

REGULARITY OF QUASI-MINIMIZERS FOR NON-UNIFORMLY ELLIPTIC INTEGRALS

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ABSTRACT. In this paper we consider a class of non-uniformly elliptic integral functionals \mathcal{F} and we prove the local boundedness of the quasi-minimizers of \mathcal{F} . As regards the integrand function f defining \mathcal{F} , we require that

$$\lambda(x)|\xi|^p \leq f(x, u, \xi) \leq \mu(x)(|\xi|^p + |u|^q) + a(x),$$

where λ, μ, a are measurable functions satisfying suitable integrability assumptions. Our approach is based on a suitable adaptation of the celebrated De Giorgi proof, and it relies on an appropriate Caccioppoli-type inequality.

1. INTRODUCTION

In the present paper we shall be concerned with a class of non-uniformly elliptic functionals \mathcal{F} of the Calculus of Variation taking the following form

$$\mathcal{F} : W_{\text{loc}}^{1,1}(\Omega) \rightarrow [0, \infty], \quad \mathcal{F}(u) := \int_{\Omega} f(x, u, \nabla u) \, dx, \quad (1.1)$$

where $\Omega \subseteq \mathbb{R}^n$ is a bounded open set and f is a real-valued Carathéodory function on $\Omega \times \mathbb{R} \times \mathbb{R}^n$ (precise assumptions on f will be given below). Our main aim is to prove, via a suitable adaptation of the celebrated De Giorgi's method, the (local) boundedness of any quasi-minimizer of \mathcal{F} (see Definition 2.1 in Section 2).

As is well-known, the development of the theory of regularity for single equations and scalar integrals is strictly connected with the pioneering result established by De Giorgi [8] in 1957. Indeed, in his celebrated paper De Giorgi considered a linear elliptic equation (with associated quadratic functional) of the form

$$\sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j}(x) \right) = 0 \quad (x \in \Omega \subseteq \mathbb{R}^n); \quad (1.2)$$

where the functions a_{ij} 's are measurable and bounded in Ω , and the $n \times n$ matrix $A(x) = (a_{ij}(x))_{ij}$ is symmetric (for all $x \in \Omega$). Assuming that, for some $\lambda > 0$,

$$\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geq \lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^n$$

and for a.e. $x \in \Omega$, De Giorgi proved that any weak solution u of (1.2) is locally bounded and, further, locally Hölder continuous. One of the crucial points in the De Giorgi proof is the following Caccioppoli inequality on the super-level sets of u :

$$\int_{A_{\beta,\rho}} |\nabla u|^2 \, dx \leq \frac{c}{(R-\rho)^p} \int_{A_{\beta,R}} (u-\beta)^2 \, dx \quad (\text{valid for all } 0 < \rho < R), \quad (1.3)$$

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where $R > 0$ is such that $B(x_0, R) \Subset \Omega$ and, for any $\beta \in \mathbb{R}$, we have set

$$A_{\beta, R} := \{x \in B(x_0, R) : u(x) \geq \beta\}.$$

It is out of doubts that De Giorgi's regularity result [8], together with the subsequent results by Moser [20] and Stampacchia [21], produced a decisive progress in the theory of regularity. Starting from these seminal works, several improvements (regarding the scalar case) have been achieved: for example, in the series of papers [11, 12, 13] Giaquinta and Giusti proved the Hölder regularity for the minimizers of a class of non-differentiable scalar functionals as in (1.1) (where no Euler-Lagrange equation is available), assuming that the integrand function f satisfies

$$|\nabla u|^p \leq f(x, u, \nabla u) \leq C(1 + |\nabla u|^p) \quad (\text{with } p > 1).$$

We refer the reader to [14] for a comprehensive treatment of this topic.

In [22], Trudinger started the study of the regularity for non-uniformly elliptic equations. More precisely, he considered the equation (1.2) and assumed that the measurable functions a_{ij} satisfy the non-uniform elliptic condition

$$\lambda(x)|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \leq n^2\mu(x)|\xi|^2 \quad (1.4)$$

for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^n$. Here, $\lambda(x)$ is the minimum eigenvalue of the symmetric matrix $A(x) = (a_{ij}(x))$ and $\mu := \sup_{i,j} |a_{ij}|$. The main novelty in Trudinger's result (which also highlights the non-uniformity of (1.4)) relies in the assumption that

$$\lambda^{-1} \in L_{\text{loc}}^r(\Omega) \quad \text{and} \quad \mu_1 = \lambda^{-1}\mu^2 \in L_{\text{loc}}^\sigma(\Omega) \quad \text{with} \quad \frac{1}{r} + \frac{1}{\sigma} < \frac{2}{n}, \quad (1.5)$$

hence the ratio μ/λ is not necessarily bounded in Ω . Under conditions (1.4) and (1.5), Trudinger proved that any weak solution of (1.2) is locally bounded in Ω .

As is well-known, the equation (1.2) is usually called *degenerate* when the a_{ij} 's fulfill (1.4) but $\lambda^{-1} \notin L^\infty(\Omega)$, whereas it is called *singular* when $\mu \notin L^\infty(\Omega)$. It is worth emphasizing that the theory of regularity for (1.2) under assumption (1.4) is significantly complicated by the x -dependence of λ and μ ; in fact the problem of finding optimal additional conditions on λ and μ which guarantee the Hölder continuity of the solutions of (1.2) is still an open problem.

The literature concerning non-uniformly elliptic problems is quite extended, and it is beyond our scopes to present here an exhaustive list of references on this topic; we limit ourselves to highlight the series of papers [1, 5, 6, 7, 9, 10, 15, 17, 18, 19, 23] and the references therein.

We now turn to describe more closely the main result of this paper. As anticipated, we aim to prove the (local) boundedness of any quasi-minimizer for the functional \mathcal{F} in (1.1). In our context, we require the integrand function f to satisfy a generalized version of Trudinger's assumption (1.4), namely we assume that

- (a) f is a Carathéodory function on $\Omega \times \mathbb{R} \times \mathbb{R}^n$ (where $\Omega \subseteq \mathbb{R}^n$ is an open set);
- (b) $f(x, \cdot, \cdot)$ is convex on $\mathbb{R} \times \mathbb{R}^n$ for every fixed $x \in \Omega$;
- (c) f satisfies the growth condition

$$\lambda(x)|\xi|^p \leq f(x, u, \xi) \leq \mu(x) \left(|\xi|^p + |u|^q \right) + a(x), \quad (1.6)$$

where $q \geq p > 1$ and $\lambda, \mu, a : \Omega \rightarrow \mathbb{R}$ are non-negative measurable functions such that $\lambda(x) \leq k\mu(x)$ for a.e. $x \in \Omega$ and some positive constant k .

While we postpone to Section 2 the statement of our result in its full generality, we now give the statement in the particular case when (1.6) holds with $q = p$.

Theorem 1.1. *Assume (1.6) with $q = p$, and*

$$\lambda^{-1} \in L_{\text{loc}}^r(\Omega), \quad \mu \in L_{\text{loc}}^\sigma(\Omega), \quad a \in L_{\text{loc}}^\tau(\Omega), \quad (1.7)$$

for some $r \in [1, \infty]$, $\sigma \in (1, \infty]$ and $\tau \in (1, \infty]$ fulfilling

$$\frac{1}{r} + \max \left\{ \frac{1}{\sigma}, \frac{1}{\tau} \right\} < \frac{p}{n} \quad \text{and} \quad r \geq \frac{1}{p-1} \quad \text{if } p \in (1, 2). \quad (1.8)$$

If $u \in W_{\text{loc}}^{1,1}(\Omega)$ is a quasi-minimizer for the functional \mathcal{F} in (1.1), then u is locally bounded in Ω .

