ON THE CONVERGENCE RATE OF SOME NONLOCAL ENERGIES

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Abstract. We study the rate of convergence of some nonlocal functionals recently considered by Bourgain, Brezis, and Mironescu, and, after a suitable rescaling, we establish the Γ-convergence of the corresponding rate functionals to a limit functional of second order.

Contents

1. Introduction

We are interested in the rate of converge, as $h \searrow 0$, of the nonlocal functionals

$$\mathcal{F}_h(u) \coloneqq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K_h(y-x) f\left(\frac{|u(y)-u(x)|}{|y-x|} \right) dydx,$$

to the limit functional

$$\mathcal{F}_0(u) \coloneqq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(z) f(|\nabla u(x) \cdot \hat{z}|) dzdx.$$

Here and in the sequel, we set $K_h(z) \coloneqq h^{-d} K(z/h)$ with $K \colon \mathbb{R}^d \to [0,+\infty)$ an even function in $L^1(\mathbb{R}^d)$ that has finite second moment, see (27) below. We let $f \colon [0,+\infty) \to [0,+\infty)$ be a convex function of class C^2 satisfying $f(0) = f'(0) = 0$, and, for $z \in \mathbb{R}^d \setminus \{ 0 \}$, we put $\hat{z} = z/|z|$.

Functionals similar to $\mathcal{F}_h$ and $\mathcal{F}_0$ were considered by Bourgain, Brezis, and Mironescu in [5]. For K radial and $f(t) = |t|^p$ with $p \geq 1$, they established convergence as $h \searrow 0$ to a multiple of $\|u\|^p_{W^{1,p}(\Omega)}$ whenever $u \in W^{1,p}(\Omega)$ with Ω a smooth, bounded domain in $\mathbb{R}^d$. Their result has been extended in several directions ([3, 9, 12, 13], see also [2, 4, 11]), and, among others, we would like to spend some words on the contributions by Ponce [16]. The author studied the case in which $\{ K_h \}$ is a suitable family of functions in $L^1(\mathbb{R}^d)$ that approaches the Dirac delta in 0 and $f \colon [0,+\infty) \to [0,+\infty)$ is a generic convex function. When $u \in L^p(\Omega)$

for some $p \geq 1$ and the boundary of Ω is compact and Lipschitz, he showed pointwise convergence of some functionals that generalize the ones in [5]. The limit is a first order functional, which is given by a variant of $\mathcal{F}_0$ if $K_h(z) = h^{-d}K(z/h)$ for some $K \in L^1(\mathbb{R}^d)$ and $u \in W^{1,p}(\Omega)$. Further, when Ω is also bounded, in [16] Γ-convergence to the pointwise limit with respect to the L^1-topology is proved too. For the definition and the properties of Γ-convergence, we refer to the monographs [6, 8].

Let now

(1)
$$\begin{aligned}\mathcal{E}_h(u) &:= \frac{\mathcal{F}_0(u) - \mathcal{F}_h(u)}{h^2} \\ &= \frac{1}{h^2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left[K(z)f(|\nabla u(x)\cdot\hat{z}|) - K_h(z)f\left(\frac{|u(x+z)-u(x)|}{|z|}\right)\right]dzdx\end{aligned}$$

be the functional which measures the rate of convergence of $\mathcal{F}_h$ to $\mathcal{F}_0$. In this paper, under the assumptions that the function f is strongly convex (see condition (3) below) and that we restrict to functions that vanish outside a bounded, Lipschitz set Ω, we prove that the family $\{\mathcal{E}_h\}$ Γ-converges, with respect to the $H^1(\mathbb{R}^d)$-topology, to the *second order* limit functional

$$\mathcal{E}_0(u) := \begin{cases} \dfrac{1}{24}\displaystyle\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(z)\,|z|^2\, f''(|\nabla u(x)\cdot\hat{z}|)\,\left|\nabla^2 u(x)\hat{z}\cdot\hat{z}\right|^2 dzdx & \text{if } u \in H^2(\mathbb{R}^d), \\ +\infty & \text{otherwise.}\end{cases}$$

The uniform convexity assumption on f, which is needed for the Γ-inferior limit inequality, excludes some interesting cases such as $f(x) = |x|^p$ with $p \geq 1$, $p \neq 2$. In particular, when $f(x) = |x|$ and K is radially symmetric, the analysis is related to a geometric problem considered in [14] in the context of a physical model for liquid drops with dipolar repulsion. We also observe that our study differs from a higher order Γ-limit of $\mathcal{F}_h$ (see [7]), which would rather correspond to deal with the Γ-limit of the functionals

$$\frac{\mathcal{F}_h - \min \mathcal{F}_0}{h^\alpha} \qquad \text{for some } \alpha > 0\,.$$

As a consequence of our result (see Remark 5) we also get that, if the rate of convergence of $\mathcal{F}_h(u)$ to $\mathcal{F}_0(u)$ is fast enough, more precisely if $|\mathcal{E}_h(u)| \leq M$ for all h's sufficiently small, then $u \in H^2(\mathbb{R}^d)$.

We notice that our result is reminiscent to the one obtained by Peletier, Planqué, and Röger in [15]. There, motivated by a model for bilayer membranes, the authors considered the convolution functionals

$$\mathcal{G}_h(u) := \int_{\mathbb{R}^d} f\left(K_h * u\right) dx,$$

which converge to the functional $\mathcal{G}_0(u) = c\int_{\mathbb{R}^d} f\,(u)\,dx$ as $h \searrow 0$, with $c = c(K, d)$ a positive constant, and they showed that the corresponding rate functionals

$$\frac{\mathcal{G}_0(u) - \mathcal{G}_h(u)}{h^2} = \frac{1}{h^2}\int_{\mathbb{R}^d}\big(c\,f\,(u) - f\left(K_h * u\right)\big)dx \tag{2}$$

converge pointwise to the limit functional

$$\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(z)\,|z|^2\, f''(u(x))|\nabla u(x)\cdot\hat{z}|^2 dzdx \qquad \text{for } u \in H^1(\mathbb{R}^d).$$

In particular, the rate functionals are uniformly bounded if and only if $u \in H^1(\mathbb{R}^d)$.

In the proof of our convergence result, we follow a strategy similar to the one in [10, 11]: we first consider a related 1-dimensional problem, and then reduce the general case to it by a *slicing* procedure. More precisely, in Section 2 we study the functionals

$$E_h(u) \coloneqq \frac{1}{h^2} \int_{\mathbb{R}} \left[f(u(x)) - f\left(\fint_x^{x+h} u(y)dy \right) \right] dx,$$

which are a particular case of (2), and we show their convergence (see Theorem 1) to the limit energy

$$E_0(u) \coloneqq \frac{1}{24} \int_{\mathbb{R}} f''(u(x)) \left|u'(x)\right|^2 dx \qquad \text{for } u \in H^1(\mathbb{R}).$$

Then, in Section 3 we consider the general functionals in (1) and we establish the Γ-convergence to $\mathcal{E}_0$ (see Theorem 2), which is the main result of the present paper. We first show the convergence for $d = 1$, using the result of Section 2, and then we reduce to the 1-dimensional case by means of a delicate slicing technique.

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2. Finite difference functionals in the 1-dimensional case

Let $f \colon \mathbb{R} \to [0, +\infty)$ be a strongly convex function of class C^2 such that $f(0) = f'(0) = 0$. By saying that f is strongly convex, we mean that

$$\text{there exists } \gamma > 0 \text{ such that } f''(t) \geq \gamma \text{ for all } t \in \mathbb{R}. \tag{3}$$

Let us fix an open interval $I \coloneqq (a, b) \subset \mathbb{R}$. We introduce the closed subspace $Y \subset L^2(\mathbb{R})$ defined as

$$Y \coloneqq \{ u \in L^2(\mathbb{R}) : u = 0 \text{ in } \mathbb{R} \setminus I \}, \tag{4}$$

and, for $u \in Y$ and $h > 0$, we define the energy

$$E_h(u) \coloneqq \frac{1}{h^2} \int_{\mathbb{R}} \left[f(u(x)) - f\left(D_h U(x)\right) \right] dx, \tag{5}$$

where

$$U(x) \coloneqq \int_0^x u(y)dy \quad \text{and} \quad D_h U(x) \coloneqq \frac{U(x+h) - U(x)}{h} = \fint_x^{x+h} u(y)dy. \tag{6}$$

By using some simple changes of variable and the positivity of f, one can find

$$\begin{aligned} \int_{\mathbb{R}} f(u(x))dx = \int_a^b f(u(x))dx &= \int_{a-h}^b \fint_x^{x+h} f(u(y))dydx \\ &= \int_{\mathbb{R}} \fint_x^{x+h} f(u(y))dydx, \end{aligned}$$

the integrals possibly diverging to $+\infty$. Then, by combining the previous identity with Jensen's inequality

$$\fint_x^{x+h} f(u(y))dy \geq f(D_h U(x)),$$

we see that $E_h(u)$ ranges in $[0,+\infty]$ when $u \in Y$.

In the present section we compute the Γ-limit of $\{ E_h \}$ regarded as a family of functionals on Y endowed with the L^2-topology. Let us set

$$E_0(u) \coloneqq \begin{cases} \dfrac{1}{24}\displaystyle\int_{\mathbb{R}} f''(u(x))\,|u'(x)|^2\,dx & \text{if } u \in Y \cap H^1(\mathbb{R}), \\ +\infty & \text{otherwise.} \end{cases} \tag{7}$$

We prove the following:

Theorem 1. *Let us assume that $f\colon \mathbb{R} \to [0,+\infty)$ is a function of class C^2 such that $f(0) = f'(0) = 0$ and (3) holds. Then, the restriction to Y of the family $\{ E_h \}$ Γ-converges, as $h \searrow 0$, to E_0 w.r.t. the $L^2(\mathbb{R})$-topology, that is, for every $u \in Y$ the following properties hold:*

(1) For any family $\{ u_h \} \subset Y$ that converges to u in $L^2(\mathbb{R})$ we have

$$E_0(u) \leq \liminf_{h\searrow 0} E_h(u_h).$$

(2) There exists a sequence $\{ u_h \} \subset Y$ converging to u in $L^2(\mathbb{R})$ such that

$$\limsup_{h\searrow 0} E_h(u_h) \leq E_0(u).$$

The Γ-upper limit is established in Proposition 1, while Proposition 2 takes care of the lower limit. In turn, the latter is achieved by exploiting a suitable lower bound on the energy (see Lemma 1) and a compactness result (see Lemma 2), which are a consequence of the strong convexity of f.

2.1. **Pointwise limit and upper bound.** We now compute the limit of $E_h(u)$, as $h \searrow 0$, for a function $u \in Y \cap C^2(\mathbb{R})$. We observe that strong convexity of f is not needed for the next proposition to hold.

