ON THE ANISOTROPIC KIRCHHOFF-PLATEAU PROBLEM

ANTONIO DE ROSA AND LUCA LUSSARDI

Abstract. We extend to the anisotropic setting the existence of solutions for the Kirchhoff-Plateau problem and its dimensional reduction.

Keywords: Kirchhoff-Plateau, anisotropic energies.
2010 Mathematics Subject Classification: 49Q20, 49Q10, 49J45, 74K10.

1. Introduction

The study of minimal surfaces spanning elastic boundaries dates back to Courant [13] and Lewy [33]. They studied the Plateau problem under the assumption that the boundary of the minimal surface is not fixed, but is constrained to lie on a prescribed manifold. The generalization to minimal surfaces spanning non-constrained elastic boundaries has been recently addressed by Giomi & Mahadevan [26]. These results have been complemented with an investigation of the stability of flat circular solutions by Chen & Fried [11], Biria & Fried [9, 10], Giusteri, Franceschini & Fried [27], and Hoang & Fried [32]. A similar problem has been treated by Bernatzky & Ye [6] employing the theory of currents, however the elastic energy used therein fails to satisfy the physical requirement of invariance under superposed rigid transformations.

The Kirchhoff-Plateau problem differs from the aforementioned works, because the spanning boundary is assumed to lay on the surface of an elastic loop, referred to as the rod, which is modeled as a deformable manifold. On the contrary, in all of the studies above the boundary of the spanning surface was assumed to coincide with the loop midline. In the Kirchhoff-Plateau problem the filament forming the loop is assumed to be thin enough to be modeled faithfully by a Kirchhoff rod, that is an unshearable inextensible rod which can sustain bending of its midline and twisting of its cross-sections, see Antman's work [5]. To model the flexible rods, some physical constraints are imposed, such as local and global non-interpenetration of matter introduced by Schuricht [34]. The *isotropic* Kirchhoff-Plateau problem, that is minimizing the area functional, has been investigated by Giusteri, Lussardi & Fried [28] with only one filament, and by Bevilacqua, Lussardi & Marzocchi [7] taking into account a system of linked rods. The authors utilize the boundary condition via linking number introduced by Harrison [30] and further investigated by De Lellis, Ghiraldin & Maggi

[15]. Moreover, a dimensional reduction of the aforementioned variational problem has been treated by Bevilacqua, Lussardi & Marzocchi [8].

In view of the works of Almgren [3, 4], Taylor [36] and Allard [1, 2], a natural question is whether the isotropic results [7, 8] generalize to anisotropic surface energies. Indeed, an increasing interest has been recently devoted to the study of the anisotropic Plateau problem: see for instance the results by De Philippis, De Rosa & Ghiraldin [16, 17, 18], De Rosa [20], De Rosa & Kolasinski [23] and Harrison & Pugh [31]. We also refer the reader to [21, 22, 24].

The aim of this paper is to address this question, considering the anisotropic Kirchhoff-Plateau problem for systems of linked rods. The energy functional we minimize is given by the sum of the elastic and the potential energy for the link and the anisotropic surface energy of the film. As for the isotropic Kirchhoff-Plateau problem we prescribe the linking type of the system of rods as well as the non-interpenetration of matter for each rod. Furthermore, each midline has a prescribed knot-type.

To conclude, we perform a dimensional reduction in the spirit of the analysis carried out in the isotropic setting.

2. Notation and preliminaries

In this section we recall notation for the geometry of curves. If $\boldsymbol{x}_1, \boldsymbol{x}_2 \colon [0, L] \to \mathbb{R}^3$ are two continuous and closed curves, their *linking number* is the integer value

$$\mathrm{Link}(\boldsymbol{x}_1, \boldsymbol{x}_2) := \frac{1}{4\pi} \int_0^L \int_0^L \frac{\boldsymbol{x}_1(s) - \boldsymbol{x}_2(t)}{|\boldsymbol{x}_1(s) - \boldsymbol{x}_2(t)|^3} \cdot \boldsymbol{x}_1'(s) \times \boldsymbol{x}_2'(t)\, ds dt.$$

We say that $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are *isotopic*, and we use the notation $\boldsymbol{x}_1 \simeq \boldsymbol{x}_2$, if there exists an open neighborhood N_1 of $\boldsymbol{x}_1([0, L])$, an open neighborhood N_2 of $\boldsymbol{x}_2([0, L])$ and a continuous map $\Phi \colon N_1 \times [0, 1] \to \mathbb{R}^3$ such that $\Phi(N_1, \tau)$ is homeomorphic to N_1 for all τ in $[0, 1]$ and

$$\Phi(\cdot, 0) = \mathrm{Identity}\,, \quad \Phi(N_1, 1) = N_2\,, \quad \Phi(\boldsymbol{x}_1([0, L]), 1) = \boldsymbol{x}_2([0, L])\,.$$

