From Harnack inequality to heat kernel estimates on metric measure spaces and applications

Luca Tamanini *

July 16, 2019

Abstract

Aim of this short note is to show that a dimension-free Harnack inequality on an infinitesimally Hilbertian metric measure space where the heat semigroup admits an integral representation in terms of a kernel is sufficient to deduce a sharp upper Gaussian estimate for such kernel. As intermediate step, we prove the local logarithmic Sobolev inequality (known to be equivalent to a lower bound on the Ricci curvature tensor in smooth Riemannian manifolds). Both results are new also in the more regular framework of \( \text{RCD}(K,\infty) \) spaces.

Contents

1 Introduction 1
2 Preliminaries and setting 3
3 Auxiliary results 5
4 Main result and applications 10

1 Introduction

In a smooth Riemannian manifold \((M,g,\text{vol})\) with Ricci curvature bounded from below it is well-known [23] that the heat kernel \(r_t\), namely the positive fundamental solution of the heat equation defined by the Laplace-Beltrami operator (or, from a probabilistic viewpoint, the transition probability of the Brownian motion), satisfies two-sided Gaussian estimates, which read as

\[
\frac{1}{C_1 \text{vol}(B_{\sqrt{t}}(y))} \exp \left( -\frac{d^2(x,y)}{(4-\varepsilon)t} - C_2 t \right) \leq r_t[x](y) \leq \frac{C_1}{\text{vol}(B_{\sqrt{t}}(y))} \exp \left( -\frac{d^2(x,y)}{(4+\varepsilon)t} + C_2 t \right) \quad (1.1)
\]

for every \( \varepsilon > 0 \), \( x, y \in X \) and \( t > 0 \), for suitable positive constants \( C_1 = C_1(K,N,\delta) \) and \( C_2 = C_2(K,N,\delta) \), where \( K \) is a lower bound on the Ricci curvature and \( N \) an upper bound for the dimension. But because of the great interest in the study of metric (measure) spaces

*Institut für Angewandte Mathematik, Universität Bonn. email: tamanini@iam.uni-bonn.de
(we refer to [17] for an overview on the topic and detailed bibliography) and the essentially metric-measure nature of (1.1), it is rather natural to investigate possible generalizations of these kind of bounds to more abstract settings. For regular, symmetric, strongly local Dirichlet forms on locally compact separable Hausdorff spaces satisfying doubling & Poincaré, this has been achieved by K.-T. Sturm in [24], while more recently R. Jiang, H. Li and H. Zhang studied the problem on finite-dimensional $\text{RCD}^*(K,N)$ spaces [19], which are still locally compact but only locally doubling for $K < 0$. This class of metric measure spaces with Ricci curvature bounded from below, introduced in [5], [13] starting from the seminal papers [21], [25], [26], is the natural framework where bounds like (1.1) can be expected to hold, since it enjoys many analytical and geometric properties and the existence of the heat kernel is well understood.

As concerns weighted Riemannian manifolds $(M,g,\mu)$, the picture is not so clear, as to the best of our knowledge no lower bounds are known (unless $M$ has finite volume, see [28]), whereas an upper estimate can still be deduced [15], [14] and reads as

$$r_t(x,y) \leq \frac{1}{\sqrt{\mu(B_{\sqrt{t}}(x))\mu(B_{\sqrt{t}}(y))}} \exp \left( C_\varepsilon (1 + C_K t) - \frac{d^2(x,y)}{(4 + \varepsilon) t} \right)$$

for every $\varepsilon > 0$, $x, y \in X$ and $t > 0$; notice however that in general it is not possible to get rid of one of the two volumes on the right-hand side, since $M$ does not need to be doubling.

As in the previous discussion, also (1.2) has an essentially metric-measure nature and if on the one hand the non-smooth counterpart of smooth Riemannian manifolds with lower Ricci bounds is given by $\text{RCD}^*(K,N)$ spaces, on the other hand $\text{RCD}(K,\infty)$ spaces represent the natural alternative to weighted manifolds and in this setting (1.2) is still missing; therefore it would be natural, as a first attempt, to prove (1.2) on any $\text{RCD}(K,\infty)$ space.

Yet, after a careful look at [19] one can observe that the key role in the proof of (1.1) is played by the dimension-dependent Harnack inequality [11], [18] and that not surprisingly in [14] the dimension-free Harnack inequality, which is known to hold on $\text{RCD}(K,\infty)$ spaces after [20], is a crucial ingredient as well. For this reason, instead of establishing (1.2) on an $\text{RCD}(K,\infty)$ space, we prefer to work in a more general environment, namely a metric measure space $(X,d,m)$ supporting a dimension-free Harnack inequality where the heat semigroup admits an integral representation (see Setting 2.1 below). In order to achieve this goal we obtain a local logarithmic Sobolev inequality, that to the best of our knowledge was missing also in the context of $\text{RCD}(K,\infty)$ spaces, and the $L^\infty$-LIP regularization for the heat flow (see Proposition 3.2).

The paper is structured as follows. In Section 2 we recall all the relevant notions, results and bibliographical references related to calculus on metric measure spaces and point out the precise framework we shall work within. In Section 3 we collect some auxiliary results, most notably the local logarithmic Sobolev inequality. Finally, Section 4 is devoted to the proof of the upper Gaussian estimate for the heat kernel and some consequences.

**Acknowledgements**

The author gratefully acknowledges support by the European Union through the ERC-AdG “RicciBounds” for Prof. K. T. Sturm and would like to thank Prof. M. Gordina for useful suggestions.
2 Preliminaries and setting

By LIP(\(X\)) we denote the space of Lipschitz continuous functions and by \(C([0,1],X)\) the space of continuous curves with values in the metric space \((X,d)\). For the notion of absolutely continuous curve in a metric space and of metric speed see for instance Section 1.1 in [2]. The collection of absolutely continuous curves on \([0,1]\) is denoted by \(AC([0,1],X)\). By \(\mathcal{P}(X)\) we denote the space of Borel probability measures on \((X,d)\) and by \(\mathcal{P}_2(X) \subset \mathcal{P}(X)\) the subclass of those with finite second moment.