Proposition 1. *Let $f\colon \mathbb{R} \to [0,+\infty)$ be a C^2 function such that $f(0) = f'(0) = 0$, and let $u \in Y \cap C^2(\mathbb{R})$. Then, there exists a continuous, bounded, and increasing function $m\colon [0,+\infty) \to [0,+\infty)$ such that $m(0) = 0$ and*

$$|E_h(u) - E_0(u)| \leq c\, m(h), \tag{8}$$

where $c \coloneqq c(b-a, \|u\|_{C^2(\mathbb{R})}, \|f\|_{C^2([-\|u\|_{C^2(\mathbb{R})}, \|u\|_{C^2(\mathbb{R})}])}) > 0$ is a constant. In particular, $\lim_{h\searrow 0} E_h(u) = E_0(u)$ and for every $u \in Y$ there exists a sequence $\{ u_h \} \subset Y$ that converges to u in $L^2(\mathbb{R})$ and satisfies

$$\limsup_{h\searrow 0} E_h(u_h) \leq E_0(u).$$

Proof. Since $u \in Y \cap C^2(\mathbb{R})$ and $f \in C^2(\mathbb{R})$, it is easy to see that $h^2 E_h(u)$ and $E_0(u)$ are uniformly bounded in h. Thus, there exists a constant $c_\infty > 0$ such that

$$|E_h(u) - E_0(u)| \leq c_\infty \qquad \text{for } h > 1. \tag{9}$$

Next, we focus on the case $h \in (0,1]$. If $x \notin (a-h,b)$, then $D_hU(x) = 0$, and hence

$$h^2 E_h(u) = \int_a^b \left[f(u(x)) - f\left(D_hU(x)\right)\right] dx - \int_{a-h}^a f(D_hU(x))dx.$$

Being u regular, for any $x \in (a-h,b)$ we have the Taylor's expansion

$$D_hU(x) = u(x) + \frac{h}{2}u'(x) + \frac{h^2}{6}u''(x_h), \qquad \text{with } x_h \in (x, x+h),$$

which we rewrite as

$$D_hU(x) = u(x) + hv_h(x), \qquad \text{with } \ v_h(x) := \frac{u'(x)}{2} + \frac{h}{6}u''(x_h); \tag{10}$$

note that v_h converges uniformly to $u'/2$ as $h \searrow 0$.

Plugging (10) into the definition of E_h, we get

$$\begin{aligned} h^2 E_h(u) =& -\int_a^b \left[f\big(u(x) + hv_h(x)\big) - f(u(x))\right] dx - \int_{a-h}^a f\left(\frac{h^2}{6}u''(x_h)\right) dx \\ =& -h\int_a^b f'(u(x))v_h(x)dx - \frac{h^2}{2}\int_a^b f''(w_h(x))v_h(x)^2 dx \\ & -\int_{a-h}^a f\left(\frac{h^2}{6}u''(x_h)\right) dx, \end{aligned}$$

where w_h fulfils $w_h(x) \in (u(x), u(x) + hv_h(x_h))$ for all $x \in (a,b)$.

In view of the regularity of f and u, we can utilize the Mean Value Theorem to obtain

$$\left|\int_{a-h}^a f\left(\frac{h^2}{6}u''(x_h)\right) dx\right| \le c_1 h^5$$

for a constant $c_1 > 0$ that depends only on $N := \|u\|_{C^2(\mathbb{R})}$ and on $\|f''\|_{L^\infty([-N,N])}$. Moreover, recalling the definition of v_h, we have

$$\int_a^b f'(u(x))v_h(x)dx = \frac{h}{6}\int_a^b f'(u(x))u''(x_h)dx,$$

and therefore

(11)

$$\begin{aligned} |E_h(u) - E_0(u)| \le& \frac{1}{6}\left|-\int_a^b f'(u(x))u''(x_h)dx - \int_a^b f''(u(x))u'(x)^2 dx\right| \\ & + \frac{1}{2}\left|\frac{1}{4}\int_a^b f''(u(x))u'(x)^2 dx - \int_a^b f''(w_h(x))v_h(x)^2 dx\right| + c_1 h^5. \end{aligned}$$

Since $u \in Y \cap C^2(\mathbb{R})$, u'' admits a uniform modulus of continuity $m_{u''} \colon [0,+\infty) \to [0,\infty)$. An integration by parts gives that

$$\begin{aligned} \left|-\int_a^b f'(u(x))u''(x_h)dx - \int_a^b f''(u(x))u'(x)^2 dx\right| &\le \int_a^b |f'(u(x))|\,|u''(x) - u''(x_h)|\, dx \\ &\le c_2 m_{u''}(h), \end{aligned}$$

where $c_2 := (b-a)\|f'\|_{L^\infty([-N,N])}$.

In a similar manner, denoting by $m_{f''}$ the modulus of continuity of the restriction of f'' to the interval $[-N, N]$, we also find

$$\begin{aligned}&\left|\frac{1}{4}\int_a^b f''(u(x))u'(x)^2dx - \int_a^b f''(w_h(x))v_h(x)^2dx\right| \\ &\leq \int_a^b |f''(u(x))| \left|\frac{1}{4}u'(x)^2 - v_h(x)^2\right| dx + \int_a^b |f''(u(x)) - f''(w_h(x))|\, v_h(x)^2dx \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \leq c_3(h + m_{f''}(h)),\end{aligned}$$

with c_3 depending on $b-a$, N, and $\|f''\|_{L^\infty([-N,N])}$.

By combining (11) with the inequalities above, we obtain

$$|E_h(u) - E_0(u)| \leq c_0\big(m_{u''}(h) + m_{f''}(h) + h + h^5\big) \qquad \text{for } h \in (0, 1], \tag{12}$$

for a suitable constant $c_0 > 0$. At this stage, (8) follows by combining (9) and (12).

As for the existence of a family that fulfils the upper limit inequality, we apply a standard density argument that we sketch in the following lines. Let $u \in Y$. If $u \notin H^1(\mathbb{R})$, the inequality holds trivially; otherwise, by rescaling the domain and mollifying, we construct a sequence of smooth functions $\{ u_\ell \} \subset Y$ that converges to u both uniformly and in $H^1(\mathbb{R})$. Then, since f'' is a continuous function, we get $\lim_{\ell \nearrow +\infty} E_0(u_\ell) = E_0(u)$. Besides, we know that $\lim_{h\searrow 0} E_h(u_\ell) = E_0(u_\ell)$ for any $\ell \in \mathbb{N}$, because u_ℓ is smooth. We conclude that there exists a subsequence $\{ h_\ell \}$ such that

$$\lim_{\ell \nearrow +\infty} E_{h_\ell}(u_\ell) = E_0(u).$$

□

Remark 1. *Notice that, as a consequence of Proposition 1, the* Γ*-limit of the rate functionals*

$$hE_h(u) = \frac{1}{h}\int_{\mathbb{R}} [f(u(x)) - f(D_hU(x))]\, dx$$

is equal to zero.

2.2. **Lower bound in the strongly convex case.** In view of Proposition 1, to accomplish the proof of Theorem 1, it only remains to establish statement (1), that is, for any $u \in Y$ and for any family $\{ u_h \} \subset Y$ converging to u in $L^2(\mathbb{R})$ it holds

$$E_0(u) \leq \liminf_{h\searrow 0} E_h(u_h).$$

In the current subsection we utilize the strong convexity of the function f. We exploit this hypothesis to provide a lower bound on the energy E_h, by means of which we prove that sequences with equibounded energy are relatively compact w.r.t. the L^2-topology.

Lemma 1 (Lower bound on the energy)**.** *Let us assume that* $f\colon \mathbb{R} \to [0, +\infty)$ *is a function of class* C^2 *such that* $f(0) = f'(0) = 0$ *and* (3) *is fulfilled. Then, for any* $u \in Y$*, it holds*

$$E_h(u) \geq \sup_{\varphi \in C_c^\infty(\mathbb{R}^2)} \left\{ \int_{\mathbb{R}} \fint_x^{x+h} \left(\frac{u(y) - D_hU(x)}{h}\varphi(x, y) - \frac{\varphi(x, y)^2}{4\lambda_h(x, y)} \right) dydx \right\}, \tag{13}$$

with

$$\lambda_h(x,y) := \int_0^1 (1-\vartheta) f''\big((1-\vartheta)D_hU(x) + \vartheta u(y)\big)d\vartheta. \tag{14}$$

Moreover,

$$E_h(u) \geq \frac{\gamma}{4}\int_{\mathbb{R}}\int_{-h}^{h} J_h(r)\left(\frac{u(y+r)-u(y)}{h}\right)^2 drdy, \tag{15}$$

where

$$J(r) := (1-|r|)^+ \quad \textit{and} \quad J_h(r) := \frac{1}{h}J\left(\frac{r}{h}\right).$$

Proof. For a given $h>0$, let us consider $u \in Y$ such that $E_h(u)$ is finite. We write

$$E_h(u) = \frac{1}{h^2}\int_{\mathbb{R}} e_h(x)dx, \quad \text{where } e_h(x) := \fint_x^{x+h} [f(u(y)) - f(D_hU(x))]dy.$$

Thanks to the identity

$$f(s) - f(t) = f'(t)(s-t) + (s-t)^2\int_0^1 (1-\vartheta)f''((1-\vartheta)t + \vartheta s))d\vartheta,$$

we find

$$e_h(x) = \fint_x^{x+h} \lambda_h(x,y)\left(u(y) - D_hU(x)\right)^2 dy, \tag{16}$$

where $\lambda_h(x,y)$ is as in (14). Observe that for any $\varphi \in C_c^\infty\left(\mathbb{R}^2\right)$ we have the pointwise inequality

$$\lambda_h(x,y)\left(u(y)-D_hU(x)\right)^2 \geq \frac{u(y)-D_hU(x)}{h}\varphi(x,y) - \frac{\varphi(x,y)^2}{4\lambda_h(x,y)},$$

from which we infer (13).

The strong convexity of f grants that $\lambda_h(x,y) \geq \gamma/2$ for all $(x,y) \in \mathbb{R}^2$ and $h>0$, thus we also deduce that

$$e_h(x) \geq \frac{\gamma}{2}\fint_x^{x+h}\left(u(y) - D_hU(x)\right)^2.$$

Hence, we get

$$\begin{aligned} E_h(u) \geq & \frac{\gamma}{2}\int_{\mathbb{R}}\fint_x^{x+h}\left(\frac{u(y)-D_hU(x)}{h}\right)^2 dydx \\ \geq & \frac{\gamma}{4}\int_{\mathbb{R}}\fint_x^{x+h}\fint_x^{x+h}\left(\frac{u(z)-u(y)}{h}\right)^2 dzdydx, \end{aligned}$$

where the last inequality follows from the identity

$$\int |\varphi(y)|^2 d\mu(y) = \left|\int \varphi(y)d\mu(y)\right|^2 + \frac{1}{2}\int\int |\varphi(z)-\varphi(y)|^2 d\mu(z)d\mu(y),$$

which holds whenever μ is a probability measure and $\varphi \in L^2(\mu)$. By Fubini's Theorem and neglecting contributions near the boundary, we find the lower bound on the energy:

$$\begin{aligned} E_h(u) \geq& \frac{\gamma}{4} \int_{\mathbb{R}} \fint_{y-h}^{y} \fint_{x}^{x+h} \left(\frac{u(z) - u(y)}{h} \right)^2 dz dx dy \\ =& \frac{\gamma}{4h} \int_{\mathbb{R}} \int_{y-h}^{y+h} \left(1 - \frac{|z-y|}{h} \right) \left(\frac{u(z) - u(y)}{h} \right)^2 dz dy. \end{aligned}$$

The conclusion (15) is now achieved by the change of variables $r = z - y$. □

Remark 2. *Let $u \in Y \cap H^1(\mathbb{R})$. If along with the previous assumptions we also require that f'' is bounded above by a constant $c > 0$, then the family $\{ E_h(u) \}$ is uniformly bounded. Indeed, it follows from* (16) *and the definition of λ_h that*

$$E_h(u) \leq \frac{c}{2h^2} \int_{\mathbb{R}} \fint_{x}^{x+h} (u(y) - D_h U(x))^2 \, dy dx.$$

Then, since $u \in H^1(\mathbb{R})$, when $y \in (x, x+h)$ it holds

$$|u(y) - D_h U(x)|^2 \leq h \int_{x}^{x+h} |u'(z)|^2 \, dz = h^2 \int_0^1 |u'(x+hz)|^2 \, dz,$$

and we derive the estimate

$$E_h(u) \leq \frac{c}{2} \int_{\mathbb{R}} |u'(x)|^2 \, dx. \tag{17}$$

Lemma 2 (Compactness)**.** *Let the function f be as in Lemma 1 and let $\{ u_h \} \subset Y$ be a sequence of functions such that $E_h(u_h) \leq M$ for some $M \geq 0$. Then, there exist a subsequence $\{ u_{h_\ell} \}$ and a function $u \in Y \cap H^1(\mathbb{R})$ such that $u_{h_\ell} \to u$ in $L^2(\mathbb{R})$.*

Proof. We adapt the strategy of [1, Theorem 3.1].