Following Gonzalez et al. [29], we define the *minimal global radius of curvature* of a closed curve $\boldsymbol{x} \in W^{1,p}((0, L); \mathbb{R}^3)$, with $p > 1$, by

$$\Delta(\boldsymbol{x}) := \inf_{s \neq \sigma \neq \tau \neq s \in [0,L)} R(\boldsymbol{x}(s), \boldsymbol{x}(\sigma), \boldsymbol{x}(\tau))$$

where $R(x, y, z)$ denotes the radius of the unique circle containing $x \neq y \neq z \neq x$, with the convention $R(x, y, z) = +\infty$ if x, y, z are collinear. The global radius of curvature determines the self-intersections of the tubular neighborhoods of a curve. More precisely, for every $r > 0$ we define the *r-tubular neighborhood of $\boldsymbol{x}$* by

$$U_r(\boldsymbol{x}) = \bigcup_{s \in [0,L]} B_r(\boldsymbol{x}(s)).$$

Accordingly to Ciarlet et al. [12] we say that $U_r(\boldsymbol{x})$ is *not self-intersecting* if for any $\boldsymbol{p} \in \partial U_r(\boldsymbol{x})$ there exists a unique $s \in [0, L]$ such that $\|\boldsymbol{p} - \boldsymbol{x}(s)\| = r$. It turns out (see Gonzalez et al. [29])

that $\Delta(\boldsymbol{x}) \geq r$ if and only if $U_r(\boldsymbol{x})$ is not self-intersecting. In particular, if $\Delta(\boldsymbol{x}) > 0$ then $\boldsymbol{x}$ is simple, that is $\boldsymbol{x}\colon [0,L) \to \mathbb{R}^3$ is injective.

3. The anisotropic Plateau problem

First we recall that a set $S \subset \mathbb{R}^3$ is said to be 2-*rectifiable* if it can be covered, up to an $\mathcal{H}^2$-negligible set, by countably many 2-dimensional submanifolds of class C^1, see [35, Chapter 3]. Given a 2-rectifiable set S, we denote by T_xS the approximate tangent space of $S \subset \mathbb{R}^3$ at x, which exists for $\mathcal{H}^2$-almost every point $x \in S$ [35, Chapter 3]. We also denote by G the Grassmannian of unoriented 2-dimensional planes in $\mathbb{R}^3$. The anisotropic integrands considered in the rest of the note will be continuous maps

$$F\colon \mathbb{R}^3 \times G \ni (x,\pi) \mapsto F(x,\pi) \in (0,+\infty),$$

verifying the lower and upper bounds

$$0 < \lambda \leq F(x,\pi) \leq \Lambda, \qquad \forall (x,\pi) \in \mathbb{R}^3 \times G. \tag{3.1}$$

We also require that F is *elliptic* [25, 5.1.2-5.1.5], namely its even and positively 1-homogeneous extension to $\mathbb{R}^3 \times (\Lambda_2(\mathbb{R}^3) \setminus \{0\})$ is C^2 and it is convex in the π variable. Given a 2-rectifiable set $S \subset \mathbb{R}^3$ we define:

$$\mathbf{F}(S) := \int_S F(x, T_xS)\, d\mathcal{H}^2(x). \tag{3.2}$$

Next, we need to define the spanning condition. For any closed set $H \subset \mathbb{R}^3$, let $\mathcal{C}(H)$ be the class of all smooth embeddings $\gamma\colon \mathbb{S}^1 \to \mathbb{R}^3 \setminus H$. We say that $\mathcal{C} \subset \mathcal{C}(H)$ is *closed by homotopy* if for every $\gamma \in \mathcal{C}$ then $\tilde{\gamma} \in \mathcal{C}$ for any $\tilde{\gamma} \in [\gamma] \in \pi_1(\mathbb{R}^3 \setminus H)$. We denote by $\mathcal{P}(H,\mathcal{C})$ the family of all 2-rectifiable relatively closed sets $S \subset \mathbb{R}^3 \setminus H$ such that

$$S \cap \gamma(\mathbb{S}^1) \neq \emptyset, \quad \forall \gamma \in \mathcal{C}.$$

We recall the following result, see [21, Theorem 2.7]:

Theorem 3.1. *The problem*

$$\min\{\mathbf{F}(S) : S \in \mathcal{P}(H,\mathcal{C})\}$$

has a solution $S \in \mathcal{P}(H,\mathcal{C})$ and the set S is an $(\mathbf{F},0,\infty)$-minimal set in $\mathbb{R}^3 \setminus H$ in the sense of Almgren [4].

4. The anisotropic Kirchhoff-Plateau problem

4.1. **The system of linked rods.** Let $N \in \mathbb{N}\setminus\{0\}$ and $p \in (1,+\infty)$. For every $i = 1,\dots,N$, let $L^i > 0$ and $\boldsymbol{x}_0^i, \boldsymbol{t}_0^i, \boldsymbol{d}_0^i \in \mathbb{R}^3$ be such $\boldsymbol{t}_0^i \perp \boldsymbol{d}_0^i$ and $|\boldsymbol{t}_0^i| = |\boldsymbol{d}_0^i| = 1$. Moreover let $\kappa_1^i, \kappa_2^i, \omega^i \in L^p(0,L^i)$ such that

$$w_1^i := (\kappa_1^i, \kappa_2^i, \omega^i) \in L^p((0,L^i);\mathbb{R}^3),$$
$$w^i := (w_1^i, \boldsymbol{x}_0^i, \boldsymbol{t}_0^i, \boldsymbol{d}_0^i) \in L^p((0,L^i);\mathbb{R}^3) \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3,$$

and

$$w := (w_1^1, w^2, \dots, w^N) \in L^p((0,L^1);\mathbb{R}^3) \times \prod_{i=2}^N ((L^p((0,L^i);\mathbb{R}^3) \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3) =: V.$$