Let \((X,d,m)\) be a complete and separable metric measure space endowed with a Borel non-negative measure which is finite on bounded sets.

For the definition of the Sobolev class \(S^2(X)\) and of minimal weak upper gradient \(|Df|\) see [4] (and the previous works [9], [22] for alternative - but equivalent - definitions of Sobolev functions). The Sobolev space \(W^{1,2}(X)\) is defined as \(L^2(X) \cap S^2(X)\). When endowed with the norm \(\|f\|_{W^{1,2}} := \|f\|_{L^2}^2 + \|Df\|_{L^2}^2\), \(W^{1,2}(X)\) is a Banach space. The Cheeger energy is the convex and lower-semicontinuous functional \(E:L^2(X) \to [0,\infty]\) given by

\[
E(f) := \begin{cases} 
\frac{1}{2} \int |Df|^2 \, dm & \text{for } f \in W^{1,2}(X) \\
\infty & \text{otherwise}
\end{cases}
\]

\((X,d,m)\) is infinitesimally Hilbertian (see [13]) if \(W^{1,2}(X)\) is Hilbert. In this case the cotangent module \(L^2(T^*X)\) (see [12]) and its dual, the tangent module \(L^2(TX)\), are canonically isomorphic, the differential is a well-defined linear map \(d\) from \(S^2(X)\) with values in \(L^2(T^*X)\) and the isomorphism sends the differential \(df\) to the gradient \(\nabla f\). Furthermore \(E\) is a Dirichlet form admitting a carré du champ given by \(\langle \nabla f, \nabla g \rangle\), where \(\langle \cdot, \cdot \rangle\) is the pointwise scalar product on the Hilbert module \(L^2(TX)\). The infinitesimal generator \(\Delta\) of \(E\), which is a closed self-adjoint linear operator on \(L^2(X)\), is called Laplacian on \((X,d,m)\) and its domain denoted by \(D(\Delta) \subset W^{1,2}(X)\). A function \(f \in W^{1,2}(X)\) belongs to \(D(\Delta)\) and \(g = \Delta f\) if and only if

\[
\int \phi g \, dm = -\int \langle \nabla \phi, \nabla f \rangle \, dm, \quad \forall \phi \in W^{1,2}(X).
\]

The flow \((h_t)\) associated to \(E\) is called heat flow (see [4]), and for any \(f \in L^2(X)\) the curve \(t \mapsto h_t f \in L^2(X)\) is continuous on \([0,\infty)\), locally absolutely continuous on \((0,\infty)\) and the only solution of

\[
\frac{d}{dt} h_t f = \Delta h_t f, \quad h_t f \to f \text{ as } t \downarrow 0.
\]

After this preliminary part, we can describe the framework we shall work within throughout this note.

**Setting 2.1.** \((X,d,m)\) is a complete and separable metric space equipped with a non-negative Borel measure which is finite on bounded sets and supports the following dimension-free Harnack inequality: for any \(p \in (1,\infty)\), \(x,y \in X\) and for any \(f \in L^1 \cap L^\infty(X)\) it holds

\[
|h_t f(x)|^p \leq (h_t |f|^p)(y) \exp\left(\frac{pKd^2(x,y)}{2(p-1)(e^{2Kt} - 1)}\right)
\]

with \(K \in \mathbb{R}\). We also assume that there exists a function

\[
(0,\infty) \times X^2 \ni (t,x,y) \quad \mapsto \quad r_t[x](y) \in (0,\infty)
\]

(2.2)
called heat kernel, such that for every \( x \in X \) and \( t > 0 \), \( r_t[x] \) is a probability density, and the following identity holds

\[
h_t f(x) = \int f(y) r_t[x](y) \, dm(y) \quad \forall t > 0, \forall f \in L^2(X). \tag{2.3}
\]

By the arguments in [5] and [1] with slight adaptations, the semigroup property of \( h_t \) and the representation formula (2.3) entail that the heat kernel is symmetric, i.e. \( r_t[x](y) = r_t[y](x) \) \( m \otimes m \)-a.e. in \( X^2 \), and satisfies the Chapman-Kolmogorov formula

\[
r_{t+s}[x](y) = \int r_t[x](z) r_s[z](y) \, dm(z) \quad \text{for } m \otimes m \text{-a.e. } (x, y) \in X^2, \forall t, s \geq 0. \tag{2.4}
\]

Moreover, (2.3) and the fact that \( r_t[x] \) is a probability density can also be used to extend the heat flow to \( L^1(X) \), show that it is mass preserving and satisfies the maximum principle, i.e.

\[
f \leq c \quad m - \text{a.e.} \quad \Rightarrow \quad h_t f \leq c \quad m\text{-a.e.}, \forall t > 0.
\]

Finally, by Proposition 4.1 in [10] the dimension-free Harnack inequality (2.1) implies the strong Feller property for \( h_t \) for all \( t > 0 \), namely if \( f \in L^\infty(X) \), then \( h_t f \) is continuous and bounded.

Let us recall that the minimal weak upper gradient is local (i.e. \( |Df| = |Dg| \) \( m \)-a.e. on \( \{ f = g \} \) ), lower semicontinuous w.r.t. \( m \)-a.e. convergence and that Lipschitz functions with bounded support are dense in \( L^p(X) \), \( p \in (1, \infty) \) (see [3], where the density is actually proved in \( W^{1,p}(X) \)).

As regards the properties of the differential, the following calculus rules (see [12] for the proof) will be used extensively without further notice:

\[
|df| = |Df| \quad m \text{-a.e.} \quad \forall f \in S^2(X)
\]

\[
df = dg \quad m \text{-a.e. on } \{ f = g \} \quad \forall f, g \in S^2(X)
\]

\[
d(\varphi \circ f) = \varphi' \circ f \, df \quad \forall f \in S^2(X), \varphi : \mathbb{R} \rightarrow \mathbb{R} \text{ Lipschitz}
\]

\[
d(fg) = g \, df + f \, dg \quad \forall f, g \in L^\infty \cap S^2(X)
\]

where it is part of the properties the fact that \( \varphi \circ f, fg \in S^2(X) \) for \( \varphi, f, g \) as above.