By Lemma 1, we infer that

$$\frac{\gamma}{4} \int_{\mathbb{R}} \int_{-h}^{h} J_h(r) \left(\frac{u_h(y+r) - u_h(y)}{h} \right)^2 dr dy \leq M. \tag{18}$$

Observe that $J_h(r)dr$ is a probability measure on $[-h, h]$.

We now introduce the mollified functions $v_h := \rho_h * u_h$, where $\{ \rho_h \}$ is the family

$$\rho_h(r) := \frac{1}{ch} \rho\left(\frac{r}{h} \right), \qquad \text{with } c := \int_{\mathbb{R}} \rho(r) dr.$$

Here, $\rho \in C_c^\infty(\mathbb{R})$ is an even kernel, and it is chosen in such a way that its support is contained in $[-1, 1]$,

$$0 \leq \rho \leq J, \quad \text{and} \quad |\rho'| \leq J.$$

Note that, for all $h > 0$, $v_h \colon \mathbb{R} \to \mathbb{R}$ is a smooth function whose support is a subset of $(a-h, b+h)$. Moreover, the family of derivatives $\{ v_h' \}_{h \in (0,1)}$ is uniformly bounded in $L^2(\mathbb{R})$; indeed, since $\int_{\mathbb{R}} \rho'(r)dr = 0$, it holds

$$\begin{aligned}\int_{\mathbb{R}} |v_h'(y)|^2\, dy &= \int_{\mathbb{R}} \left| \int_{-h}^{h} \rho_h'(r)[u_h(y+r) - u_h(y)]dr \right|^2 dy\\ &\le \int_{\mathbb{R}} \left(\int_{-h}^{h} |\rho_h'(r)|\, |u_h(y+r) - u_h(y)|\, dr \right)^2 dy\\ &\le \frac{1}{c^2} \int_{\mathbb{R}} \left(\int_{-h}^{h} J_h(r) \left| \frac{u_h(y+r) - u_h(y)}{h} \right| dr \right)^2 dy\\ &\le \frac{1}{c^2} \int_{\mathbb{R}} \int_{-h}^{h} J_h(r) \left| \frac{u_h(y+r) - u_h(y)}{h} \right|^2 drdy,\end{aligned}$$

and thus

$$\int_{\mathbb{R}} |v_h'(y)|^2\, dy \le \frac{4M}{c^2\gamma}. \tag{19}$$

For all $h \in (0,1)$, let $\tilde{v}_h$ be the restriction of v_h to the interval $(a-1, b+1)$. By Poincaré inequality, (19) entails boundedness in $H_0^1((a-1,b+1))$ of the family $\{\, \tilde{v}_h \,\}_{h\in(0,1)}$, and, in view of Sobolev's Embedding Theorem, this grants in turn that there exists a subsequence $\{\, \tilde{v}_{h_\ell} \,\}$ uniformly converging to some $\tilde{u} \in H_0^1([a-1, b+1])$. Since each $\tilde{v}_{h_\ell}$ is supported in $(a - h_\ell, b + h_\ell)$, we see that $\tilde{u} \in H_0^1(\bar{I})$; therefore, if we set

$$u(x) \coloneqq \begin{cases} \tilde{u}(x) & \text{if } x \in \bar{I},\\ 0 & \text{otherwise,}\end{cases}$$

we deduce that $\{\, v_{h_\ell} \,\}$ converges uniformly to $u \in Y \cap H^1(\mathbb{R})$.

Lastly, to achieve the conclusion, we provide a bound on the L^2-distance between u_h and v_h. Similarly to the previous computations, we have

$$\begin{aligned}\int_{\mathbb{R}} |v_h(y) - u_h(y)|^2\, dy &= \int_{\mathbb{R}} \left| \int_{-h}^{h} \rho_h(r)[u_h(y+r) - u_h(y)]dr \right|^2 dy\\ &\le \int_{\mathbb{R}} \int_{-h}^{h} \rho_h(r)\, |u_h(y+r) - u_h(y)|^2\, drdy\\ &\le \frac{1}{c} \int_{\mathbb{R}} \int_{-h}^{h} J_h(r)\, |u_h(y+r) - u_h(y)|^2\, drdy,\end{aligned}$$

and, by (18), we get

$$\int_{\mathbb{R}} |v_h(y) - u_h(y)|^2 dy \le \frac{4M}{c\gamma} h^2. \tag{20}$$

Since there exists a subsequence $\{\, v_{h_\ell} \,\}$ uniformly converging to a function $u \in Y \cap H^1(\mathbb{R})$, (20) gives the conclusion. □

Now we can prove statement (1) of Theorem 1.

Proposition 2. *Let the function f be as in Lemma 1. Then, for any $u \in Y$ and for any family $\{\, u_h \,\} \subset Y$ that converges to u in $L^2(\mathbb{R})$, it holds*

$$E_0(u) \le \liminf_{h\searrow 0} E_h(u_h). \tag{21}$$

Proof. Fix $u, u_h \in Y$ in such a way that $u_h \to u$ in $L^2(\mathbb{R})$. We can suppose that the inferior limit in (21) is finite, otherwise the conclusion holds trivially. Consequently, up to extracting a subsequence, which we do not relabel, there exists $\lim_{h\searrow 0} E_h(u_h)$ and it is finite. In particular, there exists $M \geq 0$ such that $E_h(u_h) \leq M$ for all $h > 0$, and, by Lemma 2, this yields that $u \in Y \cap H^1(\mathbb{R})$.

We use formula (13) for each u_h, choosing, for $(x, y) \in \mathbb{R}^2$,

$$\varphi(x,y) = \psi\left(x, \frac{y-x}{h}\right), \qquad \text{with } \psi \in C_c^\infty(\mathbb{R}^2).$$

We get

$$\begin{aligned} E_h(u_h) \geq & \int_{\mathbb{R}} \fint_x^{x+h} \frac{u_h(y) - \fint_x^{x+h} u_h}{h} \psi\left(x, \frac{y-x}{h}\right) dydx \\ & - \frac{1}{4} \int_{\mathbb{R}} \fint_x^{x+h} \frac{\psi\left(x, \frac{y-x}{h}\right)^2}{\lambda_h(x,y)} dydx, \end{aligned} \tag{22}$$

where, coherently with (14),

$$\lambda_h(x,y) \coloneqq \int_0^1 (1-\vartheta) f''\left((1-\vartheta)\fint_x^{x+h} u_h(z)dz + \vartheta u_h(y)\right) d\vartheta \geq \frac{\gamma}{2}.$$

Let us focus on the first quantity on the right-hand side of (22). We have

$$\begin{aligned} & \frac{1}{h}\int_{\mathbb{R}} \fint_x^{x+h} \left(\fint_x^{x+h} u_h(z)dx\right) \psi\left(x, \frac{y-x}{h}\right) dydx \\ & \quad = \frac{1}{h^3} \int_{\mathbb{R}} \int_x^{x+h} \int_x^{x+h} u_h(z) \psi\left(x, \frac{y-x}{h}\right) dydzdx \\ & \quad = \frac{1}{h^3} \int_{\mathbb{R}} \int_{z-h}^{z} \int_x^{x+h} u_h(z) \psi\left(x, \frac{y-x}{h}\right) dydxdz, \end{aligned}$$

and, by similar computations, we obtain

$$\begin{aligned} & \int_{\mathbb{R}} \fint_x^{x+h} \frac{u_h(y) - \fint_x^{x+h} u_h}{h} \psi\left(x, \frac{y-x}{h}\right) dydx \\ & \quad = \frac{1}{h} \int_{\mathbb{R}} \fint_{y-h}^{y} \fint_x^{x+h} u_h(y) \left[\psi\left(x, \frac{y-x}{h}\right) - \psi\left(x, \frac{z-x}{h}\right)\right] dzdxdy. \end{aligned} \tag{23}$$

By a simple change of variable, we get

$$\begin{aligned} \fint_{y-h}^{y} \fint_x^{x+h} \psi\left(x, \frac{y-x}{h}\right) dzdx &= \int_0^1 \int_0^1 \psi(y - hr, r) dqdr, \\ \fint_{y-h}^{y} \fint_x^{x+h} \psi\left(x, \frac{z-x}{h}\right) dzdx &= \fint_{y-h}^{y} \int_0^1 \psi(x, r) drdx \\ &= \int_0^1 \int_0^1 \psi(y - hq, r) dqdr, \end{aligned}$$

hence

$$\begin{aligned}\frac{1}{h}\fint_{y-h}^{y}\fint_{x}^{x+h}&\left[\psi\left(x,\frac{y-x}{h}\right)-\psi\left(x,\frac{z-x}{h}\right)\right]dzdx\\&=\int_0^1\int_0^1\frac{\psi(y-hr,r)-\psi(y-hq,r)}{h}dqdr\\&=-\int_0^1\int_0^1\int_q^r\partial_1\psi(y-hs,r)dsdqdr\\&=-\int_0^1\int_0^1(r-q)\fint_q^r\partial_1\psi(y-hs,r)dsdqdr.\end{aligned}$$

Being ψ smooth, we have that $\partial_1\psi(y-hs,r)=\partial_1\psi(y,r)+O(h)$ as $h\searrow 0$, uniformly for $s\in[0,1]$. Consequently,

$$\frac{1}{h}\fint_{y-h}^{y}\fint_{x}^{x+h}\left[\psi\left(x,\frac{y-x}{h}\right)-\psi\left(x,\frac{z-x}{h}\right)\right]dzdx=-\int_0^1\left(r-\frac{1}{2}\right)\partial_1\psi(y,r)dr+O(h).$$

Plugging this equality in (23) yields

$$\int_{\mathbb{R}}\fint_x^{x+h}\frac{u_h(y)-\fint_x^{x+h}u_h}{h}\psi\left(x,\frac{y-x}{h}\right)dydx=-\int_{\mathbb{R}}u_h(y)\int_0^1\left(r-\frac{1}{2}\right)\partial_1\psi(y,r)dr+O(h).$$

It is possible to take the limit $h\searrow 0$ in the previous formula, since $u_h\to u$ in $L^2(\mathbb{R})$. We then get

(24)

$$\lim_{h\searrow 0}\int_{\mathbb{R}}\fint_x^{x+h}\frac{u_h(y)-\fint_x^{x+h}u_h}{h}\psi\left(x,\frac{y-x}{h}\right)dydx=-\int_{\mathbb{R}}\int_0^1 u(y)(r-\tfrac{1}{2})\partial_1\psi(y,r)drdy.$$

Now, we turn to the second addendum on the right-hand side of (22). By Fubini's Theorem and a change of variables, we have

$$\int_{\mathbb{R}}\fint_x^{x+h}\frac{\psi(x,\frac{y-x}{h})^2}{\lambda_h(x,y)}dydx=\int_{\mathbb{R}}\int_0^1\frac{\psi\left(y-hr,r\right)^2}{\lambda_h(y-hr,y)}drdy.$$

The function ψ has compact support and $\lambda_h\geq\gamma/2$ for all $h>0$, therefore we can apply Lebesgue's Convergence Theorem to let $h\searrow 0$ in the previous expression, and we get

$$\begin{aligned}\lim_{h\searrow 0}\int_{\mathbb{R}}\int_0^1&\frac{\psi\left(y-hr,r\right)^2}{\int_0^1(1-\vartheta)f''\left((1-\vartheta)\fint_{y-hr}^{y+(1-r)h}u_h(z)dz+\vartheta u_h(y)\right)d\vartheta}drdy\\&=\int_{\mathbb{R}}\int_0^1\frac{\psi(y,r)^2}{\int_0^1(1-\vartheta)f''(u(y))d\vartheta}drdy,\end{aligned}$$

thus

$$\lim_{h\searrow 0}\int_{\mathbb{R}}\fint_x^{x+h}\frac{\psi(x,\frac{y-x}{h})^2}{\lambda_h(x,y)}dydx=2\int_{\mathbb{R}}\int_0^1\frac{\psi(y,r)^2}{f''(u(y))}drdy. \tag{25}$$

Summing up, by (24) and (25), we deduce

(26)

$$\liminf_{h\searrow 0}E_h(u_h)\geq-\int_{\mathbb{R}}\int_0^1\left[u(y)\left(r-\frac{1}{2}\right)\partial_1\psi(y,r)drdy+\frac{1}{2}\int_{\mathbb{R}}\int_0^1\frac{\psi(y,r)^2}{f''(u(y))}\right]drdy,$$

for all $\psi\in C_c^\infty(\mathbb{R}^2)$.