We refer to Theorem 2.2 for the general case $1 < p \leq q$, and for a somehow quantitative estimate of the L^∞ -norm of the quasi-minimizers.

Although being inspired by Trudinger's result, our Theorem 1.1 is significantly different from the one exhibited in [22]: in fact, we consider quasi-minimizers of integral functionals and our proof relies on a suitable version of De Giorgi's approach in [8], the crucial step being a suitable Caccioppoli-type inequality, see Section 3.

We conclude this Introduction by explicitly highlighting that our result applies to several energy integrals which are nowadays receiving great attention in the literature; this is the case, e.g., of the double phase energy integral

$$\mathcal{F}(v) = \int_{\Omega} \left(h_1(x) |\nabla v(x)|^s + h_2(x) |\nabla v(x)|^p \right) dx \quad 1 < s < p, \quad (1.9)$$

with non-negative functions h_1 and h_2 . A deep study of such functionals was recently carried out under many different aspects, see e.g. [2, 3, 4].

It is straightforward to check that the integrand function

$$f(x, \xi) := h_1(x) |\xi|^s + h_2(x) |\xi|^p$$

fulfills the growth condition (1.6) with

$$\lambda(x) := h_2(x), \quad \mu(x) := (h_1(x) + h_2(x)), \quad a(x) := h_1(x).$$

As a consequence, if we assume that

$$h_2^{-1} \in L_{\text{loc}}^r(\Omega), \quad (h_1 + h_2) \in L_{\text{loc}}^\sigma(\Omega) \quad \text{and} \quad h_1 \in L_{\text{loc}}^\tau(\Omega)$$

for some $r \in [1, \infty)$, $\sigma \in (1, \infty)$, $\tau \in (1, \infty)$ satisfying the relations (1.8), we infer from Theorem 1.1 that any quasi-minimizer of \mathcal{F} is locally bounded.

Another application of Theorem 1.1 is the following (see also the more general Example 2.3 in Section 2): consider the functional

$$\mathcal{F}(v) = \int_{\Omega} h(x) (|\nabla v(x)|^p + |u(x)|^p) dx, \quad (1.10)$$

where $p > 1$ and h is non-negative on Ω . If we assume that

$$h^{-1} \in L_{\text{loc}}^r(\Omega) \quad \text{and} \quad h \in L_{\text{loc}}^\sigma(\Omega)$$

for some $r \in [1, \infty)$, $\sigma \in (1, \infty)$ satisfying

$$\frac{1}{r} + \frac{1}{\sigma} < \frac{p}{n} \quad \text{and} \quad r \geq \max \left\{ 2, \frac{1}{p-1} \right\}$$

then the quasi-minimizer of \mathcal{F} are locally bounded.

Plan of the paper. A short plan of the paper is now in order.

- In Section 2 we shall give the needed preliminary definitions, the statement of our result in its full generality, Theorem 2.2, together with some useful remarks concerning our main assumptions.
- In Section 3 we shall prove a Caccioppoli-type inequality for the quasi-minimizers of \mathcal{F} , which is the fundamental tool for the proof of Theorem 2.2.

- In Section 4 we shall set up the De Giorgi iterative method, by means of which we shall be able to provide the proof of Theorem 2.2.

2. PRELIMINARY DEFINITIONS AND STATEMENT OF THE MAIN RESULT

As anticipated in the Introduction, throughout this paper we shall be concerned with integral functionals of the following form

$$\mathcal{F} : W_{\text{loc}}^{1,1}(\Omega) \longrightarrow [0, \infty], \quad \mathcal{F}(u) := \int_{\Omega} f(x, u, \nabla u) dx, \quad (2.1)$$

where $\Omega \subseteq \mathbb{R}^n$ is a fixed open set and $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a Carathéodory function satisfying the following assumptions (already presented in the Introduction):

$$f(x, \cdot, \cdot) \text{ is convex on } \mathbb{R} \times \mathbb{R}^n \text{ for every fixed } x \in \Omega; \quad (2.2)$$

and, for every $x \in \Omega$ and every $(u, \xi) \in \mathbb{R} \times \mathbb{R}^n$, one has

$$\lambda(x) |\xi|^p \leq f(t, u, \xi) \leq \mu(x) \left(|\xi|^p + |u|^q \right) + a(x), \quad (2.3)$$

where $q \geq p > 1$ and $a, \lambda, \mu : \Omega \rightarrow (0, \infty)$ are measurable functions such that

$$\lambda(x) \leq k \mu(x) \quad \text{a.e. in } \Omega, \quad (2.4)$$

for a suitable constant $k > 0$. Furthermore, we require that

$$\lambda^{-1} \in L_{\text{loc}}^r(\Omega), \quad \mu \in L_{\text{loc}}^\sigma(\Omega), \quad a \in L_{\text{loc}}^\tau(\Omega), \quad (2.5)$$

for some $r \in [1, \infty]$, $\sigma \in (1, \infty]$ and $\tau \in (1, \infty]$ fulfilling the relations

$$\frac{1}{r} + \max \left\{ \frac{1}{\sigma}, \frac{1}{\tau} \right\} < \frac{p}{n}, \quad \text{and} \quad r \geq \frac{1}{p-1} \quad \text{if } p \in (1, 2). \quad (2.6)$$

Since we aim at proving the local boundedness of the quasi-minimizers of the functional \mathcal{F} , we remind the precise definition of this concept.

Definition 2.1. We say that a function $u \in W_{\text{loc}}^{1,1}(\Omega)$ is a *quasi-minimizer* of \mathcal{F} in (2.1) if $x \mapsto f(x, u, \nabla u) \in L_{\text{loc}}^1(\Omega)$ and there exists a real $Q \geq 1$ such that

$$\int_{\text{supp } \varphi} f(x, u, \nabla u) dx \leq Q \int_{\text{supp } \varphi} f(x, u + \varphi, \nabla(u + \varphi)) dx, \quad (2.7)$$

for every $\varphi \in W^{1,1}(\Omega)$ with $\text{supp } \varphi \Subset \Omega$. In the particular case when (2.7) holds with $Q = 1$, we say that u is a *local minimizer* of \mathcal{F} .

With simple computations (see e.g. Proposition 3.1 in [7]), for every open set Ω' compactly contained in Ω we deduce that

$$\|\lambda^{-1}\|_{L^r(\Omega')}^{-1} \|\nabla v\|_{L^{\frac{pr}{r+1}}(\Omega'; \mathbb{R}^n)}^p \leq \int_{\Omega'} f(x, v, \nabla v) dx. \quad (2.8)$$

From (2.8) and the classical Sobolev embedding theorem, we then get

$$W^{1, \frac{pr}{r+1}}(\Omega') \supset W^{1, \mathcal{F}}(\Omega') \quad (2.9)$$

where $W^{1, \mathcal{F}}(\Omega')$ denotes the set of maps u with finite \mathcal{F} -energy on Ω' , that is,

$$W^{1, \mathcal{F}}(\Omega') := \{u \in W^{1,1}(\Omega') : \mathcal{F}(u; \Omega') < \infty\}.$$