Finally, given \( f : X \rightarrow \mathbb{R} \), the local Lipschitz constant \( \text{lip}(f) : X \rightarrow [0, \infty] \) is defined as 0 on isolated points and otherwise as

\[
\text{lip} f(x) := \limsup_{y \rightarrow x} \frac{|f(x) - f(y)|}{d(x, y)}.
\]

If \( f \) is Lipschitz, then its Lipschitz constant is denoted by \( \text{Lip}(f) \). It is worth stressing that if \( f \) is Lipschitz with \( \text{lip}(f) \in L^2(X) \), then \( f \in S^2(X) \) with

\[
|Df| \leq \text{lip}(f), \quad m \text{-a.e.} \tag{2.5}
\]

**Remark 2.2** (Lipschitz cut-off function). Given a complete and separable metric measure space \((X, d, m)\) equipped with a non-negative Borel measure which is finite on bounded sets, for any \( x \in X \) and \( r > 0 \) there exists a Lipschitz function \( \chi_r : X \rightarrow [0, 1] \) such that \( \chi_r \equiv 1 \) on \( B_r(x) \), \( \text{supp}(\chi_r) \subseteq B_{r+1}(x) \) and \( \| \text{lip}(\chi_r) \|_{L^\infty(X)} \) does not depend on \( r \).

Indeed, if \( \eta \in \text{LIP}(\mathbb{R}) \) with bounded support, \( \eta \equiv 1 \) on \([0, 1/3]\), \( \eta \equiv 0 \) on \([2/3, \infty)\) and set \( \chi_r := \eta \circ d(\cdot, B_r(x)) \), where \( d(\cdot, B_r(x)) := \inf_{y \in B_r(x)} d(\cdot, y) \), then it is easy to see that \( \chi_r \equiv 1 \) on \( B_{r+1/3}(x) \), \( \text{supp}(\chi_r) \subseteq B_{r+2/3}(x) \) and is Lipschitz with \( \text{lip}(\chi_r) \leq |\eta'(d(\cdot, B_r(x)))| \leq C \) \( m \)-a.e. with \( C \) independent of \( r \).
3 Auxiliary results

In this section we collect all technical results that are required in view of the proof of Theorem 4.1. We begin with a continuity statement for the heat semigroup.

**Lemma 3.1.** With the same assumptions and notations as in Setting 2.1, let \( f \in L^1 \cap L^\infty(X) \). Then \((h_t f) \in C([0, \infty), L^p(X))\) for all \( p \in [1, \infty) \).

In particular, for all \( T > 0 \), \( p \in [1, \infty) \) and \( \varepsilon > 0 \) there exists a bounded open set \( B \) such that

\[
\int_{X \setminus B} |h_t f|^p \, dm < \varepsilon, \quad \forall t \in [0, T].
\]

**proof** As a first step, recall that the heat flow satisfies the maximum principle and \((h_t f) \in C([0, \infty), L^2(X))\). Therefore, if \( p \geq 2 \) it is sufficient to observe that

\[
\int |h_s f - h_t f|^p \, dm \leq \|h_s f - h_t f\|_{L^\infty(X)}^p \int |h_s f - h_t f|^2 \, dm \leq 2\|f\|_{L^\infty(X)}^2 \|h_s f - h_t f\|^2_{L^2(X)}
\]

for all \( s, t \geq 0 \) to deduce the \( L^p \)-continuity. For \( p = 1 \) we rely on Brezis-Lieb’s lemma (see [8]); for our purposes it is enough to know that if \((u_n) \subset L^1(X)\) is a bounded sequence with \( u_n \to u \) m.a.e. for some measurable function \( u \), then \( u \in L^1(X) \) and

\[
\lim_{n \to \infty} \int \left( |u_n| - |u_n - u| \right) \, dm = \int |u| \, dm.
\]

Hence pick any \( t \geq 0 \), any sequence \((t_n)\) converging to \( t \) and set \( u_n := h_{t_n} f \). From \((h_t f) \in C([0, \infty), L^2(X))\) we know that, up to pass to a subsequence, \( u_n \to u = h_t f \) m.a.e. and from the mass-preserving property of the heat flow \( \|u_n\|_{L^1(X)} = \|u\|_{L^1(X)} \). Therefore (3.2) yields \( \|u_n - u\|_{L^1(X)} \to 0 \) as \( n \to \infty \) and by the arbitrariness of \((t_n)\) we conclude that \((h_t f) \in C([0, \infty), L^1(X))\). For \( p \in (1, 2) \), it is sufficient to notice that, by interpolation, there exists \( C_p \) such that

\[
\|h_s f - h_t f\|_{L^p(X)} \leq C_p \left( \|h_s f - h_t f\|_{L^1(X)} + \|h_s f - h_t f\|_{L^2(X)} \right), \quad \forall s, t \geq 0.
\]

As regards (3.1), we first observe that if we set \( u_n := |h_{t_n} f|^p \) and \( u := |h_t f|^p \) with \( t_n \to t \), then the \( L^p \)-continuity of the heat flow implies that, up to subsequences, \( u_n \to u \) m.a.e. and \( \|u_n\|_{L^1(X)} \to \|u\|_{L^1(X)} \) as \( n \to \infty \). Plugging this information into (3.2) we deduce that \(|h_t f|^p \in C([0, \infty), L^1(X))\) and in turn this implies that \( K_T := \{ |h_t f|^p : t \in [0, T] \} \subset L^1(X) \) is compact for all \( T \geq 0 \). Indeed, if \((u_n) \subset K_T \), then \( u_n = |h_{t_n} f|^p \) for some \( t_n \in [0, T] \) and \((t_n)\) admits a convergent subsequence, say \( t_{n_k} \to t \), so that \( u_{n_k} \to u := |h_t f|^p \in K_T \) in \( L^1(X) \). In particular \( K_T \) is weakly compact in \( L^1(X) \) and, by the Dunford-Pettis theorem for \( \sigma \)-finite measures, this implies that for every \( \varepsilon > 0 \) there exists a measurable set \( B \) with finite measure such that

\[
\int_{X \setminus B} |h_t f|^p \, dm < \varepsilon, \quad \forall t \in [0, T].
\]