We endow V with the natural L^p-norm, that we denote by $\|\cdot\|_V$. For any $i = 1,\dots,N$ and for any $w \in V$, we denote by $\boldsymbol{x}^i[w] \in W^{2,p}((0,L^i);\mathbb{R}^3)$ and $\boldsymbol{t}^i[w], \boldsymbol{d}^i[w] \in W^{1,p}((0,L^i);\mathbb{R}^3)$ the unique solutions (as proved in [29, Lemma 6]) of the Cauchy problem

$$\begin{cases} \boldsymbol{x}^i[w]'(s) = \boldsymbol{t}^i[w](s) \\ \boldsymbol{t}^i[w]'(s) = \kappa_1^i(s)\boldsymbol{d}^i[w](s) + \kappa_2^i(s)\boldsymbol{t}^i[w](s) \times \boldsymbol{d}^i[w](s) \\ \boldsymbol{d}^i[w]'(s) = \omega^i(s)\boldsymbol{t}^i[w](s) \times \boldsymbol{d}^i[w](s) - \kappa_1^i(s)\boldsymbol{t}^i[w](s) \\ \boldsymbol{x}^i[w](0) = \boldsymbol{x}_0^i \\ \boldsymbol{t}^i[w](0) = \boldsymbol{t}_0^i \\ \boldsymbol{d}^i[w](0) = \boldsymbol{d}_0^i. \end{cases}$$

It is easy to see that $\boldsymbol{t}^i[w](s) \perp \boldsymbol{d}^i[w](s)$ and $|\boldsymbol{t}^i[w](s)| = |\boldsymbol{d}^i[w](s)| = 1$ and consequently that

$$(\boldsymbol{t}^i[w](s), \boldsymbol{d}^i[w](s), \boldsymbol{t}^i[w](s) \times \boldsymbol{d}^i[w](s))$$

is an orthonormal frame in $\mathbb{R}^3$, for any $s \in [0, L^i]$ and for any $i = 1,\dots,N$. Let $\eta, \nu > 0$ and consider $\mathcal{A}^i(s) \subset \mathbb{R}^2$ be compact and simply connected such that

$$B_\eta(\mathbf{0}) \subset \mathcal{A}^i(s) \subset B_\nu(\mathbf{0}), \qquad \forall s \in [0, L^i],\ i = 1,\dots,N.$$

For any $i = 1,\dots,N$ we define

$$\Omega^i := \{(s,\zeta_1,\zeta_2) \in \mathbb{R}^3 : s \in [0,L^i],\ (\zeta_1,\zeta_2) \in \mathcal{A}^i(s)\},$$

$$\Lambda^i[w] := \{\boldsymbol{x}^i[w](s) + \zeta_1\boldsymbol{d}^i[w](s) + \zeta_2\boldsymbol{t}^i[w](s) \times \boldsymbol{d}^i[w](s) : (s,\zeta_1,\zeta_2) \in \Omega^i\}, \tag{4.1}$$

and

$$\Lambda[w] := \bigcup_{i=1}^N \Lambda^i[w].$$

The system of closed rods is subjected to some constraints on w, enumerated below, which will identify the admissible subset $W \subset V$: First of all we assume that the midlines are closed and sufficiently smooth, that is

(C1) $\boldsymbol{x}^i[w](L^i) = \boldsymbol{x}^i[w](0) = \boldsymbol{x}_0^i$, for any $i = 1,\dots,N$

and

(C2) $\boldsymbol{t}^i[w](L^i) = \boldsymbol{t}^i[w](0) = \boldsymbol{t}_0^i$, for any $i = 1,\dots,N$.

To prescribe how many times the ends of the rods are twisted before being glued together, we prescribe the linking number between the midline and a closed curve close to the midline.

More precisely, for any $i = 1, \dots, N$ we close up the curve $\boldsymbol{x}^i[w] + \tau \boldsymbol{d}^i[w]$, for $\tau > 0$ fixed and small enough, defining as in Schuricht [34]

$$\tilde{\boldsymbol{x}}^i{}_\tau[w](s) := \begin{cases} \boldsymbol{x}^i[w](s) + \tau \boldsymbol{d}^i[w](s) & \text{if } s \in [0, L^i] \\ \\ \boldsymbol{x}^i[w](L^i) + \tau(\cos(\varphi^i(s - L^i))\boldsymbol{d}^i[w](L^i) \\ +\sin(\varphi^i(s - L^i))\boldsymbol{t}^i[w](L^i) \times \boldsymbol{d}^i[w](L^i)) & \text{if } s \in [L^i, L^i + 1] \end{cases} \tag{4.2}$$

where $\varphi^i \in [0, 2\pi)$ is the unique angle between $\boldsymbol{d}_0^i$ and $\boldsymbol{d}^i[w](L^i)$ such that $\varphi^i - \pi$ has the same sign as $\boldsymbol{d}_0^i \times \boldsymbol{d}^i[w](L^i) \cdot \boldsymbol{t}_0^i$. We trivially identify $\boldsymbol{x}^i[w]$ with its extension $\boldsymbol{x}^i[w](s) = \boldsymbol{x}^i(L^i)$ for any $s \in [L^i, L^i + 1]$ and therefore we require that for any $i = 1, \dots, N$ there is some $l^i \in \mathbb{Z}$ such that

(C3) $\mathrm{Link}(\boldsymbol{x}^i[w], \tilde{\boldsymbol{x}}^i{}_\tau[w]) = l^i$.

To encode the knot type of the midlines, for any $i = 1, \dots, N$ we fix a continuous mapping $\boldsymbol{\ell}^i \colon [0, L^i] \to \mathbb{R}^3$ such that $\boldsymbol{\ell}^i(L^i) = \boldsymbol{\ell}^i(0)$ and we require that

(C4) $\boldsymbol{x}^i[w] \simeq \boldsymbol{\ell}^i$.