It is clear that if we impose appropriate conditions at the boundary of a fixed open set $\Omega' \Subset \Omega$, then from standard direct methods of the Calculus of Variations we derive the existence of minimizers in the Sobolev space

$$W_{\text{loc}}^{1, \frac{pr}{r+1}}(\Omega'),$$

provided that $pr > r + 1$. Furthermore, if $u \in W_{\text{loc}}^{1,1}(\Omega)$ is a quasi-minimizer of \mathcal{F} , it follows from (2.9) that

$$u \in W_{\text{loc}}^{1, \frac{pr}{r+1}}(\Omega).$$

For the sake of simplicity, we will explicitly assume this in our statements and we also introduce the following notation:

$$\nu := \frac{pr}{r+1}, \quad (2.10)$$

with the convention $\nu := p$ if $r = +\infty$. It should be noticed that, from our assumptions on r (that is, $r \geq 1$ and $r \geq 1/(p-1)$ if $1 < p < 2$), we easily infer that

$$\nu = \frac{pr}{r+1} \geq 1. \quad (2.11)$$

Throughout the sequel, given any $\vartheta \in [1, \infty]$, we shall adopt the following notation: we indicate by ϑ' the conjugate exponent of ϑ , that is,

$$\vartheta' := \begin{cases} \frac{\vartheta}{\vartheta-1}, & \text{if } \vartheta \in (1, \infty); \\ \infty, & \text{if } \vartheta = 1; \\ 1, & \text{if } \vartheta = \infty. \end{cases}$$

Moreover, we denote by ϑ^* the Sobolev exponent of ϑ , that is,

$$\vartheta^* := \begin{cases} \frac{n\vartheta}{n-\vartheta}, & \text{if } \vartheta < n, \\ \text{any number } t > \vartheta, & \text{if } \vartheta \geq n. \end{cases}$$

Our main result is contained in the following theorem.

Theorem 2.2. *Let us consider the functional \mathcal{F} in (2.1), and let us suppose that its integrand function f satisfies (2.2), (2.3), (2.4), (2.5) and (2.6), with*

$$q\sigma' < \nu^*. \quad (2.12)$$

Let $u \in W_{\text{loc}}^{1, \frac{\nu^*}{r+1}}(\Omega)$ be a quasi-minimizer of \mathcal{F} .

Then u is locally bounded in Ω . Furthermore, it is possible to find a constant $c > 0$, only depending on n, p, q, r, σ, τ and on k and Q , with the following property: for every fixed $x_0 \in \Omega$ and every $R_0 \in (0, 1]$ satisfying $B(x_0, R_0) \Subset \Omega$, one has

$$\|u\|_{L^\infty(B_{R_0/2}(x_0))} \leq c \left(\frac{H}{R_0} \right)^{\frac{\nu^*}{\nu^* - q\sigma'}} (1 + \|u\|_{L^{\nu^*}(B_{R_0})})^{\frac{\nu^*(q-p+p\alpha)}{p(\nu^* - q\sigma')}}. \quad (2.13)$$

where

$$H := (1 + \|\lambda^{-1}\|_{L^\tau(B(x_0, R_0))}^{1/p}) (1 + \|a\|_{L^\tau(B(x_0, R_0))}^{1/p}) (1 + \|\mu\|_{L^\sigma(B(x_0, R_0))}^{1/p}) \quad (2.14)$$

and

$$\alpha := \min \left\{ 1 - \frac{q\sigma'}{\nu^*}, \sigma' \left(1 - \frac{1}{\tau} - \frac{p}{\nu^*} \right) \right\}. \quad (2.15)$$

We observe that, in the particular case when $q = p$, Theorem 2.2 reduces to Theorem 1.1 stated in the Introduction.

Example 2.3. Let us consider the functional

$$\mathcal{F}(v) = \int_{\Omega} h(x) (|\nabla v(x)|^p + |u(x)|^q) dx, \quad (2.16)$$

where $1 < p \leq q$ are real and h is non-negative on Ω . If we assume that

(i) $h^{-1} \in L_{\text{loc}}^r(\Omega)$ and $h \in L_{\text{loc}}^\sigma(\Omega)$, with

$$\frac{1}{r} + \frac{1}{\sigma} < \frac{p}{n} \quad \text{and} \quad r \geq \max \left\{ 2, \frac{1}{p-1} \right\};$$

(ii) $q < \nu^*/\sigma'$ (remind that $\nu = pr/(r+1)$ and ν^* is Sobolev exponent of ν),

we infer from Theorem 2.2 that the quasi-minimizer of \mathcal{F} are locally bounded.

Remark 2.4. Here, we list a few consequences of (2.6) and (2.12) which are of general interest.

- (1) The first inequality in (2.6) implies

$$p\sigma' < \nu^* \quad \text{and} \quad 1 - \frac{1}{\tau} - \frac{p}{\nu^*} > 0. \quad (2.17)$$

- (2) The assumption on q , see (2.12), together with the last inequality in (2.17), implies $\alpha > 0$.
(3) If the function a in the growth condition (2.3) is a locally bounded function, i.e., $\tau = +\infty$, we obtain

$$\frac{1}{r} + \frac{1}{\sigma} < \frac{p}{n}. \quad (2.18)$$

It should be noticed that, in the particular case $p = 2$, the above (2.18) is precisely the assumption made by Trudinger in [22] (see indeed (1.4)).

- (4) From the first inequality in (2.6) we get $\tau > n/p$. This bound from below for τ coincides with the condition in order to gain regularity for the minimizer in the standard case when λ, μ are constants.
(5) We notice that if $r = \sigma = +\infty$, i.e., $\lambda^{-1}, \mu \in L^\infty$, then (2.12) reduces to the well-known assumption $q < p^*$ (see for example Giusti [14]).
(6) If $r = \sigma = \tau = +\infty$ and $p = q$, then

$$\frac{\nu^*(q - p + p\alpha)}{p(\nu^* - q\sigma')} = 1$$

and the estimate (2.13) becomes

$$\|u\|_{L^\infty(B_{R_0/2}(x_0))} \leq c \left(\frac{H}{R_0} \right)^{\frac{p^*}{p^*-p}} (1 + \|u\|_{L^{p^*}(B_{R_0})}).$$

3. CACCIOPPOLI-TYPE INEQUALITY FOR \mathcal{F}

Throughout the sequel, we shall make use of the following notation: given any function u in $W_{\text{loc}}^{1,1}(\Omega)$, if $x_0 \in \Omega$ and $R > 0$ are such that $B(x_0, R) \Subset \Omega$ we set

$$A_{\beta,R} = A_{\beta,R}(u) := \{x \in B(x_0, R) : u(x) > \beta\}, \quad \beta \in \mathbb{R}. \quad (3.1)$$

A fundamental tool for proving the local boundedness of the quasi-minimizers of \mathcal{F} is the Caccioppoli-type inequality contained in the next Proposition 3.1. In the proof of this proposition we shall exploit a very classical lemma, see Lemma 3.2 below.

Proposition 3.1. *Let $u \in W_{\text{loc}}^{1, \frac{pr}{r+1}}(\Omega)$ be a quasi-minimizer of the functional \mathcal{F} in (2.1); moreover, let $x_0 \in \Omega$ and $R \in (0, 1]$ be such that $B(x_0, R) \Subset \Omega$. Then, for every $\rho \in (0, R)$ and every $\beta \in \mathbb{R}$ with $\beta \geq 1$ we have*

$$\int_{A_{\beta,\rho}} f(x, u, \nabla u) \, dx \leq \Lambda \left\{ \frac{1}{(R-\rho)^p} \int_{A_{\beta,R}} \mu(x) \left((u-\beta)^p + \beta^q \right) \, dx + \|a\|_{L^\tau(B(x_0,R))} \cdot |A_{\beta,R}|^{1-\frac{1}{\tau}} \right\}, \quad (3.2)$$

where $\Lambda > 0$ is a constant only depending on p and Q .