In order to see that \( B \) can be replaced by a bounded set, by the maximum principle and the fact that \( f \in L^\infty(X) \) we know that, for some constant \( C > 0 \), \( |h_t f|^p \leq C \) for all \( t \geq 0 \) and, since \( m(B) < \infty \), for all \( x \in X \) there exists \( R > 0 \) such that \( m(B \setminus B_R(x)) < \varepsilon/C \). Therefore, fixing \( x \in X \) and setting \( B' := B \cap B_R(x) \) we see that

\[
\int_{X \setminus B'} |h_t f|^p \, dm = \int_{X \setminus B} |h_t f|^p \, dm + \int_{B \setminus B_R(x)} |h_t f|^p \, dm < 2\varepsilon,
\]

whence the conclusion by the arbitrariness of \( \varepsilon \). \( \square \)
Then we show that the dimension-free Harnack inequality implies the local logarithmic Sobolev inequality, which is the non-smooth counterpart of the main result contained in [6]. On smooth Riemannian manifolds [27] and in the setting of Bakry-Émery Γ-calculus [7], both (2.1) and (3.4) are known to be equivalent to the curvature-dimension condition $\text{CD}(K, \infty)$ ([25], see also Section 4 for the definition), but in the present framework the implication (2.1) $\Rightarrow$ (3.4) was still missing. As a byproduct we improve the strong Feller property of $h_t$ to an $L^\infty$-LIP regularization.

**Proposition 3.2.** With the same assumptions and notations as in Setting 2.1, $h_t$ maps $L^\infty(X)$ into $\text{LIP}(X)$ and for any $f \in L^\infty$ positive, $t > 0$ it holds

\[
\text{lip}(h_t f)^2 \leq \frac{2K}{e^{2Kt} - 1} h_t f (h_t (f \log f) - h_t f \log h_t f) \quad \text{pointwise.} \quad (3.3)
\]

In addition, for any $f \in L^p(X)$ positive, $p \in (1, \infty)$, and for any $t > 0$ it holds

\[
|Dh_t f|^2 \leq \frac{2K}{e^{2Kt} - 1} h_t f (h_t (f \log f) - h_t f \log h_t f) \quad m \text{-a.e.} \quad (3.4)
\]

**Proof.** Let $x \in X$, $y_n \to x$ and for sake of brevity set $d_n := d(x, y_n)$; let us also fix $\delta > 0$ and take $f \in L^\infty(X)$ positive (not necessarily in $L^p(X)$). Since the function $\Phi : (0, \infty) \times (0, \infty) \to \mathbb{R}$ defined by

\[
\Phi(z, \alpha) := \begin{cases} \frac{z^\alpha - 1}{\alpha} & \text{if } \alpha > 0 \\ \log z & \text{if } \alpha = 0 \end{cases}
\]

is continuous, by the weak Feller property of $h_t$ the function $y \mapsto \Phi(h_t f(y), \delta d(x, y))$ is continuous as well, whence

\[
\limsup_{n \to \infty} \frac{h_t f(y_n) - h_t f(x)}{d_n} + \delta(h_t f \log h_t f)(x) = \limsup_{n \to \infty} \frac{(h_t f)^{1+\delta d_n}(y_n) - h_t f(x)}{d_n}. \quad (3.5)
\]

Moreover by (2.1) we have

\[
(h_t f)^{1+\delta d_n}(y_n) - h_t f(x) \leq h_t |f|^{1+\delta d_n} \exp\left(\frac{K(1+\delta d_n)d_n}{2\delta(e^{2Kt} - 1)}\right) - h_t f(x)
\]

\[
= \exp\left(\frac{K(1+\delta d_n)d_n}{2\delta(e^{2Kt} - 1)}\right)(h_t |f|^{1+\delta d_n}(x) - h_t f(x)) + h_t f(x)\left(\exp\left(\frac{K(1+\delta d_n)d_n}{2\delta(e^{2Kt} - 1)}\right) - 1\right)
\]

and on the other hand

\[
\limsup_{n \to \infty} \exp\left(\frac{K(1+\delta d_n)d_n}{2\delta(e^{2Kt} - 1)}\right) \frac{|h_t f|^{1+\delta d_n}(x) - h_t f(x)}{d_n} = \limsup_{n \to \infty} \int \frac{|f|^{1+\delta d_n} - f}{d_n} r_t[x]d\mu
\]

\[
\leq \delta \int f \log f r_t[x]d\mu = \delta h_t(f \log f)(x),
\]

where the inequality is motivated by Fatou’s lemma (indeed $(f \Phi(f, \delta d_n))_n$ is bounded from above, since so is $f$), while on the other hand

\[
\limsup_{n \to \infty} \frac{h_t f(x)}{d_n}\left(\exp\left(\frac{K(1+\delta d_n)d_n}{2\delta(e^{2Kt} - 1)}\right) - 1\right) = \frac{K}{2\delta(e^{2Kt} - 1)} h_t f(x).
\]
Plugging these observations into (3.5) yields
\[
\limsup_{n \to \infty} \frac{h_tf(y_n) - h_tf(x)}{d_n} \leq \delta h_t(f \log f)(x) - \delta(h_tf \log h_tf)(x) + \frac{K}{2\delta(e^{2Kt} - 1)} h_tf(x). \tag{3.6}
\]

Next, assume that there exists \( c > 0 \) such that \( f \geq c \); since by smoothness \( \Phi(z, \alpha) \) converges uniformly to \( \log z \) on \([z_0, z_1]\) with \( z_0 > 0 \) as \( \alpha \downarrow 0 \), this implies that \( f\Phi(f, \delta d_n) \) converges uniformly to \( f \log f \) as \( n \to \infty \). As on the other hand \( r[y_n] \to r[x] \) in \( L^1(X) \) by the strong Feller property of \( h_t \), this yields
\[
\lim_{n \to \infty} h_tf(y_n) - h_tf(y_n) = -\delta h_t(f \log f)(x),
\]
whence
\[
\limsup_{n \to \infty} \frac{h_tf(x) - h_tf(y_n)}{d_n} - \delta h_t(f \log f)(x) = \limsup_{n \to \infty} \frac{h_tf(x) - h_tf(y_n)}{d_n}.
\]

