

Homogenization of cohesive fracture in masonry structures

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Abstract

We derive a homogenized mechanical model of a masonry-type structure constituted by a periodic assemblage of blocks with interposed mortar joints. The energy functionals in the model under investigation consist in (i) a linear elastic contribution within the blocks, (ii) a Barenblatt's cohesive contribution at contact surfaces between blocks and (iii) a suitable unilateral condition on the strain across contact surfaces, and are governed by a small parameter representing the typical ratio between the length of the blocks and the dimension of the structure. Using the terminology of Γ -convergence and within the functional setting supplied by the functions of bounded deformation, we analyze the asymptotic behavior of such energy functionals when the parameter tends to zero, and derive a simple homogenization formula for the limit energy. Furthermore, we highlight the main mathematical and mechanical properties of the homogenized energy, including its non-standard growth conditions under tension or compression. The key point in the limit process is the definition of macroscopic tensile and compressive stresses, which are determined by the unilateral conditions on contact surfaces and the geometry of the blocks.

1 Introduction

In this work we examine the mechanical modeling of a class of materials referred to as *masonry-like materials*. Such materials are characterized by different behaviors in tension or compression, possibly undergoing fracture. A noteworthy instance in that class is represented by dry masonry, typical of historical buildings, consisting in an assemblage of blocks that are in unilateral contact with each other. The fact that the blocks can be detached at no energy expense results in a fully degenerate overall behavior, with vanishing resistance of the material under tension and the possibility of an unresisted cracking process. Alternative to dry masonry, in building practice it is widespread to interpose mortar joints between bricks. In that case, blocks are still in contact with each other, but a (small) resistance to their detachment is offered by the mortar, which is responsible for cohesive tractions resisting the cracking process.

A first general mathematical treatment of the dry-masonry problem goes back to the works by Giaquinta and Giusti [27] and Anzellotti [5]. In those works, in the framework of infinitesimal strain theory and under the assumption of a linear elastic behavior, energy of deformations of the form

$$G(u) = \int_{\Omega} g(P_{K^{\perp}} \mathcal{E}u) \, dx$$

are considered, where $\Omega \subset \mathbb{R}^n$ is a reference configuration domain, g is a linear elastic energy density and $\mathcal{E}u$ is the strain associated to the displacement field u . The degeneracy of the material under tension is modeled by the introduction of a *cone K of tensile strains*, such that only the projection $P_{K^{\perp}} \mathcal{E}u$ of the strain onto the dual *cone K^{\perp} of compressive strains* determines energy storage. This degenerate behavior implies that the energy of deformation G is not coercive on any reasonable normed space of admissible displacements. However, by replacing the energy of deformation with the total potential energy (i.e. considering also the contribution of the work of the external forces) and introducing suitable safety conditions (indeed prescribing the external forces to be compressive), coerciveness can be recovered in the space $BD(\Omega)$ of *functions of bounded deformation* on Ω ; that is, the space of functions whose distributional strain $\mathcal{E}u$ is a measure [42]. Following the direct method of the Calculus of Variations, existence theorems are derived by showing that the total potential energy fulfills a lower-semicontinuity property on $BD(\Omega)$ as well. A mechanical insight in the subject can be found e.g. in [22, 33].

As an alternative to that *macroscopic* description of a masonry material, a *microscopic* approach has been adopted in [11]. In that approach, the assemblage of blocks interacting through their contact surfaces is regarded as the periodic microstructure of a material whose macroscopic properties are determined by *homogenization*; i.e., by analyzing its asymptotic behavior as the characteristic size of the microstructure vanishes. In mathematical terms, a $(n - 1)$ -dimensional periodic closed set B is introduced, to be scaled by the characteristic size ε of the microstructure, and the reference configuration Ω is subdivided into a periodic collection of disconnected sets $\Omega \setminus \varepsilon B$. A space of admissible displacements is then considered as a prescription for the block kinematics:

$$\mathcal{U}_\varepsilon(\Omega) = \{u \in SBD(\Omega) : J_u \subseteq \varepsilon B, (u^+ - u^-) \odot \nu_u \in K_0 \text{ } \mathcal{H}^{n-1}\text{-a.e}\}, \quad (1.1)$$