Finally, in order to prevent the interpenetration of matter, following Ciarlet et al. [12] we require that

(C5)

$$\int_{\Omega^i} (1 - \zeta_1 k_2^i(s) + \zeta_2 k_1^i(s))\, ds d\zeta_1 d\zeta_2 \leq |\Lambda^i[w]| \quad \forall i = 1, \dots, N, \quad \text{and} \quad \bigcap_{i=1}^N \mathrm{int}(\Lambda^i[w]) = \emptyset.$$

We now require that our system of rods has a prescribed chain structure. We fix $L^{ij} \in \mathbb{Z}^{N \times N}$, with the property that $|L^{i(i+1)}| = 1$ for every $i = 1, \dots, N - 1$ and we assume that:

(C6) $\mathrm{Link}(\boldsymbol{x}^i[w], \boldsymbol{x}^j[w]) = L^{ij}$.

We finally denote by W the set of all constraints, namely

$$W := \big\{ w \in V : \text{(C1)–(C6) hold true} \big\}.$$

It turns out that W is weakly closed in V (see Gonzalez et al. [29] and Schuricht [34]).

4.2. **Energy contributions and existence of a minimizer.** In what follows we will prescribe an elastic energy of the system of rods, which is a proper function

$$E^{\mathrm{el}} \colon W \to \mathbb{R} \cup \{+\infty\}, \qquad \text{satisfying} \qquad E^{\mathrm{el}}(w) \geq c\|w\|_V, \tag{4.3}$$

for some $c > 0$. The second energy contribution we want to take into account is the weight of the rods. Let $\rho^i \in L^\infty(\Omega^i)$ with $\rho \geq 0$ be the mass density functions and $\boldsymbol{g}$ be the gravitational acceleration. Let us define $E^{\mathrm{g}} \colon W \to \mathbb{R} \cup \{+\infty\}$ as

$$E^{\mathrm{g}}(w) := \sum_{i=0}^N \int_{\Omega^i} \rho^i(s, \zeta_1, \zeta_2) \boldsymbol{g} \cdot (\boldsymbol{x}^i[w](s) + \zeta_1 \boldsymbol{d}^i[w](s) + \zeta_2 \boldsymbol{t}^i[w](s) \times \boldsymbol{d}^i[w](s))\, ds d\zeta_1 d\zeta_2.$$

The last contribution we want to take into account is the surface energy. Let $\mathcal{C}_w \subset \mathcal{C}(\Lambda[w])$ be the class of all $\gamma \in \mathcal{C}(\Lambda[w])$ such that there exists $i = 1, \ldots, N$ with

$$|\mathrm{Link}(\gamma, \boldsymbol{x}^i[w])| = 1, \quad \mathrm{Link}(\gamma, \boldsymbol{x}^j[w]) = 0, \forall j \neq i.$$

$\mathcal{C}_w$ is closed by homotopy, see [30]. We define $E^{\mathrm{sf}} \colon W \to \mathbb{R} \cup \{+\infty\}$ as

$$E^{\mathrm{sf}}(w) := \inf \big\{ \mathbf{F}(S) : S \in \mathcal{P}(\Lambda[w], \mathcal{C}_w) \big\}.$$

We define the energy functional of our variational problem as

$$E(w) := E^{\mathrm{el}}(w) + E^{\mathrm{g}}(w) + E^{\mathrm{sf}}(w), \quad w \in W. \tag{4.4}$$

The first main result of the paper is given by the following existence theorem.

Theorem 4.1. *Let $\overline{E^{\mathrm{el}}}$ be the lower semicontinuous envelope of E^{el} with respect to the weak topology of V. Assume that $\inf_W E < +\infty$. Then the problem*

$$\min_{w \in W} \overline{E^{\mathrm{el}}}(w) + E^{\mathrm{g}}(w) + E^{\mathrm{sf}}(w)$$

has a solution $w_0 \in W$ and there exists $S_\infty \in \mathcal{P}(\Lambda[w_0], \mathcal{C}_{w_0})$ which is an $(\mathbf{F}, 0, \infty)$-minimal set in $\mathbb{R}^3 \setminus \Lambda[w_0]$ in the sense of Almgren such that

$$E^{\mathrm{el}}(w_0) + E^{\mathrm{g}}(w_0) + \mathbf{F}(S_\infty) = \min_{w \in W} \overline{E^{\mathrm{el}}}(w) + E^{\mathrm{g}}(w) + E^{\mathrm{sf}}(w) = \inf_{w \in W} E(w).$$

4.3. **Proof of Theorem 4.1.** First of all we prove that the weight and the soap film energy are weakly continuous.

Lemma 4.2. *The functional E^{g} is weakly continuous on W.*

Proof. Let (w_h) be a sequence in W with $w_h \rightharpoonup w$ in W for some $w \in W$. Then $\boldsymbol{x}^i[w_h] \rightharpoonup \boldsymbol{x}^i[w]$ in $W^{2,p}$ and $\boldsymbol{t}^i[w_h] \rightharpoonup \boldsymbol{t}^i[w]$, $\boldsymbol{d}^i[w_h] \rightharpoonup \boldsymbol{d}^i[w]$ in $W^{1,p}$. Then by Sobolev embedding we deduce that $\boldsymbol{x}^i[w_h] \to \boldsymbol{x}^i[w]$ in $C^{1,\alpha}$ and $\boldsymbol{t}^i[w_h] \to \boldsymbol{t}^i[w]$, $\boldsymbol{d}^i[w_h] \to \boldsymbol{d}^i[w]$ in $C^{0,\alpha}$ for some $\alpha \in (0,1)$. This is enough to pass to the limit under the integral and get the claim. □

The continuity of the soap film energy follows from the next theorem.