Proof. Let ρ, β be as in the statement of the proposition, and let $s, t \in (0, R)$ be such that $\rho \leq s < t$. Moreover, let $\eta \in C_c^\infty(\mathbb{R}^n, \mathbb{R})$ be a cut-off function satisfying the following properties:

- (i) $0 \leq \eta \leq 1$ on \mathbb{R}^n ;
- (ii) $\text{supp } \eta \Subset B(x_0, t)$ and $\eta \equiv 1$ on $B(x_0, s)$;
- (iii) $|\nabla \eta(x)| \leq 2/(t-s)$ for every $x \in \mathbb{R}^n$.

Setting $w := (u - \beta)_+ = \max\{u - \beta, 0\}$, we finally define

$$\varphi : \Omega \rightarrow \mathbb{R}, \quad \varphi(x) := -\eta^p(x) w(x).$$

It is easy to check that

$$\varphi \in W^{1,1}(\Omega) \text{ and } A_{\beta,s} \subseteq \text{supp } \varphi \subseteq A_{\beta,t} \Subset \Omega; \quad (3.3)$$

furthermore, we have

$$\nabla w = \nabla u \text{ on } \text{supp } \varphi. \quad (3.4)$$

Since u is a quasi-minimizer of \mathcal{F} with constant $Q \geq 1$, and since $f \geq 0$, one has

$$\begin{aligned} \int_{A_{\beta,s}} f(x, u, \nabla u) \, dx &\leq \int_{\text{supp } \varphi} f(x, u, \nabla u) \, dx \\ &\leq Q \int_{\text{supp } \varphi} f(x, u + \varphi, \nabla(u + \varphi)) \, dx \\ &\stackrel{(3.4)}{=} Q \int_{\text{supp } \varphi} f\left(x, (1 - \eta^p)u + \eta^p \beta, (1 - \eta^p)\nabla u + \eta^p(pw\eta^{-1}\nabla\eta)\right) \, dx \\ &\stackrel{(2.2)}{\leq} Q \int_{\text{supp } \varphi} \left\{ (1 - \eta^p) f(x, u, \nabla u) + \eta^p f(x, \beta, pw\eta^{-1}\nabla\eta) \right\} \, dx. \end{aligned}$$

We now introduce the notation

$$\begin{aligned} I_1 &:= \int_{\text{supp } \varphi} (1 - \eta^p) f(x, u, \nabla u) \, dx, \\ I_2 &:= \int_{\text{supp } \varphi} \eta^p f(x, \beta, pw\eta^{-1}\nabla\eta) \, dx, \end{aligned}$$

and we proceed by giving an estimate of them.

As regards I_1 , since $\eta \equiv 1$ on $B(x_0, s)$ and since $0 \leq \eta \leq 1$ on the whole of \mathbb{R}^n , we immediately obtain

$$\begin{aligned} I_1 &\stackrel{(3.3)}{\leq} \int_{A_{\beta,t}} (1 - \eta^p) f(x, u, \nabla u) \, dx \\ &= \int_{A_{\beta,t} \setminus A_{\beta,s}} (1 - \eta^p) f(x, u, \nabla u) \, dx \\ &\leq \int_{A_{\beta,t} \setminus A_{\beta,s}} f(x, u, \nabla u) \, dx. \end{aligned} \quad (3.5)$$

We then turn our attention to I_2 . On account of (2.3), one has

$$\begin{aligned} I_2 &\leq \int_{\text{supp } \varphi} \eta^p \left\{ \mu(x) \left((pw\eta^{-1})^p |\nabla\eta|^p + \beta^q \right) + a(x) \right\} \, dx \\ &\leq \int_{\text{supp } \varphi} \mu(x) \left((pw)^p |\nabla\eta|^p + \beta^q \right) \, dx + \int_{\text{supp } \varphi} a(x) \, dx =: (\star); \end{aligned}$$

moreover, by using the estimate for $|\nabla\eta|$ in (iii) and since $0 < t - s \leq R \leq 1$ and $p > 1$, we get

$$\begin{aligned} (\star) &\leq \int_{\text{supp } \varphi} \mu(x) \left(\frac{(2p)^p}{(t-s)^p} w^p + \beta^q \right) \, dx + \int_{\text{supp } \varphi} a(x) \, dx \\ &\leq \frac{(2p)^p}{(t-s)^p} \int_{\text{supp } \varphi} \mu(x) \left(w^p + \beta^q \right) \, dx + \int_{\text{supp } \varphi} a(x) \, dx =: (2\star). \end{aligned}$$

By applying Hölder's inequality, using (3.3), (2.5) and noting that $w = (u - \beta)_+$, we finally get

$$\begin{aligned}
(2\star) &\leq \frac{(2p)^p}{(t-s)^p} \int_{A_{\beta,t}} \mu(x) (w^p + \beta^q) dx + \int_{A_{\beta,t}} a(x) dx \\
&\leq \frac{(2p)^p}{(t-s)^p} \int_{A_{\beta,t}} \mu(x) (w^p + \beta^q) dx + \|a\|_{L^\tau(B(x_0,R))} |A_{\beta,R}|^{1-\frac{1}{\tau}} \\
&= \frac{(2p)^p}{(t-s)^p} \int_{A_{\beta,R}} \mu(x) \left((u-\beta)^p + \beta^q \right) dx + \|a\|_{L^\tau(B(x_0,R))} |A_{\beta,R}|^{1-\frac{1}{\tau}}. \quad (3.6)
\end{aligned}$$

Gathering together (3.5) and (3.6), we then obtain

$$\begin{aligned}
\int_{A_{\beta,s}} f(x, u, \nabla u) dx &\leq Q \int_{A_{\beta,t} \setminus A_{\beta,s}} f(x, u, \nabla u) dx \\
&\quad + Q \left(\frac{(2p)^p}{(t-s)^p} \int_{A_{\beta,R}} \mu(x) (|u-\beta|^p + \beta^q) dx \right. \\
&\quad \left. + \|a\|_{L^\tau(B(x_0,R))} |A_{\beta,R}|^{1-\frac{1}{\tau}} \right).
\end{aligned}$$

By the hole-filling method, this last estimate clearly implies that

$$\begin{aligned}
\int_{A_{\beta,s}} f(x, u, \nabla u) dx &\leq \frac{Q}{Q+1} \int_{A_{\beta,t}} f(x, u, \nabla u) dx \\
&\quad + \left(\frac{(2p)^p}{(t-s)^p} \int_{A_{\beta,R}} \mu(x) \left((u-\beta)^p + \beta^q \right) dx \right. \\
&\quad \left. + \|a\|_{L^\tau(B(x_0,R))} |A_{\beta,R}|^{1-\frac{1}{\tau}} \right). \quad (3.7)
\end{aligned}$$

We are now entitled to apply Lemma 3.2 below, with

$$\omega : [\rho, R] \rightarrow \mathbb{R}, \quad \omega(t) := \int_{A_{\beta,t}} f(x, u, \nabla u) dx,$$

and the constants ϑ, κ, A and B given, respectively, by

$$\vartheta = \frac{Q}{Q+1}, \quad \kappa = p, \quad A := (2p)^p \int_{A_{\beta,R}} \mu(x) \left((u-\beta)^p + \beta^q \right) dx,$$

$$\text{and } B := \|a\|_{L^\tau(B(x_0,R))} |A_{\beta,R}|^{1-\frac{1}{\tau}}.$$

Thus, we find a constant $\Lambda > 0$, only depending on $\kappa = p$ and on Q , such that

$$\begin{aligned}
\omega(\rho) &= \int_{A_{\beta,\rho}} f(x, u, \nabla u) dx \\
&\leq \Lambda \left\{ \frac{1}{(R-\rho)^p} \int_{A_{\beta,R}} \mu(x) \left((u-\beta)^p + \beta^q \right) dx \right. \\
&\quad \left. + \|a\|_{L^\tau(B(x_0,R))} |A_{\beta,R}|^{1-\frac{1}{\tau}} \right\}.
\end{aligned}$$

The proof is complete. \square

We now state the classical lemma used in the proof above. We refer, e.g., to [14, Lemma 6.1] for a demonstration of this result.

