq c\int_{p_{\nu_k}(i)+[0,\varepsilon)^{n-1}}\Big(1+|M_{l'_k}p_{\nu_k}(i)+b_{l'_k}-(M_{l_k}p_{\nu_k}(i)+b_{l_k})|+\varepsilon m_\varepsilon|M_{l'_k}-M_{l_k}|\Big)\,d\mathcal{H}^{n-1}(y)$$
$$\leq c\int_{p_{\nu_k}(i)+[0,\varepsilon)^{n-1}}\Big(1+|M_{l'_k}y+b_{l'_k}-(M_{l_k}y+b_{l_k})|+\varepsilon(m_\varepsilon+1)|M_{l'_k}-M_{l_k}|\Big)\,d\mathcal{H}^{n-1}(y).$$

Note that $M_{l'_k}y+b_{l'_k}=u^+(y)$, $M_{l_k}y+b_{l_k}=u^-(y)$ for $\mathcal{H}^{n-1}$-a.e. $y\in S_{l_k,l'_k}$. Hence, we obtain

$$\sum_{i\in\mathcal{I}_\varepsilon^{l'_k}\cap\mathcal{L}_\varepsilon^{l'_k}}\ \sum_{\substack{j\in Z_\varepsilon(\Omega_i)\\ |j|<\frac{\varepsilon m_\varepsilon\sin\theta}{4\cos\theta}}}\ \sum_{\substack{\xi\in\mathbb{Z}^n\\ |\xi|<\frac{m_\varepsilon\sin\theta}{4\cos\theta}}} c^{j,\xi}_{\varepsilon,\alpha^l_\varepsilon(i)}\varepsilon^n\mathbf{m}^{j,\xi}_{\varepsilon,l}u(i)$$
$$\leq c\sum_{\alpha\in\mathbb{N}}\sum_{j\in Z_\varepsilon(\mathbb{R}^n)}\sum_{\xi\in\mathbb{Z}^n}c^{j,\xi}_{\varepsilon,\alpha}\sum_{\substack{i\in\mathcal{I}_\varepsilon^{l'_k}\cap\mathcal{L}_\varepsilon^{l'_k}\\ \alpha_\varepsilon^{l_k}(i)=\alpha}}\int_{p_{\nu_k}(i)+[0,\varepsilon)^{n-1}}\Big(1+|u^+(y)-u^-(y)|+c(u)\varepsilon m_\varepsilon\Big)\,d\mathcal{H}^{n-1}(y)$$
$$\leq\Big(c\int_{S_{l_k,l'_k}}(1+|u^+(y)-u^-(y)|)\,d\mathcal{H}^{n-1}(y)+c(u)\varepsilon m_\varepsilon\mathcal{H}^{n-1}(S_{l_k,l'_k})\Big)\sum_{\alpha\in\mathbb{N}}\sum_{j\in\mathbb{Z}_\varepsilon(\mathbb{R}^n)}\sum_{\xi\in\mathbb{Z}^n}c^{j,\xi}_{\varepsilon,\alpha}.\quad(3.20)$$

Eventually, summing up over l and gathering (3.12)-(3.20), thanks to the choice of M^ε_η and m_ε we deduce that

$$\limsup_{\varepsilon\to0}F_\varepsilon(u_\varepsilon,A)\leq c\left(\int_A(|\nabla u|^p+1)\ dx+\int_{S_u\cap\overline{A}}(1+|u^+(y)-u^-(y)|)\,d\mathcal{H}^{n-1}(y)\right)+c(u)\eta,$$

hence (3.11) follows by the arbitrariness of $\eta>0$.

In the general case $A\in\mathcal{A}^{reg}(\Omega)$ we choose A' polyhedral with $A\subset\subset A'\subset\subset\tilde{\Omega}$. Since F'' is increasing in A we then obtain

$$F''(u,A)\leq F''(u,A')\leq c\left(\int_{A'}(|\nabla u|^p+1)\ dx+\int_{S_u\cap\overline{A'}}(1+|u^+(y)-u^-(y)|)\,d\mathcal{H}^{n-1}(y)\right),$$

and (3.11) follows by letting $A'\searrow A$.

Step 2: We now prove (3.10) for $A\in\mathcal{A}^{reg}(\Omega)$ and $u\in SBV^p(A;\mathbb{R}^d)\cap L^\infty(A;\mathbb{R}^d)$. Thanks to the Lipschitz-regularity of A, using a local reflection argument we can extend u to a function $\tilde{u}\in SBV^p(\tilde{\Omega})\cap L^\infty(\tilde{\Omega})$ in such a way that $\mathcal{H}^{n-1}(S_{\tilde{u}}\cap\partial A)=0$. Thus Step 1 together with Proposition 3.2 give

$$F''(u,A)=F''(\tilde{u}_{|A},A)\leq c\left(\int_A(|\nabla u|^p+1)\ dx+\int_{S_u\cap\overline{A}}(1+|u^+(y)-u^-(y)|)\,d\mathcal{H}^{n-1}(y)\right)$$
$$=c\left(\int_A(|\nabla u|^p+1)\ dx+\int_{S_u\cap A}(1+|u^+(y)-u^-(y)|)\,d\mathcal{H}^{n-1}(y)\right).$$

Step 3: We finally remove the assumption $u\in SBV^p(A;\mathbb{R}^d)\cap L^\infty(A;\mathbb{R}^d)$ by considering the truncated functions introduced in Remark 2.2. More precisely, for any $u\in GSBV^p(A;\mathbb{R}^d)\cap L^1(\Omega;\mathbb{R}^d)$ and any $k\in\mathbb{N}$ consider the truncation $T_ku\in SBV^p(A;\mathbb{R}^d)\cap L^\infty(\Omega;\mathbb{R}^d)$. Combining Step 2 with (2.10) we then obtain

$$F''(u,A)=\lim_{k\to+\infty}F''(u_k,A)$$
$$\leq c\limsup_{k\to+\infty}\left(\int_A(|\nabla T_ku|^p+1)\ dx+\int_{S_{T_ku}\cap\overline{A}}(1+|(T_ku)^+(y)-(T_ku)^-(y)|)\,d\mathcal{H}^{n-1}(y)\right),$$

hence (3.10) follows by Properties (ii) and (iii) in Remark 2.2. □

As a next step we establish an almost subadditivity of the functionals F''. As a preliminary step we prove a version of [1, Lemma 3.6] adapted to our setting.

Lemma 3.6. *There exists $c>0$ depending only on n such that for any $u\in\mathcal{A}_\varepsilon(\Omega;\mathbb{R}^d)$ and any $\xi\in\mathbb{Z}^n$ we have*

$$\sum_{\substack{i\in Z_\varepsilon(\Omega)\\ i+\varepsilon\xi\in\Omega}}\min\Big\{|D^\xi_\varepsilon u^i|^p,\frac{1}{\varepsilon|\xi|}\Big\}\leq c\sum_{i\in Z_\varepsilon(B_R)}\min\Big\{\sum_{k=1}^n|D^k_\varepsilon u^i|^p,\frac{1}{\varepsilon}\Big\},$$

where $B_R\subset\mathbb{R}^n$ is any open ball with $\Omega\subset\subset B_R$.

Proof. Following the same procedure as in [1, Lemma 3.6] for $\xi\in\mathbb{Z}^n$ and $i\in Z_\varepsilon(\mathbb{R}^n)$ we set

$$\mathcal{I}^\xi_\varepsilon(i):=\{j\in Z_\varepsilon(\mathbb{R}^n)\colon (j+[-\varepsilon,\varepsilon]^n)\cap[i,i+\varepsilon\xi]\neq\emptyset\},$$

and for $i\in Z_\varepsilon(\Omega)$ with $i+\varepsilon\xi\in\Omega$ we choose a sequence $(i_h)_{h=0}^{|\xi|_1}\subset\mathcal{I}^\xi_\varepsilon(i)$ satisfying

$$i_0=i,\quad i_{|\xi|_1}=i+\varepsilon\xi,\quad i_h=i_{h-1}+\varepsilon e_{i(h)}\text{ for some }i(h)\in\{1,\dots,n\},$$

so that

$$D^\xi_\varepsilon u(i)=\frac{1}{|\xi|}\sum_{h=1}^{|\xi|_1}D^{i(h)}_\varepsilon u(i_{h-1}).$$

As in [1, Lemma 3.6], applying Jensen's inequality we obtain

$$|D^\xi_\varepsilon u(i)|^p\leq\frac{n^{\frac{p}{2}}}{|\xi|_1}\sum_{h=1}^{|\xi|_1}|D^{i(h)}_\varepsilon u(i_{h-1})|^p,$$

hence the fact that min is non-decreasing yields

$$\begin{aligned}\min\Big\{|D^\xi_\varepsilon u(i)|^p,\frac{1}{\varepsilon|\xi|}\Big\}&\leq\min\Big\{\frac{n^{\frac{p}{2}}}{|\xi|_1}\sum_{h=1}^{|\xi|_1}|D^{i(h)}_\varepsilon u(i_{h-1})|^p,\frac{1}{\varepsilon|\xi|}\Big\}\\&=\frac{n^{\frac{p}{2}}}{|\xi|_1}\min\Big\{\sum_{h=1}^{|\xi|_1}|D^{i(h)}_\varepsilon u(i_{h-1})|^p,\frac{|\xi|_1}{\varepsilon|\xi|n^{\frac{p}{2}}}\Big\}\leq\frac{n^{\frac{p}{2}}}{|\xi|_1}\min\Big\{\sum_{j\in\mathcal{I}^\xi_\varepsilon(i)}\sum_{k=1}^n|D^k_\varepsilon u(j)|^p,\frac{|\xi|_1}{\varepsilon|\xi|n^{\frac{p}{2}}}\Big\}\\&\leq\frac{n^{\frac{p}{2}}}{|\xi|_1}\sum_{j\in\mathcal{I}^\xi_\varepsilon(i)}\min\Big\{\sum_{k=1}^n|D^k_\varepsilon u(j)|^p,\frac{|\xi|_1}{\varepsilon|\xi|n^{\frac{p}{2}}}\Big\},\end{aligned}\tag{3.21}$$

where in the last step we have used the subadditivity of min. Let $B_R\subset\mathbb{R}^n$ be any open ball with $\Omega\subset\subset B_R$. Notice that for $\xi\in\mathbb{Z}^n$, $i\in Z_\varepsilon(\Omega)$ with $i+\varepsilon\xi\in\Omega$ and ε sufficiently small there holds $\mathcal{I}^\xi_\varepsilon(i)\subset Z_\varepsilon(B_R)$. Thus, from (3.21) together with the fact that $\frac{|\xi|_1}{|\xi|n^{\frac{p}{2}}}\leq 1$ we deduce

$$\sum_{\substack{i\in Z_\varepsilon(\Omega)\\ i+\varepsilon\xi\in\Omega}}\Big\{|D^\xi_\varepsilon u(i)^p,\frac{1}{\varepsilon|\xi|}\Big\}\leq\frac{n^{\frac{p}{2}}}{|\xi|_1}\sum_{j\in Z_\varepsilon(B_R)}\#\mathcal{J}^\xi_\varepsilon(j)\min\Big\{\sum_{k=1}^n|D^k_\varepsilon u(j)|^p,\frac{1}{\varepsilon}\Big\},\tag{3.22}$$

where for any $j\in Z_\varepsilon(B_R)$ we have set

$$\mathcal{J}^\xi_\varepsilon(j):=\{i\in Z_\varepsilon(\Omega)\colon i+\varepsilon\xi\in\Omega,\ j\in\mathcal{I}^\xi_\varepsilon(i)\}.$$

In [1, Lemma 3.6] it has been proved that $\#\mathcal{J}^{\xi}_{\varepsilon}(j) \leq c(n)|\xi|$ for some $c(n) > 0$ independent of ε, j, ξ, hence the result follows from (3.22) taking $c = c(n)n^{\frac{p}{2}}$ upon noticing that $|\xi| \leq |\xi|_1$. □

Proposition 3.7 (Subadditivity). *Let $u \in GSBV^p(\Omega;\mathbb{R}^d) \cap L^1(\Omega;\mathbb{R}^d)$ and $A, B \in \mathcal{A}(\Omega)$ and suppose that ϕ_i^{ε} satisfy (H1)-(H6). For every $A', B' \in \mathcal{A}^{reg}(\Omega)$ with $A' \subset\subset A$ and $B' \subset\subset B$ we have*

$$F''(u, A' \cup B') \leq F''(u, A) + F''(u, B). \tag{3.23}$$

Proof. It suffices to prove the result for $u \in SBV^p(\Omega;\mathbb{R}^d) \cap L^\infty(\Omega;\mathbb{R}^d)$, then the general case follows by arguing as in Step 3 of Proposition 3.5. Moreover, we can assume that $F''(u, A) + F''(u, B) < +\infty$, otherwise the inequality trivially holds. Let $(u_\varepsilon), (v_\varepsilon) \subset \mathcal{A}_\varepsilon(\Omega;\mathbb{R}^d)$ be two sequences converging in $L^1(\Omega;\mathbb{R}^d)$ to u with

$$\limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, A) = F''(u, A) < +\infty, \tag{3.24}$$

$$\limsup_{\varepsilon\to 0} F_\varepsilon(v_\varepsilon, B) = F''(u, B) < +\infty. \tag{3.25}$$

Thanks to (H2), upon considering the truncated sequences $(T_M u_\varepsilon), (T_M v_\varepsilon)$ with $M = \|u\|_{L^\infty(\Omega;\mathbb{R}^d)}$ we can always assume that $\|u_\varepsilon\|_{L^\infty(\Omega;\mathbb{R}^d)}, \|v_\varepsilon\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq 3\|u\|_{L^\infty(\Omega;\mathbb{R}^d)}$ for every $\varepsilon > 0$, which implies that $u_\varepsilon \to u$, $v_\varepsilon \to u$ also in $L^p(\Omega;\mathbb{R}^d)$. Moreover, in view of (H4) we get

$$\sup_{\varepsilon>0} \sum_{i\in Z_\varepsilon(A'')} \varepsilon^n \min\left\{\sum_{k=1}^n |D^k_\varepsilon u^i_\varepsilon|^p, \frac{1}{\varepsilon}\right\} < +\infty, \tag{3.26}$$

$$\sup_{\varepsilon>0} \sum_{i\in Z_\varepsilon(B'')} \varepsilon^n \min\left\{\sum_{k=1}^n |D^k_\varepsilon v^i_\varepsilon|^p, \frac{1}{\varepsilon}\right\} < +\infty, \tag{3.27}$$

for every $A'' \subset\subset A$, $B'' \subset\subset B$.

Step 1: We first replace (u_ε) and (v_ε) by sequences $(\tilde{u}_\varepsilon), (\tilde{v}_\varepsilon)$ satisfying (3.26) and (3.27) with B_R in place A'' (respectively B''), where $B_R \subset \mathbb{R}^n$ is an open ball with $\Omega \subset\subset B_R$. To do so, arguing as in Proposition 3.5 we extend $u \in SBV^p(\Omega;\mathbb{R}^d) \cap L^\infty(\Omega;\mathbb{R}^d)$ to a function $\tilde{u} \in SBV^p(B_R;\mathbb{R}^d) \cap L^\infty(B_R;\mathbb{R}^d)$ with

$$F''(\tilde{u}_{|\Omega}, \Omega) \leq c\left(\int_\Omega (|\nabla u|^p + 1)\,dx + \int_{S_u} (1 + |u^+(y) - u^-(y)|)\,d\mathcal{H}^{n-1}(y)\right) < +\infty. \tag{3.28}$$

In view of (3.28) there exists a sequence $(w_\varepsilon) \subset \mathcal{A}_\varepsilon(\Omega;\mathbb{R}^d)$ converging in $L^1(\Omega;\mathbb{R}^d)$ to $\tilde{u}_{|\Omega} = u$ with

$$\limsup_{\varepsilon\to 0} F_\varepsilon(w_\varepsilon, \Omega) = F''(\tilde{u}_{|\Omega}, \Omega) < +\infty.$$

Arguing again by truncation we can assume that $\|w_\varepsilon\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq 3\|u\|_{L^\infty(\Omega;\mathbb{R}^d)}$ for every $\varepsilon > 0$ and thus $w_\varepsilon \to u$ in $L^p(\Omega;\mathbb{R}^d)$. Moreover, appealing once more to (H4), upon extending w_ε by 0 outside of Ω we get

$$\sup_{\varepsilon>0} \sum_{i\in Z_\varepsilon(B_R)} \varepsilon^n \min\left\{\sum_{k=1}^n |D^k_\varepsilon w^i_\varepsilon|^p, \frac{1}{\varepsilon}\right\} < +\infty. \tag{3.29}$$

We now choose $A'', A''', B'', B''' \in \mathcal{A}^{reg}(\Omega)$ with $A' \subset\subset A'' \subset\subset A''' \subset\subset A$ and $B' \subset\subset B'' \subset\subset B''' \subset\subset B$ and cut-off functions φ_A between A'' and A''' and φ_B between B'' and B'''. Set

$$\tilde{u}_\varepsilon := \varphi_A u_\varepsilon + (1 - \varphi_A) w_\varepsilon$$

$$\tilde{v}_\varepsilon := \varphi_B v_\varepsilon + (1 - \varphi_B) w_\varepsilon.$$

We still have $\tilde{u}_\varepsilon, \tilde{v}_\varepsilon \to u$ in $L^p(\Omega;\mathbb{R}^d)$, so that

$$\lim_{\varepsilon\to 0}\sum_{i\in Z_\varepsilon(\Omega)}\varepsilon^n|\tilde{u}_\varepsilon-\tilde{v}_\varepsilon|^p=0. \tag{3.30}$$

Further, for every $i\in Z_\varepsilon(B_R)$ and every $k\in\{1,\dots,n\}$ there holds

$$D_\varepsilon^k\tilde{u}_\varepsilon^i=\varphi_A(i+\varepsilon e_k)D_\varepsilon^k u_\varepsilon^i+(1-\varphi_A(i+\varepsilon e_k))D_\varepsilon^k w_\varepsilon^i+D_\varepsilon^k\varphi_A(i)(v_\varepsilon^i-w_\varepsilon^i).$$

Thus, (3.27) and (3.29) together with the equi-boundedness of $\|v_\varepsilon\|_{L^p(\Omega;\mathbb{R}^d)}, \|w_\varepsilon\|_{L^p(\Omega;\mathbb{R}^d)}$ and the fact that $\{\varphi_A>0\}\subset\subset A$ yield

$$\sup_{\varepsilon>0}\sum_{i\in Z_\varepsilon(B_R)}\varepsilon^n\min\left\{\sum_{k=1}^n|D_\varepsilon^k\tilde{u}_\varepsilon^i|^p,\frac{1}{\varepsilon}\right\}<+\infty. \tag{3.31}$$

Anlogously we also obtain

$$\sup_{\varepsilon>0}\sum_{i\in Z_\varepsilon(B_R)}\varepsilon^n\min\left\{\sum_{k=1}^n|D_\varepsilon^k\tilde{v}_\varepsilon^i|^p,\frac{1}{\varepsilon}\right\}<+\infty. \tag{3.32}$$

Step 2: For fixed $\eta>0$ we now construct a sequence $(\tilde{w}_\varepsilon)\subset\mathcal{A}_\varepsilon(\Omega;\mathbb{R}^d)$ converging to u in $L^1(\Omega;\mathbb{R}^d)$ and satisfying

$$\limsup_{\varepsilon\to 0}F_\varepsilon(\tilde{w}_\varepsilon,A'\cup B')\le(1+\eta)\big(F''(u,A)+F''(u,B)\big)+c(u,A',B')\eta, \tag{3.33}$$

then (3.23) follows by the arbitrariness of $\eta>0$.

Let $\eta>0$ be arbitrary and for every $\varepsilon>0$ let $M_\eta^\varepsilon>0$ be as in (2.5) in (H5) with

$$\limsup_{\varepsilon\to 0}\sum_{\max\left\{\alpha,\frac{1}{\varepsilon}|j|,|\xi|\right\}>M_\eta^\varepsilon}c_{\varepsilon,\alpha}^{j,\xi}<\eta.$$

Moreover, set $\mathrm{d}_A:=\mathrm{dist}(A',\mathbb{R}^n\setminus A'')$, choose $L\in\mathbb{N}$ and for every $l\in\{1,\dots,L\}$ set

$$A_l:=\Big\{x\in A''\colon\ \mathrm{dist}(x,A')<\frac{l\,\mathrm{d}_A}{L}\Big\},$$

and let $A_0:=A'$. Notice that up to choosing A'' such that d_A is small enough the sets A_l have Lipschitz-boundary for every $l\in\{1,\dots,L\}$ and satisfy $\mathcal{H}^{n-1}(\partial A_l)\le\mathcal{H}^{n-1}(\partial A')+1$.

For every $l\in\{1,\dots,L-1\}$ let φ_l be a cut-off function between A_l and A_{l+1}, so that $\varphi_l\equiv 1$ on A_l, $\varphi_l\equiv 0$ on $\Omega\setminus A_{l+1}$ and $\|\nabla\varphi_l\|_{L^\infty(\Omega,\mathbb{R}^n)}\le\frac{2L}{\mathrm{d}_A}$.

We also set $\mathrm{d}_B:=\mathrm{dist}(B',\mathbb{R}^n\setminus B'')$ and we choose $\varepsilon_0>0$ such that $\varepsilon\sqrt{n}M_\eta^\varepsilon<\min\{\mathrm{d}_B,\frac{\mathrm{d}_A}{L}\}$ for every $\varepsilon\in(0,\varepsilon_0)$. For every $l\in\{1,\dots,L-3\}$ and $\varepsilon\in(0,\varepsilon_0)$ we then define a function $w_{\varepsilon,l}\in\mathcal{A}_\varepsilon(\Omega;\mathbb{R}^d)$ by setting

$$w_{\varepsilon,l}^i:=\varphi_l(i)\tilde{u}_\varepsilon^i+(1-\varphi_l(i))\tilde{v}_\varepsilon^i,$$

and we notice that $w_{\varepsilon,l}\to u$ in $L^1(\Omega;\mathbb{R}^d)$ as $\varepsilon\to 0$. Moreover,

$$F_\varepsilon(w_{\varepsilon,l},A'\cup B')=F_\varepsilon(w_{\varepsilon,l},A_{l-1})+F_\varepsilon(w_{\varepsilon,l},(A_{l+2}\setminus A_{l-1})\cap B')+F_\varepsilon(w_{\varepsilon,l},B'\setminus A_{l+2}). \tag{3.34}$$

We estimate the three terms on the right hand side of (3.34) separately. We start with the estimate for $F_\varepsilon(w_{\varepsilon,l},A_{l-1})$. To this end, for every $i\in Z_\varepsilon(A_{l-1})$ we set

$$\alpha_\varepsilon^l(i):=\sup\{\alpha\in\mathbb{N}\colon i+\varepsilon\alpha Q\subset A_l\}.$$

Since $\varepsilon\sqrt{n}M_\eta^\varepsilon < \frac{\mathrm{d}_A}{L}$, we have $\alpha_\varepsilon^l(i) > M_\eta^\varepsilon$ for every $i \in Z_\varepsilon(A_{l-1})$. Further,

$$w_{\varepsilon,l}^{i+j} = \tilde{u}_\varepsilon^{i+j} = u_\varepsilon^{i+j} \text{ for every } j \in Z_\varepsilon(\varepsilon\alpha_\varepsilon^l(i)Q),$$

and for every $\alpha \in \mathbb{N}$ we have

$$\varepsilon^{n-1}\#\{i \in Z_\varepsilon(A_{l-1})\colon \alpha_\varepsilon^l(i) = \alpha\} \le c\mathcal{H}^{n-1}(\partial A_l) + o_\varepsilon(1) \le c(\mathcal{H}^{n-1}(\partial A') + 1).$$

Hence, (H3) yields

$$\begin{aligned}
F_\varepsilon(w_{\varepsilon,l}, A_{l-1}) \le& \sum_{i\in Z_\varepsilon(A_{l-1})} \varepsilon^n \phi_i^\varepsilon(\{u_\varepsilon^{i+j}\}_{j\in Z_\varepsilon(\Omega_i)}) \\
&+ \sum_{i\in Z_\varepsilon(A_{l-1})}\sum_{j\in Z_\varepsilon(\Omega_i)}\sum_{\substack{\xi\in\mathbb{Z}^n\\ j+\varepsilon\xi\in\Omega_i}} c_{\varepsilon,\alpha_\varepsilon^l(i)}^{j,\xi} \min\left\{|D_\varepsilon^\xi w_{\varepsilon,l}(i+j)|^p, \frac{1+|w_{\varepsilon,l}(i+j+\varepsilon\xi) - u_\varepsilon(i+j+\varepsilon\xi)|}{\varepsilon}\right\} \\
&\le F_\varepsilon(u_\varepsilon, A) + (1+6\|u\|_{L^\infty}) \sum_{\alpha>M_\eta^\varepsilon}\sum_{j\in Z_\varepsilon(\mathbb{R}^n)}\sum_{\xi\in\mathbb{Z}^n} c_{\varepsilon,\alpha}^{j,\xi}\varepsilon^{n-1}\#\{i\in Z_\varepsilon(A_{l-1})\colon \alpha_\varepsilon^l(i)=\alpha\} \\
&\le F_\varepsilon(u_\varepsilon, A) + c(1+6\|u\|_{L^\infty})(\mathcal{H}^{n-1}(\partial A')+1)\sum_{\alpha>M_\eta^\varepsilon}\sum_{j\in Z_\varepsilon(\mathbb{R}^n)}\sum_{\xi\in\mathbb{Z}^n} c_{\varepsilon,\alpha}^{j,\xi}.
\end{aligned} \tag{3.35}$$

Analogously, for every $i \in Z_\varepsilon(B' \setminus A_{l+2})$ we set

$$\beta_\varepsilon^l(i) := \sup\{\beta \in \mathbb{N}\colon i + \varepsilon\beta Q \subset B'' \setminus A_{l+1}\},$$

and we observe that $\beta_\varepsilon^l(i) > M_\eta^\varepsilon$ for every $i \in Z_\varepsilon(B' \setminus A_{l+2})$ and

$$w_{\varepsilon,l}^{i+j} = \tilde{v}_\varepsilon^{i+j} = v_\varepsilon^{i+j} \text{ for every } j \in Z_\varepsilon(\varepsilon\beta_\varepsilon^l(i)Q).$$

Thus, an analogous computation as in (3.35) leads to

$$\begin{aligned}
&F_\varepsilon(w_{\varepsilon,l}, B'\setminus A_{l+2}) \\
&\le F_\varepsilon(v_\varepsilon, B) + (1+6\|u\|_{L^\infty})\sum_{\beta>M_\eta^\varepsilon}\sum_{j\in Z_\varepsilon(\mathbb{R}^n)}\sum_{\xi\in\mathbb{Z}^n} c_{\varepsilon,\beta}^{j,\xi}\varepsilon^{n-1}\#\{i\in Z_\varepsilon(B'\setminus A_{l+2})\colon \beta_\varepsilon^l(i)=\beta\} \\
&\le F_\varepsilon(v_\varepsilon, B) + c(1+6\|u\|_{L^\infty})(\mathcal{H}^{n-1}(\partial A') + \mathcal{H}^{n-1}(\partial B')+1)\sum_{\beta>M_\eta^\varepsilon}\sum_{j\in Z_\varepsilon(\mathbb{R}^n)}\sum_{\xi\in\mathbb{Z}^n} c_{\varepsilon,\beta}^{j,\xi}.
\end{aligned} \tag{3.36}$$

Finally, in view of (H6) we have

$$\begin{aligned}
F_\varepsilon(w_{\varepsilon,l}, (A_{l+2}\setminus A_{l-1})\cap B') \le c_3\Bigg(&\sum_{i\in Z_\varepsilon(S_l)}\varepsilon^n\phi_i^\varepsilon(\{\tilde{u}_\varepsilon^{i+j}\}_{j\in Z_\varepsilon(\Omega_i)}) + \sum_{i\in Z_\varepsilon(S_l)}\varepsilon^n\phi_i^\varepsilon(\{\tilde{v}_\varepsilon^{i+j}\}_{j\in Z_\varepsilon(\Omega_i)})\Bigg) \\
&+ \sum_{i\in Z_\varepsilon(S_l)}\varepsilon^n R_i^\varepsilon(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon, \varphi_l),
\end{aligned} \tag{3.37}$$

where $S_l := (A_{l+2}\setminus A_{l-1})\cap B'$ and

$$\begin{aligned}
R_i^\varepsilon(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon, \varphi_l) = \left(\frac{2L}{\mathrm{d}_A}\right)^p &\sum_{j\in Z_\varepsilon(\Omega)}\sum_{\substack{\xi\in\mathbb{Z}^n\\ j+\varepsilon\xi\in\Omega}} c_\varepsilon^{j-i,\xi}|\tilde{u}_\varepsilon(j+\varepsilon\xi) - \tilde{v}_\varepsilon(j+\varepsilon\xi)|^p \\
&+ \sum_{j\in Z_\varepsilon(\Omega)}\sum_{\substack{\xi\in\mathbb{Z}^n\\ j+\varepsilon\xi\in\Omega}} c_\varepsilon^{j-i,\xi}\left(\min\left\{|D_\varepsilon^\xi\tilde{u}_\varepsilon^j|^p, \frac{1}{\varepsilon|\xi|}\right\} + \min\left\{|D_\varepsilon^\xi\tilde{v}_\varepsilon^j|^p, \frac{1}{\varepsilon|\xi|}\right\}\right).
\end{aligned}$$