Using again (2.1) we obtain
\[
\begin{align*}
h_tf(x) - h_tf(y_n) &\leq h_tf(x) - (h_tf)^{1+\delta}(y_n) \\
&= h_tf(x) \left( 1 - \exp \left( - \frac{K(1+\delta d_n)}{2\delta(e^{2Kt} - 1)} \right) \right) \exp \left( - \frac{K(1+\delta d_n)}{2\delta(e^{2Kt} - 1)} \right) (h_tf(x) - h_tf(y_n)),
\end{align*}
\]
and on the one hand
\[
\limsup_{n \to \infty} \frac{h_tf(x) \left( 1 - \exp \left( - \frac{K(1+\delta d_n)}{2\delta(e^{2Kt} - 1)} \right) \right)}{d_n} = \frac{K}{2\delta(e^{2Kt} - 1)} h_tf(x)
\]
while on the other hand
\[
\limsup_{n \to \infty} \exp \left( - \frac{K(1+\delta d_n)}{2\delta(e^{2Kt} - 1)} \right) \frac{h_tf(x) - h_tf(y_n)}{d_n} = -\delta(h_tf \log h_tf)(x),
\]
so that
\[
\limsup_{n \to \infty} \frac{h_tf(x) - h_tf(y_n)}{d_n} \leq \delta h_t(f \log f)(x) + \frac{K}{2\delta(e^{2Kt} - 1)} h_tf(x) - \delta(h_tf \log h_tf)(x). \tag{3.7}
\]

The assumption \( f \geq c \) for some \( c > 0 \) can now be removed, since if we define \( f_k := f + 1/k \) then (3.7) holds for \( f_k \) and passing to the limit as \( k \to \infty \) yields the validity of (3.7) for \( f \); indeed, the left-hand side is easily seen to be constant w.r.t. \( k \), while on the right-hand side it is sufficient to use the representation formula (2.3) to deduce that \( h_tf_k \to h_tf \) and \( h_t(f_k \log f_k) \to h_t(f \log f) \) pointwise.

Combining (3.7) with (3.6) and recalling the definition of \( \text{lip}(f) \) imply
\[
\text{lip}(h_tf) \leq \delta h_t(f \log f) - \delta(h_tf \log h_tf) + \frac{K}{2\delta(e^{2Kt} - 1)} h_tf
\]
pointwise, as the right-hand side is continuous by the strong Feller property of \( h_t \), and it is now sufficient to optimize the right-hand side w.r.t. \( \delta \) to get (3.3) and the \( L^\infty\)-LIP regularization for \( h_t \). In order to prove (3.4) it is sufficient to consider (3.3) for \( f \in \text{LIP}(X) \) positive with bounded support, recall (2.5) and use the density of Lipschitz functions with bounded support in \( L^p(X), p \in (1, \infty) \), together with the lower semicontinuity of the minimal weak upper gradient. \( \square \)
Finally we prove the integral maximum principle for the heat semigroup. On Riemannian manifolds several different proofs are possible (see for instance [16] and references therein); here we adapt to the metric measure framework the one proposed in [14].

**Lemma 3.3** (Integral maximum principle). With the same assumptions and notations as in Setting 2.1, let $T > 0, p \in (1, \infty), x \in X$ and set

\[
\xi_t(y) := -\frac{d^2(x, y)}{2(T - qt)}, \quad \text{for } y \in X, t \in [0, T/q),
\]

where $q := \frac{p}{2(p-1)}$. Then, for any non-negative $f \in L^p(X)$ and $t \in [0, T/q)$, it holds

\[
\int (h_t f)^p e^{\xi_t} \, d\mu \leq \int f^p \exp \left( -\frac{d^2(x, \cdot)}{2T} \right) \, d\mu.
\]

**Proof** As a preliminary remark, by standard approximation argument it is not restrictive to assume that $f \in L^\infty(X)$ with bounded support: by Lemma 3.1 this implies that $(h_t f) \in C([0, \infty), L^p(X))$. Hence for all $t \geq 0$ and any sequence $(t_n)$ converging to $t$ there exists $g \in L^p(X)$ such that, up to pass to a subsequence, $h_{t_n} f \leq g$ for all $n \in \mathbb{N}$. As $e^{\xi_t} \leq 1$ and $\xi_t$ smoothly depends on $t \in [0, T/q)$, by dominated convergence we deduce that

\[
I(t) := \int (h_t f)^p e^{\xi_t} \, d\mu
\]

is continuous on $[0, T/q)$. In order to see that it is also locally absolutely continuous on $(0, T/q)$, let $R > 0$ and take a cut-off function $\chi_R$ as in Remark 2.2. Notice that $(h_t f) \in AC_{loc}((0, \infty), L^2(X))$ and, by the maximum principle, it is uniformly bounded in space and time. As $\xi_t$ smoothly depends on $t \in [0, T/q)$ and $d^2(x, \cdot) \in W^{1,2}_{loc}(X)$, we deduce that $(0, T/q) \ni t \mapsto \chi_R(h_t f)^p e^{\xi_t} \in L^2(X)$ is absolutely continuous. In particular, so is $\int \chi_R(h_t f)^p e^{\xi_t} \, d\mu$ and it is then clear that

\[
\frac{d}{dt} \int \chi_R(h_t f)^p e^{\xi_t} \, d\mu = \int \chi_R \left( p e^{\xi_t} (h_t f)^{p-1} \Delta h_t f - \frac{q d_x^2}{2(T - qt)^2} e^{\xi_t} (h_t f)^p \right) \, d\mu, \quad \text{a.e. } t
\]

where $d_x := d(x, \cdot)$. As regards the first summand on the right-hand side, using integration by parts and the fact that $|\nabla d_x| \leq 1$ m-a.e. we can rewrite it as

