PLATEAU'S PROBLEM AS A SINGULAR LIMIT OF CAPILLARITY PROBLEMS

DARREN KING, FRANCESCO MAGGI, AND SALVATORE STUVARD

Abstract. Soap films at equilibrium are modeled, rather than as surfaces, as regions of small total volume through the introduction of a capillarity problem with a homotopic spanning condition. This point of view introduces a length scale in the classical Plateau's problem, which is in turn recovered in the vanishing volume limit. This approximation of area minimizing hypersurfaces leads to an energy based selection principle for Plateau's problem, points at physical features of soap films that are unaccessible by simply looking at minimal surfaces, and opens several challenging questions.

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1. Introduction

1.1. **Overview.** The theory of minimal surfaces with prescribed boundary data provides the basic model for soap films hanging from a wire frame: given an $(n-1)$-dimensional surface $\Gamma \subset \mathbb{R}^{n+1}$ without boundary, one seeks n-dimensional surfaces M such that

$$H_M = 0\,, \qquad \partial M = \Gamma\,, \tag{1.1}$$

where H_M is the mean curvature of M (and $n = 2$ in the physical case). A limitation of (1.1) as a physical model is that, in general, (1.1) may be non-uniquely solvable, including unstable (and thus, not related to observable soap films) solutions. Area minimization can be used to construct stable (and thus, physical) solutions, providing a strong motivation for the study of *Plateau's problem.* Here we are concerned with a more elementary physical limitation of (1.1), namely, the absence of a length scale: if M solves (1.1) for Γ, then $t\,M$ solves (1.1) for $t\,\Gamma$, no matter how large $t > 0$ is.

Following [MSS19], we introduce a length scale in the modeling of soap films by thinking of them as regions $E \subset \mathbb{R}^{n+1}$ with small volume $|E| = \varepsilon$. At equilibrium, the isotropic pressure at a point y interior to the liquid but immediately close to its boundary ∂E is

$$p(y) = p_0 + \sigma\,\vec{H}_{\partial E}(y) \cdot \nu_E(y)\,, \tag{1.2}$$

where p_0 is the atmospheric pressure, σ is the surface tension, ν_E the outer unit normal to E, and $\vec{H}_{\partial E}$ the mean curvature vector of ∂E; at the same time, for any two points y, z inside the film we have

$$p(y) - p(z) = \rho\, g\,(z - y) \cdot e_{n+1}\,, \tag{1.3}$$

FIGURE 1.1. Minimizers of the capillarity problem in the unusual container Ω consisting of the complement of a δ-neighborhood $I_\delta(\Gamma)$ of a curve Γ (depicted in light gray). The shape of E is drastically different depending on whether or not a homotopic spanning condition is prescribed: (a) without a $\mathcal{C}$-spanning condition, we observe tiny droplets sitting near points of maximal mean curvature of $\partial\Omega$; (b) with a $\mathcal{C}$-spanning condition, small rounds droplets will not be admissible, and a different region of the energy landscape is explored; minimizers are now expected to stretch out and look like soap films.

where ρ is the density of the fluid, g the gravity of Earth and e_{n+1} is the vertical direction. In the absence of gravity, (1.2) and (1.3) imply that $H_E = \vec{H}_{\partial E} \cdot \nu_E$ is *constant* along ∂E. A heuristic analysis shows that if ∂E is representable, locally, by the two graphs $\{x \pm (h(x)/2)\, \nu_M(x) : x \in M\}$ defined by a positive function h over an ideal mid-surface M, then H_M should be *small, but non-zero*, with $H_M = \mathrm{O}(h^3)$, see [MSS19, Section 2]. As it is well-known, one cannot prescribe non-vanishing mean curvature with arbitrarily large boundary data, see, e.g. [Giu78, DF90]. Hence this point of view can potentially capture physical features of soap films that are not accessible by modeling them as minimal surfaces.

The goal of this paper is starting the analysis of the variational problem playing for (1.2) and (1.3) the role that Plateau's problem plays for (1.1). The new aspect is not in the energy minimized, but in the boundary conditions under which the minimization occurs. Indeed, the equivalence between the constancy of H_E and the balance equations (1.2) and (1.3), leads us to work in the classical framework of Gauss' capillarity model for liquid droplets in a container. Given an open set $\Omega \subset \mathbb{R}^{n+1}$ (the container), the surface tension energy[1] of a droplet occupying the open region $E \subset \Omega$ is given by

$$\sigma\, \mathcal{H}^n(\Omega \cap \partial E)\,,$$

where $\mathcal{H}^n$ denotes n-dimensional Hausdorff measure (surface area if $n = 2$, length if $n = 1$). In the case of soap films hanging from a wire frame Γ, we choose as container Ω the set

$$\Omega = \mathbb{R}^{n+1} \setminus I_\delta(\Gamma)\,,$$

corresponding to the complement of the "solid wire" $I_\delta(\Gamma)$, where I_δ denotes the closed δ-neighborhood of a set. The minimization of $\mathcal{H}^n(\Omega \cap \partial E)$ among open sets $E \subset \Omega$ with $|E| = \varepsilon$ leads indeed to finding minimizers whose boundaries have constant mean curvature. However, these boundaries will not resemble soap films at all, but will rather consist of small "droplets" sitting at points of maximal curvature for $I_\delta(\Gamma)$; see Figure 1.1, and [BR05, Fal10, MM16] for more information.

To observe soap films, rather than droplets, we must require that ∂E stretches out to span $I_\delta(\Gamma)$. To this end, we exploit a beautiful idea introduced by Harrison and Pugh in [HP16a], as slightly generalized in [DLGM17]. The idea is fixing a **spanning class**, i.e. a homotopically closed family $\mathcal{C}$ of smooth embeddings of $\mathbb{S}^1$ into $\Omega = \mathbb{R}^{n+1} \setminus I_\delta(\Gamma)$, and to say that a relatively closed set $S \subset \Omega$ is **$\mathcal{C}$-spanning** $I_\delta(\Gamma)$ if

$$S \cap \gamma \neq \emptyset \qquad \forall \gamma \in \mathcal{C}\,. \tag{1.4}$$

[1]For simplicity, we are setting to zero the adhesion coefficient with the container; see, e.g. [Fin86].

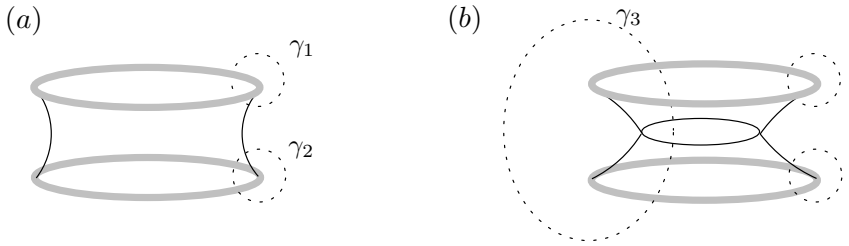

FIGURE 1.2. The variational problem (1.5) with Γ given by two parallel circles centered on the same axis at a mutual distance smaller than their common radius. Different choices of $\mathcal{C}$ lead to different minimizers S in ℓ: (a) if $\mathcal{C}$ is generated by the loops γ_1 and γ_2, then S is the area minimizing catenoid; (b) if we add to $\mathcal{C}$ the homotopy class of γ_3, then S is the *singular* area minimizing catenoid, consisting of two catenoidal necks, meeting at equal angles along a circle of Y-points bounding a "floating" disk. Such singular catenoid cannot be approximated in energy by smooth surfaces: hence the choice of casting ℓ in a class of non-smooth surfaces.

Given a choice of $\mathcal{C}$, we have a corresponding version of Plateau's problem

$$\ell = \inf\Big\{\mathcal{H}^n(S) : S \text{ is relatively closed in } \Omega, \text{ and } S \text{ is } \mathcal{C}\text{-spanning } I_\delta(\Gamma)\Big\}\,, \tag{1.5}$$

as illustrated in Figure 1.2. The variational problem $\psi(\varepsilon)$ studied here is thus a reformulation of ℓ as a capillarity problem with a homotopic spanning condition, namely:

$$\psi(\varepsilon) = \inf\Big\{\mathcal{H}^n(\Omega\cap\partial E) : E\subset\Omega,\, |E|=\varepsilon,\, \Omega\cap\partial E \text{ is } \mathcal{C}\text{-spanning } I_\delta(\Gamma)\Big\}\,, \qquad \varepsilon>0\,.$$

We now give *informal* statements of our main results (e.g., we make no mention to singular sets or comment on reduced vs topological boundaries); see section 1.2 for the formal ones.

Existence of generalized minimizers and Euler-Lagrange equations (Theorem 1.4 and Theorem 1.6): *There always exists a generalized minimizer (K,E) for $\psi(\varepsilon)$: that is, there exists a set $K\subset\Omega$, relatively closed in Ω and $\mathcal{C}$-spanning $I_\delta(\Gamma)$, and there exists an open set $E\subset\Omega$ with $\Omega\cap\partial E\subset K$ and $|E|=\varepsilon$, such that*

$$\psi(\varepsilon) = \mathcal{F}(K,E) = 2\,\mathcal{H}^n(K\setminus\partial E) + \mathcal{H}^n(\Omega\cap\partial E)\,.$$

Moreover, (K,E) minimizes $\mathcal{F}$ with respect to all its diffeomorphic images: in particular, $\Omega\cap\partial E$ has constant mean curvature $\lambda\in\mathbb{R}$ and $K\setminus\partial E$ has zero mean curvature.

Convergence to the Plateau's problem (Theorem 1.9): *We always have $\psi(\varepsilon)\to 2\,\ell$ when $\varepsilon\to 0^+$, and if (K_j,E_j) are generalized minimizers for $\psi(\varepsilon_j)$ with $\varepsilon_j\to 0^+$, then, up to extracting subsequences, we can find a minimizer S for ℓ with*

$$2\int_{K_j\setminus\partial E_j}\varphi + \int_{\partial E_j}\varphi \to 2\int_S\varphi \qquad \forall\varphi\in C^0_c(\Omega)\,,$$

as $j\to\infty$; in other words, generalized minimizers in $\psi(\varepsilon_j)$ with $\varepsilon_j\to 0^+$ converge as Radon measures to minimizers in the Harrison-Pugh formulation of Plateau's problem.

Example 1.1 (Volume and thickness in the non-collapsed case)**.** Consider the case when Γ consists of two points at distance r in the plane, or, more generally, of an $(n-1)$-dimensional sphere of radius r in $\mathbb{R}^{n+1}$. For ε small enough, $\psi(\varepsilon)$ should admit a unique generalized minimizer (K,E), consisting of two almost flat spherical caps meeting orthogonally along the torus $I_\delta(\Gamma)$ (so that $K=\partial E$ and collapsing does not occur); see Figure 1.3-(a). More generally, we expect that *when all the minimizers S in ℓ are non-singular, then*

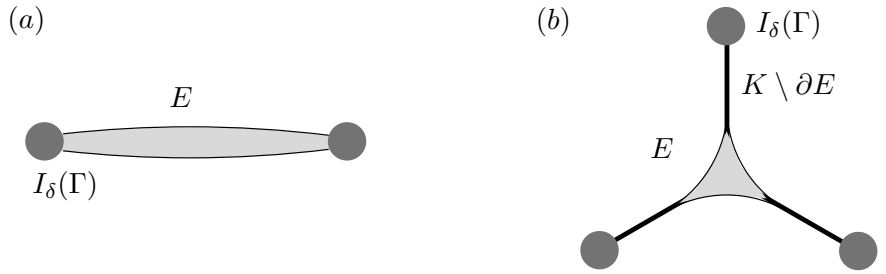

FIGURE 1.3. (a) If Γ consists of two points, then the minimizer is not collapsed, and is bounded by two very flat circular arcs; (b) when Γ consists of the vertexes of an equilateral triangle, the generalized minimizer is indeed collapsed. The three segments defining $K \setminus \partial E$ are depicted in bold, and E is a negatively curved curvilinear triangle nested around the singular point of the unique minimizer of ℓ.

generalized minimizers in $\psi(\varepsilon)$ are not collapsed, and, for small ε, $K = \partial E$ is a two-sided approximation of S, with constant mean curvature $\psi'(\varepsilon) \to 0$ as $\varepsilon \to 0^+$ and with

$$\psi(\varepsilon) = 2\,\ell + C\,\varepsilon^2 + \mathrm{o}(\varepsilon^2)\,, \qquad \text{as } \varepsilon \to 0^+\,, \tag{1.6}$$

for a positive constant C. This insight is consistent with the idea, proposed in [MSS19], that *almost minimal surfaces* arise in studying soap films with a thickness. In particular, *volume and thickness will be directly related* in terms of the geometry of Γ. Sending $\varepsilon \to 0^+$ with Γ fixed or, equivalently, considering $t\,\Gamma$ for large t at ε fixed, will make the thickness decrease until it reaches a threshold below which we do not expect soap films to form. A critical thickness can definitely be identified with the characteristic length scale of the molecules of surfactant, around which the model itself stops making sense. But depending on temperatures, actual soap films with even larger thicknesses should burst out due to the increased probability of fluctuations towards unstable configurations.

Example 1.2 (Volume and thickness in the collapsed case)**.** At small volumes, and in presence of singularities in the minimizers of ℓ, collapsing is energetically convenient, and allows $\psi(\varepsilon)$ to approximate $2\,\ell$ from below. If $\Gamma \subset \mathbb{R}^2$ consists of the three vertexes of an equilateral triangle, for small δ the unique minimizer of ℓ consists of a Y-configuration. For small ε, we expect generalized minimizers (K, E) of $\psi(\varepsilon)$ to be collapsed, see Figure 1.3-(b): there, E is a curvilinear triangle made up of three circular arcs whose length is $\mathrm{O}(\sqrt{\varepsilon})$, and whose (negative) curvature is $\mathrm{O}(1/\sqrt{\varepsilon})$. The thickness of an actual soap film in this configuration should thus be considerably larger near the singularity than along the collapsed region, and the volume and the thickness of the film are somehow *independent* geometric quantities. This suggests that, in presence of singularities, the use of the Allen-Cahn model (or of another diffused interface model with an additional length scale built into the definition of the surface energy) should capture features of soap films that are not accessible by $\psi(\varepsilon)$.

Example 1.3 (Capillarity as a selection principle for Plateau's problem)**.** The following statement holds (as a heuristic principle): *Generalized minimizers of $\psi(\varepsilon)$ converge to those minimizers of Plateau's problem* (1.5) *with larger singular set, and when no singular minimizers are present, they select those whose second fundamental form has maximal L^2-norm.* Since the second part of this selection principle is justified by standard second variation arguments, we illustrate the first part only. In Figure 1.4, Γ is either given by four or by six points, that are suitably spaced so that ℓ has different minimizers. As $\varepsilon \to 0^+$, $\psi(\varepsilon)$ selects those ℓ-minimizers with singularities over the ones without singularities; and when more minimizers with singularities are present, it selects the ones with the largest number of singularities. Indeed, the approximation of a smooth minimizer in ℓ will require

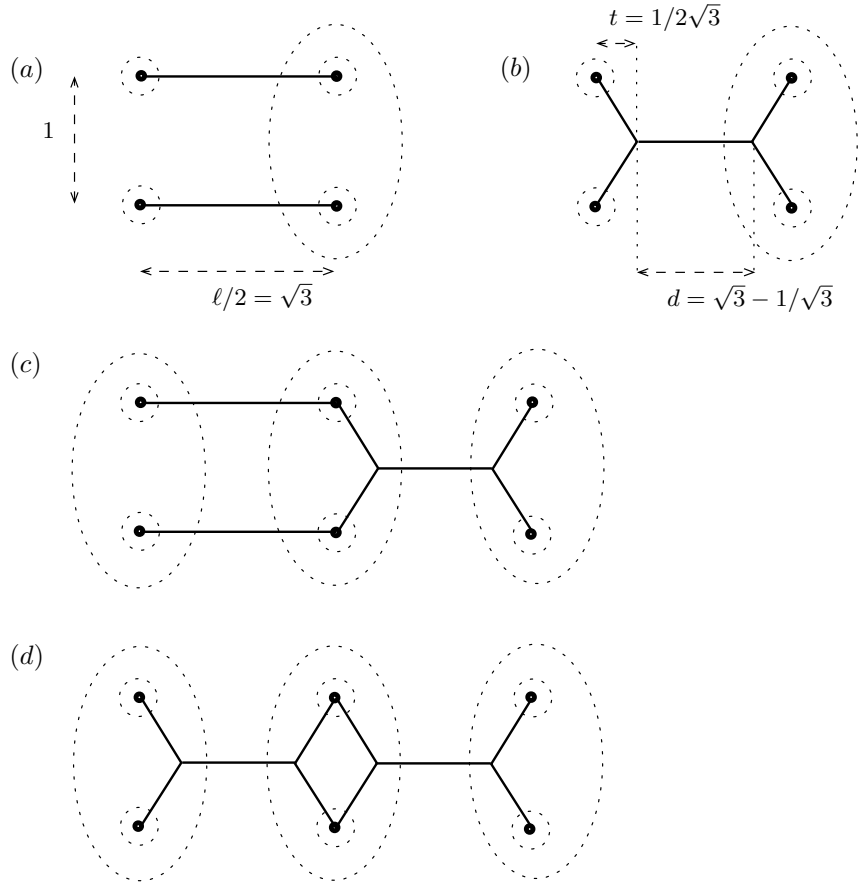

FIGURE 1.4. (a) and (b): a four points configuration Γ with a choice of $\mathcal{C}$ such that ℓ admits two minimizers, one with and one without singularities; (c) and (d): a six points configuration Γ with a choice of $\mathcal{C}$ such that ℓ admits many minimizers, possibly with a variable number of singularities; here we have depicted two of them, including the one with four singular points that is selected by the $\psi(\varepsilon)$ problems.

an energy cost larger than $2\,\ell$. At the same time, each time a singularity is present, minimizers of $\psi(\varepsilon)$ can save length in the approximation, thus paying less than $2\,\ell$ in energy, and the more the singularities, the bigger the gain. To check this claim, pick N singularities, and denote by ε_i the volume placed near the i-th singularity and by r_i the radius of the three circular arcs enclosing ε_i. Each wetted singularity has area $c_1\, r_i^2$, while the total relaxed energy of the approximating configuration is $\mathcal{F} = 2\,\ell - c_2 \sum_{i=1}^N r_i$. Minimizing under the constraint $\varepsilon = c_1 \sum_{i=1}^N r_i^2$, we must take $r_i = \sqrt{\varepsilon/Nc_1}$, thus finding

$$\psi(\varepsilon) = 2\,\ell - c_2 \sqrt{\frac{\varepsilon\, N_{\max}}{c_1}}\,,$$

if $N_{\max}$ is the maximal number of singularities available among minimizers of ℓ. This example suggests that (in every dimension) *in the presence of singular minimizers of ℓ, one should have*

$$\psi'(\varepsilon) \to -\infty \text{ as } \varepsilon \to 0^+\,. \tag{1.7}$$

This is of course markedly different from what we expect to be the situation when ℓ has only smooth minimizers, see (1.6). We finally notice that a selection principle for the capillarity model (without homotopic spanning conditions) via its Allen-Cahn approximation has been recently obtained by Leoni and Murray, see [LM16, LM17].

1.2. **Statements of the results.** We now give a more technical introduction to our paper, with precise statements, more bibliographical references, and comments on the proofs.

Plateau's problem with homotopic spanning: We fix a compact set $W \subset \mathbb{R}^{n+1}$ (the "wire frame") and denote the region accessible by the soap film as

$$\Omega = \mathbb{R}^{n+1} \setminus W\,.$$

The typical case we have in mind is $W = I_\delta(\Gamma)$, as discussed in section 1.1, but this is not necessary. We fix a **spanning class** $\mathcal{C}$, that is a non-empty family of smooth embeddings of $\mathbb{S}^1$ into Ω which is closed by homotopy in Ω. We assume that W and $\mathcal{C}$ are such that the **Plateau's problem defined by** $\mathcal{C}$

$$\ell = \inf\big\{\mathcal{H}^n(S) : S \in \mathcal{S}\big\} \tag{1.8}$$

is such that $\ell < \infty$. Here, for the sake of brevity, we have introduced

$$\mathcal{S} = \big\{S \subset \Omega : S \text{ is relatively closed in } \Omega \text{ and } S \text{ is } \mathcal{C}\text{-spanning } W\big\}\,.$$

As proved in [HP16a, DLGM17], if $\ell < \infty$, then there exists a compact, $\mathcal{H}^n$-rectifiable set S such that $\mathcal{H}^n(S) = \ell$; see also [Har14, Dav14, Fan16, HP16b, HP16c, DPDRG16, DLDRG19, GLF17, HP17, FK18, DR18] for related existence results. In addition, S minimizes $\mathcal{H}^n$ with respect to Lipschitz perturbations of the identity localized in Ω, so that: (i) S is a classical minimal surface outside of an $\mathcal{H}^n$-negligible, relatively closed set in Ω by [Alm76]; (ii) if $n = 1$, S consists of finitely many segments, possibly meeting in three equal angles at singular Y-points in Ω; (iii) if $n = 2$, S satisfies **Plateau's laws** by [Tay76]: namely, S is locally diffeomorphic either to a plane, or to a cone $Y = T^1 \times \mathbb{R}$, or to a cone T^2, where T^n is the cone over the origin defined by the $(n-1)$-dimensional faces of a regular tetrahedron in $\mathbb{R}^{n+1}$. The validity of Plateau's laws in this context makes (1.8) more suitable when one is motivated by physical considerations: indeed, minimizers of the codimension one Plateau's problem in the class of rectifiable currents are necessarily smooth if $n \le 6$. Although smoothness is desirable for geometric applications, it creates an *a priori* limitation when studying actual soap films; see also [Dav14, HP16a, DLGM17].

The capillarity problem and the relaxed energy: Next, we give a precise formulation of the capillarity problem $\psi(\varepsilon)$ at volume $\varepsilon > 0$, which is defined as

$$\psi(\varepsilon) = \inf\Big\{\mathcal{H}^n(\Omega \cap \partial E) : E \in \mathcal{E},\ |E| = \varepsilon,\ \Omega \cap \partial E \text{ is } \mathcal{C}\text{-spanning } W\Big\}\,. \tag{1.9}$$

Here we have introduced the family of sets

$$\mathcal{E} = \Big\{E \subset \Omega :\ E \text{ is an open set and } \partial E \text{ is } \mathcal{H}^n\text{-rectifiable}\Big\}\,. \tag{1.10}$$

If $E \in \mathcal{E}$, then ∂E is $\mathcal{H}^n$-finite and covered by countably many Lipschitz images of $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$. Thus, E is of finite perimeter in Ω by a classical result of Federer, and its (distributional) perimeter $P(E;U)$ in an open set $U \subset \Omega$ is equal to $\mathcal{H}^n(U \cap \partial^* E)$, where $\partial^* E$ is the reduced boundary of E (notice that, in general, $P(E;U) \le \mathcal{H}^n(U \cap \partial E)$). The relaxed energy $\mathcal{F}$ is defined by

$$\begin{aligned}
\mathcal{F}(K,E;U) &= \mathcal{H}^n(U \cap \partial^* E) + 2\,\mathcal{H}^n\big(U \cap (K \setminus \partial^* E)\big)\,, \qquad U \subset \Omega \text{ open}\,,\\
\mathcal{F}(K,E) &= \mathcal{F}(K,E;\Omega)\,,
\end{aligned}$$

on every pair (K,E) in the family $\mathcal{K}$ given by

$$\mathcal{K} = \Big\{(K,E) : E \subset \Omega \text{ is open with } \Omega \cap \mathrm{cl}\,(\partial^* E) = \Omega \cap \partial E \subset K\,,$$
$$K \in \mathcal{S} \text{ and } K \text{ is } \mathcal{H}^n\text{-rectifiable in } \Omega\Big\}\,.$$

By the requirement $K \in \mathcal{S}$, K is $\mathcal{C}$-spanning W, while $\Omega \cap \partial E$, which is always a subset of K, may be not be $\mathcal{C}$-spanning W; we expect this when collapsing occurs, see Figure 1.3.

Assumptions on Ω: We make two main assumptions on W and $\mathcal{C}$. Firstly, we assume that $\partial\Omega = \partial W$ is a C^2-regular hypersurface in $\mathbb{R}^{n+1}$ and that

$$\exists\, \tau_0 > 0 \text{ such that, for every } \tau < \tau_0,\ \mathbb{R}^{n+1} \setminus I_\tau(W) \text{ is connected}\,. \tag{1.11}$$

This is compatible with the idea that, in the physical case $n = 2$, W represents a "solid wire". For example, when Γ is a piecewise smooth curve we can simply let W be a smoothing of (the non-smooth set) $I_\delta(\Gamma)$. Secondly, we require that

$$\exists\, \eta_0 > 0 \text{ and a minimizer } S \text{ in } \ell \text{ s.t. } \gamma \setminus I_{\eta_0}(S) \neq \emptyset \text{ for every } \gamma \in \mathcal{C}\,, \tag{1.12}$$

to verify the finiteness of $\psi(\varepsilon)$. We easily see that (1.12) is very general: for example, by the convex hull property of stationary varifolds, denoting by Z the closed convex hull of W, we see that (1.12) holds if $\gamma \setminus Z \neq \emptyset$ for every $\gamma \in \mathcal{C}$; at the same time, it is easy to think about situations where the latter property does not hold, but still (1.12) does. Since (1.12) seems to be quite generic, we shall just accept it as a reasonable assumption, without further discussing possible necessary conditions for it.

Existence of minimizers and Euler-Lagrange equations: We start with the existence of generalized minimizers:

Theorem 1.4 (Existence of generalized minimizers)**.** *Let $\ell < \infty$, ∂W be of class C^2 and let* (1.11) *and* (1.12) *hold. If $\{E_j\}_j$ is a minimizing sequence for $\psi(\varepsilon)$, then there exists a pair $(K, E) \in \mathcal{K}$ with $|E| = \varepsilon$ such that, possibly up to extracting subsequences,*

$$\begin{aligned} &E_j \to E \text{ in } L^1(\Omega)\,, \\ &\mathcal{H}^n \llcorner (\Omega \cap \partial E_j) \overset{*}{\rightharpoonup} \theta\, \mathcal{H}^n \llcorner K \qquad \text{as Radon measures in } \Omega\,, \end{aligned} \tag{1.13}$$

as $j \to \infty$, where $\theta : K \to \mathbb{R}$ is an upper semicontinuous function with

$$\theta = 2 \ \mathcal{H}^n\text{-a.e. on } K \setminus \partial^* E, \qquad \theta = 1 \text{ on } \Omega \cap \partial^* E\,. \tag{1.14}$$

Moreover, $\psi(\varepsilon) = \mathcal{F}(K, E)$ and, for a suitable constant C, $\psi(\varepsilon) \le 2\,\ell + C\,\varepsilon^{n/(n+1)}$.

Remark 1.5. Whenever $(K, E) \in \mathcal{K}$ is such that $|E| = \varepsilon$, $\mathcal{F}(K, E) = \psi(\varepsilon)$ and there exists a minimizing sequence $\{E_j\}_j$ for $\psi(\varepsilon)$ which converges to (K, E) as in (1.13), we say that (K, E) is a **generalized minimizer of** $\psi(\varepsilon)$. We say that (K, E) is **collapsed** if $K \setminus \partial E \neq \emptyset$. If (K, E) is not collapsed, then E is a (standard) minimizer of $\psi(\varepsilon)$.

Next we derive the Euler-Lagrange equations for generalized minimizers, showing that the collapsed region is stationary, and the non-collapsed region has constant mean curvature. Smoothness outside of a closed singular set follows by Allard's theorem.

Theorem 1.6 (Euler-Lagrange equation for generalized minimizers)**.** *Let $\ell < \infty$, ∂W be of class C^2 and let* (1.11) *and* (1.12) *hold. If (K, E) is a generalized minimizer of $\psi(\varepsilon)$ and $f : \Omega \to \Omega$ is a diffeomorphism such that $|f(E)| = |E|$, then*

$$\mathcal{F}(K, E) \le \mathcal{F}(f(K), f(E))\,. \tag{1.15}$$

In particular:

(i) *there exists $\lambda \in \mathbb{R}$ such that*

$$\lambda \int_{\partial^* E} X \cdot \nu_E \, d\mathcal{H}^n = \int_{\partial^* E} \operatorname{div}^K X \, d\mathcal{H}^n + 2 \int_{K \setminus \partial^* E} \operatorname{div}^K X \, d\mathcal{H}^n\,, \tag{1.16}$$

for every $X \in C^1(\Omega; \mathbb{R}^{n+1})$ with $X \cdot \nu_\Omega = 0$ on $\partial\Omega$, where div^K denotes the tangential divergence along K;

(ii) *there exists $\Sigma \subset K$, closed and with empty interior in K, such that $K \setminus \Sigma$ is a smooth hypersurface, $K \setminus (\Sigma \cup \partial E)$ is a smooth embedded minimal hypersurface, $\mathcal{H}^n(\Sigma \setminus \partial E) = 0$, $\Omega \cap (\partial E \setminus \partial^* E) \subset \Sigma$ has empty interior in K, and $\Omega \cap \partial^* E$ is a smooth embedded hypersurface with constant scalar (w.r.t. ν_E) mean curvature λ.*

Remark 1.7. Although we do not pursue this point here, we mention that we would expect (K, E) to be a proper minimizer of $\mathcal{F}$ among pairs $(K', E') \in \mathcal{K}$ with $|E'| = \varepsilon$ (and not just when $K' = f(K)$ for a diffeomorphism f, as proved in (1.15)). To show this we would need to approximate in energy a generic (K', E') by competitors $\{F_j\}_j$ for $\psi(\varepsilon)$. The natural *ansatz* for this approximation would be taking $F_j = U_{\eta_j}(K' \cup E') \setminus I_{\eta_j}(K' \cap E')$ for $\eta_j \to 0^+$, where U_η denotes the *open* η-neighborhood of a set. The convergence of this approximation is delicate, and can be made to work by elaborating on the ideas contained in [ACV08, Vil09] at least for (K', E') in certain subclasses of $\mathcal{K}$.

Remark 1.8. Theorem 1.6 points at two interesting free boundary problems. The first one concerns the transition region $\partial E \setminus \partial^* E$ which, at least in principle, could have positive $\mathcal{H}^n$-measure. The second one concerns the wetted region $\partial\Omega \cap \partial E$, which could either be $\mathcal{H}^n$-negligible or not, recall Figure 1.3: in the former case, $\partial\Omega \cap \partial E$ should be $(n-1)$-dimensional, while in the latter case $\partial\Omega \cap \partial E$ should be a set of finite perimeter inside $\partial\Omega$, and Young's law $\nu_\Omega \cdot \nu_E = 0$ should hold at generic boundary points of $\partial\Omega \cap \partial E$ relative to $\partial\Omega$; see for example [DPM15, DPM17].

Convergence towards Plateau's problem: The next theorem establishes the nature of Plateau's problem ℓ as the singular limit of the capillarity problems $\psi(\varepsilon)$ as $\varepsilon \to 0^+$.