Notice that the same computations as in (3.35) and (3.36) lead to

$$\sum_{i\in Z_\varepsilon(S_l)} \varepsilon^n \phi_i^\varepsilon(\{\tilde{u}_\varepsilon^{i+j}\}_{j\in Z_\varepsilon(\Omega_i)}) \leq F_\varepsilon(u_\varepsilon, S_l) + c(u,A') \sum_{\alpha>M_\eta^\varepsilon}\sum_{j\in Z_\varepsilon(\mathbb{R}^n)}\sum_{\xi\in\mathbb{Z}^n} c_{\varepsilon,\alpha}^{j,\xi} \tag{3.38}$$

and

$$\sum_{i\in Z_e(S_l)} \varepsilon^n \phi_i^\varepsilon(\{\tilde{v}_\varepsilon^{i+j}\}_{j\in Z_\varepsilon(\Omega_i)}) \leq F_\varepsilon(v_\varepsilon, S_l) + c(u,A',B') \sum_{\alpha>M_\eta^\varepsilon}\sum_{j\in Z_\varepsilon(\mathbb{R}^n)}\sum_{\xi\in\mathbb{Z}^n} c_{\varepsilon,\alpha}^{j,\xi}, \tag{3.39}$$

respectively. Moreover, Lemma 3.6 together with (3.31) and (3.32) give

$$\sup_{\varepsilon>0}\sup_{\xi\in\mathbb{Z}^n} \sum_{\substack{j\in Z_\varepsilon(\Omega)\\ j+\varepsilon\xi\in\Omega}} \varepsilon^n \left(\min\left\{|D_\varepsilon^\xi \tilde{u}_\varepsilon^j|^p, \frac{1}{\varepsilon|\xi|}\right\} + \min\left\{|D_\varepsilon^\xi \tilde{v}_\varepsilon^j|^p, \frac{1}{\varepsilon|\xi|}\right\}\right) \leq M \tag{3.40}$$

for some $M>0$. For every l we have $\#\{l'\neq l\colon S_l\cap S_{l'}\neq\emptyset\}\leq 5$. Thus, gathering (3.34)-(3.40), summing up over l and averaging we find $l(\varepsilon)\in\{1,\dots,L-3\}$ such that

$$\begin{aligned}
&F_\varepsilon(w_{\varepsilon,l(\varepsilon)}, A'\cup B') \leq \frac{1}{L-4}\sum_{l=1}^{L-3} F_\varepsilon(w_{\varepsilon,l}, A'\cup B')\\
&\leq \left(1+\frac{5c_3}{L-4}\right)(F_\varepsilon(u_\varepsilon,A)+F_\varepsilon(v_\varepsilon,B)) + c(u,A',B')\sum_{\alpha>M_\eta^\varepsilon}\sum_{j\in Z_\varepsilon(\mathbb{R}^n)}\sum_{\xi\in\mathbb{Z}^n} c_{\varepsilon,\alpha}^{j,\xi}\\
&+\frac{5}{L-4}\left(\frac{2L}{\mathrm{d}_A}\right)^p \sum_{i\in Z_\varepsilon(A''\cap B')}\sum_{j\in Z_\varepsilon(\Omega)}\sum_{\substack{\xi\in\mathbb{Z}^n\\ j+\varepsilon\xi\in\Omega}} \varepsilon^n c_\varepsilon^{j-i,\xi}|\tilde{u}_\varepsilon(j+\varepsilon\xi)-\tilde{v}_\varepsilon(j+\varepsilon\xi)|^p\\
&+\frac{5}{L-4}\sum_{i\in Z_\varepsilon(A''\cap B')}\varepsilon^n\sum_{j\in Z_\varepsilon(\Omega)}\sum_{\substack{\xi\in\mathbb{Z}^n\\ j+\varepsilon\xi\in\Omega}} c_\varepsilon^{j-i,\xi}\left(\min\left\{|D_\varepsilon^\xi\tilde{u}_\varepsilon^j|^p,\frac{1}{\varepsilon|\xi|}\right\}+\min\left\{|D_\varepsilon^\xi\tilde{v}_\varepsilon^j|^p,\frac{1}{\varepsilon|\xi|}\right\}\right)\\
&\leq \left(1+\frac{5c_3}{L-4}\right)(F_\varepsilon(u_\varepsilon,A)+F_\varepsilon(v_\varepsilon,B)) + c(u,A',B')\sum_{\alpha>M_\eta^\varepsilon}\sum_{j\in Z_\varepsilon(\mathbb{R}^n)}\sum_{\xi\in\mathbb{Z}^n} c_{\varepsilon,\alpha}^{j,\xi}\\
&+\frac{5}{L-4}\left(\frac{2L}{\mathrm{d}_A}\right)^p\sum_{\xi\in\mathbb{Z}^n}\sum_{z\in Z_\varepsilon(\mathbb{R}^n)} c_\varepsilon^{z,\xi}\sum_{\substack{j\in Z_\varepsilon(\Omega)\\ j+\varepsilon\xi\in\Omega}} |\tilde{u}_\varepsilon(j+\varepsilon\xi)-\tilde{v}_\varepsilon(j+\varepsilon\xi)|^p\\
&+\frac{5}{L-4}\sum_{\xi\in\mathbb{Z}^n}\sum_{z\in Z_\varepsilon(\mathbb{R}^n)} c_\varepsilon^{z,\xi}\sum_{\substack{j\in Z_\varepsilon(\Omega)\\ j+\varepsilon\xi\in\Omega}}\varepsilon^n\left(\min\left\{|D_\varepsilon^\xi\tilde{u}_\varepsilon^j|^p,\frac{1}{\varepsilon|\xi|}\right\}+\min\left\{|D_\varepsilon^\xi\tilde{v}_\varepsilon^j|^p,\frac{1}{\varepsilon|\xi|}\right\}\right)\\
&\leq \left(1+\frac{5c_3}{L-4}\right)(F_\varepsilon(u_\varepsilon,A)+F_\varepsilon(v_\varepsilon,B)) + c(u,A',B')\sum_{\alpha>M_\eta^\varepsilon}\sum_{j\in Z_\varepsilon(\mathbb{R}^n)}\sum_{\xi\in\mathbb{Z}^n} c_{\varepsilon,\alpha}^{j,\xi}\\
&+\frac{5}{L-4}\left(\frac{2L}{\mathrm{d}_A}\right)^p\left(\sum_{\xi\in\mathbb{Z}^n}\sum_{z\in Z_\varepsilon(\mathbb{R}^n)} c_\varepsilon^{z,\xi}\right)\sum_{i\in Z_\varepsilon(\Omega)}\varepsilon^n|\tilde{u}_\varepsilon^i-\tilde{v}_\varepsilon^i|^p + \frac{5M}{L-4}\sum_{\xi\in\mathbb{Z}^n}\sum_{z\in Z_\varepsilon(\mathbb{R}^n)} c_\varepsilon^{z,\xi},
\end{aligned}$$

hence (3.24),(3.25) and (3.30) together with the choice of M_η yield

$$\limsup_{\varepsilon\to 0} F_\varepsilon(w_{\varepsilon,l(\varepsilon)},A'\cup B')\leq\Big(1+\frac{5c_3}{L-4}\Big)\left(F''(u,A)+F''(u,B)\right)+c(u,A',B')\eta+\frac{c}{L-4}.$$

It remains to choose $L\in\mathbb{N}$ sufficiently large such that $\frac{5c_3}{L-4}<\eta$ and $\frac{c}{L-4}<\eta$, then $\tilde{w}_\varepsilon:=w_{\varepsilon,l(\varepsilon)}$ is the required sequence satisfying (3.33). □

Remark 3.8 (Extension). As a last step we establish the inner regularity of $F''(u,\cdot)$ on Lipschitz sets. To this end it is convenient to extend the functionals $F_\varepsilon(\cdot,\cdot)$ to $\mathcal{A}_\varepsilon(\tilde{\Omega};\mathbb{R}^d)\times\mathcal{A}(\tilde{\Omega})\to[0,+\infty)$ for $\tilde{\Omega}\subset\mathbb{R}^n$ open bounded and with Lipschitz boundary such that $\Omega\subset\subset\tilde{\Omega}$ similar as in [17, Proposition 3.6]. More precisely, for every $\varepsilon>0$ and $i\in Z_\varepsilon(\tilde{\Omega})$ set $\tilde{\Omega}_i:=\tilde{\Omega}-i$ and define $\tilde{\phi}_i^\varepsilon:(\mathbb{R}^d)^{Z_\varepsilon(\tilde{\Omega}_i)}$ by setting

$$\tilde{\phi}_i^\varepsilon(\{z^j\}_{j\in Z_\varepsilon(\tilde{\Omega}_i)}):=\begin{cases}\phi_i^\varepsilon(\{(z_{|\Omega})^j\}_{j\in Z_\varepsilon(\Omega_i)}) & \text{if } i\in Z_\varepsilon(\Omega),\\ \min\left\{\sum_{k=1}^n|D_\varepsilon^k z(0)|^p,\frac{1}{\varepsilon}\right\} & \text{if } i\in Z_\varepsilon(\tilde{\Omega}\setminus\Omega).\end{cases}$$

Then, for every $(u,A)\in\mathcal{A}_\varepsilon(\tilde{\Omega};\mathbb{R}^d)\times\mathcal{A}(\tilde{\Omega})$ we set

$$\tilde{F}_\varepsilon(u,A):=\sum_{i\in Z_\varepsilon(\tilde{\Omega})}\varepsilon^n\tilde{\phi}_i^\varepsilon(\{u^{i+j}\}_{j\in Z_\varepsilon(\tilde{\Omega}_i)}).\tag{3.41}$$

Notice that the functions $\tilde{\phi}_i^\varepsilon$ still satisfy (H1)-(H6) with $\tilde{\Omega}$ in place of Ω and c_1,c_2,c_3 replaced by $\max\{c_1,\sqrt{n}\}$, $\min\{c_2,1\}$ and $\max\{c_3,3^{p-1}\}$. In particular, Propositions 3.5 and 3.7 hold true also with $\tilde{\Omega}$ and $\tilde{F}$ in place of Ω and F. Moreover, for every $u\in\mathcal{A}_\varepsilon(\Omega;\mathbb{R}^d)$, $\tilde{u}\in\mathcal{A}_\varepsilon(\tilde{\Omega};\mathbb{R}^d)$ with $\tilde{u}^i=u^i$ for every $i\in Z_\varepsilon(\Omega)$ and $A\in\mathcal{A}(\Omega)$ the definition of $\tilde{\phi}_i^\varepsilon$ implies that

$$\tilde{F}_\varepsilon(\tilde{u},A)=F_\varepsilon(u,A).$$

Thus, for every $u\in GSBV^p(\Omega;\mathbb{R}^d)\cap L^1(\Omega;\mathbb{R}^d)$, $\tilde{u}\in GSBV^p(\tilde{\Omega};\mathbb{R}^d)\cap L^1(\tilde{\Omega};\mathbb{R}^d)$ with $\tilde{u}=u$ a.e. in Ω and every $A\in\mathcal{A}(\Omega)$ we obtain

$$\tilde{F}''(\tilde{u},A)=F''(u,A).\tag{3.42}$$

The extension described above allows us to proof the following result.

Proposition 3.9 (Inner regularity)**.** *Suppose that $\phi_i^\varepsilon:(\mathbb{R}^d)^{Z_\varepsilon(\Omega_i)}\to[0,+\infty)$ satisfy (H1)-(H6). Then for every $(u,A)\in GSBV^p(\Omega;\mathbb{R}^d)\cap L^1(\Omega;\mathbb{R}^d)\times\mathcal{A}^{reg}(\Omega)$ there holds*

$$F''(u,A)=F''_-(u,A),$$

where $F''_-(u,A)$ is as in (2.11).

Proof. Let $(u,A)\in GSBV^p(\Omega;\mathbb{R}^d)\cap L^1(\Omega;\mathbb{R}^d)\times\mathcal{A}^{reg}(\Omega)$. Since F'' is increasing as a set function it suffices to prove $F''(u,A)\leq\sup\{F''(u,A')\colon A'\subset\subset A\}$. A standard way to prove this inequality consists in using the subadditivity together with the upper bound. In order to apply the same reasoning in our case we need to consider an open bounded set $\tilde{\Omega}\subset\mathbb{R}^n$ with Lipschitz boundary such that $\Omega\subset\subset\tilde{\Omega}$ and extend F_ε to a functional $\tilde{F}_\varepsilon:\mathcal{A}_\varepsilon(\tilde{\Omega};\mathbb{R}^d)\times\mathcal{A}(\tilde{\Omega})\to[0,+\infty)$ as described in Remark 3.8. Then we apply Proposition 3.5 and Proposition 3.7 to $\tilde{F}$.

Let $\tilde{\Omega}$ be as above; arguing as in Step 2 and Step 3 in the proof of Proposition 3.5 we can assume that $u\in SBV^p(A;\mathbb{R}^d)\cap L^\infty(A;\mathbb{R}^d)$ and extend u to a function $\tilde{u}\in SBV^p(\tilde{\Omega};\mathbb{R}^d)\cap L^\infty(\tilde{\Omega};\mathbb{R}^d)$ satisfying $\mathcal{H}^{n-1}(S_{\tilde{u}}\cap\partial A)=0$.

Let $\eta > 0$ be fixed; since A has Lipschitz boundary and $\mathcal{H}^{n-1}(S_{\tilde u} \cap \partial A) = 0$ we can find open bounded Lipschitz sets

$$U' \subset\subset U'' \subset\subset V' \subset\subset V'' \subset\subset A \subset\subset \tilde A \subset \tilde \Omega$$

such that $A \setminus \overline{U''} \in \mathcal{A}^{reg}(\Omega)$, $\tilde A \setminus \overline{U'} \in \mathcal{A}^{reg}(\tilde\Omega)$ and

$$\int_{\tilde A \setminus U'} (|\nabla \tilde u|^p + 1)\, dx + \int_{S_{\tilde u} \cap (\tilde A \setminus \overline{U'})} (1 + |\tilde u^+(y) - \tilde u^-(y)|)\, d\mathcal{H}^{n-1}(y) \le \eta.$$

Notice that $A \setminus \overline{U''} \subset\subset \tilde A \setminus \overline{U'}$. Thus, appealing to Propositions 3.5 and 3.7 with $\tilde F$ and $\tilde \Omega$ in place of F and Ω we obtain

$$\begin{aligned}
\tilde F''(\tilde u, A) &\le \tilde F''(\tilde u, (A \setminus \overline{U''}) \cup V') \le \tilde F''(\tilde u, \tilde A \setminus \overline{U'}) + \tilde F(\tilde u, V'') \\
&\le c \left(\int_{\tilde A \setminus U'} (|\nabla \tilde u|^p + 1)\, dx + \int_{S_{\tilde u} \cap (\tilde A \setminus \overline{U'})} (1 + |\tilde u^+(y) - \tilde u^-(y)|)\, d\mathcal{H}^{n-1}(y) \right) + \tilde F(\tilde u, V'') \\
&\le \sup\{\tilde F''(\tilde u, A') \colon A' \subset\subset A\} + c\eta.
\end{aligned}$$

Thanks to (3.42) we deduce that

$$F''(u, A) \le \sup\{F''(u, A') \colon A' \subset\subset A\} + c\eta$$

and we conclude by the arbitrariness of $\eta > 0$. □

Remark 3.10. Notice that Proposition 3.9 holds true also when $F''_-(u, A)$ is replaced by

$$\sup\{F''(u, A') \colon A' \in \mathcal{A}^{reg}(\Omega),\ A' \subset\subset A\}.$$

On account of Propositions 3.2, 3.3, 3.5, 3.7 and 3.9 we can now prove the following compactness result.

Theorem 3.11 (Compactness by Γ-convergence)**.** *Let F_ε be as in (2.3) and suppose that $\phi_i^\varepsilon : (\mathbb{R}^d)^{Z_\varepsilon(\Omega_i)} \to [0, +\infty)$ satisfy (H1)-(H6). For every sequence of positive numbers converging to 0 there exist a subsequence (ε_j) and a functional $F : L^1(\Omega; \mathbb{R}^d) \times \mathcal{A}(\Omega) \to [0, +\infty)$ with*

$$F(\cdot, A) = F'_-(\cdot, A) = F''_-(\cdot, A) \quad \text{on } GSBV^p(\Omega; \mathbb{R}^d) \cap L^1(\Omega; \mathbb{R}^d). \tag{3.43}$$

Moreover, F satisfies the following properties:

(i) *For every $A \in \mathcal{A}(\Omega)$ the functional $F(\cdot, A)$ is lower semicontinuous in the strong $L^1(\Omega; \mathbb{R}^d)$-topology and local;*

(ii) *there exists $c > 0$ such that for every $(u, A) \in GSBV^p(\Omega; \mathbb{R}^d) \cap L^1(\Omega; \mathbb{R}^d) \times \mathcal{A}(\Omega)$ we have*

$$\begin{aligned}
\frac{1}{c} \left(\int_A |\nabla u|^p\, dx + \mathcal{H}^{n-1}(S_u \cap A) \right) &\le F(u, A) \\
&\le c \left(\int_A (|\nabla u|^p + 1)\, dx + \int_{S_u \cap A} (1 + |[u]|)\, d\mathcal{H}^{n-1} \right);
\end{aligned}$$

(iii) *for every $u \in GSBV^p(\Omega; \mathbb{R}^d) \cap L^1(\Omega; \mathbb{R}^d)$ the set function $F(u, \cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure;*

(iv) *for every $A \in \mathcal{A}^{reg}(\Omega)$ there holds*

$$F(\cdot, A) = F'(\cdot, A) = F''(\cdot, A) \quad \text{on } GSBV^p(\Omega; \mathbb{R}^d) \cap L^1(\Omega; \mathbb{R}^d).$$

(v) *F is invariant under translations in u.*

Proof. Thanks to the general compactness theorem [28, Theorem 16.9] we obtain a subsequence (ε_j) and a functional F satisfying (3.43). Moreover, Remark 2.3 yields the $(L^1(\Omega;\mathbb{R}^d)$-lower semicontinuity, while Proposition 3.2 combined with Remark 3.10 ensures that $F(\cdot,A)$ is local for every $A\in\mathcal{A}(\Omega)$. Further, for every $u\in GSBV^p(\Omega;\mathbb{R}^d)\cap L^1(\Omega;\mathbb{R}^d)$ the estimates in (ii) are a consequence of the corresponding estimates for regular sets in Propositions 3.3 and 3.5 together with the inner regularity of the set functions $F_1(u,\cdot),F_2(u,\cdot)$ defined as $F_1(u,A):=\int_A|\nabla u|^p\,dx+\mathcal{H}^{n-1}(S_u\cap A)$ and $F_2(u,A):=\int_A(|\nabla u|^p+1)\,dx+\int_{S_u\cap A}(1+|[u]|)\,d\mathcal{H}^{n-1}$.

Since the set function $F(u,\cdot)$ is inner regular by construction, increasing and superadditive (Remark 2.3), in order to obtain (iii) it suffices to prove that $F(u,\cdot)$ is also subadditive, then the claim follows thanks to the De Giorgi and Letta measure criterion and the upper bound in (ii). Let $u\in GSBV^p(\Omega;\mathbb{R}^d)\cap L^1(\Omega;\mathbb{R}^d)$ and $A,B\in\mathcal{A}(\Omega)$ and $U\in\mathcal{A}(\Omega)$ with $U\subset\subset A\cup B$. We now show that $F''(u,U)\leq F(u,A)+F(u,B)$, then the subadditivity follows by passing to the supremum over U. To this end we notice that we can find $A',A'',B',B''\in\mathcal{A}^{reg}(\Omega)$ with $A'\subset\subset A''\subset\subset A$ and $B'\subset\subset B''\subset\subset B$ such that $U\subset\subset A'\cup B'$. Thus, since F'' is increasing as a set function from Proposition 3.7 we deduce

$$F''(u,U)\leq F''(u,A'\cup B')\leq F''(u,A'')+F''(u,B'')\leq F(u,A)+F(u,B).$$

Finally, in view of Proposition 3.7 we have $F''(u,A)=F(u,A)$ for every $(u,A)\in GSBV^p(\Omega;\mathbb{R}^d)\cap L^1(\Omega;\mathbb{R}^d)\times\mathcal{A}^{reg}(\Omega)$, hence (iv) follows by (3.43) together with the trivial inequality $F'_-(u,A)\leq F'(u,A)\leq F''(u,A)$. It remains to notice that (v) is a direct consequence of the fact that thanks to (H2) the functionals F_ε are invariant under translation in u. ☐

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. Let (ε_j) and F be as in Theorem 3.11. Then Propositions 3.3 and 3.5 ensure that the domain of F coincides with $GSBV^p(\Omega;\mathbb{R}^d)\times L^1(\Omega;\mathbb{R}^d)$. Moreover, in view of Theorem 3.11 the restriction of the functional F to $SBV^p(\Omega;\mathbb{R}^d)\times\mathcal{A}(\Omega)$ satisfies all hypotheses of [9, Theorem 1] except for the lower bound. In order to recover the lower bound we use a standard perturbation argument, that is, for every $\sigma>0$ we consider the functional $F_\sigma:SBV^p(\Omega;\mathbb{R}^d)\times\mathcal{A}(\Omega)\to[0,+\infty)$ defined as

$$F_\sigma(u,A):=F(u,A)+\sigma\int_{S_u\cap A}|[u]|\,d\mathcal{H}^{n-1}.$$

We observe that for every $\sigma>0$ F_σ satisfies all hypotheses of [9, Theorem 1] which thus provides us with two functions $f_0^\sigma:\Omega\times\mathbb{R}^d\times\mathbb{R}^{d\times n}\to[0,+\infty)$ and $g_0^\sigma:\Omega\times\mathbb{R}^d\times\mathbb{R}^d\times S^{n-1}\to[0,+\infty)$ such that

$$F_\sigma(u,A)=\int_A f_0^\sigma(x,u,\nabla u)\,dx+\int_{S_u\cap A}g_0^\sigma(x,u^+,u^-,\nu_u)\,d\mathcal{H}^{n-1},$$

for every $u\in SBV^p(\Omega;\mathbb{R}^d)$ and $A\in\mathcal{A}(\Omega)$. Moreover, since F and then also F_σ is invariant under translation in u, formulas (2) and (3) in [9, Theorem 1] imply that f_0^σ does not depend on u and g_0^σ depends on the values u^+ and u^- only through their difference $[u]$, i.e., $f_0^\sigma(x,u,\xi)=f^\sigma(x,\xi)$ and $g_0^\sigma(x,a,b,\nu)=g^\sigma(x,a-b,\nu)$ for some functions $f^\sigma:\Omega\times\mathbb{R}^{d\times n}\to[0,+\infty)$, $g^\sigma:\Omega\times\mathbb{R}^d\times S^{n-1}\to[0,+\infty)$. Finally, formulas (2) and (3) in [9, Theorem 1] also imply that f^σ and g^σ decrease as σ decreases. Hence, setting $f(x,\xi):=\lim_{\sigma\to0^+}f^\sigma(x,\xi)$, $g(x,t,\nu):=\lim_{\sigma\to0^+}g^\sigma(x,t,\nu)$, from the pointwise convergence of F_σ to F and the Monotone Convergence Theorem we deduce

$$F(u,A)=\int_A f(x,\nabla u)\,dx+\int_{S_u\cap A}g(x,[u],\nu_u)\,d\mathcal{H}^{n-1},$$

for every $u \in SBV^p(\Omega;\mathbb{R}^d)$ and $A \in \mathcal{A}(\Omega)$. In particular, thanks to Theorem 3.11 (iv) we deduce that (3.4) holds for every $u \in SBV^p(\Omega;\mathbb{R}^d)$ and $A \in \mathcal{A}^{reg}(\Omega)$, and choosing $A = \Omega$ in the formula above we obtain the desired integral representation on $SBV^p(\Omega;\mathbb{R}^d)$. We finally observe that formulas (2) and (3) in [9, Theorem] imply that the integrands f and g are given by (3.2).

Eventually, we show that the integral representation also extends to $GSBV^p(\Omega;\mathbb{R}^d)\cap L^1(\Omega;\mathbb{R}^d)$. To this end, for every $u \in GSBV^p(\Omega;\mathbb{R}^d)\cap L^1(\Omega;\mathbb{R}^d)$ and every $k \in \mathbb{N}$ we consider again the truncation $T_k u$ as in Remark 2.2. Using (ii) and (iii) in Remark 2.2 together with (2.10) and appealing to the Monotone Convergence Theorem we get

$$\begin{aligned}\Gamma\text{-}\lim_{j\to+\infty} F_{\varepsilon_j}(u) = \lim_{k\to+\infty} F(T_k u) &= \lim_{k\to+\infty}\left(\int_\Omega f(x,\nabla T_k u)\,dx + \int_{S_{T_k u}} g(x,[T_k u],\nu_{T_k u})\,d\mathcal{H}^{n-1}\right)\\ &= \int_\Omega f(x,\nabla u)\,dx + \int_{S_u} g(x,[u],\nu_u)\,d\mathcal{H}^{n-1}.\end{aligned}$$

□

3.1. **Treatment of Dirichlet problems.**

For further use in Section 4 we study here the asymptotic behavior of minimum problems for F_ε when suitable Dirichlet boundary conditions are taken into account. More precisely, for every $\delta > 0$, every $A \in \mathcal{A}^{reg}(\Omega)$ and every pointwise well-defined function $\bar u \in L^1(\Omega;\mathbb{R}^d)$ we consider the minimization problem

$$\mathbf{m}^\delta_\varepsilon(\bar u, A) := \inf\{F_\varepsilon(u,A)\colon u \in \mathcal{A}^\delta_\varepsilon(\bar u, A)\},$$

where

$$\mathcal{A}^\delta_\varepsilon(\bar u, A) := \{u \in \mathcal{A}_\varepsilon(\Omega;\mathbb{R}^d)\colon u(i) = \bar u(i) \text{ if } \operatorname{dist}(i,\mathbb{R}^n\setminus A) < \delta\},$$

and we study the asymptotic behavior of $\mathbf{m}^\delta_\varepsilon(\bar u, A)$ when first $\varepsilon \to 0$ and then $\delta \to 0$. For our purpose it is sufficient to consider boundary data $\bar u \in SBV^p(\Omega;\mathbb{R}^d)\cap L^\infty(\Omega;\mathbb{R}^d)$ satisfying $\mathcal{H}^{n-1}(S_{\bar u}\cap\partial A) = 0$ and such that

$$\bar u_\varepsilon \to \bar u \quad \text{in } L^1(\Omega;\mathbb{R}^d), \qquad \limsup_{\varepsilon\to 0} F_\varepsilon(\bar u_\varepsilon, B) \leq c\left(\int_B |\nabla \bar u|^p\,dx + \mathcal{H}^{n-1}(S_{\bar u}\cap \overline{B})\right) \tag{3.44}$$

where $\bar u_\varepsilon \in \mathcal{A}_\varepsilon(\Omega;\mathbb{R}^d)$ is defined by setting $\bar u^i_\varepsilon := \bar u(i)$ and $B \in \mathcal{A}^{reg}(\Omega)$. For $\bar u$ as above we can prove the following convergence result.

Lemma 3.12. *Let $\phi^\varepsilon_i : (\mathbb{R}^d)^{Z_\varepsilon(\Omega_i)} \to [0,+\infty)$ satisfy hypotheses (H1)–(H6) and let F_{ε_j} be the subsequence provided by Theorem 3.1. Moreover, let $A \in \mathcal{A}^{reg}(\Omega)$ with $A \subset\subset \Omega$. For every pointwise well-definede function $\bar u \in SBV^p(\Omega;\mathbb{R}^d)\cap L^\infty(\Omega;\mathbb{R}^d)$ with $\mathcal{H}^{n-1}(S_{\bar u}\cap\partial A) = 0$ and satisfying (3.44) we have*

$$\lim_{\delta\to 0}\liminf_{j\to+\infty}\mathbf{m}^\delta_{\varepsilon_j}(\bar u, A) = \lim_{\delta\to 0}\limsup_{j\to+\infty}\mathbf{m}^\delta_{\varepsilon_j}(\bar u, A) = \mathbf{m}(\bar u, A),$$

where $\mathbf{m}(\bar u, A)$ is as in (3.3).