where $SBD(\Omega)$ is the space of *special functions of bounded deformation* on Ω (e.g., see [2]), J_u is the jump set of u , u^\pm are the traces of u on both sides of J_u and ν_u is the unit normal to J_u . This analytical description translates the fact that the admissible displacements are functions whose jumps localize at the interfaces of the blocks and fulfill a unilateral condition prescribed by the cone of matrices K_0 . Possible choices for that cone include $K_0 = \{a \odot b : b = \lambda a, \lambda \geq 0\}$ or $K_0 = \{a \odot b : (a, b) \geq 0\}$, respectively implying an infinitesimal no-sliding condition (i.e., infinite friction assumption) and a detachment condition on the opening of a crack (i.e., vanishing friction assumption). Upon noticing that admissible displacements u are such that $u \in H^1(\Omega \setminus \varepsilon B)$, and hence the strain Eu reduces to its absolutely continuous part $\mathcal{E}u$ with respect to the Lebesgue measure on $\Omega \setminus \varepsilon B$, the following microscopic linearly elastic energy is considered in [11]:

$$G_\varepsilon(u) = \frac{1}{2} \int_{\Omega \setminus \varepsilon B} (A\mathcal{E}u, \mathcal{E}u) dx, \quad u \in \mathcal{U}_\varepsilon(\Omega),$$

with A as a fixed fourth-order tensor. A *homogenized energy density* g_{hom} can be defined from A and B by a homogenization formula optimizing over periodic perturbations of a given strain. The function g_{hom} may vanish on a set of matrices, which we denote as K_{hom} , the cone of the *homogenized tensile strains*. Under the assumption that $g_{\text{hom}}(\cdot) = g_{\text{hom}}(P_{K_{\text{hom}}}^\perp \cdot)$ (which is satisfied in usual examples), it is shown that this microscopic approach leads to a masonry-type energy; more precisely, that the family (G_ε) Γ -converges as $\varepsilon \rightarrow 0^+$ to a homog-

enized energy G_{hom} on $BD(\Omega)$ of the form

$$G_{\text{hom}}(u) = \int_{\Omega} g_{\text{hom}}\left(P_{K_{\text{hom}}^{\perp}} \mathcal{E}u\right) dx, \quad u \in \mathcal{U}_{\text{hom}}(\Omega). \quad (1.2)$$

The space of *homogenized admissible displacements* $\mathcal{U}_{\text{hom}}(\Omega)$ is the set of those functions $u \in BD(\Omega)$ such that the projection $P_{K_{\text{hom}}^{\perp}} E^{\text{s}}u$, with $E^{\text{s}}u$ as the singular part of the strain measure Eu and K_{hom}^{\perp} as the cone orthogonal to K_{hom} , vanishes.

As regards the problem of masonry with mortar joints between bricks, many contributions from the mechanical literature could be mentioned, including: macromechanical continuum models based on phenomenological constitutive laws (e.g., [35, 28]), micromechanical models (e.g., [26, 34]), and homogenized multiscale models (e.g., [32, 31, 1]). However, a rigorous mathematical treatment seems to be missing in the framework of homogenization theory for periodic masonry with mortar joints on the space of functions of bounded deformation, i.e. explicitly considering fractures as stemming from discontinuous displacement fields.

To develop such a homogenization theory, the approach discussed in [11] can be taken as departing point. In particular, accounting for the mortar requires to include some surface energy contribution on the discontinuity set εB at the microscopic level. Here we focus on Barenblatt's model of cohesive fracture, prescribing an isotropic surface energy, positively homogeneous of degree one [9]. Accordingly, we are led to considering the family of functionals (F_{ε}) given by

$$F_{\varepsilon}(u) = \frac{1}{2} \int_{\Omega \setminus \varepsilon B} (A\mathcal{E}u, \mathcal{E}u) dx + \int_{J_u \cap \varepsilon B} |u^+ - u^-| d\mathcal{H}^{n-1}, \quad u \in \mathcal{U}_{\varepsilon}(\Omega),$$

with the space of admissible displacements $\mathcal{U}_{\varepsilon}(\Omega)$ given in (1.1). The main result of the present work consists in proving that the functionals (F_{ε}) Γ -converge as $\varepsilon \rightarrow 0^+$ to a homogenized energy F_{hom} on $BD(\Omega)$ given by

$$F_{\text{hom}}(u) = \int_{\Omega} f_{\text{hom}}(\mathcal{E}u) dx + \int_{\Omega} f_{\text{hom}}^{\infty} \left(\frac{dE^{\text{s}}u}{d|E^{\text{s}}u|} \right) d|E^{\text{s}}u| dx, \quad u \in \mathcal{U}_{\text{hom}}(\Omega), \quad (1.3)$$

where f_{hom} is a *cohesive homogenized energy density*, f_{hom}^{∞} is the recession function of f_{hom} , $E^{\text{s}}u$ denotes the singular part of the strain Eu with respect to the Lebesgue measure and the space of homogenized admissible displacements