Lemma 3.2. *Let $[a, b] \subseteq \mathbb{R}$ and let $\omega : [a, b] \rightarrow \mathbb{R}$ be non-negative and bounded. We assume that there exist constants $\kappa, A, B > 0$ and $\vartheta \in (0, 1)$ such that*

$$\omega(s) \leq \vartheta \omega(t) + \frac{A}{(t-s)^\kappa} + B, \quad \text{for every } a \leq s \leq t \leq b.$$

It is then possible to find a constant $C = C(\vartheta, \kappa) > 0$ such that

$$\omega(\rho) \leq C \left\{ \frac{A}{(R-\rho)^\kappa} + B \right\}, \quad \text{for every } a \leq \rho \leq R \leq b.$$

4. PROOF OF THEOREM 2.2

Now we proceed towards the proof of our Theorem 2.2 by setting up the celebrated De Giorgi iterative method. In what follows, we tacitly understand that all the assumptions and the notation introduced in the previous sections do apply.

Let $u \in W_{\text{loc}}^{1, \frac{pr}{r+1}}(\Omega)$ be a quasi-minimizer of \mathcal{F} in (2.1). Taking into account Remark 2.4, we fix a point $x_0 \in \Omega$ and a real $R_0 \in (0, 1]$ in such a way that

$$B(x_0, R_0) \Subset \Omega \quad \text{and} \quad \int_{B(x_0, R_0)} |u|^{p\sigma'} dx \leq 1. \quad (4.1)$$

For every fixed $R \in (0, R_0]$, we then define the following (decreasing) sequences:

$$\rho_h := \frac{R}{2} \left(1 + \frac{1}{2^h} \right) \quad \text{and} \quad \bar{\rho}_h := \frac{\rho_{h+1} + \rho_h}{2}, \quad h \in \mathbb{N} \cup \{0\}. \quad (4.2)$$

Moreover, given any real number $\beta \geq 1$, we consider the (increasing) sequence

$$\beta_h := \beta \left(1 - \frac{1}{2^{h+1}} \right), \quad h \in \mathbb{N} \cup \{0\}. \quad (4.3)$$

Finally, we define a sequence $(J_h)_{h \geq 0}$ of non-negative numbers as follows:

$$J_h := \int_{A_{\beta_h, \rho_h}} (u - \beta_h)^{p\sigma'} dx. \quad (4.4)$$

We explicitly observe that, since $(\rho_h)_h$ is decreasing and $(\beta_h)_h$ is increasing, the sequence $(J_h)_h$ is decreasing: in fact, we have

$$\begin{aligned} J_{h+1} &= \int_{A_{\beta_{h+1}, \rho_{h+1}}} (u - \beta_{h+1})^{p\sigma'} dx \leq \int_{A_{\beta_{h+1}, \rho_h}} (u - \beta_{h+1})^{p\sigma'} dx \\ &\leq \int_{A_{\beta_{h+1}, \rho_h}} (u - \beta_h)^{p\sigma'} dx \leq \int_{A_{\beta_h, \rho_h}} (u - \beta_h)^{p\sigma'} dx = J_h. \end{aligned} \quad (4.5)$$

Moreover, on account of (4.1) (and since $\rho_h \leq R \leq R_0$), we have

$$\begin{aligned} J_h &= \int_{A_{\beta_h, \rho_h}} (u - \beta_h)^{p\sigma'} dx \leq \int_{A_{\beta_h, \rho_h}} |u|^{p\sigma'} dx \\ &\leq \int_{B(x_0, R_0)} |u|^{p\sigma'} dx \leq 1. \end{aligned} \quad (4.6)$$

Finally, by taking into account the definitions of J_h , β_h and ρ_h , we have

$$\begin{aligned} J_h &= \int_{A_{\beta_h, \rho_h}} (u - \beta_h)^{p\sigma'} dx \geq \int_{A_{\beta_{h+1}, \rho_h}} (u - \beta_h)^{p\sigma'} dx \\ &\geq (\beta_{h+1} - \beta_h)^{p\sigma'} |A_{\beta_{h+1}, \rho_h}| = \left(\frac{\beta}{2^{h+2}} \right)^{p\sigma'} |A_{\beta_{h+1}, \rho_h}|, \end{aligned} \quad (4.7)$$

The following proposition is the key tool for the proof of Theorem 2.2.

Proposition 4.1. *Let the above assumptions and notation do apply. Then, there exists a constant $C > 0$ such that, for every $R \in (0, R_0]$ and every $\beta \geq 1$, one has*

$$J_{h+1} \leq C \left(1 + \|u\|_{L^{\nu^*}(B(x_0, R))}^{(q-p)\sigma'}\right) \cdot \frac{8^{p\sigma'^2 h}}{R^{p\sigma'} \beta^{p\sigma'(1-\frac{q\sigma'}{\nu^*})}} J_h^{1+\alpha} \quad \forall h \in \mathbb{N} \cup \{0\}, \quad (4.8)$$

where ν is as in (2.10) and α is as in (2.15). Furthermore, the dependence of C with respect to λ, μ, a can be explicated as follows:

$$C = c \left\{ \|\lambda^{-1}\|_{L^r(B(x_0, R_0))} \cdot (1 + \|a\|_{L^\tau(B(x_0, R_0))}) \cdot (1 + \|\mu\|_{L^\sigma(B(x_0, R_0))}) \right\}^{\sigma'}, \quad (4.9)$$

where $c > 0$ is constant depending on n, p, q, r, σ, τ , on k in (2.4) and on Q .

Before proving Proposition 4.1 we need a couple of preliminary lemmas. The first one, whose proof can be found, e.g., in [7], can be stated as follows.

Lemma 4.2. *Let $u \in W_{\text{loc}}^{1, \frac{pr}{r+1}}(\Omega)$. Then, for every open set $\Omega' \Subset \Omega$ we have*

$$\int_{\Omega'} |\nabla u|^{\frac{pr}{r+1}} dx \leq \|\lambda^{-1}\|_{L^r(\Omega')}^{\frac{r}{r+1}} \left(\int_{\Omega'} f(x, u, \nabla u) dx \right)^{\frac{r}{r+1}}. \quad (4.10)$$

The second preliminary lemma is the following (see [7], Lemma 3.2).

Lemma 4.3. *Let $p > 1$ be fixed and let $r \geq \max\{1, \frac{1}{p-1}\}$. As usual, we set*

$$\nu := \frac{pr}{r+1}.$$

Then, there exists a constant $c_S > 0$ such that

$$\left(\int_B |v|^{\nu^*} dx \right)^{p/\nu^*} \leq c_S \|\lambda^{-1}\|_{L^r(B)} \int_B \lambda(x) |\nabla v|^p dx, \quad (4.11)$$

for every ball $B \Subset \Omega$ and every function $v \in W_0^{1, \frac{pr}{r+1}}(B)$.

Now, we can provide the proof of Proposition 4.1.

Proof of Proposition 4.1. Let $(\eta_h)_{h \geq 0}$ be a sequence in $C_c^\infty(\mathbb{R}^n)$ such that

- (i) $0 \leq \eta_h \leq 1$ on \mathbb{R}^n ;
- (ii) $\text{supp } \eta_h \subseteq B(x_0, \bar{\rho}_h)$ and $\eta_h \equiv 1$ on $B(x_0, \rho_{h+1})$;
- (iii) $|\nabla \eta_h| \leq \frac{2^{h+4}}{R}$.