\[
\int \chi_R e^{\xi_t} (h_t f)^{p-1} \Delta h_t f \, d\mu = -\int \langle \nabla (\chi_R e^{\xi_t} (h_t f)^{p-1}), \nabla h_t f \rangle \, d\mu
\]

\[
= -\int \chi_R \left( (p - 1)(h_t f)^{p-2} |\nabla h_t f|^2 + (h_t f)^{p-1} \langle \nabla \xi_t, \nabla h_t f \rangle \right) e^{\xi_t} \, d\mu
\]

\[
- \int e^{\xi_t} (h_t f)^{p-1} \langle \nabla \chi_R, \nabla h_t f \rangle \, d\mu
\]

\[
\leq \int \chi_R \left( - (p - 1)(h_t f)^{p-2} |\nabla h_t f|^2 + \frac{d_x}{T - qt} (h_t f)^{p-1} |\nabla h_t f| \right) e^{\xi_t} \, d\mu
\]

\[
- \int (h_t f)^{p-1} \langle \nabla \chi_R, \nabla h_t f \rangle e^{\xi_t} \, d\mu
\]

so that

\[
\frac{d}{dt} \int \chi_R(h_t f)^p e^{\xi_t} \, d\mu \leq -p \int \left( (h_t f)^{p-1} \langle \nabla \chi_R, \nabla h_t f \rangle + (p - 1)\chi_R(h_t f)^{p-2} |\nabla h_t f|^2 \right) e^{\xi_t} \, d\mu
\]

\[
+ \int \chi_R \left( \frac{pd_x}{T - qt} (h_t f)^{p-1} |\nabla h_t f| - \frac{qd_x^2}{2(T - qt)^2} (h_t f)^p \right) e^{\xi_t} \, d\mu.
\]
The fact that $h_t f \in L^p(X)$ for all $t \geq 0, 0 \leq \chi_R \leq 1$ and $\chi_R \to 1$ m.a.e. as $R \to \infty$ are sufficient to deduce by dominated convergence

$$
\lim_{R \to \infty} \int \chi_R(h_t f)^p e^{\xi t} \, dm = \int (h_t f)^p e^{\xi t} \, dm, \quad \forall t \in (0, T/q).
$$

Moreover, since $d_x^2 e^{\xi t} \in L^\infty(X)$ uniformly in $t \in [0, T/q)$, from (3.1) we see that for all $\varepsilon > 0$ there exists $R$ sufficiently large such that

$$
\int (1 - \chi_R)(h_t f)^p d_x^2 e^{\xi t} \, dm < \varepsilon, \quad \forall t \in [0, T/q)
$$

whence

$$
\lim_{R \to \infty} \int \frac{\chi_R d_x^2}{(T - qt)^2} (h_t f)^p e^{\xi t} \, dm = \int \frac{d_x^2}{(T - qt)^2} (h_t f)^p e^{\xi t} \, dm \quad \text{loc. uniformly in } t \in [0, T/q)
$$

by the arbitrariness of $\varepsilon$. Then (3.4) entails that for any $\mathcal{C} \subset (0, T/q)$ compact there exists $C > 0$ such that $|\nabla h_t f|^2 \leq C h_t f(f \log f - h_t f \log h_t f)$ for all $t \in \mathcal{C}$, whence

$$
(h_t f)^{p - 2} |\nabla h_t f|^2 e^{\xi t} \leq C(h_t f)^{p - 1} h_t f(f \log f)e^{\xi t} + C(h_t f)^p |\log h_t f|e^{\xi t}
$$

$$
C(h_t f)^{p - 1} h_t f(f \log f)e^{\xi t} + C(h_t f)^{\frac{p + 1}{2}} (h_t f)^{\frac{p + 1}{2}} |\log h_t f|e^{\xi t} =: g_t
$$

On the one hand $(h_t f)^{p - 1} e^{\xi t}$, $(h_t f)^{(p - 1)/2} |\log h_t f|$ is bounded on $[0, \|f\|_{\infty}(X)]$ uniformly in $t \in \mathcal{C}$, since $z \mapsto z^\alpha |\log z|$ is bounded on $[0, \|f\|_{\infty}(X)]$ for all $\alpha > 0$ and $(p - 1)/2 > 0$; on the other hand (3.1) applies to $h_t(f \log f)$ and $(h_t f)^{(p + 1)/2}$, since for the former the fact that $f \in L^\infty(X)$ with bounded support entails $f \log f \in L^1 \cap L^\infty(X)$ while for the latter $(p + 1)/2 \geq 1$. Thus, arguing as above, by (3.1) we have that for all $\varepsilon > 0$ there exists $R$ large enough so that

$$
\int (1 - \chi_R) (h_t f)^{p - 2} |\nabla h_t f|^2 e^{\xi t} \, dm = \int_{X \setminus B_R(x)} (1 - \chi_R)(h_t f)^{p - 2} |\nabla h_t f|^2 e^{\xi t} \, dm
$$

$$
\leq \int_{X \setminus B_R(x)} g_t \, dm < \varepsilon, \quad \forall t \in [0, T/q),
$$

whence

$$
\lim_{R \to \infty} \int \chi_R (h_t f)^{p - 2} |\nabla h_t f|^2 e^{\xi t} \, dm = \int (h_t f)^{p - 2} |\nabla h_t f|^2 e^{\xi t} \, dm \quad \text{loc. uniformly in } t \in (0, T/q).
$$