Remark 3.13. Lemma 3.12 together with (3.2) provide us with asymptotic formulas for the integrands f and g given by Theorem 3.1. Indeed, for $x_0 \in \Omega$, $\nu \in S^{n-1}$ and $\rho > 0$ sufficiently small we have $Q^\nu_\rho(x_0) \subset\subset \Omega$. Moreover, for every $\zeta \in \mathbb{R}^d$ and $M \in \mathbb{R}^{d\times n}$ the functions $u_{M,x_0}, u^\nu_{\zeta,x_0}$ as in (2.1) satisfy the hypotheses of Lemma 3.12. Thus, passing to the upper limit as $\rho \to 0$ we obtain the following formulas for f and g

$$f(x_0,M) = \limsup_{\rho\to 0}\frac{1}{\rho^n}\lim_{\delta\to 0}\liminf_{j\to+\infty}\mathbf{m}^\delta_{\varepsilon_j}(u_{M,x_0}) = \limsup_{\rho\to 0}\frac{1}{\rho^n}\lim_{\delta\to 0}\limsup_{j\to+\infty}\mathbf{m}^\delta_{\varepsilon_j}(u_{M,x_0}),$$

$$g(x_0,\zeta,\nu) = \limsup_{\rho\to 0}\frac{1}{\rho^{n-1}}\lim_{\delta\to 0}\liminf_{j\to+\infty}\mathbf{m}^\delta_{\varepsilon_j}(u^\nu_{\zeta,x_0}) = \limsup_{\rho\to 0}\frac{1}{\rho^{n-1}}\lim_{\delta\to 0}\limsup_{j\to+\infty}\mathbf{m}^\delta_{\varepsilon_j}(u^\nu_{\zeta,x_0}).$$

Proof of Lemma 3.12. Let $A,\bar u$ be as in the statement. Observe that due to monotonicity the limit as $\delta\to 0$ exists. We show that $\mathbf{m}(\bar u, A)$ is both an asymptotic lower and an asymptotic upper bound for $\mathbf{m}^\delta_{\varepsilon_j}(\bar u, A)$.

Step 1: We first establish the inequality

$$\mathbf{m}(\bar u, A) \le \lim_{\delta\to 0}\liminf_{j\to+\infty}\mathbf{m}^\delta_{\varepsilon_j}(\bar u, A). \tag{3.45}$$

To this end, let $\delta>0$ be fixed and let $u_j\in\mathcal{A}_{\varepsilon_j}(\Omega;\mathbb{R}^d)$ be admissible for $\mathbf{m}^\delta_{\varepsilon_j}(\bar u, A)$ with

$$F_{\varepsilon_j}(u_j, A) = \mathbf{m}^\delta_{\varepsilon_j}(\bar u, A).$$

Thanks to Remark 2.2 we can assume that $\|u_j\|_{L^\infty}\le 3\|\bar u\|_{L^\infty}$. In particular, the sequence (u_j) is equi-integrable, hence (H3) together with [32, Lemma 5.6] yield the existence of a subsequence (not relabeled) converging in $L^1(\Omega;\mathbb{R}^d)$ to some $u\in GSBV^p(A;\mathbb{R}^d)\cap L^1(A;\mathbb{R}^d)$. Since $u_j=\bar u_{\varepsilon_j}$ ond $\partial A + B_\delta(0)$, (3.44) ensures that $u=\bar u$ on $\partial A+B_\delta(0)$, hence u is admissible for $\mathbf{m}(\bar u, A)$. Thus, Theorem 3.1 yields

$$\mathbf{m}(\bar u, A)\le F(u,A)\le \liminf_{j\to+\infty}F_{\varepsilon_j}(u_j,A) = \liminf_{j\to+\infty}\mathbf{m}^\delta_{\varepsilon_j}(\bar u, A)$$

hence (3.45) follows by letting $\delta\to 0$.

Step 2: We now prove that

$$\lim_{\delta\to 0}\limsup_{j\to+\infty}\mathbf{m}^\delta_{\varepsilon_j}(\bar u, A)\le \mathbf{m}(\bar u, A).$$

To this end, for fixed $\eta>0$ we choose $u\in SBV^p(A;\mathbb{R}^d)$ with $u=\bar u$ in a neighborhood of ∂A and $F(u,A)\le \mathbf{m}(\bar u, A)+\eta$. Thanks to Proposition 3.2 we can extend u to $\Omega\setminus A$ by $\bar u$ without changing $F(u,A)$. Moreover, Theorem 3.1 provides us with a sequence of functions $u_j\in A_{\varepsilon_j}(\Omega;\mathbb{R}^d)$ converging to u in $L^1(\Omega;\mathbb{R}^d)$ and satisfying

$$\limsup_{j\to+\infty}F_{\varepsilon_j}(u_j,A) = F(u,A). \tag{3.46}$$

We now modify u_j to fulfill the required discrete boundary condition. Since $u=\bar u$ in a neighborhood of ∂A, we can find $A'\in\mathcal{A}^{reg}(\Omega)$, $A'\subset\subset A$ such that $u=\bar u$ on $A\setminus\overline{A'}$ (and by extension $u=\bar u$ on $\Omega\setminus\overline{A'}$). Since moreover $\mathcal{H}^{n-1}(S_{\bar u}\cap\partial A)=0$ we can choose further sets $A'',\tilde A\in\mathcal{A}^{reg}(\Omega)$ with $A'\subset\subset A''\subset\subset A\subset\subset\tilde A$ and

$$\int_{\tilde A\setminus A'}|\nabla\bar u|^p\,dx + \mathcal{H}^{n-1}(S_{\bar u}\cap\overline{\tilde A\setminus A'})\le\eta.$$

Arguing as in the proof of Proposition 3.7 we can construct a sequence (w_j) with $w_j=u_j$ on A'', $w_j=\bar u$ on $\Omega\setminus\overline{A}$ and

$$\begin{aligned}\limsup_{j\to+\infty}F_{\varepsilon_j}(w_j,A) &= \limsup_{j\to+\infty}F_{\varepsilon_j}(w_j,A''\cup A\setminus\overline{A''})\\ &\le (1+\eta)\left(\limsup_{j\to+\infty}F_{\varepsilon_j}(u_j,A)+\limsup_{j\to+\infty}F_{\varepsilon_j}(\bar u_j,\tilde A\setminus\overline{A'})\right)+\eta,\end{aligned}\tag{3.47}$$

where $\bar u_j$ is as in (3.44) with ε_j in place of ε. In view of (3.44) and the choice of $A', \tilde A$ we have

$$\limsup_{j\to+\infty} F_{\varepsilon_j}(\bar u_j, \tilde A\setminus\overline{A'}) \le c\left(\int_{\tilde A\setminus A'}|\nabla\bar u|^p\,dx + \mathcal{H}^{n-1}(S_{\bar u}\cap\overline{\tilde A\setminus A'})\right)\le c\eta.$$

Moreover, for δ sufficiently small w_j is admissible for $\mathbf{m}^\delta_{\varepsilon_j}(\bar u, A)$. Thus, gathering (3.46)–(3.47) thanks to the choice of u we deduce that

$$\lim_{\delta\to0}\limsup_{j\to+\infty}\mathbf{m}^\delta_{\varepsilon_j}(\bar u, A)\le\limsup_{j\to+\infty}F_{\varepsilon_j}(w_j, A)\le(1+\eta)\mathbf{m}(\bar u, A)+c\eta$$

and we conclude by the arbitrariness of $\eta>0$. □

4. Homogenization

In this section we consider a special class of periodic interaction-energy densities ϕ_i^ε for which we can show that the Γ-limit provided by Theorem 3.1 does not depend on the Γ-converging subsequence, which in turn implies that the whole sequence (F_ε) Γ-converges. We first need to specify what periodicity means in the case of interaction-energy densities $\phi_i^\varepsilon:(\mathbb{R}^d)^{Z_\varepsilon(\Omega_i)}\to[0,+\infty)$ that may depend on the whole state $\{z^j\}_{j\in Z_\varepsilon(\Omega_i)}$. This difficulty is also present in [17, Section 5]. To avoid the dependence of ϕ_i^ε on Ω_i in [17] the authors use a sequence of periodic finite-range interactions ϕ_i^k defined on the entire lattice $(\mathbb{R}^d)^{\mathbb{Z}^n}$ whose range increases as k increases and which converge for every $i\in\mathbb{Z}^n$ to a long-range interaction-energy density $\phi_i:(\mathbb{R}^d)^{\mathbb{Z}^n}\to[0,+\infty)$ as $k\to+\infty$. For $i\in Z_\varepsilon(\Omega)$ the functions ϕ_i^ε are then obtained by a rescaling of a suitably chosen $\phi^{k(\varepsilon)}_{\frac{i}{\varepsilon}}$, where $\varepsilon k(\varepsilon)$ is proportional to the distance of i to the boundary of Ω. Since the energy densities ϕ_i^ε that we consider here contain both a bulk and a surface scaling the approach in [17] cannot be adapted to our setting. Instead, here we consider functions $\psi_i^\varepsilon:(\mathbb{R}^d)^{Z_\varepsilon(\mathbb{R}^n)}\to[0,+\infty)$ defined on the entire scaled lattice $\varepsilon\mathbb{Z}^n$ which have only finite range. This finite-range assumption will be crucial to decouple the bulk and the surface scaling in the Γ-limit.

We now state our precise hypotheses. Let $K\in\mathbb{N}$, $L\in\mathbb{N}$ and consider functions $\psi_i^\varepsilon:(\mathbb{R}^d)^{Z_\varepsilon(\mathbb{R}^n)}\to[0,+\infty)$ which are εK-periodic in i and satisfy hypotheses (H1)–(H6) with $Z_\varepsilon(\Omega_i)$ replaced by $Z_\varepsilon(\mathbb{R}^n)$; where in addition the sequences $(c^{j,\xi}_{\varepsilon,\alpha})$ and $(c^{j,\xi}_\varepsilon)$ provided by (H5) and (H6), respectively, satisfy

$$\begin{aligned}c^{j,\xi}_{\varepsilon,\alpha}&=0\qquad\text{if }\max\{\alpha, 2|\tfrac{j}{\varepsilon}|_\infty, 2|\xi|_\infty, 2|\tfrac{j}{\varepsilon}+\xi|_\infty\}\ge L,\\ c^{j,\xi}_{\varepsilon}&=0\qquad\text{if }\max\{2|\tfrac{j}{\varepsilon}|_\infty, 2|\xi|_\infty, 2|\tfrac{j}{\varepsilon}+\xi|_\infty\}\ge L.\end{aligned}\tag{4.1}$$

In particular, whenever $z,w:Z_\varepsilon(\mathbb{R}^n)\to\mathbb{R}^d$ are such that $z^j=w^j$ for all $j\in Z_\varepsilon(\varepsilon LQ)$, we have

$$\psi_i^\varepsilon(\{z^j\}_{j\in Z_\varepsilon(\varepsilon LQ)})=\psi_i^\varepsilon(\{w^j\}_{j\in Z_\varepsilon(\varepsilon LQ)}).\tag{4.2}$$

We also set

$$\Omega^L_\varepsilon:=\{x\in\Omega\colon\ \mathrm{dist}_\infty(x,\partial\Omega)>L\varepsilon\}$$

and we define $\phi_i^\varepsilon:(\mathbb{R}^d)^{Z_\varepsilon(\Omega_i)}\to[0,+\infty)$ by setting

$$\phi_i^\varepsilon(\{z^j\}_{j\in Z_\varepsilon(\Omega_i)}):=\begin{cases}\psi_i^\varepsilon(\{z^j\chi^j_{\varepsilon LQ}\}_{j\in Z_\varepsilon(\mathbb{R}^n)}) & \text{if } i\in Z_\varepsilon(\Omega^L_\varepsilon),\\[2ex] \min\Big\{\displaystyle\sum_{\substack{k=1\\ \varepsilon e_k\in\Omega_i}}^n|D^k_\varepsilon z(0)|^p,\frac{1}{\varepsilon}\Big\} & \text{if } i\in Z_\varepsilon(\Omega\setminus\Omega^L_\varepsilon),\end{cases}\tag{4.3}$$

which is well-defined thanks to (4.2). By construction, $\phi_i^\varepsilon : (\mathbb{R}^d)^{Z_\varepsilon(\Omega_i)} \to [0,+\infty)$ satisfy hypotheses (H1)–(H6). We now aim to prove that for $\phi_i^\varepsilon : (\mathbb{R}^d)^{Z_\varepsilon(\Omega_i)} \to [0,+\infty)$ defined as in (4.3) the integrands f and g provided by Theorem 3.1 are independent of the position x.

Proposition 4.1. *Let F_ε be as in (2.3) with $\phi_i^\varepsilon : (\mathbb{R}^d)^{Z_\varepsilon(\Omega_i)} \to [0,+\infty)$ given by (4.3), where $\psi_i^\varepsilon : (\mathbb{R}^d)^{Z_\varepsilon(\mathbb{R}^n)} \to [0,+\infty)$ are εK-periodic in i, satisfy (H1)–(H6) with $Z_\varepsilon(\Omega_i)$ replaced by $Z_\varepsilon(\mathbb{R}^n)$, and (4.1). Let (ε_j) and F be the subsequence and the functional provided by Theorem 3.1. Then F is of the form*

$$F(u) = \int_\Omega \bar{f}(\nabla u)\,dx + \int_{S_u} \bar{g}([u],\nu_u)\,d\mathcal{H}^{n-1}, \qquad u \in GSBV^2(\Omega;\mathbb{R}^d), \tag{4.4}$$

for some functions $\bar{f} : \mathbb{R}^{d\times n} \to [0,+\infty)$ and $\bar{g} : \mathbb{R}^d \times S^{n-1} \to [0,+\infty)$ possibly depending on the Γ-converging subsequence. Moreover, for every $A \in \mathcal{A}^{reg}(\Omega)$ and $u \in GSBV^2(\Omega;\mathbb{R}^d)$ there holds

$$\Gamma\text{-}\lim_{j\to+\infty} F_{\varepsilon_j}(u,A) = \int_A \bar{f}(\nabla u)\,dx + \int_{S_u\cap A} \bar{g}([u],\nu_u)\,d\mathcal{H}^{n-1}.$$

We prove 4.1 by adapting a well-known argument (see, *e.g.*, [18, Lemma 2.7]) to our setting showing that the minimization problem $\mathbf{m}(\bar{u},A)$ defined in (3.3) is invariant under translation for a suitable class of functions $\bar{u}$. We start by introducing some notation. For every $A \in \mathcal{A}(\Omega)$ and $y \in \mathbb{R}^n$ we set $\tau_y A := A + y$. Moreover, for every $u : \Omega \to \mathbb{R}^d$ and every $A \in \mathcal{A}(\Omega)$ with $\tau_y A \subset \Omega$ we define $\tau_y u : \tau_y A \to \mathbb{R}^d$ by setting $\tau_y u(x) := u(x-y)$ for every $x \in \tau_y A$. For our purpose it is sufficient to consider pointwise well-defined functions $\bar{u} \in SBV^p_{\rm loc}(\mathbb{R}^n;\mathbb{R}^d)$ which satisfy

$$\tau_y \bar{u}_\varepsilon^i \to \tau_y \bar{u} \quad \text{in } L^1(\Omega;\mathbb{R}^d) \quad \text{for every } y \in \mathbb{R}^n, \tag{4.5}$$

where for every $y \in \mathbb{R}^n$ the function $\tau_y \bar{u}_\varepsilon \in \mathcal{A}_\varepsilon(\Omega;\mathbb{R}^d)$ is defined by setting $\tau_y \bar{u}_\varepsilon^i := \tau_y \bar{u}(i)$ for every $i \in Z_\varepsilon(\mathbb{R}^n)$. We now prove the following auxiliary lemma.

Lemma 4.2. *Suppose that $\phi_i^\varepsilon : (\mathbb{R}^d)^{Z_\varepsilon(\Omega_i)} \to [0,+\infty)$ are given by (4.3), where $\psi_i^\varepsilon : (\mathbb{R}^d)^{Z_\varepsilon(\mathbb{R}^n)} \to [0,+\infty)$ are εK-periodic in i, satisfy (H1)–(H6) with $Z_\varepsilon(\Omega_i)$ replaced by $Z_\varepsilon(\mathbb{R}^n)$, and (4.1). Let $A \in \mathcal{A}^{reg}(\Omega)$ with $A \subset\subset \Omega$ and let $\bar{u} \in SBV^p_{\rm loc}(\mathbb{R}^n;\mathbb{R}^d)$ be a pointwise well-defined function satisfying (4.5). For any $y \in \mathbb{R}^n$ with $\tau_y A \subset\subset \Omega$ there holds*

$$\mathbf{m}(\bar{u},A) = \mathbf{m}(\tau_y\bar{u},\tau_y A),$$

where $\mathbf{m}(\bar{u},A), \mathbf{m}(\tau_y\bar{u},\tau_y A)$ are defined according to (3.3).

Proof. Let $A, \bar{u}$ and y be as in the statement and let us prove that

$$\mathbf{m}(\tau_y\bar{u},\tau_y A) \leq \mathbf{m}(\bar{u},A). \tag{4.6}$$

To this end let $u \in SBV^p(A;\mathbb{R}^d)$ be admissible for $\mathbf{m}(\bar{u},A)$ and $A' \subset\subset A$ with $u = \bar{u}$ in $A \setminus \overline{A'}$. In view of Proposition 3.2 we can extend u to $\Omega \setminus A$ by $\bar{u}$ without changing $F(u,A)$. In order to simplify notation we still denote the subsequence provided by Theorem 3.1 by ε and we choose a sequence $(u_\varepsilon) \subset \mathcal{A}_\varepsilon(\Omega;\mathbb{R}^d)$ converging to u in $L^1(\Omega;\mathbb{R}^d)$ and satisfying

$$\lim_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon,A) = F(u,A).$$

We now construct a suitable sequence (v_ε) converging to $\tau_y u$ in $L^1(\Omega;\mathbb{R}^d)$. We choose $A'', A''' \in \mathcal{A}^{reg}(\Omega)$ with $A' \subset\subset A'' \subset\subset A''' \subset\subset A$ and ε_0 sufficiently small such that for all $\varepsilon \in (0,\varepsilon_0)$ the following conditions are satisfied.

(i) $A \cup \tau_y A \subset \Omega_\varepsilon^L$;

(ii) $\tau_y A'' \subset \tau_{y_\varepsilon} A'''$ and $\tau_y A''' \subset \tau_{y_\varepsilon} A$, where $y_\varepsilon := \varepsilon K \lfloor \frac{y}{\varepsilon K} \rfloor$;
(iii) $\varepsilon L < \mathrm{dist}_\infty(A'', \partial A''')$.

For $\varepsilon \in (0, \varepsilon_0)$ we then define $v_\varepsilon \in \mathcal{A}_\varepsilon(\Omega; \mathbb{R}^d)$ by setting

$$v_\varepsilon^i := \begin{cases} u_\varepsilon^{i-y_\varepsilon} & \text{if } i \in Z_\varepsilon(\tau_y A'''), \\ \tau_y \bar{u}(i) & \text{if } i \in Z_\varepsilon(\Omega \setminus \tau_y A'''), \end{cases}$$

which is well-defined thanks to the second inclusion in (ii).

Since $u = \bar{u}$ in $\Omega \setminus \overline{A'}$, thanks to (4.5) it is immediate to see that $v_\varepsilon \to \tau_y u$ in $L^1(\Omega; \mathbb{R}^d)$. Moreover, for all $i \in Z_\varepsilon(\tau_y A'')$ and $j \in Z_\varepsilon(\varepsilon L Q)$ assumption (iii) yields $i + j \in \tau_y A'''$, and hence

$$v_\varepsilon^{i+j} = u_\varepsilon^{i-y_\varepsilon+j}.$$

Thanks to the locality property (4.2) and the periodicity assumption we thus obtain

$$\begin{aligned} F_\varepsilon(v_\varepsilon, \tau_y A'') &= \sum_{i \in Z_\varepsilon(\tau_y A'')} \varepsilon^n \psi_i^\varepsilon(\{u_\varepsilon^{i-y_\varepsilon+j} \chi_{\varepsilon LQ}^j\}_{j \in Z_\varepsilon(\mathbb{R}^n)}) \leq \sum_{i \in Z_\varepsilon(A''')} \varepsilon^n \psi_i^\varepsilon(\{u_\varepsilon^{i+j} \chi_{\varepsilon LQ}^j\}_{j \in Z_\varepsilon(\mathbb{R}^n)}) \\ &\leq \sum_{i \in Z_\varepsilon(A)} \varepsilon^n \phi_i^\varepsilon(\{u_\varepsilon^{i+j}\}_{j \in \mathbb{Z}^n}) = F_\varepsilon(u_\varepsilon, A), \end{aligned}$$

where in the second inequality we have used the first inclusion in (ii). Together with the fact that $F(\cdot, \tau_y A'') = \Gamma\text{-}\lim_\varepsilon F_\varepsilon(\cdot, \tau_y A'')$ and $v_\varepsilon \to \tau_y u$ in $L^1(\Omega; \mathbb{R}^d)$ the above inequality allows us to deduce that

$$F(\tau_y u, \tau_y A'') \leq \liminf_{\varepsilon \to 0} F_\varepsilon(v_\varepsilon, \tau_y A'') \leq \lim_\varepsilon F_\varepsilon(u_\varepsilon, A) = F(u, A).$$

In view of Proposition 3.9, Remark 3.10 and the arbitrariness of $A'' \subset\subset A$ we finally get

$$F(\tau_y u, \tau_y A) \leq F(u, A). \tag{4.7}$$

Hence, since $\tau_y u$ is admissible for $\mathbf{m}(\tau_y \bar{u}, A)$ and u was arbitrarily chosen we obtain (4.6) by passing to the infimum on both sides of (4.7). To deduce the result it then suffices to notice that the opposite inequality follows by applying (4.6) with τ_{-y}. □

On account of Lemma 4.2 we now prove Proposition 4.1.

Proof of Proposition 4.1. Let F be as in Theorem 3.1. We claim that the integrands f and g as in (3.2) are independent of the position x_0, then F can be written in the form (4.4). To prove the claim we fix $x_0, y_0 \in \Omega$ and choose $\rho > 0$ sufficiently small such that $Q_\rho^\nu(x_0) \cup Q_\rho^\nu(y_0) \subset\subset \Omega$. For every $M \in \mathbb{R}^{d \times n}$ and every $(\zeta, \nu) \in \mathbb{R}^d \times S^{n-1}$ the functions u_{M,x_0} and u_{ζ,x_0}^ν defined as in (2.1) satisfy the hypotheses of Lemma 4.2. Thus, we obtain

$$\mathbf{m}(u_{\zeta,y_0}^\nu, Q_\rho^\nu(y_0)) = \mathbf{m}(\tau_{y_0-x_0} u_{\zeta,x_0}^\nu, \tau_{y_0-x_0} Q_\rho^\nu(x_0)) = \mathbf{m}(u_{\zeta,x_0}^\nu, Q_\rho^\nu(x_0))$$

and

$$\mathbf{m}(u_{M,y_0}, Q_\rho^\nu(y_0)) = \mathbf{m}(\tau_{y_0-x_0} u_{M,x_0}, \tau_{y_0-x_0} Q_\rho^\nu(x_0)) = \mathbf{m}(u_{M,x_0}, Q_\rho^\nu(x_0)).$$

We conclude by letting $\rho \to 0$. □

4.1. **Separation of bulk and surface effects.** In this subsection we give sufficient conditions on the functions ψ_i^ε under which a separation of energy contributions takes place in the limit. We state the precise hypotheses after introducing some notation. For every $\varepsilon > 0$, every $u : Z_\varepsilon(\mathbb{R}^n) \to \mathbb{R}^d$ and every $i \in Z_\varepsilon(\mathbb{R}^n)$ set

$$|\nabla_\varepsilon u|(i) := \sum_{k=1}^n \big(|D_\varepsilon^{e_k} u^i| + |D_\varepsilon^{-e_k} u^i|\big), \qquad |\nabla_{\varepsilon,L} u|(i) := \sum_{\xi\in Z_1(LQ)} \left|\frac{u^i - u^{i+\varepsilon\xi}}{\varepsilon}\right|.$$

We then assume that for every $i \in \mathbb{Z}^n$ there exist $\psi_i^b, \psi_i^s : (\mathbb{R}^d)^{\mathbb{Z}^n} \to [0,+\infty)$ such that the following holds.

(H$_\psi$1) For every $\eta > 0$ and every $\Lambda > 0$ there exists $\bar\varepsilon = \bar\varepsilon(\eta,\Lambda) > 0$ such that for every $\varepsilon \in (0,\bar\varepsilon)$, for every $i \in \mathbb{Z}^n$ and for every $z : \mathbb{Z}^n \to \mathbb{R}^d$ with $|\nabla_{1,L} z|(0) < \Lambda$ we have

$$|\psi_{\varepsilon i}^\varepsilon(\{\varepsilon z^{\frac{j}{\varepsilon}}\}_{j\in Z_\varepsilon(\mathbb{R}^n)}) - \psi_i^b(\{z^j\}_{j\in\mathbb{Z}^n})| < \eta.$$

(H$_\psi$2) For every $\eta > 0$ there exist $\Lambda(\eta) > 0$ and $\hat\varepsilon = \hat\varepsilon(\eta) > 0$ such that for every $\varepsilon \in (0,\hat\varepsilon)$, for every $i \in \mathbb{Z}^n$ and every $z : \mathbb{Z}^n \to \mathbb{R}^d$ with $\varepsilon^{\frac{1-p}{p}}|\nabla_{1,L} z|(0) \ge \Lambda(\eta)$ or $|\nabla_{1,L} z|(0) = 0$ we have

$$|\varepsilon\psi_{\varepsilon i}^\varepsilon(\{z^{\frac{j}{\varepsilon}}\}_{j\in Z_\varepsilon(\mathbb{R}^n)}) - \psi_i^s(\{z^j\}_{j\in\mathbb{Z}^n})| < \eta.$$

Moreover, we assume that the functions ψ_i^s satisfy the following continuity hypotheses.

(H$_\psi$3) There exists a constant $c_s > 0$ such that for every $z, w : \mathbb{Z}^n \to \mathbb{R}^d$ with $|\nabla_{1,L} z|(0) > 0$, $|\nabla_{1,L} w|(0) > 0$ and for every $i \in \mathbb{Z}^n$ there holds

$$|\psi_i^s(\{z^j\}_{j\in\mathbb{Z}^n}) - \psi_i^s(\{w^j\}_{j\in\mathbb{Z}^n})| \le c_s \sum_{j\in Z_1(Q_L(i))} \sum_{\substack{\xi\in Z_1(Q_L(i))\\ j+\xi\in Q_L(i)}} |z^{j+\xi} - w^{j+\xi}|.$$

The main result of this section is the following theorem which states that under the additional assumptions (H$_\psi$1)–(H$_\psi$3) the bulk and surface interactions decouple in the Γ-limit. As a consequence we obtain asymptotic minimization formulas for the bulk and the surface energy density that are independent of the Γ-converging subsequence.

Theorem 4.3 (Homogenization)**.** *Assume that $\phi_i^\varepsilon : (\mathbb{R}^d)^{Z_\varepsilon(\Omega_i)} \to [0,+\infty)$ are given by (4.3), where $\psi_i^\varepsilon : (\mathbb{R}^d)^{Z_\varepsilon(\mathbb{R}^n)} \to [0,+\infty)$ are εK-periodic in i, satisfy (H1)–(H6) with $Z_\varepsilon(\Omega_i)$ replaced by $Z_\varepsilon(\mathbb{R}^n)$, and (4.1), and suppose that in addition (H$_\psi$1)–(H$_\psi$3) are satisfied. Then the functionals $F_\varepsilon : L^1(\Omega;\mathbb{R}^d) \to [0,+\infty]$ defined as in (2.3) Γ-converge in the strong $L^1(\Omega;\mathbb{R}^d)$-topology to the functional $F_{\rm hom} : L^1(\Omega;\mathbb{R}^d) \to [0,+\infty]$ given by*

$$F_{\rm hom}(u) = \begin{cases} \displaystyle\int_\Omega f_{\rm hom}(\nabla u)\,dx + \int_{S_u} g_{\rm hom}([u],\nu_u)\,d\mathcal{H}^{n-1} & \text{if } u \in GSBV^p(\Omega;\mathbb{R}^d),\\ +\infty & \text{otherwise in } L^1(\Omega;\mathbb{R}^d),\end{cases}$$

where $f_{\rm hom} : \mathbb{R}^{d\times n} \to [0,+\infty)$ and $g_{\rm hom} : \mathbb{R}^d \times S^{n-1} \to [0,+\infty)$ are given by

$$f_{\rm hom}(M) = \lim_{T\to+\infty} \frac{1}{T^n} \inf\Big\{ \sum_{i\in Z_1(TQ)} \psi_i^b(\{u^{i+j}\}_{j\in\mathbb{Z}^n}) \colon u \in \mathcal{A}_1^{\sqrt{n}L}(u_M, TQ)\Big\} \tag{4.8}$$

and

$$g_{\rm hom}(\zeta,\nu) = \lim_{T\to+\infty} \frac{1}{T^{n-1}} \inf\Big\{ \sum_{i\in Z_1(TQ^\nu)} \psi_i^s(\{u^{i+j}\}_{j\in\mathbb{Z}^n}) \colon u \in \mathcal{A}_1^{\sqrt{n}L}(u_{\zeta,\nu}, TQ^\nu)\Big\}. \tag{4.9}$$

The proof of Theorem 4.3 will be established in Sections 4.1.1 and 4.1.2 below in which we treat separately the bulk and the surface energy density. As a preliminary step it is useful to compare the two operators $|\nabla_{\varepsilon,L}|$ and $|\nabla_\varepsilon|$.