Theorem 1.9 (Plateau's problem as a singular limit of capillarity problems)**.** *If $\ell < \infty$, ∂W is of class C^2, and* (1.11) *and* (1.12) *hold, then ψ is lower semicontinuous on* $(0, \infty)$ *and*

$$\lim_{\varepsilon \to 0^+} \psi(\varepsilon) = 2\,\ell\,. \tag{1.17}$$

In addition, if $\{(K_h, E_h)\}_h$ is a sequence of generalized minimizers of $\psi(\varepsilon_h)$ for $\varepsilon_h \to 0^+$ as $h \to \infty$, then there exists a minimizer S in ℓ such that, up to extracting subsequences and as $h \to \infty$,

$$\mathcal{H}^n \llcorner (\Omega \cap \partial^* E_h) + 2\,\mathcal{H}^n \llcorner (K_h \setminus \partial^* E_h) \overset{*}{\rightharpoonup} 2\,\mathcal{H}^n \llcorner S\,, \qquad \text{as Radon measures in } \Omega\,. \tag{1.18}$$

Remark 1.10. The behavior of $\psi(\varepsilon) - 2\,\ell$ as $\varepsilon \to 0^+$ is expected to depend heavily on whether minimizers of ℓ have or do not have singularities, as noticed in (1.6) and (1.7).

Remark 1.11. Showing the Hausdorff convergence of K_h to S seems, in general, delicate. Given (1.18), Hausdorff convergence would follow from an area lower bound on K_h. In turn, this could be deduced (thanks to area monotonicity) from a uniform L^p-bound, for some $p > n$, on the mean curvature vectors $\vec{H}_{V_h}$ of the integer varifolds V_h supported on K_h, with multiplicity 2 on $K_h \setminus \partial^* E_h$, and multiplicity 1 on $\partial^* E_h$. Notice however that, by (1.16), if λ_h is the Lagrange multiplier of (K_h, E_h), then $\vec{H}_{V_h} = \lambda_h\, \nu_{E_h}\, 1_{\partial^* E_h}$, so that, even when $n = 1$, the only uniform L^p-bound that can hold is the one with $p = 1$; see Example 1.2.

Proofs: The approach to the existence of generalized minimizers of $\psi(\varepsilon)$ is similar in spirit to the one adopted in [DLGM17] when studying ℓ. The idea there is addressing the convergence of minimizing sequences for ℓ by exploiting two canonical type of local competitors, the so-called cone and cup competitors. To tackle $\psi(\varepsilon)$, a richer class of canonical competitors is needed. One reason is that our competitors have a boundary structure, an aspect that, for example, forces us to consider two different kinds of cup competitors. Another reason is that, having to deal with generalized minimizers, in order to prove that the limit energy is equal to $\psi(\varepsilon)$ we need to exclude various kinds of concentration phenomena: by multiple folding, near the boundary, or at infinity (with possible loss of volume). In addition to that, after taking local variations, we need to adjust volumes: this issue requires the construction of uniform volume-fixing variations along the minimizing sequence, a technique inspired by [Alm76]. These different tools are presented in section 2,

while the proof of Theorem 1.4 is discussed in section 3. Theorem 1.6 is proved in section 4 starting from some ideas originating in [DPH03], and further developed in [DLGM17] when addressing David's sliding minimizers. The key difference is that, in our setting, one cannot use comparison by maps that locally project competitors onto planes, and has thus to resort to certain canonical competitors (called "slab" competitors), which also turn out handy in excluding concentration by folding in Theorem 1.4. In section 5 we prove Theorem 1.9: the main difficulty, explained there in more detail, is that, at vanishing volume, we have no non-trivial local limit sets to be used for constructing uniform volume-fixing variations.

Structure of generalized minimizers: Theorem 1.4, Theorem 1.6 and Theorem 1.9 lay the foundations to study the properties of generalized minimizers of $\psi(\varepsilon)$. The most intriguing questions are concerned with the relations between the properties of minimizers in Plateau's problem ℓ, like the presence or the absence of singularities, and the properties of minimizers in $\psi(\varepsilon)$ at small ε: collapsing vs non-collapsing and the sign of λ, limiting behavior of λ as $\varepsilon \to 0^+$, dimensionality of the wetted part of the wire, etc. This is of course a very large set of problems, which will require further investigations. In the forthcoming paper [KMS19], we start this kind of study by proving that collapsed minimizers have non-positive Lagrange multipliers, and deduce from this property that they satisfy the convex hull property.

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2. Cone, cup and slab competitors, nucleation and collapsing

Section 2.1 contains the notation and terminology used in the paper. Section 2.2 collects some basic properties of $\mathcal{C}$-spanning sets. In section 2.3 we construct the three types of cup competitors needed in our arguments, and similarly sections 2.4 and 2.5 deal with slab and cone competitors. Section 2.6 contains a basic nucleation lemma, needed to find uniform chunks of volume along sequences of sets, while section 2.7 concerns density lower bounds for collapsing sequences of sets of finite perimeter.

2.1. **Notation and terminology.** We denote by $|A|$ and $\mathcal{H}^s(A)$ the Lebesgue and the s-dimensional Hausdorff measures of $A \subset \mathbb{R}^{n+1}$, by $I_\eta(A)$ and $U_\eta(A)$ the closed and open η-neighborhoods of A, by $B_r(x)$ the open ball of center at x and radius r. We work in the framework of [Sim83, AFP00, Mag12]. Given $k \in \mathbb{N}$, $1 \le k \le n$, a Borel set $M \subset \mathbb{R}^{n+1}$ is **countably $\mathcal{H}^k$-rectifiable** if it is covered by countably many Lipschitz images of $\mathbb{R}^k$; it is **(locally) $\mathcal{H}^k$-rectifiable** if, in addition, M is (locally) $\mathcal{H}^k$-finite. If M is locally $\mathcal{H}^k$-rectifiable, then for $\mathcal{H}^k$-a.e. $x \in M$ there exists a unique k-plane T_xM such that, as $r \to 0^+$, $\mathcal{H}^k \llcorner (M-x)/r \overset{*}{\rightharpoonup} \mathcal{H}^k \llcorner T_xM$ as Radon measures in $\mathbb{R}^{n+1}$; T_xM is called the **approximate tangent plane to M at x**. Given a Lipschitz map $f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, we denote by $J^M f$ its **tangential jacobian along** M, so that if f is smooth and $f(x) = x + t\,X(x) + \mathrm{o}(t)$ in C^1 as $t \to 0^+$, then $J^M f = 1 + t \operatorname{div}^M X + \mathrm{o}(t)$ where $\operatorname{div}^M X$ is the **tangential divergence of X along M**; moreover, M has **distributional mean curvature vector $\vec{H} \in L^1_{\mathrm{loc}}(U; \mathcal{H}^k \llcorner M)$ in** U open, if

$$\int_M \operatorname{div}^M X \, d\mathcal{H}^k = \int_M X \cdot \vec{H} \, d\mathcal{H}^k \,, \qquad \forall X \in C^\infty_c(U; \mathbb{R}^{n+1}) \,,$$

see [Sim83, Sections 8 and 9]. A Borel set $E \subset \mathbb{R}^{n+1}$ has **finite perimeter** if there exists an $\mathbb{R}^{n+1}$-valued Radon measure on $\mathbb{R}^{n+1}$, denoted by μ_E, such that $\langle \mu_E, X\rangle = \int_E \operatorname{div} X$ whenever $X \in C^1_c(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$ and $P(E;\mathbb{R}^{n+1}) = |\mu_E|(\mathbb{R}^{n+1}) < \infty$. The set of points $x \in \mathbb{R}^{n+1}$ such that $|\mu_E|(B_r(x))^{-1}\,\mu_E(B_r(x)) \to \nu_E(x) \in \mathbb{S}^n$ as $r \to 0^+$ is denoted by $\partial^* E$, and called the **reduced boundary** $\partial^* E$ of E. Then $\mu_E = \nu_E\,\mathcal{H}^n \llcorner \partial^* E$, $\partial^* E$ is $\mathcal{H}^n$-rectifiable in $\mathbb{R}^{n+1}$, and $T_x\partial^* E = \nu_E(x)^\perp$ for every $x \in \partial^* E$. The **set** $E^{(t)}$ **of points of density** $t \in [0,1]$ **of** E is given by those $x \in \mathbb{R}^{n+1}$ with $|E \cap B_r(x)|/|B_r(x)| \to t$ as $r \to 0^+$, and (see, e.g., see [Mag12, Theorem 16.2]),

$$\{\partial^* E, E^{(0)}, E^{(1)}\} \text{ is a partition of } \mathbb{R}^{n+1} \text{ modulo } \mathcal{H}^n\,. \tag{2.1}$$

Federer's criterion [Fed69, 4.5.11] states that if the **essential boundary** $\partial^{\mathrm{e}} E = \mathbb{R}^{n+1} \setminus (E^{(0)} \cup E^{(1)})$ is $\mathcal{H}^n$-finite, then E is of finite perimeter in $\mathbb{R}^{n+1}$. If E is open, then $\partial^{\mathrm{e}} E \subset \partial E$: hence, if $E \in \mathcal{E}$ and $\mathcal{H}^n(\partial\Omega) < \infty$, then E is of finite perimeter.

2.2. **Some preliminary results.** In the following, W is a compact set, $\mathcal{C}$ a spanning class for W and $\Omega = \mathbb{R}^{n+1} \setminus W$.

Lemma 2.1. *If $\{K_j\}_j$ are relatively closed sets in Ω, such that each K_j is $\mathcal{C}$-spanning W and $\mathcal{H}^n \llcorner K_j \overset{*}{\rightharpoonup} \mu$ as Radon measures in Ω, then $K = \Omega \cap \operatorname{spt}\mu$ is $\mathcal{C}$-spanning W.*

Proof. See [DLGM17, Step 2, proof of Theorem 4]. □

Lemma 2.2. *Let K be relatively closed in Ω and let $B_r(x) \subset\subset \Omega$. Then K is $\mathcal{C}$-spanning W if and only if, whenever $\gamma \in \mathcal{C}$ is such that $\gamma \cap K \setminus B_r(x) = \emptyset$, then there exists a connected component of $\gamma \cap \operatorname{cl}(B_r(x))$ which is diffeomorphic to an interval, and whose end-points belong to distinct connected components of $\operatorname{cl}(B_r(x)) \setminus K$, as well as to distinct components of $\partial B_r(x) \setminus K$.*

Proof. This is [DLGM17, Lemma 10]. □

Lemma 2.3. *If K is $\mathcal{C}$-spanning W, $B_r(x) \subset\subset \Omega$, and $f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is a bi-Lipschitz map with $\{f \neq \mathrm{id}\} \subset\subset B_r(x)$ and $f(B_r(x)) \subset B_r(x)$, then $f(K)$ is $\mathcal{C}$-spanning W.*

Proof. By $f(K) \setminus B_r(x) = K \setminus B_r(x)$, if $f(K)$ is not $\mathcal{C}$-spanning W, then there exists $\gamma \in \mathcal{C}$ with $\gamma \cap K \setminus B_r(x) = \emptyset$ such that $\gamma \cap f(K) = \emptyset$. Hence, the curve $\tilde\gamma := f^{-1} \circ \gamma$ is a continuous embedding of $\mathbb{S}^1$ in Ω, homotopic to γ in Ω, and such that $\tilde\gamma \cap K = \emptyset$. Since $\tilde\gamma$ and W are compact and K is closed, $\tilde\gamma$ has positive distance from $K \cup W$, and by smoothing out $\tilde\gamma$ we define a smooth embedding $\hat\gamma$ of $\mathbb{S}^1$ into Ω, disjoint from K, and homotopic to $\tilde\gamma$ (and therefore to γ) in Ω, a contradiction. □

In the proof of Theorem 1.4 we will need the following boundary version of Lemma 2.3.

Lemma 2.4. *If Ω has smooth boundary, then there exists $r_0 > 0$ with the following property. If $x \in \partial\Omega$, $\Omega \subset \Omega'$, $f : \operatorname{cl}(\Omega) \to \operatorname{cl}(\Omega') = f(\operatorname{cl}(\Omega))$ is a diffeomorphism with $f(\partial\Omega) = \partial\Omega'$, $\{f \neq \mathrm{id}\} \subset\subset B_{r_0}(x)$, and $f(B_{r_0}(x)) = B_{r_0}(x)$, and if K is $\mathcal{C}$-spanning W, then $K' = f(K \cap \Omega^*)$ is relatively closed in Ω and is $\mathcal{C}$-spanning W, where $\Omega^* = f^{-1}(\Omega)$.*

Proof. Step one: We show that, for K relatively closed in Ω and $B_{r_0}(x)$ as in the statement, K is $\mathcal{C}$-spanning W if and only if, whenever $\gamma \in \mathcal{C}$ is such that $\gamma \cap K \setminus B_{r_0}(x) = \emptyset$, then there exists a connected component of $\gamma \cap \operatorname{cl}(B_{r_0}(x))$, diffeomorphic to an interval, and whose end-points belong to distinct connected components of $\Omega \cap \operatorname{cl}(B_{r_0}(x)) \setminus K$. We only prove the "only if" part. Arguing as in [DLGM17, Step two, proof of Lemma 10] we can assume that γ and $\partial B_{r_0}(x)$ intersect transversally, so that there exist finitely many disjoint $I_i = [a_i, b_i] \subset \mathbb{S}^1$ such that $\gamma \cap \operatorname{cl}(B_{r_0}(x)) = \bigcup_i \gamma(I_i)$ with $\gamma \cap \partial B_{r_0}(x) = \bigcup_i \{\gamma(a_i), \gamma(b_i)\}$ and $\gamma \cap B_{r_0}(x) = \bigcup_i \gamma((a_i, b_i))$. Assume by contradiction that for each i there exists a connected component A_i of $\Omega \cap \operatorname{cl}(B_{r_0}(x)) \setminus K$ such that $\gamma(a_i), \gamma(b_i) \in A_i$. If r_0 is small enough, then $\operatorname{cl}(\Omega \cap B_{r_0}(x))$ is diffeomorphic to $\operatorname{cl}(B_1(0) \cap \{x_1 > 0\})$ through a

diffeomorphism mapping $B_{r_0}(x) \cap \partial\Omega$ into $B_1(0) \cap \{x_1 = 0\}$. Using this fact and the connectedness of each A_i, we define smooth embeddings $\tau_i : I_i \to A_i$ with $\tau_i(a_i) = \gamma(a_i)$, $\tau_i(b_i) = \gamma(b_i)$ and τ_i homotopic in $\Omega \cap B_{r_0}(x)$ to the restriction of γ to I_i. Moreover, this can be done with $\tau_i(I_i) \cap \tau_j(I_j) = \emptyset$. The new embedding $\bar\gamma$ of $\mathbb{S}^1$ obtained by replacing γ with τ_i on I_i is thus homotopic to γ in Ω, and such that $\bar\gamma \cap K = \emptyset$, a contradiction.

Step two: Since $K \cap \Omega^*$ is relatively closed in Ω^*, $K' = f(K \cap \Omega^*)$ is relatively closed in $\Omega = f(\Omega^*)$. Should K' not be $\mathcal{C}$-spanning W, given that $K' \setminus B_{r_0}(x) = K \setminus B_{r_0}(x)$, we could find $\gamma \in \mathcal{C}$ with $\gamma \cap K \setminus B_{r_0}(x) = \emptyset$ and $\gamma \cap K' = \emptyset$. By step one, there would be a connected component σ of $\gamma \cap \mathrm{cl}\,(B_{r_0}(x))$, diffeomorphic to an interval, and such that: (i) the end-points p and q of σ (which lie on $\partial B_{r_0}(x)$) belong to distinct connected components of $\Omega \cap \mathrm{cl}\,(B_{r_0}(x)) \setminus K$; and (ii) p and q belong to the same connected component of $\Omega \cap \mathrm{cl}\,(B_{r_0}(x)) \setminus K'$. Since f is a diffeomorphism, $f(p) = p$, and $f(q) = q$, by (i) we would find that p and q belong to *distinct* connected components of

$$f\big(\Omega \cap \mathrm{cl}\,(B_{r_0}(x)) \setminus K\big) = \Omega' \cap \mathrm{cl}\,(B_{r_0}(x)) \setminus f(K)$$

while, by (ii), there would be an arc connecting them in $\Omega \cap \mathrm{cl}\,(B_{r_0}(x)) \setminus K'$, where

$$\begin{aligned} \Omega \cap \mathrm{cl}\,(B_{r_0}(x)) \setminus K' &= \Omega \cap \mathrm{cl}\,(B_{r_0}(x)) \setminus f(K \cap \Omega^*) \\ &= \Omega \cap \mathrm{cl}\,(B_{r_0}(x)) \setminus f(K) \ \subset\ \Omega' \cap \mathrm{cl}\,(B_{r_0}(x)) \setminus f(K)\,, \end{aligned}$$

and hence p and q would belong to a *same* component of $\Omega' \cap \mathrm{cl}\,(B_{r_0}(x)) \setminus f(K)$. □

2.3. **Cup competitors.** Given $E \in \mathcal{E}$, $B_r(x) \subset\subset \Omega$ and a connected component A of $\partial B_r(x) \setminus \partial E$, cup competitors are used to compare $\mathcal{H}^n(B_r(x) \cap \partial E)$ with $\mathcal{H}^n(\partial B_r(x) \setminus A)$. The construction is more involved than in the case of Plateau's problem considered in [DLGM17] as we need to construct cup competitors as *boundaries*, and we have to argue differently depending on whether $A \cap E = \emptyset$ or $A \subset E$. Given $Y \subset \partial B_r(x)$, we set

$$N_\eta(Y) = \Big\{ y - t\,\nu_{B_r(x)}(y) : y \in Y\,, t \in (0,\eta) \Big\}\,, \qquad 0 < \eta < r\,.$$

Lemma 2.5 (Cup competitors)**.** *Let $E \in \mathcal{E}$ be such that $\Omega \cap \partial E$ is $\mathcal{C}$-spanning W, let $x \in \Omega$, $r < \mathrm{dist}(x, \partial\Omega)$, $\eta \in (0, r/2)$, and let A be a connected component of $\partial B_r(x) \setminus \partial E$. Assume that $\partial E \cap \partial B_r(x)$ is $\mathcal{H}^{n-1}$-rectifiable. (i) If $A \cap E = \emptyset$, then the cup competitor defined by E and A is given by*

$$F = \big(E \setminus \mathrm{cl}\,(B_r(x))\big) \cup N_\eta(Y)\,, \qquad Y = \partial B_r(x) \setminus \big(\mathrm{cl}\,(E \cap \partial B_r(x)) \cup \mathrm{cl}\,(A)\big)\,, \tag{2.2}$$

see Figure 2.1-(a), and satisfies

$$\limsup_{\eta \to 0^+} \mathcal{H}^n(B_r(x) \cap \partial F) \le \mathcal{H}^n\Big(\partial B_r(x) \setminus \big(A \cup (E \cap \partial B_r)\big)\Big)\,; \tag{2.3}$$

(ii) if $A \subset E$, then the cup competitor defined by E and A is given by

$$F = \big(E \cup B_r(x)\big) \setminus \mathrm{cl}\,\big(N_\eta(Y)\big)\,, \qquad Y = (E \cap \partial B_r(x)) \setminus \mathrm{cl}\,(A)\,, \tag{2.4}$$

see Figure 2.1-(b), and satisfies

$$\limsup_{\eta \to 0^+} \mathcal{H}^n(B_r(x) \cap \partial F) \le \mathcal{H}^n\big(E \cap \partial B_r(x) \setminus A\big)\,. \tag{2.5}$$

In both cases, $F \in \mathcal{E}$, $\Omega \cap \partial F$ is $\mathcal{C}$-spanning W, and

$$\partial F \setminus \mathrm{cl}\,(B_r(x)) = \partial E \setminus \mathrm{cl}\,(B_r(x))\,, \tag{2.6}$$

$$\partial B_r \cap \partial F = \partial B_r \setminus A \qquad (\textit{modulo } \mathcal{H}^n)\,, \tag{2.7}$$

$$\limsup_{\eta \to 0^+} \mathcal{H}^n(\Omega \cap \partial F) \le \mathcal{H}^n\big(\partial E \setminus B_r(x)\big) + 2\,\mathcal{H}^n(\partial B_r(x) \setminus A)\,. \tag{2.8}$$

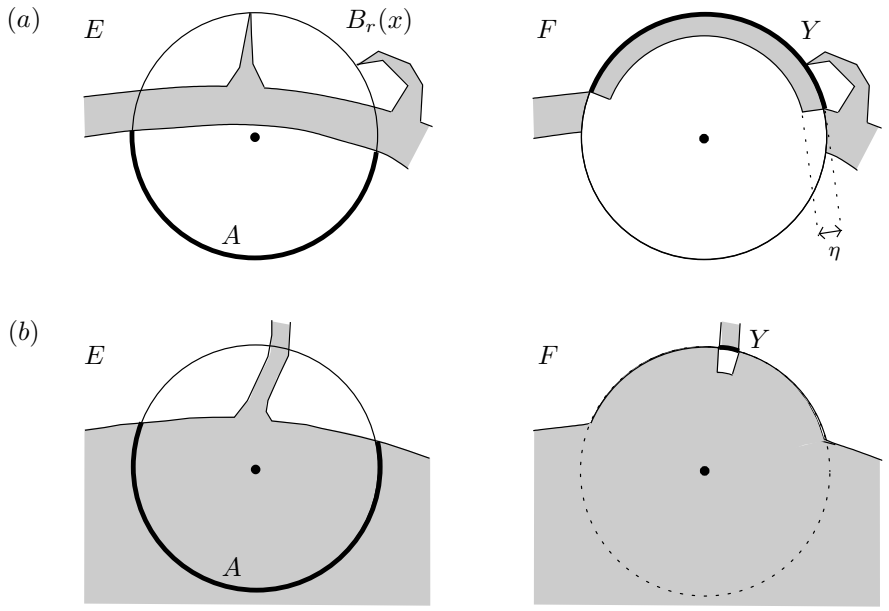

FIGURE 2.1. Cup competitors when: (a) $A \cap E = \emptyset$; (b) $A \subset E$.

Proof. Step one: We assume that $A \cap E = \emptyset$ and F is given by (2.2). For the sake of brevity we set $B_r = B_r(x)$. We claim that (2.6) hold, as well as

$$\begin{aligned}
B_r \cap \partial F &= B_r \cap \partial N_\eta(Y)\,, && (2.9)\\
Y &\subset \partial F \cap \partial B_r\,, && (2.10)\\
E \cap \partial B_r &\subset \partial F \cap \partial B_r\,, && (2.11)\\
\partial B_r \setminus \mathrm{cl}\,(A) &\subset \partial F \cap \partial B_r\,, && (2.12)\\
A,\ E \cap \partial B_r,\ Y &\text{ are open and disjoint in } \partial B_r && (2.13)\\
\partial F \cap \partial B_r &\subset \partial B_r \setminus A\,, && (2.14)\\
\partial B_r \setminus \mathrm{cl}\,(E) &\subset A \cup Y\,, && (2.15)\\
\mathrm{cl}\,(Y) \setminus Y &\subset \partial B_r \cap \partial E\,, && (2.16)\\
\mathrm{cl}\,(A) \setminus A &\subset \partial B_r \cap \partial E\,, && (2.17)\\
\mathrm{cl}\,(E \cap \partial B_r) \setminus (E \cap \partial B_r) &\subset \partial B_r \cap \partial E\,. && (2.18)
\end{aligned}$$

Indeed, (2.6) and (2.9) follow from $F \cap B_r = N_\eta(Y) \cap B_r$ and $F \setminus \mathrm{cl}\,(B_r) = E \setminus \mathrm{cl}\,(B_r)$. To prove (2.10): $Y \subset \mathrm{cl}\,(N_\eta(Y))$ gives $Y \subset \mathrm{cl}\,(F)$, and $F \cap \partial B_r = \emptyset$ implies $Y \cap F = \emptyset$. To prove (2.11): $E \cap \partial B_r \subset \mathrm{cl}\,(E \setminus \mathrm{cl}\,(B_r))$, so that $E \cap \partial B_r \subset \mathrm{cl}\,(F)$, while $F \cap \partial B_r = \emptyset$ gives $(E \cap \partial B_r) \cap F = \emptyset$. (2.13) is obvious, and (2.12) follows from (2.10) and (2.11). To prove (2.14): A is open in $\partial B_r \setminus \partial E$ and $A \cap E = \emptyset$, thus $A \cap \mathrm{cl}\,(E) = \emptyset$; moreover, $A \cap \mathrm{cl}\,(Y) = \emptyset$ by (2.13), hence

$$\partial F \cap \partial B_r \ \subset\ \mathrm{cl}\,(F) \cap \partial B_r \ \subset\ \mathrm{cl}\,(E) \cup \Big(\mathrm{cl}\,(N_\eta(Y)) \cap \partial B_r\Big) \ = \mathrm{cl}\,(E) \cup \mathrm{cl}\,(Y)\,,$$

and we deduce (2.14). To prove (2.15): if $y \in \partial B_r \setminus \mathrm{cl}\,(E)$, then y belongs to one of the open connected components of $\partial B_r \setminus \partial E$, so it is either $y \in A$, or $y \in \partial B_r \setminus \mathrm{cl}\,(A) \subset Y$. To prove (2.16): by (2.13) we have $A \cap \mathrm{cl}\,(Y) = \emptyset$, so that by (2.15)

$$\mathrm{cl}\,(Y) \setminus Y \ \subset\ \partial B_r \setminus (A \cup Y) \ \subset\ \partial B_r \cap \mathrm{cl}\,(E)\,,$$

and we conclude by $(E \cap \partial B_r) \cap \mathrm{cl}\,(Y) = \emptyset$ (again, thanks to (2.13)). Finally, (2.17) and (2.18) are obvious. Having proved the claim, we complete the proof. First, (2.7) follows by (2.12), (2.14), (2.17) and $\mathcal{H}^n(\partial E \cap \partial B_r) = 0$. To show that $\Omega \cap \partial F$ is $\mathcal{C}$-spanning W:

given $\gamma \in \mathcal{C}$, if $\gamma \cap \partial E \setminus \mathrm{cl}\,(B_r) \neq \emptyset$, then $\gamma \cap \partial F \neq \emptyset$ by (2.6); if instead $\gamma \cap \partial E \setminus \mathrm{cl}\,(B_r) = \emptyset$, then, by Lemma 2.2, γ intersects two distinct connect components of $\partial B_r \setminus \partial E$, and at least one of them is contained in $\partial F \cap \partial B_r$: indeed, $\partial F \cap \partial B_r$ contains $\partial F \setminus \mathrm{cl}\,(A)$ by (2.12), where $\mathrm{cl}\,(A)$ is disjoint from all the connected components of $\partial B_r \setminus \partial E$ that are different from A. Finally, we show that

$$\begin{aligned} \mathcal{H}^n(\Omega \cap \partial F) \;&\leq\; \mathcal{H}^n\big(\partial E \setminus B_r\big) + \mathcal{H}^n(E \cap \partial B_r) \\ &\quad + \big(2 + C(n)\,\eta\big)\,\mathcal{H}^n\Big(\partial B_r \setminus \big(A \cup (E \cap \partial B_r)\big)\Big) + C(n)\,\eta\,\mathcal{H}^{n-1}\big(\partial E \cap \partial B_r\big)\,. \end{aligned} \tag{2.19}$$

First, by $\mathcal{H}^n(\partial E \cap \partial B_r) = 0$, (2.6) and (2.14) we have

$$\begin{aligned} \mathcal{H}^n(\Omega \cap \partial F) \;&=\; \mathcal{H}^n(\partial E \setminus B_r) + \mathcal{H}^n(\mathrm{cl}\,(B_r) \cap \partial F) \\ &\leq\; \mathcal{H}^n(\partial E \setminus B_r) + \mathcal{H}^n(\partial B_r \setminus A) + \mathcal{H}^n(B_r \cap \partial F)\,. \end{aligned} \tag{2.20}$$

If $g(y,t) = y - t\,\nu_{B_r}(y)$, then

$$B_r \cap \partial F = B_r \cap \partial N_\eta(Y) = g(Y,\eta) \,\cup\, g\Big(\mathrm{cl}\,(Y) \setminus Y; [0,\eta]\Big)\,,$$

so that, (2.9), (2.16), the $\mathcal{H}^{n-1}$-rectifiability of $\partial E \cap \partial B_r$, and the area formula give us

$$\mathcal{H}^n(B_r \cap \partial F) \leq (1 + C(n)\,\eta)\,\mathcal{H}^n(Y) + C(n)\,\eta\,\mathcal{H}^{n-1}(\partial E \cap \partial B_r)\,. \tag{2.21}$$

By $\mathcal{H}^n(\partial E \cap \partial B_r) = 0$, (2.17) and (2.18) we have

$$\mathcal{H}^n(Y) = \mathcal{H}^n\Big(\partial B_r \setminus \big(A \cup (E \cap \partial B_r)\big)\Big)\,, \tag{2.22}$$

so that (2.20), (2.21) and (2.22) imply (2.19). Letting $\eta \to 0^+$ in (2.20) we find (2.8), and doing the same in (2.21) and (2.22), we deduce (2.3).