We can reach the conclusion from the last inequality by a suitable choice of the test function ψ. To see this, we let $\eta \in C_c^\infty(\mathbb{R})$ and we choose a standard sequence of mollifiers $\{ \rho_k \}$. We then set

$$\psi(x,y) = \psi_k(x,y) \coloneqq \eta(x) \left(\zeta_k(y) - \frac{1}{2} \right), \quad \text{with } \zeta_k(y) \coloneqq \int_{\mathbb{R}} \rho_k(z-y) z dz,$$

so that (26) reads

$$\begin{aligned} \liminf_{h \searrow 0} E_h(u_h) \geq & - \int_0^1 \left(r - \frac{1}{2} \right) \left(\zeta_k(r) - \frac{1}{2} \right) dr \int_{\mathbb{R}} u(y) \eta'(y) dy \\ & - \frac{1}{2} \int_0^1 \left(\zeta_k(r) - \frac{1}{2} \right)^2 dr \int_{\mathbb{R}} \frac{\eta(y)^2}{f''(u(y))} dy. \end{aligned}$$

Because of the identity $\int_0^1 (r-1/2)^2 dr = 1/12$, letting $k \to +\infty$ yields

$$\begin{aligned} \liminf_{h \searrow 0} E_h(u_h) \geq & - \frac{1}{12} \left[\int_{\mathbb{R}} u(y) \eta'(y) dy + \frac{1}{2} \int_{\mathbb{R}} \frac{\eta(y)^2}{f''(u(y))} dy \right] \\ = & \frac{1}{12} \left[\int_{\mathbb{R}} u'(y) \eta(y) dy - \frac{1}{2} \int_{\mathbb{R}} \frac{\eta(y)^2}{f''(u(y))} dy \right], \end{aligned}$$

where $u' \in L^2(\mathbb{R})$ is the distributional derivative of u, which exists since $u \in H^1(\mathbb{R})$. Recall that, in the previous formula, the test function η is arbitrary, thus, to recover (21), it suffices to take the supremum w.r.t. $\eta \in C_c^\infty(\mathbb{R})$. $\square$

3. Γ-limit in arbitrary dimension

Let us fix the assumptions and the notation that we use in the current section. We consider an open, bounded set $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary and a measurable function $K \colon \mathbb{R}^d \to [0,+\infty)$ such that

$$\int_{\mathbb{R}^d} K(z) \left(1 + |z|^2 \right) dz < +\infty. \tag{27}$$

We require that $K(z) = K(-z)$ for a.e. $z \in \mathbb{R}^d$ and that the support of K contains a sufficiently large annulus centered at the origin. More precisely, let us set

$$\sigma_d \coloneqq \begin{cases} 1 & \text{when } d = 2, \\ \dfrac{d-2}{d-1} & \text{when } d > 2; \end{cases} \tag{28}$$

we suppose that there exist $r_0 \geq 0$ and $r_1 > 0$ such that $r_0 < \sigma_d r_1$ and

$$\operatorname{ess\,inf} \{ K(z) : z \in B(0,r_1) \setminus B(0,r_0) \} > 0. \tag{29}$$

The simplest case for which (29) holds is when there exists $k > 0$ such that $K(z) \geq k$ for all $z \in B(0,r_1)$. Finally, we let $f \colon [0,+\infty) \to [0,+\infty)$ be a C^2 function such that $f(0) = f'(0) = 0$ and the strong convexity condition (3) is satisfied.

For $u \in H^1(\mathbb{R}^d)$, we define the functionals

$$\begin{aligned} \mathcal{F}_h(u) &\coloneqq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K_h(y-x) f \left(\frac{|u(y)-u(x)|}{|y-x|} \right) dy dx, \\ \mathcal{F}_0(u) &\coloneqq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(z) f(|\nabla u(x) \cdot \hat{z}|) dz dx, \end{aligned}$$

where $\hat{z} \coloneqq z/|z|$ for $z \neq 0$ and $K_h(z) \coloneqq h^{-d} K(z/h)$.

Remark 3. *By appealing to the results in* [16], *one can show that* $\mathcal{F}_h(u)$ *tends to* $\mathcal{F}_0(u)$ *as* $h \searrow 0$ *when* u *is smooth enough and vanishes in* $\mathbb{R}^d \setminus \Omega$, *and also that* $\mathcal{F}_0$ *is the* Γ*-limit of the family* $\{ \mathcal{F}_h \}$. *Indeed, if* $u = 0$ *a.e. in* $\mathbb{R}^d \setminus \Omega$, *we have*

$$\begin{aligned}\mathcal{F}_h(u) \coloneqq& \int_\Omega \int_\Omega K_h(y-x) f\left(\frac{|u(y)-u(x)|}{|y-x|}\right) dydx \\ &+ 2\int_\Omega \int_{\mathbb{R}^d\setminus\Omega} K_h(y-x) f\left(\frac{|u(x)|}{|y-x|}\right) dydx.\end{aligned}$$

By [16], *we know that the first addendum on the right-hand side converges and* Γ*-converges to* $\mathcal{F}_0$. *It is clear that this is also the* Γ*-inferior limit of* $\{ \mathcal{F}_h \}$, *because the term*

$$\tilde{\mathcal{F}}_h(u) \coloneqq 2\int_\Omega \int_{\mathbb{R}^d\setminus\Omega} K_h(y-x) f\left(\frac{|u(x)|}{|y-x|}\right) dydx$$

is positive and may be dropped. As for the pointwise limit, we pick a function $u \in C^1(\mathbb{R}^d)$ *that equals* 0 *in* $\mathbb{R}^d \setminus \Omega$, *and we observe that the quotient* $|u(x)| / |y-x|$ *is bounded above by* $\|u\|_{C^1(\Omega)}$. *It follows that*

$$\tilde{F}_h(u) \leq c \int_\Omega \int_{\mathbb{R}^d\setminus\Omega} K_h(y-x) dydx,$$

with $c \coloneqq 2\|f\|_{L^\infty([0,\|u\|_{C^1(\mathbb{R}^d)}])}$, *whence* $\lim_{h\searrow 0} \tilde{F}_h(u) = 0$ *(recall that* $K \in L^1(\mathbb{R}^d)$*).*

Analogously to the 1-dimensional case, we define

$$\mathcal{E}_h(u) \coloneqq \frac{\mathcal{F}_0(u) - \mathcal{F}_h(u)}{h^2}$$

and we study the asymptotics of this family as $h \searrow 0$. Notice that the functionals $\mathcal{E}_h$ are positive (see Lemma 3 below). Let us set

$$X \coloneqq \{ u \in H^1(\mathbb{R}^d) : u = 0 \text{ a.e. in } \mathbb{R}^d \setminus \Omega \} \tag{30}$$

and

$$\mathcal{E}_0(u) \coloneqq \begin{cases} \dfrac{1}{24}\displaystyle\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(z)\,|z|^2 f''(|\nabla u(x)\cdot \hat{z}|)\left|\nabla^2 u(x)\hat{z}\cdot\hat{z}\right|^2 dzdx & \text{if } u \in X \cap H^2(\mathbb{R}^d), \\ +\infty & \text{otherwise.}\end{cases} \tag{31}$$

We observe that if $u \in X \cap H^2(\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d)$ or if f'' has quadratic growth at infinity and $u \in X \cap H^2(\mathbb{R}^d)$, then $\mathcal{E}_0(u)$ is finite.

Remark 4 (Radial case)**.** *When* K *is radial, that is* $K(z) = \bar{K}(|z|)$ *for some* $\bar{K}\colon [0,+\infty) \to [0,+\infty)$, *we have*

$$\mathcal{F}_0(u) = \|K\|_{L^1(\mathbb{R}^d)} \int_{\mathbb{R}^d} \fint_{\mathbb{S}^{d-1}} f(|\nabla u(x)\cdot e|) d\mathcal{H}^{d-1}(e)dx,$$

$$\mathcal{E}_0(u) = \frac{1}{24}\left(\int_{\mathbb{R}^d} K(z)\,|z|^2\,dz\right) \int_{\mathbb{R}^d} \fint_{\mathbb{S}^{d-1}} f''(|\nabla u(x)\cdot e|)\left|\nabla^2 u(x) e\cdot e\right|^2 d\mathcal{H}^{d-1}(e)dx.$$

This Section is devoted to the proof of the following:

Theorem 2. *Let* Ω, K, *and* f *satisfy the assumptions stated at the beginning of the current section. Then, there hold:*

(1) *For any family* $\{ u_h \} \subset X$ *such that* $\mathcal{E}_h(u_h) \le M$ *for some* $M > 0$*, there exists a subsequence* $\{ u_{h_\ell} \}$ *and a function* $u \in X \cap H^2(\mathbb{R}^d)$ *such that* $\nabla u_{h_\ell} \to \nabla u$ *in* $L^2(\mathbb{R}^d)$*.*

(2) *For any family* $\{ u_h \} \subset X$ *that converges to* $u \in X$ *in* $H^1(\mathbb{R}^d)$

$$\mathcal{E}_0(u) \le \liminf_{h \searrow 0} \mathcal{E}_h(u_h).$$

(3a) *For any* $u \in X \cap W^{1,\infty}(\mathbb{R}^d)$ *there exists a family* $\{ u_h \} \subset X$ *that converges to* u *in* $H^1(\mathbb{R}^d)$ *with the property that*

$$\limsup_{h \searrow 0} \mathcal{E}_h(u_h) \le \mathcal{E}_0(u).$$

(3b) *If* f'' *is bounded, for any* $u \in X$ *there exists a family* $\{ u_h \} \subset X$ *that converges to* u *in* $H^1(\mathbb{R}^d)$ *with the property that*

$$\limsup_{h \searrow 0} \mathcal{E}_h(u_h) \le \mathcal{E}_0(u).$$

Statements 2, (3a), and (3b) amounts to saying that $\mathcal{E}_0$ is the Γ-limit of $\{ \mathcal{E}_h \}$ with respect to the $H^1(\mathbb{R}^d)$-convergence if either we restrict to functions in $X \cap W^{1,\infty}(\mathbb{R}^d)$ or f'' is bounded.