Lemma 4.4. *There exist constants $\hat{c}_1, \hat{c}_2 > 0$ depending only on n, p and L such that for every $u : Z_\varepsilon(\mathbb{R}^n) \to \mathbb{R}^d$ and every $i \in Z_\varepsilon(\mathbb{R}^n)$ there holds*

$$|\nabla_{\varepsilon,L} u|^p(i) \leq \hat{c}_1 \sum_{j \in Z_\varepsilon(Q_{\varepsilon L}(i))} |\nabla_\varepsilon u|^p(j), \tag{4.10}$$

and for every $A \subset \mathbb{R}^n$ we have

$$\sum_{i \in Z_\varepsilon(A)} \min\Big\{|\nabla_{\varepsilon,L} u|^p(i), \frac{1}{\varepsilon}\Big\} \leq \hat{c}_2 \sum_{i \in Z_\varepsilon(A+\varepsilon L[-1,1]^n)} \min\Big\{|\nabla_\varepsilon u|^p(i), \frac{1}{\varepsilon}\Big\}. \tag{4.11}$$

Proof. Let $u : \mathbb{Z}_\varepsilon(\mathbb{R}^n) \to \mathbb{R}^d$ and $i \in Z_\varepsilon(\mathbb{R}^n)$. By Jensen's inequality we have

$$|\nabla_{\varepsilon,L} u|^p(i) \leq (\# Z_1(LQ))^{p-1} \sum_{\xi \in Z_1(LQ)} \Big|\frac{u^i - u^{i+\varepsilon\xi}}{\varepsilon}\Big|^p. \tag{4.12}$$

Moreover, for any $\xi \in Z_1(LQ)$ there exists a sequence of lattice points $i_0, \dots, i_{|\xi|_1} \in Z_\varepsilon(Q_{\varepsilon L}(i))$ with the following properties: $i_0 = i$, $i_{|\xi|_1} = \xi$ and for every $h \in \{1, \dots, n\}$ there exists $i(h) \in \{1, \dots, n\}$ such that $i_h \in \{i_{h-1} + e_{i(h)}, i_{h-1} - e_{i(h)}\}$. Thus, using again Jensen's inequality we obtain

$$\Big|\frac{u^i - u^{i+\varepsilon\xi}}{\varepsilon}\Big|^p = \Big|\sum_{h=1}^{|\xi|_1} D_\varepsilon^{\pm e_{i(h)}}(i_{h-1})\Big|^p \leq |\xi|_1^{p-1} \sum_{h=1}^{|\xi|_1} |D_\varepsilon^{\pm e_{i(h)}}(i_{h-1})|^p \leq |\xi|_1^p \sum_{j \in Z_\varepsilon(Q_{\varepsilon l}(i))} |\nabla_\varepsilon u|^p(j).$$

Summing the above estimate over $\xi \in Z_1(LQ)$ from (4.12) we deduce

$$|\nabla_{\varepsilon,L} u|^p(i) \leq (\#(Z_1(LQ))^{p-1} \sum_{\xi \in Z_1(LQ)} |\xi|_1^p \sum_{j \in Z_\varepsilon(Q_{\varepsilon L}(i))} |\nabla_\varepsilon u|^p(j) \leq \frac{n^p L^p}{2^p} (\#(Z_1(LQ))^p \sum_{j \in Z_\varepsilon(Q_{\varepsilon L}(i))} |\nabla_\varepsilon u|^p(j),$$

which gives (4.10) with $\hat{c}_1 := \frac{n^p L^p}{2^p}(\#(Z_1(LQ))^p$.

Now (4.11) is a direct consequence of (4.10). In fact, using (4.10) together with the subadditvity of the min, for any $A \subset \mathbb{R}^n$ we obtain

$$\begin{aligned}
&\sum_{i \in Z_\varepsilon(A)} \min\Big\{|\nabla_{\varepsilon,L} u|^p(i), \frac{1}{\varepsilon}\Big\} \leq \sum_{i \in Z_\varepsilon(A)} \min\Big\{\hat{c}_1 \sum_{j \in Z_\varepsilon(Q_{\varepsilon L}(i))} |\nabla_\varepsilon u|^p(j), \frac{1}{\varepsilon}\Big\} \\
&\quad\leq \sum_{i \in Z_\varepsilon(A)} \sum_{j \in Z_\varepsilon(Q_{\varepsilon L}(i))} \min\Big\{\hat{c}_1 |\nabla_\varepsilon u|^p(j), \frac{1}{\varepsilon}\Big\} \leq \max\{\hat{c}_1, 1\} \sum_{j \in Z_1(LQ)} \sum_{i \in Z_\varepsilon(A)} \min\Big\{|\nabla_\varepsilon u|^p(i+\varepsilon j), \frac{1}{\varepsilon}\Big\} \\
&\quad\leq \max\{\hat{c}_1, 1\} \# Z_1(LQ) \sum_{i \in Z_\varepsilon(A+\varepsilon L[-1,1]^n)} \min\Big\{|\nabla_\varepsilon u|^p(i), \frac{1}{\varepsilon}\Big\},
\end{aligned}$$

hence (4.11) follows by setting $\hat{c}_2 := \max\{\hat{c}_1, 1\}\# Z_1(LQ)$. $\square$

4.1.1. *The bulk energy density.* In this section we show that the bulk energy density $\bar{f}$ in (4.4) coincides with $f_{\rm hom}$ as in (4.8). This will be done by comparing our functionals with a class of functionals that fall into the framework of [17]. More precisely, we introduce rescaled interaction-energy densities $\psi_i^{\varepsilon,b} : (\mathbb{R}^d)^{Z_\varepsilon(\Omega_i)} \to [0,+\infty)$ given by

$$\psi_i^{\varepsilon,b}(\{z^j\}_{j\in Z_\varepsilon(\Omega_i)}) := \begin{cases} \psi^b_{\frac{i}{\varepsilon}}(\{\frac{1}{\varepsilon}z^{\varepsilon j}\chi^{\varepsilon j}_{LQ}\}_{j\in\mathbb{Z}^n}) & \text{if } i \in Z_\varepsilon(\Omega^L_\varepsilon), \\ \displaystyle\sum_{\substack{k=1\\ \varepsilon e_k\in\Omega_i}}^{n} |D^k_\varepsilon z(0)|^p & \text{if } i\in Z_\varepsilon(\Omega\setminus\Omega^L_\varepsilon), \end{cases}$$

and we consider the functionals $G_\varepsilon : L^1(\Omega;\mathbb{R}^d)\times\mathcal{A}(\Omega)\to[0,+\infty]$ defined by setting

$$G_\varepsilon(u,A) := \sum_{i\in Z_\varepsilon(A)} \varepsilon^n \psi_i^{\varepsilon,b}(\{u^{i+j}\}_{j\in Z_\varepsilon(\Omega_i)}), \quad \text{for } u\in\mathcal{A}_\varepsilon(\Omega;\mathbb{R}^d), \tag{4.13}$$

and extended to $+\infty$ on $L^1(\Omega;\mathbb{R}^d)\setminus\mathcal{A}_\varepsilon(\Omega;\mathbb{R}^d)$.

We show that the functions ψ_i^b have the same properties as the functions $\phi_i^k : (\mathbb{R}^d)^{\mathbb{Z}^n}\to[0,+\infty)$ defined in [17, Section 5] for $k = L$ fixed. In addition, they satisfy a suitable upper bound (see ($\mathrm{H_b}$7) below).

Lemma 4.5 (Properties of ψ_i^b)**.** *Suppose that $\psi_i^\varepsilon : (\mathbb{R}^d)^{Z_\varepsilon(\mathbb{R}^n)} \to [0,+\infty)$ are εK-periodic in i, satisfy (H1)–(H6) with $Z_\varepsilon(\Omega_i)$ replaced by $Z_\varepsilon(\mathbb{R}^n)$, and suppose that in addition (4.1) is satisfied. Assume moreover that there exists $\psi_i^b : (\mathbb{R}^d)^{\mathbb{Z}^n}\to[0,+\infty)$ such that (H_ψ1) holds true. Then the functions ψ_i^b are K-periodic in i and satisfy conditions (H1)–(H3) with $Z_\varepsilon(\Omega_i)$ replaced by $\mathbb{Z}^n$. Moreover, the following holds true for every $i\in\mathbb{Z}^n$.*

($\mathrm{H_b}$4) (lower bound) *For every $z:\mathbb{Z}^n\to\mathbb{R}^d$ there holds*

$$\psi_i^b(\{z^j\}_{j\in\mathbb{Z}^n}) \geq c_2\sum_{k=1}^n |D^k_1 z(0)|^p;$$

($\mathrm{H_b}$5) (locality) *for all $z,w:\mathbb{Z}^n\to\mathbb{R}^d$ with $z^j = w^j$ for all $j\in Z_1(LQ)$ we have*

$$\psi_i^b(\{z^j\}_{j\in\mathbb{Z}^n}) = \psi_i^b(\{w^j\}_{j\in\mathbb{Z}^n});$$

($\mathrm{H_b}$6) (controlled non-convexity) *there exists $c_4>0$ such that for all $z,w:\mathbb{Z}^n\to\mathbb{R}^d$ and every cut-off $\varphi:\mathbb{R}^n\to[0,1]$ we have*

$$\begin{aligned}\psi_i^b(\{\varphi^j z^j + (1-\varphi^j)w^j\}_{j\in\mathbb{Z}^n}) &\leq c_3\big(\psi_i^b(\{z^j\}_{j\in\mathbb{Z}^n}+\psi_i^b(\{w^j\}_{j\in\mathbb{Z}^n})\\ &+ c_4\sum_{j\in Z_1(LQ)}\sum_{\substack{\xi\in Z_1(LQ)\\ j+\xi\in LQ}}\Big(\sup_{\substack{l\in Z_1(LQ)\\ k\in\{1,\dots,n\}}}|D^k_1\varphi(l)|^p|z(j+\xi)-w(j+\xi)|^p + |D^\xi_1 z(j)|^p + |D^\xi_1 w(j)|^p\Big);\end{aligned}$$

($\mathrm{H_b}$7) (upper bound) *there exists $c_5 = c_5(n,L,p)>0$ such that for all $z:\mathbb{Z}^n\to\mathbb{R}^d$ there holds*

$$\psi_i^b(\{z^j\}_{j\in\mathbb{Z}^n}) \leq c_5(|\nabla_{1,L}z|^p(0)+1).$$

Proof. We first show that ψ_i^b is K-periodic in i. Fix $\eta>0$ and let $z:\mathbb{Z}^n\to\mathbb{R}^d$ be arbitrary. We find $\bar\varepsilon = \bar\varepsilon(z,\eta)>0$ corresponding to (H_ψ1) with $\Lambda_z = |\nabla_{1,L}z|(0)<+\infty$ such that for all $\varepsilon\in(0,\bar\varepsilon)$ and for all $i\in\mathbb{Z}^n$ we have

$$\psi^\varepsilon_{\varepsilon i}(\{\varepsilon z^{\frac{j}{\varepsilon}}\}_{j\in Z_\varepsilon(\mathbb{R}^n)}) - \eta < (\psi_i^b(\{z^j\}_{j\in\mathbb{Z}^n}) < \psi^\varepsilon_{\varepsilon i}(\{\varepsilon z^{\frac{j}{\varepsilon}}\}_{j\in Z_\varepsilon(\mathbb{R}^n)}) + \eta. \tag{4.14}$$

Thus, for all $k \in \{1,\dots,n\}$ the K-periodicity of $\psi^\varepsilon_{\varepsilon i}$ together with the fact that (4.14) holds uniformly in i ensure that

$$\psi^b_{i+Ke_k}(\{z^j\}_{j\in\mathbb{Z}^n}) < \psi^\varepsilon_{\varepsilon(i+Ke_k)}(\{\varepsilon z^{\frac{j}{\varepsilon}}\}_{j\in Z_\varepsilon(\mathbb{R}^n)}) + \eta = \psi^\varepsilon_{\varepsilon i}(\{\varepsilon z^{\frac{j}{\varepsilon}}\}_{j\in Z_\varepsilon(\mathbb{R}^n)}) + \eta < \psi^b_i(\{z^j\}_{j\in\mathbb{Z}^n}) + 2\eta.$$

Using the first inequality in (4.14) the same argument as above then leads to

$$\psi^b_i(\{z^j\}_{j\in\mathbb{Z}^n}) - 2\eta < \psi^b_{i+Ke_k}(\{z^j\}_{j\in\mathbb{Z}^n}) < \psi^b_i(\{z^j\}_{j\in\mathbb{Z}^n}) + 2\eta,$$

and we conclude by the arbitrariness of $\eta > 0$.

An analogous argument shows that (H1)–(H3) transfer from $\psi^\varepsilon_{\varepsilon i}$ to ψ^b_i and that $(\mathrm{H_b}5)$ follows from (4.2). Moreover, for every $\eta > 0$ and $z : \mathbb{Z}^n \to \mathbb{R}^d$ there exists $\bar\varepsilon = \bar\varepsilon(z,\eta) > 0$ such that for all $\varepsilon \in (0,\bar\varepsilon)$ and every $i \in \mathbb{Z}^n$ we have

$$\psi^b_i(\{z^j\}_{j\in\mathbb{Z}^n}) > \psi^\varepsilon_{\varepsilon i}(\{\varepsilon z^{\frac{j}{\varepsilon}}\}_{j\in Z_\varepsilon(\varepsilon LQ)}) - \eta \geq c_2 \min\Big\{\sum_{k=1}^n |D^k_1 z(0)|^p, \frac{1}{\varepsilon}\Big\} - \eta = c_2 \sum_{k=1}^n |D^k_1 z(0)|^p - \eta,$$

hence $(\mathrm{H_b}4)$ follows again by the arbitrariness of $\eta > 0$.

We continue proving $(\mathrm{H_b}6)$. Let $(c^{j,\xi}_\varepsilon)$ be the sequence provided by (H6). In view of (2.6) there exists $\varepsilon_0 > 0$ such that

$$c_4 := \sup_{\varepsilon\in(0,\varepsilon_0)} \sum_{j\in Z_\varepsilon(\varepsilon LQ)} \sum_{\substack{\xi\in Z_1(LQ)\\ j+\varepsilon\xi\in\varepsilon LQ}} c^{j,\xi}_\varepsilon < +\infty.$$

Fix $\eta > 0$; for any $z, w : \mathbb{Z}^n \to \mathbb{R}^d$ and $\varphi : \mathbb{Z}^n \to [0,1]$ we find $\bar\varepsilon = \bar\varepsilon(z,w,\varphi,\eta) \in (0,\varepsilon_0)$ such that for all $\varepsilon \in (0,\bar\varepsilon)$ and for all $i \in \mathbb{Z}^n$ there holds

$$\psi^b_i(\{\varphi^j z^j + (1-\varphi^j) w^j\}_{j\in\mathbb{Z}^n}) \leq \psi^\varepsilon_{\varepsilon i}(\{\varphi^{\frac{j}{\varepsilon}} \varepsilon z^{\frac{j}{\varepsilon}} + (1-\varphi^{\frac{j}{\varepsilon}}) \varepsilon w^{\frac{j}{\varepsilon}}\}_{j\in Z_\varepsilon(\mathbb{R}^n)}) + \eta,$$
$$\psi^\varepsilon_{\varepsilon i}(\{\varepsilon z^{\frac{j}{\varepsilon}}\}_{j\in Z_\varepsilon(\mathbb{R}^n)}) + \psi^\varepsilon_{\varepsilon i}(\{\varepsilon w^{\frac{j}{\varepsilon}}\}_{j\in Z_\varepsilon(\mathbb{R}^n)}) \leq \psi^b_i(\{z^j\}_{j\in\mathbb{Z}^n}) + \psi^b_i(\{w^j\}_{j\in\mathbb{Z}^n}) + \eta.$$

Then (H6) together with (4.1) yield

$$\psi^b_i(\{\varphi^j z^j + (1-\varphi^j) w^j\}_{j\in\mathbb{Z}^n}) \leq c_3\big(\psi^b_i(\{z^j\}_{j\in\mathbb{Z}^n}) + \psi^b_i(\{w^j\}_{j\in\mathbb{Z}^n}) + \eta\big) + R^\varepsilon(z,w,\varphi) + \eta,$$

where

$$R^\varepsilon(z,w,\varphi) = \sum_{j\in Z_\varepsilon(\varepsilon LQ)} \sum_{\substack{\xi\in Z_1(LQ)\\ j+\varepsilon\xi\in\varepsilon LQ}} c^{j,\xi}_\varepsilon\Big(\sup_{\substack{l\in Z_1(LQ)\\ k\in\{1,\dots,n\}}} |D^k_1\varphi(l)|^p |z(\tfrac{j}{\varepsilon}+\xi) - w(\tfrac{j}{\varepsilon}+\xi)|^p\Big)$$
$$+ c^{j,\xi}_\varepsilon\big(|D^\xi_1 z(\tfrac{j}{\varepsilon})|^p + |D^\xi_1 w(\tfrac{j}{\varepsilon})|^p\big).$$

Since $\bar\varepsilon \in (0,\varepsilon_0)$ we have $c^{j,\xi}_\varepsilon \leq c_4$ for all $\varepsilon \in (0,\bar\varepsilon)$, $j \in Z_\varepsilon(\varepsilon LQ)$ and $\xi \in Z_1(LQ)$. Hence

$$R^\varepsilon(z,w,\varphi) \leq c_4 \sum_{j\in Z_1(LQ)} \sum_{\substack{\xi\in Z_1(LQ)\\ j+\xi\in LQ}} \Big(\sup_{\substack{l\in Z_1(LQ)\\ k\in\{1,\dots,n\}}} |D^k_1\varphi(l)|^p |z(j+\xi) - w(j+\xi)|^p + |D^\xi_1 z(j)|^p + |D^\xi_1 w(j)|^p \Big)$$

and $(\mathrm{H_b}6)$ follows by the arbitrariness of $\eta > 0$.

Using a similar argument we eventually verify $(\mathrm{H_b}7)$. We consider the sequence $(c^{j,\xi}_{\varepsilon,\alpha})$ provided by (H5) and we notice that thanks to (2.4) there exists $\varepsilon_0 > 0$ such that

$$\bar c_5 := \sup_{\varepsilon\in(0,\varepsilon_0)} \sum_{j\in Z_\varepsilon(\varepsilon LQ)} \sum_{\substack{\xi\in Z_1(LQ)\\ j+\varepsilon\xi\in\varepsilon LQ}} c^{j,\xi}_{\varepsilon,1} < +\infty. \tag{4.15}$$

For any $z:\mathbb{Z}^n\to\mathbb{R}^d$ we choose $\bar\varepsilon=\bar\varepsilon(z)\in(0,\varepsilon_0)$ such that $\psi_i^b(\{z^j\}_{j\in\mathbb{Z}^n})<\psi_{\varepsilon i}^\varepsilon(\{\varepsilon z^{j/\varepsilon}\}_{j\in Z_\varepsilon(\mathbb{R}^n)})+1$ for every $\varepsilon\in(0,\bar\varepsilon)$. Moreover we define a constant function $\hat z:\mathbb{Z}^n\to\mathbb{R}^d$ by setting $\hat z^j:=z^0$ for every $j\in\mathbb{Z}^n$. Since $\bar\varepsilon<\varepsilon_0$, (H5) and (2.7) in Remark 2.1 yield for any $\varepsilon\in(0,\bar\varepsilon)$ the estimate

$$\psi_i^b(\{z^j\}_{j\in\mathbb{Z}^n})\leq c_1+2+\sum_{j\in Z_\varepsilon(\varepsilon LQ)}\sum_{\substack{\xi\in Z_1(LQ)\\ j+\varepsilon\xi\in\varepsilon LQ}}c_{\varepsilon,1}^{j,\xi}|D_1^\xi z(\tfrac{j}{\varepsilon})|^p\leq c_1+2+\bar c_5\sum_{j\in Z_1(LQ)}\sum_{\substack{\xi\in Z_1(LQ)\\ j+\xi\in LQ}}|D_1^\xi z(j)|^p.$$

Finally, the last term in the estimate above can be bounded via

$$\sum_{j\in Z_1(LQ)}\sum_{\substack{\xi\in Z_1(LQ)\\ j+\xi\in LQ}}|D_1^\xi z(j)|^p\leq 2^{p-1}(1+\#Z_1(LQ))|\nabla_{1,L}z|^p(0),$$

hence we obtain ($\mathrm{H_b}7$) by setting $c_5:=\max\{c_1+2,\bar c_5 2^{p-1}(1+\#Z_1(LQ))\}$. □

Remark 4.6. The arguments used to verify ($\mathrm{H_b}7$) also show that for all $\varepsilon\in(0,\varepsilon_0)$ with ε_0 as in (4.15), for all $i\in Z_\varepsilon(\mathbb{R}^n)$ and for all $z:Z_\varepsilon(\mathbb{R}^n)\to\mathbb{R}^d$ there holds

$$\psi_i^\varepsilon(\{z^j\}_{j\in Z_\varepsilon(\mathbb{R}^n)})\leq c_5(|\nabla_{\varepsilon,L}z|^p(0)+1).$$

Thanks to Lemma 4.5 the following is a consequence of [17, Theorem 5.1].

Theorem 4.7. *Let $G_\varepsilon:L^1(\Omega;\mathbb{R}^d)\times\mathcal{A}(\Omega)\to[0,+\infty]$ be given by (4.13) and suppose that the functions $\psi_i^\varepsilon:(\mathbb{R}^d)^{Z_\varepsilon(\mathbb{R}^n)}\to[0,+\infty)$ are εK-periodic in i, satisfy (H1)–(H6) with $Z_\varepsilon(\Omega_i)$ replaced by $Z_\varepsilon(\mathbb{R}^n)$, and the locality condition (4.1). Assume that in addition Hypotheses ($\mathrm{H}_\psi1$) holds true. Then G_ε Γ-converges in the strong $L^p(\Omega;\mathbb{R}^d)$-topology to the functional $G:L^p(\Omega;\mathbb{R}^d)\to[0,+\infty]$ given by*

$$G(u)=\int_\Omega f_{\mathrm{hom}}(\nabla u)\,dx,\qquad u\in W^{1,p}(\Omega;\mathbb{R}^d)$$

and extended by $+\infty$ in $L^p(\Omega;\mathbb{R}^d)\setminus W^{1,p}(\Omega;\mathbb{R}^d)$, where the integrand f_{hom} is given by (4.8). In particular, the limit defining f_{hom} exists and is independent of the Γ-converging subsequence.

Remark 4.8. Notice that Theorem 4.7 holds also locally, *i.e.*, for every $A\in\mathcal{A}(\Omega)$ and every $u\in W^{1,p}(\Omega;\mathbb{R}^d)$ we have

$$\Gamma\text{-}\lim_{\varepsilon\to0}G_\varepsilon(u,A)=\int_A f_{\mathrm{hom}}(\nabla u)\,dx.$$

Moreover, thanks to the finite-range assumption (4.1) the width of the boundary layer in the definition of f_{hom} can be chosen as $\sqrt{n}L$ (instead of $\sqrt{T}$ as in [17, Theorem 5.1]).

Thanks to ($\mathrm{H}_\psi1$) we can compare the two discrete energies F_ε and G_ε following a smimilar strategy as in [32]. To this end it is convenient to recall the notion of discrete maximal function and some of its properties that have been proved in [32] (see also [31]).

Given $\varepsilon>0$, $v:Z_\varepsilon(\mathbb{R}^n)\to\mathbb{R}$ and $r>0$ we define the maximal function $\mathcal{M}_\varepsilon^r v:Z_\varepsilon(\mathbb{R}^n)\to[0,+\infty)$ by setting

$$\mathcal{M}_\varepsilon^r v(i):=\sup_{s\in(0,r)}\frac{1}{\#Z_\varepsilon(\overline{B}_s^{|\cdot|_1}(i))}\sum_{j\in Z_\varepsilon(\overline{B}_s^{|\cdot|_1}(i))}|v^j|,$$

where $\overline{B}_s^{|\cdot|_1}(i)$ is the closed ball of radius s around i with respect to the $|\cdot|_1$-norm. The following lemma is a consequence of [32, Lemma 5.16 and Remark 5.17].

Lemma 4.9. *There exists a constant $\bar{c}>0$ such that for all $\varepsilon>0$ and for every $u:Z_\varepsilon(\mathbb{R}^n)\to\mathbb{R}^d$ there holds*

$$|u^i-u^j|\leq \bar{c}|i-j|_1\Big(\mathcal{M}_\varepsilon^{\bar{c}|i-j|_1}|\nabla_\varepsilon u|(i)+\mathcal{M}_\varepsilon^{\bar{c}|i-j|_1}|\nabla_\varepsilon u|(j)\Big)\quad \textit{for every } i,j\in Z_\varepsilon(\mathbb{R}^n).$$

Moreover, the following result has been established in [32, Lemma 5.18].

Lemma 4.10. *Let $x_0\in\mathbb{R}^n$, $\lambda>0$ and suppose that $u_\varepsilon:Z_\varepsilon(\mathbb{R}^n)\to\mathbb{R}^d$ satisfy*

$$\sup_{\varepsilon>0}\sum_{i\in Z_\varepsilon(B_{(3+6\bar{c}\sqrt{n})\lambda}(x_0))}|\nabla_\varepsilon u_\varepsilon|^p(i)<+\infty,$$

where $\bar{c}$ is as in Lemma 4.9. Then there exist a subsequence (ε_h) and functions $w_j:Z_{\varepsilon_h}(\mathbb{R}^n)\to\mathbb{R}^d$ such that $|\nabla_{\varepsilon_h}w_h|^p$ is equiintegrable on $B_{2\lambda}(x_0)$ and

$$\lim_{h\to+\infty}\varepsilon_h^n\#\{i\in Z_{\varepsilon_h}(B_{2\lambda}(x_0))\colon u_{\varepsilon_h}\not\equiv w_h \textit{ on } \overline{B}^{|\cdot|_1}_{\varepsilon_h}(i)\}=0. \tag{4.16}$$

Remark 4.11. Let the sequences (u_ε), (ε_h) and (w_h) be as in Lemma 4.10. Then we also have

$$\lim_{h\to+\infty}\varepsilon_h^n\#\{i\in Z_{\varepsilon_h}(B_\lambda(x_0))\colon u_{\varepsilon_h}\not\equiv w_{\varepsilon_h}\text{ on } Z_{\varepsilon_h}(Q_{\varepsilon_h L}(i))\}=0. \tag{4.17}$$

To verify (4.17) we denote by $\mathcal{U}_h$ the set in (4.16) and by $\mathcal{U}_h^L$ the set in (4.17) and we notice that for every $i\in\mathcal{U}_h^L$ there exists $j_i\in Q_{\varepsilon_h L}(i)$ such that $u_{\varepsilon_h}^{j_i}\neq w_h^{j_i}$. Since $i\in B_\lambda(x_0)$ we have $j_i\in B_{2\lambda}(x_0)$ for h sufficiently large, so that $j_i\in\mathcal{U}_h$. Hence for h sufficiently large we get

$$\varepsilon_h^n\#\mathcal{U}_h^L\leq\varepsilon_h^n\sum_{j\in\mathcal{U}_h}\#\{i\in\mathcal{U}_h^L: j\in Q_{\varepsilon_h L}(i)\}\leq cL^n\varepsilon_h^n\#\mathcal{U}_h\ \to 0 \text{ as } h\to+\infty.$$

We are now in a position to prove the following result.

Proposition 4.12. *Let the sequence (F_ε) be defined according to (2.3) with $\phi_i^\varepsilon:(\mathbb{R}^d)^{Z_\varepsilon(\Omega_i)}\to[0,+\infty)$ as in (4.3) and assume the functions $\psi_i^\varepsilon:(\mathbb{R}^d)^{Z_\varepsilon(\mathbb{R}^n)}\to[0,+\infty)$ are εK-periodic in i, satisfy (H1)–(H6) with $Z_\varepsilon(\Omega_i)$ replaced by $Z_\varepsilon(\mathbb{R}^n)$, the locality condition (4.1), and (H$_\psi$1). Then $\bar{f}(M)=f_{\rm hom}(M)$ for every $M\in\mathbb{R}^{d\times n}$, where $\bar{f}$ is as in (4.4).*

Proof. The strategy used to derive the formula for $\bar{f}$ follows closely the one used in [32, Proposition 5.19]. A main difference with respect to the situation in [32] is the fact that the interaction-energy densities ψ_i^ε are bounded from below only in terms of $|\nabla_\varepsilon u|$, while they can be bounded from above in terms of the finite-range gradient $|\nabla_{\varepsilon,L}u|$. To circumvent this additional difficulty we will frequently use Lemma 4.4.

The proof is divided into two major steps establishing separately a lower and an upper bound of $\bar{f}$ in terms of $f_{\rm hom}$.