Step two: We now assume that $A \subset E$, and define F by (2.4), that is

$$F = \big(E \cup B_r\big) \setminus \mathrm{cl}\,\big(N_\eta(Y)\big)\,, \qquad Y = (E \cap \partial B_r) \setminus \mathrm{cl}\,(A)\,. \tag{2.23}$$

We claim that (2.6) hold, as well as

$$Y \;\subset\; \partial F \cap \partial B_r\,, \tag{2.24}$$

$$\partial B_r \setminus E \;\subset\; \partial F \cap \partial B_r\,, \tag{2.25}$$

$$\partial B_r \setminus \mathrm{cl}\,(A) \;\subset\; \partial F \cap \partial B_r\,, \tag{2.26}$$

$$B_r \cap \partial F \;\subset\; B_r \cap \partial N_\eta(Y)\,, \tag{2.27}$$

$$A, \partial B_r \setminus \mathrm{cl}\,(E), Y \text{ are open and disjoint in } \partial B_r\,, \tag{2.28}$$

$$\partial F \cap \partial B_r \;\subset\; \partial B_r \setminus A\,, \tag{2.29}$$

$$\mathrm{cl}\,(A) \setminus A \;\subset\; \partial B_r \cap \partial E\,, \tag{2.30}$$

$$\mathrm{cl}\,(Y) \setminus Y \;\subset\; \partial B_r \cap \partial E\,. \tag{2.31}$$

First, $F \setminus \mathrm{cl}\,(B_r) = E \setminus \mathrm{cl}\,(B_r)$ implies (2.6). To prove (2.24): since E is open we have $E \cap \partial B_r \subset \mathrm{cl}\,(E \setminus \mathrm{cl}\,(B_r)) = \mathrm{cl}\,(F \setminus \mathrm{cl}\,(B_r))$ (by (2.23)), thus $Y \subset \mathrm{cl}\,(F)$; we conclude as $Y \cap F = \emptyset$. As $F \cap \partial B_r \subset E \cap \partial B_r$, to prove (2.25) we just need to show that $\partial B_r \setminus E \subset \mathrm{cl}\,(F)$: since $\mathrm{cl}\,(U) \setminus \mathrm{cl}\,(V) \subset \mathrm{cl}\,(U \setminus \mathrm{cl}\,(V))$ for every $U, V \subset \mathbb{R}^{n+1}$, by $\partial B_r \cap \mathrm{cl}\,(N_\eta(Y)) \subset \mathrm{cl}\,(E)$,

$$\begin{aligned} \partial B_r \setminus \mathrm{cl}\,(E) \;&\subset\; \mathrm{cl}\,(B_r) \setminus \mathrm{cl}\,(N_\eta(Y)) \subset \mathrm{cl}\,\big(B_r \setminus \mathrm{cl}\,(N_\eta(Y)\big) \subset \mathrm{cl}\,(F)\,, \\ (\partial B_r \cap \partial E) \setminus \mathrm{cl}\,(N_\eta(Y)) \;&\subset\; \mathrm{cl}\,(E) \setminus \mathrm{cl}\,(N_\eta(Y)) \;\subset\; \mathrm{cl}\,(E \setminus \mathrm{cl}\,(N_\eta(Y))) \;\subset\; \mathrm{cl}\,(F)\,, \\ \partial B_r \cap \partial E \cap \mathrm{cl}\,(N_\eta(Y)) \;&\subset\; \partial E \cap \mathrm{cl}\,(Y) \;\subset\; \partial F\,, \end{aligned}$$

where the last inclusion follows by (2.24). Next, (2.26) follows by (2.24), (2.25) and

$$\partial B_r \setminus \mathrm{cl}\,(A) = [(E \cap \partial B_r) \setminus \mathrm{cl}\,(A)] \cup [\partial B_r \setminus (E \cup \mathrm{cl}\,(A))] \subset Y \cup (\partial B_r \setminus E)\,.$$

To prove (2.27): setting $V^c = \mathbb{R}^{n+1} \setminus V$, by $B_r \cap F = B_r \cap \mathrm{cl}\,(N_\eta(Y))^c$ we find $B_r \cap \partial F = B_r \cap \partial\,[\mathrm{cl}\,(N_\eta(Y))^c]$, where, as a general fact on open set $U \subset \mathbb{R}^{n+1}$, we have

$$\partial[\mathrm{cl}\,(U)^c] = \mathrm{cl}\,(\mathrm{cl}\,(U)^c) \setminus \mathrm{cl}\,(U)^c = \mathrm{cl}\,(U) \cap \mathrm{cl}\,(\mathrm{cl}\,(U)^c)\,, \qquad \mathrm{cl}\,(\mathrm{cl}\,(U)^c) \subset U^c\,,$$

and thus $\partial[\mathrm{cl}\,(U)^c] \subset \partial U$. Next, (2.28) is obvious, and implies $A \cap \mathrm{cl}\,(Y) = \emptyset$ where $\mathrm{cl}\,(Y) = \mathrm{cl}\,(N_\eta(Y)) \cap \partial B_r$, so that $A \cap \partial B_r \subset E \cap \partial B_r \setminus \mathrm{cl}\,(N_\eta(Y)) = F \cap \partial B_r$, and (2.29) follows. To prove (2.30), just notice that $A \subset E$ and A is a connected component of $\partial B_r \setminus \partial E$. To prove (2.31): trivially, $\mathrm{cl}\,(Y) \setminus Y \subset \mathrm{cl}\,(Y) \subset \partial B_r \cap \mathrm{cl}\,(E)$, while by definition of Y and by $\mathrm{cl}\,(Y) \cap A = \emptyset$

$$\begin{aligned} E \cap (\mathrm{cl}\,(Y) \setminus Y) &= \big(\mathrm{cl}\,(Y) \cap (E \cap \partial B_r)\big) \setminus Y = \mathrm{cl}\,(Y) \cap (E \cap \partial B_r) \cap \mathrm{cl}\,(A) \\ &= (E \cap \partial B_r) \cap \mathrm{cl}\,(Y) \cap \partial A \subset E \cap (\mathrm{cl}\,(A) \setminus A) = \emptyset\,, \end{aligned}$$

thanks to (2.30). We have completed the claim. Next, by (2.26), (2.29), (2.30), and by $\mathcal{H}^n(\partial B_r \cap \partial E) = 0$, we deduce (2.7), while $\Omega \cap \partial F$ is $\mathcal{C}$-spanning W thanks to (2.6) and (2.26). Finally,

$$\begin{aligned} \mathcal{H}^n(\Omega \cap \partial F) &\le \mathcal{H}^n\big(\partial E \setminus B_r\big) + \mathcal{H}^n(\partial B_r \setminus E) \\ &\quad + \big(2 + C(n)\,\eta\big)\,\mathcal{H}^n(E \cap \partial B_r \setminus A) + C(n)\,\eta\,\mathcal{H}^{n-1}\big(\partial E \cap \partial B_r\big)\,. \end{aligned} \tag{2.32}$$

Indeed, by $\mathcal{H}^n(\partial E \cap \partial B_r) = 0$, (2.6), and (2.29)

$$\begin{aligned} \mathcal{H}^n(\Omega \cap \partial F) &\le \mathcal{H}^n(\partial E \setminus B_r) + \mathcal{H}^n(\partial F \cap \mathrm{cl}\,(B_r)) \\ &\le \mathcal{H}^n(\partial E \setminus B_r) + \mathcal{H}^n(\partial B_r \setminus A) + \mathcal{H}^n(B_r \cap \partial F) \\ &\le \mathcal{H}^n(\partial E \setminus B_r) + \mathcal{H}^n(\partial B_r \setminus E) + \mathcal{H}^n((E \cap \partial B_r) \setminus A) + \mathcal{H}^n(B_r \cap \partial F)\,; \end{aligned} \tag{2.33}$$

by (2.27), (2.31), the $\mathcal{H}^{n-1}$-rectifiability of $\partial E \cap \partial B_r$, and the area formula

$$\begin{aligned} \mathcal{H}^n(B_r \cap \partial F) &\le \mathcal{H}^n(B_r \cap \partial N_\eta(Y)) \\ &\le (1 + C(n)\,\eta)\mathcal{H}^n(Y) + C(n)\,\eta\,\mathcal{H}^{n-1}(\partial E \cap \partial B_r)\,, \end{aligned} \tag{2.34}$$

while (2.30) and $\mathcal{H}^n(\partial B_r \cap \partial E) = 0$ give

$$\mathcal{H}^n(Y) = \mathcal{H}^n((E \cap \partial B_r) \setminus \mathrm{cl}\,(A)) = \mathcal{H}^n((E \cap \partial B_r) \setminus A)\,.$$

We thus deduce (2.32). As $\eta \to 0^+$ in (2.32) and in (2.34) we get (2.8) and (2.5). □

In the following lemma we introduce the notion of exterior cup competitor. We set

$$M_\eta(Y) = \Big\{y + t\,\nu_B(y) : y \in Y\,, t \in (0, \eta)\Big\}\,, \qquad \eta > 0\,,$$

whenever B is an open ball and $Y \subset \partial B$.

Lemma 2.6 (Exterior cup competitor)**.** *Let $E \in \mathcal{E}$ be such that $\Omega \cap \partial E$ is $\mathcal{C}$-spanning W, let $R > 0$ be such that $W \subset\subset B_R(0)$ and $\partial E \cap \partial B_R(0)$ is $\mathcal{H}^{n-1}$-rectifiable, let $\eta \in (0, 1)$, and let A be a connected component of $\partial B_R(0) \setminus \partial E$ such that $A \cap E = \emptyset$. Let the exterior cup competitor defined by E and A be given by*

$$F = \big(E \cap B_R(0)\big) \cup M_\eta(Y)\,, \qquad Y = \partial B_R(0) \setminus \big(\mathrm{cl}\,(E \cap \partial B_R(0)) \cup \mathrm{cl}\,(A)\big)\,, \tag{2.35}$$

see Figure 2.2. Then $F \in \mathcal{E}$, $\Omega \cap \partial F$ is $\mathcal{C}$-spanning W and

$$\limsup_{\eta \to 0^+} \mathcal{H}^n(\Omega \cap \partial F) \le \mathcal{H}^n\big(\Omega \cap B_R(0) \cap \partial E\big) + 2\,\mathcal{H}^n(\partial B_R(0) \setminus A)\,. \tag{2.36}$$

Proof. If $\gamma \in \mathcal{C}$ is such that $\gamma \cap \partial E \cap B_R(0) = \emptyset$, then an adaptation of step one in the proof of Lemma 2.4 shows that there exists a connected component of $\gamma \setminus B_R(0)$ which is diffeomorphic to an interval, and whose end-points belong to distinct connected components of $(\mathbb{R}^{n+1} \setminus B_R(0)) \setminus \partial E$. Using this fact, we just need to show that $\partial B_R(0) \cap \partial F$ contains $\partial B_R(0) \setminus \mathrm{cl}\,(A)$ in order to show that F is $\mathcal{C}$-spanning W. This is done by repeating with minor variations the considerations contained in step one of the proof of Lemma 2.5. The proof of (2.36) is obtained in a similar way, and the details are omitted. □

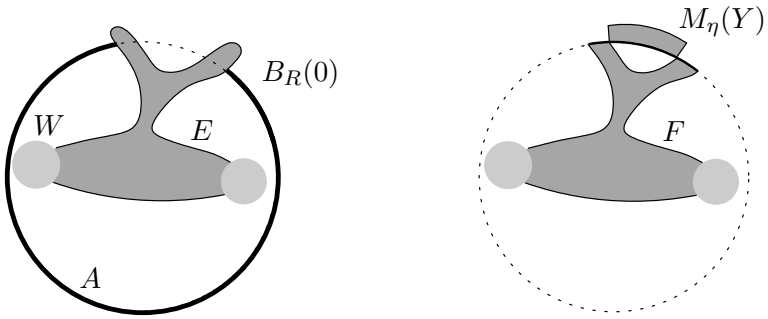

FIGURE 2.2. An exterior cup competitor.

2.4. **Slab competitors.** Bi-Lipschitz deformations of cup competitors can be used to generate new competitors thanks to Lemma 2.3. We will crucially use this remark to replace balls with "slabs" (see Figures 3.2, 3.3 and 3.4) and obtain sharp area concentration estimates in step five of the proof of Theorem 1.4, as well as in the proof of Theorem 1.6, see e.g. (4.7). Given $\tau \in (0,1)$, $x \in \mathbb{R}^{n+1}$, $r > 0$, and $\nu \in \mathbb{S}^n$, we set

$$S^\nu_{\tau,r}(x) = \big\{ y \in B_r(x) : |(y-x)\cdot\nu| < \tau\, r \big\}\,,$$

and we claim the existence of a bi-Lipschitz map $\Phi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with

$$\{\Phi \neq \mathrm{id}\} \subset\subset B_{2\,r}(x)\,, \qquad \Phi\big(B_{2\,r}(x)\big) \subset B_{2\,r}(x)\,, \qquad \Phi\big(\partial S^\nu_{\tau,t}(x)\big) = \partial B_t(x) \quad \forall t \in (0,r)\,,$$

and such that $\mathrm{Lip}\,\Phi$ and $\mathrm{Lip}\,\Phi^{-1}$ depend only on n and τ. The construction of Φ is simple, and therefore is omitted.

Lemma 2.7 (Slab competitors)**.** *Let $E \in \mathcal{E}$ be such that $\Omega \cap \partial E$ is $\mathcal{C}$-spanning W, and let $B_{2r}(x) \subset\subset \Omega$, $\nu \in \mathbb{S}^n$, $\tau \in (0,1)$ with $\partial S^\nu_{\tau,r}(x) \cap \partial E$ $\mathcal{H}^{n-1}$-rectifiable. Let A be an open connected component of $\partial S^\nu_{\tau,r}(x) \setminus \partial E$. Then for every $\eta \in (0, r/2)$, there exists $F \in \mathcal{E}$ such that $\Omega \cap \partial F$ is $\mathcal{C}$-spanning W,*

$$F \setminus \mathrm{cl}\,(S^\nu_{\tau,r}(x)) \;=\; E \setminus \mathrm{cl}\,(S^\nu_{\tau,r}(x))\,, \tag{2.37}$$

$$\partial F \cap \partial S^\nu_{\tau,r}(x) \;=\; \partial S^\nu_{\tau,r}(x) \setminus A\,, \qquad \textit{modulo } \mathcal{H}^n\,, \tag{2.38}$$

and such that if $A \cap E = \emptyset$, then

$$\limsup_{\eta\to 0^+} \mathcal{H}^n(S^\nu_{\tau,r}(x) \cap \partial F) \;\le\; C(n,\tau)\,\mathcal{H}^n\big(\partial S^\nu_{\tau,r}(x) \setminus (A \cup E)\big)\,; \tag{2.39}$$

while, if $A \subset E$, then

$$\limsup_{\eta\to 0^+} \mathcal{H}^n(S^\nu_{\tau,r}(x) \cap \partial F) \;\le\; C(n,\tau)\,\mathcal{H}^n\big(E \cap \partial S^\nu_{\tau,r}(x) \setminus A\big)\,. \tag{2.40}$$

Proof. Let us set for brevity $S_r = S^\nu_{\tau,r}(x)$ and $B_r = B_r(x)$. By Lemma 2.3, $\Phi(E) \in \mathcal{E}$ and $\Omega \cap \partial\Phi(E)$ is $\mathcal{C}$-spanning W. Since Φ is an homeomorphism between ∂S_r and ∂B_r, $\Phi(A)$ is an open connected component of $\partial B_r \setminus \partial\Phi(E)$. Depending on whether $A \cap E = \emptyset$ or $A \subset E$, and thus, respectively, depending on whether $\Phi(A) \cap \Phi(E) = \emptyset$ or $\Phi(A) \cap \Phi(E) = \emptyset$, we consider the cup competitor G defined by $\Phi(E)$ and $\Phi(A)$, so that

$$G = \big(\Phi(E) \setminus \mathrm{cl}\,(B_r)\big) \cup N_\eta(Y)\,, \qquad Y = \partial B_r \setminus \big(\mathrm{cl}\,(\Phi(E) \cap \partial B_r) \cup \mathrm{cl}\,(\Phi(A))\big)\,,$$

if $A \cap E = \emptyset$, see (2.2), and

$$G = \big(\Phi(E) \cup B_r\big) \setminus \mathrm{cl}\,\big(N_\eta(Y)\big)\,, \qquad Y = \big(\Phi(E) \cap \partial B_r\big) \setminus \mathrm{cl}\,(\Phi(A))\,,$$

if $A \subset E$, see (2.4). Finally, we set $F = \Phi^{-1}(G)$. Since $G \in \mathcal{E}$ and $\Omega \cap \partial G$ is $\mathcal{C}$-spanning W, by Lemma 2.3 we find that $F \in \mathcal{E}$ and that $\Omega \cap \partial F$ is $\mathcal{C}$-spanning W. By construction $G \setminus \mathrm{cl}\,(B_r) = \Phi(E) \setminus \mathrm{cl}\,(B_r)$, so that (2.37) follows by

$$F \setminus \mathrm{cl}\,(S_r) = \Phi^{-1}\big(G \setminus \mathrm{cl}\,(B_r)\big) = \Phi^{-1}\big(\Phi(E) \setminus \mathrm{cl}\,(B_r)\big) = E \setminus \mathrm{cl}\,(S_r)\,.$$

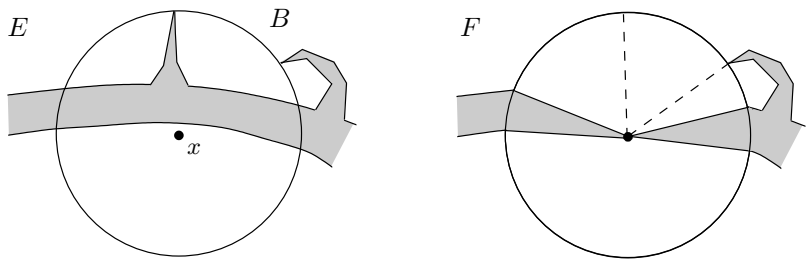

FIGURE 2.3. In this picture, the cone competitor F defined by $E \cap \partial B_r$ as in (2.41) may fail to be $\mathcal{C}$-spanning W. Notice that the dashed lines are part of the cone competitor K' defined by $K = \Omega \cap \partial E$ in $B_r(x)$, which is indeed strictly larger than $\Omega \cap \partial F$.

By (2.7), $\partial B_r \cap \partial G$ is $\mathcal{H}^n$-equivalent to $\partial B_r \setminus \Phi(A)$, which gives (2.38) by the area formula. Finally, (2.39) and (2.40) are deduced by the area formula, (2.3) and (2.5). □

2.5. **Cone competitor.** As customary in the analysis of area minimization problems, we want to compare $\mathcal{H}^n(B_r(x) \cap \partial E)$ with $\mathcal{H}^n(B_r(x) \cap \partial F)$, where F is the cone spanned by $E \cap \partial B_r(x)$ over x,

$$F = \big(E \setminus \mathrm{cl}\,(B_r(x))\big) \cup \big\{(1-t)\,x + t\,y : y \in E \cap \partial B_r(x)\,, t \in (0,1]\big\}\,. \tag{2.41}$$

Following the terminology of [DLGM17], given $K \in \mathcal{S}$, the cone competitor K' of K in $B_r(x)$ is similarly defined as

$$K' = (K \setminus B_r(x)) \cup \big\{(1-t)\,x + t\,y : y \in K \cap \partial B_r(x)\,, t \in [0,1]\big\}\,,$$

and is indeed $\mathcal{C}$-spanning W (since K was). However, for a generic radius r, $\partial F \cap B_r$ may be strictly smaller than the cone competitor defined by $K = \Omega \cap \partial E$ in $B_r(x)$, and thus may fail to be $\mathcal{C}$-spanning; see Figure 2.3. By Sard's lemma, if E has smooth boundary in Ω this issue can be avoided as, for a.e. r, ∂E and ∂B_r intersect transversally, and thus $\partial E \cap \partial B_r(x)$ is the boundary of $E \cap \partial B_r(x)$ relative to $\partial B_r(x)$; but working with smooth boundary leads to other difficulties when constructing cup competitors. We thus approximate F (as defined in (2.41)) in energy by means of diffeomorphic images of E.

Lemma 2.8. *Let $E \in \mathcal{E}$ be such that $\Omega \cap \partial E$ is $\mathcal{C}$-spanning W, and let $B = B_r(x) \subset\subset \Omega$ be such that $E \cap \partial B_r(x)$ is $\mathcal{H}^n$-rectifiable, $\partial E \cap \partial B_r(x)$ is $\mathcal{H}^{n-1}$-rectifiable and r is a Lebesgue point of the maps $t \mapsto \mathcal{H}^n(E \cap \partial B_t(x))$ and $t \mapsto \mathcal{H}^{n-1}(\partial E \cap \partial B_t(x))$. Then for each $\eta \in (0, r/2)$ there exists $F \in \mathcal{E}$ such that $F \Delta E \subset B_r(x)$, $\Omega \cap \partial F$ is $\mathcal{C}$-spanning W, and*

$$\limsup_{\eta \to 0^+} \mathcal{H}^n(\Omega \cap \partial F) \le \mathcal{H}^n(\partial E \setminus B_r(x)) + \frac{r}{n}\,\mathcal{H}^{n-1}(\partial E \cap \partial B_r(x))\,, \tag{2.42}$$

$$\liminf_{\eta \to 0^+} |F| \ge |E \setminus B_r(x)| + \frac{r}{n+1}\,\mathcal{H}^n(E \cap \partial B_r(x))\,. \tag{2.43}$$

Proof. Let $x = 0$, $r = 1$, $B_r = B_r(0)$, and define a bi-Lipschitz map f_η by $f_\eta(0) = 0$ and $f_\eta(x) = u_\eta(|x|)\,\hat{x}$ if $x \neq 0$, where $\hat{x} = x/|x|$ and $u_\eta : \mathbb{R} \to [0,\infty)$ is given by

$$u_\eta(t) := \begin{cases} \max\{0, \eta\,t\}\,, & \text{for } t \le 1-\eta\,, \\ \eta(1-\eta) + \frac{t-(1-\eta)}{\eta}\,(1-\eta(1-\eta))\,, & \text{for } t \in [1-\eta, 1]\,, \\ t\,, & \text{for } t \ge 1\,. \end{cases} \tag{2.44}$$

Clearly, $\{f_\eta \neq \mathrm{id}\} \subset B_1$ and $f_\eta(B_1) \subset B_1$. The open set $F = f_\eta(E)$ is such that $\Omega \cap \partial F = f_\eta(\Omega \cap \partial E)$, so that $\Omega \cap \partial F$ is $\mathcal{H}^n$-rectifiable and, by Lemma 2.3, $\mathcal{C}$-spanning

W. Thanks to the area formula, (2.42) will follow by showing

$$\limsup_{\eta\to 0^+} \int_{B_1\cap\partial E} J^{\partial E} f_\eta \, d\mathcal{H}^n \le \frac{1}{n}\, \mathcal{H}^{n-1}(\partial E \cap \partial B_1)\,. \tag{2.45}$$

Trivially, the integral over $B_{1-\eta}\cap\partial E$ is bounded by $C(n)\,\eta^n\,\mathcal{H}^n(\Omega\cap\partial E)$. The integral over $B_1\setminus B_{1-\eta}$ is treated as in [DLGM17, Step two, Theorem 7]; by the coarea formula,

$$\begin{aligned}\int_{(B_1\setminus B_{1-\eta})\cap\partial E} J^{\partial E} f_\eta \, d\mathcal{H}^n = &\int_{1-\eta}^1 dt \int_{\partial B_t\cap\partial E\cap\{|\nu_E\cdot\hat{x}|<1\}} \frac{J^{\partial E} f_\eta}{\sqrt{1-(\nu_E\cdot\hat{x})^2}}\, d\mathcal{H}^{n-1} \\ &+ \int_{(B_1\setminus B_{1-\eta})\cap\partial E\cap\{|\nu_E\cdot\hat{x}|=1\}} J^{\partial E} f_\eta \, d\mathcal{H}^n\,,\end{aligned} \tag{2.46}$$

where $\nu_E(x)\in T_x(\partial E)\cap\mathbb{S}^n$ at $\mathcal{H}^n$-a.e. $x\in\partial E$. By

$$\nabla f_\eta(x) = \frac{u_\eta(|x|)}{|x|}\,\mathrm{Id} + \left(u_\eta'(|x|) - \frac{u_\eta(|x|)}{|x|}\right)\hat{x}\otimes\hat{x}\,, \tag{2.47}$$

if $|\nu_E(x)\cdot\hat{x}|=1$, then $J^{\partial E} f_\eta = (u_\eta(|x|)/|x|)^n \le (1-\eta)^{-n}\to 1$ as $\eta\to 0^+$. Since

$$\lim_{\eta\to 0^+} \mathcal{H}^n(\partial E\cap(B_1\setminus B_{1-\eta})) = 0\,, \tag{2.48}$$

the second term on the right-hand side of (2.46) converges to 0 as $\eta\to 0^+$. As for the first term, by (2.47)

$$J^{\partial E} f_\eta(x) \le C(n) + \sqrt{1-(\nu_E(x)\cdot\hat{x})^2}\, u_\eta'(|x|)\left(\frac{u_\eta(|x|)}{|x|}\right)^{n-1} \quad \text{for } \mathcal{H}^n\text{-a.e. } x\in\partial E\,.$$

The term corresponding to $C(n)$ converges to 0 as $\eta\to 0^+$ by (2.48). At the same time,

$$\limsup_{\eta\to 0^+}\left|\int_{1-\eta}^1 \Big(\mathcal{H}^{n-1}(\partial E\cap\partial B_t) - \mathcal{H}^{n-1}(\partial E\cap\partial B_1)\Big)\, u_\eta'\left(\frac{u_\eta}{t}\right)^{n-1} dt\right| = 0$$

since $t=1$ is a Lebesgue point of $t\mapsto\mathcal{H}^{n-1}(\partial B_t\cap\partial E)$, and since $u_\eta'(t)\le 1/\eta$ and $(u_\eta(t)/t)\le(1-\eta)^{-1}$ for $t\in[1-\eta,1]$. Finally,

$$\int_{1-\eta}^1 u_\eta'\left(\frac{u_\eta}{t}\right)^{n-1} dt \le \frac{1}{(1-\eta)^{n-1}}\,\frac{1-\eta^n(1-\eta)^n}{n} \to \frac{1}{n}$$

as $\eta\to 0^+$, thus completing the proof of (2.42). The proof of (2.43) follows an analogous argument. The goal is to show that

$$\liminf_{\eta\to 0^+}\int_{E\cap B_1} J f_\eta \, dx \ge \frac{1}{n+1}\,\mathcal{H}^n(E\cap\partial B_1)\,, \tag{2.49}$$

and by the coarea formula and (2.47) it is immediate to see that

$$\int_{E\cap B_1} J f_\eta \, dx \ge \int_{1-\eta}^1 u_\eta'(t)\left(\frac{u_\eta(t)}{t}\right)^n \mathcal{H}^n(E\cap\partial B_t)\, dt\,.$$

The estimate in (2.49) then readily follows using that $t=1$ is a Lebesgue point for the map $t\mapsto\mathcal{H}^n(E\cap\partial B_t)$, together with

$$\int_{1-\eta}^1 u_\eta'(t)\left(\frac{u_\eta(t)}{t}\right)^n dt \ge \frac{1-\eta^{n+1}(1-\eta)^{n+1}}{n+1} \to \frac{1}{n+1} \qquad \text{as } \eta\to 0^+\,. \qquad \square$$

2.6. **Nucleation lemma.** The following nucleation lemma can be found, with slightly different statements, in [Alm76, VI(13)] or in [Mag12, Lemma 29.10].

Lemma 2.9. *Let $\xi(n)$ be the constant of Besicovitch's covering theorem in $\mathbb{R}^{n+1}$. If T is closed, $A = \mathbb{R}^{n+1} \setminus T$, $0 < |E| < \infty$, $P(E;A) < \infty$, $\tau > 0$, and*

$$\sigma = \min\Big\{\frac{|E \setminus I_\tau(T)|}{\tau\, P(E;A)}, \frac{\xi(n)}{n+1}\Big\} > 0$$

then there exists $x \in E^{(1)} \setminus I_\tau(T)$ such that

$$|E \cap B_\tau(x)| \geq \Big(\frac{\sigma}{2\xi(n)}\Big)^{n+1} \tau^{n+1}\,.$$

Proof. By contradiction one assumes that

$$|E \cap B_\tau(x)| < \Big(\frac{\sigma}{2\xi(n)}\Big)^{n+1} \tau^{n+1} \qquad \forall x \in E^{(1)} \setminus I_\tau(T)\,. \tag{2.50}$$

Setting $\alpha = \xi(n)/\sigma$, so that $\alpha \geq n+1$, we claim that (2.50) implies the existence, for each $x \in E^{(1)} \setminus I_\tau(T)$, of $\tau_x \in (0,\tau)$ such that

$$P(E; B_{\tau_x}(x)) > \frac{\alpha}{\tau}\, |E \cap B_{\tau_x}(x)|\,. \tag{2.51}$$

In turn (2.51) is in contradiction with (2.50): indeed, by applying Besicovitch's theorem to $\{\mathrm{cl}\,(B_{\tau_x}(x)) : x \in E^{(1)} \setminus I_\tau(T)\}$ we find an at most countable subset I of $E^{(1)} \setminus I_\tau(T)$ such that $\{\mathrm{cl}\,(B_{\tau_x}(x))\}_{x\in I}$ is disjoint and

$$\begin{aligned} |E \setminus I_\tau(T)| &\leq \xi(n) \sum_{x\in I} |E \cap B_{\tau_x}(x)| < \frac{\xi(n)\,\tau}{\alpha} \sum_{x\in I} P(E; B_{\tau_x}(x)) \\ &\leq \frac{\xi(n)\,\tau\, P(E;A)}{\alpha} = \tau\,\sigma\, P(E;A) \leq |E \setminus I_\tau(T)|\,, \end{aligned}$$

a contradiction. We show that (2.50) implies (2.51): indeed, if (2.50) holds but (2.51) fails, then there exists $x \in E^{(1)} \setminus I_\tau(T)$ such that, setting $m(r) = |E \cap B_r(x)|$ for $r > 0$,

$$m > 0 \text{ on } (0,\infty)\,, \qquad m(\tau) < \Big(\frac{\tau}{2\alpha}\Big)^{n+1} \tag{2.52}$$

and $(\alpha/\tau)\, m(r) \geq P(E; B_r(x))$ for every $r \in (0,\tau)$. Adding up $\mathcal{H}^n(\partial B_r(x) \cap E)$ we obtain

$$m'(r) + \frac{\alpha}{\tau}\, m(r) \geq P(E \cap B_r(x)) \geq m(r)^{n/(n+1)}\,, \qquad \text{for a.e. } r \in (0,\tau)\,. \tag{2.53}$$

where in the last inequality we have used that $P(F) \geq |F|^{n/(n+1)}$ whenever $0 < |F| < \infty$. Since $m > 0$ on $(0,\infty)$ we find

$$\begin{cases} \dfrac{\alpha}{\tau} m(r) \leq (1/2) m(r)^{n/(n+1)} \\ \forall r \in (0,\tau) \end{cases} \quad \text{iff} \quad \begin{cases} m(r) \leq (\tau/2\alpha)^{n+1} \\ \forall r \in (0,\tau) \end{cases} \quad \text{if} \quad m(\tau) \leq \Big(\frac{\tau}{2\alpha}\Big)^{n+1},$$

where the last condition holds by (2.52). Thus (2.53) gives $m'(r) \geq (1/2) m(r)^{n/(n+1)}$ for a.e. $r \in (0,\tau)$, thus $m(\tau) \geq (\tau/2(n+1))^{n+1} \geq (\tau/2\alpha)^{n+1}$ as $\alpha \geq n+1$, a contradiction. □

2.7. **Isoperimetry and a geometric lower density bound.** Given an L^1-converging sequence of sets of finite perimeter $\{E_j\}_j$, the boundary of the L^1-limit set E will be (in general) strictly included in $K = \mathrm{spt}\,\mu$, where μ is the weak-star limit of the Radon measures defined by the boundaries of the E_j's. In the next lemma we show that, under some mild bounds on μ and E_j, if μ is absolutely continuous with respect to $\mathcal{H}^n \llcorner K$ then the Radon-Nikodým density θ of μ is everywhere larger than 1, and is actually larger than 2 at a.e. point of $K \setminus \partial^* E$ (that is, a cancellation can happen only when boundaries are collapsing).