Finally, since $f \log f \leq f \log(f + 2)$ and $f \log(f + 2) > 0$ as $f > 0$, by (3.4) we have that for any $\mathcal{C} \subset (0, T/q)$ compact there exists $C > 0$ such that

$$
(h_t f)^{p - 1} |\nabla h_t f| \leq C(h_t f)^{p - 1/2} \sqrt{h_t(f \log(f + 2))} - h_t f \log h_t f
$$

$$
\leq C \frac{(h_t f)^{p - 1/2}}{\sqrt{h_t(f \log(f + 2))}} h_t(f \log(f + 2)) + C(h_t f)^p |\log h_t f|
$$

$$
\leq C \frac{1}{\log 2} (h_t f)^{p - 1} h_t(f \log(f + 2)) + C(h_t f)^p |\log h_t f|
$$

for all $t \in \mathcal{C}$, so that arguing as above, using the fact that $d_x e^{\xi t} \in L^\infty(X)$ uniformly in $t \in [0, T/q)$ and $|\nabla \chi_R| \in L^\infty(X)$ uniformly in $R > 0$ with $\chi_R, |\nabla \chi_R|$ converging m.a.e. to 0,1
respectively, we see that
\[
\lim_{R \to \infty} \int (h_t f)^{(p-1)}(\nabla R, \nabla h_t f) e^{\xi t} \, dm = 0,
\]
\[
\lim_{R \to \infty} \int \chi_{R} \frac{p d_x}{T - qt} (h_t f)^{(p-1)} |\nabla h_t f| e^{\xi t} \, dm = \int \frac{p d_x}{T - qt} (h_t f)^{(p-1)} |\nabla h_t f| e^{\xi t} \, dm,
\]
locally uniformly in \( t \in (0, T/q) \). We thus obtain that \( I \in AC_{\text{loc}}((0, T/q)) \) with
\[
I'(t) \leq \int \left( -p(p - 1)(h_t f)^{(p-2)} |\nabla h_t f|^2 + \frac{p d_x}{T - qt} (h_t f)^{(p-1)} |\nabla h_t f| - \frac{q d_x^2}{2(T - qt)^2} (h_t f)^p \right) e^{\xi t} \, dm
\]
\[
= -p(p - 1) \int (h_t f)^p \left( \frac{|\nabla h_t f|}{h_t f} - \frac{d_x}{2(p - 1)(T - t)} \right)^2 e^{\xi t} \, dm \leq 0
\]
and since \( I \) is continuous up to \( t = 0 \), this yields the conclusion. \( \square \)

4 Main result and applications

We are now in the position to prove the upper Gaussian estimate for the heat kernel, the proof being partially inspired by [15] and [14].

**Theorem 4.1** (Gaussian upper bound). With the same assumptions and notations as in Setting 2.1, for any \( \varepsilon > 0 \) there exist constants \( C_\varepsilon > 0 \) and \( C_K \geq 0 \) such that, for every \( x, y \in X \) and \( t > 0 \), it holds
\[
r_t[x](y) \leq \frac{1}{\sqrt{m(B_{\sqrt{t}}(x))m(B_{\sqrt{t}}(y))}} \exp \left( C_\varepsilon (1 + C_K t) - \frac{d^2(x, y)}{(4 + \varepsilon) t} \right). \tag{4.1}
\]
If \( K \geq 0 \), then \( C_K \) can be chosen equal to 0.

**proof** As a first step, for any \( x \in X \) and \( t, D > 0 \) define
\[
E_D(x, t) := \int (r_t[x](y))^2 \exp \left( \frac{d^2(x, y)}{D t} \right) dm(y) \in [0, \infty]
\]
and observe that by (2.4) and the triangle inequality
\[
r_t[x](y) = \exp \left( -\frac{d^2(x, y)}{2 D t} \right) \int r_{t/2}[x](z) \exp \left( \frac{d^2(x, z)}{D t} \right) r_{t/2}[z](y) \exp \left( \frac{d^2(z, y)}{D t} \right) dm(z),
\]
whence by the Cauchy-Schwarz inequality
\[
r_t[x](y) \leq \sqrt{E_D(x, t/2)E_D(y, t/2)} \exp \left( -\frac{d^2(x, y)}{2 D t} \right). \tag{4.2}
\]
In order to estimate \( E_D(x, t/2) \) and \( E_D(y, t/2) \), let \( f \in L^1 \cap L^\infty(X) \) be non-negative, \( T > 0 \) and \( p \in (1, 2) \), consider the Harnack inequality (2.1) with exponent \( 2/p \), multiply both sides by \( e^{\xi t} \) with \( \xi_t \) defined as in Lemma 3.3 and integrate w.r.t. \( m \) in the \( y \) variable to get
\[
\int (h_t f(x))^p e^{\xi_t(y)} \exp \left( -\frac{p K d^2(x, y)}{(2 - p)(e^{2Kt} - 1)} \right) dm(y) \leq \int (h_t (f^{2/p}(y)))^p e^{\xi_t(y)} dm(y).
\]
On the one hand, as $f^{2/p} \in L^p(X)$ Lemma 3.3 can be applied, whence
\[
\int (h_t(f^{2/p}))^p \exp \frac{e^{\xi_t}}{2T} \, \text{d}m \leq \int f^2 \exp \left(- \frac{d^2(x, \cdot)}{2T}\right) \, \text{d}m, \quad \forall t \in [0, T/q)
\]
with $q$ also defined as in Lemma 3.3. On the other hand
\[
\int (h_t f(x))^2 \exp \left(- \frac{pKd^2(x, y)}{(2 - p)(e^{2Kt} - 1)}\right) \, \text{d}m(y)
\geq (h_t f(x))^2 \int_{B_{\sqrt{T}}(x)} \exp \left(- \frac{pKd^2(x, y)}{(2 - p)(e^{2Kt} - 1)}\right) \, \text{d}m(y)
\geq (h_t f(x))^2 \exp \left(- \frac{t}{T-qt} - \frac{2pKt}{(2 - p)(e^{2Kt} - 1)}\right) m(B_{\sqrt{T}}(x)).
\]
By combining these inequalities we obtain
\[
(h_t f(x))^2 \leq \frac{1}{m(B_{\sqrt{T}}(x))} \exp \left(\frac{t}{T-qt} + \frac{2pKt}{(2 - p)(e^{2Kt} - 1)}\right) \int f^2 \exp \left(- \frac{d^2(x, \cdot)}{2T}\right) \, \text{d}m
\]
for all $t \in (0, T/q)$ and in particular this is true when $f$ is equal to
\[
f_n := (n \land r_t[x]) \exp \left(n \land \frac{d^2(x, \cdot)}{2T}\right),
\]
where $a \land b := \min\{a, b\}$. By the representation formula (2.3) and by the monotone convergence theorem, as the $f_n$'s are monotonically increasing, we deduce that (4.3) actually holds with $f = r_t[x] e^{\frac{d^2(x, \cdot)}{2T}}$ and this is equivalent to
\[
\int (r_t[x])^2 \exp \left(\frac{d^2(x, \cdot)}{2T}\right) \, \text{d}m \leq \frac{1}{m(B_{\sqrt{T}}(x))} \exp \left(\frac{t}{T-qt} + \frac{2pKt}{(2 - p)(e^{2Kt} - 1)}\right)
\]
for all $t \in (0, T/q)$. Recalling that $q := \frac{p}{2(p-1)}$ and $p \in (1, 2)$ is arbitrary, we observe that for all $D > 2$ there exists $p'$ sufficiently close to 2 such that $D > 2q' > 2$ with $q' = \frac{p'}{2(p-1)}$. Hence, since also $T$ is arbitrary, if we pick any $t > 0$ and set $T := Dt/2$, in such a way that $t = 2T/D < T/q'$, we deduce that (4.4) holds and
\[
\frac{t}{T-qt} = \frac{2}{D-2q'} =: C_D,
\]
so that we have just shown that for all $D > 2$ there exists $p' \in (1, 2)$ and $C_D > 0$ both depending only on $D$ such that
\[
\int (r_t[x])^2 \exp \left(\frac{d^2(x, \cdot)}{Dt}\right) \, \text{d}m \leq \frac{1}{m(B_{\sqrt{T}}(x))} \exp \left(C_D + \frac{2p'Kt}{(2 - p')(e^{2Kt} - 1)}\right), \quad \forall t > 0.
\]
Now observe that
\[
\frac{2p'Kt}{(2 - p')(e^{2Kt} - 1)} \leq C_D'(1 + C_K t), \quad \forall t > 0
\]
for suitable constants $C_D', C_K$ and it is easily seen that $C_K$ can be chosen equal to 0 when $K \geq 0$. Plugging this inequality into the previous one and recalling the definition of $E_D(x, t)$ it follows that for any $D > 2$
\[
E_D(x, t) \leq \frac{1}{m(B_{\sqrt{T}}(x))} \exp \left(C_D''(1 + C_K t)\right), \quad \forall t > 0
\]
for some $C_D'' > 0$ depending only on $D$ and combining this inequality with (4.2) the bound (4.1) follows. \[\square\]
Remark 4.2. In the case of smooth Riemannian manifolds the Varadhan asymptotic formula for short-time behaviour of the heat kernel states that
\[
\lim_{t \downarrow 0} t \log r_t[x](y) = -\frac{d^2(x, y)}{4}, \quad x \neq y.
\]
Hence (4.1) is sharp for short time. □