Step 1: $\bar{f}\geq f_{\rm hom}$

Fix $M\in\mathbb{R}^{d\times n}$ and let $x_0\in\Omega$ and $\rho>0$ with $B_\rho(x_0)\subset\subset\Omega$. Then

$$|B_1|\bar{f}(M)=\frac{1}{\rho^n}F(u_{M,x_0},B_\rho(x_0)).$$

We now estimate $F(u_{M,x_0},B_\rho(x_0))$ from below. Without loss of generality we assume $x_0=0$ and for fixed $\rho_0>0$ with $B_{\rho_0}\subset\subset\Omega$ we choose functions $u_\varepsilon\in\mathcal{A}_\varepsilon(\Omega;\mathbb{R}^d)$ converging in $L^1(\Omega;\mathbb{R}^d)$ to u_M and satisfying

$$\lim_{\varepsilon\to 0}F_\varepsilon(u_\varepsilon,B_{\rho_0})=F(u_M,B_{\rho_0}).$$

Then (u_ε) is a recovery sequence for u_M on B_ρ for every $\rho \in (0,\rho_0)$, since

$$\begin{aligned} F(u_M,B_\rho) &= F(u_M,B_{\rho_0}) - F(u_M,B_{\rho_0}\setminus\overline{B_\rho}) \\ &\geq \lim_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon,B_{\rho_0}) - \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon,B_{\rho_0}\setminus\overline{B_\rho}) \geq \limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon,B_\rho), \end{aligned}$$

where in the first step we used that $F(u_M,B_\rho)$ does not concentrate on the boundary of B_ρ. In particular, we have

$$|B_1|\bar f(M) \geq \frac{1}{\rho^n}\limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon,B_\rho) \quad \text{for every } \rho\in(0,\rho_0). \tag{4.18}$$

We now introduce a constant $\bar k > 0$ satisfying

$$\bar k > 3 + 6\bar c\sqrt{n} + |M|,$$

where $\bar c$ is as in Remark 4.9. Since $|u_M| \leq |M|\rho \leq \bar k\rho$ on B_ρ, the truncated functions $T_{\bar k\rho}u_\varepsilon$ converge to u_M in $L^1(B_\rho,\mathbb{R}^d)$. In particular, in view of Remark 2.2 they still provide a recovery sequence for u_M on B_ρ.

Fix $\eta > 0$ and for every $\rho \in (0,(3\bar k^2)^{-1}\rho_0)$ let $\bar\varepsilon_\rho = \bar\varepsilon(\eta,\frac{\sqrt{n}L}{2}\bar k\Lambda_\rho\#Z_1(LQ))$ be given by (H$_\psi$1) with Λ_ρ to be chosen later. We choose

$$\varepsilon_\rho < \min\Big\{\rho^2,\rho^{\frac{p}{p-1}},\bar\varepsilon_\rho,\frac{\operatorname{dist}_\infty(B_{\rho_0},\partial\Omega)}{L}\Big\}$$

non-decreasing in ρ and satisfying

$$F_{\varepsilon_\rho}(T_{\bar k\rho}u_{\varepsilon_\rho},B_{3\bar k^2\rho}) \leq c(|M|^p+1)\rho^n, \tag{4.19}$$

$$\frac{1}{|B_1|\rho^n}\int_{B_\rho}|T_{\bar k\rho}u_{\varepsilon_\rho} - u_M|^p\,dx \leq \rho^{p+1}. \tag{4.20}$$

Here the first estimate can be realized thanks to (4.18) and the fact that $\bar f(M) \leq c(|M|^p+1)$. Observe that since $\rho < (3\bar k^2)^{-1}\rho_0$ our choice of ε_ρ implies that $B_{3\bar k^2\rho}\subset B_{\rho_0}\subset\Omega^L_{\varepsilon_\rho}$ and hence

$$F_{\varepsilon_\rho}(T_{\bar k\rho}u_{\varepsilon_\rho},B_{3\bar k^2\rho}) = \sum_{i\in Z_{\varepsilon_\rho}(B_{3\bar k^2\rho})}\varepsilon_\rho^n\psi_i^{\varepsilon_\rho}(\{u^{i+j}_{\varepsilon_\rho}\chi^j_{\varepsilon_\rho LQ}\}_{j\in Z_{\varepsilon_\rho}(\mathbb{R}^n)}) = \sum_{i\in Z_1(B_{3\bar k^2\frac{\rho}{\varepsilon_\rho}})}\varepsilon_\rho^n\psi^{\varepsilon_\rho}_{\varepsilon_\rho i}(\{\varepsilon_\rho v_\rho^{i+j/\varepsilon_\rho}\}_{j\in Z_{\varepsilon_\rho}(\mathbb{R}^n)}), \tag{4.21}$$

where $v_\rho : \mathbb{Z}^n \to \mathbb{R}^d$ is defined by setting

$$v^i_\rho := \frac{1}{\varepsilon_\rho}T_{\bar k\rho}u^{\varepsilon_\rho i}_{\varepsilon_\rho}\chi^{\varepsilon_\rho i}_\Omega, \quad \text{for every } i\in\mathbb{Z}^n.$$

Substep 1a: Construction of Lipschitz-competitors
We now aim to replace v_ρ by a Lipschitz function $\bar v_\rho$ with Lipschitz constant at most $\bar k\Lambda_\rho$. To this end we introduce the sets of regular and singular points defined as

$$\mathcal{R}_\rho := \{i\in Z_1(B_{\bar k\rho/\varepsilon_\rho})\colon \mathcal{M}_1^{\bar k^2\rho/\varepsilon_\rho}|\nabla_1 v_\rho| \leq \Lambda_\rho\}, \qquad \mathcal{S}_\rho := \{i\in\mathbb{Z}^n\colon |\nabla_1 v_\rho|(i) \geq \Lambda_\rho/2\},$$

respectively. Notice that for every $i,j\in\mathcal{R}_\rho$ thanks to Lemma 4.9 we have the Lipschitz estimate

$$|v^i_\rho - v^j_\rho| \leq \bar c\sqrt{n}|i-j|\Big(\mathcal{M}_1^{\bar k^2\rho/\varepsilon_\rho}|\nabla_1 v_\rho|(i) + \mathcal{M}_1^{\bar k^2\rho/\varepsilon_\rho}|\nabla_1 v_\rho|(j)\Big) \leq \bar k\Lambda_\rho|i-j|.$$

Using Kirszbraun's extension theorem we thus find a function $\bar{v}_\rho : \mathbb{Z}^n \to \mathbb{R}^d$ coinciding with v_ρ on $\mathcal{R}_\rho$ and satisfying $|\bar{v}_\rho^i - \bar{v}_\rho^j| \leq \bar{k}\Lambda_\rho|i-j|$ for every $i,j \in \mathbb{Z}^n$. In particular, we have

$$|\nabla_{1,L}\bar{v}_\rho|(i) \leq \frac{\sqrt{n}L}{2}\bar{k}\Lambda_\rho \# Z_1(LQ) \quad \text{for every } i \in \mathbb{Z}^n. \tag{4.22}$$

In addition, by truncation with the operator $T_{3\bar{k}\rho/\varepsilon_\rho}$ we can assume that $\|\bar{v}_\rho\|_\infty \leq 9\bar{k}\rho/\varepsilon_\rho$.

In the remaining part of this substep we bound the number of points in which v_ρ and $\bar{v}_\rho$ do not coincide, that is the cardinality of $Z_1(B_{\bar{k}\rho/\varepsilon_\rho})\backslash\mathcal{R}_\rho$. We first observe that for every $i \in Z_1(B_{\bar{k}\rho/\varepsilon_\rho})\backslash\mathcal{R}_\rho$ there exists $s_i \in (0, \bar{k}^2\rho/\varepsilon_\rho)$ such that

$$\Lambda_\rho \# Z_1(\overline{B}^{|\cdot|_1}_{s_i}(i)) \leq \sum_{j \in Z_1(\overline{B}^{|\cdot|_1}_{s_i}(i))} |\nabla_1 v_\rho|(j).$$

Applying Vitali's covering lemma we find $\mathcal{I}_\rho \subset Z_1(B_{\bar{k}\rho/\varepsilon_\rho}) \setminus \mathcal{R}_\rho$ (finite) such that the family $(\overline{B}^{|\cdot|_1}_{s_i}(i))_{i\in\mathcal{I}_\rho}$ is disjoint and

$$Z_1(B_{\bar{k}\rho/\varepsilon_\rho}) \setminus \mathcal{R}_\rho \subset \bigcup_{i\in\mathcal{I}_\rho} \overline{B}^{|\cdot|_1}_{5s_i}(i),$$

hence

$$\# Z_1(B_{\bar{k}\rho/\varepsilon_\rho}) \setminus \mathcal{R}_\rho \leq \# Z_1\Big(\bigcup_{i\in\mathcal{I}_\rho} \overline{B}^{|\cdot|_1}_{5s_i}(i)\Big) \leq 5^n \# Z_1\Big(\bigcup_{i\in\mathcal{I}_\rho} \overline{B}^{|\cdot|_1}_{s_i}(i)\Big). \tag{4.23}$$

To estimate the cardinality of $Z_1\big(\bigcup_i \overline{B}^{|\cdot|_1}_{s_i}(i)\big)$ we distinguish between the lattice points in $\bigcup_i \overline{B}^{|\cdot|_1}_{s_i}(i)$ belonging to $\mathcal{S}_\rho$ and those that belong to its complement. In fact, since the balls $\overline{B}^{|\cdot|_1}_{s_i}(i)$ are disjoint, the definition of $\mathcal{S}_\rho$ implies that

$$\Lambda_\rho \# Z_1\Big(\bigcup_{i\in\mathcal{I}_\rho} \overline{B}^{|\cdot|_1}_{s_i}(i)\Big) \leq \sum_{j\in Z_1(\bigcup_i \overline{B}^{|\cdot|_1}_{s_i}(i))} |\nabla_1 v_\rho|(j) \leq \sum_{j\in\bigcup_i \overline{B}^{|\cdot|_1}_{s_i}(i)\cap\mathcal{S}_\rho} |\nabla_1 v_\rho|(j) + \frac{\Lambda_\rho}{2} \# Z_1\Big(\bigcup_{i\in\mathcal{I}_\rho} \overline{B}^{|\cdot|_1}_{s_i}(i)\Big),$$

hence

$$\# Z_1\Big(\bigcup_{i\in\mathcal{I}_\rho} \overline{B}^{|\cdot|_1}_{s_i}(i)\Big) \leq \frac{2}{\Lambda_\rho} \sum_{j\in\bigcup_i \overline{B}^{|\cdot|_1}_{s_i}(i)\cap\mathcal{S}_\rho} |\nabla_1 v_\rho|(j). \tag{4.24}$$

We aim to bound the term on the right-hand side of (4.24) via $F_{\varepsilon_\rho}(T_{\bar{k}\rho}u_{\varepsilon_\rho}, B_{3\bar{k}^2\rho})$. To this end we introduce the set of jump points

$$\mathcal{J}_\rho := \Big\{ i \in \mathbb{Z}^n \colon |\nabla_1 v_\rho|^p(i) \geq 1/\varepsilon_\rho \Big\}$$

and we use Hölder's inequality to obtain the estimate

$$\sum_{j\in\bigcup_i \overline{B}^{|\cdot|_1}_{s_i}(i)\cap\mathcal{S}_\rho\setminus\mathcal{J}_\rho} |\nabla_1 v_\rho|(j) \leq \Big(\#\big(\bigcup_{i\in\mathcal{I}_\rho} \overline{B}^{|\cdot|_1}_{s_i}(i) \cap \mathcal{S}_\rho \setminus \mathcal{J}_\rho\big)\Big)^{\frac{p-1}{p}} \Big(\sum_{i\in\bigcup_i \overline{B}^{|\cdot|_1}_{s_i}(i)\cap\mathcal{S}_\rho\setminus\mathcal{J}_\rho} |\nabla_1 v_\rho|^p(j)\Big)^{\frac{1}{p}}. \tag{4.25}$$

Then by definition for every $j \in \bigcup_i \overline{B}^{|\cdot|_1}_{s_i}(i) \cap \mathcal{S}_\rho \setminus \mathcal{J}_\rho$ we have

$$|\nabla_1 v_\rho|^p(j) = \min\big\{|\nabla_1 v_\rho|^p(j), \frac{1}{\varepsilon_\rho}\big\} \leq (2n)^{p-1}\Big(\min\big\{\sum_{k=1}^n |D^k_\varepsilon v^j_\rho|^p, \frac{1}{\varepsilon_\rho}\big\} + \min\big\{\sum_{k=1}^n |D^k_\varepsilon v^{j-e_k}_\rho|^p, \frac{1}{\varepsilon_\rho}\big\}\Big),$$

where in the second step we used the subadditivity of min. Moreover, for every $j \in \bigcup_i \overline{B}^{|\cdot|_1}_{s_i}(i)$ there holds

$$j - e_k \in \bigcup_{i\in\mathcal{I}_\rho} \overline{B}^{|\cdot|_1}_{s_i+\varepsilon_\rho}(i) \subset B_{3\bar{k}^2\rho/\varepsilon_\rho} \quad \text{for every } k \in \{1,\dots,n\}. \tag{4.26}$$

Thus, from (H4) together the energy bound (4.19) we infer

$$\sum_{i\in\bigcup_i \overline{B}^{|\cdot|_1}_{s_i}(i)\cap\mathcal{S}_\rho\setminus\mathcal{J}_\rho} |\nabla_1 v_\rho|^p(j) \leq 2(2n)^{p-1} \sum_{j\in Z_1(B_{3\bar{k}^2\rho/\varepsilon_\rho})} \min\Big\{\sum_{k=1}^n |D_1^k v_\rho|^p, \frac{1}{\varepsilon_\rho}\Big\} \leq c\frac{\rho^n}{\varepsilon_\rho^n}, \tag{4.27}$$

where the additional factor 2 comes from the fact that each term is counted at most twice. Finally, since $|\nabla_1 v_\rho| \geq \frac{\Lambda_\rho}{2}$ on $\mathcal{S}_\rho$, (4.27) gives

$$\Big(\frac{\Lambda_\rho}{2}\Big)^p \# \bigcup_{i\in\mathcal{I}_\rho} \overline{B}^{|\cdot|_1}_{s_i}(i)\cap\mathcal{S}_\rho\setminus\mathcal{J}_\rho \leq \sum_{j\in\bigcup_i \overline{B}^{|\cdot|_1}_{s_i}(i)\cap\mathcal{S}_\rho\setminus\mathcal{J}_\rho} |\nabla_1 v_\rho|^p(j) \leq c\frac{\rho^n}{\varepsilon_\rho^n}. \tag{4.28}$$

Gathering (4.25), (4.27) and (4.28) we eventually deduce that

$$\frac{2}{\Lambda_\rho} \sum_{j\in\bigcup_i \overline{B}^{|\cdot|_1}_{s_i}(i)\cap\mathcal{S}_\rho\setminus\mathcal{J}_\rho} |\nabla_1 v_\rho|(j) \leq c\Lambda_\rho^{-p}\frac{\rho^n}{\varepsilon_\rho^n}. \tag{4.29}$$

To estimate the remaining contributions in (4.24) we observe that for every $j \in \mathcal{J}_\rho$ there exists $k(j) \in \{1,\dots,n\}$ such that either $|D_1^k v_\rho^j|^p \geq 1/\varepsilon_\rho(2n)^p$ or $|D_1^k v_\rho^{j-e_k}|^p \geq 1/\varepsilon_\rho(2n)^p$. Using once more the inclusion in (4.26) we then obtain

$$\frac{1}{\varepsilon_\rho(2n)^p}\#\Big(\bigcup_{i\in\mathcal{I}_\rho} \overline{B}^{|\cdot|_1}_{s_i}(i)\cap\mathcal{J}_\rho\Big) \leq 2\sum_{j\in Z_1(B_{3\bar{k}^2\rho/\varepsilon_\rho})} \min\Big\{\sum_{k=1}^n |D_1^k v_\rho^j|^p, \frac{1}{\varepsilon_\rho}\Big\} \leq c\frac{\rho^n}{\varepsilon_\rho^n},$$

where the additional factor 2 results again from a possible double counting of interactions. Moreover, the uniform bound on v_ρ implies $|D_1^k v_\rho^j| \leq c\bar{k}\rho/\varepsilon_\rho$ for every $j \in \mathbb{Z}^n$, so that the above estimate yields

$$\frac{2}{\Lambda_\rho}\sum_{j\in\bigcup_i \overline{B}^{|\cdot|_1}_{s_i}(i)\cap\mathcal{J}_\rho} |\nabla_1 v_\rho|(j) \leq c\bar{k}\Lambda_\rho^{-1}\frac{\rho}{\varepsilon_\rho}\#\Big(\bigcup_{i\in\mathcal{I}_\rho} \overline{B}^{|\cdot|_1}_{s_i}(i)\cap\mathcal{J}_\rho\Big) \leq c\rho\Lambda_\rho^{-1}\frac{\rho^n}{\varepsilon_\rho^n}. \tag{4.30}$$

Combining (4.23), (4.24), (4.29) and (4.30) and choosing $\Lambda_\rho = \rho^{\frac{1}{1-p}}$ we finally deduce that

$$\#(Z_1(B_{\bar{k}\rho/\varepsilon_\rho})\setminus\mathcal{R}_\rho) \leq c(\rho\Lambda_\rho^{-1} + \Lambda_\rho^{-p})\frac{\rho^n}{\varepsilon_\rho^n} = c\rho^{\frac{p}{p-1}}\frac{\rho^n}{\varepsilon_\rho^n}. \tag{4.31}$$

Substep 1b: From Lipschitz continuity to equiintegrable gradients
In this substep we show that the rescaled functions $\tilde{v}_\rho$ obtained by setting

$$\tilde{v}_\rho^i := \frac{\varepsilon_\rho}{\rho}\bar{v}_\rho^{\frac{\rho}{\varepsilon_\rho}i} \quad \text{for every } i \in Z_{\frac{\varepsilon_\rho}{\rho}}(\mathbb{R}^n)$$

satisfy the hypotheses of Lemma 4.10 with $\lambda = 1$ and $x_0 = 0$ along the vanishing sequence $\sigma_\rho := \frac{\varepsilon_\rho}{\rho}$. We start observing that $\tilde{v}_\rho$ satisfy the following conditions.

(i) $\|\tilde{v}_\rho\|_\infty \leq 9\bar{k}$;
(ii) $|\tilde{v}_\rho^i - \tilde{v}_\rho^j| \leq \bar{k}\Lambda_\rho|i-j|$ for all $i,j \in Z_{\sigma_\rho}(\mathbb{R}^n)$;

(iii) $\tilde{v}_\rho^i = \frac{1}{\rho} T_{\bar{k}\rho} u_{\varepsilon_\rho}^{\rho i}$ if $\frac{\rho}{\varepsilon_\rho} i \in \mathcal{R}_\rho$.

Note that (ii) implies that $|\nabla_{\sigma_\rho}\tilde{v}_\rho|^p(i) \leq c\Lambda_\rho^p$ for every $i \in Z_{\sigma_\rho}(\mathbb{R}^n)$. We thus obtain the estimate

$$\sum_{i\in Z_{\sigma_\rho}(B_{\bar{k}})} \sigma_\rho^n |\nabla_{\sigma_\rho}\tilde{v}_\rho|^p(i) \leq \sum_{\substack{i\in Z_{\varepsilon_\rho}(B_{\bar{k}\rho})\\ \frac{i}{\varepsilon_\rho}\in\mathcal{R}_\rho}} \frac{\varepsilon_\rho^n}{\rho^n} \sum_{\substack{k=1\\ \frac{i}{\varepsilon_\rho}+e_k\in\mathcal{R}_\rho}}^{n} |D_{\varepsilon_\rho}^k T_{\bar{k}\rho} u_{\varepsilon_\rho}^i|^p + c\Lambda_\rho^p \frac{\varepsilon_\rho^n}{\rho^n} \#(Z_1(B_{\bar{k}\rho/\varepsilon_\rho}) \setminus \mathcal{R}_\rho). \tag{4.32}$$

Thanks to (4.31) we can bound the second term on the right-hand side of (4.32) by a constant. Moreover, the definition of the maximal function together with the choice of the set $\mathcal{R}_\rho$ implies that for all $i \in Z_{\varepsilon_\rho}(B_{\bar{k}\rho})$ with $i/\varepsilon_\rho \in \mathcal{R}_\rho$ we have

$$\sum_{k=1}^n |D_{\varepsilon_\rho}^k T_{\bar{k}\rho} u_{\varepsilon_\rho}^i|^p \leq |\nabla_{\varepsilon_\rho} T_{\bar{k}\rho} u_{\varepsilon_\rho}|^p(i) = |\nabla_1 v_\rho|^p \leq \Lambda_\rho^p = \rho^{\frac{p}{1-p}} < \frac{1}{\varepsilon_\rho},$$

where in the last step we have used that $\varepsilon_\rho < \rho^{\frac{p}{p-1}}$. Hence we can bound the first term on the right-hand side of (4.32) by the energy and use (4.19) to deduce that

$$\sum_{i\in Z_{\sigma_\rho}(B_{\bar{k}})} \sigma_\rho^n |\nabla_{\sigma_\rho}\tilde{v}_\rho|^p(i) \leq \frac{c}{\rho^n} F_{\varepsilon_\rho}(T_{\bar{k}\rho} u_{\varepsilon_\rho}, B_{\bar{k}\rho}) + c \leq c.$$

Thanks to our choice of $\bar{k}$ Lemma 4.10 then provides us with a subsequence (ρ_h) and functions $w_h : Z_{\sigma_h}(\mathbb{R}^n) \to \mathbb{R}^d$ such that $|\nabla_{\sigma_h} w_h|^p$ is equiintegrable on B_2 and

$$\lim_{h\to+\infty} \sigma_h^n \#\{i \in Z_{\sigma_h}(B_2)\colon \tilde{v}_{\rho_h} \not\equiv w_h \text{ on } \overline{B}_{\sigma_h}^{|\cdot|_1}(i)\} = 0, \tag{4.33}$$

where we have set $\sigma_h := \sigma_{\rho_h}$. Moreover, upon truncation we can assume that $\|w_h\|_\infty \leq 27\bar{k}$.

Substep 1c: Conclusion of the lower-bound inequality

We continue by proving that the sequence (w_h) obtained in Substep 1b converges to u_M in $L^p(B_1;\mathbb{R}^d)$. To simplify notation we set $\varepsilon_h := \varepsilon_{\rho_h}$. We start estimating

$$\|w_h - u_m\|_{L^p(B_1;\mathbb{R}^d)} \leq \|w_h - \frac{1}{\rho_h} T_{\bar{k}\rho_h} u_{\varepsilon_h}(\rho_h\cdot)\|_{L^p(B_1;\mathbb{R}^d)} + \|\frac{1}{\rho_h} T_{\bar{k}\rho_h} u_{\varepsilon_h}(\rho_h\cdot) - u_M\|_{L^p(B_1;\mathbb{R}^d)}.$$

By a change of variables and (4.20) we obtain

$$\|\frac{1}{\rho_h} T_{\bar{k}\rho_h} u_{\varepsilon_h}(\rho_h\cdot) - u_M\|^p_{L^p(B_1;\mathbb{R}^d)} \leq \frac{1}{\rho_h^{n+p}} \int_{B_{\rho_h}} |T_{\bar{k}\rho_h} u_{\varepsilon_h} - u_M|^p\,dx \leq \rho_h \ \to 0 \text{ as } h \to +\infty.$$

Moreover, we denote by $\mathcal{U}_h$ the set in (4.33) and we notice that for all $i \in Z_{\sigma_h}(B_2) \setminus \mathcal{U}_h$ with $i/\sigma_h \in \mathcal{R}_{\rho_h}$ we have $w_h^i = 1/\rho_h T_{\bar{k}\rho_h} u_{\varepsilon_h}^{\rho_h i}$. Thus, the uniform bound on $\|w_h\|_\infty$ together with (4.31), (4.33) yield

$$\|w_h - \frac{1}{\rho_h} T_{\bar{k}\rho_h} u_{\varepsilon_h}(\rho_h\cdot)\|^p_{L^p(B_1;\mathbb{R}^d)} \leq c|M|^p \sigma_h^n \big(\#\mathcal{U}_h + \#(Z_1(B_{2\rho_h/\varepsilon_h}) \setminus \mathcal{R}_{\rho_h})\big) \leq c|M|^p\big(\sigma_h^n \#\mathcal{U}_h + \rho_h^{\frac{p}{p-1}}\big),$$

where the second inequality follows from (4.31). Thanks to (4.33) we conclude that $w_h \to u_M$ in $L^p(B_1;\mathbb{R}^d)$.

We finally show that up to a small error $1/\rho_h^n F_{\varepsilon_h}(u_{\varepsilon_h}, B_{\rho_h})$ is asymptotically bounded from below by $|B_1| f_{\mathrm{hom}}$. Then the required inequality follows from (4.18) by letting $h \to +\infty$. We start by introducing the sets

$$\mathcal{U}_h^L := \{i \in Z_{\sigma_h}(B_1)\colon \tilde{v}_{\rho_h} \not\equiv w_h \text{ on } Q_{\sigma_h L}(i)\},$$

$$\mathcal{V}_h := \{i \in Z_1(B_{\rho_h/\varepsilon_h})\colon Z_1(Q_L(i)) \subset \mathcal{R}_{\rho_h},\ \sigma_h i \in Z_{\sigma_h}(B_1) \setminus \mathcal{U}_h^L\}.$$

and by observing that Remark 4.11 and (4.31) yield

$$\sigma_h^n \#(Z_1(B_{\rho_h/\varepsilon_h}) \setminus \mathcal{V}_h) \leq \sigma_h^n\big(\#\mathcal{U}_h^L + cL^n\#(Z_1(B_{\bar{k}\rho_h/\varepsilon_h}) \setminus \mathcal{R}_{\rho_h})\big) \to 0 \text{ as } h \to +\infty. \tag{4.34}$$

Moreover, thanks to the locality property (4.2) we have

$$\begin{aligned}\frac{1}{\rho_h^n}F_{\varepsilon_h}(u_{\varepsilon_h}, B_{\rho_h}) &\geq \frac{1}{\rho_h^n}F_{\varepsilon_h}(T_{\bar{k}\rho_h}u_{\varepsilon_h}, B_{\rho_h}) \geq \sum_{i\in\mathcal{V}_h}\sigma_h^n\psi_{\varepsilon_h i}^{\varepsilon_h}(\{\varepsilon_h\bar{v}_{\rho_h}^{i+\frac{j}{\varepsilon_h}}\}_{j\in Z_{\varepsilon_h}(\mathbb{R}^n)})\\ &\geq \sum_{i\in\mathcal{V}_h}\sigma_h^n\psi_i^b(\{\bar{v}_{\rho_h}^{i+j}\}_{j\in\mathbb{Z}^n}) - \eta,\end{aligned}$$

where the last inequality follows from (4.22) and ($\mathrm{H}_\psi 1$) together with the fact that $\varepsilon_h < \bar{\varepsilon}_{\rho_h}$. By construction $\bar{v}_{\rho_h}^{i+j} = 1/\sigma_h w_h^{\sigma_h(i+j)}$ for every $i \in \mathcal{V}_h$ and every $j \in Z_1(LQ)$, hence we obtain

$$\begin{aligned}\frac{1}{\rho_h^n}F_{\varepsilon_h}(u_{\varepsilon_h}, B_{\rho_h}) &\geq \sum_{i\in Z_{\sigma_h}(B_1)}\sigma_h^n\psi_{\frac{i}{\sigma_h}}^b(\{\frac{1}{\sigma_h}w_h^{i+\sigma_h j}\}_{j\in\mathbb{Z}^n}) - \sum_{i\in Z_1(B_{\rho_h/\varepsilon_h})\setminus\mathcal{V}_h}\sigma_h^n\psi_i^b(\{\frac{1}{\sigma_h}w_h^{\sigma_h(i+j)}\}_{j\in\mathbb{Z}^n}) - \eta\\ &\geq G_{\sigma_h}(w_h, B_1) - c_5\sum_{i\in Z_1(B_{\rho_h/\varepsilon_h})\setminus\mathcal{V}_h}\sigma_h^n\big(|\nabla_{\sigma_h,L}w_h|^p(\sigma_h i) + 1\big) - \eta\\ &\geq G_{\sigma_h}(w_h, B_1) - c_5\sum_{i\in Z_1(B_{\rho_h/\varepsilon_h})\setminus\mathcal{V}_h}\sigma_h^n\Big(\hat{c}_1\sum_{j\in Z_{\sigma_h}(\sigma_h Q_L(i))}|\nabla_{\sigma_h}w_h|^p(j) + 1\Big) - \eta,\end{aligned} \tag{4.35}$$

where $\hat{c}_1$ is given by (4.10). In order to further estimate the second term in (4.35) we consider the set

$$\mathcal{W}_h := \{j \in Z_{\sigma_h}(B_{3/2})\colon \exists\ i \in Z_1(B_{\rho_h/\varepsilon_h}) \setminus \mathcal{V}_h \text{ s.t. } j \in \sigma_h LQ(i)\}$$

and for every $j \in \mathcal{W}_h$ we define

$$\gamma_h(j) := \#\{i \in Z_1(B_{\rho_h/\varepsilon_h}) \setminus \mathcal{V}_h\colon j \in \sigma_h LQ(i)\}.$$

Then for h sufficiently large we have

$$\sum_{i\in Z_1(B_{\rho_h/\varepsilon_h})\setminus\mathcal{V}_h}\sigma_h^n\sum_{j\in Z_{\sigma_h}(\sigma_h Q_L(i))}|\nabla_{\sigma_h}w_h|^p(j) \leq \sum_{j\in\mathcal{W}_h}\sigma_h^n\gamma_h(j)|\nabla_{\sigma_h}w_h|^p(j) \leq c(n,L)\sum_{j\in\mathcal{W}_h}\sigma_h^n|\nabla_{\sigma_h}w_h|^p(j), \tag{4.36}$$

where in the second step we used that $\gamma_h(j) \leq \#Z_{\sigma_h}(Q_{\sigma_h L}(j)) \leq cL^n$ for every $j \in \mathcal{W}_h$. We eventually observe that $\#\mathcal{W}_h \leq cL^n\#(Z_1(B_{\rho_h/\varepsilon_h}) \setminus \mathcal{V}_h) \to 0$ as $h \to +\infty$. Hence the equiintegrability of $|\nabla_{\sigma_h}w_h|^p$ on B_2 yields the existence of some $h_\eta > 0$ such that

$$c_5\hat{c}_1 c(n,L)\sum_{j\in\mathcal{W}_h}\sigma_h^n|\nabla_{\sigma_h}w_h|^p(j) < \eta \quad \text{for every } h \geq h_\eta.$$

As a consequence, combining (4.35) and (4.36) we obtain

$$\frac{1}{\rho_h^n}F_{\varepsilon_h}(u_{\varepsilon_h}, B_{\rho_h}) \geq G_{\sigma_h}(w_h, B_1) - c_5\sigma_h^n\#(Z_1(B_{\rho_h/\varepsilon_h}) \setminus \mathcal{V}_h) - 2\eta, \tag{4.37}$$

for all $h \geq h_\eta$. Thus, since $w_h \to u_M$ in $L^p(B_1, \mathbb{R}^d)$, from (4.18), (4.34) and (4.37) together with Theorem 4.7 and Remark 4.8 we deduce that

$$|B_1|\bar{f}(M) \geq \liminf_{h\to+\infty}G_{\sigma_h}(w_h, B_1) - 2\eta \geq G(u_M, B_1) - 2\eta = |B_1|f_{\mathrm{hom}}(M) - 2\eta$$

and we conclude by letting $\eta \to 0$.