Theorem 4.3. *Let (w_h) be a sequence in W with $w_h \rightharpoonup w$ in W for some $w \in W$. Assume that*

(a) *$S_h \in \mathcal{P}(\Lambda[w_h], \mathcal{C}_{w_h})$, for every $h \in \mathbb{N}$;*
(b) *$\sup_{h \in \mathbb{N}} \mathbf{F}(S_h) = \sup_{h \in \mathbb{N}} \inf\{\mathbf{F}(S) : S \in \mathcal{P}(\Lambda[w_h], \mathcal{C}_{w_h})\} < +\infty$.*

Let $\mu_h := F\mathcal{H}^2 \llcorner S_h$. Then the following three statements hold true:

$$\mu_h \rightharpoonup^* \mu \quad \textit{(up to subsequences)}, \tag{4.5}$$

$$\mu \geq F\mathcal{H}^2 \llcorner S_\infty, \ \textit{where } S_\infty = (\mathit{spt}\, \mu) \setminus \Lambda[w] \ \textit{is 2-rectifiable}; \tag{4.6}$$

$$S_\infty \in \mathcal{P}(\Lambda[w], \mathcal{C}_w). \tag{4.7}$$

Proof. We first observe that the classes $\mathcal{P}(\Lambda[w_h], \mathcal{C}_{w_h})$ and $\mathcal{P}(\Lambda[w], \mathcal{C}_w)$ are good classes in the sense of De Lellis et al. [21, Def. 2.2], as proved in [21, Thm. 2.7(a)]. Then the proof of (4.5) and (4.6) follows verbatim the proof of Theorem 2.5 of [21]. It is sufficient to observe that the convergence of $\{\Lambda[w_h]\}$ ensures that, whenever $x \in S_\infty$, we have $d(x, \Lambda[w_h]) > 0$ for h large enough. We are left to prove (4.7), namely that $S_\infty \cap \gamma(\mathbb{S}^1) \neq \emptyset$ for any $\gamma \in \mathcal{C}_w$. Assume by contradiction that there exists $\gamma \in \mathcal{C}_w$ with $S_\infty \cap \gamma(\mathbb{S}^1) = \emptyset$. Since γ is compact and contained in $\mathbb{R}^3 \setminus \Lambda[w]$ and S_∞ is relatively closed in $\mathbb{R}^3 \setminus \Lambda[w]$, there exists a positive ε such that the tubular neighborhood $U_{2\varepsilon}(\gamma)$ does not intersect S_∞ and is contained in $\mathbb{R}^3 \setminus \Lambda[w]$. Hence $\mu(U_{2\varepsilon}(\gamma)) = 0$, and consequently

$$\lim_{h\to\infty} \mathcal{H}^2(S_h \cap U_\varepsilon(\gamma)) = 0. \tag{4.8}$$

Denote by B_ε the open disk of $\mathbb{R}^2$ with radius ε and centered at the origin of $\mathbb{R}^2$, and consider a diffeomorphism $\Phi\colon \mathbb{S}^1 \times B_\varepsilon \to U_\varepsilon(\gamma)$ such that $\Phi_{|_{\mathbb{S}^1\times\{0\}}} = \gamma$. Let y belong to B_ε and set $\gamma_y := \Phi_{|_{\mathbb{S}^1\times\{y\}}}$. Then γ_y in $[\gamma]$ represents an element of $\pi_1(\mathbb{R}^3 \setminus \Lambda[w])$. Since $w_h \rightharpoonup w$ in W then $(\boldsymbol{x}^i[w_h])$ converges to $\boldsymbol{x}^i[w]$ strongly in $W^{1,p}((0,L);\mathbb{R}^3)$ for every $i = 1, \dots, N$. In particular, $(\boldsymbol{x}^i[w_h])$ converges to $\boldsymbol{x}^i[w]$ uniformly on $[0, L^i]$ for every $i = 1, \dots, N$, which implies the existence of $\delta > 0$ such that, for h sufficiently large, $\Lambda[w_h]$ is contained in $U_\delta(\Lambda[w])$ with $U_\delta(\Lambda[w]) \cap U_\varepsilon(\gamma) = \emptyset$. Hence, for such h and ε it follows that, for any $y \in B_\varepsilon$, $\gamma_y(\mathbb{S}^1) \subset \mathbb{R}^3 \setminus U_\delta(\Lambda[w])$. This implies that $\|\boldsymbol{x}^i[w_h] - \gamma_y\|_\infty \geq \delta$ for any $y \in B_\varepsilon$ and for every $i = 1, \dots, N$. This estimate, together with the $W^{1,p}$ convergence of $\boldsymbol{x}^i[w_h]$ to $\boldsymbol{x}^i[w]$, implies that

$$\lim_{h\to+\infty} \mathrm{Link}(\boldsymbol{x}^i[w_h], \gamma_y) = \mathrm{Link}(\boldsymbol{x}^i[w], \gamma_y), \qquad \forall y \in B_\varepsilon,\ \forall i = 1, \dots, N.$$

As a consequence, for h large enough, $\gamma_y \in \mathcal{C}_{w_h}$ which, combined with $S_h \in \mathcal{P}(\Lambda[w_h], \mathcal{C}_{w_h})$, yields $S_h \cap \gamma_y(\mathbb{S}^1) \neq \emptyset$. Take now $\tilde{\pi} : \mathbb{S}^1 \times B_\varepsilon \to B_\varepsilon$ as the projection on the second factor and let $\hat{\pi} := \tilde{\pi} \circ \Phi^{-1}$. Then, $\hat{\pi}$ is Lipschitz-continuous and B_ε is contained in $\hat{\pi}(S_h \cap U_\varepsilon(\gamma))$, which entails that

$$\pi\varepsilon^2 = \mathcal{H}^2(B_\varepsilon) \leq \mathcal{H}^2(\hat{\pi}(S_h \cap U_\varepsilon(\gamma)) \leq (\mathrm{Lip}\,\hat{\pi})^2 \mathcal{H}^2(S_h \cap U_\varepsilon(\gamma))\,.$$