3.1. **Slicing.** When the dimension is 1, by virtue of the analysis in Section 2, it is not difficult to derive the Γ-convergence of the functionals $\mathcal{E}_h$.

Corollary 1. *Let* $K \colon \mathbb{R} \to [0, +\infty)$ *be an even function such that* (27) *holds. For* $h > 0$ *and* $u \in H^1(\mathbb{R})$*, we define the family*

$$\mathcal{E}_h(u) \coloneqq \frac{1}{h^2} \int_{\mathbb{R}} \int_{\mathbb{R}} K_h(z) \left[f(|u'(x)|) - f\left(\left| \frac{u(x+z) - u(x)}{z} \right| \right) \right] dz dx.$$

We also let $\Omega = (a, b)$ *be an open interval,* $f \colon [0, +\infty) \to [0, +\infty)$ *be a* C^2 *function satisfying* $f(0) = f'(0) = 0$ *and* $f''(t) \ge \gamma$ *with* $\gamma > 0$ *for all* $t \in \mathbb{R}$*, and* $X \subset H^1(\mathbb{R})$ *be as in* (30)*. Then, the restrictions of the functionals* $\mathcal{E}_h$ *to* X Γ*-converge w.r.t. the* $H^1(\mathbb{R})$*-topology to*

$$\mathcal{E}_0(u) \coloneqq \begin{cases} \dfrac{1}{24} \left(\displaystyle\int_{\mathbb{R}} K(z) z^2 dz \right) \displaystyle\int_{\mathbb{R}} f''(u'(x)) \left| u''(x) \right|^2 dx & \text{if } u \in X \cap H^2(\mathbb{R}), \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. A change of variables gives

$$\mathcal{E}_h(u) = \int_{\mathbb{R}} K(z) z^2 \left[\frac{1}{(hz)^2} \int_{\mathbb{R}} f(|u'(x)|) - f\left(\left| \frac{u(x+hz) - u(x)}{hz} \right| \right) dx \right] dz.$$

Recalling (5), we notice that the quantity between square brackets is equal to $E_{hz}(u')$, therefore the conclusion follows by a straightforward adaptation of the proof of Theorem 1 (see also the proof of Proposition 3). □

Corollary 1 concludes the analysis when $d = 1$, so we may henceforth assume that $d \ge 2$. Our aim is proving that the restrictions to X of the functionals $\mathcal{E}_h$ Γ-converge w.r.t. the $H^1(\mathbb{R}^d)$-topology to $\mathcal{E}_0$. The gist of our proof is a slicing procedure, which amounts to express the d-dimensional energies $\mathcal{E}_h$ as superpositions of the 1-dimensional energies E_h, regarded as functionals on each line of $\mathbb{R}^d$.

Hereafter we tacitly assume that Ω, K, and f satisfy the hypotheses made at the beginning of the section. When $z \in \mathbb{R}^d \setminus \{ 0 \}$, we set

$$\hat{z}^\perp \coloneqq \left\{ \xi \in \mathbb{R}^d : \xi \cdot \hat{z} = 0 \right\}.$$

Lemma 3 (Slicing)**.** *For $u \in X$, $z \in \mathbb{R}^d \setminus \{ 0 \}$, and $\xi \in \hat{z}^{\perp}$, we define $w_{\hat{z},\xi} \colon \mathbb{R} \to \mathbb{R}$ as $w_{\hat{z},\xi}(t) \coloneqq u(\xi + t\hat{z})$. Then, $w'_{\hat{z},\xi}(t) = \nabla u(\xi + t\hat{z}) \cdot \hat{z}$ and*

$$\mathcal{E}_h(u) = \int_{\mathbb{R}^d} \int_{z^{\perp}} K(z) \, |z|^2 \, E_{h|z|}(w'_{\hat{z},\xi}) d\mathcal{H}^{d-1}(\xi) dz, \tag{32}$$

where $E_{h|z|}$ is as in (5) *(note that the function f in* (5) *must be replaced here by $f(|t|)$).*

Proof. Formula (32) is an easy consequence of Fubini's Theorem. Indeed, once the direction $\hat{z} \in \mathbb{S}^{d-1}$ is fixed, we can write $x \in \mathbb{R}^d$ as $x = \xi + t\hat{z}$ for some $\xi \in \mathbb{R}^d$ such that $\xi \cdot z = 0$ and $t \in \mathbb{R}$. Using this decomposition, we have

$$\begin{aligned} \mathcal{F}_h(u) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(z) f\left(\frac{|u(x+hz) - u(x)|}{h\,|z|} \right) dz dx \\ &= \int_{\mathbb{R}^d} \int_{z^{\perp}} \int_{\mathbb{R}} K(z) f\left(\frac{|w_{\hat{z},\xi}(t + h\,|z|) - w_{\hat{z},\xi}(t)|}{h\,|z|} \right) dt d\mathcal{H}^{d-1}(\xi) dz, \end{aligned}$$

whence

$$\mathcal{E}_h(u) = \frac{1}{h^2} \int_{\mathbb{R}^d} \int_{z^{\perp}} \int_{\mathbb{R}} K(z) \left[f\left(\left| w'_{\hat{z},\xi}(t) \right| \right) - f\left(\frac{|w_{\hat{z},\xi}(t+h|z|) - w_{\hat{z},\xi}(t)|}{h\,|z|} \right) \right] dt d\mathcal{H}^{d-1}(\xi) dz.$$

To obtain (32), it now suffices to multiply and divide the integrands by $|z|^2$. □

The connection with the 1-dimensional case provided by Lemma 3 suggests that the Γ-convergence of the functionals E_h might be exploited to prove Theorem 2. Though, to be able to apply the results of Section 2, we need the functions $w_{\hat{z},\xi}$ in (32) to admit a second order weak derivative for a.e. z and ξ. This poses no real problem for the proof of the upper limit inequality, because we may reason on regular functions; as for the lower limit one, we shall tackle the difficulty in the next subsection by means of a compactness criterion, see Lemma 6 below. For the moment being, we are able to establish the following:

Proposition 3. *Let $u \in X \cap H^2(\mathbb{R}^d)$. Then:*

1. *For any family $\{ u_h \} \subset X$ that converges to u in $H^1(\mathbb{R}^d)$, there holds*
$$\mathcal{E}_0(u) \le \liminf_{h \searrow 0} \mathcal{E}_h(u_h).$$
2. *If $u \in X \cap C^3(\mathbb{R}^d)$, then*
$$\mathcal{E}_0(u) = \lim_{h \searrow 0} \mathcal{E}_h(u).$$

Proof. We prove both the assertions by using the slicing formula (32).

1. For all $h > 0$, $z \in \mathbb{R}^d \setminus \{ 0 \}$, and $\xi \in \hat{z}^{\perp}$, we let $w_{h;\hat{z},\xi} \colon \mathbb{R} \to \mathbb{R}$ be defined as $w_{h;\hat{z},\xi}(t) \coloneqq u_h(\xi + t\hat{z})$. Then,
$$\mathcal{E}_h(u_h) = \int_{\mathbb{R}^d} \int_{z^{\perp}} K(z) \, |z|^2 \, E_{h|z|}(w'_{h;\hat{z},\xi}) d\mathcal{H}^{d-1}(\xi) dz,$$
and, by Fatou's Lemma,
$$\liminf_{h \searrow 0} \mathcal{E}_h(u_h) \ge \int_{\mathbb{R}^d} \int_{z^{\perp}} K(z) \, |z|^2 \left[\liminf_{h \searrow 0} E_{h|z|}(w'_{h;\hat{z},\xi}) \right] d\mathcal{H}^{d-1}(\xi) dz. \tag{33}$$

Let $w_{\hat{z},\xi}$ be as in Lemma 3. Note that for any kernel $\rho\colon \mathbb{R}^d \to [0,+\infty)$ such that $\|\rho\|_{L^1(\mathbb{R}^d)} = 1$ we may write

$$\begin{aligned}\int_{\mathbb{R}^d} |\nabla u_h - \nabla u|^2 &\geq \int_{\mathbb{R}^d} \rho(z) \int_{\hat{z}^\perp} \int_{\mathbb{R}} \left| \big(\nabla u_h(\xi + t\hat{z}) - \nabla u(\xi + t\hat{z})\big) \cdot \hat{z} \right|^2 dt d\mathcal{H}^{d-1}(\xi) dz \\ &= \int_{\mathbb{R}^d} \rho(z) \int_{\hat{z}^\perp} \int_{\mathbb{R}} \left| w'_{h;\hat{z},\xi}(t) - w'_{\hat{z},\xi}(t) \right|^2 dt d\mathcal{H}^{d-1}(\xi) dz.\end{aligned}$$

Since the left-hand side vanishes as $h \searrow 0$, it follows that there exists a subsequence of $\{ w'_{h;\hat{z},\xi} \}$, which we do not relabel, that converges in $L^2(\mathbb{R})$ to $w'_{\hat{z},\xi}$ for $\mathcal{L}^d$-a.e. $z \in \mathbb{R}^d$ and $\mathcal{H}^{d-1}$-a.e. $\xi \in \hat{z}^\perp$. In particular, by assumption, $w'_{\hat{z},\xi} \in H^1(\mathbb{R})$ for a.e. (z,ξ) and it equals 0 on the complement of some open interval $I_{\hat{z},\xi}$.

From the previous considerations, we see that Proposition 2 can be applied on the right-hand side of (33), yielding

$$\liminf_{h\searrow 0} \mathcal{E}_h(u_h) \geq \int_{\mathbb{R}^d} \int_{z^\perp} K(z) \, |z|^2 \, E_0(w'_{\hat{z},\xi}) d\mathcal{H}^{d-1}(\xi) dz = \mathcal{E}_0(u).$$

(2) For any fixed $z \in \mathbb{R}^d \setminus \{ 0 \}$ and $\xi \in \hat{z}^\perp$, we define the function $w_{\hat{z},\xi} \in C^3(\mathbb{R})$ as above. Since Ω is bounded, there exists $r > 0$ such that, for any choice of z, $w'_{\hat{z},\xi}(t) = \nabla u(\xi + t\hat{z}) \cdot \hat{z} = 0$ whenever $\xi \in z^\perp$ satisfies $|\xi| \geq r$, while $w'_{\hat{z},\xi}(t)$ is supported in an open interval $I_{\hat{z},\xi}$ if $|\xi| < r$.

By virtue of the slicing formula (32), we obtain

$$|\mathcal{E}_h(u) - \mathcal{E}_0(u)| \leq \int_{\mathbb{R}^d} \int_{z^\perp} K(z) \, |z|^2 \left| E_{h|z|}(w'_{\hat{z},\xi}) - E_0(w'_{\hat{z},\xi}) \right| d\mathcal{H}^{d-1}(\xi) dz$$

Proposition 1 gives the existence of a constant $c > 0$ and of a continuous, bounded, and increasing function $m\colon [0,+\infty) \to [0,+\infty)$ such that $m(0) = 0$ and

$$|\mathcal{E}_h(u) - \mathcal{E}_0(u)| \leq c \int_{\mathbb{R}^d} \int_{z^\perp} K(z) \, |z|^2 \, m(h\,|z|) d\mathcal{H}^{d-1}(\xi) dz.$$

We remark that here m can be chosen depending only on ∇u, and not on $\hat{z}$ and ξ.

Recalling (27), to achieve the conclusion it now suffices to appeal to Lebesgue's Convergence Theorem.