We remind here that

$$\nu := \frac{pr}{r+1}.$$

Since $p\sigma' < \nu^*$ (see Remark 2.4-(1)), and reminding that

$$\eta_h \equiv 1 \text{ on } A_{\beta_{h+1}, \rho_{h+1}} \quad \text{and} \quad \text{supp } \eta_h \subseteq B(x_0, \bar{\rho}_h)$$

we have

$$\begin{aligned} J_{h+1} &= \int_{A_{\beta_{h+1}, \rho_{h+1}}} (u - \beta_{h+1})^{p\sigma'} dx \\ &\leq |A_{\beta_{h+1}, \rho_{h+1}}|^{1-\frac{p\sigma'}{\nu^*}} \left(\int_{A_{\beta_{h+1}, \rho_{h+1}}} (u - \beta_{h+1})^{\nu^*} dx \right)^{\frac{p\sigma'}{\nu^*}} \\ &= |A_{\beta_{h+1}, \rho_{h+1}}|^{1-\frac{p\sigma'}{\nu^*}} \left(\int_{A_{\beta_{h+1}, \rho_{h+1}}} \eta_h (u - \beta_{h+1})^{\nu^*} dx \right)^{\frac{p\sigma'}{\nu^*}} \end{aligned}$$

$$\leq |A_{\beta_{h+1}, \rho_{h+1}}|^{1-\frac{p\sigma'}{\nu^*}} \left(\int_{B(x_0, \bar{\rho}_h)} \eta_h (u - \beta_{h+1})_+^{\nu^*} dx \right)^{\frac{p\sigma'}{\nu^*}} =: (\star).$$

To proceed further we observe that, by the fact that $\eta_h \in C_c^\infty(B(x_0, \bar{\rho}_h))$ (see (ii)), we certainly have $v := \eta_h (u - \beta_{h+1})_+ \in W_0^{1, \nu}(B(x_0, \bar{\rho}_h))$. We are then entitled to apply Lemma 4.3, with $\Omega' := B(x_0, \bar{\rho}_h) \Subset B(x_0, R)$. If we denote

$$K := \|\lambda^{-1}\|_{L^\tau(B(x_0, R_0))}^{\sigma'},$$

we have (here c_S is the embedding constant in Lemma 4.3)

$$\begin{aligned} (\star) &\leq c_S^{\sigma'} K |A_{\beta_{h+1}, \rho_{h+1}}|^{1-\frac{p\sigma'}{\nu^*}} \left(\int_{B(x_0, \bar{\rho}_h)} \lambda(x) |\nabla(\eta_h (u - \beta_{h+1})_+)|^p dx \right)^{\sigma'} \\ &\leq c_S^{\sigma'} K |A_{\beta_{h+1}, \rho_{h+1}}|^{1-\frac{p\sigma'}{\nu^*}} \times \\ &\quad \times \left(\int_{B(x_0, \bar{\rho}_h)} \lambda(x) \left\{ |\nabla \eta_h| (u - \beta_{h+1})_+ + \eta_h |\nabla u| \chi_{A_{\beta_{h+1}, \bar{\rho}_h}} \right\}^p dx \right)^{\sigma'} =: (2\star). \end{aligned}$$

By the estimate for $|\nabla \eta_h|$ in (iii), using $\eta_h \leq 1$ and inequality (2.3) we get (here and throughout, we indicate by c_1, c_2, \dots different positive constants only depending on the Sobolev constant c_S , on n, p, q, r, σ, τ , on k in (2.4) and on Q)

$$\begin{aligned} &\int_{B(x_0, \bar{\rho}_h)} \lambda(x) \left\{ |\nabla \eta_h| (u - \beta_{h+1})_+ + \eta_h |\nabla u| \chi_{A_{\beta_{h+1}, \bar{\rho}_h}} \right\}^p dx \\ &\leq c_1 \left\{ \left(\frac{2^{h+4}}{R} \right)^p \int_{A_{\beta_{h+1}, \bar{\rho}_h}} \lambda(x) (u - \beta_{h+1})^p dx + \int_{A_{\beta_{h+1}, \bar{\rho}_h}} \lambda(x) |\nabla u|^p dx \right\} \\ &\leq c_1 \left\{ \left(\frac{2^{h+4}}{R} \right)^p \int_{A_{\beta_{h+1}, \bar{\rho}_h}} \lambda(x) (u - \beta_{h+1})^p dx + \int_{A_{\beta_{h+1}, \bar{\rho}_h}} f(x, u, \nabla u) dx \right\}. \end{aligned}$$

Collecting the above inequalities and by exploiting the Caccioppoli-type inequality (3.2) established in Section 3, we then obtain

$$\begin{aligned} (2\star) &\stackrel{(2.3)}{\leq} c_1 K |A_{\beta_{h+1}, \rho_{h+1}}|^{1-\frac{p\sigma'}{\nu^*}} \left[\left(\frac{2^{h+4}}{R} \right)^p \int_{A_{\beta_{h+1}, \bar{\rho}_h}} \lambda(x) (u - \beta_{h+1})^p dx \right. \\ &\quad \left. + \int_{A_{\beta_{h+1}, \bar{\rho}_h}} f(x, u, \nabla u) dx \right]^{\sigma'} \end{aligned}$$

(by (3.2), with $\rho = \bar{\rho}_h$ and $R = \rho_h$)

$$\begin{aligned} &\leq c_1 K |A_{\beta_{h+1}, \rho_{h+1}}|^{1-\frac{p\sigma'}{\nu^*}} \left\{ \left(\frac{2^{h+4}}{R} \right)^p \int_{A_{\beta_{h+1}, \bar{\rho}_h}} \lambda(x) (u - \beta_{h+1})^p dx \right. \\ &\quad \left. + \Lambda \left[\left(\frac{2^{h+3}}{R} \right)^p \int_{A_{\beta_{h+1}, \rho_h}} \mu(x) \left((u - \beta_{h+1})^p + \beta_{h+1}^q \right) dx \right. \right. \\ &\quad \left. \left. + \|a\|_{L^\tau(B(x_0, R))} |A_{\beta_{h+1}, \rho_h}|^{1-\frac{1}{\tau}} \right] \right\}^{\sigma'} =: (3\star). \end{aligned}$$

Let us now recall that (2.4) holds and $R \leq 1$. Therefore, setting

$$K_1 := K \cdot (1 + \|a\|_{L^\tau(B(x_0, R_0))})^{\sigma'} \quad \text{and} \quad K_2 := K_1 \cdot (1 + \|\mu\|_{L^\sigma(B(x_0, R_0))})^{\sigma'},$$

by Hölder's inequality we obtain

$$\begin{aligned}
(3\star) &\leq c_2 K_1 |A_{\beta_{h+1}, \rho_{h+1}}|^{1-\frac{p\sigma'}{\nu^*}} \left(\frac{2^h}{R}\right)^{p\sigma'} \times \\
&\quad \times \left[\int_{A_{\beta_{h+1}, \rho_h}} \mu(x) \left((u - \beta_{h+1})^p + \beta_{h+1}^q \right) dx + |A_{\beta_{h+1}, \rho_h}|^{1-\frac{1}{\tau}} \right]^{\sigma'} \\
&\leq c_2 K_1 |A_{\beta_{h+1}, \rho_{h+1}}|^{1-\frac{p\sigma'}{\nu^*}} \left(\frac{2^h}{R}\right)^{p\sigma'} \times \\
&\quad \times \left[\|\mu\|_{L^\sigma(B(x_0, R))} \left(\int_{A_{\beta_{h+1}, \rho_h}} \left((u - \beta_{h+1})^p + \beta_{h+1}^q \right)^{\sigma'} dx \right)^{\frac{1}{\sigma'}} \right. \\
&\quad \left. + |A_{\beta_{h+1}, \rho_h}|^{1-\frac{1}{\tau}} \right]^{\sigma'} \\
&\leq c_3 K_2 |A_{\beta_{h+1}, \rho_{h+1}}|^{1-\frac{p\sigma'}{\nu^*}} \left(\frac{2^h}{R}\right)^{p\sigma'} \times \\
&\quad \left\{ \int_{A_{\beta_{h+1}, \rho_h}} \left((u - \beta_{h+1})^p + \beta_{h+1}^q \right)^{\sigma'} dx + |A_{\beta_{h+1}, \rho_h}|^{\sigma'(1-\frac{1}{\tau})} \right\} =: (4\star)
\end{aligned}$$