We thus conclude that

$$\mathcal{H}^2(S_h \cap U_\varepsilon(\gamma)) \geq \frac{\pi\varepsilon^2}{(\mathrm{Lip}\,\hat{\pi})^2}$$

which contradicts (4.8). □

Proof of Theorem 4.1. Thanks to the weak continuity of E^{g} and E^{sf}, proved in Lemma 4.2 and Theorem 4.3, we deduce that $\overline{E^{\mathrm{el}}}(w) + E^{\mathrm{g}}(w) + E^{\mathrm{sf}}(w)$ is the lower semicontinuous envelope of E, from which we get

$$\inf_{w\in W} \overline{E^{\mathrm{el}}}(w) + E^{\mathrm{g}}(w) + E^{\mathrm{sf}}(w) = \inf_{w\in W} E(w).$$

Let $\{w_h\}$ be a minimizing sequence for $E^{\rm el}+E^{\rm g}+E^{\rm sf}$. Since $\inf_W E<+\infty$ we can say that $E(w_h)\le c$ for some $c>0$. In particular, $E^{\rm el}(w_h)\le c$ and, by coercivity of $E^{\rm el}$, we have $w_h \rightharpoonup w_0$ in W. We deduce, using again Lemma 4.2 and Theorem 4.3, that

$$\begin{aligned}\overline{E^{\rm el}}(w_0)+E^{\rm g}(w_0)+E^{\rm sf}(w_0) &\le \liminf_h \overline{E^{\rm el}}(w_h)+E^{\rm g}(w_h)+E^{\rm sf}(w_h)\\ &\le \liminf_h E(w_h)=\inf_W E=\inf_W \overline{E^{\rm el}}+E^{\rm g}+E^{\rm sf}.\end{aligned}$$

Moreover, since $E^{\rm sf}(w_0)<+\infty$, applying Theorem 2.7 of [21] we deduce the claim. □

5. Dimensional reduction of the anisotropic Kirchhoff-Plateau problem

The second main result of the paper concerns the dimensional reduction. In this section we consider cross sections with vanishing diameter. The set of constraints is almost the same, but in order to prevent the non-selfintersection in the limit configurations (otherwise the knot-type is not well defined) we replace the constraint (C5) by (C5)'. Precisely, we require that:

(C5)' $\Delta(\boldsymbol{x}^i[w])\ge\Delta_0$ for some prescribed $\Delta_0>0$.

We denote by W' the set of all constraints, namely

$$W' := \big\{w\in L^p([0,L];\mathbb{R}^3) : \text{(C1)-(C2)-(C3)-(C4)-(C5)'-(C6) hold true}\big\}.$$

It turns out that W' is weakly closed in V (see again Gonzalez et al. [29] and Schuricht [34]). For every $i=1,\dots,N$, every $\varepsilon>0$ small enough and every $w\in W'$ we let

$$\Lambda^i_\varepsilon[w] := \{\boldsymbol{x}^i[w](s)+\zeta_1\boldsymbol{d}^i[w](s)+\zeta_2\boldsymbol{t}^i[w](s)\times\boldsymbol{d}^i[w](s) : (s,\zeta_1,\zeta_2)\in\Omega^i_\varepsilon\} \tag{5.1}$$

where

$$\Omega^i_\varepsilon := \{(s,\zeta_1,\zeta_2)\in\mathbb{R}^3 : s\in[0,L^i],\,(\zeta_1,\zeta_2)\in\varepsilon\mathcal{A}^i(s)\}.$$

We also let

$$\Lambda_\varepsilon[w] := \bigcup_{i=1}^N \Lambda^i_\varepsilon[w].$$

The main goal of this section is to prove that as ε approaches 0, we recover by Γ-convergence the anisotropic Plateau problem with elastic one dimensional boundary. The first two energy contributions to take into account are the elastic energy $E^{\rm el}$ as in (4.3) and the scaled weight

$$E^{\rm g}_\varepsilon(w) := \frac{1}{\varepsilon^2}\sum_{i=1}^N\int_{\Omega^i_\varepsilon}\rho^i(s,\zeta_1,\zeta_2)\boldsymbol{g}\cdot(\boldsymbol{x}^i[w](s)+\zeta_1\boldsymbol{d}^i[w](s)+\zeta_2\boldsymbol{t}^i[w](s)\times\boldsymbol{d}^i[w](s))\,ds d\zeta_1 d\zeta_2$$

where $\rho^i\in L^\infty(\Omega^i_1)$ and $\rho^i\ge 0$. Concerning the soap film energy, similarly to the previous section, we define $\mathcal{C}_{\varepsilon,w}\subset\mathcal{C}(\Lambda_\varepsilon[w])$ as the class of all $\gamma\in\mathcal{C}(\Lambda_\varepsilon[w])$ such that there exists $i=1,\dots,N$ with

$$|\mathrm{Link}(\gamma,\boldsymbol{x}^i[w])|=1,\quad \mathrm{Link}(\gamma,\boldsymbol{x}^j[w])=0\quad\forall j\ne i.$$

We define $E^{\mathrm{sf}}_\varepsilon \colon W' \to \mathbb{R} \cup \{+\infty\}$ as

$$E^{\mathrm{sf}}_\varepsilon(w) := \inf\big\{\mathbf{F}(S) : S \in \mathcal{P}(\Lambda_\varepsilon[w], \mathcal{C}_{\varepsilon,w})\big\}.$$