Lemma 2.10 (Collapsing lemma)**.** *Let K be a relatively compact and $\mathcal{H}^n$-rectifiable set in Ω, let $E \subset \Omega$ be a set of finite perimeter with $\Omega \cap \partial^* E \subset K$, and let $\{E_j\}_j \subset \mathcal{E}$ such that $E_j \to E$ in $L^1_{\rm loc}(\Omega)$, and $\mu_j \overset{*}{\rightharpoonup} \mu$ as Radon measures in Ω, where $\mu_j = \mathcal{H}^n \llcorner (\Omega \cap \partial E_j)$ and $\mu = \theta\, \mathcal{H}^n \llcorner K$ for a Borel function θ. If $\Omega' \subset \Omega$ and $r_* > 0$ are such that for every $x \in K \cap \Omega'$ and a.e. $r < r_*$ with $B_r(x) \subset\subset \Omega'$ we have*

$$\mu(B_r(x)) \;\geq\; c(n)\, r^n\,, \tag{2.54}$$

$$\liminf_{j\to\infty} \mathcal{H}^n(B_r(x) \cap \partial E_j) \;\leq\; C(n) \liminf_{j\to\infty} \mathcal{H}^n(\partial B_r(x) \setminus A^0_{r,j})\,, \tag{2.55}$$

where $A^0_{r,j}$ denotes an $\mathcal{H}^n$-maximal connected component of $\partial B_r(x) \setminus \partial E_j$, then $\theta(x) \geq 1$ for $\mathcal{H}^n$-a.e. $x \in K \cap \Omega'$, and $\theta(x) \geq 2$ for $\mathcal{H}^n$-a.e. $x \in (K \setminus \partial^* E) \cap \Omega'$.

The bound $\theta \geq 1$ follows by arguing exactly as in [DLGM17, Proof of Theorem 2, Step three], and has nothing to do with the fact that the measures μ_j are defined by boundaries; the latter information is in turn crucial in obtaining the bound $\theta \geq 2$, and requires a new argument. For the sake of clarity, we also give the details of the $\theta \geq 1$ bound, which in turn is based on spherical isoperimetry.

Lemma 2.11 (Spherical isoperimetry)**.** *Let $\Sigma \subset \mathbb{R}^{n+1}$ denote a spherical cap in the n-dimensional unit sphere $\mathbb{S}^n$, possibly with $\Sigma = \mathbb{S}^n$. If K is a compact set in $\mathbb{R}^{n+1}$ and $\{A^h\}_{h=0}^\infty$ is the family of the open connected components of $\Sigma \setminus K$, ordered so to have $\mathcal{H}^n(A^h) \geq \mathcal{H}^n(A^{h+1})$, then*

$$\mathcal{H}^n(\Sigma \setminus A^0) \leq C(n)\, \mathcal{H}^{n-1}(\Sigma \cap K)^{n/(n-1)}\,. \tag{2.56}$$

Moreover, if $\Sigma = \mathbb{S}^n$, $\sigma_n = \mathcal{H}^n(\mathbb{S}^n)$ and $\mathcal{H}^{n-1}(\mathbb{S}^n \cap K) < \infty$, then each A^h is a set of finite perimeter in $\mathbb{S}^n$ and for every $\tau > 0$ there exists $\sigma > 0$ such that

$$\min\Big\{\mathcal{H}^n(A^0), \mathcal{H}^n(A^1)\Big\} = \mathcal{H}^n(A^1) \geq \frac{\sigma_n}{2} - \sigma \tag{2.57}$$

implies

$$\min\Big\{\mathcal{H}^{n-1}(\partial^* A^0), \mathcal{H}^{n-1}(\partial^* A^1)\Big\} \geq \sigma_{n-1} - \tau\,. \tag{2.58}$$

Here $\partial^* A^h$ denotes the reduced boundary of A^h in $\mathbb{S}^n$.

Proof. This is [DLGM17, Lemma 9]. However, (2.58) is stated in a weaker form in [DLGM17, Lemma 9], so we give the details. Arguing by contradiction, we can find $\tau > 0$ and $\{K_j\}_j$ such that, for $\alpha = 0, 1$, $\mathcal{H}^{n-1}(\partial^* A^\alpha_j) \leq \sigma_{n-1} - \tau$ for every j, but $\mathcal{H}^n(A^\alpha_j) \to \sigma_n/2$ as $j \to \infty$. Since $\sigma_n = \mathcal{H}^n(\mathbb{S}^n)$ and $A^0_j \cap A^1_j = \emptyset$, we find that, for $\alpha = 0, 1$, $A^\alpha_j \to A^\alpha$ in $L^1(\mathbb{S}^n)$ where $A^0 \cap A^1 = \emptyset$ and $A^0 \cup A^1$ is $\mathcal{H}^n$-equivalent to $\mathbb{S}^n$. Therefore $\mathcal{H}^{n-1}(\partial^* A^0) = \mathcal{H}^{n-1}(\partial^* A^1) \leq \sigma_{n-1} - \tau$, where we have used lower semicontinuity of perimeter. Since $\inf \mathcal{H}^{n-1}(\partial^* A)$ with $\mathcal{H}^n(A) = \sigma_n/2$ is equal to σ_{n-1} we have reached a contradiction. □

Proof of Lemma 2.10. Step one: We fix $x \in K \cap \Omega'$ such that $\mathcal{H}^n \llcorner (K - x)/r \overset{*}{\rightharpoonup} \mathcal{H}^n \llcorner T_x K$ as $r \to 0^+$. Setting $\nu(x)^\perp = T_x K$ for $\nu(x) \in \mathbb{S}^n$, by the lower density estimate (2.54) we easily find that for every $\sigma > 0$ there exists $r_0 = r_0(\sigma, x) \in (0, \min\{r_*, \mathrm{dist}(x, \partial\Omega')\})$ such that $|(y - x) \cdot \nu(x)| < \sigma\, r$ for every $y \in K \cap B_r(x)$ and every $r < r_0$. In particular,

$$\lim_{j\to\infty} \mathcal{H}^n(\partial E_j \cap \{y \in B_r(x) :\, |(y - x) \cdot \nu(x)| > \sigma\, r\}) = 0 \qquad \text{for every } r \leq r_0\,,$$

and thus by the coarea formula (see [DLGM17, Equation (2.13)])

$$\lim_{j\to\infty} \mathcal{H}^{n-1}(\Sigma^\pm_{r,\sigma} \cap \partial E_j) = 0 \qquad \text{for a.e. } r \leq r_0\,, \tag{2.59}$$

where we have set

$$\begin{aligned}\Sigma^+_{r,\sigma} &= \big\{y \in \partial B_r(x) : (y-x)\cdot\nu(x) > \sigma\, r\big\}\,,\\ \Sigma^-_{r,\sigma} &= \big\{y \in \partial B_r(x) : (y-x)\cdot\nu(x) < -\sigma\, r\big\}\,.\end{aligned}$$

Let $A^+_{r,j}$ be an $\mathcal{H}^n$-maximal connected component of $\Sigma^+_{r,\sigma} \setminus \partial E_j$, and define similarly $A^-_{r,j}$. Equations (2.59) and (2.56) imply that, for a.e. $r < r_0$,

$$\lim_{j\to\infty} \mathcal{H}^n(A^\pm_{r,j}) = \mathcal{H}^n(\Sigma^\pm_{r,\sigma})\,. \tag{2.60}$$

Now let $\{A^h_{r,j}\}_{h=0}^\infty$ denote the open connected components of $\partial B_r(x) \setminus \partial E_j$, ordered by decreasing $\mathcal{H}^n$-measure. We claim that

$$\text{if (2.60) holds, then either } A^+_{r,j} \text{ or } A^-_{r,j} \text{ is not contained in } A^0_{r,j}\,. \tag{2.61}$$

Indeed, if for some r we have $A^+_{r,j} \cup A^-_{r,j} \subset A^0_{r,j}$, then by (2.55) and (2.60) we find

$$\mu(B_r(x)) \le \liminf_{j\to\infty} \mu_j(B_r(x)) \le C(n) \liminf_{j\to\infty} \mathcal{H}^n(\partial B_r \setminus A^0_{r,j}) \le C(n)\, r^n\, \sigma\,, \tag{2.62}$$

a contradiction to (2.54) if $\sigma \le \sigma_0(n)$ for a suitable $\sigma_0(n)$. By (2.61) and (2.60),

$$\min\Big\{\mathcal{H}^n(A^0_{r,j}), \mathcal{H}^n(A^1_{r,j})\Big\} \ge \Big(\frac{\sigma_n}{2} - C(n)\,\sigma\Big)\, r^n \qquad \text{for a.e. } r < r_0\,. \tag{2.63}$$

By Lemma 2.11 and (2.63), given $\tau > 0$, if σ is small enough in terms of n and τ, then

$$\min\Big\{\mathcal{H}^{n-1}(\partial^* A^0_{r,j}), \mathcal{H}^{n-1}(\partial^* A^1_{r,j})\Big\} \ge \big(\sigma_{n-1} - \tau\big)\, r^{n-1} \qquad \text{for a.e. } r < r_0\,, \tag{2.64}$$

where $\partial^* A^\alpha_{r,j}$ is the reduced boundary of $A^\alpha_{r,j}$ as a subset of $\partial B_r(x)$. Since $A^0_{r,j}$ is a connected component of $\partial B_r(x) \setminus \partial E_j$ we have

$$\big(\sigma_{n-1} - \tau\big)\, r^{n-1} \le \mathcal{H}^{n-1}(\partial^* A^0_{r,j}) \le \mathcal{H}^{n-1}(\partial B_r(x) \cap \partial E_j)\,. \tag{2.65}$$

Now if $f_j(r) = \mu_j(B_r(x))$ and $f(r) = \mu(B_r(x))$ then by the coarea formula we easily find that $f_j \to f$ a.e. with $\liminf_{j\to\infty} f_j'(r) \le f'(r) \le Df$, where Df denotes the distributional derivative of f. Hence, letting $j \to \infty$ and $\tau \to 0^+$ in (2.65) we obtain $Df \ge \sigma_{n-1}\, r^{n-1}\, dr$ on $(0, r_0)$. As $\omega_n = n\,\sigma_{n-1}$, we conclude that $\theta(x) \ge 1$. We stress once more that so far we have just followed the argument of [DLGM17, Proof of Theorem 2, Step three].

Step two: We use the boundary structure to show that $\theta \ge 2$ $\mathcal{H}^n$-a.e. on $\Omega' \cap (K \setminus \partial^* E)$. Since $\{E^{(0)}, E^{(1)}\, \partial^* E\}$ is an $\mathcal{H}^n$-a.e. partition of $\mathbb{R}^{n+1}$, we can assume that $x \in (E^{(0)} \cup E^{(1)}) \cap K \cap \Omega'$. We consider first the case $x \in E^{(0)}$. Given $\sigma > 0$, up to decrease r_0,

$$\sigma\, r_0^{n+1} \ge \lim_{j\to\infty} |E_j \cap B_{r_0}(x)| = \lim_{j\to\infty} \int_0^{r_0} \mathcal{H}^n(E_j \cap \partial B_r(x))\, dr\,. \tag{2.66}$$

Let us consider the measurable set $I_j \subset (0, r_0)$

$$I_j = \big\{r \in (0, r_0) : A^0_{r,j} \cup A^1_{r,j} \subset \partial B_r(x) \setminus \mathrm{cl}\,(E_j)\big\}\,.$$

We claim that

$$\mathcal{H}^{n-1}(\partial^* A^0_{r,j} \cap \partial^* A^1_{r,j}) = 0 \qquad \forall r \in I_j\,. \tag{2.67}$$

Indeed, if $r \in I_j$, then $A^0_{r,j}$, $A^1_{r,j}$, and $\partial B_r(x) \cap E_j$ are disjoint sets of finite perimeter in $\partial B_r(x)$, and in particular

$$\begin{aligned}\nu_{A^0_{r,j}} &= -\nu_{A^1_{r,j}}\,, && \mathcal{H}^{n-1}\text{-a.e. on } \partial^* A^0_{r,j} \cap \partial^* A^1_{r,j}\,,\\ \nu_{A^0_{r,j}} &= -\nu_{\partial B_r(x)\cap E_j} && \mathcal{H}^{n-1}\text{-a.e. on } \partial^* A^0_{r,j} \cap \partial^*[\partial B_r(x) \cap E_j]\,,\\ \nu_{A^1_{r,j}} &= -\nu_{\partial B_r(x)\cap E_j} && \mathcal{H}^{n-1}\text{-a.e. on } \partial^* A^1_{r,j} \cap \partial^*[\partial B_r(x) \cap E_j]\,.\end{aligned}$$

At the same time, since $\{A^h_{r,j}\}_{h=0}^\infty$ are connected components of $\partial B_r(x) \setminus \partial E_j$,

$$\partial^* A^h_{r,j} \subset \partial^*[\partial B_r(x) \cap E_j] \qquad \text{modulo } \mathcal{H}^n$$

and thus $\mathcal{H}^{n-1}$-a.e. on $\partial^* A^0_{r,j} \cap \partial^* A^1_{r,j}$ we have

$$\nu_{\partial B_r(x)\cap E_j} = -\nu_{A^0_{r,j}} = \nu_{A^1_{r,j}} = -\nu_{\partial B_r(x)\cap E_j}$$

a contradiction. By (2.64) and (2.67), given $\tau > 0$ and provided σ is small enough in terms of n and τ, for a.e. $r \in I_j$ we find

$$\begin{aligned} f_j'(r) &\geq \mathcal{H}^{n-1}(\partial B_r(x) \cap \partial E_j) \geq \mathcal{H}^{n-1}(\partial^* A^0_{r,j} \cup \partial^* A^1_{r,j}) \\ &= \mathcal{H}^{n-1}(\partial^* A^0_{r,j}) + \mathcal{H}^{n-1}(\partial^* A^1_{r,j}) \geq 2\left(\sigma_{n-1} - \tau\right) r^{n-1}\,. \end{aligned}$$

Hence,

$$\begin{aligned} f_j(r_0) &\geq 2\left(\sigma_{n-1} - \tau\right)\frac{r_0^n}{n} - C(n)\int_{(0,r_0)\setminus I_j} r^{n-1}\,dr \\ &\geq 2\left(\sigma_{n-1} - \tau\right)\frac{r_0^n}{n} - C(n)\, r_0^{1/n}\Big(\int_{(0,r_0)\setminus I_j} r^n\,dr\Big)^{(n-1)/n}. \end{aligned} \tag{2.68}$$

We notice that for a.e. $r \in (0, r_0) \setminus I_j$, (2.63) gives

$$\mathcal{H}^n(E_j \cap \partial B_r(x)) \geq \min\Big\{\mathcal{H}^n(A^0_{r,j}), \mathcal{H}^n(A^1_{r,j})\Big\} \geq \Big(\frac{\sigma_n}{2} - C(n)\,\sigma\Big)\, r^n\,,$$

so that (2.66) implies

$$\sigma\, r_0^{n+1} \geq c(n) \limsup_{j\to\infty} \int_{(0,r_0)\setminus I_j} r^n\,dr\,. \tag{2.69}$$

If we combine (2.68) and (2.69) and let $j \to \infty$, then we find

$$f(r_0) = \lim_{j\to\infty} f_j(r_0) \geq 2\left(\sigma_{n-1} - \tau\right)\frac{r_0^n}{n} - C(n)\, r_0^{1/n}\left(\sigma\, r_0^{n+1}\right)^{(n-1)/n}$$

Dividing by r_0^n and letting $r_0 \to 0^+$, $\sigma \to 0^+$ and $\tau \to 0^+$ we find $\theta(x) \geq 2$ whenever $x \in E^{(0)} \cap K \cap \Omega'$. The case when $x \in E^{(1)}$ is analogous and the details are omitted. □

3. Existence of generalized minimizers: Proof of Theorem 1.4

We first give an overview of the proof. It is useful to recall the existence proof for Plateau's problem ℓ from [DLGM17, Theorem 2 and Theorem 4]. Given a minimizing sequence $\{S_j\}_j$ for ℓ, the key idea in [DLGM17] is comparing S_j (through the minimizing sequence inequality $\mathcal{H}^n(S_j) \leq \mathcal{H}^n(S') + 1/j$ for every $S' \in \mathcal{S}$) with competitors S' obtained by replacing $S_j \cap B_r(x)$ either with $\partial B_r(x) \setminus A_j$, where A_j is a $\mathcal{H}^n$-maximal connected component of $\partial B_r(x) \setminus S_j$ (cup competitors); or with the cone over $\partial B_r(x) \cap S_j$ centered at x (cone competitors). The resulting comparison inequalities are then used to show that any Radon measure μ arising as a subsequential weak-star limit of $\mathcal{H}^n \llcorner S_j$ has the form $\mu = \theta\, \mathcal{H}^n \llcorner S$ for an $\mathcal{H}^n$-rectifiable set S that $\mathcal{C}$-spans W, while $\theta \geq 1$ is proved as in step one of Lemma 2.10 by spherical isoperimetry. The minimality of S in ℓ is then immediate from $\theta \geq 1$ and from the lower semicontinuity of Radon measures, since $S \in \mathcal{S}$.

Coming to Theorem 1.4, one first needs to check that $\psi(\varepsilon) < \infty$ if $\ell < \infty$. This is done in step one by using the open neighborhoods of a minimizer S of ℓ as comparison sets for $\psi(\varepsilon)$. We remark that this is the only point of the proof where (1.12) is used. It is important here to allow for sufficiently non-smooth sets in the competition class $\mathcal{E}$: indeed, minimizers of ℓ are known to be smooth only outside of a close $\mathcal{H}^n$-negligible set

in arbitrary dimension. Once $\psi(\varepsilon) < \infty$ is established, we consider a minimizing sequence $\{E_j\}_j$ for $\psi(\varepsilon)$, so that $E_j \in \mathcal{E}$, $|E_j| = \varepsilon$, $\Omega \cap \partial E_j$ is $\mathcal{C}$-spanning W and

$$\mathcal{H}^n(\Omega \cap \partial E_j) \le \mathcal{H}^n(\Omega \cap \partial F) + \frac{1}{j} \qquad \forall F \in \mathcal{E}\,,\ |F| = \varepsilon\,,\ \Omega \cap \partial F \text{ is } \mathcal{C}\text{-spanning } W\,. \tag{3.1}$$

We want to apply (3.1) to the comparison sets constructed in section 2, but, in general, those local variations do not preserve the volume constraint. A family of volume-fixing variations acting uniformly on $\{E_j\}_j$ is constructed through the nucleation lemma (Lemma 2.9) following some ideas introduced by Almgren in the existence theory of minimizing clusters [Alm76]; see steps two and three. In step four we exploit cup and cone competitors to show that, up to extracting subsequences, $\mathcal{H}^n \llcorner (\Omega \cap \partial E_j) \overset{*}{\rightharpoonup} \mu = \theta\, \mathcal{H}^n \llcorner K$ as Radon measures in Ω, and $E_j \to E$ in $L^1_{\rm loc}(\Omega)$, for a pair $(K, E) \in \mathcal{K}$ and for an upper semicontinuous function $\theta \ge 1$ on K. An application of Lemma 2.10 shows that $\theta \ge 2$ $\mathcal{H}^n$-a.e. on $K \setminus \partial^* E$, thus proving $\psi(\varepsilon) \ge \mathcal{F}(K, E)$. In order to show that $\psi(\varepsilon) = \mathcal{F}(K, E)$, and thus that (K, E) is a generalized minimizer of $\psi(\varepsilon)$, we need to exclude that $\Omega \cap \partial E_j$ concentrates area by folding against K, at infinity, or against the wire frame. By using slab competitors we prove that $\Omega \cap \partial E_j$, in its convergence towards K, cannot fold at all near points in $\partial^* E$, and can fold at most twice near points in $K \cap (E^{(0)} \cup E^{(1)})$ (step five). In step six, concentration of area at the boundary is ruled out by a deformation argument based on Lemma 2.4. Finally, in step seven, we exclude area (and volume) concentration at infinity by using exterior cup competitors to construct a uniformly bounded minimizing sequence.

Proof of Theorem 1.4. Step one: We show that

$$\psi(\varepsilon) \le 2\,\ell + C(n)\,\varepsilon^{n/(n+1)} \qquad \forall \varepsilon > 0\,. \tag{3.2}$$

Let S be a minimizer of ℓ, and let $\eta_0 > 0$ be such that (1.12) holds. If $\eta \in (0, \eta_0)$, then the open η-neighborhood $U_\eta(S)$ of S is such that $\Omega \cap \partial U_\eta(S)$ is $\mathcal{C}$-spanning W: otherwise we could find $\eta \in (0, \eta_0)$ and $\gamma \in \mathcal{C}$ such that $\gamma \cap \partial U_\eta(S) = \emptyset$. Since γ is connected, we would either have $\gamma \subset \{x : \mathrm{dist}(x, S) > \eta\}$, against the fact that S is $\mathcal{C}$-spanning; or we would have $\gamma \subset U_\eta(S)$, against (1.12). Hence $\Omega \cap \partial U_\eta(S)$ is $\mathcal{C}$-spanning W.

As proved in [DLGM17], S is $\mathcal{H}^n$-rectifiable, and, because of area monotonicity, such that $\mathcal{H}^n(S \cap B_r(x)) \ge \omega_n\, r^n$ for every $B_r(x) \subset\subset \Omega$. Thus we can use [AFP00, Theorem 2.104] to find

$$|U_\eta(S)| = 2\,\eta\,\mathcal{H}^n(S) + \mathrm{o}(\eta) \qquad \text{as } \eta \to 0^+\,. \tag{3.3}$$

By the coarea formula for Lipschitz maps applied to the distance function from S, see [Mag12, Theorem 18.1, Remark 18.2], we have

$$|U_\eta(S) \cap A| = \int_0^\eta P(U_t(S); A)\, dt = \int_0^\eta \mathcal{H}^n(A \cap \partial U_t(S))\, dt\,, \qquad \forall A \subset \mathbb{R}^{n+1} \text{ open}\,,$$

so that $U_\eta(S)$ is a set of finite perimeter in $\mathbb{R}^{n+1}$ and $\mathcal{H}^n(\partial U_\eta(S) \setminus \partial^* U_\eta(S)) = 0$ for a.e. $\eta > 0$. Summarizing, we have proved that, for a.e. $\eta \in (0, \eta_0)$,

$$F_\eta = \Omega \cap U_\eta(S) \in \mathcal{E}\,, \qquad \Omega \cap \mathrm{cl}\,(\partial^* F_\eta) = \Omega \cap \partial F_\eta \text{ is } \mathcal{C}\text{-spanning } W\,,$$

and, by (3.3),

$$f(\eta) = |F_\eta| = \int_0^\eta P(F_t; \Omega)\, dt = \int_0^\eta P(U_t(S); \Omega)\, dt \le 2\,\eta\,\mathcal{H}^n(S) + \mathrm{o}(\eta)\,.$$

Notice that $f(s)$ is absolutely continuous with $f(\eta) = \int_0^\eta f'(t)\, dt$ and $f'(t) = P(F_t; \Omega)$ for a.e. $t \in (0, \eta)$. Hence, for every $\eta > 0$ there exist $t_1(\eta), t_2(\eta) \in (0, \eta)$ such that $f'(t_1(\eta)) \le f(\eta)/\eta \le f'(t_2(\eta))$. Setting $F_j = F_{t_1(\eta_j)}$ for a suitable $\eta_j \to 0^+$, we get

$$\limsup_{j \to \infty} P(F_j; \Omega) \le 2\,\ell\,,$$

where $|F_j| \to 0^+$. Finally, given $\varepsilon > 0$, we pick j such that $|F_j| < \varepsilon$, and construct a competitor for $\psi(\varepsilon)$ by adding to F_j a disjoint ball of volume $\varepsilon - |F_j|$. In this way, $\psi(\varepsilon) \le P(F_j;\Omega) + C(n)\,\big(\varepsilon - |F_j|\big)^{n/(n+1)}$, and (3.2) is found by letting $j \to \infty$.

Since $\psi(\varepsilon) < \infty$, we can now consider a minimizing sequence $\{E_j\}_{j=1}^\infty$ for $\psi(\varepsilon)$. Given that $P(E_j) \le \mathcal{H}^n(\partial\Omega) + \mathcal{H}^n(\Omega \cap \partial E_j) \le \mathcal{H}^n(\partial\Omega) + \psi(\varepsilon) + 1$ for j large, and that $|E_j| = \varepsilon$ for every j, there exist a set of finite perimeter $E \subset \Omega$ and a Radon measure μ in Ω such that, up to extracting subsequences,

$$E_j \to E \quad \text{in } L^1_{\rm loc}(\Omega)\,, \quad \mu_j = \mathcal{H}^n \llcorner (\Omega \cap \partial E_j) \overset{*}{\rightharpoonup} \mu \quad \text{as Radon measures on } \Omega\,, \tag{3.4}$$

as $j \to \infty$. We consider the set, relatively closed in Ω, defined by

$$K = \Omega \cap {\rm spt}\mu = \big\{x \in \Omega : \mu(B_r(x)) > 0 \quad \forall r > 0\big\}\,,$$

and claim that

$$K \text{ is } \mathcal{C}\text{-spanning } W\,, \qquad \Omega \cap \partial^* E \subset K\,. \tag{3.5}$$

Indeed, the first claim in (3.5) is obtained by applying Lemma 2.1 to $K_j = \Omega \cap \partial E_j$; and if $x \in \Omega \cap \partial^* E$ and $B_r(x) \subset \Omega$, then

$$0 < P(E; B_r(x)) \le \liminf_{j\to\infty} P(E_j; B_r(x)) \le \liminf_{j\to\infty} \mu_j(B_r(x)) \le \mu({\rm cl}\,(B_r(x)))$$

so that $x \in K$. Notice that, at this stage, we still do not know if $(K, E) \in \mathcal{K}$: we still need to show that K is $\mathcal{H}^n$-rectifiable and, possibly up to Lebesgue negligible modifications, that E is open with $\Omega \cap {\rm cl}\,(\partial^* E) = \Omega \cap \partial E$. Moreover, we just have $|E| \le \varepsilon$ (possible volume loss at infinity), and we know nothing about the structure of μ.

Step two: We show the existence of $\tau > 0$ such that for every E_j there exist $x_j^1, x_j^2 \in \mathbb{R}^{n+1}$ such that $\{{\rm cl}\,(B_{2\tau}(x_j^1)), {\rm cl}\,(B_{2\tau}(x_j^2)), W\}$ is disjoint and

$$|E_j \cap B_\tau(x_j^1)| = \kappa_1\,, \qquad |E_j \cap B_\tau(x_j^2)| = \kappa_2\,, \tag{3.6}$$

for some $\kappa_1, \kappa_2 \in (0, |B_\tau|/2]$ depending on n, τ, ε and ℓ only. With τ_0 as in (1.11), for $M \in \mathbb{N} \setminus \{0\}$ to be chosen later on, and by compactness of W, we can pick $\tau > 0$ so that

$$(M+1)\,\tau < \tau_0\,, \qquad |B_{M\,\tau}| < \frac{\varepsilon}{4}\,, \qquad |I_{(M+1)\tau}(W) \setminus W| < \frac{\varepsilon}{2}\,. \tag{3.7}$$

The value σ in Lemma 2.9 corresponding to E_j and $T = I_{M\,\tau}(W)$ is given by

$$\min\Big\{\frac{|E_j \setminus I_\tau(T)|}{\tau\, P(E_j; \mathbb{R}^{n+1} \setminus T)}, \frac{\xi(n)}{n+1}\Big\} \ge \min\Big\{\frac{\varepsilon/2}{\tau\,(\psi(\varepsilon)+1)}, \frac{\xi(n)}{n+1}\Big\} > 0\,,$$

since $|E_j \setminus I_\tau(T)| \ge \varepsilon/2$ by (3.7), and since $P(E_j;\Omega) \le \psi(\varepsilon) + 1$. Therefore, setting

$$\sigma_1 = \min\Big\{\frac{\varepsilon/2}{\tau\,(\psi(\varepsilon)+1)}, \frac{\xi(n)}{n+1}\Big\}\,,$$

an application of Lemma 2.9 yields $y_j \in \mathbb{R}^{n+1} \setminus I_{(M+1)\tau}(W)$ such that

$$|E_j \cap B_\tau(y_j)| \ge \min\Big\{\Big(\frac{\sigma_1}{2\xi(n)}\Big)^{n+1} \tau^{n+1}\,, \frac{|B_\tau|}{2}\Big\} = \kappa_1\,,$$

so that $\kappa_1 \in (0, |B_\tau|/2]$ depends on n, ℓ, ε, and τ only (observe that this is a consequence of (3.2)). The continuous map $x \mapsto |E_j \cap B_\tau(x)|$ takes a value larger than κ_1 at $y_j \in \mathbb{R}^{n+1} \setminus I_{(M+1)\,\tau}(W)$; at the same time, by (1.11), $\mathbb{R}^{n+1} \setminus I_{(M+1)\,\tau}(W)$ is pathwise connected, and $|E_j \cap B_\tau(x)| \to 0$ as $|x| \to \infty$ in $\mathbb{R}^{n+1} \setminus I_{(M+1)\,\tau}(W)$. Therefore we can find $x_j^1 \in \mathbb{R}^{n+1} \setminus I_{(M+1)\,\tau}(W)$ such that the first identity in (3.6) holds and $\{{\rm cl}\,(B_{(M+1)\,\tau}(x_j^1)), W\}$

is disjoint. Setting $B = \mathrm{cl}\,(B_{(M-2)\tau}(x_j^1))$, the value σ in Lemma 2.9 corresponding to E_j and $T = I_\tau(W) \cup B$ is given by

$$\min\Big\{\frac{|E_j \setminus I_\tau(T)|}{\tau\, P(E_j;\mathbb{R}^{n+1}\setminus T)}, \frac{\xi(n)}{n+1}\Big\} \ge \min\Big\{\frac{\varepsilon/4}{\tau\,(\psi(\varepsilon)+1)}, \frac{\xi(n)}{n+1}\Big\} > 0\,,$$

so that, after setting

$$\sigma_2 = \min\Big\{\frac{\varepsilon/4}{\tau\,(\psi(\varepsilon)+1)}, \frac{\xi(n)}{n+1}\Big\}\,,$$

we can find $z_j \in \mathbb{R}^{n+1} \setminus (I_{2\tau}(W) \cup \mathrm{cl}\,(B_{(M-1)\,\tau}(x_j^1)))$ such that

$$|E_j \cap B_\tau(z_j)| \ge \min\Big\{\Big(\frac{\sigma_2}{2\xi(n)}\Big)^{n+1}\tau^{n+1}\,, \frac{|B_\tau|}{2}\Big\} = \kappa_2\,,$$

with $\kappa_2 \in (0, |B_\tau|/2]$ depending on n, ℓ, ε, and τ only. Since $I_{2\tau}(W)$ and $\mathrm{cl}\,(B_{(M-1)\tau}(x_j^1))$ are disjoint and since $\mathbb{R}^{n+1} \setminus I_{2\tau}(W)$ is pathwise connected by (1.11), we easily check that $\mathbb{R}^{n+1} \setminus (I_{2\tau}(W) \cup \mathrm{cl}\,(B_{(M-1)\tau}(x_j^1)))$ is pathwise connected. By continuity,

$$\exists\, x_j^2 \in \mathbb{R}^{n+1} \setminus (I_{2\tau}(W) \cup \mathrm{cl}\,(B_{(M-1)\,\tau}(x_j^1))) \tag{3.8}$$

such that the second identity in (3.6) holds. Finally, (3.8) implies that the family of sets $\{\mathrm{cl}\,(B_{(M-3)\tau}(x_j^1)), \mathrm{cl}\,(B_{2\tau}(x_j^2)), W\}$ is disjoint. We pick $M = 5$ to conclude the proof.