Step 2: $\bar{f} \le f_{\rm hom}$

In order to prove the opposite inequality we choose a sequence (u_ε) converging to u_M in $L^p(\Omega;\mathbb{R}^d)$ and satisfying

$$\lim_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon, B_{\rho_0}) = G(u_M, B_{\rho_0}).$$

Fix $\rho \in (0,(3\bar{k}^2)^{-1}\rho_0)$; then the truncated functions $T_{\bar{k}\rho}u_\varepsilon$ still provide a recovery sequence for u_M on B_ρ. In particular, we obtain

$$|B_1| f_{\rm hom}(M) = \frac{1}{\rho^n} G(u_M, B_\rho) \ge \frac{1}{\rho^n} \limsup_{\varepsilon\to 0} G_\varepsilon(T_{\bar{k}\rho}u_\varepsilon, B_\rho). \tag{4.38}$$

In order to use (H$_\psi$1) to pass from G_ε to F_ε we need replace $T_{\bar{k}\rho u_\varepsilon}$ by a sequence of functions with equiintegrable discrete gradients. This can be done by using Lemma 4.10 along the vanishing sequence ε with $\lambda = \rho$. We start observing that thanks to (H$_{\rm b}$4) the functions $T_{\bar{k}\rho}u_\varepsilon$ satisfy the assumptions of Lemma 4.10. In fact,

$$\frac{c_2}{(2n)^p} \sum_{i\in Z_\varepsilon(B_{\bar{k}\rho})} \varepsilon^n |\nabla_\varepsilon T_{\bar{k}\rho}u_\varepsilon|^p(i) \le c_2 \sum_{i\in Z_\varepsilon(B_{2\bar{k}\rho})} \varepsilon^n \sum_{k=1}^n |D_\varepsilon^k T_{\bar{k}\rho} u_\varepsilon^i|^p \le G_\varepsilon(u_\varepsilon, B_{\rho_0}) \le c\rho^n,$$

for some $c > 0$ uniformly with respect to ε. Thus, Lemma 4.10 ensures the existence of a subsequence ε_h and functions $w_h : Z_{\varepsilon_h}(\mathbb{R}^n) \to \mathbb{R}^d$ (possibly depending on ρ) such that $|\nabla_{\varepsilon_h} w_h|^p$ is equiintegrable on B_{2_ρ} and such that

$$\lim_{h\to+\infty} \varepsilon_h^n \#\{i \in Z_{\varepsilon_h}(B_{2\rho})\colon T_{\bar{k}\rho}u_{\varepsilon_h} \not\equiv w_h \text{ on } \overline{B}_{\varepsilon_h}^{|\cdot|_1}(i)\} = 0. \tag{4.39}$$

Moreover, upon truncation we can assume that $\|w_h\|_\infty \le 9\bar{k}$. Denoting by $\mathcal{U}_{\varepsilon_h}$ the set in (4.39) the uniform bound on $\|T_{\bar{k}\rho}u_{\varepsilon_h}\|_\infty$ and $\|w_h\|_\infty$ together with (4.39) give

$$\begin{aligned}\|w_h - u_M\|_{L^p(B_\rho;\mathbb{R}^d)} &\le \|w_h - T_{\bar{k}\rho}u_{\varepsilon_h}\|_{L^p(B_\rho;\mathbb{R}^d)} + \|T_{\bar{k}\rho}u_{\varepsilon_h} - u_M\|_{L^p(B_\rho;\mathbb{R}^d)} \\ &\le c|M|(\varepsilon_h^n \#\mathcal{U}_{\varepsilon_h})^{\frac{1}{p}} + \|T_{\bar{k}\rho}u_{\varepsilon_h} - u_M\|_{L^p(B_\rho;\mathbb{R}^d)} \ \to 0 \text{ as } h \to +\infty.\end{aligned}$$

Hence, Theorem 3.1 implies that

$$|B_1|\bar{f}(M) = \frac{1}{\rho^n} F(u_M, B_\rho) \le \frac{1}{\rho^n} \liminf_{h\to+\infty} F_{\varepsilon_h}(w_h, B_\rho), \tag{4.40}$$

and it remains to compare $F_{\varepsilon_h}(w_h, B_\rho)$ and $G_{\varepsilon_h}(T_{\bar{k}\rho}u_{\varepsilon_h}, B_\rho)$. We start comparing $G_{\varepsilon_h}(T_{\bar{k}\rho}u_{\varepsilon_h}, B_\rho)$ and $G_{\varepsilon_h}(w_h, B_\rho)$. To this end we introduce the sets

$$\begin{aligned}\mathcal{U}_{\varepsilon_h}^L &:= \{i \in Z_{\varepsilon_h}(B_\rho)\colon T_{\bar{k}\rho}u_{\varepsilon_h} \not\equiv w_h \text{ on } Q_{\varepsilon_h L}(i)\}, \\ \mathcal{V}_{\varepsilon_h}^L &:= \{j \in Z_{\varepsilon_h}(B_{3\rho/2})\colon \exists\, i \in \mathcal{U}_{\varepsilon_h}^L \text{ s.t. } j \in Q_{\varepsilon_h L}(i)\},\end{aligned}$$

and we notice that as in Substep 1c one can show that

$$\lim_{h\to+\infty} \varepsilon_h^n \#\mathcal{U}_{\varepsilon_h}^L = 0, \qquad \lim_{h\to+\infty} \varepsilon_h^n \#\mathcal{V}_{\varepsilon_h}^L = 0.$$

Thus, arguing as in (4.36) and using the equiintegrability of $|\nabla_{\varepsilon_h} w_h|^p$ on $B_{2\rho}$ we deduce that there exists $h_1 = h_1(\eta, \rho) > 0$ such that for all $h \ge h_1$ we have

$$\frac{c_5}{\rho^n} \sum_{i\in\mathcal{U}_{\varepsilon_h}^l} \varepsilon_h^n (|\nabla_{\varepsilon_h,L} w_h|^p(i) + 1) \le \frac{c_5}{\rho^n} \hat{c}_1 c(n,L) \sum_{i\in\mathcal{V}_{\varepsilon_h}^L} \varepsilon_h^n \big(|\nabla_{\varepsilon_h} w_h|^p(i) + 1\big) < \eta.$$

As a consequence, thanks to the upper bound (H$_\mathrm{b}$7) we obtain

$$\begin{aligned}\frac{1}{\rho^n}G_{\varepsilon_h}(T_{\bar k\rho}u_{\varepsilon_h},B_\rho) &\geq \frac{1}{\rho^n}G_{\varepsilon_h}(w_h,B_\rho)-\frac{1}{\rho^n}\sum_{i\in\mathcal{U}^L_{\varepsilon_h}}\varepsilon_h^n\psi^b_{\frac{i}{\varepsilon_h}}(\{\tfrac{1}{\varepsilon_h}w_h^{i+\varepsilon_h j}\}_{j\in\mathbb{Z}^n})\\ &\geq \frac{1}{\rho^n}G_{\varepsilon_h}(w_h,B_\rho)-\eta \quad \text{for all } h\geq h_1. \end{aligned} \tag{4.41}$$

We finally estimate from below $G_{\varepsilon_h}(w_h,B_\rho)$ in terms of $F_{\varepsilon_h}(w_h,B_\rho)$. For every $\Lambda>0$ we set

$$\mathcal{S}_{\varepsilon_h}(\Lambda):=\{i\in Z_{\varepsilon_h}(B_{2\rho})\colon |\nabla_{\varepsilon_h,L}w_h|^p(i)\geq\Lambda\}.$$

For every $i\in\mathcal{S}_{\varepsilon_h}(\Lambda)$ Lemma 4.4 gives

$$\Lambda\leq|\nabla_{\varepsilon_h,L}w_h|^p(i)\leq\hat c_1\sum_{j\in Z_{\varepsilon_h}(Q_{\varepsilon_h L}(i))}|\nabla_{\varepsilon_h}w_h|^p(j)\leq\hat c_1\#Z_1(LQ)\max_{j\in Z_{\varepsilon_h}(Q_{\varepsilon_h L}(i))}|\nabla_{\varepsilon_h}w_h|^p(j).$$

In particular, for every $i\in\mathcal{S}_{\varepsilon_h}(\Lambda)\cap B_\rho$ there exists $j_i\in Z_{\varepsilon_h}(B_{2\rho})$ with $|\nabla_{\varepsilon_h}w_h|^p(j_i)\geq\Lambda/(\hat c_1\#Z_1(LQ))$. Setting $\hat c:=\hat c_1\#Z_1(LQ)$ this gives

$$\begin{aligned}\sum_{i\in\mathcal{S}_{\varepsilon_h}(\Lambda)\cap B_\rho}|\nabla_{\varepsilon_h,L}w_h|^p(i)&\leq\sum_{j\in\mathcal{S}_{\varepsilon_h}(\Lambda/\hat c)}|\nabla_{\varepsilon_h}w_h|^p(j)\#\{i\in\mathcal{S}_{\varepsilon_h}(\Lambda)\colon j\in Q_{\varepsilon_h L}(i)\}\\ &\leq\#Z_1(LQ)\sum_{j\in\mathcal{S}_{\varepsilon_h}(\Lambda/\hat c)}|\nabla_{\varepsilon_h}w_h|^p(j).\end{aligned}$$

Thus, for fixed $\eta>0$ the equiintegrability of $|\nabla_{\varepsilon_h}w_h|^p$ on $B_{2\rho}$ ensures the existence of $\bar\Lambda=\bar\Lambda(\eta,\rho)>0$ and $h_2=h_2(\eta,\rho)>0$ such that for every $h\geq h_2$ we have

$$\frac{c_5}{\rho^n}\sum_{i\in\mathcal{S}_{\varepsilon_h}(\bar\Lambda)\cap B_\rho}\varepsilon_h^n(|\nabla_{\varepsilon_h,L}w_h|^p(i)+1)\leq\frac{c_5}{\rho^n}\#Z_1(LQ)\sum_{j\in\mathcal{S}_{\varepsilon_h}(\bar\Lambda/\hat c)}\varepsilon_h^n(|\nabla_{\varepsilon_h}w_h|^p(i)+1)<\eta. \tag{4.42}$$

In addition, since $|\nabla_{\varepsilon_h,L}w_h|(i)<\bar\Lambda^{\frac1p}$ for all $i\in Z_{\varepsilon_h}(B_\rho)\setminus S_{\varepsilon_h}(\bar\Lambda)$, in view of (H$_\psi$1) there exists $h_3=h_3(\eta,\rho)>0$ such that for all $h\geq h_3$ and for all $i\in Z_{\varepsilon_h}(B_\rho)\setminus S_{\varepsilon_h}(\bar\Lambda)$ there holds

$$|\psi_i^{\varepsilon_h}(\{w_h^{i+j}\}_{j\in Z_{\varepsilon_h}(\mathbb{R}^n)})-\psi^b_{\frac{i}{\varepsilon_h}}(\{\tfrac{1}{\varepsilon_h}w_h^{i+\varepsilon_h j}\}_{j\in\mathbb{Z}^n})|<\frac{\eta}{|B_1|}. \tag{4.43}$$

Combining (4.42) and (4.43) in view of Remark 4.6 we deduce that for all $h\geq\max\{h_2,h_3\}$ we have

$$\begin{aligned}\frac{1}{\rho^n}G_{\varepsilon_h}(w_h,B_\rho)&\geq\frac{1}{\rho^n}\sum_{i\in Z_{\varepsilon_h}(B_\rho)\setminus\mathcal{S}_{\varepsilon_h}(\Lambda_{\eta,\rho})}\varepsilon_h^n(\psi_i^{\varepsilon_h}(\{w_h^{i+j}\}_{j\in Z_{\varepsilon_h}(\mathbb{R}^n)})-\eta)\\ &\geq\frac{1}{\rho^n}F_{\varepsilon_h}(w_h,B_\rho)-\eta-o(\varepsilon_h)-\frac{c_5}{\rho^n}\sum_{i\in\mathcal{S}_{\varepsilon_h}(\bar\Lambda)\cap B_\rho}\varepsilon_h^n(|\nabla_{\varepsilon_h,L}w_h|^p(i)+1)\\ &\geq\frac{1}{\rho^n}F_{\varepsilon_h}(w_h,B_\rho)-2\eta-o(\varepsilon_h).\end{aligned} \tag{4.44}$$

Eventually, gathering (4.40), (4.38), (4.41) and (4.44) we obtain

$$|B_1|f_{\rm hom}(M)\geq\frac{1}{\rho^n}\liminf_{h\to+\infty}F_{\varepsilon_h}(w_h,B_\rho)-3\eta\geq|B_1|\bar f(M)-3\eta,$$

hence we may conclude letting $\eta\to0$. □

4.1.2. *The surface energy density.* In this section we finally characterize the surface-energy density of the Γ-limit. We start proving some properties of the unscaled interaction-energy densities ψ_i^s. Since these properties can be obtained in a similar way as the corresponding properties of ψ_i^b in Lemma 4.5 we only sketch the proof.

Lemma 4.13. *Suppose that $\psi_i^\varepsilon : (\mathbb{R}^d)^{Z_\varepsilon(\mathbb{R}^n)} \to [0,+\infty)$ are εK-periodic in i, satisfy (H1)–(H6) with $Z_\varepsilon(\Omega_i)$ replaced by $Z_\varepsilon(\mathbb{R}^n)$, and suppose that in addition (4.1) is satisfied. Assume moreover that there exists $\psi_i^s : (\mathbb{R}^d)^{\mathbb{Z}^n} \to [0,+\infty)$ such that $(\mathrm{H}_\psi 2)$ holds true. Then the functions ψ_i^s are K-periodic in i and satisfy Hypotheses (H1)–(H2) with $Z_\varepsilon(\Omega_i)$ replaced by $\mathbb{Z}^n$. Moreover, the following holds for every $i \in \mathbb{Z}^n$.*

- $(\mathrm{H_s}3)$ (upper bound for constant functions) *For all $z : \mathbb{Z}^n \to \mathbb{R}^d$ with $z \equiv w$ for some $w \in \mathbb{R}^d$ we have $\psi_i^s(\{z^j\}_{j\in\mathbb{Z}^n}) = 0$;*
- $(\mathrm{H_s}4)$ (upper bound) *there exists $c_6 = c_6(n, L) > 0$ such that for all $z : \mathbb{Z}^n \to \mathbb{R}^d$ there holds*
$$\psi_i^s(\{z^j\}_{j\in\mathbb{Z}^n}) \le c_6(\|z\|_{L^\infty(LQ)} + 1);$$
- $(\mathrm{H_s}5)$ (locality) *for all $z, w : \mathbb{Z}^n \to \mathbb{R}^d$ with $z^j = w^j$ for all $j \in Z_1(LQ)$ we have*
$$\psi_i^s(\{z^j\}_{j\in\mathbb{Z}^n}) = \psi_i^s(\{w^j\}_{j\in\mathbb{Z}^n}).$$
In particular, $\psi_i^s(\{z^j\}_{j\in\mathbb{Z}^n}) = 0$ for all $z : \mathbb{Z}^n \to \mathbb{R}^d$ with $z \equiv w$ on $Z_1(LQ)$ for some $w \in \mathbb{R}^d$.

Proof. The periodicity of ψ_i^s, (H1)–(H2) and $(\mathrm{H_s}5)$ follow from the corresponding properties of ψ_i^ε as in the case of ψ_i^b. Thus, we only prove $(\mathrm{H_s}3)$ and $(\mathrm{H_s}4)$ here. To this end, fix $\eta > 0$ and suppose that $z : \mathbb{Z}^n \to \mathbb{R}^d$ is such $z \equiv w$ for some $w \in \mathbb{R}^d$. Then $|\nabla_{1,L} z|(0) = 0$ and according to $(\mathrm{H}_\psi 2)$ we find $\hat\varepsilon = \hat\varepsilon(\eta) > 0$ such that $\psi_i^s(\{z^j\}_{j\in\mathbb{Z}^n}) < \varepsilon\psi_{\varepsilon i}^\varepsilon(\{z^{\frac{j}{\varepsilon}}\}_{j\in Z_\varepsilon(\mathbb{R}^n)}) + \eta$ for every $\varepsilon \in (0, \hat\varepsilon)$ and every $i \in \mathbb{Z}^n$. Thus, (2.7) gives
$$\psi_i^s(\{z^j\}_{j\in\mathbb{Z}^n}) < \varepsilon(c_1 + 1) + \eta,$$
and we obtain $(\mathrm{H_s}3)$ by letting first $\varepsilon \to 0$ and then $\eta \to 0$.

We continue proving $(\mathrm{H_s}4)$. Let ε_0 and $\bar c_5$ be as in (4.15) and let $z : \mathbb{Z}^n \to \mathbb{R}^d$. Note that either $|\nabla_{1,L} z|(0) = 0$ or we can find $\varepsilon(z) \in (0, \varepsilon_0)$ such that $\varepsilon^{\frac{1-p}{p}} |\nabla_{1,L} z|(0) \ge \Lambda(1)$ for any $\varepsilon \in (0, \varepsilon(z))$. Thanks to $(\mathrm{H}_\psi 2)$ there exists $\hat\varepsilon \in (0, \varepsilon(z))$ such that $\psi_i^s(\{z^j\}_{j\in\mathbb{Z}^n}) < \varepsilon\psi_{\varepsilon i}^\varepsilon(\{z^{\frac{j}{\varepsilon}}\}_{j\in Z_\varepsilon(\mathbb{R}^n)}) + 1$ for every $\varepsilon \in (0, \hat\varepsilon)$ and every $i \in \mathbb{Z}^n$. Arguing as in the proof of Lemma 4.5 to obtain $(\mathrm{H_b}7)$ we deduce
$$\begin{aligned}\psi_i^s(\{z^j\}_{j\in\mathbb{Z}^n}) &< \varepsilon(c_1 + 1) + 1 + \varepsilon\bar c_5 \sum_{j\in Z_1(LQ)} \sum_{\substack{\xi\in Z_1(LQ)\\ j+\xi\in LQ}} \frac{1 + |z^{\frac{j}{\varepsilon}+\xi} - z^0|}{\varepsilon}\\ &\le \varepsilon(c_1 + 1) + 1 + \bar c_5(1 + 2\|z\|_{L^\infty(LQ;\mathbb{R}^d)})(\#Z_1(LQ))^2,\end{aligned}$$
hence $(\mathrm{H_s}4)$ follows by setting $c_6 := 2\max\{\bar c_5(\#Z_1(LQ))^2, 1\}$ and letting $\varepsilon \to 0$. □

Remark 4.14. Thanks to $(\mathrm{H_s}3)$ and $(\mathrm{H_s}5)$ the continuity assumption $(\mathrm{H}_\psi 3)$ reads as follows. For every $z, w : \mathbb{Z}^n \to \mathbb{R}^d$ with $|\nabla_{1,L} z|(0) > 0$ there holds
$$\psi_i^s(\{z^j\}_{j\in\mathbb{Z}^n}) \ge \psi_i^s(\{w^j\}_{j\in\mathbb{Z}^n}) - c_s \sum_{j\in Z_1(Q_L(i))} \sum_{\substack{\xi\in Z_1(Q_L(i))\\ j+\xi\in Q_L(i)}} |z^{j+\xi} - w^{j+\xi}| \text{ for every } i \in \mathbb{Z}^n.$$

On account of Lemma 4.13 we now prove the following proposition.

Proposition 4.15. *Let F_ε be given by (2.3) with $\phi_i^\varepsilon : (\mathbb{R}^d)^{Z_\varepsilon(\Omega_i)} \to [0,+\infty)$ as in (4.3) and assume that the functions $\psi_i^\varepsilon : (\mathbb{R}^d)^{Z_\varepsilon(\mathbb{R}^n)} \to [0,+\infty)$ are εK-periodic in i, satisfy (H1)–(H6) with $Z_\varepsilon(\Omega_i)$ replaced by $Z_\varepsilon(\mathbb{R}^n)$, and the locality condition (4.1). Suppose in addition that there exist ψ_i^s : $(\mathbb{R}^d)^{Z^n} \to [0,+\infty)$ such that (H$_\psi$2) and (H$_\psi$3) are satisfied. Then for each pair $(\zeta,\nu) \in \mathbb{R}^d \times S^{n-1}$ there exists the limit defining $g_{\rm hom}$ in (4.9) and $g_{\rm hom}(\zeta,\nu) = \bar g(\zeta,\nu)$, where $\bar g$ is as in (4.4).*

Proof. Having at hand Lemma 4.13 the existence of the limit in (4.9) can proved as in [13, Proposition 4.5] and we thus omit its prove here.

Let $(\zeta,\nu) \in \mathbb{R}^d \times S^{n-1}$ be fixed and let us show that $\bar g(\zeta,\nu) = g_{\rm hom}(\zeta,\nu)$. To reduce notation for every $T > 0$ we set

$$g_T(\zeta,\nu) := \inf\Big\{ \sum_{i\in Z_1(TQ^\nu)} \psi_i^s(\{u^{i+j}\}_{j\in\mathbb{Z}^n})\colon u \in \mathcal{A}_1^{\sqrt{n}L}(u_{\zeta,\nu}, TQ^\nu)\Big\},$$

so that $g_{\rm hom}(\zeta,\nu) = \lim_T 1/T^{n-1} g_T(\zeta,\nu)$.

Step 1: $\bar g(\zeta,\nu) \geq g_{\rm hom}(\zeta,\nu)$

Let $\bar g$ be as in (4.4); thanks to formula 3.2 in Theorem 3.1 and Proposition 4.1 there exists $x_0 \in \Omega$ such that

$$\bar g(\zeta,\nu) = \limsup_{\rho\to 0} \lim_{\delta\to 0} \limsup_{\varepsilon\to 0} \frac{1}{\rho^{n-1}} \inf\{F_\varepsilon(u, Q_\rho^\nu(x_0))\colon u \in \mathcal{A}_\varepsilon^\delta(u_{\zeta,x_0}^\nu, Q_\rho^\nu(x_0))\}.$$

Note that to simplify notation we do not relabel the Γ-converging subsequence. Moreover, from now on we assume $x_0 = 0$. We fix a number $\alpha \in (0,(p-1)/p)$ whose meaning will become clear later and for every $\rho > 0$ we denote by $N_\rho := \lfloor \rho^{-\alpha} \rfloor$ the integer part of $\rho^{-\alpha}$. We further write $\zeta = (\zeta^1,\ldots,\zeta^d)$ and we choose $\rho \in (0,1)$ with $Q_{2\rho} \subset\subset \Omega$ such that $2/N_\rho < |\zeta^m|$ for every $m \in \{1,\ldots,d\}$ with $\zeta^m \neq 0$. Let $\delta \in (0,\rho/2)$ and for every $\varepsilon > 0$ with $\varepsilon\sqrt{n}L < \delta$ let $u_\varepsilon \in \mathcal{A}_\varepsilon^\delta(u_\zeta^\nu, Q_\rho^\nu)$ be such that

$$F_\varepsilon(u_\varepsilon, Q_\rho^\nu) \leq F_\varepsilon(u_\zeta^\nu, Q_\rho^\nu) \leq c\rho^{n-1}. \tag{4.45}$$

Since $\varepsilon\sqrt{n}L < \delta < \rho/2$ and $Q_{2\rho} \subset\subset \Omega$ we can extend u_ε by 0 outside Ω without modifying the energy or changing the boundary conditions. Moreover, by truncation we can assume that $\|u_\varepsilon\|_{L^\infty} \leq 3|\zeta|$.

Let us fix $\eta > 0$; in the remaining part of this step we construct functions $w_\varepsilon : \mathbb{Z}^n \to \mathbb{R}^d$ which are admissible for the minimum problem defining $g_{T_\varepsilon}(\zeta,\nu)$ with $T_\varepsilon = \rho/\varepsilon$ and satisfying for ε sufficiently small (depending on η) the estimate

$$\frac{1}{\rho^{n-1}} F_\varepsilon(u_\varepsilon, Q_\rho^\nu) \geq \frac{1}{T_\varepsilon^{n-1}} \sum_{i\in Z_1(T_\varepsilon Q^\nu)} \psi_i^s(\{w_\varepsilon^{i+j}\}_{j\in\mathbb{Z}^n}) - R(\varepsilon,\rho) - c\eta, \tag{4.46}$$

where the reminder $R(\varepsilon,\rho)$ is such that $\lim_\rho \lim_\varepsilon R(\varepsilon,\rho) = 0$ and the constant c depends only on n, L and ζ. Passing to the limit first in ε then in δ and finally in ρ, thanks to the arbitrariness of $u_\varepsilon \in \mathcal{A}_\varepsilon^\delta(u_\zeta^\nu, Q_\rho^\nu)$ we may then deduce that

$$\bar g(\zeta,\nu) \geq \liminf_{\varepsilon\to 0} \frac{1}{T_\varepsilon^{n-1}} g_{T_\varepsilon}(\zeta,\nu) - c\eta = g_{\rm hom}(\zeta,\nu) - c\eta, \tag{4.47}$$

which will eventually give the desired inequality by letting $\eta \to 0$.