□

3.2. **Lower bound, compactness, and proof of the main result.** Similarly to the 1-dimensional case, we shall prove the compactness of functions with equibounded energy by establishing at first a lower bound on the functionals $\mathcal{E}_h$. More precisely, Lemma 4 below shows that, when f is strongly convex, $\mathcal{E}_h(u)$ is greater than a double integral which takes into account, for each $z \in \mathbb{R}^d \setminus \{ 0 \}$, the squared projection of the difference quotients of ∇u in the direction of z. Thanks to the slicing formula, the inequality follows with no effort by applying Lemma 1 on each line of $\mathbb{R}^d$.

We point out that our approach results in the appearance of an effective kernel $\tilde{K}$ in front of the difference quotients. This function stands as a multidimensional counterpart of the kernel J in Lemma 1; actually, $\tilde{K}$ depends both on K and on J (see (34) for the precise definition). In Lemma 5, we shall collect some properties of the effective kernel that will be useful in the proof of Lemma 6.

Lemma 4 (Lower bound on the energy)**.** *Let us set*

$$\tilde{K}(z) := \int_{-1}^{1} J(r) K_{|r|}(z) dr \qquad \text{for a.e. } z \in \mathbb{R}^d, \tag{34}$$

with J as in Lemma 1. Then, it holds

$$\mathcal{E}_h(u) \geq \frac{\gamma}{4} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{K}(z) \left[\frac{\big(\nabla u(x+hz) - \nabla u(x)\big) \cdot \hat{z}}{h} \right]^2 dx dz. \tag{35}$$

Proof. Thanks to Lemma 3, we can reduce to the 1-dimensional case, and we take advantage of the lower bound provided by Lemma 1. Keeping the notation of Lemma 3, we find

$$\begin{aligned}
\mathcal{E}_h(u) \geq & \frac{\gamma}{4} \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} \int_{-h|z|}^{h|z|} J_{h|z|}(r) K(z) \, |z|^2 \left(\frac{w'_{\hat{z},\xi}(t+r) - w'_{\hat{z},\xi}(t)}{h\,|z|} \right)^2 dr dt d\mathcal{H}^{d-1}(\xi) dz \\
= & \frac{\gamma}{4} \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} \int_{-h|z|}^{h|z|} J_{h|z|}(r) K(z) \left(\frac{w'_{\hat{z},\xi}(t+r) - w'_{\hat{z},\xi}(t)}{h} \right)^2 dr dt \mathcal{H}^{d-1}(\xi) dz.
\end{aligned}$$

To cast this bound in the form of (35), we change variables and use Fubini's Theorem:

$$\begin{aligned}
I :=& \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} \int_{-h|z|}^{h|z|} J_{h|z|}(r) K(z) \left(\frac{w'_{\hat{z},\xi}(t+r) - w'_{\hat{z},\xi}(t)}{h} \right)^2 dr dt d\mathcal{H}^{d-1}(\xi) dz \\
=& \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} \int_{-1}^{1} J(r) K(z) \left(\frac{w'_{\hat{z},\xi}(t+h\,|z|\,r) - w'_{\hat{z},\xi}(t)}{h} \right)^2 dr dt d\mathcal{H}^{d-1}(\xi) dz \\
=& \int_{-1}^{0} \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} J(r) K_{-r}(z) \left(\frac{w'_{\hat{z},\xi}(t-h\,|z|) - w'_{\hat{z},\xi}(t)}{h} \right)^2 dt d\mathcal{H}^{d-1}(\xi) dz dr \\
&+ \int_{0}^{1} \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} J(r) K_{r}(z) \left(\frac{w'_{\hat{z},\xi}(t+h\,|z|) - w'_{\hat{z},\xi}(t)}{h} \right)^2 dt d\mathcal{H}^{d-1}(\xi) dz dr
\end{aligned}$$

Note that

$$\begin{aligned}
&\int_{-1}^{0} \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} J(r) K_{-r}(z) \left(\frac{w'_{\hat{z},\xi}\big(t+h\,|z|\big) - w'_{\hat{z},\xi}(t)}{h} \right)^2 dt d\mathcal{H}^{d-1}(\xi) dz dr \\
=& \int_{-1}^{0} \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} J(r) K_{-r}(z) \left(\frac{w'_{-\hat{z},\xi}\big(-(t+h\,|z|)\big) - w'_{-\hat{z},\xi}(-t)}{h} \right)^2 dt d\mathcal{H}^{d-1}(\xi) dz dr \\
=& \int_{-1}^{0} \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} J(r) K_{-r}(z) \left(\frac{w'_{\hat{z},\xi}\big(t+h\,|z|\big) - w'_{\hat{z},\xi}(t)}{h} \right)^2 dt d\mathcal{H}^{d-1}(\xi) dz dr,
\end{aligned}$$

because $w'_{-\hat{z},\xi}(-s) = -w'_{\hat{z},\xi}(s)$ for all $s \in \mathbb{R}$. Thus, we conclude that

$$\begin{aligned}
I =& \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} \left(\int_{-1}^{1} J(r) K_{|r|}(z) dr \right) \left(\frac{w'_{\hat{z},\xi}\big(t+h\,|z|\big) - w'_{\hat{z},\xi}(t)}{h} \right)^2 dt d\mathcal{H}^{d-1}(\xi) dz \\
=& \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{K}(z) \left[\frac{\big(\nabla u(x+hz) - \nabla u(x)\big) \cdot \hat{z}}{h} \right]^2 dx dz,
\end{aligned}$$

which concludes the proof. $\square$

Let us remind that, by assumption, the kernel K is bounded away from 0 in a suitable annulus. The next lemma shows that the effective kernel appearing $\tilde{K}$ in (35) inherits a similar property.

Lemma 5. *Let $\tilde{K}\colon \mathbb{R}^d \to [0,+\infty)$ be as in* (34)*. Then,*

$$\int_{\mathbb{R}^d} \tilde{K}(z)\left(1+|z|^2\right) dz < +\infty. \tag{36}$$

Moreover, if σ_d and r_1 are the constants in (28) *and* (29)*, then,*

$$\operatorname{ess\,inf}\left\{\tilde{K}(z) : z \in B(0,\sigma_d r_1)\right\} > 0. \tag{37}$$

Proof. The convergence of the integral in (36) follows easily from (27). Indeed, by the definition of $\tilde{K}$, we see that

$$\int_{\mathbb{R}^d} \tilde{K}(z)dz = \int_{-1}^{1}\int_{\mathbb{R}^d} J(r)K_{|r|}(z)dzdr = \int_{\mathbb{R}^d} K(z)dz;$$

analogously, one finds that

$$\int_{\mathbb{R}^d} \tilde{K}(z)\,|z|^2\,dz = c\int_{\mathbb{R}^d} K(z)\,|z|^2\,dz,$$

for some $c>0$.

For what concerns (37), let us set $k \coloneqq \operatorname{ess\,inf}\{\, K(z) : z \in B(0,r_1)\setminus B(0,r_0)\,\}$. In view of (29), $k>0$.

We distinguish between the case $z \in B(0,r_0)$ and the case $z \in B(0,r_1)\setminus B(0,r_0)$. In the first situation, for a.e. $z \in \mathbb{R}^d$,

$$\begin{aligned}\tilde{K}(z) \geq & 2\int_{\frac{|z|}{r_1}}^{\frac{|z|}{r_0}} J(r)K_r(z)dr \geq 2k\int_{\frac{|z|}{r_1}}^{\frac{|z|}{r_0}} \frac{1}{r^d}J(r)dr \\ = & \frac{2k}{|z|^{d-1}}\int_{r_0}^{r_1} s^{d-2}\left(1-\frac{|z|}{s}\right)ds.\end{aligned}$$

When $z \in B(0,r_1)\setminus B(0,r_0)$, instead, similar computations get

$$\tilde{K}(z) \geq 2\int_{\frac{|z|}{r_1}}^{1} J(r)K_r(z)dr = \frac{2k}{|z|^{d-1}}\int_{|z|}^{r_1} s^{d-2}\left(1-\frac{|z|}{s}\right)ds \qquad \text{for a.e. } z\in\mathbb{R}^d,$$

so that we obtain

$$\tilde{K}(z) \geq \frac{2k}{|z|^{d-1}}\int_{\max(r_0,|z|)}^{r_1} s^{d-2}\left(1-\frac{|z|}{s}\right)ds \qquad \text{for a.e. } z\in\mathbb{R}^d. \tag{38}$$

When $d=2$, the estimate above becomes

$$\tilde{K}(z) \geq 2k\left[\frac{r_1-\max(r_0,|z|)}{|z|} - \log\left(\frac{r_1}{\max(r_0,|z|)}\right)\right] \qquad \text{for a.e. } z\in\mathbb{R}^d.$$

Exploiting the concavity of the logarithm, we see that the lower bound that we have obtained is strictly positive if $|z| < r_1 = \sigma_2 r_1$.

On the other hand, putting $M \coloneqq \max(r_0,|z|)$ for shortness, if $d\geq 3$, the right-hand side in (38) equals

$$\frac{2k}{(d-1)(d-2)\,|z|^{d-1}}\left[(d-2)\left(r_1^{d-1}-M^{d-1}\right)-(d-1)\,|z|\left(r_1^{d-2}-M^{d-2}\right)\right],$$

and therefore

$$\begin{aligned}\tilde{K}(z) \geq & \frac{2kM^{d-2}}{(d-1)(d-2)\,|z|^{d-1}} \\ & \cdot \left\{ \left(\frac{r_1}{M}\right)^{d-2} [(d-2)r_1 - (d-1)\,|z|] - [(d-2)M - (d-1)\,|z|] \right\}\end{aligned}$$

for a.e. $z \in \mathbb{R}^d$. When $|z| < \frac{d-2}{d-1} r_1 = \sigma_d r_1$, the quantity between braces is strictly positive if

$$\frac{(M - |z|)d - (2M - |z|)}{(r_1 - |z|)d - (2r_1 - |z|)} < \left(\frac{r_1}{M}\right)^{d-2}.$$

Observe that both the left-hand side and the right-hand one are strictly increasing in d; also, the left-hand side is bounded above by $(M - |z|)/(r_1 - |z|)$, so the last inequality holds if

$$\frac{M - |z|}{r_1 - |z|} < \frac{r_1}{M},$$

which, in turn, is true for all $z \in B(0, r_1)$. □

We are now in the position to prove that families with equibounded energy are compact in $H^1(\mathbb{R}^d)$, and that their accumulation points admit second order weak derivatives.

Lemma 6 (Compactness)**.** *If $\{ u_h \} \subset X$ satisfies $\mathcal{E}_h(u_h) \leq M$ for some $M \geq 0$, there exist a subsequence $\{ u_{h_\ell} \}$ and a function $u \in X \cap H^2(\mathbb{R}^d)$ such that $u_{h_\ell} \to u$ in $H^1(\mathbb{R}^d)$.*

Proof. Let $\tilde{k} := \operatorname{ess\,inf} \{ \tilde{K}(z) : z \in B(0, \sigma_d r_1) \}$; Lemma 5 ensures that $\tilde{k} > 0$. We consider a function $\rho \in C_c^\infty([0, +\infty))$ such that

$$\rho(r) = 0 \qquad \text{if } r \in \left[\frac{\sigma_d r_1}{\sqrt{2}}, +\infty\right),$$

and we further require that

$$0 \leq \rho(r) \leq \tilde{k} \quad \text{and} \quad |\rho'(r)| \leq \tilde{k}.$$

For $h > 0$ and $y \in \mathbb{R}^d$, we set

$$\rho_h(y) := \frac{1}{ch^d} \rho\left(\frac{|y|}{h}\right), \qquad \text{with } c := \int_{\mathbb{R}^d} \rho(|y|) dy,$$

and we introduce the functions $v_h := \rho_h * u_h$, as before.