Step three: In this step we show that (3.1) can be modified to allow for comparison with local variations F_j of E_j that do not necessarily preserve the volume constraint. More precisely, we prove the existence of positive constants r_* and C_* (depending on the whole sequence $\{E_j\}_j$, and thus uniform in j) such that if $x \in \Omega$, $r < r_*$ and $\{F_j\}_j$ is an **admissible local variation of** $\{E_j\}_j$ **in** $B_r(x)$, in the sense that

$$F_j \in \mathcal{E}\,, \qquad F_j \Delta E_j \subset\subset B_r(x)\,, \qquad \Omega \cap \partial F_j \text{ is } \mathcal{C}\text{-spanning } W\,, \tag{3.9}$$

(notice that we do not require $B_r(x) \subset \Omega$), then

$$\mathcal{H}^n(\Omega \cap \partial E_j) \le \mathcal{H}^n(\Omega \cap \partial F_j) + C_*\,\Big||E_j| - |F_j|\Big| + \frac{1}{j}\,. \tag{3.10}$$

We first claim that if $B_j \subset \Omega$ is a ball with $\mathrm{dist}(B_j, B_r(x)) > 0$, $\psi : \Omega \to \Omega$ is a diffeomorphism with $\psi(B_j) \subset B_j$ and $\{\psi \ne \mathrm{id}\,\} \subset\subset B_j$, and if

$$G_j = \Big(F_j \cap B_r(x)\Big) \cup \Big(\psi(E_j) \cap B_j\Big) \cup \Big(E_j \setminus (B_j \cup B_r(x))\Big) \tag{3.11}$$

then $G_j \in \mathcal{E}$ and $\Omega \cap \partial G_j$ is $\mathcal{C}$-spanning W. The fact that G_j is open is obvious since G_j is equal to E_j in a neighborhood of $\Omega \setminus (B_r(x) \cup B_j)$, to F_j in a neighborhood of $B_r(x)$, and to $\psi(E_j)$ in a neighborhood of B_j, where E_j, F_j and $\psi(E_j)$ are open, and where $\mathrm{dist}(B_j, B_r(x)) > 0$; this also shows that ∂G_j is equal to ∂E_j in a neighborhood of $\Omega \setminus (B_r(x) \cup B_j)$, to ∂F_j in a neighborhood of $B_r(x)$, and to $\partial\psi(E_j) = \psi(\partial E_j)$ in a neighborhood of B_j, so that $\Omega \cap \partial G_j$ is $\mathcal{H}^n$-rectifiable and, thanks to (3.9) and Lemma 2.3, that $\Omega \cap \partial G_j$ is $\mathcal{C}$-spanning W. Having proved the claim, we only have to construct sets G_j as in (3.11) and such that

$$|G_j| = \varepsilon\,, \qquad \mathcal{H}^n(\Omega \cap \partial G_j) \le \mathcal{H}^n(\Omega \cap \partial F_j) + C_*\,\big||E_j| - |F_j|\big|\,, \tag{3.12}$$

in order to deduce (3.10) from (3.1). To this aim, let $\{x_j^k\}_{k=1,2}$ be as in step two: the sets $\{(E_j - x_j^k) \cap B_\tau(0)\}_j$ are bounded in $B_\tau(0)$, and have uniformly bounded perimeters, so that, up to extracting a subsequence, for each $k = 1, 2$ there exists a set of finite perimeter $E_*^k \subset B_\tau(0)$ such that $(E_j - x_j^k) \cap B_\tau(0) \to E_*^k$ in $L^1(\mathbb{R}^{n+1})$. The crucial point is that, by (3.6) and since $\kappa_k \in (0, |B_\tau(0)|/2]$, we must have

$$B_\tau(0) \cap \partial^* E_*^k \ne \emptyset\,.$$

Hence, by arguing as in [Mag12, Section 29.6], we can find positive constants C'_* and ε_* such that for every set of finite perimeter $E' \subset B_\tau(0)$ with

$$|E' \Delta E^k_*| < \varepsilon_* \,,$$

there exists a C^1-map $\Phi_k : (-\varepsilon_*, \varepsilon_*) \times B_\tau(0) \to B_\tau(0)$ such that, for each $|v| < \varepsilon_*$: (i) $\Phi_k(v, \cdot)$ is a diffeomorphism with $\{\Phi_k(v, \cdot) \neq \mathrm{Id}\} \subset\subset B_\tau(0)$; (ii) $|\Phi_k(v, E')| = |E'| + v$; (iii) if Σ is an $\mathcal{H}^n$-rectifiable set in $B_\tau(0)$, then

$$\Big|\mathcal{H}^n(\Phi_k(v, \Sigma)) - \mathcal{H}^n(\Sigma)\Big| \le C'_* \, \mathcal{H}^n(\Sigma) \, |v| \,.$$

By taking $E' = (E_j - x^k_j) \cap B_\tau(0)$ (for j large enough), by composing the maps Φ_k with a translation by x^k_j, and then by extending the resulting maps as the identity map outside of $B_\tau(x^k_j)$, we prove the existence of C^1-maps $\Psi_k : (-\varepsilon_*, \varepsilon_*) \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that, for each $|v| < \varepsilon_*$: (i) $\Psi_k(v, \cdot)$ is a diffeomorphism with $\{\Psi_k(v, \cdot) \neq \mathrm{Id}\} \subset\subset B_\tau(x^k_j)$; (ii) $|\Psi_k(v, E_j)| = |E_j| + v$; (iii) if Σ is an $\mathcal{H}^n$-rectifiable set in $\mathbb{R}^{n+1}$, then

$$\Big|\mathcal{H}^n(\Psi_k(v, \Sigma)) - \mathcal{H}^n(\Sigma)\Big| \le C'_* \, \mathcal{H}^n(\Sigma) \, |v| \,.$$

Finally, we set

$$r_* = \min\Big\{\tau, \Big(\frac{\varepsilon_*}{2\,\omega_{n+1}}\Big)^{1/(n+1)}\Big\}\,, \qquad B_j = B_\tau(x^{k(j)}_j)$$

where $k = k(j) \in \{1, 2\}$ is selected so that $\mathrm{dist}(B_r(x), B_j) > 0$ (this is possible because $r_* \le \tau$ and $\{\mathrm{cl}\,(B_{2\tau}(x^1_j)), \mathrm{cl}\,(B_{2\tau}(x^2_j))\}$ are disjoint). We finally define G_j by (3.11) with

$$\psi = \Psi_{k(j)}(v_j, \cdot)\,, \qquad v_j = |E_j \cap B_r(x)| - |F_j \cap B_r(x)|\,,$$

as we are allowed to do since $E_j \Delta F_j \subset\subset B_r(x)$ and thus $|v_j| \le \omega_{n+1}\, r^{n+1}_* \le \varepsilon_*/2$. To prove (3.12): first, we have $G_j \Delta F_j \subset\subset \Omega \setminus \mathrm{cl}\,(B_r(x))$, while property (ii) of $\Psi_{k(j)}$ gives

$$\begin{aligned} |G_j| - |E_j| &= |\Psi_{k(j)}(v_j, E_j) \cap B_j| + |F_j \cap B_r(x)| - |E_j \cap B_j| - |E_j \cap B_r(x)| \\ &= |\Psi_{k(j)}(v_j, E_j) \cap B_j| - v_j - |E_j \cap B_j| = 0\,; \end{aligned}$$

second, property (iii) applied to the $\mathcal{H}^n$-rectifiable set $\Sigma = B_j \cap \partial E_j$ gives

$$\begin{aligned} &\mathcal{H}^n(\Omega \cap \partial G_j) - \mathcal{H}^n(\Omega \cap \partial F_j) \\ = \;& \mathcal{H}^n\Big(\Psi_{k(j)}(v_j, B_j \cap \partial E_j)\Big) - \mathcal{H}^n(B_j \cap \partial E_j) \le C'_* \, |v_j| \, \mathcal{H}^n(B_j \cap \partial E_j) \end{aligned}$$

so that (3.12) follows by taking $C_* = C'_* \, (\psi(\varepsilon) + 1)$.

Step four: In this step we apply (3.10) to the cup and cone competitors constructed in section 2 and show that $K = \Omega \cap \mathrm{spt}\mu$ is relatively compact in Ω and $\mathcal{H}^n$-rectifiable, that $\mu = \theta\, \mathcal{H}^n \llcorner K$ with $\theta \ge 1$ on K and $\theta \ge 2$ $\mathcal{H}^n$-a.e. on $K \setminus \partial^* E$, and, finally, that $(K, E) \in \mathcal{K}$. To this end, pick $x \in K$, set $d(x) = \mathrm{dist}(x, W) > 0$, and let

$$f_j(r) = \mu_j(B_r(x)) = \mathcal{H}^n(B_r(x) \cap \partial E_j)\,, \qquad f(r) = \mu(B_r(x))\,, \qquad \text{for every } r \in (0, d(x))\,.$$

Denoting by Df the distributional derivative of f, and by f' its classical derivative, the coarea formula (see [DLGM17, Step one, proof of Theorem 2]) gives

$$f_j \to f \text{ a.e. on } (0, d(x))\,, \quad Df_j \ge f'_j \, dr\,, \quad Df \ge f' \, dr\,, \quad f' \ge g = \liminf_{j\to\infty} f'_j\,, \tag{3.13}$$

$$f'_j(r) \ge \mathcal{H}^{n-1}(\partial B_r(x) \cap \partial E_j) \qquad \forall j \text{ and for a.e. } r \in (0, d(x))\,. \tag{3.14}$$

Now let $\eta \in (0, r/2)$, let A_j denote an $\mathcal{H}^n$-maximal open connected component of $\partial B_r(x) \setminus \partial E_j$, and let F_j be the cup competitor defined by E_j and A_j as in Lemma 2.5:

$$\begin{aligned} F_j &= \big(E_j \setminus \mathrm{cl}\,(B_r(x))\big) \cup N_\eta(Y_j)\,, && (3.15)\\ Y_j &= \partial B_r(x) \setminus \big(\mathrm{cl}\,(E_j \cap \partial B_r(x)) \cup \mathrm{cl}\,(A_j)\big)\,, \qquad \text{if } E_j \cap A_j = \emptyset\,, \\ F_j &= \big(E_j \cup B_r(x)\big) \setminus \mathrm{cl}\,\big(N_\eta(Y_j)\big)\,, && (3.16)\\ Y_j &= (E_j \cap \partial B_r(x)) \setminus \mathrm{cl}\,(A_j)\,, \qquad \text{if } A_j \subset E_j\,, \end{aligned}$$

see Figure 2.1. In both cases, $\{F_j\}_j$ is an admissible local variation of $\{E_j\}_j$ in $B_{r'}(x)$ for some $r' > r$, and by (2.8), for a.e. $r < d(x)$ we have

$$\limsup_{\eta \to 0^+} \mathcal{H}^n(\Omega \cap \partial F_j) \le \mathcal{H}^n(\partial E_j \setminus B_r(x)) + 2\,\mathcal{H}^n(\partial B_r(x) \setminus A_j)$$

so that, by (3.10), for a.e. $r < \min\{d(x), r_*\}$, we have

$$f_j(r) \le 2\,\mathcal{H}^n(\partial B_r(x) \setminus A_j) + C_* \limsup_{\eta \to 0^+} \big||E_j| - |F_j|\big| + \frac{1}{j}\,. \tag{3.17}$$

The estimate of $||E_j| - |F_j||$ is different depending on whether F_j is given by (3.15) or by (3.16). In both cases we make use of the Euclidean isoperimetric inequality

$$(n+1)\,|B_1|^{1/(n+1)}\,|U|^{n/(n+1)} \le P(U)\,, \qquad \forall U \subset \mathbb{R}^{n+1}\,,$$

and we also need the perimeter identities

$$\begin{aligned} P(E_j \cap B_r(x)) &= P(E_j; B_r(x)) + \mathcal{H}^n(E_j \cap \partial B_r(x))\,, \\ P(B_r(x) \setminus E_j) &= P(E_j; B_r(x)) + \mathcal{H}^n(\partial B_r(x) \setminus E_j)\,, \end{aligned} \tag{3.18}$$

which hold for a.e. $r > 0$, with the exceptional set of r-values that can be made independent from j. We now take F_j as in (3.15): up to further decrease the value of r_* so to entail $C_*\, r_*/(n+1) \le 1/2$, and assuming that $r < r_*$, we have

$$\begin{aligned} C_* \Big||E_j| - |F_j|\Big| &\le C_*\,|E_j \cap B_r(x)| + C_*\,c(n)\,r^n\,\eta \\ &\le C_*\,|B_1|^{1/(n+1)}\,r\,|E_j \cap B_r(x)|^{n/(n+1)} + C_*\,c(n)\,r^n\,\eta \\ &\le \frac{C_*}{n+1}\,r_*\,P(E_j \cap B_r(x)) + C_*\,c(n)\,r^n\,\eta \\ &\le \frac{1}{2}\Big\{P(E_j; B_r(x)) + \mathcal{H}^n(E_j \cap \partial B_r(x))\Big\} + C_*\,c(n)\,r^n\,\eta \\ &\le \frac{1}{2}\Big\{f_j(r) + \mathcal{H}^n(\partial B_r(x) \setminus A_j)\Big\} + C_*\,c(n)\,r^n\,\eta\,, \end{aligned} \tag{3.19}$$

where in the last inequality we have used $\partial^* E_j \subset \partial E$ and $A_j \cap E_j = \emptyset$ (that is the assumption under which F_j is chosen as in (3.15)). If instead we take F_j as in (3.16), then

$$\begin{aligned} C_* \Big||E_j| - |F_j|\Big| &= C_* \Big||E_j \cap B_r(x)| - |F_j \cap B_r(x)|\Big| = C_* \Big||B_r(x) \setminus E_j| - |B_r(x) \setminus F_j|\Big| \\ &\le C_*\,|B_1|^{1/(n+1)}\,r\,|B_r(x) \setminus E_j|^{n/(n+1)} + C_*\,|N_\eta(\partial B_r(x) \cap E_j \setminus \mathrm{cl}\,(A_j))| \\ &\le \frac{1}{2}\Big\{P(E_j; B_r(x)) + \mathcal{H}^n(\partial B_r(x) \setminus E_j)\Big\} + C_*\,c(n)\,r^n\,\eta \\ &\le \frac{1}{2}\Big\{f_j(r) + \mathcal{H}^n(\partial B_r(x) \setminus A_j)\Big\} + C_*\,c(n)\,r^n\,\eta\,, \end{aligned} \tag{3.20}$$

where in the last inequality we have used $\partial^* E_j \subset \partial E_j$ and $A_j \subset E_j$ (the assumption corresponding to (3.16)). By combining (3.17) with (3.19) and (3.20), we conclude that

$$\frac{f_j(r)}{2} \le 3\,\mathcal{H}^n(\partial B_r(x) \setminus A_j) + \frac{1}{j}\,, \qquad \text{for a.e. } r < \min\{r_*, d(x)\}\,. \tag{3.21}$$

By the spherical isoperimetric inequality, Lemma 2.11, and by (3.14), for a.e. $r < d(x)$,

$$\mathcal{H}^n(\partial B_r(x) \setminus A_j) \le C(n)\, \mathcal{H}^{n-1}(\partial B_r(x) \cap \partial E_j)^{n/(n-1)} \le C(n)\, f_j'(r)^{n/(n-1)}\,,$$

which combined with (3.21) and (3.13), allows us to conclude (letting $j \to \infty$), that

$$f(r) \le C(n)\, f'(r)^{n/(n-1)}\,, \qquad \text{for a.e. } r < \min\{r_*, d(x)\}\,. \tag{3.22}$$

Since $x \in \mathrm{spt}\mu$, f is positive, and thus (3.22) implies the existence of $\theta_0(n) > 0$ such that

$$\mu(B_r(x)) \ge \theta_0\, \omega_n\, r^n \qquad \forall x \in K\,, r < r_*\,, B_r(x) \subset\subset \Omega\,. \tag{3.23}$$

Since $K = \Omega \cap \mathrm{spt}\mu$, by [Mat95, Theorem 6.9] and (3.23) we obtain

$$\mu \ge \theta_0\, \mathcal{H}^n \llcorner K \qquad \text{on } \Omega\,. \tag{3.24}$$

As a consequence of $\mu(\Omega) < \infty$ and of (3.23) we deduce that K is bounded, thus relatively compact in Ω. In turn, $\partial^* E \subset K$ implies the boundedness of E. Notice that we have not excluded $|E| < \varepsilon$ yet.

To further progress in the analysis of μ, given $\eta \in (0, r/2)$ let use now denote by F_j the set corresponding to η constructed in Lemma 2.8, so that, by (2.42), for a.e. $r < d(x)$,

$$\limsup_{\eta \to 0^+} \mathcal{H}^n(\Omega \cap \partial F_j) \le \mathcal{H}^n(\partial E_j \setminus B_r(x)) + \frac{r}{n}\, \mathcal{H}^{n-1}(\partial E_j \cap \partial B_r(x))\,. \tag{3.25}$$

Using that $\{F_j\}_j$ is an admissible local variation of $\{E_j\}_j$ in $B_r(x)$, and combining (3.10) and (3.25) with $||E_j| - |F_j|| \le C(n)\, r^{n+1}$, we find that

$$\mathcal{H}^n(B_r(x) \cap \partial E_j) \le \frac{r}{n}\, f_j'(r) + C_*\, r^{n+1} + \frac{1}{j}\,,$$

so that, as $j \to \infty$, $f(r) \le (r/n)\, f'(r) + C_*\, r^{n+1}$. By combining this last inequality with $Df \ge f'(r)\, dr$ and (3.23) we find that

$$\begin{aligned} D\,(e^{\Lambda r} f(r)/r^n) &= \frac{n\, e^{\Lambda r}}{r^{n+1}} \Big\{ \frac{r}{n}\, Df + \Big(\frac{r\,\Lambda}{n}\, f(r) - f(r) \Big)\, dr \Big\} \\ &\ge \frac{n\, e^{\Lambda r}}{r^{n+1}} \Big\{ f(r) - C_*\, r^{n+1} + \frac{r\,\Lambda}{n}\, f(r) - f(r) \Big\}\, dr \\ &= \frac{n\, e^{\Lambda r}}{r^n} \Big\{ -C_*\, r^n + \frac{\Lambda\, f(r)}{n} \Big\}\, dr \ge n\, e^{\Lambda r} \Big\{ -C_* + \frac{\Lambda\, \theta_0\, \omega_n}{n} \Big\}\, dr \end{aligned}$$

so that, setting $\Lambda \ge n\, C_*/(\theta_0 \omega_n)$, we have proved

$$e^{\Lambda r}\, \frac{\mu(B_r(x))}{r^n} \qquad \text{is non-decreasing on } r < \min\{r_*, d(x)\}\,. \tag{3.26}$$

By (3.26) and (3.24) we find that

$$\theta(x) = \lim_{r \to 0^+} \frac{\mu(B_r(x))}{\omega_n\, r^n} \quad \text{exists in } (0, \infty) \text{ for every } x \in K\,.$$

By Preiss' theorem, $\mu = \theta\, \mathcal{H}^n \llcorner K^*$ for a Borel function θ and a countably $\mathcal{H}^n$-rectifiable set $K^* \subset \Omega$. Since $K = \Omega \cap \mathrm{spt}\mu$, we have $\mathcal{H}^n(K^* \setminus K) = 0$, while (3.24) gives $\mathcal{H}^n(K \setminus K^*) = 0$. Thus K is countably $\mathcal{H}^n$-rectifiable and $\mu = \theta\, \mathcal{H}^n \llcorner K$. Moreover, θ is upper semicontinuous on K thanks to (3.26). Finally, consider the open set

$$E^* = \big\{ x \in \Omega : \exists r > 0 \text{ s.t. } |B_r(x)| = |E \cap B_r(x)| \big\}\,.$$

The topological boundary of E^* is equal to

$$\partial E^* = \big\{ x \in \mathrm{cl}\,(\Omega) : 0 < |E \cap B_r(x)| < |B_r(x)| \quad \forall r > 0 \big\}\,,$$

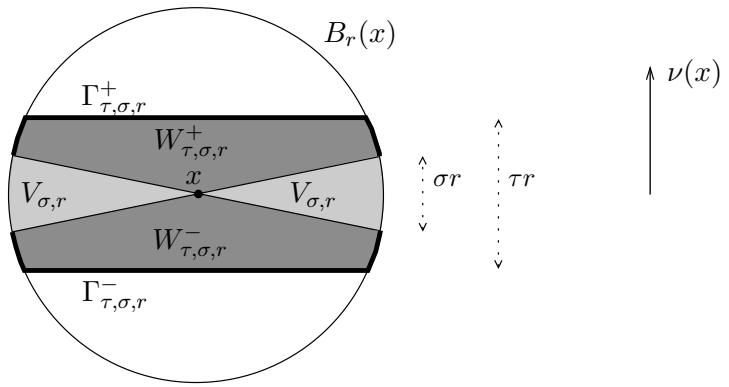

FIGURE 3.1. The sets defined in (3.27). Here $\sigma < \tau < 1$, and $S_{\tau,r}$ is decomposed into a central open cone $V_{\sigma,r}$ of small amplitude σ, the upper and lower open regions $W^{\pm}_{\tau,\sigma,r}$, and the closed cone $S_{\tau,r} \cap \partial V_{\sigma,r}$. For $r \le r_0(\sigma, x)$, $B_r(x) \cap K$ lies inside $V_{\sigma,r}$ by approximate differentiability of K at x and by the density estimate (3.23). When $x \in \partial^* E$, if we choose $\nu(x) = \nu_E(x)$, then the divergence theorem implies that E fills up the whole $W^-_{\tau,\sigma,r}$, and leaves empty $W^+_{\tau,\sigma,r}$.

so that $\Omega \cap \mathrm{cl}\,(\partial^* E) = \Omega \cap \partial E^*$ by [Mag12, Proposition 12.19]. Clearly $E^* \subset E^{(1)}$: moreover, if $x \in E^{(1)} \setminus E^*$, then $0 < |E \cap B_r(x)| < |B_r(x)|$ for every $r > 0$, and thus $x \in \partial E^*$. In particular,

$$\Omega \cap (E^{(1)} \setminus E^*) \subset \Omega \cap \partial E^* = \Omega \cap \mathrm{cl}\,(\partial^* E) \subset K\,,$$

where K is $\mathcal{H}^n$-rectifiable, and thus Lebesgue negligible. Since $\mathcal{H}^n(\partial\Omega) < \infty$, we have proved $\mathcal{H}^n(E^{(1)} \setminus E^*) < \infty$, and thus $|E^{(1)} \Delta E^*| = 0$. By the Lebesgue's points theorem, E^* is equivalent to E, so that $\partial^* E = \partial^* E^*$. Replacing E with E^* we find $(K, E) \in \mathcal{K}$. Finally, the lower bounds $\theta \ge 1$ $\mathcal{H}^n$-a.e. on K and $\theta \ge 2$ $\mathcal{H}^n$-a.e. on $K \setminus \partial^* E$ follow by applying Lemma 2.10 with $\Omega' = \Omega$: notice indeed that assumptions (2.54) and (2.55) in Lemma 2.10 hold by (3.23) and by (3.21).

Step five: We show that $\theta(x) \le 1$ at every $x \in \Omega \cap \partial^* E$ and that $\theta(x) \le 2$ at every $x \in K \cap (E^{(0)} \cup E^{(1)})$ such that K admits an approximate tangent plane at x (thus, that $\theta \le 2$ $\mathcal{H}^n$-a.e. on $K \setminus \partial^* E$). We set $T_x K = \nu(x)^\perp$ (and by default $\nu(x) = \pm\nu_E(x)$ for $\mathcal{H}^n$-a.e. $x \in \partial^* E$), and let $B_{2r}(x) \subset\subset \Omega$. For $\tau \in (0,1)$ and $\sigma \in (0, \tau)$ we set

$$\begin{aligned} S_{\tau,r} &= \big\{ y \in B_r(x) : |(y-x)\cdot\nu(x)| < \tau\, r \big\}\,, \\ V_{\sigma,r} &= \big\{ y \in B_r(x) : |(y-x)\cdot\nu(x)| < \sigma\, |y-x| \big\} \subset S_{\sigma,r} \subset S_{\tau,r}\,, \\ W^{\pm}_{\tau,\sigma,r} &= \big(S_{\tau,r} \setminus \mathrm{cl}\,(V_{\sigma,r})\big) \cap \{ y : (y-x)\cdot\nu_E(x) \gtrless 0 \}\,, \\ \Gamma^{\pm}_{\tau,\sigma,r} &= \partial S_{\tau,r} \cap \partial W^{\pm}_{\tau,\sigma,r}\,, \end{aligned} \tag{3.27}$$

that are depicted in Figure 3.1. By (3.23) and since $\mathcal{H}^n \llcorner (K-x)/\rho \overset{*}{\rightharpoonup} \mathcal{H}^n \llcorner T_x K$ as $\rho \to 0^+$, a standard argument shows the existence of $r_0 = r_0(\sigma, x) > 0$ such that $K \cap B_r(x) \subset S_{\sigma,r}$ for every $r < r_0$, or, equivalently,

$$K \cap B_{r_0}(x) \subset V_{\sigma,r_0} \cup \{x\}\,. \tag{3.28}$$

In particular

$$\mu(S_{\tau,r}) = \mu(B_r(x))\,, \qquad \forall r < r_0\,. \tag{3.29}$$

We also notice that for a.e. value of r we have

$$\partial S_{\tau,r} \cap \partial E_j \text{ is } \mathcal{H}^{n-1}\text{-rectifiable} \qquad \forall j\,. \tag{3.30}$$

We now introduce the family of open sets

$$\begin{aligned}\mathcal{A}^{\rm out}_{r,j} &= \Big\{A\subset\partial S_{\tau,r}: A \text{ is an open connected component} \\ &\qquad\qquad \text{of } \partial S_{\tau,r}\setminus\partial E_j \text{ and } A \text{ is } \textbf{disjoint} \text{ from } E_j\Big\}\,,\\ \mathcal{A}^{\rm in}_{r,j} &= \Big\{A\subset\partial S_{\tau,r}: A \text{ is an open connected component} \\ &\qquad\qquad \text{of } \partial S_{\tau,r}\setminus\partial E_j \text{ and } A \text{ is } \textbf{contained} \text{ in } E_j\Big\}\,,\end{aligned}$$

and denote by $A^{\rm out}_{r,j}$ and $A^{\rm in}_{r,j}$ $\mathcal{H}^n$-maximal elements of $\mathcal{A}^{\rm out}_{r,j}$ and $\mathcal{A}^{\rm in}_{r,j}$ respectively. Finally, given $\eta\in(0,r/2)$, we let $F^\star_j$ be the slab competitor defined by E_j, $A^\star_{r,j}$ and τ in $B_{2r}(x)$ for $\star\in\{\text{out},\text{in}\}$ as in Lemma 2.7: accordingly, $F^\star_j\in\mathcal{E}$, $\Omega\cap\partial F^\star_j$ is $\mathcal{C}$-spanning W,

$$F^\star_j\setminus\text{cl}\,(S_{\tau,r}) = E_j\setminus\text{cl}\,(S_{\tau,r})\,, \tag{3.31}$$

$$\partial S_{\tau,r}\cap\partial F^\star_j = \partial S_{\tau,r}\setminus A^\star_{r,j} \qquad \text{modulo } \mathcal{H}^n\,, \tag{3.32}$$

and

$$\limsup_{\eta\to0^+}\mathcal{H}^n(S_{\tau,r}\cap\partial F^\star_j)\le C(n,\tau)\begin{cases}\mathcal{H}^n\big(\partial S_{\tau,r}\setminus(A^{\rm out}_{r,j}\cup E_j)\big)\,, & \text{if } \star=\text{out}\,,\\ \mathcal{H}^n\big((E_j\cap\partial S_{\tau,r})\setminus A^{\rm in}_{r,j}\big)\,, & \text{if } \star=\text{in}\,;\end{cases} \tag{3.33}$$

see (2.37), (2.38), (2.39) and (2.40). By (3.10), $\mathcal{H}^n(\partial S_{\tau,r}\cap\partial E_j)=0$ and (3.31),

$$\mathcal{H}^n(S_{\tau,r}\cap\partial E_j)\le\mathcal{H}^n(\text{cl}\,(S_{\tau,r})\cap\partial F^\star_j)+C_*\,c(n)\,r^{n+1}+\frac{1}{j}\,,\qquad\forall\star\in\{\text{out},\text{in}\}\,.$$

By (3.32) and (3.33), taking the limit first as $\eta\to0^+$ and then as $j\to\infty$, and by taking also into account that $\mu_j\overset{*}{\rightharpoonup}\mu$ and that (3.29) holds, we find, in the case $\star=\text{out}$, that

$$\begin{aligned}\mu(B_r(x)) &\le \limsup_{j\to\infty}\mathcal{H}^n(E_j\cap\partial S_{\tau,r}) \\ &\quad+C(n,\tau)\limsup_{j\to\infty}\mathcal{H}^n\big(\partial S_{\tau,r}\setminus(A^{\rm out}_{r,j}\cup E_j)\big)+C_*\,c(n)\,r^{n+1}\,,\end{aligned} \tag{3.34}$$

and, in the case $\star=\text{in}$, that

$$\begin{aligned}\mu(B_r(x)) &\le \limsup_{j\to\infty}\mathcal{H}^n(\partial S_{\tau,r}\setminus E_j) \\ &\quad+C(n,\tau)\limsup_{j\to\infty}\mathcal{H}^n\big((E_j\cap\partial S_{\tau,r})\setminus A^{\rm in}_{r,j}\big)+C_*\,c(n)\,r^{n+1}\,.\end{aligned} \tag{3.35}$$

We now discuss the cases $x\in\partial^*E$, $x\in K\cap E^{(0)}$ and $x\in K\cap E^{(1)}$ separately.