It holds that

$$\int_{A_{\beta_{h+1}, \rho_h}} \left((u - \beta_{h+1})^p + \beta_{h+1}^q \right)^{\sigma'} dx \leq c_{\sigma'} \left\{ \int_{A_{\beta_{h+1}, \rho_h}} (u - \beta_{h+1})^{p\sigma'} dx + \int_{A_{\beta_{h+1}, \rho_h}} \beta_{h+1}^{q\sigma'} dx \right\}.$$

Let us now estimate the last integral. Since

$$\beta_{h+1}^{\nu^*} |A_{\beta_{h+1}, \rho_h}| \leq \int_{A_{\beta_{h+1}, \rho_h}} |u|^{\nu^*} dx \leq \|u\|_{L^{\nu^*}(B(x_0, R))}^{\nu^*},$$

we have (remind that $\beta_h \leq \beta$ for any $h \geq 0$)

$$\begin{aligned}
\int_{A_{\beta_{h+1}, \rho_h}} \beta_{h+1}^{q\sigma'} dx &= \left(\beta_{h+1}^{\nu^*} |A_{\beta_{h+1}, \rho_h}| \right)^{\frac{(q-p)\sigma'}{\nu^*}} \beta_{h+1}^{p\sigma'} |A_{\beta_{h+1}, \rho_h}|^{1-\frac{(q-p)\sigma'}{\nu^*}} \\
&\leq \|u\|_{L^{\nu^*}(B(x_0, R))}^{(q-p)\sigma'} \beta^{p\sigma'} |A_{\beta_{h+1}, \rho_h}|^{1-\frac{(q-p)\sigma'}{\nu^*}}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_{A_{\beta_{h+1}, \rho_h}} \left((u - \beta_{h+1})^p + \beta_{h+1}^q \right)^{\sigma'} dx \\
&\leq c_{\sigma'} \int_{A_{\beta_{h+1}, \rho_h}} (u - \beta_{h+1})^{p\sigma'} dx + c_{\sigma'} \|u\|_{L^{\nu^*}(B(x_0, R))}^{(q-p)\sigma'} \beta^{p\sigma'} |A_{\beta_{h+1}, \rho_h}|^{1-\frac{(q-p)\sigma'}{\nu^*}} \\
&\leq c_{\sigma'} J_h + c_{\sigma'} \|u\|_{L^{\nu^*}(B(x_0, R))}^{(q-p)\sigma'} \beta^{p\sigma'} |A_{\beta_{h+1}, \rho_h}|^{1-\frac{(q-p)\sigma'}{\nu^*}}, \tag{4.12}
\end{aligned}$$

where in the last inequality we used (4.5). Now, owing to (4.7) we get

$$|A_{\beta_{h+1}, \rho_{h+1}}| \leq |A_{\beta_{h+1}, \rho_h}| \leq \left(\frac{2^{h+2}}{\beta}\right)^{p\sigma'} J_h = 4^{p\sigma'} \left(\frac{2^h}{\beta}\right)^{p\sigma'} J_h; \tag{4.13}$$

this, together with (4.12) and the assumptions $q \geq p$ and $J_h \leq 1$, implies

$$\begin{aligned} & \int_{A_{\beta_{h+1}, \rho_h}} \left((u - \beta_{h+1})^p + \beta_{h+1}^q \right)^{\sigma'} dx + |A_{\beta_{h+1}, \rho_h}|^{\sigma'(1-\frac{1}{\tau})} \\ & \leq c_4 \left(1 + \|u\|_{L^{\nu^*}(B(x_0, R))}^{(q-p)\sigma'} \right) \left\{ J_h + 2^{hp\sigma'} \left(1 - \frac{(q-p)\sigma'}{\nu^*} \right) \beta^{p\sigma' - p\sigma'} \left(1 - \frac{(q-p)\sigma'}{\nu^*} \right) J_h^{1 - \frac{(q-p)\sigma'}{\nu^*}} \right. \\ & \quad \left. + \left(\frac{2^h}{\beta} \right)^{p\sigma'^2(1-\frac{1}{\tau})} J_h^{\sigma'(1-\frac{1}{\tau})} \right\} \\ & \leq c_4 \left(1 + \|u\|_{L^{\nu^*}(B(x_0, R))}^{(q-p)\sigma'} \right) 2^{hp\sigma'^2} \beta^{\frac{p(q-p)\sigma'^2}{\nu^*}} J_h^\theta, \end{aligned}$$

where we have introduced the constant

$$\theta := \min \left\{ 1 - \frac{(q-p)\sigma'}{\nu^*}, \sigma' \left(1 - \frac{1}{\tau} \right) \right\}.$$

Thus, we get

$$\begin{aligned} (4\star) & \leq c_5 K_2 \left(1 + \|u\|_{L^{\nu^*}(B(x_0, R))}^{(q-p)\sigma'} \right) \cdot \frac{2^{2p\sigma'^2 h}}{R^{p\sigma'}} |A_{\beta_{h+1}, \rho_{h+1}}|^{1 - \frac{p\sigma'}{\nu^*}} \beta^{\frac{p(q-p)\sigma'^2}{\nu^*}} J_h^\theta \\ & \stackrel{(4.13)}{\leq} c_6 K_2 \left(1 + \|u\|_{L^{\nu^*}(B(x_0, R))}^{(q-p)\sigma'} \right) \cdot \frac{2^{3p\sigma'^2 h}}{R^{p\sigma'}} \frac{\beta^{\frac{p(q-p)\sigma'^2}{\nu^*}}}{\beta^{p\sigma'(1-\frac{p\sigma'}{\nu^*})}} J_h^{1+\theta - \frac{p\sigma'}{\nu^*}} \\ & \leq c_6 K_2 \left(1 + \|u\|_{L^{\nu^*}(B(x_0, R))}^{(q-p)\sigma'} \right) \cdot \frac{8^{p\sigma'^2 h}}{R^{p\sigma'}} \frac{1}{\beta^{p\sigma'(1-\frac{p\sigma'}{\nu^*})}} J_h^{1+\theta - \frac{p\sigma'}{\nu^*}}. \end{aligned}$$

Since, by the very definition of θ , we have

$$\alpha = \theta - \frac{p\sigma'}{\nu^*} = \min \left\{ 1 - \frac{(q-p)\sigma'}{\nu^*} - \frac{p\sigma'}{\nu^*}, \sigma' \left(1 - \frac{1}{\tau} \right) - \frac{p\sigma'}{\nu^*} \right\},$$

by collecting the previous chain of inequalities we get (4.8) with

$$\begin{aligned} C & = c_6 K_2 \\ & = c_6 \left\{ \|\lambda^{-1}\|_{L^\tau(B(x_0, R_0))} \cdot \left(1 + \|a\|_{L^\tau(B(x_0, R_0))} \right) \cdot \left(1 + \|\mu\|_{L^\sigma(B(x_0, R_0))} \right) \right\}^{\sigma'}. \end{aligned}$$

This ends the demonstration. \square

With Proposition 4.1 at hand, we are ready to provide the proof of our main result, namely Theorem 2.2. Before doing this, we remind the following very classical lemma of Real Analysis (see, e.g., [14, Lemma 7.1]).