Each function v_h is a smooth function and, for all $\tilde{h} \in (0, 1)$, its support is contained in

$$\Omega_{\tilde{h}} := \{ x : \operatorname{dist}(x, \Omega) \leq 2^{-1/2} \tilde{h} \sigma_d r_1 \}$$

if $h \in (0, \tilde{h})$. In particular, we can choose $\tilde{h}$ so small that $\partial\Omega_{\tilde{h}}$ is still Lipschitz. For such an $\tilde{h}$, we assert that the family $\{ v_h \}_{h \in (0, \tilde{h})}$ is relatively compact in $H_0^1(\Omega_{\tilde{h}})$. In order to prove this, we first remark that

$$\int_{\Omega_{\tilde{h}}} \left|\nabla^2 v_h\right|^2 = \int_{\Omega_{\tilde{h}}} |\Delta v_h|^2, \tag{39}$$

and next we show that the right-hand side is uniformly bounded.

We observe that $\int_{\mathbb{R}^d} \nabla\rho_h(y)dy = 0$ for all $h > 0$, because ρ is compactly supported. Hence,

$$\begin{aligned}\|\Delta v_h\|^2_{L^2(\Omega_{\tilde h})} &= \int_{\mathbb{R}^d} |\Delta v_h|^2 \\ &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \nabla\rho_h(y)\cdot\big(\nabla u_h(x+y) - \nabla u_h(x)\big)dy\right|^2 dx \\ &\le \int_{\mathbb{R}^d}\left[\frac{1}{ch^{d+1}}\int_{\mathbb{R}^d}\left|\rho'\left(\frac{|y|}{h}\right)\right|\left|\big(\nabla u_h(x+y)-\nabla u_h(x)\big)\cdot\hat y\right| dy\right]^2 dx.\end{aligned}$$

By our choice of ρ and (37), we find

$$\begin{aligned}\|\Delta v_h\|^2_{L^2(\Omega_{\tilde h})} &\le \int_{\mathbb{R}^d}\left[\frac{1}{ch}\int_{\mathbb{R}^d}\tilde K_h(y)\left|\big(\nabla u_h(x+y)-\nabla u_h(x)\big)\cdot\hat y\right| dy\right]^2 dx \\ &\le \int_{\mathbb{R}^d}\left[\frac{1}{ch}\int_{\mathbb{R}^d}\tilde K(z)\left|\big(\nabla u_h(x+hz)-\nabla u_h(x)\big)\cdot\hat z\right| dz\right]^2 dx\end{aligned}$$

Further, since $\tilde K \in L^1(\mathbb{R}^d)$, Jensen's inequality and Fubini's Theorem yield

$$\|\Delta v_h\|^2_{L^2(\Omega_{\tilde h})} \le \frac{\|\tilde K\|_{L^1(\mathbb{R}^d)}}{c^2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\tilde K(z)\left[\frac{\big(\nabla u_h(x+hz)-\nabla u_h(x)\big)\cdot\hat z}{h}\right]^2 dxdz.$$

The lower bound (35) entails

$$\|\Delta v_h\|^2_{L^2(\Omega_{\tilde h})} \le \frac{4}{c^2\gamma}\|\tilde K\|_{L^1(\mathbb{R}^d)}\mathcal{E}_h(u_h),$$

so that, in view of the assumption $\mathcal{E}_h(u_h) \le M$ and of (39), we get

$$\|\nabla^2 v_h\|^2_{L^2(\Omega_{\tilde h})} \le \frac{4M}{c^2\gamma}\|\tilde K\|_{L^1(\mathbb{R}^d)}. \tag{40}$$

We argue as in the proof of Lemma 2. We recall that, for $h \in (0,\tilde h)$, each v_h vanishes on the complement of $\Omega_{\tilde h}$, and thus, by Poincaré inequality, (40) implies a uniform bound on the norms $\|v_h\|_{H^2_0(\Omega_{\tilde h})}$. As a consequence, by Rellich-Kondrachov Theorem, the family $\{\,\tilde v_h\,\}_{h\in(0,\tilde h)}$ of the restrictions of the functions v_h to $\Omega_{\tilde h}$ admits a subsequence $\{\,\tilde v_{h_\ell}\,\}$ that converges in $H^1_0(\Omega_{\tilde h})$ to a function $\tilde u \in H^2_0(\Omega_{\tilde h})$. Actually, the support of $\tilde u$ is contained in $\bar\Omega$, and, if we put,

$$u(x) \coloneqq \begin{cases}\tilde u(x) & \text{if } x \in \bar\Omega,\\ 0 & \text{otherwise,}\end{cases}$$

we infer that $\{\,v_{h_\ell}\,\}$ converges in $H^1(\mathbb{R}^d)$ to $u \in X \cap H^2(\mathbb{R}^d)$.

To accomplish the proof, it suffices to show that the L^2 distance between ∇u_h and ∇v_h vanishes when $h \searrow 0$. Since ρ_h has unit $L^1(\mathbb{R}^d)$-norm and is radial, we have

$$\begin{aligned}\int_{\mathbb{R}^d}|\nabla v_h(x)-\nabla u_h(x)|^2dx &= \int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}\rho_h(y)\big(\nabla u_h(x+y)-\nabla u_h(x)\big)dy\right|^2 dx \\ &= \frac14\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}\rho_h(y)\big(\nabla u_h(x+y)+\nabla u_h(x-y)-2\nabla u_h(x)\big)dy\right|^2 dx \\ &\le \frac14\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\rho_h(y)\left|\nabla u_h(x+y)+\nabla u_h(x-y)-2\nabla u_h(x)\right|^2 dydx.\end{aligned}$$

We remark that for any fixed $y \in \mathbb{R}^d \setminus \{0\}$ and for all $p \in \mathbb{R}^d$, the identity $|p|^2 = |p \cdot y|^2 + |(\mathrm{Id} - y \otimes y)p|^2$ can be reformulated as

$$
\begin{aligned}
|p|^2 &= |p \cdot y|^2 + \int_{\hat{y}^\perp} \pi(|\eta|)\, |p \cdot \eta|^2 \, d\mathcal{H}^{d-1}(\eta) \\
&= |p \cdot y|^2 + \frac{1}{h^2} \int_{\hat{y}^\perp} \pi_h(\eta)\, |p \cdot \eta|^2 \, d\mathcal{H}^{d-1}(\eta),
\end{aligned} \tag{41}
$$

where $\pi\colon [0, +\infty) \to [0, +\infty)$ is a continuous function such that

$$
\int_{e_d^\perp} \pi(|\eta|)\, |\eta|^2 \, d\mathcal{H}^{d-1}(\eta) = 1,
$$

and $\pi_h(\eta) \coloneqq h^{-d+1} \pi(|\eta|/h)$. We further prescribe that

$$
\pi(r) = 0 \qquad \text{if } r \in \left[\frac{\sigma_d r_1}{\sqrt{2}}, +\infty\right)
$$

and that $\lim_{r \searrow 0} \pi(r)/r \in \mathbb{R}$.

We apply the formula (41) to $p_h(x,y) \coloneqq \nabla u_h(x+y) + \nabla u_h(x-y) - 2\nabla u_h(x)$ and we find that

$$
\int_{\mathbb{R}^d} |\nabla v_h(x) - \nabla u_h(x)|^2 \, dx \le \frac{1}{4}\left(I_1 + I_2\right), \tag{42}
$$

where

$$
\begin{aligned}
I_1 &\coloneqq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho_h(y)\, |y|^2\, |p_h(x,y) \cdot \hat{y}|^2 \, dy dx, \\
I_2 &\coloneqq \frac{1}{h^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\hat{y}^\perp} \rho_h(y) \pi_h(\eta)\, |p_h(x,y) \cdot \eta|^2 \, d\mathcal{H}^{d-1}(\eta) dy dx.
\end{aligned}
$$

We first consider I_1. Keeping in mind that ρ is compactly supported and $\rho(|y|) \le \tilde{k} \le \tilde{K}(y)$ for a.e. $y \in B(0, 2^{-1/2}\sigma_d r_1)$, we get

$$
\begin{aligned}
I_1 \le \frac{(\sigma_d r_1)^2}{c} \Bigg[&\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{K}_h(y) \left| \big(\nabla u_h(x+y) - \nabla u_h(x)\big) \cdot \hat{y} \right|^2 dy dx \\
&+ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{K}_h(y) \left| \big(\nabla u_h(x-y) - \nabla u_h(x)\big) \cdot \hat{y} \right|^2 dy dx \Bigg],
\end{aligned}
$$

and, by (35),

$$
I_1 \le \frac{8(\sigma_d r_1)^2 M}{c\gamma} h^2. \tag{43}
$$

As for I_2, we assert that there exist a constant $L > 0$, depending on d, σ_d, r_1, $\tilde{k}$, and c, such that

$$
I_2 \le \frac{LM}{\gamma} h^2. \tag{44}
$$

To prove the claim, we write the integrand appearing in I_2 as follows:

$$\begin{aligned}
p_h(x,y)\cdot\eta =&\big(\nabla u_h(x+y)+\nabla u_h(x-y)-2\nabla u_h(x)\big)\cdot\eta\\
=&\big(\nabla u_h(x+y)+\nabla u_h(x-y)-2\nabla u_h(x-\eta)\big)\cdot\eta\\
&+2\big(\nabla u_h(x-\eta)-\nabla u_h(x)\big)\cdot\eta\\
=&\big(\nabla u_h(x+y)-\nabla u_h(x-\eta)\big)\cdot(\eta+y)\\
&+\big(\nabla u_h(x-y)-\nabla u_h(x-\eta)\big)\cdot(\eta-y)\\
&-\big(\nabla u_h(x+y)-\nabla u_h(x-y)\big)\cdot y+2\big(\nabla u_h(x-\eta)-\nabla u_h(x)\big)\cdot\eta.
\end{aligned}$$

We plug this identity in the definition of I_2 and we find that

$$\begin{aligned}
I_2\leq&\frac{4}{c}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\hat{y}^\perp}\rho(|y|)\pi(|\eta|)\left|\big(\nabla u_h(x+hy)-\nabla u_h(x-h\eta)\big)\cdot(\eta+y)\right|^2 d\mathcal{H}^{d-1}(\eta)dydx\\
&+\frac{4}{c}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\hat{y}^\perp}\rho(|y|)\pi(|\eta|)\left|\big(\nabla u_h(x-hy)-\nabla u_h(x-h\eta)\big)\cdot(\eta-y)\right|^2 d\mathcal{H}^{d-1}(\eta)dydx\\
&+\frac{8}{c}\|\pi\|_{L^1(e_d^\perp)}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\rho(|y|)\,|y|^2\left|\big(\nabla u_h(x+hy)-\nabla u_h(x)\big)\cdot\hat{y}\right|^2 dydx\\
&+\frac{16}{c}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\hat{y}^\perp}\rho(|y|)\pi(|\eta|)\left|\big(\nabla u_h(x+h\eta)-\nabla u_h(x)\big)\cdot\eta\right|^2 d\mathcal{H}^{d-1}(\eta)dydx.
\end{aligned}$$

We estimate separately each of the contributions on the right-hand side.

Let us set $\mathbb{S}^{d-1}_+ \coloneqq \{\, e\in\mathbb{S}^{d-1} : e\cdot e_d>0\,\}$ and $\mathbb{S}^{d-1}_- \coloneqq \{\, e\in\mathbb{S}^{d-1} : e\cdot e_d<0\,\}$. Hereafter, we denote by L any strictly positive constant depending only on d, σ_d, r_1, and on the norms of ρ and π.