The case $x\in\partial^*E$: We claim that, in this case, for every $\sigma\in(0,\tau)$ and for a.e. $r<r_0(\sigma,x)$,

$$\limsup_{j\to\infty}\mathcal{H}^n\Big(\partial S_{\tau,r}\setminus\big(A^{\rm out}_{r,j}\cup E_j\big)\Big) \le C(n)\,\sigma\,r^n\,, \tag{3.36}$$

$$\limsup_{j\to\infty}\Big|\mathcal{H}^n\big(E_j\cap\partial S_{\tau,r}\big)-\omega_n\,r^n\Big| \le C(n)\,\tau\,r^n\,; \tag{3.37}$$

see Figure 3.2. We notice that (3.36) and (3.37) combined with (3.34) imply

$$\frac{\mu(B_r(x))}{r^n}\le\omega_n+C(n)\,\tau+C(n,\tau)\,\sigma+C_*\,c(n)\,r\,,\qquad\text{for a.e. } r<r_0\,,$$

which gives $\theta(x)\le1$ by letting, in the order, $r\to0^+$, $\sigma\to0^+$ and then $\tau\to0^+$. We now prove (3.36) and (3.37). Since $x\in\partial^*E$, we can set $\nu(x)=\nu_E(x)$. As $\nu_E(x)$ is the outer normal to E, by $\partial^*E\subset K$, (3.28) and the divergence theorem, we obtain

$$|W^-_{\tau,\sigma,r_0}\setminus E|=|W^+_{\tau,\sigma,r_0}\cap E|=0\,.$$

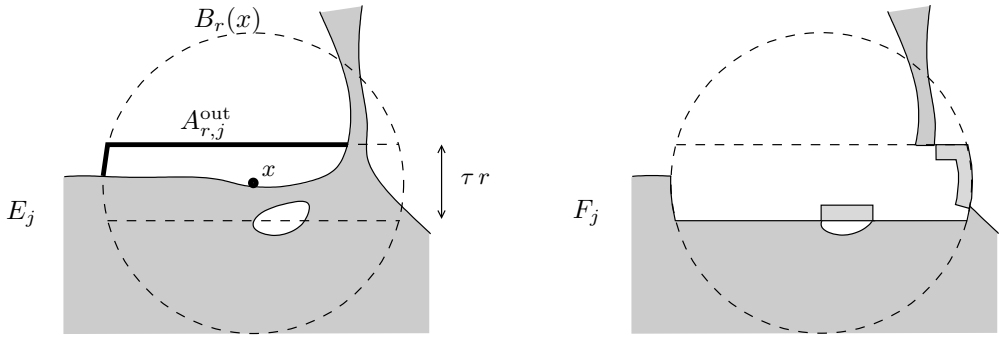

FIGURE 3.2. The slab competitor $F_j^{\rm out}$ is used in proving that $\theta(x) \le 1$. The fact that $x \in \partial^* E$ is used to show that $E_j \cap \partial S_{\tau,r}$ consists of a large connected component whose area is close to $\omega_n\, r^n$ up to a $\mathrm{o}(r^n)$ error as $r \to 0^+$.

By $|W^-_{\tau,\sigma,r_0} \setminus E| = 0$, the coarea formula and Fatou's lemma, we deduce

$$\begin{aligned} 0 &= \lim_{j\to\infty} |W^-_{\tau,\sigma,r_0} \setminus E_j| = \lim_{j\to\infty} \int_0^{r_0} \mathcal{H}^n\Big(\partial S_{\tau,r} \cap \big(W^-_{\tau,\sigma,r_0} \setminus E_j\big)\Big)\, dr \\ &\ge \int_0^{r_0} \liminf_{j\to\infty} \mathcal{H}^n\Big(\Gamma^-_{\tau,\sigma,r} \setminus E_j\Big)\, dr\,, \end{aligned}$$

and by arguing similarly with $|W^+_{\tau,\sigma,r_0} \cap E| = 0$ we conclude that, for a.e. $r < r_0$,

$$\lim_{j\to\infty} \mathcal{H}^n\big(\Gamma^+_{\tau,\sigma,r} \cap E_j\big) = 0\,, \tag{3.38}$$

$$\lim_{j\to\infty} \mathcal{H}^n\big(\Gamma^-_{\tau,\sigma,r} \setminus E_j\big) = 0\,. \tag{3.39}$$

By (3.38), (3.39), and since

$$\partial S_{\tau,r} = \Gamma^+_{\tau,\sigma,r} \cup \Gamma^-_{\tau,\sigma,r} \cup \big(\partial S_{\tau,r} \cap \partial S_{\sigma,r}\big) \tag{3.40}$$

we find that, as $j \to \infty$,

$$\begin{aligned} \big|\mathcal{H}^n(\partial S_{\tau,r} \cap E_j) - \omega_n\, r^n\big| &\le \mathcal{H}^n(\partial S_{\tau,r} \cap \partial S_{\sigma,r}) + \big|\mathcal{H}^n(\Gamma^-_{\tau,\sigma,r} \cap E_j) - \omega_n\, r^n\big| + \mathrm{o}(1) \\ &\le C(n)\,\sigma\, r^n + \big|\mathcal{H}^n(\Gamma^-_{\tau,\sigma,r}) - \omega_n\, r^n\big| + \mathrm{o}(1) \\ &\le C(n)\,\tau\, r^n + \mathrm{o}(1)\,, \end{aligned}$$

that is (3.37). At the same time, again by (3.28) and by the coarea formula, assuming without loss of generality that $r_0 = r_0(\sigma, x)$ also satisfies $\mathcal{H}^n(K \cap \partial B_{r_0}(x)) = 0$ in addition to (3.28), we get

$$\begin{aligned} 0 &= \mu(K \cap \mathrm{cl}\,(B_{r_0}(x)) \setminus V_{\sigma,r_0}) = \lim_{j\to\infty} \mathcal{H}^n\big(B_{r_0}(x) \cap \partial E_j \setminus V_{\sigma,r_0}\big) \\ &\ge \lim_{j\to\infty} \mathcal{H}^n\big(S_{\tau,r_0} \cap \partial E_j \setminus V_{\sigma,r_0}\big) \\ &\ge \lim_{j\to\infty} \int_0^{r_0} \mathcal{H}^{n-1}\big(\partial S_{\tau,r} \cap \partial E_j \setminus V_{\sigma,r_0}\big)\, dr\,, \end{aligned}$$

that is

$$\lim_{j\to\infty} \mathcal{H}^{n-1}\big(\partial S_{\tau,r} \cap \partial E_j \setminus V_{\sigma,r_0}\big) = 0 \qquad \text{for a.e. } r < r_0\,. \tag{3.41}$$

Notice that (3.41) implies in particular that

$$\lim_{j\to\infty} \mathcal{H}^{n-1}\big(\Gamma^+_{\tau,\sigma,r} \cap \partial E_j\big) = 0 \qquad \text{for a.e. } r < r_0\,. \tag{3.42}$$

Since $\Gamma^+_{\tau,\sigma,r}$ is a bi-Lipschitz image of a hemisphere, by Lemma 2.11,

$$\mathcal{H}^{n-1}(\Gamma^+_{\tau,\sigma,r} \cap J)^{n/(n-1)} \ge c(n,\tau,\sigma)\, \mathcal{H}^n(\Gamma^+_{\tau,\sigma,r} \setminus A)\,, \tag{3.43}$$

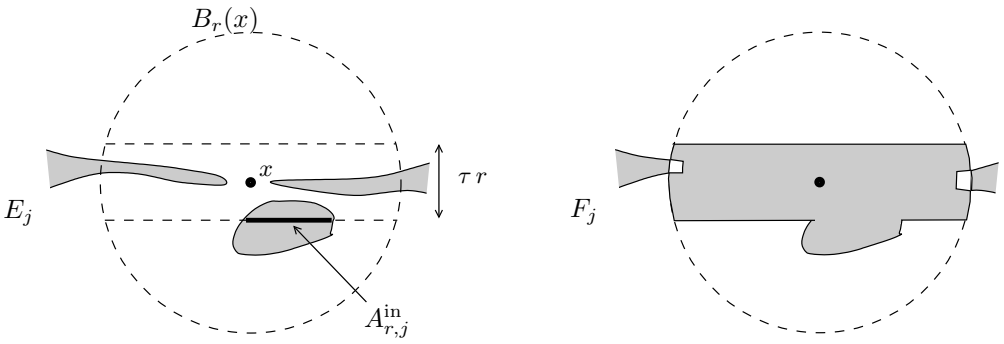

FIGURE 3.3. The slab competitor used in proving that $\theta(x) \le 2$ when $x \in E^{(0)}$ is the one defined by $A^{\rm in}_{r,j}$. Since $x \in E^{(0)}$ we can show that $E_j \cap \partial S_{\tau,r}$ is $\mathrm{o}(r^n)$ as $r \to 0^+$.

whenever J is relatively closed in $\Gamma^+_{\tau,\sigma,r}$, and A is an $\mathcal{H}^n$-maximal connected component of $\Gamma^+_{\tau,\sigma,r} \setminus J$. By (3.42) and (3.43) we find that, if

$$A^+_{r,j} \text{ is a maximal } \mathcal{H}^n\text{-component of } \Gamma^+_{\tau,\sigma,r} \setminus \partial E_j\,,$$

then

$$\lim_{j\to\infty} \mathcal{H}^n(\Gamma^+_{\tau,\sigma,r} \setminus A^+_{r,j}) = 0\,, \qquad \text{for a.e. } r < r_0\,. \tag{3.44}$$

By connectedness, $A^+_{r,j}$ is either contained in $A^{\rm out}_{r,j}$, or in E_j, or in

$$Y_{r,j} = \bigcup \big\{ A : A \in \mathcal{A}^{\rm out}_{r,j}\,, A \neq A^{\rm out}_{r,j} \big\}\,.$$

By combining (3.38) with (3.44) we find that for a.e. $r < r_0$, if j is large enough, then

$$A^+_{r,j} \cap E_j = \emptyset\,.$$

Similarly, should there be a non-negligible set of values of r such that for infinitely many value of j the inclusion $A^+_{r,j} \subset Y_{r,j}$ holds, then by (3.39) and (3.44) there would be an element of $\mathcal{A}^{\rm out}_{r,j}$ different from $A^{\rm out}_{r,j}$ with $\mathcal{H}^n$-measure arbitrarily close to $\mathcal{H}^n(\Gamma^+_{\tau,\sigma,r})$; thanks to (3.39), we would then have $\mathcal{H}^n(A^{\rm out}_{r,j}) \to 0$, against the $\mathcal{H}^n$-maximality of $A^{\rm out}_{r,j}$ itself. In conclusion, it must be

$$A^+_{r,j} \subset A^{\rm out}_{r,j} \text{ for a.e. } r < r_0 \text{ and for } j \text{ large enough}\,. \tag{3.45}$$

By combining (3.45) and (3.44) we conclude that

$$\lim_{j\to\infty} \mathcal{H}^n\big(\Gamma^+_{\tau,\sigma,r} \setminus A^{\rm out}_{r,j}\big) = 0\,. \tag{3.46}$$

By (3.40), (3.39) and (3.46) we conclude that

$$\limsup_{j\to\infty} \mathcal{H}^n\Big(\partial S_{\tau,r} \setminus \big(A^{\rm out}_{r,j} \cup E_j\big)\Big) \le \mathcal{H}^n(\partial S_{\tau,r} \cap \partial S_{\sigma,r}) \le C(n)\,\sigma\, r^n\,,$$

that is (3.36). This completes the proof of $\theta(x) \le 1$ for $x \in \partial^* E$.

The case $x \in E^{(0)}$: We claim that, in this case, for every $\sigma \in (0,\tau)$,

$$\limsup_{j\to\infty} \mathcal{H}^n(E_j \cap \partial S_{\tau,r}) \le C(n)\,\sigma\, r^n\,, \tag{3.47}$$

$$\limsup_{j\to\infty} \big|\mathcal{H}^n\big(\partial S_{\tau,r} \setminus E_j\big) - 2\,\omega_n\, r^n\big| \le C(n)\,\tau\, r^n\,, \tag{3.48}$$

for a.e. $r < r_0(\sigma, x)$, see Figure 3.3. The idea is using the competitor defined by $A^{\rm in}_{r,j}$:

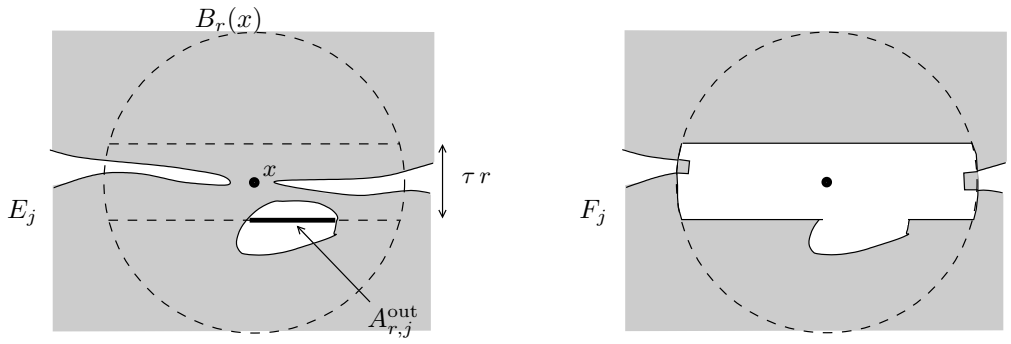

FIGURE 3.4. The slab competitor used in proving that $\theta(x) \le 2$ when $x \in E^{(1)}$ is the one defined by $A^{\rm out}_{r,j}$.

indeed, (3.47), (3.48), and (3.35) give

$$\begin{aligned}\frac{\mu(B_r(x))}{r^n} &\le \limsup_{j\to\infty} \frac{\mathcal{H}^n(\partial S_{\tau,r} \setminus E_j)}{r^n} \\ &\quad + C(n,\tau) \limsup_{j\to\infty} \frac{\mathcal{H}^n\big((E_j \cap \partial S_{\tau,r}) \setminus A^{\rm in}_{r,j}\big)}{r^n} + C_*\, c(n)\, r \\ &\le 2\,\omega_n + C(n)\,\tau + C(n,\tau)\,\sigma + C_*\, c(n)\, r\,,\end{aligned}$$

and then $\theta(x) \le 2$ by letting, in the order, $r \to 0^+$, $\sigma \to 0^+$ and then $\tau \to 0^+$. The proof of (3.47) and (3.48) is simple: since $x \in E^{(0)}$ and $\partial^* E \subset K$, by (3.28) and by the divergence theorem we find that

$$|E \cap B_{r_0}(x) \setminus V_{\sigma,r_0}| = 0\,.$$

In particular, by the coarea formula we find that for a.e. $r < r_0$,

$$0 = \lim_{j\to\infty} \mathcal{H}^n\Big((E_j \setminus V_{\sigma,r_0}) \cap \partial S_{\tau,r}\Big) = \lim_{j\to\infty} \mathcal{H}^n\Big(E_j \cap \big(\Gamma^+_{\tau,\sigma,r} \cup \Gamma^-_{\tau,\sigma,r}\big)\Big)\,,$$

so that, by (3.40),

$$\mathcal{H}^n(E_j \cap \partial S_{\tau,r}) = \mathcal{H}^n(\partial S_{\tau,r} \cap \partial S_{\sigma,r}) + \mathrm{o}(1) \le C(n)\,\sigma\, r^n + \mathrm{o}(1)\,,$$

as $j \to \infty$, that is (3.47), and

$$\begin{aligned}\big|\mathcal{H}^n(\partial S_{\tau,r} \setminus E_j) - 2\,\omega_n\, r^n\big| &\le \mathcal{H}^n(\partial S_{\tau,r} \cap \partial S_{\sigma,r}) + \big|\mathcal{H}^n(\Gamma^+_{\tau,\sigma,r} \cup \Gamma^-_{\tau,\sigma,r}) - 2\,\omega_n\, r^n\big| + \mathrm{o}(1) \\ &\le C(n)\,\tau\, r^n + \mathrm{o}(1)\end{aligned}$$

as $j \to \infty$, that is (3.48).

The case $x \in E^{(1)}$: We claim that for every $\sigma \in (0,\tau)$,

$$\limsup_{j\to\infty} \big|\mathcal{H}^n(E_j \cap \partial S_{\tau,r}) - 2\,\omega_n\, r^n\big| \le C(n)\,\tau\, r^n\,, \tag{3.49}$$

$$\limsup_{j\to\infty} \mathcal{H}^n\big(\partial S_{\tau,r} \setminus E_j\big) \le C(n)\,\sigma\, r^n\,, \tag{3.50}$$

for a.e. $r < r_0(\sigma, x)$, see Figure 3.4. Indeed, by using as in the case $x \in \partial^* E$ the competitor defined by $A^{\rm out}_{r,j}$, (3.49), (3.50) are combined with (3.34) to obtain

$$\begin{aligned}\frac{\mu(B_r(x))}{r^n} &\le \limsup_{j\to\infty} \frac{\mathcal{H}^n(E_j \cap \partial S_{\tau,r})}{r^n} \\ &\quad + C(n,\tau) \limsup_{j\to\infty} \frac{\mathcal{H}^n\big(\partial S_{\tau,r} \setminus (A^{\rm out}_{r,j} \cup E_j)\big)}{r^n} + C_*\, c(n)\, r \\ &\le 2\,\omega_n + C(n)\,\tau + C(n,\tau)\,\sigma + C_*\, c(n)\, r\,, \end{aligned} \tag{3.51}$$

which gives $\theta(x) \le 2$ by letting once again $r \to 0^+$, $\sigma \to 0^+$ and finally $\tau \to 0^+$. To prove (3.49) and (3.50), we notice that by $x \in E^{(1)}$, $\partial^* E \subset K$, (3.28) and the divergence theorem, we have

$$\big|B_{r_0}(x) \setminus \big(V_{\sigma,r_0} \cup E\big)\big| = 0\,.$$

By the coarea formula, for a.e. $r < r_0$ we find

$$0 = \lim_{j\to\infty} \mathcal{H}^n\big(\big(\Gamma^+_{\tau,\sigma,r} \cup \Gamma^-_{\tau,\sigma,r}\big) \setminus E_j\big)\,,$$

and conclude as in the previous case by exploiting (3.40).

Remark: We make an important remark on the constructions of step five, which will be needed in the proof of Theorem 1.6. We claim that, under the assumptions on x considered in step five, for a.e. $r < r_0(\sigma, x)$ we have

$$\begin{aligned}\limsup_{\eta\to 0^+} \Big|\mathcal{H}^n\Big(\Big\{y \in \mathrm{cl}\,(S_{\tau,r}) \cap \partial F_j^\star : T_y(\partial F_j^\star) = T_x K\Big\}\Big) - \theta(x)\,\omega_n\, r^n\Big| \qquad (3.52)\\ \le C(n)\,\tau\, r^n + C(n,\tau)\,\sigma\, r^n + \mathrm{o}(1)\,, \qquad \text{as } j \to \infty\,.\end{aligned}$$

Here $\star = \mathrm{out}$ if $x \in \partial^* E \cup (K \cap E^{(1)})$, $\star = \mathrm{in}$ if $x \in K \cap E^{(0)}$, and $\theta(x) = 1$ if $x \in \partial^* E$ and $\theta(x) = 2$ if $x \in K \cap (E^{(0)} \cup E^{(1)})$. Consider, for example, the case when $x \in \partial^* E$. By (3.32), $\partial S_{\tau,r} \cap \partial F_j^{\mathrm{out}}$ is $\mathcal{H}^n$-equivalent to $\partial S_{\tau,r} \setminus A^{\mathrm{out}}_{r,j}$: thus, by taking into account that

$$T_y(\partial F_j^{\mathrm{out}}) = T_y(\partial S_{\tau,r}) \qquad \mathcal{H}^n\text{-a.e. on } \partial F_j^{\mathrm{out}} \cap \partial S_{\tau,r}$$

and that

$$\big\{y \in \partial S_{\tau,r} : T_y(\partial S_{\tau,r}) = T_x K\big\} = \partial S_{\tau,r} \setminus \partial B_r(x)\,,$$

(recall that $T_x K = \nu(x)^\perp$), we have

$$\begin{aligned}&\Big|\mathcal{H}^n\Big(\Big\{y \in \mathrm{cl}\,(S_{\tau,r}) \cap \partial F_j^{\mathrm{out}} : T_y(\partial F_j^{\mathrm{out}}) = T_x K\Big\}\Big) - \omega_n\, r^n\Big|\\ \le\ &\Big|\mathcal{H}^n\Big(\Big\{y \in \partial S_{\tau,r} \cap \partial F_j^{\mathrm{out}} : T_y(\partial F_j^{\mathrm{out}}) = T_x K\Big\}\Big) - \omega_n\, r^n\Big| + \mathcal{H}^n(S_{\tau,r} \cap \partial F_j^{\mathrm{out}})\\ \le\ &\Big|\mathcal{H}^n\Big(\Big\{y \in \partial S_{\tau,r} \setminus A^{\mathrm{out}}_{r,j} : T_y(\partial S_{\tau,r}) = T_x K\Big\}\Big) - \omega_n\, r^n\Big| + \mathcal{H}^n(S_{\tau,r} \cap \partial F_j^{\mathrm{out}})\\ =\ &\Big|\mathcal{H}^n\big(\partial S_{\tau,r} \setminus (\partial B_r(x) \cup A^{\mathrm{out}}_{r,j})\big) - \omega_n\, r^n\Big| + \mathcal{H}^n(S_{\tau,r} \cap \partial F_j^{\mathrm{out}})\end{aligned}$$

so that, by (3.33), (3.36), and $\mathcal{H}^n(\partial S_{\tau,r} \cap \partial B_r(x)) \le C(n)\,\tau\, r^n$,

$$\begin{aligned}&\limsup_{\eta\to 0^+} \Big|\mathcal{H}^n\Big(\Big\{y \in \mathrm{cl}\,(S_{\tau,r}) \cap \partial F_j^{\mathrm{out}} : T_y(\partial F_j^{\mathrm{out}}) = T_x K\Big\}\Big) - \omega_n\, r^n\Big|\\ \le\ &\Big|\mathcal{H}^n(\partial S_{\tau,r} \cap E_j) - \omega_n\, r^n\Big| + C(n,\tau)\mathcal{H}^n\big(\partial S_{\tau,r} \setminus (A^{\mathrm{out}}_{r,j} \cup E_j)\big) + C(n)\,\tau\, r^n\,.\end{aligned}$$

By (3.36) and (3.37) we deduce (3.52) when $x \in \partial^* E$. The case when $x \in K \cap (E^{(0)} \cup E^{(1)})$ is treated analogously and the details are omitted.

Step six: We exclude area concentration near $\partial\Omega$, by showing that

$$\limsup_{\eta\to 0^+} \limsup_{j\to\infty} \mu_j(\Omega \cap U_\eta(\partial\Omega)) = 0\,. \qquad (3.53)$$

Exploiting the smoothness and boundedness of $\partial\Omega$, we can find $r_0 > 0$ such that Lemma 2.4 holds, and such that for every $x \in \partial\Omega$ there exists an open set Ω' with $\Omega \subset \Omega'$ and a diffeomorphism $f : \mathrm{cl}\,(\Omega) \to \mathrm{cl}\,(\Omega') = f(\mathrm{cl}\,(\Omega))$ with $f(\partial\Omega) = \partial\Omega'$, $\{f \ne \mathrm{id}\} \subset\subset B_{r_0}(x)$, $f(B_{r_0}(x)) = B_{r_0}(x)$, and

$$f\Big(\Omega \cap U_\eta(\partial\Omega) \cap B_{r_0/2}(x)\Big) \subset \Omega' \setminus \Omega\,, \qquad \|f - \mathrm{id}\,\|_{C^1(\Omega)} \le C\,\eta\,; \qquad (3.54)$$

see Figure 3.5. Let $\Omega^* = f^{-1}(\Omega)$ and let $F_j = f(E_j \cap \Omega^*) = f(E_j) \cap \Omega$. Clearly $F_j \in \mathcal{E}$,

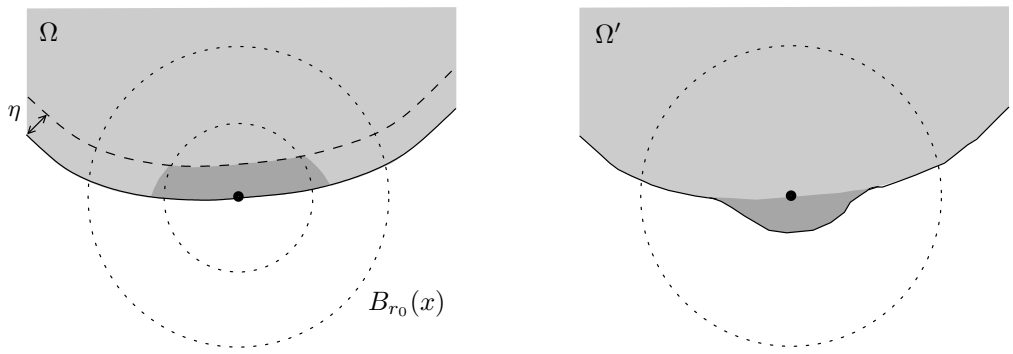

FIGURE 3.5. The boundary diffeomorphism f pushes out Ω into a larger open set Ω'. Regions depicted with the same color are mapped one into the other. Notice that the dark region on the left contains $\Omega \cap U_\eta(\partial\Omega) \cap B_{r_0/2}(x)$, and is mapped outside of Ω. The diffeomorphism f can be formally constructed by exploiting the local graphicality of Ω, and the simple details are omitted.

and $f(\partial\Omega^*) = \partial\Omega$ and $\Omega^* \cap \partial(E_j \cap \Omega^*) = \Omega^* \cap \partial E_j$ give

$$\Omega \cap \partial F_j = f(\Omega^*) \cap f\big(\partial(E_j \cap \Omega^*)\big) = f\big(\Omega^* \cap \partial E_j\big)\,,$$

so that $\Omega \cap \partial F_j$ is $\mathcal{C}$-spanning W by Lemma 2.4. Assuming without loss of generality that $r_0 < r_*$, by (3.10), $\{f \neq \mathrm{id}\} \subset\subset B_{r_0}(x)$ and $f(B_{r_0}(x)) = B_{r_0}(x)$ we have

$$\begin{aligned}\mathcal{H}^n(\Omega \cap B_{r_0}(x) \cap \partial E_j) &\leq \mathcal{H}^n\big(f(B_{r_0}(x) \cap \Omega^* \cap \partial E_j)\big) + C_* \big||F_j| - |E_j|\big| + \frac{1}{j} \\ &\leq \big(1 + C\,\eta\big)\,\mathcal{H}^n\big(B_{r_0}(x) \cap \Omega^* \cap \partial E_j\big) + C_* \big||F_j| - |E_j|\big| + \frac{1}{j}\,,\end{aligned}$$

where

$$\big||F_j| - |E_j|\big| \leq \big||E_j \cap \Omega^*| - |E_j|\big| + \int_{E_j \cap \Omega^*} |Jf - 1| \leq \big|\Omega \setminus \Omega^*\big| + C\,\varepsilon\,\eta \leq C\,\eta\,,$$

so that

$$\mathcal{H}^n(\Omega \cap B_{r_0}(x) \cap \partial E_j \setminus \Omega^*) \leq C\,\eta\left\{\mathcal{H}^n\big(\Omega \cap \partial E_j\big) + 1\right\} + \frac{1}{j} \leq C\,\eta\left\{\psi(\varepsilon) + 2\right\} + \frac{1}{j}\,.$$

Since $\Omega \cap U_\eta(\partial\Omega) \cap B_{r_0/2}(x) \subset \Omega \setminus \Omega^*$, by letting $j \to \infty$ we conclude that

$$\mu\big(B_{r_0/2}(x) \cap U_\eta(\partial\Omega)\big) \leq C\,\eta\,, \qquad \forall x \in \partial\Omega\,.$$

By a covering argument we find $\mu(\Omega \cap U_\eta(\partial\Omega)) \leq C\,\eta$, and thus (3.53) follows.