Lemma 4.4. *Let $(z_h)_{h \geq 0}$ be a sequence of positive real numbers satisfying the following recursive relation*

$$z_{h+1} \leq L \zeta^h z_h^{1+\gamma} \quad (h \in \mathbb{N} \cup \{0\}), \quad (4.14)$$

where $L, \gamma > 0$ and $\zeta > 1$. If $z_0 \leq L^{-\frac{1}{\gamma}} \zeta^{-\frac{1}{\gamma^2}}$, then

$$z_h \leq \zeta^{-\frac{h}{\gamma}} z_0 \quad \text{for every } h \geq 0.$$

In particular, $z_h \rightarrow 0$ as $h \rightarrow \infty$.

We now prove Theorem 2.2.

Proof of Theorem 2.2. Let $u \in W_{\text{loc}}^{1, \frac{pr}{r+1}}(\Omega)$ be a quasi-minimizer of the functional \mathcal{F} in (2.1), let $x_0 \in \Omega$ be fixed and let $R_0 \in (0, 1]$ be such that $B(x_0, R_0) \Subset \Omega$. Moreover, let $\beta \geq 1$ (to be chosen later on) and let $(J_h)_{h \geq 0}$ be the sequence defined in (4.4). Owing to Proposition 4.1 (with $R = R_0$), we have the estimate

$$J_{h+1} \leq L (8^{p\sigma'^2})^h J_h^{1+\alpha} \quad (h \in \mathbb{N} \cup \{0\}), \quad (4.15)$$

where α is as in (2.15), that is,

$$\alpha := \min \left\{ 1 - \frac{q\sigma'}{\nu^*}, \sigma' \left(1 - \frac{1}{\tau} - \frac{p}{\nu^*} \right) \right\},$$

and the constant L is given by

$$L := C' \frac{1}{\beta^{p\sigma'(1-\frac{q\sigma'}{\nu^*})}} \left(\frac{1}{R_0} \right)^{p\sigma'}.$$

We remind that, in its turn, we have

$$\begin{aligned} C' &= C \left(1 + \|u\|_{L^{\nu^*}(B(x_0, R_0))}^{(q-p)\sigma'} \right) \\ &= c \left\{ \|\lambda^{-1}\|_{L^r(B(x_0, R_0))} \cdot \left(1 + \|a\|_{L^r(B(x_0, R_0))} \right) \cdot \left(1 + \|\mu\|_{L^\sigma(B(x_0, R_0))} \right) \right\}^{\sigma'} \times \\ &\quad \times \left(1 + \|u\|_{L^{\nu^*}(B(x_0, R_0))}^{(q-p)\sigma'} \right), \end{aligned}$$

where $c > 0$ only depends on n, p, q, r, σ, τ and on k and Q . We now claim that it is possible to choose $\beta \geq 1$ in such a way that $J_0 \leq L^{-1/\alpha} (8^{p\sigma'^2})^{-1/\alpha^2}$, that is,

$$\begin{aligned} J_0 &= \int_{A_{\frac{\beta}{2}, R_0}} \left(u - \frac{\beta}{2} \right)^{p\sigma'} dx \\ &\leq \left\{ C \left(1 + \|u\|_{L^{\nu^*}(B(x_0, R_0))}^{(q-p)\sigma'} \right) \cdot \frac{1}{\beta^{p\sigma'(1-\frac{q\sigma'}{\nu^*})}} \left(\frac{1}{R_0} \right)^{p\sigma'} \right\}^{-1/\alpha} (8^{p\sigma'^2})^{-1/\alpha^2}. \end{aligned} \quad (4.16)$$

In fact, by definition of J_0 (and since $u \in L_{\text{loc}}^{p\sigma'}(\Omega)$, see Remark 2.4), we have

$$J_0 \leq \int_{B(x_0, R_0)} |u|^{p\sigma'} dx = \|u\|_{L^{p\sigma'}(B(x_0, R_0))}^{p\sigma'} < \infty;$$

thus, reminding that $R_0 \leq 1$, the claimed (4.16) is clearly fulfilled if we choose

$$\beta := \left\{ 1 + \frac{C}{R_0^{p\sigma'}} (8^{p\sigma'^2})^{1/\alpha} \left(1 + \|u\|_{L^{\nu^*}(B(x_0, R_0))}^{(q-p)\sigma'} \right) \cdot \|u\|_{L^{p\sigma'}(B(x_0, R_0))}^{p\sigma'\alpha} \right\}^{\frac{1}{p\sigma'(1-\frac{q\sigma'}{\nu^*})}}. \quad (4.17)$$

With (4.16) at hand and $\beta \geq 1$ as in (4.17), we are entitled to apply Lemma 4.4. As a consequence, we obtain

$$\lim_{h \rightarrow \infty} J_h = \lim_{h \rightarrow \infty} \int_{A_{\beta h, \rho h}} (u - \beta h)^{p\sigma'} dx = \int_{A_{\beta, R_0/2}} (u - \beta)^{p\sigma'} dx = 0. \quad (4.18)$$

Since, by definition, $u - \beta > 0$ on $A_{\beta, R_0/2}$, from (4.18) we then conclude that

$$|A_{\beta, R_0/2}| = 0, \quad \text{whence } u \leq \beta \text{ for a.e. } x \in B(x_0, R_0/2).$$

To prove that u is also bounded from below by $-\beta$ it suffices to observe that, setting $v := -u$, then $v \in W_{\text{loc}}^{1,1}(\Omega)$ is a quasi-minimizer of the functional

$$\mathcal{G}(w) := \int_{\Omega} g(x, w, \nabla w) dx,$$

where $g : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ is the function defined by

$$g(x, w, \xi) := f(x, -w, -\xi).$$

As a consequence, since g clearly satisfies assumptions (H1) and (H2) (with the same functions λ, μ), by the above argument we deduce that

$$|A_{\beta, R_0/2}(v)| = 0, \quad \text{whence } -u \leq \beta \text{ for a.e. } x \in B(x_0, R_0/2).$$

Summing up, we have proved that, if $\beta \geq 1$ is as in (4.17), one has

$$|u(x)| \leq \beta, \quad \text{for a.e. } x \in B(x_0, R_0/2). \quad (4.19)$$

We now observe that, by Hölder's inequality (since $p\sigma' < \nu^*$), we have

$$(1 + \|u\|_{L^{\nu^*}(B_{R_0})}^{(q-p)\sigma'}) \|u\|_{L^{p\sigma'}(B(x_0, R_0))}^{p\sigma'\alpha} \leq c(1 + \|u\|_{L^{\nu^*}(B_{R_0})})^{\sigma'(q-p+\alpha)},$$

with $c > 0$ independent of u . This, together with (4.17), $R_0 \leq 1$, $\varrho^{1/\alpha} \geq 1$, allows us to conclude that there exists a constant $c > 0$, depending on n, p, q, r, σ, τ and on k and Q , such that (here, H is as in (2.14))

$$\beta \leq c \left(\frac{H}{R_0} \right)^{\frac{\nu^*}{\nu^* - q\sigma'}} (1 + \|u\|_{L^{\nu^*}(B_{R_0})})^{\frac{q-p+\alpha}{p(1-\frac{q\sigma'}{\nu^*})}}.$$

This inequality and (4.19) give exactly estimate (2.13). \square

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