Finally, $E_\varepsilon \colon W' \to \mathbb{R} \cup \{+\infty\}$ is given by

$$E_\varepsilon(w) := E^{\mathrm{el}}(w) + E^{\mathrm{g}}_\varepsilon(w) + E^{\mathrm{sf}}_\varepsilon(w).$$

We now describe the Γ-limit functional. For any $i = 1, \dots, N$, let $\rho^i_0 \colon [0, L^i] \to \mathbb{R}$ be given by

$$\rho^i_0(s) := \lim_{(\xi_1,\xi_2)\to(0,0)} \rho^i(s, \xi_1, \xi_2)$$

and let

$$E_0(w) := \overline{E^{\mathrm{el}}}(w) + \sum_{i=1}^N \int_0^{L^i} |\mathcal{A}^i(s)| \rho^i_0(s) \boldsymbol{g} \cdot \boldsymbol{x}^i[w](s)\, ds + \inf\{\mathbf{F}(S) : S \in \mathcal{P}(H_w, \mathcal{C}_w)\},$$

where

$$H_w := \bigcup_{i=1}^N \boldsymbol{x}^i[w]([0, L^i]),$$

and $\mathcal{C}_w$ is the class of all $\gamma \in \mathcal{C}(H_w)$ such that there exists $i = 1, \dots, N$ with

$$|\mathrm{Link}(\gamma, \boldsymbol{x}^i[w])| = 1, \quad \mathrm{Link}(\gamma, \boldsymbol{x}^j[w]) = 0 \quad \forall j \neq i.$$

We are ready to state our second main result.

Theorem 5.1. *Let (ε_h) be a positive and infinitesimal sequence and let (w_h) be a sequence in W' with $\sup_{h\in\mathbb{N}} E_{\varepsilon_h}(w_h) \leq c$ for some $c > 0$. Then, up to a subsequence, $w_h \rightharpoonup w$ in V and $w \in W'$. Moreover, the family $\{E_\varepsilon\}_{\varepsilon>0}$ Γ-converges to E_0 as $\varepsilon \to 0^+$ with respect to the weak topology of V, namely:*

(a) *for any sequence (ε_h) with $\varepsilon_h \to 0$, for any $w \in W'$ and for any sequence (w_h) in W' with $w_h \rightharpoonup w$ in V we have*

$$E_0(w) \leq \liminf_{h\to+\infty} E_{\varepsilon_h}(w_h); \tag{5.2}$$

(b) *for any $w \in W'$ there is a sequence (ε_h) with $\varepsilon_h \to 0$ and a sequence $(\bar{w}_h)$ in W' with $\bar{w}_h \rightharpoonup w$ in V such that*

$$E_0(w) \geq \limsup_{h\to+\infty} E_{\varepsilon_h}(\bar{w}_h). \tag{5.3}$$

As a standard consequence of Theorem 5.1 we have the next result.

Corollary 5.2. *Let (ε_h) be a positive and infinitesimal sequence. For any $h \in \mathbb{N}$ and for any $\sigma_h \to 0$ let $w_h \in W'$ be such that*

$$E_{\varepsilon_h}(w_h) \leq \inf_{W'} E_{\varepsilon_h} + \sigma_h. \tag{5.4}$$

Then up to a subsequence $w_h \rightharpoonup w_0$ in V and

$$E_0(w_0) = \min_{W'} E_0.$$

5.1. **Proof of Theorem 5.1.** Here we give some preliminary propositions and then prove Theorem 5.1. Fix a positive and infinitesimal sequence (ε_h).

Proposition 5.3. *Let (w_h) be a sequence in W' with $\sup_{h\in\mathbb{N}} E_{\varepsilon_h}(w_h) \le c$ for some $c>0$. Then, up to a subsequence, $w_h \rightharpoonup w$ in V and $w\in W'$.*

Proof. The conclusion follows from the coercivity of E^{el}. □

The study of the weight term is easy, since the weak convergence $w_h \rightharpoonup w$ implies the uniform convergence of the midlines.

Proposition 5.4. *For any $w\in W'$ and for any sequence (w_h) in W' with $w_h \rightharpoonup w$ in V we have*

$$\lim_{h\to+\infty} E^{\mathrm{g}}_{\varepsilon_h}(w_h) = \sum_{i=1}^N \int_0^{L^i} |\mathcal{A}^i(s)|\rho_0^i(s)\boldsymbol{g}\cdot\boldsymbol{x}^i[w](s)\,ds. \tag{5.5}$$

Proof. By the change of variables $\zeta_j = \varepsilon_h\eta_j$, $j=1,2$, we obtain that for any $i=1,\dots N$,

$$= \frac{1}{\varepsilon_h^2}\int_{\Omega^i_{\varepsilon_h}} \rho^i(s,\zeta_1,\zeta_2)\boldsymbol{g}\cdot(\boldsymbol{x}^i[w_h](s)+\zeta_1\boldsymbol{d}^i[w_h](s)+\zeta_2\boldsymbol{t}^i[w_h](s)\times\boldsymbol{d}^i[w_h](s))\,dsd\zeta_1d\zeta_2$$

$$= \int_{\Omega^i_1} \rho^i(s,\varepsilon_h\eta_1,\varepsilon_h\eta_2)\boldsymbol{g}\cdot(\boldsymbol{x}^i[w_h](s)+\varepsilon_h\eta_1\boldsymbol{d}^i[w_h](s)+\varepsilon_h\eta_2\boldsymbol{t}^i[w_h](s)\times\boldsymbol{d}^i w_h](s))\,dsd\eta_1d\eta_2.$$

Passing to the limit as $h\to+\infty$, using the fact that $\boldsymbol{x}^i[w_h]\to\boldsymbol{x}^i[w]$ uniformly on $[0,L^i]$ for any $i=1,\dots,N$ and applying the Dominated Convergence Theorem we conclude. □

Now we pass to the limit in the soap film part of the energy. First of all we need the following Theorem whose proof requires minor modifications of the proof of Theorem 4.3.