Step seven: Let us now pick $R > 0$ such that $W \cup K \cup E \subset\subset B_R(0)$. If $E_j \subset B_{R+1}(0)$ for infinitely many values of j, then $|E| = \varepsilon$ and $\mu_j(\Omega \setminus B_{R+1}(0)) = 0$, which combined with (3.53) implies $\mu_j(\Omega) \to \mu(\Omega) = \mathcal{F}(K, E)$ as $j \to \infty$, and thus $\psi(\varepsilon) = \mathcal{F}(K, E)$ with $(K, E) \in \mathcal{K}$ and $|E| = \varepsilon$: thus (K, E) is a generalized minimizer of $\psi(\varepsilon)$, as desired. We now assume without loss of generality that $|E_j \setminus B_{R+1}(0)| > 0$ for every j. By (3.4),

$$\limsup_{j\to\infty} |E_j \cap (B_{R+1}(0) \setminus B_R(0))| = \limsup_{j\to\infty} \mathcal{H}^n((B_{R+1}(0) \setminus B_R(0)) \cap \partial E_j) = 0\,.$$

By the coarea formula, this implies that for a.e. $s \in (R, R+1)$,

$$\limsup_{j\to\infty} \mathcal{H}^n(E_j \cap \partial B_s(0)) = \limsup_{j\to\infty} \mathcal{H}^{n-1}(\partial E_j \cap \partial B_s(0)) = 0\,. \tag{3.55}$$

We fix a value of s such that (3.55) holds, and we let A_j denote an $\mathcal{H}^n$-maximal connected component of $\partial B_s(0) \setminus \partial E_j$. It must be $A_j \cap E_j = \emptyset$: for, otherwise, by the spherical isoperimetric inequality, $A_j \subset E_j$ would imply

$$\begin{aligned} C(n)\,\mathcal{H}^{n-1}(\partial B_s(0) \cap \partial E_j)^{n/(n-1)} &\geq \mathcal{H}^n(\partial B_s(0) \setminus A_j) \geq \mathcal{H}^n(\partial B_s(0) \setminus E_j) \\ &\geq c(n)\,R^n - \mathcal{H}^n(E_j \cap \partial B_s(0))\,, \end{aligned}$$

a contradiction to (3.55). Since $A_j \cap E_j = \emptyset$, we can consider the exterior cup competitor defined by E_j and A_j,

$$F_j = \big(E_j \cap B_s(0)\big) \cup M_\eta(Y_j)\,, \qquad Y_j = \partial B_s(0) \setminus \mathrm{cl}\,(A_j \cup E_j)\,, \qquad \eta \in (0, R+1-s)\,,$$

and deduce from Lemma 2.6 that $F_j \in \mathcal{E}$, with $\Omega \cap \partial F_j$ $\mathcal{C}$-spanning W, $F_j \subset B_{R+1}$ and

$$\begin{aligned} \limsup_{\eta \to 0^+} \mathcal{H}^n(\Omega \cap \partial F_j) &\leq \mathcal{H}^n(\Omega \cap B_s(0) \cap \partial E_j) + 2\,\mathcal{H}^n(\partial B_s(0) \setminus A_j) \\ &\leq \mathcal{H}^n(\Omega \cap B_s(0) \cap \partial E_j) + C(n)\,\mathcal{H}^{n-1}(\partial B_s(0) \cap \partial E_j)^{n/(n-1)}\,. \end{aligned} \tag{3.56}$$

Since $|E_j \setminus B_{R+1}(0)| > 0$ for every j, we can select η sufficiently small (depending on j) so that $|E_j \setminus B_s(0)| > |M_\eta(Y_j)|$, and then define $\rho_j > 0$ by the equation

$$|B_{\rho_j}| = |E_j| - |F_j| = |E_j \setminus B_s(0)| - |M_\eta(Y_j)|\,.$$

In particular, $|B_{\rho_j}| \leq \varepsilon$, so that we can find $x \in \Omega$ such that $\mathrm{cl}\,(B_{\rho_j}(x)) \cap \mathrm{cl}\,(F_j) = \emptyset$ and

$$E_j^* = F_j \cup B_{\rho_j}(x) \subset B_{R+1+C(n)\,\varepsilon^{1/(n+1)}}(0) \qquad \forall j\,.$$

We notice that $E_j^* \in \mathcal{E}$ with $|E_j^*| = \varepsilon$ and $\Omega \cap \partial F_j \subset \Omega \cap \partial E_j^*$, so that $\Omega \cap \partial E_j^*$ is $\mathcal{C}$-spanning W: in particular, $\psi(\varepsilon) \leq \mathcal{H}^n(\Omega \cap \partial E_j^*)$. By the Euclidean isoperimetric inequality, and since $|B_{\rho_j}| \leq |E_j \setminus B_s(0)|$ by definition of ρ_j, we have

$$P(B_{\rho_j}) \leq P(E_j \setminus B_s(0)) = \mathcal{H}^n(\partial E_j \setminus B_s(0)) + \mathcal{H}^n(E_j \cap \partial B_s(0))\,,$$

so that by (3.55), (3.56) and $\eta = \eta_j \to 0^+$ as $j \to \infty$ we get

$$\begin{aligned} \psi(\varepsilon) &\leq \limsup_{j\to\infty} \mathcal{H}^n(\Omega \cap \partial E_j^*) \leq \limsup_{j\to\infty} \mathcal{H}^n(\Omega \cap \partial F_j) + P(B_{\rho_j}) \\ &\leq \limsup_{j\to\infty} \mathcal{H}^n(\Omega \cap \partial E_j) + 2\,C(n) \limsup_{j\to\infty} \mathcal{H}^{n-1}(\partial B_s(0) \cap \partial E_j)^{n/(n-1)} = \psi(\varepsilon)\,. \end{aligned}$$

We have thus proved that $\{E_j^*\}_j$ is a minimizing sequence for $\psi(\varepsilon)$, with $E_j^* \subset B_{R^*}(0)$ for some R^* depending only on R, n and ε. By repeating the argument of the first six steps with E_j^* in place of E_j we see that $E_j^* \to E^*$ in $L^1(\Omega)$ and $\mu_j^* = \mathcal{H}^n \llcorner (\Omega \cap \partial E_j^*) \overset{*}{\rightharpoonup} \mu^*$ where $\mu^* = 2\,\mathcal{H}^n \llcorner (K^* \setminus \partial^* E^*) + \mathcal{H}^n \llcorner \partial^* E^*$, and where $(K^*, E^*) \in \mathcal{K}$ with $|E^*| = \varepsilon$ and with

$$\limsup_{\eta\to 0^+} \limsup_{j\to\infty} \mu_j^*(\Omega \cap\ U_\eta(\partial\Omega)) = 0\,.$$

Therefore $\mu_j^*(\Omega) \to \mu^*(\Omega) = \mathcal{F}(K^*, E^*)$ and in conclusion

$$\mathcal{F}(K^*, E^*) = \mu^*(\Omega) = \lim_{j\to\infty} \mu_j^*(\Omega) = \psi(\varepsilon)$$

so that, by $|E^*| = \varepsilon$, (K^*, E^*) is indeed a generalized minimizer of $\psi(\varepsilon)$. This concludes the proof of the theorem. $\square$

4. The Euler-Lagrange equation: Proof of Theorem 1.6

Proof of Theorem 1.6. Let (K,E) be a generalized minimizer of $\psi(\varepsilon)$ and $f:\Omega\to\Omega$ be a diffeomorphism such that $|f(E)|=|E|$. We want to prove that

$$\mathcal{F}(K,E)\le\mathcal{F}(f(K),f(E))\,. \tag{4.1}$$

Let K' denote the set of points of approximate differentiability of K, so that $\mathcal{H}^n(K\setminus K')=0$, and for $x\in K'$ denote by $T_x=T_xK=\nu_x^\perp$ the approximate tangent plane to K at x, where $\nu_x\in\mathbb{S}^n$ is chosen so that $\nu_x=\nu_E(x)$ if $x\in\partial^*E$. As in step five of the proof of Theorem 1.4, for every $\sigma>0$ we introduce $r_0=r_0(\sigma,x)$ such that

$$K\cap B_r(x)\subset S^x_{\sigma,r}=\Big\{y\in B_r(x):|(y-x)\cdot\nu_x|<\sigma\,r\Big\}\qquad\forall r<r_0(\sigma,x)\,, \tag{4.2}$$

see (3.28). In fact, by Egoroff's theorem, we can find a compact set $K^*\subset K'$ with $\mathcal{H}^n(K\setminus K^*)<\sigma$ such that $r_*(\sigma)=\max\{r_0(\sigma,x):x\in K^*\}\to0^+$ as $\sigma\to0^+$, that is, such that (4.2) holds uniformly on K^*,

$$K\cap B_r(x)\subset S^x_{\sigma,r}\qquad\forall x\in K^*\,,\forall r<r_*(\sigma)\,. \tag{4.3}$$

Similarly, if G_n denotes the family of the n-planes in $\mathbb{R}^{n+1}$, endowed with a distance d, by Lusin's theorem and up to further decreasing the size of K^* while keeping $\mathcal{H}^n(K\setminus K^*)<\sigma$, we can make sure that

$$\sup_{x,y\in K_*,|x-y|<r}d(T_x,T_y)+\sup_{x,y\in K^*,|y-x|<r}|\nabla f(x)-\nabla f(y)|\le\omega_*(r)\,, \tag{4.4}$$

for a function $\omega_*(r)\to0^+$ as $r\to0^+$. Finally, since

$$\begin{cases}\mathcal{H}^n(B_r(x)\cap\partial^*E)=\mathrm{o}(r^n)\,,\\ \mathcal{H}^n\big(B_r(x)\cap(K\setminus\partial^*E)\big)=\omega_n\,r^n+\mathrm{o}(r^n)\,,\qquad\text{for }\mathcal{H}^n\text{-a.e. }x\in K\setminus\partial^*E\,,\end{cases}$$

$$\begin{cases}\mathcal{H}^n(B_r(x)\cap\partial^*E)=\omega_n\,r^n+\mathrm{o}(r^n)\,,\\ \mathcal{H}^n\big(B_r(x)\cap(K\setminus\partial^*E)\big)=\mathrm{o}(r^n)\,,\qquad\text{for }\mathcal{H}^n\text{-a.e. }x\in\partial^*E\,,\end{cases}$$

as $r\to0^+$, by Egoroff's theorem, up to decrease K^* and increase ω^*, we can also entail

$$\sup_{x\in K^*\setminus\partial^*E}\mathcal{H}^n(B_r(x)\cap\partial^*E)+\Big|\mathcal{H}^n\big(B_r(x)\cap(K\setminus\partial^*E)\big)-\omega_n\,r^n\Big|\le\omega_*(r)\,r^n\,, \tag{4.5}$$

$$\sup_{x\in K^*\cap\partial^*E}\Big|\mathcal{H}^n(B_r(x)\cap\partial^*E)-\omega_n\,r^n\Big|+\mathcal{H}^n\big(B_r(x)\cap(K\setminus\partial^*E)\big)\le\omega_*(r)\,r^n\,, \tag{4.6}$$

while still keeping $\mathcal{H}^n(K\setminus K^*)<\sigma$ and $\omega_*(r)\to0^+$ as $r\to0^+$.

Let $\{E_j\}_j$ be a minimizing sequence for $\psi(\varepsilon)$ converging to (K,E) as in (1.13), and consider a point $x\in K^*$. Given $\tau\in(0,1)$ and $\sigma\in(0,\tau)$, for a.e. $r<r_*(\sigma)$ such that $B_{2r}(x)\subset\subset\Omega$, we have that $\partial S^x_{\tau,r}\cap\partial E_j$ is $\mathcal{H}^{n-1}$-rectifiable for every j (with the exceptional set depending on x). For such values of r and for every $\eta\in(0,r/2)$, we can set

$$F^x_j=\begin{cases}F^{\rm out}_j\,,&\text{if }x\in\partial^*E\cup(K^*\cap E^{(1)})\,,\\ F^{\rm in}_j\,,&\text{if }x\in K^*\cap E^{(0)}\,,\end{cases}$$

with $F^{\rm out}_j$ and $F^{\rm in}_j$ defined as in step five of the proof of Theorem 1.4. In particular, $F^x_j\in\mathcal{E}$, $\Omega\cap\partial F^x_j$ is $\mathcal{C}$-spanning W, $F^x_j\setminus{\rm cl}\,(S^x_{\tau,r})=E_j\setminus{\rm cl}\,(S^x_{\tau,r})$ and, as proved in (3.52), for a.e. $r<r_*(\sigma)$ we have

$$\begin{aligned}\limsup_{\eta\to0^+}\Big|\mathcal{H}^n\Big(\Big\{y\in{\rm cl}\,(S^x_{\tau,r})\cap\partial F^x_j:T_y(\partial F^x_j)=T_x\Big\}\Big)-\theta(x)\,\omega_n\,r^n\Big|\\ \le C(n)\,\tau\,r^n+C(n,\tau)\,\sigma\,r^n+\mathrm{o}(1)\qquad\text{as }j\to\infty\,,\end{aligned} \tag{4.7}$$

where $\theta(x) = 1$ if $x \in \partial^* E$ and $\theta(x) = 2$ if $x \in K \cap (E^{(0)} \cup E^{(1)})$, as well as

$$\limsup_{j\to\infty} \limsup_{\eta\to 0^+} \mathcal{H}^n(S^x_{\tau,r} \cap \partial F^x_j) \le C(n,\tau)\,\sigma\, r^n\,, \tag{4.8}$$

see (3.33), (3.36), (3.47), and (3.50). By Besicovitch-Vitali's covering theorem and by Federer's theorem (2.1), we can find a *finite* disjoint family of closed balls $\{B_i = \mathrm{cl}\,(B_{r_i}(x_i))\}_i$ such that $B_i \subset\subset \Omega$ and

$$\mathcal{H}^n\Big(K^* \setminus \bigcup B_{r_i}(x_i)\Big) < \sigma\,, \qquad x_i \in K^* \cap \big(E^{(0)} \cup E^{(1)} \cup \partial^* E\big)\,, \qquad r_i < r_*(\sigma)\,. \tag{4.9}$$

We let $\eta < \min_i\{r_i/2\}$, define $F^{x_i}_j$ accordingly, and set

$$S_i = S^{x_i}_{\tau,r_i} \subset\subset B_i\,, \qquad T_i = T_{x_i}\,, \qquad F^i_j = F^{x_i}_j\,.$$

Correspondingly, we define a sequence $\{F_j\}_j \subset \mathcal{E}$ with $\Omega \cap \partial F_j$ $\mathcal{C}$-spanning W by setting

$$F_j \setminus \bigcup_i B_i = E_j \setminus \bigcup_i B_i\,, \qquad F_j \cap B_i = F^i_j \cap B_i\,. \tag{4.10}$$

Since $F^i_j \setminus \mathrm{cl}\,(S_i) = E_j \setminus \mathrm{cl}\,(S_i)$ we find that

$$F_j \setminus \bigcup_i \mathrm{cl}\,(S_i) = E_j \setminus \bigcup_i \mathrm{cl}\,(S_i)\,, \tag{4.11}$$

and, setting,

$$\theta_i = 1 \quad \text{if } x_i \in \partial^* E\,, \qquad \theta_i = 2 \quad \text{if } x_i \in E^{(0)} \cup E^{(1)} \tag{4.12}$$

we deduce from (4.7) and (4.8) that, for each i,

$$\begin{aligned} \limsup_{\eta\to 0^+} \Big|\mathcal{H}^n\big(\{y \in \mathrm{cl}\,(S_i) \cap \partial F_j : T_y(\partial F_j) = T_i\}\big) - \theta_i\,\omega_n\, r_i^n\Big| \\ \le C(n)\,\tau\, r_i^n + C(n,\tau)\,\sigma\, r_i^n + \mathrm{o}(1) \end{aligned} \tag{4.13}$$

$$\limsup_{\eta\to 0^+} \mathcal{H}^n(S_i \cap \partial F_j) \le C(n,\tau)\,\sigma\, r_i^n + \mathrm{o}(1) \tag{4.14}$$

as $j \to \infty$. Now let C_* and ε_* the volume-fixing variation constants defined by $f(E)$. By the monotonicity formula (3.26), which can be applied to $B_{r_i}(x_i)$ as $x_i \in K$, we have

$$e^{-\Lambda\, r_*(\sigma)}\,\theta_i\,\omega_n\, r_i^n \le e^{-\Lambda\, r_i}\,\theta_i\,\omega_n\, r_i^n \le \mu(B_{r_i}(x_i)) = \mu(S_i)\,, \tag{4.15}$$

where in the last identity we have used (4.3), and where Λ depends on E. By (4.15), $\theta_i \ge 1$, and $\mu = \theta\,\mathcal{H}^n \llcorner K$ with $\theta \le 2$,

$$\sum_i r_i^n \le C(n,E) \sum_i \mathcal{H}^n(K \cap B_i) \le C(n,E)\,\mathcal{H}^n(K) = C(n,E,K)\,, \tag{4.16}$$

so that, by (4.11), $|S_i| \le C(n)\,\tau\, r_i^{n+1}$ and $r_i \le r_*(\sigma) \le 1$, we find

$$|F_j \Delta E_j| \le \sum_i |S_i| \le C(n,E,K)\,\tau\,.$$

Therefore,

$$|f(F_j)\Delta f(E)| \le C\big(n,E,\mathrm{Lip}(f),\mathcal{H}^n(K)\big)\Big\{\tau + |E_j \Delta E|\Big\} < \varepsilon_*\,,$$

provided j is large enough and τ is small enough depending on ε_*. By the volume-fixing variations construction, for each j large enough there exists a smooth map $\Phi_j : (-\varepsilon_*,\varepsilon_*) \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, such that, for every $|v| < \varepsilon_*$, $\Phi_j(v,\cdot)$ is a diffeomorphism with $\Phi_j(v,\Omega) = \Omega$ and

$$|\Phi_j(v, f(F_j))| = v + |f(F_j)|\,, \qquad \mathcal{H}^n\big(\Phi_j(v,\Sigma)\big) \le \mathcal{H}^n(\Sigma) + C_*\,|v|\,\mathcal{H}^n(\Sigma)\,,$$

for every $\mathcal{H}^n$-rectifiable set $\Sigma \subset \Omega$. In particular, if we set

$$G_j = \Phi_j(v_j, f(F_j))\,, \qquad v_j = |f(E)| - |f(F_j)| = |E| - |f(F_j)|\,,$$

then we find that $G_j \in \mathcal{E}$, $|G_j| = |E| = \varepsilon$ and

$$\mathcal{H}^n(\Omega \cap \partial G_j) \le \Big(1 + C\big(n, E, \mathrm{Lip}(f), \mathcal{H}^n(K)\big)\,\Big\{\tau + |E_j \Delta E|\Big\}\Big)\,\mathcal{H}^n(\Omega \cap \partial f(F_j))\,.$$

Since $\Omega \cap \partial F_j$ is $\mathcal{C}$-spanning W, so is $\Omega \cap \partial G_j$ thanks to Lemma 2.3, so that the minimizing sequence property of E_j implies

$$\mathcal{H}^n(\Omega \cap \partial E_j) \le \Big(1 + C\,\Big\{\tau + |E_j \Delta E|\Big\}\Big)\,\mathcal{H}^n(\Omega \cap \partial f(F_j)) + \frac{1}{j}\,, \tag{4.17}$$

where, here and for the rest of the proof, C is a generic constant depending on K, E, f and n. We now claim that

$$\limsup_{\sigma \to 0^+} \limsup_{j \to \infty} \limsup_{\eta \to 0^+} \mathcal{H}^n(\Omega \cap \partial f(F_j)) \le \mathcal{F}(f(K), f(E)) + C\,\tau\,. \tag{4.18}$$

Notice that by combining (4.17) and (4.18), and by finally letting $\tau \to 0^+$, we complete the proof of (4.1).

To prove (4.18), we notice that $f(\Omega) = \Omega$, $\Omega \cap \partial f(F_j) = f(\Omega \cap \partial F_j)$, and (4.11) yield

$$\mathcal{H}^n(\Omega \cap \partial f(F_j)) \le \mathcal{H}^n\Big(f\Big(\Omega \cap \partial E_j \setminus \bigcup_i \mathrm{cl}\,(S_i)\Big)\Big) + \sum_i \int_{\mathrm{cl}\,(S_i) \cap \partial F_j} J^{\partial F_j} f\, d\mathcal{H}^n\,,$$

where

$$\limsup_{j \to \infty} \limsup_{\eta \to 0^+} \mathcal{H}^n\Big(f\Big(\Omega \cap \partial E_j \setminus \bigcup_i \mathrm{cl}\,(S_i)\Big)\Big) \le C\,\mathcal{H}^n\Big(K \setminus \bigcup_i S_i\Big) \le C\,\sigma$$

by (4.3), (4.9), and $\mathcal{H}^n(K \setminus K^*) < \sigma$. Hence, as

$$\mathcal{H}^n(\Omega \cap \partial f(F_j)) \le \sum_i \int_{\mathrm{cl}\,(S_i) \cap \partial F_j} J^{\partial F_j} f\, d\mathcal{H}^n + C\,\sigma + \mathrm{o}(1)\,, \tag{4.19}$$

where $\mathrm{o}(1) \to 0^+$ if we let first $\eta \to 0^+$ and then $j \to \infty$.

If we set

$$Z_i = \big\{y \in \partial S_i \cap \partial F_j : T_y(\partial F_j) = T_i\big\}\,,$$

then by (4.13) and (4.14) we find

$$\begin{aligned} \mathcal{H}^n\big((\mathrm{cl}\,(S_i) \cap \partial F_j) \Delta Z_i\big) &\le C(n)\,\tau\, r_i^n + C(n,\tau)\,\sigma\, r_i^n + \mathrm{o}(1)\,, \\ \big|\mathcal{H}^n(Z_i) - \theta_i\,\omega_n\, r_i^n\big| &\le C(n)\,\tau\, r_i^n + C(n,\tau)\,\sigma\, r_i^n + \mathrm{o}(1)\,, \end{aligned}$$

where $\mathrm{o}(1) \to 0^+$ if we let first $\eta \to 0^+$ and then $j \to \infty$. Also, it follows from (4.15), the characterization of μ, and (4.6) that

$$e^{-\Lambda\, r_*(\sigma)}\,\theta_i\,\omega_n\, r_i^n \le \theta_i\,\mathcal{H}^n(S_i \cap K) + \omega_*(r_i)\, r_i^n\,. \tag{4.20}$$

By (4.4), (4.20), and $r_i < r_*(\sigma)$, we thus find

$$\begin{aligned}
\int_{\mathrm{cl}\,(S_i)\cap\partial F_j} J^{\partial F_j} f &\le \int_{Z_i} J^{T_i} f + (\mathrm{Lip}\, f)^n \big\{C(n)\,\tau + C(n,\tau)\,\sigma\big\}\, r_i^n + \mathrm{o}(1) \\
&\le \theta_i\, \omega_n\, r_i^n \left\{J^{T_i} f(x_i) + C(n)\,\omega_*(r_i)\right\} + C\left\{\tau + C(n,\tau)\,\sigma\right\} r_i^n + \mathrm{o}(1) \\
&\le \Big\{J^{T_i} f(x_i) + C\left(\omega_*(r_*(\sigma)) + \tau + C(n,\tau)\,\sigma\right)\Big\} \times \\
&\quad \times \left(\theta_i\, \mathcal{H}^n(S_i\cap K) + \omega_*(r_*(\sigma))\, r_i^n\right)\, e^{\Lambda\, r_*(\sigma)} + \mathrm{o}(1) \\
&= J^{T_i} f(x_i)\big(\theta_i\, \mathcal{H}^n(S_i\cap K^*) + \alpha_i + \omega_*(r_*(\sigma))\, r_i^n\big)\, e^{\Lambda\, r_*(\sigma)} \qquad (4.21)\\
&\quad + C\left\{\omega_*(r_*(\sigma)) + \tau + C(n,\tau)\,\sigma\right\}\, \left(\mathcal{H}^n(S_i\cap K) + \omega_*(r_*(\sigma))\, r_i^n\right)\, e^{\Lambda\, r_*(\sigma)} \\
&\quad + \mathrm{o}(1)\,,
\end{aligned}$$

where we have set

$$\alpha_i = \theta_i\, \mathcal{H}^n(S_i\cap (K\setminus K^*)) \quad \text{so that} \quad \sum_i \alpha_i < 2\,\sigma\,. \qquad (4.22)$$

Now, again by (4.4) we see that

$$\begin{aligned}
\theta_i\, J^{T_i} f(x_i)\, \mathcal{H}^n(S_i\cap K^*) &\le \theta_i \int_{S_i\cap K^*} J^K f\, d\mathcal{H}^n + C(n)\,\omega_*(r_i)\, \mathcal{H}^n(S_i\cap K^*) \\
&= \theta_i\, \mathcal{H}^n(f(S_i\cap K^*)) + C(n)\,\omega_*(r_i)\, \mathcal{H}^n(S_i\cap K^*)\,.
\end{aligned}$$

By combining this last relation with (4.16), (4.19), (4.21) and $r_i < r_*(\sigma)$, we find that

$$\begin{aligned}
\mathcal{H}^n(\Omega\cap\partial f(F_j)) &\le e^{\Lambda\, r_*(\sigma)} \sum_i \theta_i\, \mathcal{H}^n(f(S_i\cap K^*)) \qquad (4.23)\\
&\quad + C\left\{\omega_*(r_*(\sigma)) + \tau + C(n,\tau)\,\sigma\right\} e^{\Lambda\, r_*(\sigma)} + \mathrm{o}(1)\,,
\end{aligned}$$

with $\mathrm{o}(1)\to 0$ as first $\eta\to 0^+$ and then $j\to\infty$. If $x_i\in K^*\setminus\partial^* E$, then $\theta_i = 2$ and by (4.5) we have

$$\begin{aligned}
\theta_i\, \mathcal{H}^n(f(S_i\cap K^*)) &\le 2\,\mathcal{H}^n\big(f\big(S_i\cap (K^*\setminus\partial^* E)\big)\big) + 2\,\mathrm{Lip}(f)^n\, \omega_*(r_i)\, r_i^n \\
&\le 2\mathcal{H}^n\big(f\big(S_i\cap (K\setminus\partial^* E)\big)\big) + C\,\omega_*(r_*(\sigma))\, r_i^n\,;
\end{aligned}$$

if, instead, $x_i\in\partial^* E$, then $\theta_i = 1$ and (4.6) give

$$\begin{aligned}
\theta_i\, \mathcal{H}^n(f(S_i\cap K^*)) &\le \mathcal{H}^n\big(f\big(S_i\cap K^*\cap\partial^* E\big)\big) + \mathrm{Lip}(f)^n\, \omega_*(r_i)\, r_i^n \\
&\le \mathcal{H}^n\big(f(S_i\cap\partial^* E)\big) + C\,\omega_*(r_*(\sigma))\, r_i^n\,;
\end{aligned}$$

combining these last two estimates with (4.16), we find

$$\begin{aligned}
\sum_i \theta_i\, \mathcal{H}^n(f(S_i\cap K^*)) &\le \sum_i 2\,\mathcal{H}^n\big(f\big(S_i\cap (K\setminus\partial^* E)\big)\big) + \mathcal{H}^n\big(f(S_i\cap\partial^* E)\big) \\
&\quad + C\,\omega_*(r_*(\sigma)) \sum_i r_i^n \\
&\le \mathcal{F}\Big(f(K), f(E); \bigcup_i f(S_i)\Big) + C\,\omega_*(r_*(\sigma))\,,
\end{aligned}$$

where $f(\partial^* E) = \partial^* f(E)$ by Lemma A.1. Combining this last estimate with (4.23) we find

$$\mathcal{H}^n(\Omega\cap\partial f(F_j)) \le e^{\Lambda\, r_*(\sigma)} \Big\{\mathcal{F}(f(K), f(E)) + C\left\{\omega_*(r_*(\sigma)) + \tau + C(n,\tau)\,\sigma\right\}\Big\} + \mathrm{o}(1)\,,$$

where $\mathrm{o}(1)\to 0$ as first $\eta\to 0^+$ and then $j\to\infty$; in particular, (4.18) holds.

We conclude the proof. As explained, (4.18) implies (4.1). By a classical first variation argument, we deduce the existence of $\lambda \in \mathbb{R}$ such that

$$\lambda \int_{\partial^* E} X \cdot \nu_E \, d\mathcal{H}^n = \int_{\partial^* E} \operatorname{div}^K X \, d\mathcal{H}^n + 2 \int_{K \setminus \partial^* E} \operatorname{div}^K X \, d\mathcal{H}^n \,, \tag{4.24}$$

for every $X \in C^1(\Omega; \mathbb{R}^{n+1})$ with $X \cdot \nu_\Omega = 0$ on $\partial\Omega$. Let us now consider the integer rectifiable varifold V supported on K, with density 2 on $K \setminus \partial^* E$ and 1 on $\partial^* E$. By (4.24), we can compute the first variation of V as

$$\delta V(X) = \int \vec{H} \cdot X \, d\|V\| \qquad \forall X \in C^1_c(\Omega; \mathbb{R}^{n+1})$$

where $\vec{H} = 0$ on $K \setminus \partial^* E$ and $\vec{H} = \lambda \nu_E$ on $\partial^* E$. In particular, $\vec{H} \in L^\infty(\|V\|)$, and by Allard's regularity theorem [Sim83, Chapter 5], we have $K = \Sigma \cup \mathrm{Reg}$, where $\Sigma \subset K$ is closed and has empty interior in K, and where for every $x \in \mathrm{Reg}$ there exists a $C^{1,\alpha}$-function u defined on $\mathbb{R}^n$ such that

$$B_{r_x/2}(x) \cap K = B_{r_x/2}(x) \cap \mathrm{Reg} = B_{r_x/2}(x) \cap \mathrm{graph}(u) \,. \tag{4.25}$$

By the divergence theorem, if $x \in \mathrm{Reg} \cap \partial E$, then, by (4.25) and by $\Omega \cap \partial E \subset K$,

$$E = \mathrm{epigraph}(u) \qquad \text{inside } B_{r_x/2}(x) \,, \tag{4.26}$$

$$K = \partial E = \mathrm{graph}(u) \qquad \text{inside } B_{r_x/2}(x) \,, \tag{4.27}$$

which imply $\mathrm{Reg} \cap \partial E \subset \Omega \cap \partial^* E$. Viceversa, if $x \in \Omega \cap \partial^* E$, then $\mathcal{H}^n(B_r(x) \cap (K \setminus \partial^* E)) = \mathrm{o}(r^n)$ and $\mathcal{H}^n(B_r(x) \cap \partial^* E) = \omega_n \, r^n + \mathrm{o}(r^n)$ as $r \to 0^+$, so that Allard's regularity theorem implies $\Omega \cap \partial^* E \subset \mathrm{Reg} \cap \partial E$. Thus $\mathrm{Reg} \cap \partial E = \Omega \cap \partial^* E$, and, in particular, $\Omega \cap (\partial E \setminus \partial^* E) \subset \Sigma$, so that $\Omega \cap (\partial E \setminus \partial^* E)$ has empty interior in K. Moreover, by (4.26), (4.24) implies that u is a graph of constant mean curvature in $B_{r_x/2}(x)$, and thus that $\partial^* E$ is a smooth hypersurface. Finally, (4.24) implies that K is the support of a multiplicity one stationary varifold in the open set $\Omega \setminus \partial E$, so that $K \setminus (\Sigma \cup \partial E)$ is a smooth hypersurface with zero mean curvature, and $\mathcal{H}^n(\Sigma \setminus \partial E) = 0$. The proof of Theorem 1.6 is complete. □

5. Convergence to Plateau's problem: Proof of Theorem 1.9

This section is devoted to showing that $\psi(\varepsilon) \to 2\,\ell$ as $\varepsilon \to 0^+$ and that a sequence $\{(K_h, E_h)\}_h$ of generalized minimizers for $\psi(\varepsilon_h)$ with $\varepsilon_h \to 0^+$ as $h \to \infty$ has to converge to a minimizer S for Plateau's problem ℓ counted with multiplicity 2 in the sense of Radon measures. If one could prove the latter assertion directly, then the former would follow at once by lower semicontinuity of weak-star converging Radon measures and by the upper bound $\psi(\varepsilon) \le 2\,\ell + C\,\varepsilon^{n/(n+1)}$ proved in (3.2). A possible direct approach to the convergence of (K_h, E_h) to a minimizer of Plateau's problem may be tried using White's compactness theorem [Whi09]. That would require proving an L^1-bound on the first variations of the varifolds V_h supported on K_h with density 1 on $\Omega \cap \partial^* E_h$ and with density 2 on $K_h \setminus \partial^* E_h$. The validity of such bound is supported by the analysis of simple examples like Example 1.1 and Example 1.2. However, Example 1.2 also indicates that when singularities are present in the limit Plateau minimizers S, then an L^1-bound for the mean curvatures of the varifolds V_h would result from a quantitative balance between the rate of divergence towards $-\infty$ of the constant mean curvatures of the reduced boundaries $\partial^* E_h$, and the rate of vanishing of the areas $\mathcal{H}^n(\Omega \cap \partial^* E_h)$. Validating a quantitative analysis of this kind in some generality would be of course very interesting per se as a way to describe the behavior of generalized minimizers; nonetheless, completing this analysis has so far eluded our attempts. Coming back to the proof of Theorem 1.9, we adopt a different approach. We prove directly that $\psi(\varepsilon) \to 2\,\ell$ as $\varepsilon \to 0^+$ by exploiting the same "compactness-by-comparison" strategy adopted in the proof of Theorem 1.4. An interesting point here is that because $|E_h| = \varepsilon_h \to 0^+$, we do not have a limit set that

we can use to uniformly adjust volumes among local competitors of the elements of the minimizing sequence, and have to use a sort of "absolute minimality at vanishing volumes" of any sequence $\{(K_h, E_h)\}_h$ of generalized minimizers such that $\lim_{h\to\infty} \mathcal{F}(K_h, E_h)$ is equal to $\liminf_{\varepsilon\to 0^+} \psi(\varepsilon)$.