In the same spirit of the remark above we can formulate the following

Corollary 4.3. With the same assumptions and notations as in Setting 2.1, if we further suppose that for any \(x \in X\) there exist constants \(R_x, c_x > 0, 0 < \alpha_x < 2\) such that
\[
m(B_r(x)) \geq c_x \exp(-1/r^{\alpha_x}), \quad r < R_x,
\]
then the upper Varadhan estimate holds
\[
\lim_{t \downarrow 0} t \log r_t[x](y) \leq -\frac{d^2(x, y)}{4}
\]
for all \(x, y \in X, x \neq y\).

As a direct consequence of Proposition 3.2 and of our main theorem we obtain the local logarithmic Sobolev inequality and an upper Gaussian estimate for the heat kernel in \(\text{RCD}(K, \infty)\) spaces, which to date were both still missing. For reader’s sake, let us recall that \((X, d, m)\) satisfies the \(\text{RCD}(K, \infty)\) condition (see [5]) if it is infinitesimally Hilbertian and the relative entropy functional \(\text{Ent}_m : \mathcal{P}(X) \to \mathbb{R} \cup \{+\infty\}\) defined as
\[
\text{Ent}_m(\mu) := \begin{cases} 
\int \rho \log(\rho) \, dm & \text{if } \mu = \rho m \text{ and } (\rho \log(\rho))^- \in L^1(X) \\
+\infty & \text{otherwise}
\end{cases}
\]
is \(K\)-convex in \((\mathcal{P}_2(X), W_2)\), namely for every \(\mu, \nu \in \mathcal{P}_2(X)\) with \(\text{Ent}_m(\mu), \text{Ent}_m(\nu) < \infty\) there exists a \(W_2\)-geodesic \((\mu_t)\) with \(\mu_0 = \mu, \mu_1 = \nu\) and such that
\[
\text{Ent}_m(\mu_t) \leq (1-t)\text{Ent}_m(\mu) + t\text{Ent}_m(\nu) - \frac{K}{2} t(1-t)W_2^2(\mu, \nu), \quad \forall t \in [0, 1].
\]
In this class of spaces, H. Li proved that the dimension-free Harnack inequality (2.1) holds [20], so that we can state the following

Corollary 4.4. Let \((X, d, m)\) be an \(\text{RCD}(K, \infty)\) space with \(K \in \mathbb{R}\. Then
\(\)
(i) for any \(f \in L^p(X)\) positive, \(p \in (1, \infty)\), and for any \(t > 0\) it holds
\[
\frac{e^{2Kt} - 1}{2K} \left| \frac{\nabla h_t f}{h_t f} \right|^2 \leq h_t(f \log f) - h_t f \log h_t f \quad m\text{-a.e.}
\]
(ii) for any \(\varepsilon > 0\) there exist constants \(C_\varepsilon > 0\) and \(C_K \geq 0\) such that, for every \(x, y \in X\) and \(t > 0\), it holds
\[
r_t[x](y) \leq \frac{1}{\sqrt{m(B_{\sqrt{t}}(x))m(B_{\sqrt{t}}(y))}} \exp \left( C_\varepsilon(1 + C_K t) - \frac{d^2(x, y)}{(4 + \varepsilon)t} \right).
\]
If \(K \geq 0\), then \(C_K\) can be chosen equal to 0.
References