Theorem 5.5. *Let (w_h) be a sequence in W' with $w_h \rightharpoonup w$ in W' for some $w\in W'$. Assume that*

(a) $\forall h\in\mathbb{N}, S_h\in\mathcal{P}(\Lambda_{\varepsilon_h}[w_h],\mathcal{C}_{\varepsilon_h,w_h})$;
(b) $\sup_{h\in\mathbb{N}}\mathbf{F}(S_h) = \sup_{h\in\mathbb{N}}\inf\{\mathbf{F}(S) : S\in\mathcal{P}(\Lambda_{\varepsilon_h}[w_h],\mathcal{C}_{\varepsilon_h,w_h})\} < +\infty$.

Let $\mu_h := F\mathcal{H}^2\llcorner S_h$. Then the following three statements hold true:

$$\mu_h \rightharpoonup^* \mu \quad \textit{(up to subsequences)}, \tag{5.6}$$

$$\mu \ge F\mathcal{H}^2\llcorner S_\infty,\ \textit{where } S_\infty = (\mathrm{spt}\,\mu)\setminus H_w \textit{ is 2-rectifiable}, \tag{5.7}$$

$$S_\infty \in \mathcal{P}(H_w,\mathcal{C}_w). \tag{5.8}$$

Now we prove the existence of a recovery sequence.

Proposition 5.6. *Consider $w\in W'$ and $(w_k)\subset W'$ such that $w_k\rightharpoonup w$ in W'. There exists (w_{k_h}) subsequence of (w_k) such that*

$$\inf\{\mathbf{F}(S) : S\in P(H_w,\mathcal{C}_w)\} \ge \limsup_{h\to+\infty} E^{\mathrm{sf}}_{\varepsilon_h}(w_{k_h}). \tag{5.9}$$

Proof. Since $w_k \rightharpoonup w$ in W', $\boldsymbol{x}^i[w_h] \to \boldsymbol{x}^i[w]$ uniformly on $[0, L^i]$ for any $i = 1, \ldots, N$. Then for every $h \in \mathbb{N}$ there exists $k_h \in \mathbb{N}$ such that

$$\|\boldsymbol{x}^i[w_{k_h}] - \boldsymbol{x}^i[w]\|_\infty \leq \frac{\varepsilon_h}{2}, \quad \forall i = 1, \ldots, N. \tag{5.10}$$

Since we can assume without loss of generality that

$$\inf\{\mathbf{F}(S) : S \in \mathcal{P}(H_w, \mathcal{C}_w)\} < +\infty,$$

again applying Theorem 2.7 of [21], we find $S_\infty \in \mathcal{P}(H_w, \mathcal{C}_w)$ such that

$$\mathbf{F}(S_\infty) = \min\{\mathbf{F}(S) : S \in \mathcal{P}(H_w, \mathcal{C}_w)\}.$$

Now we set

$$S_h := S_\infty \setminus \Lambda_{\varepsilon_h}[w_{k_h}].$$

For any $\gamma \in C(\Lambda_{\varepsilon_h}[w_{k_h}])$ not homotopic to a point in $\mathbb{R}^3 \setminus \Lambda_{\varepsilon_h}[w_{k_h}]$ we have

$$(S_\infty \setminus \Lambda_{\varepsilon_h}[w_{k_h}]) \cap \gamma(\mathbb{S}^1) \neq \emptyset.$$

As a consequence,

$$\limsup_{h\to+\infty} E^{\mathrm{sf}}_{\varepsilon_h}(w) \leq \limsup_{h\to+\infty} \mathbf{F}(S_h) \leq \mathbf{F}(S_\infty) = \min\{\mathbf{F}(S) : S \in \mathcal{P}(H_w, \mathcal{C}_w)\},$$

which concludes the proof. □

Proof of Theorem 5.1. The compactness statement is Proposition 5.3. Inequality (5.2) follows combining (5.5) and (5.7) with the subadditivity of the liminf operator. Next, for any $w \in W'$, we consider the constant sequence $w_h \equiv w$. Applying Proposition 5.6, for every $\varepsilon_h \to 0$, the (unique) subsequence $\bar{w}_h \equiv w$ of (w_h) satisfies obviously $\bar{w}_h \rightharpoonup w$ in V and (5.9). Inequality (5.3) follows easily combining (5.5) and (5.9) with the superadditivity of the limsup operator. □

Acknowledgments

A.D.R. has been partially supported by the NSF DMS Grant No. 1906451 and the NSF DMS Grant No. 2112311.

L.L. has been supported by *Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni* of the *Istituto Nazionale di Alta Matematica* (INdAM, Roma, Italy), project *GNAMPA 2019 Analysis and optimization of thin structures*.

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(A. De Rosa) Department of Mathematics, University of Maryland, 4176 Campus Dr, College Park, Maryland 20742, United States

Email address: anderosa@umd.edu

(L. Lussardi) Dipartimento di Scienze Matematiche "G.L. Lagrange", Politecnico di Torino, C.so Duca degli Abruzzi, 10129 Torino, Italy

Email address: luca.lussardi@polito.it