Taking advantage of the Coarea Formula, we rewrite the first addendum as follows:

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\hat{y}^\perp}\rho(|y|)\pi(|\eta|)\left|\big(\nabla u_h(x+hy)-\nabla u_h(x-h\eta)\big)\cdot(\eta+y)\right|^2 d\mathcal{H}^{d-1}(\eta)dydx\\
=&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\hat{y}^\perp}\rho(|y|)\pi(|\eta|)\left|\big(\nabla u_h(x+h(\eta+y))-\nabla u_h(x)\big)\cdot(\eta+y)\right|^2 d\mathcal{H}^{d-1}(\eta)dxdy\\
=&\int_{\mathbb{S}^{d-1}_+}\int_{\mathbb{R}^d}\int_{\mathbb{R}}\int_{e^\perp} r^{d-1}\rho(r)\pi(|\eta|)\left|\big(\nabla u_h(x+h(\eta+re))-\nabla u_h(x)\big)\cdot(\eta+re)\right|^2 d\mathcal{H}^{d-1}(\eta)drdxd\mathcal{H}^{d-1}(e)\\
=&\int_{\mathbb{S}^{d-1}_+}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|y|^2\,|y\cdot e|^{d-1}\rho(|y\cdot e|)\pi\big(\,|(\mathrm{Id}-e\otimes e)y|\,\big)\left|\big(\nabla u_h(x+hy)-\nabla u_h(x)\big)\cdot\hat{y}\right|^2 dydxd\mathcal{H}^{d-1}(e).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\hat{y}^\perp}\rho(|y|)\pi(|\eta|)\left|\big(\nabla u_h(x-hy)-\nabla u_h(x-h\eta)\big)\cdot(\eta-y)\right|^2 d\mathcal{H}^{d-1}(\eta)dydx\\
=&\int_{\mathbb{S}^{d-1}_-}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|y|^2\,|y\cdot e|^{d-1}\rho(|y\cdot e|)\pi\big(\,|(\mathrm{Id}-e\otimes e)y|\,\big)\left|\big(\nabla u_h(x+hy)-\nabla u_h(x)\big)\cdot\hat{y}\right|^2 dydxd\mathcal{H}^{d-1}(e),
\end{aligned}$$

and thus

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\hat{y}^\perp} \rho(|y|)\pi(|\eta|)\left|\big(\nabla u_h(x+hy)-\nabla u_h(x-h\eta)\big)\cdot(\eta+y)\right|^2 d\mathcal{H}^{d-1}(\eta)dydx\\
&\quad+\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\hat{y}^\perp} \rho(|y|)\pi(|\eta|)\left|\big(\nabla u_h(x-hy)-\nabla u_h(x-h\eta)\big)\cdot(\eta-y)\right|^2 d\mathcal{H}^{d-1}(\eta)dydx\\
&=\int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |y\cdot e|^{d-1}|y|^2\rho(|y\cdot e|)\pi\big(\,|(\mathrm{Id}-e\otimes e)y|\,\big)\left|\big(\nabla u_h(x+hy)-\nabla u_h(x)\big)\cdot\hat{y}\right|^2 dydxd\mathcal{H}^{d-1}(e).
\end{aligned}$$

Let us recall that $\rho(r)=\eta(r)=0$ if $r\notin[0,2^{-1/2}\sigma_d r_1)$, whence, for any $e\in\mathbb{S}^{d-1}$, the product $\rho(|y\cdot e|)\pi\big(\,|(\mathrm{Id}-e\otimes e)y|\,\big)$ vanishes outside the cylinder

$$C_e \coloneqq \{\, y\in\mathbb{R}^d : |y\cdot e|\,,|(\mathrm{Id}-e\otimes e)y|\in[0,2^{-1/2}\sigma_d r_1)\,\}\subset B(0,\sigma_d r_1).$$

We therefore see that the last multiple integral equals

$$\begin{aligned}
&\int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}^d}\int_{C_e} |y\cdot e|^{d-1}|y|^2\rho(|y\cdot e|)\pi\big(\,|(\mathrm{Id}-e\otimes e)y|\,\big)\left|\big(\nabla u_h(x+hy)-\nabla u_h(x)\big)\cdot\hat{y}\right|^2 dydxd\mathcal{H}^{d-1}(e)\\
&\qquad\le L\int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}^d}\int_{C_e}\tilde{K}(y)\left|\big(\nabla u_h(x+hy)-\nabla u_h(x)\big)\cdot\hat{y}\right|^2 dydxd\mathcal{H}^{d-1}(e)\\
&\qquad\le\frac{LM}{\gamma}h^2.
\end{aligned}$$

We then obtain

$$\begin{aligned}
&\frac{4}{c}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\hat{y}^\perp} \rho(|y|)\pi(|\eta|)\left|\big(\nabla u_h(x+hy)-\nabla u_h(x-h\eta)\big)\cdot(\eta+y)\right|^2 d\mathcal{H}^{d-1}(\eta)dydx\\
&+\frac{4}{c}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\hat{y}^\perp} \rho(|y|)\pi(|\eta|)\left|\big(\nabla u_h(x-hy)-\nabla u_h(x-h\eta)\big)\cdot(\eta-y)\right|^2 d\mathcal{H}^{d-1}(\eta)dydx\\
&\qquad\le\frac{LM}{\gamma}h^2.
\end{aligned}\tag{45}$$

Next, we have

$$\frac{8}{c}\|\pi\|_{L^1(e_d^\perp)}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\rho(|y|)\,|y|^2\left|\big(\nabla u_h(x+hy)-\nabla u_h(x)\big)\cdot\hat{y}\right|^2 dydx\le\frac{LM}{\gamma}h^2,\tag{46}$$

$$\frac{16}{c}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\hat{y}^\perp}\rho(|y|)\pi(|\eta|)\left|\big(\nabla u_h(x+h\eta)-\nabla u_h(x)\big)\cdot\eta\right|^2 d\mathcal{H}^{d-1}(\eta)dydx\le\frac{LM}{\gamma}h^2.\tag{47}$$

The bound in (46) may be deduced as the one in (43), so, to establish (44), we are only left to prove (47). To this aim, let $\psi\in C_c^\infty(\mathbb{R}^d\times\mathbb{R}^d)$ be a test function. By a standard argument and Fubini's Theorem we have that

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\hat{y}^\perp}\rho(|y|)\pi(|\eta|)\psi(y,\eta)d\mathcal{H}^{d-1}(\eta)dy\\
&\qquad=\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|y|}{2\varepsilon}\chi_{\{\,t<\varepsilon\,\}}(|\eta\cdot y|)\rho(|y|)\pi(|\eta|)\psi(y,\eta)d\eta dy\\
&\qquad=\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}^d}\frac{\pi(|\eta|)}{|\eta|}\left(\int_{\mathbb{R}^d}\frac{|\eta|}{2\varepsilon}\chi_{\{\,t<\varepsilon\,\}}(|\eta\cdot y|)\rho(|y|)\,|y|\,\psi(y,\eta)dy\right)d\eta\\
&\qquad=\int_{\mathbb{R}^d}\int_{\hat{\eta}^\perp}\frac{\pi(|\eta|)}{|\eta|}\rho(|y|)\,|y|\,\psi(y,\eta)d\mathcal{H}^{d-1}(y)d\eta
\end{aligned}$$

(recall that we assume $\lim_{r\searrow 0}\pi(r)/r$ to be finite). It follows that

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\hat{y}^\perp}\rho(|y|)\pi(|\eta|)\left|\big(\nabla u_h(x+h\eta)-\nabla u_h(x)\big)\cdot\eta\right|^2 d\mathcal{H}^{d-1}(\eta)dydx\\
&\quad=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\hat{\eta}^\perp}\frac{\pi(|\eta|)}{|\eta|}\rho(|y|)\,|y|\left|\big(\nabla u_h(x+h\eta)-\nabla u_h(x)\big)\cdot\eta\right|^2 d\mathcal{H}^{d-1}(y)d\eta dx\\
&\quad\leq L\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\tilde{K}(\eta)\left|\big(\nabla u_h(x+h\eta)-\nabla u_h(x)\big)\cdot\eta\right|^2 d\eta dx.
\end{aligned}$$

In view of the bound on the energy, we retrieve (47).

The proof is now concluded, because from (42), (43), and (44) we obtain

$$\int_{\mathbb{R}^d}|\nabla v_h(x)-\nabla u_h(x)|^2\,dx\leq\frac{LM}{\gamma}h^2,$$

as desired. □

Remark 5. *The choice $u_h=u$ in Lemma 6 provides a criterion for a function in $H^1(\mathbb{R}^d)$ to belong to $H^2(\mathbb{R}^d)$. Namely, when Ω, K, and f fulfil the assumptions of the current section and f'' is bounded, a function $u\in X$ is in $H^2(\mathbb{R}^d)$ if and only if $\mathcal{E}_h(u)\leq M$ for some $M>0$ and for all h's small enough. One implication is a byproduct of Lemma 6, while the other follows by exploiting the slicing formula and Remark 2: indeed, if $f''\leq c$ one finds*

$$\mathcal{E}_h(u)\leq\frac{c}{2}\left(\int_{\mathbb{R}^d}K(z)\,|z|^2\,dz\right)\int_{\mathbb{R}^d}\left|\nabla^2 u(x)\right|^2dx.$$

We can now accomplish the proof of Theorem 2.

Proof of Theorem 2. Lemma 6 provides the compactness result of statement (1) in Theorem 2.

Turning to the lower limit inequality, for any $u\in X$ and for any family $\{\,u_h\,\}\subset X$ that converges to u in $H^1(\mathbb{R}^d)$, we may focus on the situation when there exists $M\geq 0$ such that $\mathcal{E}_h(u_h)\leq M$ for all $h>0$. In view of Lemma 6, we have that $u\in H^2(\mathbb{R}^d)$, thus statement (2) follows by Proposition 3.

For what concerns the upper limit inequality, we reason as in the 1-dimensional case (see the proof of Proposition 1). In order to adapt the argument, we observe that, if $u\in X\cap H^2(\mathbb{R}^d)$, by mollification, we can construct a sequence $\{\,u_\ell\,\}\subset X$ of smooth functions that tend to u in $H^2(\mathbb{R}^d)$ and satisfy $\lim_{\ell\nearrow+\infty}\mathcal{E}_0(u_\ell)=\mathcal{E}_0(u)$, provided that f'' is bounded or $u\in X\cap H^2(\mathbb{R}^d)\cap W^{1,\infty}(\mathbb{R}^d)$. Indeed, when one of these assumptions holds, there exists $c>0$ such that $f''(|\nabla u_\ell(x)\cdot\hat{z}|)\leq c$ for a.e. x and all z, and Lebesgue's Theorem applies. Then, we can establish the upper limit inequality by combining the approximation by smooth functions and Proposition 3. □

We conclude with a couple of remarks.

Remark 6. *As in Remark 1, we see that the* Γ*-limit of*

$$h\mathcal{E}_h(u) = \frac{\mathcal{F}_0(u) - \mathcal{F}_h(u)}{h}$$

in $H^1(\mathbb{R}^d)$ *is* 0*. The same* Γ*-limit is found if one considers the* $L^2(\mathbb{R}^d)$*-topology on* X*, because* $h\mathcal{E}_h(u) \geq 0$ *for all* $u \in X$ *and Proposition 3 provides a constant recovery sequence for smooth functions.*

Remark 7. *Statements* (2)*, (3a), and (3b) in Theorem 2, that is, the* Γ*-convergence result, are not affected if we replace* X *with* $H^1(\mathbb{R}^d)$*; the proof remains essentially the same. On the other hand, if we substitute* Ω *with* $\mathbb{R}^d$*, the compactness provided by statement* (1) *of Theorem 2 may fail.*

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