Proof of Theorem 1.9. Step one: We start proving that ψ is lower semicontinuous on $(0, \infty)$. Given $\varepsilon_0 > 0$, let $\varepsilon_j \to \varepsilon_0 > 0$ as $j \to \infty$ be such that

$$\lim_{j\to\infty} \psi(\varepsilon_j) = \liminf_{\varepsilon\to\varepsilon_0} \psi(\varepsilon)\,,$$

and let $E_j \in \mathcal{E}$ be such that $|E_j| = \varepsilon_j$ and $\mathcal{H}^n(\Omega \cap \partial E_j) \le \psi(\varepsilon_j) + 1/j$. By (3.2), $\psi(\varepsilon_j)$ is bounded in j, and thus by the compactness criteria for sets of finite perimeter and for Radon measures we have that, up to extracting subsequences, $\mu_j = \mathcal{H}^n \llcorner (\Omega \cap \partial E_j) \overset{*}{\rightharpoonup} \mu$ as Radon measures in Ω and $E_j \to E$ in $L^1_{\rm loc}(\Omega)$, where μ is a Radon measure in Ω, and where $E \subset \Omega$ is a set of finite perimeter. We now repeat the proof of Theorem 1.4, with the only difference that while $|E_j|$ was constant in that proof, we know have that $|E_j| = \varepsilon_j \to \varepsilon_0$ for some $\varepsilon_0 > 0$. The modifications are minimal. In step two (nucleation of the sequence E_j), we repeat *verbatim* the argument, using the facts that $|E_j| \ge \varepsilon_0/2$ and that $\mathcal{H}^n(\Omega \cap \partial E_j) \le 2\,\ell + C\,\varepsilon_0^{n/(n-1)} + 1$ in place of $|E_j| = \varepsilon$ and $\mathcal{H}^n(\Omega \cap \partial E_j) \le \psi(\varepsilon) + 1$. Based on step two, in step three we construct volume-fixing variations with uniform constant ε_* and C_*, and then repeat the rest of the argument without modifications. As a consequence, we can show that $\mu = \theta\,\mathcal{H}^n \llcorner K$ and $(K, E) \in \mathcal{K}$ is a generalized minimizer of $\psi(\varepsilon_0)$, with

$$\psi(\varepsilon_0) = \mu(\Omega) = \lim_{j\to\infty} \mu_j(\Omega) \le \lim_{j\to\infty} \psi(\varepsilon_j) = \liminf_{\varepsilon\to\varepsilon_0} \psi(\varepsilon)\,,$$

as claimed. The key information here is of course that $|E_j| \ge \varepsilon_0/2$ where $\varepsilon_0 > 0$. If $\varepsilon_0 = 0$, then the nucleation lemma is inconsequential, and the argument cannot be used.

Step two: Thanks to (3.2), to prove $\psi(\varepsilon) \to 2\,\ell$ as $\varepsilon \to 0^+$ we just need to show that

$$\liminf_{\varepsilon\to 0^+} \psi(\varepsilon) \ge 2\,\ell\,. \tag{5.1}$$

To this end, we pick a sequence $\varepsilon_h \to 0^+$ such that

$$\liminf_{\varepsilon\to 0^+} \psi(\varepsilon) = \lim_{h\to\infty} \psi(\varepsilon_h)\,.$$

Notice that, in this way, given an arbitrary sequence $\sigma_h \to 0^+$, we have

$$\limsup_{h\to\infty} [\psi(\varepsilon_h) - \psi(\sigma_h)] \le 0\,. \tag{5.2}$$

Let $\{E_{h,j}\}_j$ be a minimizing sequence in $\psi(\varepsilon_h)$. By Theorem 1.4, there exists a generalized minimizer (K_h, E_h) in $\psi(\varepsilon_h)$ such that, up to extracting subsequences,

$$\begin{aligned}
&E_{h,j} \to E_h && \text{in } L^1(\Omega) \text{ as } j \to \infty\,,\\
&\mu_{h,j} = \mathcal{H}^n \llcorner (\Omega \cap \partial E_{h,j}) \overset{*}{\rightharpoonup} \mu_h && \text{as Radon measures in } \Omega \text{ as } j \to \infty\,,\\
&|E_{h,j}| = \varepsilon_h \text{ and } \mathcal{H}^n(\Omega \cap \partial E_{h,j}) \le \psi(\varepsilon_h) + \frac{1}{j}\,, && \forall j \in \mathbb{N}\,,
\end{aligned}$$

where, by (3.2) and up to extracting a further subsequence,

$$\mu_h = 2\,\mathcal{H}^n \llcorner (K_h \setminus \partial^* E_h) + \mathcal{H}^n \llcorner (\Omega \cap \partial^* E_h) \overset{*}{\rightharpoonup} \mu \qquad \text{as Radon measures in } \Omega \tag{5.3}$$

for some Radon measure μ in Ω. Given $x \in \Omega \cap \operatorname{spt} \mu$, we set $d(x) = \operatorname{dist}(x, \partial\Omega)$, and let

$$H_{x,r} = \big\{h \in \mathbb{N} : |E_h \setminus B_r(x)| > 0\big\}\,, \qquad I_x = \big\{r \in (0, d(x)) : H_{x,r} \text{ is infinite}\big\}\,. \tag{5.4}$$

We now look at local variations $F_{h,j}$ of $E_{h,j}$ such that $|F_{h,j}|$ has a positive limit volume σ_h as $j \to \infty$, which in turn satisfies $\sigma_h \to 0^+$ as $h \to \infty$. The idea is that, by (5.2), we will be able to use such variations to gather information on μ.

Claim: for every $r \in I_x$, if $\{F_{h,j}\}_{h\in H_{x,r},\, j\in\mathbb{N}} \subset \mathcal{E}$ is such that $\Omega \cap \partial F_{h,j}$ is $\mathcal{C}$-spanning W and $F_{h,j}\Delta E_{h,j} \subset \mathrm{cl}\,(B_r(x))$ for every $h \in H_{x,r}$ and every $j \in \mathbb{N}$, and if

$$\exists\ \sigma_h = \lim_{j\to\infty} |F_{h,j}| > 0\,, \qquad \text{and} \qquad \lim_{h\in H_{x,r}\,, h\to\infty} \sigma_h = 0\,, \tag{5.5}$$

then

$$\mu(B_r(x)) \le \liminf_{h\in H_{x,r}\,, h\to\infty} \liminf_{j\to\infty} \mathcal{H}^n\big(\mathrm{cl}\,(B_r(x)) \cap \partial F_{h,j}\big)\,. \tag{5.6}$$

To prove this claim, we first notice that, for every $h \in H_{x,r}$,

$$\sigma_h = \lim_{j\to\infty} |F_{h,j}| \ge |E_h \setminus B_r(x)| > 0\,. \tag{5.7}$$

In particular, for j large enough, $|F_{h,j}| > 0$, $\psi(|F_{h,j}|)$ is well-defined, and $F_{h,j}$ is a competitor for $\psi(|F_{h,j}|)$, so that

$$\begin{aligned}
\psi\big(|F_{h,j}|\big) &\le \mathcal{H}^n(\Omega \cap \partial F_{h,j}) = \mathcal{H}^n(\mathrm{cl}\,(B_r(x)) \cap \partial F_{h,j}) + \mathcal{H}^n(\partial E_{h,j} \cap \Omega \setminus \mathrm{cl}\,(B_r(x))) \\
&\le \mathcal{H}^n(\mathrm{cl}\,(B_r(x)) \cap \partial F_{h,j}) + \psi(\varepsilon_h) + \frac{1}{j} - \mathcal{H}^n(\partial E_{h,j} \cap B_r(x))
\end{aligned}$$

which can be recombined into

$$\mu_{h,j}(B_r(x)) \le \mathcal{H}^n(\mathrm{cl}\,(B_r(x)) \cap \partial F_{h,j}) + \psi(\varepsilon_h) - \psi\big(|F_{h,j}|\big) + \frac{1}{j}\,.$$

Letting $j \to \infty$, by $\mu_{h,j} \overset{*}{\rightharpoonup} \mu_h$, $|F_{h,j}| \to \sigma_h > 0$, and the lower semicontinuity of ψ on $(0,\infty)$, we find that

$$\mu_h(B_r(x)) \le \liminf_{j\to\infty} \mathcal{H}^n(\mathrm{cl}\,(B_r(x)) \cap \partial F_{h,j}) + \psi(\varepsilon_h) - \psi(\sigma_h)\,.$$

Since $\sigma_h \to 0^+$ as $h \to \infty$ with $h \in H_{x,r}$, by $\mu_h \overset{*}{\rightharpoonup} \mu$ and (5.2) we deduce (5.6), and thus prove the claim.

Step three: We now fix $x \in \mathrm{spt}\mu$, set $f(r) = \mu(B_r(x))$, and prove that, for a.e. $r \in I_x$,

$$\begin{cases} \text{either } f'(r) \ge c(n)\, r^{n-1}\,, \\ \text{or } (f^{1/n})'(r) \ge c(n)\,, \end{cases} \tag{5.8}$$

$$f(r) \le \frac{r}{n}\, f'(r)\,. \tag{5.9}$$

By using the coarea formula together with $|E_h| \to 0$ as $h \to \infty$ and $E_{h,j} \to E_h$ as $j \to \infty$, we find that for a.e. $r < d(x)$,

$$\partial E_{h,j} \cap \partial B_r(x) \text{ is } \mathcal{H}^{n-1}\text{-rectifiable}\,, \tag{5.10}$$

$$\lim_{j\to\infty} \mathcal{H}^n(E_{h,j} \cap \partial B_r(x)) = \mathcal{H}^n(E_h \cap \partial B_r(x))\,, \tag{5.11}$$

$$\lim_{h\to\infty} \lim_{j\to\infty} \mathcal{H}^n(E_{h,j} \cap \partial B_r(x)) = 0\,, \tag{5.12}$$

for every $h, j \in \mathbb{N}$. Moreover, if we set

$$f_{h,j}(r) = \mu_{h,j}(B_r(x))\,, \qquad f_h(r) = \mu_h(B_r(x))\,.$$

then, again by the coarea formula and by Fatou's lemma, for a.e. $r < d(x)$ we find

$$\begin{aligned}
\mathcal{H}^{n-1}(\partial E_{h,j} \cap \partial B_r(x)) &\le f'_{h,j}(r)\,, \\
g_h(r) = \liminf_{j\to\infty} f'_{h,j}(r) &\le f'_h(r)\,, \\
g(r) = \liminf_{h\in H_{x,r}, h\to\infty} f'_h(r) &\le f'(r)\,,
\end{aligned} \tag{5.13}$$

for every $h, j \in \mathbb{N}$. We first prove (5.8). Let $r \in I_x$ be such that (5.10), (5.11), (5.12) and (5.13) hold, and let $A_{h,j}$ denote an $\mathcal{H}^n$-maximal connected component of $\partial B_r(x) \setminus \partial E_{h,j}$.

If $A_{h,j} \subset E_{h,j}$, then, by spherical isoperimetry, by (5.13), and since the relative boundary to $A_{h,j}$ in $\partial B_r(x)$ is contained in $\partial B_r(x) \cap \partial E_{h,j}$, we find

$$f'_{h,j}(r) \geq c(n)\, \mathcal{H}^n(\partial B_r(x) \setminus A_{h,j})^{(n-1)/n}\,,$$

where the lower bound converges to $c(n)\, r^{n-1}$ if we let first $j \to \infty$ and then $h \to \infty$ thanks to (5.12); hence, if $A_{h,j} \subset E_{h,j}$, the first alternative in (5.8) holds. We now assume that $A_{h,j} \cap E_{h,j} = \emptyset$, and consider the corresponding cup competitor $F_{h,j}$ as defined in Lemma 2.5 starting from $E_{h,j}$, $A_{h,j}$ and $\eta = \eta_j = r/2j$, so that

$$F_{h,j} = \big(E_{h,j} \setminus \mathrm{cl}\,(B_r(x))\big) \cup N_{\eta_j}(Y_{h,j})\,, \qquad Y_{h,j} = \partial B_r(x) \setminus \mathrm{cl}\,\big(A_{h,j} \cup E_{h,j}\big)\,.$$

By Lemma 2.5, $F_{h,j} \in \mathcal{E}$, $\Omega \cap \partial F_{h,j}$ is $\mathcal{C}$-spanning W and $E_{h,j} \Delta F_{h,j} \subset \mathrm{cl}\,(B_r(x))$. Since $\eta_j \to 0$ as $j \to \infty$, we find

$$\sigma_h = \lim_{j\to\infty} |F_{h,j}| = \lim_{j\to\infty} |E_{h,j} \setminus B_r(x)| = |E_h \setminus B_r(x)|\,,$$

so that $\sigma_h > 0$ if $h \in H_{x,r}$, and $\sigma_h \to 0^+$ if we let $h \to \infty$. Thus $F_{h,j}$ satisfies (5.5), and we can apply (5.6) to $F_{h,j}$. To estimate the upper bound in (5.6), we look back at (2.20), and find that

$$\begin{aligned} \mathcal{H}^n(\mathrm{cl}\,(B_r(x)) \cap \partial F_{h,j}) \quad &\leq \quad (2 + C(n)\,\eta_j)\, \mathcal{H}^n(\partial B_r(x) \setminus A_{h,j}) \\ &\quad + C(n)\,\eta_j\, \mathcal{H}^{n-1}(\partial B_r(x) \cap \partial E_{h,j})\,. \end{aligned} \tag{5.14}$$

By (5.6) and (5.14) we deduce that

$$\begin{aligned} f(r) = \mu(B_r(x)) \quad &\leq \quad \liminf_{h\in H_{x,r}\,, h\to\infty} \liminf_{j\to\infty} \mathcal{H}^n\big(\mathrm{cl}\,(B_r(x)) \cap \partial F_{h,j}\big) \\ &\leq \quad \liminf_{h\in H_{x,r}\,, h\to\infty} \liminf_{j\to\infty} 2\, \mathcal{H}^n(\partial B_r(x) \setminus A_{h,j}) \qquad (5.15) \\ &\leq \quad C(n) \liminf_{h\in H_{x,r}\,, h\to\infty} \liminf_{j\to\infty} f'_{h,j}(r)^{n/(n-1)} \leq C(n)\, f'(r)^{n/(n-1)}\,, \end{aligned}$$

We have thus proved that the second alternative in (5.8) holds, as claimed. We now prove (5.9): let us now denote by $F_{h,j}$ the set defined by Lemma 2.8 as approximation of the cone competitor corresponding to $E_{h,j}$ in $B_r(x)$ with $\eta = \eta_j = r/2j$. We have that $F_{h,j} \in \mathcal{E}$ and that $\Omega \cap \partial F_{h,j}$ is $\mathcal{C}$-spanning W; furthermore, by (2.43) and (5.11) we find

$$\sigma_h = \lim_{j\to\infty} |F_{h,j}| \geq |E_h \setminus B_r(x)| + \frac{r}{n+1}\, \mathcal{H}^n(E_h \cap \partial B_r(x))$$

(in particular, $\sigma_h > 0$ if $h \in H_{x,r}$) and, by (5.12), $\sigma_h \to 0^+$ as $h \to \infty$. Thus (5.5) holds, and we can deduce from (5.6) and (2.42) that

$$f(r) = \mu(B_r(x)) \leq \liminf_{h\in H_{x,r}\,, h\to\infty} \liminf_{j\to\infty} \frac{r}{n}\, \mathcal{H}^{n-1}(\partial E_{h,j} \cap \partial B_r(x)) \leq \frac{r}{n}\, f'(r)\,,$$

that is (5.9).

Step four: We now define a function $g : \Omega \to (0,\infty) \cup \{-\infty\}$ by letting

$$\begin{aligned} g(x) \quad &= \quad \sup\Big\{ s > 0 : (0,s) \subset I_x \Big\} \\ &= \quad \sup\Big\{ t > 0 : \text{if } s < t, \text{ then } |E_h \setminus B_s(x)| > 0 \text{ for infinitely many } h \Big\}\,. \end{aligned}$$

We notice that

$$g \text{ is lower semicontinuous on } \Omega\,, \tag{5.16}$$

$$\{g = -\infty\} \text{ contains at most one point}\,. \tag{5.17}$$

(Notice that $\{g = -\infty\}$ may indeed contain one point: this is the case of the singular point of a triple junction, see Figure 1.3-(b)). To prove (5.16): if $g(x) \neq -\infty$, then $g(x) > 0$, and

for every $s \in (0, g(x))$, $|E_h \setminus B_s(x)| > 0$ for infinitely many h. Thus, if $\eta \in (0, g(x))$ and m_η is such that $|x - x_m| < \eta$ for every $m \geq m_\eta$, then, for every $m \geq m_\eta$ and $s \in (0, g(x) - \eta)$,

$$|E_h \setminus B_s(x_m)| \geq |E_h \setminus B_{s+\eta}(x)| > 0\,, \qquad \text{for intinitely many } h\,,$$

that is $g(x) - \eta \leq g(x_m)$ for every $m \geq m_\eta$; this proves (5.16). Next, if $g(x_1) = g(x_2) = -\infty$, then for every $s > 0$ there exists $h(s)$ such that

$$|E_h \setminus B_s(x_1)| = |E_h \setminus B_s(x_2)| = 0 \qquad \forall h \geq h(s)\,.$$

If $x_1 \neq x_2$ we can take $s = |x_1 - x_2|/2$ and deduce $|E_h| = 0$; thus (5.17) holds. Let us now consider the open set $\{g > s\} \subset \Omega$, $s > 0$, and set

$$Z(s) = \mathrm{spt}\mu \cap \{g > s\}\,, \qquad Z = \mathrm{spt}\mu \cap \{g > 0\}\,.$$

We claim that if $x \in Z(s)$, then

$$f(r) \geq c_0(n)\, r^n \qquad \forall r \in (0, s)\,, \qquad r^{-n} f(r) \text{ is increasing over } r \in (0, s)\,. \tag{5.18}$$

The second assertion is immediate from (5.9). To prove the first one, set

$$L_1 = \big\{r \in (0, s) : f'(r) \geq c(n)\, r^{n-1}\big\}\,, \qquad L_2 = (0, s) \setminus L_1\,,$$

with $c(n)$ as in (5.8). If $x \in Z(s)$ is such that $\mathcal{H}^1(L_1) \geq s/2$, then for every $r \in (0, s)$

$$f(r) \geq \int_{L_1 \cap (0,r)} f' \geq c(n) \int_{L_1 \cap (0,r)} t^{n-1}\, dt \geq c(n) \int_0^{\min\{r, s/2\}} t^{n-1}\, dt \geq \frac{c(n)}{n\, 2^n}\, r^n\,;$$

if instead $\mathcal{H}^1(L_2) \geq s/2$, then for every $r \in (0, s)$,

$$f(r)^{1/n} \geq \int_{L_2 \cap (0,r)} (f^{1/n})' \geq c(n)\, \mathcal{H}^1(L_2 \cap (0, r)) \geq c(n)\, \min\big\{r, \frac{s}{2}\big\} \geq \frac{c(n)}{2}\, r\,,$$

where we have used the fact that, by (5.8), we have $(f^{1/n})' \geq c(n)$ on L_2. Thanks to (5.18) we are in the position of using [Mat95, Theorem 6.9] and Preiss' theorem (as done in step four of the proof of Theorem 1.4) on each $Z(s)$, to find that Z is $\mathcal{H}^n$-rectifiable with

$$\mu \llcorner Z = \theta\, \mathcal{H}^n \llcorner Z\,, \tag{5.19}$$

where the density

$$\theta(x) = \lim_{r \to 0^+} \frac{\mu(B_r(x))}{\omega_n\, r^n} \quad \text{exists in } [c_0(n), \infty) \text{ for every } x \in Z\,.$$

Moreover, by (5.17),

$$\mathcal{H}^0(\mathrm{spt}\mu \setminus Z) \leq 1\,. \tag{5.20}$$

By combining (5.19) and (5.20) we find that $K = \mathrm{spt}\mu$ is $\mathcal{H}^n$-rectifiable and such that $\mu = \theta\, \mathcal{H}^n \llcorner K$. Since $K_h = \mathrm{spt}\, \mu_h$ is $\mathcal{C}$-spanning W and $\mu_h \overset{*}{\rightharpoonup} \mu$, by Lemma 2.1 we find that K is $\mathcal{C}$-spanning W, and thus admissible in ℓ, so that

$$\liminf_{\varepsilon \to 0^+} \psi(\varepsilon) = \lim_{h \to \infty} \mu_h(\Omega) \geq \mu(\Omega) = \int_K \theta\, d\mathcal{H}^n \geq \min_K \theta\, \mathcal{H}^n(K) \geq \ell \min_K \theta\,. \tag{5.21}$$

Thus, to complete the proof of (5.1) we just need to show that

$$\theta \geq 2 \ \mathcal{H}^n\text{-a.e. on } K\,. \tag{5.22}$$

Since $\mu_{h,j} = \mathcal{H}^n \llcorner (\Omega \cap \partial E_{h,j}) \overset{*}{\rightharpoonup} \mu_h$ as $j \to \infty$, with $\mu_h \overset{*}{\rightharpoonup} \theta\, \mathcal{H}^n \llcorner K$ as $h \to \infty$, we can extract a diagonal subsequence $j = j(h)$ so that, denoting $E_h^* = E_{h,j(h)}$, $\{E_h^*\}_h \subset \mathcal{E}$, $\Omega \cap \partial E_h^*$ $\mathcal{C}$-spanning W, and

$$\mu_h^* = \mathcal{H}^n \llcorner (\Omega \cap \partial E_h^*) \overset{*}{\rightharpoonup} \theta\, \mathcal{H}^n \llcorner K\,, \qquad \text{as } h \to \infty\,.$$

Moreover, $\mu(B_r(x)) \geq c(n)\, r^n$ for every $r \in (0, s)$ if $x \in K \cap \{g > s\}$ and, thanks to (5.14),

$$\liminf_{h \to \infty} \mathcal{H}^n(B_r(x) \cap \partial E_h^*) \leq C(n) \liminf_{h \to \infty} \mathcal{H}^n(\partial B_r(x) \setminus A_{r,h}^0)\,,$$

where $A^0_{r,h}$ denotes an $\mathcal{H}^n$-maximal connected component of $\partial B_r(x) \setminus \partial E^*_h$, this time for every $x \in K$ and $B_r(x) \subset\subset \Omega$. We can thus apply Lemma 2.10 with the open set $\Omega' = \{g > s\}$ to deduce that

$$\theta \geq 2\ \mathcal{H}^n\text{-a.e. on } \{g > s\} \cap K \setminus \partial^* E^*$$

where $E^* = \emptyset$ is the L^1-limit of the sets E^*_h. Since $\partial^* E^* = \emptyset$, taking the union over $s > 0$ and recalling (5.20), we conclude that (5.22) holds.

Step five: Now that $\psi(\varepsilon) \to 2\,\ell$ as $\varepsilon \to 0^+$ has been proved, let (K_h, E_h) be a sequence of generalized minimizers of $\psi(\varepsilon_h)$ for an arbitrary sequence $\varepsilon_h \to 0^+$. Since the limit of $\psi(\varepsilon)$ as $\varepsilon \to 0^+$ exists, ε_h automatically satisfies (5.2), and the arguments of step two to four can be repeated *verbatim*. Correspondingly, up to extracting subsequences, (5.3) holds with $\mu = \theta\,\mathcal{H}^n \llcorner K$, $\theta \geq 2\ \mathcal{H}^n$-a.e. on K, and K a relatively compact subset of Ω, $\mathcal{H}^n$-rectifiable, and $\mathcal{C}$-spanning W. By plugging $\psi(\varepsilon) \to 2\,\ell$ as $\varepsilon \to 0^+$ in (5.21), we find that $\theta = 2\ \mathcal{H}^n$-a.e. on K, $2\,\mathcal{H}^n(K) = 2\,\ell$, so that K is a minimizer of ℓ, and thus, looking back at (5.3), we conclude that (1.18) holds. □

Appendix A. A technical fact on sets of finite perimeter

Lemma A.1. *If Ω is an open set in $\mathbb{R}^{n+1}$, E is a set of finite perimeter in Ω, and $f : \Omega \to \Omega$ is a diffeomorphism, then $f(E)$ is a set of finite perimeter in Ω with $\partial^* f(E) = f(\partial^* E)$ and*

$$\nu_{f(E)}(f(x)) = \frac{(\nabla g(f(x)))^T \nu_E(x)}{|(\nabla g(f(x)))^T \nu_E(x)|} \qquad \forall x \in \partial^* E\,, \tag{A.1}$$

where $g = f^{-1}$.

Proof. In [Mag12, Proposition 17.1, Remark 17.2] it is shown that $f(E)$ is a set of finite perimeter with

$$\mu_{f(E)} = f_\#\Big(Jf\,(\nabla g(f))^T\,\mu_E\Big)\,.$$

and that mapping by f preserves essential boundaries (thus just the $\mathcal{H}^n$-equivalence of $\partial^* f(E)$ and $f(\partial^* E)$ is deduced there). In order to prove $\partial^* f(E) = f(\partial^* E)$, we pick a ball $B_r(f(x)) \subset\subset \Omega$, and look at

$$\begin{aligned}\frac{\mu_{f(E)}(B_r(f(x)))}{|\mu_{f(E)}|(B_r(f(x)))} &= \frac{\int_{g(B_r(f(x)))\cap\partial^* E} Jf\,\nabla g(f)^T \nu_E\, d\mathcal{H}^n}{\int_{g(B_r(f(x)))\cap\partial^* E} Jf\,|\nabla g(f)^T \nu_E|\, d\mathcal{H}^n} \\ &= \frac{\int_{(\partial^* E - x)/r} u_r(z)\nu_E(x + r\,z)\, d\mathcal{H}^n_z}{\int_{(\partial^* E - x)/r} |u_r(z)\,\nu_E(x + r\,z)|\, d\mathcal{H}^n_z}\,.\end{aligned} \tag{A.2}$$

where we have set

$$F_r = \frac{g(B_r(f(x))) - x}{r}\,, \qquad u_r(z) = 1_{F_r}(z)\,Jf(x + r\,z)\,\nabla g(f(x + r\,z))^T\,.$$

If we set $F = L(B_1(0))$ for the linear map $L = \nabla g(f(x))$, then for every $\varepsilon > 0$ we have

$$L(B_{1-\varepsilon}(0)) \subset F_r \subset L(B_{1+\varepsilon}(0)) \qquad \text{provided } r \text{ is small enough}\,,$$

and thus, as $r \to 0^+$,

$$1_{F_r} \to 1_F \qquad \text{uniformly on } \mathbb{R}^{n+1} \setminus X_\varepsilon\,,$$

where we have set

$$X_\varepsilon = L(B_{1+\varepsilon}(0) \setminus B_{1-\varepsilon}(0))\,.$$

Since $F_r, F \subset B_{\mathrm{Lip}g}(0)$, and since for every $R > 0$

$$Jf(x + r\,z)\,\nabla g(f(x + r\,z))^T \to Jf(x)\,\nabla g(f(x))^T \qquad \text{uniformly on } |z| \leq R\,,$$

as $r \to 0^+$, we conclude that

$$u_r(z) \to u(z) := 1_F(z)\, Jf(x)\, \nabla g(f(x))^T \qquad \text{uniformly on } \mathbb{R}^{n+1} \setminus X_\varepsilon\,. \tag{A.3}$$

We now decompose the integrals over $(\partial^* E - x)/r$ appearing in (A.2) through X_ε. By (A.3),

$$\begin{aligned} &\Big| \int_{[(\partial^* E - x)/r] \setminus X_\varepsilon} u_r(z) \nu_E(x + r\,z)\, d\mathcal{H}^n_z - \int_{[(\partial^* E - x)/r] \setminus X_\varepsilon} u(z) \nu_E(x + r\,z)\, d\mathcal{H}^n_z \Big| \\ \le\ & \omega(r)\, \mathcal{H}^n(B_{\mathrm{Lip} g}(0) \cap [(\partial^* E - x)/r] \setminus X_\varepsilon) \\ \le\ & \omega(r)\, P(E; B_{r\,\mathrm{Lip} g}(x)) \to 0 \end{aligned}$$

as $r \to 0^+$, while $x \in \partial^* E$ gives

$$\lim_{r \to 0^+} \int_{[(\partial^* E - x)/r] \setminus X_\varepsilon} u(z) \nu_E(x + r\,z)\, d\mathcal{H}^n_z = \int_{T_x(\partial^* E) \setminus X_\varepsilon} u(z) \nu_E(x)\, d\mathcal{H}^n_z\,.$$

At the same time, since $|u_r| \le C$ for a constant C independent of r, we have

$$\Big| \int_{X_\varepsilon \cap [(\partial^* E - x)/r]} u_r(z) \nu_E(x + r\,z)\, d\mathcal{H}^n_z \Big| \le C\, \mathcal{H}^n\Big(X_\varepsilon \cap \frac{\partial^* E - x}{r}\Big) \to C\, \mathcal{H}^n(X_\varepsilon \cap T_x(\partial^* E))$$

as $r \to 0^+$. Combining the above estimates with $|u| \le C$ we finally find

$$\begin{aligned} &\limsup_{r \to 0^+} \Big| \int_{(\partial^* E - x)/r} u_r(z) \nu_E(x + r\,z)\, d\mathcal{H}^n_z - \int_{T_x(\partial^* E)} u(z) \nu_E(x)\, d\mathcal{H}^n_z \Big| \\ \le\ & C\, \mathcal{H}^n(X_\varepsilon \cap T_x(\partial^* E))\,, \qquad \forall \varepsilon > 0\,. \end{aligned}$$

Letting $\varepsilon \to 0^+$, we find $\mathcal{H}^n(X_\varepsilon \cap T_x(\partial^* E)) \to \mathcal{H}^n(X \cap T_x(\partial^* E))$ where

$$X = L(\partial B_1(0))\,.$$

Since L is invertible, $L(\partial B_1(0))$ intersects transversally any plane through the origin, and in particular $T_x(\partial^* E)$. Therefore $\mathcal{H}^n(X \cap T_x(\partial^* E)) = 0$ and we have proved

$$\begin{aligned} \lim_{r \to 0^+} \int_{(\partial^* E - x)/r} u_r(z) \nu_E(x + r\,z)\, d\mathcal{H}^n_z &= \int_{T_x(\partial^* E)} u(z)\, \nu_E(x)\, d\mathcal{H}^n_z \\ &= Jf(x)\, L^T \nu_E(x)\, \mathcal{H}^n(F \cap T_x(\partial^* E))\,. \end{aligned}$$

An analogous argument shows

$$\lim_{r \to 0^+} \int_{(\partial^* E - x)/r} |u_r(z) \nu_E(x + r\,z)|\, d\mathcal{H}^n_z = Jf(x)\, |L^T \nu_E(x)|\, \mathcal{H}^n(F \cap T_x(\partial^* E))\,,$$

and finally we conclude that if $x \in \partial^* E$, then

$$\lim_{r \to 0^+} \frac{\mu_{f(E)}(B_r(f(x)))}{|\mu_{f(E)}|(B_r(f(x)))} = \frac{L^T \nu_E(x)}{|L^T \nu_E(x)|} \in \mathbb{S}^n\,.$$

In particular, $f(x) \in \partial^* f(E)$ and (A.1) holds. □

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Department of Mathematics, The University of Texas at Austin, 2515 Speedway, Stop C1200, Austin TX 78712-1202, USA

Email address: `king@math.utexas.edu`

Email address: `maggi@math.utexas.edu`

Email address: `stuvard@math.utexas.edu`