To obtain the required sequence (w_ε) we carefully combine the arguments used in [32, Proposition 5.21] in the discrete setting with those used in [18, Proposition 6.2] and [22, Theorem 5.2(d)] in the continuum setting. We start introducing some notation. For every $m \in \{1,\dots,d\}$ we denote by $(u^i_\varepsilon)^m$ the m-th component of u_ε and for every $t \in \mathbb{R}$ we consider the superlevel set

$$\mathcal{S}^m_\varepsilon(t) := \{i \in Z_\varepsilon(Q^\nu_\rho)\colon (u^i_\varepsilon)^m \geq t\}.$$

Further we introduce the set

$$\mathcal{R}^m_\varepsilon(t) := \{i \in Z_\varepsilon(Q^\nu_\rho)\colon \exists\, \xi \in Z_1(LQ) \text{ s.t. } i+\varepsilon\xi \in Z_\varepsilon(\mathbb{R}^n)\setminus \mathcal{S}^m_\varepsilon(t),\ i \in \mathcal{S}^m_\varepsilon(t) \text{ or vice versa}\}.$$

Let finally $N \in \mathbb{N}$ with $3|\zeta| + 1/N_\rho \leq N$; notice that for any $t \in [-N,N]$ and any $m \in \{1,\dots,d\}$ a point $i \in Z_\varepsilon(Q^\nu_\rho)$ belongs to $\mathcal{R}^m_\varepsilon(t)$ if and only if $t \in [(u^i_\varepsilon)^m, (u^{i+\varepsilon\xi}_\varepsilon)^m)$ or $t \in ((u^{i+\varepsilon\xi}_\varepsilon)^m, (u^i_\varepsilon)^m]$ for some $\xi \in Z_1(LQ)$. Thus, for any $i \in Z_\varepsilon(Q^\nu_\rho)$ we have

$$\int_{-N}^{N} \chi_{\mathcal{R}^m_\varepsilon(t)}(i)\,dt \leq \varepsilon|\nabla_{\varepsilon,L}u_\varepsilon|(i). \tag{4.48}$$

We choose $\Lambda(\eta)$ according to (H$_\psi$2) and denote by

$$\mathcal{J}_\varepsilon := \big\{i \in Z_\varepsilon(Q^\nu_\rho)\colon |\nabla_{\varepsilon,L}u_\varepsilon|^p(i) \geq \frac{\Lambda(\eta)^p}{\varepsilon}\big\}$$

the set of jump points. Without restriction we assume that $\Lambda(\eta) \geq 1$. Summing up (4.48) over all $i \in Z_\varepsilon(Q^\nu_\rho)\setminus\mathcal{J}_\varepsilon$ from Hölder's inequality we deduce that

$$\begin{aligned}
\varepsilon^{n-1}\int_{-N}^{N}\#(\mathcal{R}^m_\varepsilon(t)\setminus\mathcal{J}_\varepsilon)\,dt &\leq \sum_{i\in Z_\varepsilon(Q^\nu_\rho)\setminus\mathcal{J}_\varepsilon}\varepsilon^n|\nabla_{\varepsilon,L}u_\varepsilon|(i)\\
&\leq \varepsilon^{\frac{n(p-1)}{p}}\Big(\#\big(Z_\varepsilon(Q^\nu_\rho)\setminus\mathcal{J}_\varepsilon\big)\Big)^{\frac{p-1}{p}}\Big(\sum_{i\in Z_\varepsilon(Q^\nu_\rho)\setminus\mathcal{J}_\varepsilon}\varepsilon^n|\nabla_{\varepsilon,L}u_\varepsilon|^p(i)\Big)^{\frac{1}{p}}\\
&\leq c\Lambda(\eta)\rho^{\frac{n(p-1)}{p}}\Big(\sum_{i\in Z_\varepsilon(Q^\nu_\rho)}\varepsilon^n\min\big\{|\nabla_{\varepsilon,L}u_\varepsilon|^p(i),\frac{1}{\varepsilon}\big\}\Big)^{\frac{1}{p}}.
\end{aligned}\tag{4.49}$$

Moreover, thanks to Estimate 4.11 in Lemma 4.4 and (H4) we have

$$\begin{aligned}
\sum_{i\in Z_\varepsilon(Q^\nu_\rho)}\varepsilon^n\min\big\{|\nabla_{\varepsilon,L}u_\varepsilon|^p(i),\frac{1}{\varepsilon}\big\} &\leq 2\hat{c}_2\sum_{i\in Z_\varepsilon(Q^\nu_\rho+\varepsilon L[-1,1]^n)}\varepsilon^n\min\Big\{\sum_{k=1}^n|D^k_\varepsilon u^i|^p,\frac{1}{\varepsilon}\Big\}\\
&\leq 2\hat{c}_2\bigg(\frac{1}{c_2}F_\varepsilon(u_\varepsilon,Q^\nu_\rho) + \sum_{i\in Z_\varepsilon(Q^\nu_\rho+\varepsilon L[-1,1]^n)\setminus Q^\nu_\rho}\varepsilon^n\min\Big\{\sum_{k=1}^n|D^k_\varepsilon u^\nu_\zeta(i)|^p,\frac{1}{\varepsilon}\Big\}\bigg),
\end{aligned}\tag{4.50}$$

where in the second step we used the boundary conditions satisfied by u_ε. Notice that the last term on the right-hand side of (4.50) can be bounded by

$$\varepsilon^{n-1}\#\{i \in Z_\varepsilon(Q^\nu_\rho+\varepsilon L[-1,1])\setminus Q^\nu_\rho\colon\ \mathrm{dist}(i,\Pi_\nu)\leq\varepsilon\} \leq c(L)\varepsilon.$$

Inserting the above estimate and the energy bound (4.45) in (4.50), the estimate in (4.49) can be continued to

$$\varepsilon^{n-1}\int_{-N}^{N}\#(\mathcal{R}^m_\varepsilon(t)\setminus\mathcal{J}_\varepsilon)\,dt \leq c\Lambda(\eta)\rho^{\frac{n(p-1)}{p}}\big(\rho^{n-1}+\varepsilon\big)^{\frac{1}{p}} \leq c\Lambda(\eta)\big(\rho^{\frac{np-1}{p}} + \rho^{\frac{n(p-1)}{p}}\varepsilon^{\frac{1}{p}}\big).$$

Hence for every integer l with $-NN_\rho \leq l \leq NN_\rho$ there exists $t_l^m \in [l/N_\rho, (l+1)/N_\rho)$ such that

$$\varepsilon^{n-1}\sum_{l=-NN_\rho}^{NN_\rho-1}\#(\mathcal{R}_\varepsilon^m(t_l^m)\setminus\mathcal{J}_\varepsilon) \leq \varepsilon^{n-1}N_\rho\int_{-N}^{N}\#(\mathcal{R}_\varepsilon^m(t)\setminus\mathcal{J}_\varepsilon)\,dt \leq c\Lambda(\eta)\big(\rho^{\frac{np-1}{p}-\alpha}+\varepsilon^{\frac{1}{p}}\rho^{\frac{n(p-1)}{p}-\alpha}\big). \tag{4.51}$$

Note that α was chosen such that $(np-1)/p-\alpha > n-1$. Moreover, since $\|u_\varepsilon\|_{L^\infty} \leq N - 1/N_\rho$ the sets $\mathcal{S}_\varepsilon^m(t_l^m)\setminus\mathcal{S}_\varepsilon^m(t_{l+1}^m)$, $m\in\{1,\ldots,d\}$, $l = -NN_\rho,\ldots,NN_\rho-1$ form a partition of $Z_\varepsilon(Q_\rho^n)$. Thus, we can define a discrete function v_ε componentwise by its restriction to $\mathcal{S}_\varepsilon^m(t_l^m)\setminus\mathcal{S}_\varepsilon^m(t_{l+1}^m)$ setting

$$(v_\varepsilon^i)^m_{|\mathcal{S}_\varepsilon^m(t_l^m)\setminus\mathcal{S}_\varepsilon^m(t_{l+1}^m)} := \begin{cases}0 & \text{if } t_l^m \leq 0 < t_{l+1}^m,\\ \zeta^m & \text{if } t_l^m \leq \zeta^m < t_{l+1}^m,\\ t_l^m & \text{otherwise.}\end{cases}$$

Notice that v_ε is well-defined since $2/N_\rho < |\zeta^m|$ if $\zeta^m \neq 0$, so that in this case ζ^m and 0 can not belong to the same interval $[t_l^m, t_{l+1}^m)$.

We claim that the required sequence (w_ε) is obtained by setting $w_\varepsilon^i := v_\varepsilon^{\varepsilon i}$ for every $i\in\mathbb{Z}^n$. First notice that by construction the functions v_ε satisfy the required boundary condtions, *i.e.*, $v_\varepsilon \in \mathcal{A}_\varepsilon^\delta(u_\zeta^\nu, Q_\rho^\nu)$. Thus, since $\varepsilon L < \delta$ the rescaled functions w_ε are admissible for the minimum problem defining $g_{T_\varepsilon}(\zeta,\nu)$. We finally show that there exists $\hat\varepsilon = \hat\varepsilon(\eta) > 0$ such that for all $\varepsilon\in(0,\hat\varepsilon)$ the functions w_ε satisfy (4.46). To this end we show that $\psi_i^s(\{w_\varepsilon^{i+j}\}_{j\in\mathbb{Z}^n})$ essentially only gives a contribution to the energy when $\varepsilon i \in \mathcal{J}_\varepsilon$, in which case it will turn out to be comparable to $\varepsilon\psi_i^\varepsilon(\{u_\varepsilon^{\varepsilon i+j}\}_{j\in Z_\varepsilon(\mathbb{R}^n)})$ thanks to ($\mathrm{H}_\psi 2$) and ($\mathrm{H}_\psi 3$). We start by introducing the rescaled functions $\tilde u_\varepsilon$ defined by setting $\tilde u_\varepsilon^i := u_\varepsilon^{\varepsilon i}$ for every $i\in\mathbb{Z}^n$ and we observe that for $i\in Z_1(T_\varepsilon Q^\nu)$ with $\varepsilon i\in\mathcal{J}_\varepsilon$ we have

$$\varepsilon^{\frac{1-p}{p}}|\nabla_{1,L}\tilde u_\varepsilon|(i) = \varepsilon^{\frac{1}{p}}|\nabla_{\varepsilon,L}u_\varepsilon|(\varepsilon i) \geq \Lambda(\eta). \tag{4.52}$$

Hence, from ($\mathrm{H}_\psi 2$) we deduce the existence of $\hat\varepsilon = \hat\varepsilon(\eta) > 0$ such that for every $\varepsilon\in(0,\hat\varepsilon)$ and every $i\in\mathbb{Z}^n$ with $\varepsilon i\in\mathcal{J}_\varepsilon$ there holds

$$\varepsilon\phi_i^\varepsilon(\{u_\varepsilon^{\varepsilon i+j}\}_{j\in Z_\varepsilon(\Omega_i)}) = \varepsilon\psi_{\varepsilon i}^\varepsilon(\{u_\varepsilon^{\varepsilon i+j}\}_{j\in Z_\varepsilon(\mathbb{R}^n)}) \geq \psi_i^s(\{\tilde u_\varepsilon^{i+j}\}_{j\in\mathbb{Z}^n}) - \eta. \tag{4.53}$$

We now compare $\psi_i^s(\{\tilde u_\varepsilon^{i+j}\}_{j\in\mathbb{Z}^n})$ and $\psi_i^s(\{w_\varepsilon^{i+j}\}_{j\in\mathbb{Z}^n})$. By construction we have

$$\|w_\varepsilon - \tilde u_\varepsilon\|_{L^\infty} = \|v_\varepsilon - u_\varepsilon\|_{L^\infty} \leq \frac{2\sqrt{d}}{N_\rho} \leq 4\sqrt{d}\rho^\alpha. \tag{4.54}$$

For every $i\in\mathbb{Z}^n$ with $|\nabla_{1,L}\tilde u_e|(i) > 0$ (4.54) together with ($\mathrm{H}_\psi 3$) and Remark 4.14 gives

$$\psi_i^s(\{\tilde u_\varepsilon^{i+j}\}_{j\in\mathbb{Z}^n}) \geq \psi_i^s(\{w_\varepsilon^{i+j}\}_{j\in\mathbb{Z}^n}) - c_s\sum_{j\in Z_1(Q_L(i))}\sum_{\substack{\xi\in Z_1(Q_L(i)\\ j+\xi\in Q_L(i)}}|w_\varepsilon^{j+\xi}-\tilde u_\varepsilon^{j+\xi}| \geq \psi_i^s(\{w_\varepsilon^{i+j}\}_{j\in\mathbb{Z}^n}) - c\rho^\alpha, \tag{4.55}$$

where $c>0$ depends only on n, d and L. In particular, (4.55) holds for every $i\in\mathbb{Z}^n$ with $\varepsilon\in\mathcal{J}_\varepsilon$ thanks to (4.52). Gathering (4.55) and (4.53) we thus obtain

$$\frac{1}{\rho^{n-1}}F_\varepsilon(u_\varepsilon, Q_\rho^\nu) \geq \frac{\varepsilon^{n-1}}{\rho^{n-1}}\sum_{i\in Z_\varepsilon(Q_\rho^\nu)\cap\mathcal{J}_\varepsilon}\varepsilon\phi_i^\varepsilon(\{u_\varepsilon^{i+j}\}_{j\in Z_\varepsilon(\Omega_i)})$$

$$\geq \frac{1}{T_\varepsilon^{n-1}} \sum_{\substack{i\in Z_1(T_\varepsilon Q^\nu)\\ \varepsilon i\in \mathcal{J}_\varepsilon}} \psi_i^s(\{w_\varepsilon^{i+j}\}_{j\in\mathbb{Z}^n}) - \big(\rho^\alpha+\eta\big)\frac{\varepsilon^{n-1}}{\rho^{n-1}}\#(Z_\varepsilon(Q_\rho^\nu)\cap\mathcal{J}_\varepsilon). \tag{4.56}$$

Moreover, since $1/\varepsilon \leq |\nabla_{\varepsilon,L}u_\varepsilon|^p(i)$ for every $i\in\mathcal{J}_\varepsilon$, we can argue as in (4.50) to bound the cardinality of the set $Z_\varepsilon(Q_\rho^\nu)\cap\mathcal{J}_\varepsilon$ via

$$\frac{\varepsilon^{n-1}}{\rho^{n-1}}\#(Z_\varepsilon(Q_\rho^\nu)\cap\mathcal{J}_\varepsilon) \leq \frac{1}{\rho^{n-1}}\sum_{i\in Z_\varepsilon(Q_\rho^\nu)\cap\mathcal{J}_\varepsilon}\varepsilon^n\min\big\{|\nabla_{\varepsilon,L}u_\varepsilon|^p(i),\frac{1}{\varepsilon}\big\} \leq \frac{c}{\rho^{n-1}}\big(F_\varepsilon(u_\varepsilon,Q_\rho^\nu)+\varepsilon\big)\leq c+\frac{c\varepsilon}{\rho^{n-1}}, \tag{4.57}$$

where the last inequality follows from (4.45). It then remains to show that the contributions of $\psi_i^s(\{w_\varepsilon^{i+j}\}_{j\in\mathbb{Z}^n})$ for $\varepsilon i\notin\mathcal{J}_\varepsilon$ are negligible. First notice that for every $i\in Z_1(T_\varepsilon Q^\nu)$ with $w_\varepsilon\equiv w_\varepsilon^i$ on $Z_1(Q_L(i))$ Hypotheses $(\mathrm{H_s}5)$ gives $\psi_i^s(\{w_\varepsilon^{i+j}\}_{j\in\mathbb{Z}^n})=0$. On the other hand, if $i\in Z_1(T_\varepsilon Q^\nu)$ is such that $w_\varepsilon\not\equiv w_\varepsilon^i$ on $Z_1(Q_L(i))$ then i belongs to $\mathcal{R}_\varepsilon^m(t_l^m)$ for some $m\in\{1,\dots,d\}$ and $l\in\{-NN_\rho,\dots,NN_\rho-1\}$. Thus, we have

$$\frac{1}{T_\varepsilon^{n-1}}\sum_{\substack{i\in Z_1(T_\varepsilon Q^\nu)\\ \varepsilon i\notin\mathcal{J}_\varepsilon}}\psi_i^s(\{w_\varepsilon^{i+j}\}_{j\in\mathbb{Z}^n}) \leq \frac{1}{T_\varepsilon^{n-1}}\sum_{m=1}^d\sum_{l=-NN_\rho}^{NN_\rho-1}\sum_{\varepsilon i\in\mathcal{R}_\varepsilon^m(t_l^m)\setminus\mathcal{J}_\varepsilon}\psi_i^s(\{w_\varepsilon^{i+j}\}_{j\in\mathbb{Z}^n}). \tag{4.58}$$

We finally observe that (4.54) and our choice of ρ imply that $\|w_\varepsilon\|_{L^\infty}\leq 4|\zeta|$, so that we can use the upper bound in $(\mathrm{H_s}4)$ together with (4.51) to bound the sum on the right-hand side of (4.58). In fact, we have

$$\begin{aligned}\frac{1}{T_\varepsilon^{n-1}}\sum_{m=1}^d\sum_{l=-NN_\rho}^{NN_\rho-1}\sum_{\varepsilon i\in\mathbb{R}_\varepsilon^m(t_l^m)\setminus\mathcal{J}_\varepsilon}\psi_i^s(\{w_\varepsilon^{i+j}\}_{j\in\mathbb{Z}^n}) &\leq c_6(4|\zeta|+1)\frac{\varepsilon^{n-1}}{\rho^{n-1}}\sum_{m=1}^d\sum_{l=-NN_\rho}^{NN_\rho-1}\#(\mathcal{R}_\varepsilon^m(t_l^m)\setminus\mathcal{J}_\varepsilon)\\ &\leq c\Lambda(\eta)\big(\rho^{\frac{p-1}{p}-\alpha}+\varepsilon^{\frac{1}{p}}\rho^{\frac{p-n}{p}-\alpha}\big).\end{aligned} \tag{4.59}$$

Gathering (4.56)-(4.59) we deduce that the sequence (w_ε) satisfies (4.46) with

$$R(\varepsilon,\rho)=c\Lambda(\eta)\big(\rho^\alpha+\varepsilon\rho^{1-n}+\rho^{\frac{p-1}{p}-\alpha}+\varepsilon^{\frac{1}{p}}\rho^{\frac{p-n}{p}-\alpha}\big)\to 0\quad\text{as first }\varepsilon\to 0\text{ and then }\rho\to 0,$$

where the convergence of $R(\varepsilon,\rho)$ is guaranteed by the choice of $\alpha\in(0,(p-1)/p)$. Thus the argument in (4.47) concludes this step providing us with the inequality $\bar g\geq g_{\mathrm{hom}}$.

Step 2: $\bar g(\zeta,\nu)\leq g_{\mathrm{hom}}(\zeta,\nu)$

In order to prove the opposite inequality we construct a recovery sequence for u_{ζ,x_0}^ν on $Q_\rho^\nu(x_0)$, where $x_0\in\Omega$ and $\rho>0$ are such that $Q_\rho^\nu(x_0)\subset\subset\Omega$. To simplify the exposition we only consider the case $\nu=e_n$ here and we assume that $x_0=0$ and $\rho=1$. We fix $\eta>0$ and set

$$Q(\eta):=\big(-1/2,1/2\big)^{n-1}\times\big(-\eta/2,\eta/2\big).$$

Moreover, we choose $T=T(\eta)\in\mathbb{N}$ as a multiple of K with $1/T<\eta$ and $u_T\in\mathcal{A}_1^{\sqrt{n}L}(u_\zeta^{e_n},TQ)$ satisfying

$$\frac{1}{T^{n-1}}\sum_{i\in Z_1(TQ)}\psi_i^s(\{u_T^{i+j}\}_{j\in\mathbb{Z}^n})\leq g_{\mathrm{hom}}(\zeta,e_n)+\eta. \tag{4.60}$$

Starting from u_T we now construct a sequence (u_ε) converging in $L^1(\Omega;\mathbb{R}^d)$ to $u_\zeta^{e_n}$ and satisfying

$$\limsup_{\varepsilon\to 0}F_\varepsilon(u_\varepsilon,Q(\eta))\leq g_{\mathrm{hom}}(\zeta,e_n)+c\eta, \tag{4.61}$$

where the constant $c>0$ depends only on L, n, ζ. Then Proposition 4.1 gives

$$\bar{g}(\zeta, e_n) = F(u_\zeta^{e_n}, Q(\eta)) \leq \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, Q(\eta)) \leq g_{\rm hom}(\zeta, e_n) + c\eta,$$

and we obtain the required inequality thanks to the arbitrariness of $\eta > 0$.

As a first step we define a function $\bar{u}_T : \mathbb{Z}^n \to \mathbb{R}^d$ which is T-periodic in the directions $(e_1, \dots, e_{n-1})$ inside the stripe $\{|\langle x, e_n\rangle| < T/2\}$ by setting

$$\bar{u}_T := \begin{cases} u_T^{i-Tj'} & \text{if } i \in Z_1(Tj' + TQ) \text{ for some } j' \in \mathbb{Z}^{n-1}\times\{0\},\\ u_\zeta^{e_n}(i) & \text{otherwise in } \mathbb{Z}^n. \end{cases}$$

For every $\varepsilon > 0$ and every $i \in Z_\varepsilon(\mathbb{R}^n)$ we then set $u_\varepsilon^i := u_T^{i/\varepsilon}$ and we observe that as $\varepsilon \to 0$ the sequence (u_ε) converges in $L^1(\Omega;\mathbb{R}^d)$ to $u_\zeta^{e_n}$. It remains to show that (u_ε) satisfies (4.61). To this end, for every $\varepsilon > 0$ we consider the stripe

$$S_\varepsilon(T) := \{x \in \mathbb{R}^n \colon |\langle x, e_n\rangle| < \varepsilon T/2\}.$$

For $\varepsilon < \eta/T$ we can rewrite the energy as

$$F_\varepsilon(u_\varepsilon, Q(\eta)) = \sum_{i\in Z_1(1/\varepsilon Q)\cap S_1(T)} \varepsilon^n \psi^\varepsilon_{\varepsilon i}(\{\bar{u}_T^{i+\frac{j}{\varepsilon}}\}_{j\in Z_\varepsilon(\mathbb{R}^n)}) + \sum_{i\in Z_\varepsilon(Q(\eta))\setminus S_\varepsilon(T)} \varepsilon^n \psi_i^\varepsilon(\{u_\zeta^{e_n}(i+j)\}_{j\in Z_\varepsilon(\mathbb{R}^n)}). \tag{4.62}$$

Thanks to the upper bound for constant functions (2.7) the second term on the right hand side of (4.62) is at most proportional to η. In fact we have

$$\sum_{i\in Z_\varepsilon(Q(\eta))\setminus S_\varepsilon(T)} \varepsilon^n \psi_i^\varepsilon(\{u_\zeta^{e_n}(i+j)\}_{j\in Z_\varepsilon(\mathbb{R}^n)}) \leq (c_1+1)\varepsilon^n \#\{i \in Z_\varepsilon(Q(\eta))\} \leq c\eta \tag{4.63}$$

with c depending only on n. We continue estimating the first term on the right-hand side of (4.62). Since T is fixed, the function $\bar{u}_T$ takes only finitely many values. Thus, there exists $\varepsilon_0 = \varepsilon_0(T,\eta) > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ and every $i \in \mathbb{Z}^n$ we either have $\varepsilon^{\frac{1-p}{p}}|\nabla_{1,L}\bar{u}_T|(i) \geq \Lambda(\eta/T)$ or $|\nabla_{1,L}\bar{u}_T|(i) = 0$, where $\Lambda(\eta/T)$ is given by ($\mathrm{H}_\psi 2$). As a consequence, setting $\varepsilon_1 := \min\{\varepsilon_0, \hat{\varepsilon}(\eta/T)\}$ with $\hat{\varepsilon}(\eta/T)$ again given by ($\mathrm{H}_\psi 2$), for every $\varepsilon \in (0, \varepsilon_1)$ and every $i \in \mathbb{Z}^n$ we obtain

$$|\psi_i^s(\{\bar{u}_T^{i+j}\}_{j\in\mathbb{Z}^n} - \varepsilon\psi^\varepsilon_{\varepsilon i}(\{\bar{u}_T^{i+\frac{j}{\varepsilon}}\}_{j\in Z_\varepsilon(\mathbb{R}^n)})| < \frac{\eta}{T}.$$

Combining the above estimate with (4.62) and (4.63) we deduce that for every $\varepsilon \in (0, \varepsilon_1)$ there holds

$$F_\varepsilon(u_\varepsilon, Q(\eta)) \leq \sum_{i\in Z_1(1/\varepsilon Q)\cap S_1(T)} \varepsilon^{n-1}\psi_i^s(\{\bar{u}_T^{i+j}\}_{j\in\mathbb{Z}^n}) + \frac{\eta}{T}\varepsilon^{n-1}\#(Z_1(\frac{1}{\varepsilon}Q)\cap S_1(T)) + c\eta. \tag{4.64}$$

Notice that there exists a constant $c > 0$ depending only on n such that

$$\varepsilon^{n-1}\#(Z_1(\frac{1}{\varepsilon}Q)\cap S_1(T)) \leq cT, \quad \text{for every } \varepsilon > 0.$$

Thus, setting

$$\mathcal{Z}_\varepsilon(T) := \{j' \in \mathbb{Z}^{n-1}\times\{0\} \colon \varepsilon Tj' + \varepsilon TQ \cap Q \neq \emptyset\}$$

the estimate in (4.64) can be continued to

$$F_\varepsilon(u_\varepsilon, Q(\eta)) \leq \varepsilon^{n-1} \sum_{j'\in\mathcal{Z}_\varepsilon(T)} \sum_{i\in Z_1(Tj'+T\overline{Q})} \psi_i^s(\{\bar{u}_T^{i+j}\}_{j\in\mathbb{Z}^n}) + c\eta. \tag{4.65}$$

Notice that for every $j'\in\mathcal{Z}_\varepsilon(T)$ and for every $i\in Z_1(Tj'+T\overline{Q})$ we have

$$\bar u_T^{i+j}=u_T^{i-Tj'+j}\quad\text{for every } j\in Z_1(LQ). \tag{4.66}$$

In fact, the above equality holds true by definition of $\bar u_T$ if $i\in Z_1(Tj'+TQ)$ is such that $Q_L(i)\subset Tj'+TQ$. If instead $i\in Z_1(Tj'+TQ)$ is such that $Q_L(i)\cap(\mathbb{R}^n\setminus Tj'+TQ)\neq\emptyset$, then the boundary conditions satisfied by u_T together with the fact that $\langle j',e_n\rangle=0$ ensure that

$$\bar u_T^{i+j}=u_\zeta^{e_n}(i+j)=u_T^{i-Tj'+j}.$$

Moreover, in combination with the locality property and periodicity, (4.66) gives

$$\sum_{i\in Z_1(Tj'+T\overline{Q})}\psi_i^s(\{\bar u_T^{i+j}\}_{j\in\mathbb{Z}^n})=\sum_{i\in Z_1(Tj'+T\overline{Q})}\psi_i^s(\{u_T^{i-Tj'+j}\}_{j\in\mathbb{Z}^n})=\sum_{i\in Z_1(T\overline{Q})}\psi_i^s(\{u_T^{i+j}\}_{j\in\mathbb{Z}^n}).$$

Thus, since $\#\mathcal{Z}_\varepsilon(T)\leq(\lfloor\frac{1}{\varepsilon T}\rfloor+1)^{n-1}$, from (4.65) we deduce that

$$\begin{aligned}F_\varepsilon(u_\varepsilon,Q(\eta))&\leq(\varepsilon T)^{n-1}\Big(\Big\lfloor\frac{1}{\varepsilon T}\Big\rfloor+1\Big)^{n-1}\frac{1}{T^{n-1}}\sum_{i\in Z_1(T\overline{Q})}\psi_i^s(\{u_T^{i+j}\}_{j\in\mathbb{Z}^n})+c\eta\\&\leq(\varepsilon T)^{n-1}\Big(\Big\lfloor\frac{1}{\varepsilon T}\Big\rfloor+1\Big)^{n-1}\Big(g_{\mathrm{hom}}(\zeta,e_n)+\eta+\frac{1}{T^{n-1}}\sum_{i\in Z_1(\partial TQ)}\psi_i^s(\{u_\zeta^{e_n}(i+j)\}_{j\in\mathbb{Z}^n})\Big)+c\eta,\end{aligned}$$

where to establish the second inequality we also used (4.60) and the boundary conditions satisfied by u_T. We finally notice that for every $i\in Z_1(\partial TQ)$ with $|\langle i,e_n\rangle|\geq L/2$ the function $u_\zeta^{e_n}(i+\cdot)$ coincides with the consant function $\mathrm{sign}\langle i,e_n\rangle$ on LQ, so that $\psi_i^s(\{u^{i+j}\}_{j\in\mathbb{Z}^n})=0$. If instead $|\langle i,e_n\rangle|<L/2$ we use the upper bound in ($\mathrm{H_s}3$) to deduce that $\psi_i^s(\{u^{i+j}\}_{j\in\mathbb{Z}^n})\leq c_6(|\zeta|+1)$. Hence, we obtain

$$\frac{1}{T^{n-1}}\sum_{i\in Z_1(\partial TQ)}\psi_i^s(\{u_\zeta^{e_n}(i+j)\}_{j\in\mathbb{Z}^n})\leq c\#Z_1(\partial TQ\cap\{|\langle i,e_n\rangle|<L/2\})\leq\frac{c}{T}<c\eta,$$

where the constant c depends only on n,L,ζ. Letting $\varepsilon\to0$ we eventually find

$$\limsup_{\varepsilon\to0}F_\varepsilon(u_\varepsilon,Q(\eta))\leq g_{\mathrm{hom}}(\zeta,e_n)+c\eta,$$

that is, the sequence (u_ε) satisfies (4.61) and we may conclude. □

Proof of Theorem 4.3. The result follows combining Theorem 3.1, Proposition 4.1, Proposition 4.12 and Proposition 4.15. □

5. Examples

5.1. **Pair interactions.** In the special case of interaction-energy densities ϕ_i^ε that take into account only pairwise interactions of the point i with the remaining lattice points Theorem 3.1 provides an analogous result to [1, Theorem 3.1] in the $GSBV$-setting (see also [16] and [24] for the case of interaction-energy densities that are independent of the position i). More in detail, our result can be applied to energies of the form

$$F_\varepsilon(u)=\sum_{i\in Z_\varepsilon(\Omega)}\varepsilon^n\sum_{\substack{\xi\in\mathbb{Z}^n\\ i+\varepsilon\xi\in\Omega}}f_\varepsilon^\xi(i,D_\varepsilon^\xi u^i),$$

i.e., when $\phi_i^\varepsilon : (\mathbb{R}^d)^{Z_\varepsilon(\Omega_i)} \to [0,+\infty)$ are given by

$$\phi_i^\varepsilon(\{z^j\}_{j\in Z_\varepsilon(\Omega_i)}) := \sum_{\substack{\xi\in\mathbb{Z}^n\\ i+\varepsilon\xi\in\Omega}} f_\varepsilon^\xi(i, D_\varepsilon^\xi z^0).$$

Here we assume that for every $\varepsilon > 0$ and every $i \in Z_\varepsilon(\Omega)$ the function $f_\varepsilon^\xi(i,\cdot) : \mathbb{R}^d \to [0,+\infty)$ is increasing in the sense that

$$f_\varepsilon(i,\zeta_1) \le f_\varepsilon(i,\zeta_2) \text{ for all } \zeta_1,\zeta_2 \in \mathbb{R}^d \text{ with } |\zeta_1| \le |\zeta_2|. \tag{5.1}$$

Moreover, we suppose that there exist constants $a_\varepsilon^\xi, \hat{a}_\varepsilon^\xi \ge 0$ and $b_\varepsilon^\xi, \hat{b}_\varepsilon^{i\xi} \ge 0$ such that for every $\varepsilon > 0$ and every $\xi \in \mathbb{Z}^n$ we have

$$\min\Big\{a_\varepsilon^\xi|\zeta|^p, \frac{b_\varepsilon^\xi}{\varepsilon}\Big\} \le f_\varepsilon(i,\zeta) \le \min\Big\{\hat{a}_\varepsilon^\xi|\zeta|^p, \frac{\hat{b}_\varepsilon^\xi}{\varepsilon}\Big\} \text{ for every } (i,\zeta) \in Z_\varepsilon(\Omega)\times\mathbb{R}^d, \tag{5.2}$$

where the constants $a_\varepsilon^\xi, \hat{a}_\varepsilon^\xi, b_\varepsilon^\xi, \hat{b}_\varepsilon^\xi$ satisfy the following hypotheses.

($\mathrm{H_{pw}}1$) (upper bound) We have

$$\limsup_{\varepsilon\to0} \sum_{\xi\in\mathbb{Z}^n} (\hat{a}_\varepsilon^\xi + \hat{b}_\varepsilon^\xi) < +\infty \tag{5.3}$$

and for every $\eta > 0$ there exists $M_\eta > 0$ such that

$$\limsup_{\varepsilon\to0} \sum_{\substack{\alpha\in\mathbb{N}\\ \alpha>M_\eta}} \sum_{\substack{\xi\in\mathbb{Z}^n\\ |\xi|_\infty\ge\min\{\varepsilon\frac{\alpha}{2}, \varepsilon\frac{M_\eta}{\sqrt{n}}\}}} (\hat{a}_\varepsilon^\xi + \hat{b}_\varepsilon^\xi) < \eta; \tag{5.4}$$

($\mathrm{H_{pw}}2$) (lower bound) there exist $a, b > 0$ such that $a_\varepsilon^{e_k} \ge a$, $b_\varepsilon^{e_k} \ge b$ for every $\varepsilon > 0$ and every $k \in \{1,\dots,n\}$;

($\mathrm{H_{pw}}3$) (relative control) there exists $\gamma > 0$ such that for every $\varepsilon > 0$ and every $\xi \in \mathbb{Z}^n$ with $\hat{a}_\varepsilon^\xi \neq 0$ there holds $|\xi|\hat{b}_\varepsilon^\xi \le \gamma\hat{a}_\varepsilon^\xi$.

Under the above assumptions it can be immediately verified that ϕ_i^ε satisfy hypotheses (H1)–(H6). In fact, (H1) is automatically satisfied, since ϕ_i^ε depends on $\{z^j\}_{j\in\mathbb{Z}_\varepsilon(\Omega_i)}$ only through differences $z^j - z^l$, and (5.1) ensures that (H2) holds true. Moreover, for ε small enough the upper bound (H3) is satisfied with $c_1 := \limsup_\varepsilon \sum_\xi \hat{a}_\varepsilon^\xi + 1$, which is finite thanks to (5.3). The lower bound (H4) holds true in view of ($\mathrm{H_{pw}}2$).

To verify the mild non-locality condition (H5) we observe that for any $\varepsilon > 0$, $i \in Z_\varepsilon(\Omega)$, $\alpha \in \mathbb{N}$ and $z, w : Z_\varepsilon(\Omega_i) \to \mathbb{R}^d$ with $z^j = w^j$ for all $j \in Z_\varepsilon(\varepsilon\alpha Q)$ we have

$$\begin{aligned}\phi_i^\varepsilon(\{z^j\}_{j\in Z_\varepsilon(\Omega_i)}) &= \sum_{\xi\in Z_1(\alpha Q)} f_\varepsilon(i, D_\varepsilon^\xi w^0) + \sum_{\substack{|\xi|_\infty\ge\frac{\alpha}{2}\\ i+\varepsilon\xi\in\Omega}} f_\varepsilon^\xi(i, D_\varepsilon^\xi z^0)\\ &\le \phi_i^\varepsilon(\{w^j\}_{j\in Z_\varepsilon(\Omega_i)}) + \sum_{\substack{|\xi|_\infty\ge\frac{\alpha}{2}\\ i+\varepsilon\xi\in\Omega}} \min\Big\{\hat{a}_\varepsilon^\xi|D_\varepsilon^\xi z^0|^p, \frac{\hat{b}_\varepsilon^\xi}{\varepsilon}\Big\},\end{aligned}$$

where the second inequality follows from the positiveness of the f_ε^ξ and (5.2). Thus, the required sequence $c_{\varepsilon,\alpha}^{j,\xi}$ in (H5) is obtained by setting

$$c_{\varepsilon,\alpha}^{j,\xi} := \begin{cases} \hat{a}_\varepsilon^\xi + \hat{b}_\varepsilon^\xi & \text{if } |\xi|_\infty \geq \frac{\alpha}{2},\ j=0,\\ 0 & \text{otherwise,}\end{cases}$$

which satisfies (2.4) and (2.5) thanks to (5.3) and (5.4), respectively.

It remains to establish (H6). To this end, let $z, w : Z_\varepsilon(\Omega_i) \to \mathbb{R}^d$ and $\varphi : Z_\varepsilon(\Omega_i) \to [0,1]$ a cut-off and set $v := \varphi z + (1-\varphi)w$. Let us show that $\phi_i^\varepsilon(\{v^j\}_{j\in Z_\varepsilon(\Omega_i)}) \leq R_i^\varepsilon(z,w,\varphi)$ with $R_i^\varepsilon(z,w,\varphi)$ as in (H6). We start observing that

$$D_\varepsilon^\xi v^0 = \varphi^0 D_\varepsilon^\xi z^0 + (1-\varphi^0) D_\varepsilon^\xi w^0 + D_\varepsilon^\xi \varphi^0 (z^{\varepsilon\xi} - w^{\varepsilon\xi}) \text{ for every } \xi \in \mathbb{Z}^n. \tag{5.5}$$

Thus, (5.2) together with the convexity of $|\cdot|^p$ and the subadditivity of the min ensure that

$$\phi_i^\varepsilon(\{v^j\}_{j\in Z_\varepsilon(\Omega_i)}) \leq \sum_{\substack{\xi\in\mathbb{Z}^n\\ i+\varepsilon\xi\in\Omega}} \min\Big\{\hat{a}_\varepsilon^\xi |D_\varepsilon^\xi z^0|^p, \frac{\hat{b}_\varepsilon^\xi}{\varepsilon}\Big\} + \min\Big\{\hat{a}_\varepsilon^\xi |D_\varepsilon^\xi w^0|^p, \frac{\hat{b}_\varepsilon^\xi}{\varepsilon}\Big\} + \hat{a}_\varepsilon^\xi |D_\varepsilon^\xi \varphi^0|^p |z^{\varepsilon\xi} - w^{\varepsilon\xi}|^p.$$

Eventually, from ($\mathrm{H_{pw}}$3) we deduce that for every $\xi \in \mathbb{Z}^n$ there holds

$$\min\Big\{\hat{a}_\varepsilon^\xi |D_\varepsilon^\xi z^0|^p, \frac{\hat{b}_\varepsilon^\xi}{\varepsilon}\Big\} = \hat{a}_\varepsilon^\xi \min\Big\{|D_\varepsilon^\xi z^0|^p, \frac{\hat{b}_\varepsilon^\xi}{\hat{a}_\varepsilon^\xi \varepsilon}\Big\} \leq \hat{a}_\varepsilon^\xi \min\Big\{|D_\varepsilon^\xi z^0|^p, \frac{\gamma}{\varepsilon|\xi|}\Big\},$$

and the same estimate holds with w in place of z. Since moreover

$$|D_\varepsilon^\xi \varphi^0|^p \leq \sup_{\substack{l\in Z_\varepsilon(\Omega_i)\\ k\in\{1,\dots,n\}}} |D_\varepsilon^k \varphi^l|^p \text{ for every } \xi \in \mathbb{Z}^n \text{ with } i+\varepsilon\xi \in \Omega, \tag{5.6}$$

we obtain $\phi_i^\varepsilon(\{v^j\}_{j\in Z_\varepsilon(\Omega_i)}) \leq R_i^\varepsilon(z,w,\varphi)$ with

$$R_i^\varepsilon(z,w,\varphi) = \sum_{\substack{\xi\in\mathbb{Z}^n\\ i+\varepsilon\xi\in\Omega}} \hat{a}_\varepsilon^\xi (\gamma+1)\Big(\min\Big\{|D_\varepsilon^\xi z^0|^p, \frac{\gamma}{\varepsilon|\xi|}\Big\} + \min\Big\{|D_\varepsilon^\xi w^0|^p, \frac{\gamma}{\varepsilon|\xi|}\Big\} + \sup_{\substack{l\in Z_\varepsilon(\Omega_i)\\ k\in\{1,\dots,n}} |D_\varepsilon^k \varphi^l|^p |z^{\varepsilon\xi} - w^{\varepsilon\xi}|^p\Big).$$

It then suffices to notice that (2.6) is satisfied due to (5.3) to conclude.

5.2. **Multibody weak-membrane energies.** A prototipical example of functionals F_ε as in (2.3) where the interaction-energy densities ϕ_i^ε do not depend only on pairwise interactions of i with $i+\varepsilon\xi$ but on multiple interactions of i with $i+\varepsilon\xi_1, \dots, i+\varepsilon\xi_N$ for some $N \in \mathbb{N}$ are so-called generalized weak-membrane energies, that have been studied in detail in [32]. In our setting a generalized weak-membrane energie can be written as in (2.3) with ϕ_i^ε given by

$$\phi_i^\varepsilon(\{z^j\}_{j\in Z_\varepsilon(\Omega_i)}) := f_\varepsilon\Big(i, \sum_{\xi\in Z_1(LQ)} \sum_{\substack{j\in Z_\varepsilon(\varepsilon LQ)\\ j+\varepsilon\xi\in\varepsilon LQ}} c^\xi |D_\varepsilon^\xi z^j|^p\Big), \tag{5.7}$$

where $L \in \mathbb{N}$ is the maximal range of interaction, $c^\xi \geq 0$ for every $\xi \in Z_1(LQ)$, and for every $\varepsilon > 0$ and $i \in Z_\varepsilon(\Omega)$ the function $f_\varepsilon(i,\cdot) : [0,+\infty) \to [0,+\infty)$ is increasing and satisfies

$$\min\Big\{a_\varepsilon^i t, \frac{b_\varepsilon^i}{\varepsilon}\Big\} \leq f_\varepsilon(i,t) \leq \min\Big\{\hat{a}_\varepsilon^i t, \frac{\hat{b}_\varepsilon^i}{\varepsilon}\Big\}, \tag{5.8}$$

for some $a_\varepsilon^i, \hat{a}_\varepsilon^i, b_\varepsilon^i, \hat{b}_\varepsilon^i \geq 0$. By construction the functions ϕ_i^ε satisfy (H1) and (H2). To ensure that Hypotheses (H3)–(H6) are fulfilled we assume that the following holds.

($\mathrm{H_{wm}}1$) There exist $a, \hat{a}, b, \hat{b} \in (0,+\infty)$ such that $a^i_\varepsilon \geq a$, $b^i_\varepsilon \geq b$, $\hat{a}^i_\varepsilon \leq \hat{a}$, $\hat{b}^i_\varepsilon \leq \hat{b}$ for every $\varepsilon > 0$ and every $i \in Z_\varepsilon(\Omega)$;

($\mathrm{H_{wm}}2$) for every $k \in \{1,\dots,n\}$ there holds $c^{e_k} > 0$.

The uniform bounds on $\hat{a}^i_\varepsilon$ in ($\mathrm{H_{wm}}1$) together with the upper bound in (5.8) imply that (H3) holds true with $c_1 := \hat{a}\max\{c^\xi : \xi \in Z_1(LQ)\}(\#Z_1(LQ))^2$, while thanks to the uniform bounds on $a^i_\varepsilon, b^i_\varepsilon$ in ($\mathrm{H_{wm}}1$), ($\mathrm{H_{wm}}2$), the lower bound in (5.8) and the monotinicity of $f_\varepsilon(i,\cdot)$ Hypotheses (H4) is satisfied with $c_2 := \min\{a,b\}\min\{c^{e_k} : 1 \leq k \leq n\} > 0$.

Moreover, the mild-nonlocality condition (H5) holds true by construction, since only finite-range interactions are taken into account. More precisely, in view of ($\mathrm{H_{wm}}1$) we can choose the sequence $c^{j,\xi}_{\varepsilon,\alpha}$ in (H5) as

$$c^{j,\xi}_{\varepsilon,\alpha} := \begin{cases} \max\{\hat{a},\hat{b}\}c^\xi & \text{if } \alpha < L,\ \xi \in Z_1(LQ),\ j \in Z_\varepsilon(\varepsilon LQ), \\ 0 & \text{otherwise,} \end{cases}$$

which satisfies (2.4) and (2.5).

Eventually, for every $z, w : Z_\varepsilon(\Omega_i) \to \mathbb{R}^d$ and every cut-off $\varphi : Z_\varepsilon(\Omega_i) \to [0,1]$ we can combine (5.5) and (5.6) with the upper bounds in (5.8) and ($\mathrm{H_{wm}}1$) to deduce that

$$\phi^\varepsilon_i(\{\varphi^j z^j + (1-\varphi^j)w^j\}_{j\in Z_\varepsilon(\Omega_i)}) \leq \max\{\hat{a},\hat{b}\}\sum_{\xi\in Z_1(LQ)} c^\xi \Bigg(\sum_{\substack{j\in Z_\varepsilon(\varepsilon LQ)\\ j+\varepsilon\xi\in\varepsilon LQ}} \sup_{\substack{l\in Z_\varepsilon(\Omega_i)\\ k\in\{1,\dots,n\}}} |D^k_\varepsilon \varphi^l|^p |z^{\varepsilon\xi} - w^{\varepsilon\xi}|^p$$
$$+ \min\Big\{|D^\xi_\varepsilon z^j|^p, \frac{1}{\varepsilon}\Big\} + \min\Big\{|D^\xi_\varepsilon w^j|^p, \frac{1}{\varepsilon}\Big\}\Bigg),$$

which gives (H6) by setting $c^{j,\xi}_\varepsilon := \max\{\hat{a},\hat{b}\}c^\xi$ for $\xi \in Z_1(LQ)$, $j \in Z_\varepsilon(\varepsilon LQ)$ and $c^{j,\xi}_\varepsilon := 0$ otherwise.

Under the above assumptions the functionals F_ε defined according to (2.3) with ϕ^ε_i as in (5.7) satisfy all the assumptions of Theorem 3.1 and thus Γ-converge up to subsequences to a free-disconctinuity functional of the form (3.1). We eventually give sufficient conditions under which the sequence (F_ε) satisfies the assumptions of Theorem 4.3. The first condition is εK-periodicity of f_ε in i, that is $f_\varepsilon(i+\varepsilon K e_k,\cdot) = f_\varepsilon(i,\cdot)$ for every $k \in \{1,\dots,n\}$, every $\varepsilon > 0$ and every $i \in Z_\varepsilon(\Omega)$. We then extend f_ε to $Z_\varepsilon(\mathbb{R}^n)\times[0,+\infty)$ by periodicity and in the same way we extend ϕ^ε_i to $(\mathbb{R}^d)^{Z_\varepsilon(\mathbb{R}^n)}$. Moreover, we can assume that $a^i_\varepsilon, \hat{a}^i_\varepsilon, b^i_\varepsilon, \hat{b}^i_\varepsilon$ are εK-periodic in i. We finally show that ($\mathrm{H}_\psi 1$)–($\mathrm{H}_\psi 3$) are satisfied if we assume that in addition for every $i \in Z_1([0,K)^n)$ there exist $a^i, b^i > 0$ such that

$$a^{\varepsilon i}_\varepsilon \to a^i,\ \hat{a}^{\varepsilon i}_\varepsilon \to a^i \qquad \text{and} \qquad b^{\varepsilon i}_\varepsilon \to b^i,\ \hat{b}^{\varepsilon i}_\varepsilon \to b^i \quad \text{as } \varepsilon \to 0, \tag{5.9}$$

that is, the functions $f_\varepsilon(i,\cdot)$ approach a single truncated potential. By periodicity (5.9) extends to $i \in \mathbb{Z}^n$. We claim that the required functions $\psi^b_i, \psi^s_i : (\mathbb{R}^d)^{\mathbb{Z}^n} \to [0,+\infty)$ are obtained by setting

$$\psi^b_i(\{z^j\}_{j\in\mathbb{Z}^n}) := a^i \sum_{\xi\in Z_1(LQ)} \sum_{\substack{j\in Z_1(LQ)\\ j+\xi\in LQ}} |D^\xi_1 z^j|^p,$$

$$\psi^s_i(\{z^j\}_{j\in\mathbb{Z}^n}) := \begin{cases} 0 & \text{if } z^j = z^0 \text{ for every } j \in Z-1(LQ), \\ b^i & \text{otherwise.} \end{cases}$$

First notice that ($\mathrm{H}_\psi 3$) is automatically satisfied. We next establish ($\mathrm{H}_\psi 1$). Let $\eta > 0$, $\Lambda > 0$ and suppose that $z : \mathbb{Z}^n \to \mathbb{R}^d$ is such that $|\nabla_{1,L} z|(0) < \Lambda$. Set $z^j_\varepsilon := \varepsilon z^{\frac{j}{\varepsilon}}$ for every $j \in \mathbb{Z}_\varepsilon(\mathbb{R}^n)$. Arguing as in Lemma 4.5 to establish (H$_\mathrm{b}$7) we deduce that

$$\sum_{\xi\in Z_1(LQ)}\sum_{\substack{j\in Z_\varepsilon(\varepsilon LQ)\\ j+\varepsilon\xi\in\varepsilon LQ}} c^\xi|D^\xi_\varepsilon z^j_\varepsilon|^p=\sum_{\xi\in Z_1(LQ)}\sum_{\substack{j\in Z_1(LQ)\\ j+\xi\in LQ}}|D^\xi_1 z^j|^p<2^{p-1}\max_{\xi\in Z_1(LQ)}c^\xi(1+\#Z_1(LQ))\Lambda^p \tag{5.10}$$

Let us choose $\bar\varepsilon=\bar\varepsilon(\eta,\Lambda)>0$ sufficiently small such that

$$\Lambda_0:=2^{p-1}\max_{\xi\in Z_1(LQ)}c^\xi(1+\#Z_1(LQ))\Lambda^p\leq\frac{b}{\hat a\varepsilon},\qquad |a^{\varepsilon i}_\varepsilon-a^i|\leq\frac{\eta}{\Lambda_0},\qquad |\hat a^{\varepsilon i}_\varepsilon-a^i|\leq\frac{\eta}{\Lambda_0}, \tag{5.11}$$

for every $\varepsilon\in(0,\bar\varepsilon$ and every $i\in Z_1([0,K)^n)$. The first condition in (5.11) together with (5.10) and (H$_\mathrm{wm}$1) ensure that

$$a^{\varepsilon i}_\varepsilon\sum_{\xi\in Z_1(LQ)}\sum_{\substack{j\in Z_\varepsilon(\varepsilon LQ)\\ j+\varepsilon\xi\in\varepsilon LQ}} c^\xi|D^\xi_\varepsilon z^j_\varepsilon|^p\leq\frac{b^{\varepsilon i}_\varepsilon}{\varepsilon}\quad\text{for every } i\in\mathbb{Z}^n.$$

Thus, (5.8) gives

$$a^{\varepsilon i}_\varepsilon\sum_{\xi\in Z_1(LQ)}\sum_{\substack{j\in Z_\varepsilon(\varepsilon LQ)\\ j+\varepsilon\xi\in\varepsilon LQ}} c^\xi|D^\xi_\varepsilon z^j_\varepsilon|^p\leq\phi^\varepsilon_{\varepsilon i}(\{z^j_\varepsilon\}_{j\in Z_\varepsilon(\mathbb{R}^n)})\leq\hat a^{\varepsilon i}_\varepsilon\sum_{\xi\in Z_1(LQ)}\sum_{\substack{j\in Z_\varepsilon(\varepsilon LQ)\\ j+\varepsilon\xi\in\varepsilon LQ}} c^\xi|D^\xi_\varepsilon z^j_\varepsilon|^p,$$

which in view of the second and third estimate in (5.11) and (5.10) finally gives

$$|\psi^b_i(\{z^j\}_{j\in\mathbb{Z}^n})-\phi^\varepsilon_{\varepsilon i}(\{z^j_\varepsilon\}_{j\in Z_\varepsilon(\mathbb{R}^n)})|<\eta.$$

It remains to show that ψ^s_i satisfies (H$_\psi$2). We start choosing $\Lambda>0$ such that

$$\frac{\Lambda^p\min_{1\leq k\leq n}c^{e_k}}{\hat c_1 n^{p-1}2^p}>\frac{\hat b}{a}, \tag{5.12}$$

where $\hat c_1$ is the constant provided by Lemma 4.4. Moreover, given $\eta>0$ we choose $\hat\varepsilon=\hat\varepsilon(\eta)$ small enough such that $|b^{\varepsilon i}_\varepsilon-b^i|<\eta$, $|\hat b^{\varepsilon i}_\varepsilon-b^i|<\eta$ for every $\varepsilon\in(0,\hat\varepsilon)$ and every $i\in Z_1([0,K)^n)$. Let $\varepsilon\in(0,\hat\varepsilon)$ and suppose that $z:\mathbb{Z}^n\to\mathbb{R}^d$ satisfies $\varepsilon^{\frac{1-p}{p}}|\nabla_{1,L}z|(0)\geq\Lambda$. Then Lemma 4.4 together with Jensen's inequality yield

$$\Lambda^p\leq\varepsilon^{1-p}|\nabla_{1,L}z|^p(0)\leq\varepsilon^{1-p}\hat c_1 n^{p-1}2^p\sum_{k=1}^n\sum_{\substack{j\in Z_1(LQ)\\ j+e_k\in LQ}}|D^k_1 z^j|^p.$$

In particular, the rescaled functions $\hat z_\varepsilon$ obtained by setting $\hat z_\varepsilon:=z^{\frac{j}{\varepsilon}}$ for every $j\in Z_\varepsilon(\mathbb{R}^n)$ satisfy

$$\sum_{k=1}^n c^{e_k}\sum_{\substack{j\in Z_\varepsilon(\varepsilon LQ)\\ j+\varepsilon e_k\in\varepsilon LQ}}|D^k_\varepsilon\hat z^j_\varepsilon|^p\geq\varepsilon^{-p}\min_{1\leq k\leq n}c^{e_k}\sum_{k=1}^n\sum_{\substack{j\in Z_1(LQ)\\ j+e_k\in LQ}}|D^k_1 z^j|^p\geq\frac{\Lambda^p\min_{1\leq k\leq n}c^{e_k}}{\hat c_1 n^{p-1}2^p}\frac{1}{\varepsilon},$$

hence the choice of Λ in (5.12) and (H$_\mathrm{wm}$1) ensure that

$$\frac{b^{\varepsilon i}_\varepsilon}{\varepsilon}=\min\Big\{a^{\varepsilon i}_\varepsilon\sum_{k=1}^n c^{e_k}\sum_{\substack{j\in Z_\varepsilon(\varepsilon LQ)\\ j+\varepsilon e_k\in\varepsilon LQ}}|D^k_\varepsilon\hat z^j_\varepsilon|^p,\frac{1}{\varepsilon}\Big\}\leq\phi^\varepsilon_{\varepsilon i}(\{\hat z^j_\varepsilon\}_{j\in Z_\varepsilon(\mathbb{R}^n)})\leq\frac{\hat b^{\varepsilon i}_\varepsilon}{\varepsilon}$$

for every $i \in Z_\varepsilon(\mathbb{R}^n)$. Eventually, since $\varepsilon \in (0, \hat{\varepsilon}(\eta))$, this gives

$$b^i - \eta \leq b_\varepsilon^{\varepsilon i} \leq \varepsilon \phi_{\varepsilon i}^\varepsilon(\{\hat{z}_\varepsilon^j\}_{j \in Z_\varepsilon(\mathbb{R}^n)}) \leq \hat{b}_\varepsilon^{\varepsilon i} \leq b^i + \eta.$$

If on the other hand $z : \mathbb{Z}^n \to \mathbb{R}^d$ is such that $|\nabla_{1,L} z|(0) = 0$ we immediately obtain

$$\phi_{\varepsilon i}^\varepsilon(\{\hat{z}_\varepsilon^j\}_{j \in Z_\varepsilon(\mathbb{R}^n)}) = 0 = \psi_i^s(\{z^j\}_{j \in Z^n})$$

for every $i \in \mathbb{Z}^n$, and we conclude that the functions ψ_i^s satisfy ($\mathrm{H}_\psi 2$).

5.3. **Weak membrane with long-range small-tail interactions.** In [10] the author studies the asymptotic behavior of weak-membrane energies of the form

$$F_\varepsilon(u) = \sum_{\xi \in \mathbb{Z}} \sum_{\substack{i \in Z_\varepsilon(\Omega) \\ i + \varepsilon\xi \in \Omega}} \varepsilon \rho_\varepsilon(\varepsilon\xi - i) \min\big\{|D_\varepsilon^\xi u^i|^2, \frac{1}{\varepsilon}\big\}, \tag{5.13}$$

where $\Omega \subset \mathbb{R}$ is an open, bounded interval. Assumimg only a locally uniform summability condition for the functions $\rho_\varepsilon : \varepsilon\mathbb{Z} \to [0, +\infty)$ it is shown that the Γ-limit is a non-local integral functional. Moreover, the author provides examples of specific functions ρ_ε including very long-range interactions with small tails, for which the Γ-limit is a (local) free-discontinuity functional. Among them are the discrete functionals as in (5.13) with $\rho_\varepsilon : \varepsilon\mathbb{Z} \to [0, +\infty)$ given by

$$\rho_\varepsilon(t) := \begin{cases} 1 & \text{if } t = \varepsilon, \\ \sqrt{\varepsilon} & \text{if } t = \varepsilon\lfloor \frac{1}{\sqrt{\varepsilon}} \rfloor, \\ 0 & \text{otherwise,} \end{cases}$$

which are shown to Γ-converge to the functional

$$F(u) = \int_\Omega |u'|^2 \, dt + \sum_{t \in S_u} \min\{1 + |u^+(t) - u^-(t)|^2, 2\}.$$

We observe that thanks to our very mild non-locality condition (H5) the above example can be recast in our framework by setting

$$\phi_i^\varepsilon(\{z^j\}_{j \in Z_\varepsilon(\Omega_i)}) := \min\Big\{\Big|\frac{z^\varepsilon - z^0}{\varepsilon}\Big|^2, \frac{1}{\varepsilon}\Big\} + \sqrt{\varepsilon} \min\Big\{\Big|\frac{z^{\varepsilon\lfloor \frac{1}{\varepsilon} \rfloor} - z^0}{\varepsilon^2 \lfloor \frac{1}{\sqrt{\varepsilon}} \rfloor}\Big|^2, \frac{1}{\varepsilon}\Big\}.$$

Indeed, it is immediate to check that ϕ_i^ε satisfies (H1)–(H4) for every $\varepsilon > 0$ and every $i \in Z_\varepsilon(\Omega)$. Moreover, (H5) is satisfied with the sequence $(c_{\varepsilon,\alpha}^{j,\xi})$ defined by setting

$$c_{\varepsilon,\alpha}^{j,\xi} := \begin{cases} 1 & \text{if } \alpha \leq 2,\ j = 0,\ \xi = 1, \\ \sqrt{\varepsilon} & \text{if } \alpha \leq 2\lfloor \frac{1}{\sqrt{\varepsilon}} \rfloor,\ j = 0,\ \xi = \lfloor \frac{1}{\sqrt{\varepsilon}} \rfloor, \\ 0 & \text{otherwise.} \end{cases}$$

The sequence $(c_{\varepsilon,\alpha}^{j,\xi})$ fulfills the required summability condition (2.4), since

$$\sum_{\alpha \in \mathbb{N}} \sum_{j \in Z_\varepsilon(\mathbb{R})} \sum_{\xi \in \mathbb{Z}} c_{\varepsilon,\alpha}^{j,\xi} = 2 + \sum_{\alpha=1}^{2\lfloor \frac{1}{\sqrt{\varepsilon}} \rfloor} \sqrt{\varepsilon} \leq 4 \quad \text{for every } \varepsilon > 0.$$

Moreover, the decaying-tail condition (2.5) is satisfied since $c_{\varepsilon,\alpha}^{j,\xi} = 0$ for every $\alpha > 2\lfloor \frac{1}{\sqrt{\varepsilon}} \rfloor$. Thus, for every $\eta > 0$ the sequence (M_η^ε) can be chosen independently of η as $M_\eta^\varepsilon = 2\lfloor \frac{1}{\sqrt{\varepsilon}} \rfloor$, which satisfies the constraint $\varepsilon M_\eta^\varepsilon \to 0$ as $\varepsilon \to 0$. Eventually, (H6) can be verified by using expression (5.5) together with the convexitiy of $z \mapsto z^p$ and the subadditivity of the min.

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