$\mathcal{U}_{\text{hom}}(\Omega)$ is the same as that appearing in (1.2) in the case of dry masonry. Note that in this case no projection structure can be obtained in the limit energy. This allows to remove the assumptions on the cohesive homogenized energy density f_{hom} required for the homogenization result in [11]. We give an explicit homogenization formula for the energy density f_{hom} , which depends on both the microgeometry B and the cone K_0 . It is noteworthy to observe that such energy density satisfies a non-standard growth condition

$$c_1 (|\xi| - 1) \leq f_{\text{hom}}(\xi) \leq c_2 |\xi|^2,$$

so that the present homogenization theorem does not fit in the framework of any of the general integral representation results as in [17, 20]. As for the growth condition from below, that condition cannot be improved. In fact, we prove that the homogenized energy density f_{hom} has sublinear growth over the cone K_{hom} of homogenized tensile stresses as a natural mechanical consequence of the surface Barenblatt's energy contribution at microscopic level. Furthermore, the growth condition from below allows the homogenized energy F_{hom} to be regarded as an instance of demi-coercive functionals, as introduced in [6], thus implying an existence theorem for the related minimization problem.

Energies of the form (1.3) have been broadly investigated as the relaxation in the L^1 -topology of functionals defined on $SBD(\Omega)$ and involving interaction between bulk and surfaces energies (e.g., see [12, 15, 16]). In particular, connections can be found with the result discussed in [16], where the relaxation of elastic energies with unilateral constraints on the strains is considered. Some contact points can also be recognized with homogenization results discussed in [25, 4, 39]. Specifically, in [25], the problem of periodic homogenization in perfect elasto-plasticity is discussed. In [4] the homogenization of integral functionals involving energies concentrated on periodic multidimensional structures and defined on Sobolev spaces with respect to measures is considered. In [39], the homogenization of many-body structures undergoing large displacements and obeying a non-interpenetration constraint is dealt with. As a major difference, the large-displacement framework calls for a functional setting in the space BV (instead of BD) and the non-interpenetration constraint translates into a global condition (instead of a local condition governed by the cone K_0). To the best of the authors' knowledge, functionals of the form (1.3) are new in the context of homogenization theory for masonry structures on BD .

The present work is organized as follows. In Section 2 we set the main

notation and collect some definitions and well-known results needed in the subsequent developments. We discuss the problem statement and the main result in Section 3. Section 4 is devoted to an illustration of the main result in a one-dimensional setting. That discussion is instrumental as it highlights some properties of the homogenized energy density which are explored in the general case in Section 5. We give the proof of the main result in the technical Section 6. Conclusions and perspectives are outlined in Section 7.

2 Notation

We denote by (\cdot, \cdot) and $|\cdot|$ the scalar product and the induced norm in \mathbb{R}^n , for any $n \geq 1$. Upon identification with the space \mathbb{R}^{mn} , the same notation is also adopted for the vector space $\mathbb{M}^{m \times n}$ of $m \times n$ real matrices. The symbol $\mathbb{M}_{\text{sym}}^{n \times n}$ is used for the subspace of symmetric matrices in $\mathbb{M}^{n \times n}$. In particular, for $\xi \in \mathbb{M}^{n \times n}$, we denote by ξ^s its symmetric part. Given $\xi \in \mathbb{M}^{n \times n}$, we write u_ξ for the linear function $u_\xi(x) = \xi x$. Given $a, b \in \mathbb{R}^n$, the tensor product $a \otimes b$ is the $n \times n$ matrix with entries $a_i b_j$ for $i, j = 1, \dots, n$. The symmetric tensor product is defined as $a \odot b = (a \otimes b + b \otimes a) / 2$. Note that $|a \odot b|^2 = (|a|^2 |b|^2 + (a, b)^2) / 2$.

The open ball in \mathbb{R}^n with radius ρ and center x is denoted by $B_\rho(x)$. If Ω is an open set in \mathbb{R}^n , we denote by $\mathcal{A}(\Omega)$ the class of open subsets of Ω and by $\mathcal{A}_0(\Omega)$ the class of bounded open subsets of Ω . Moreover, $L^2(\Omega; \mathbb{R}^m)$ and $H^1(\Omega; \mathbb{R}^m)$ stand for the usual Lebesgue and Sobolev spaces of \mathbb{R}^m -valued functions. If $m = 1$, we simply write $L^2(\Omega)$ and $H^1(\Omega)$. The symbol $C_0^k(\Omega; \mathbb{R}^m)$, $0 \leq k \leq \infty$ refers to the space of compactly supported smooth functions. We denote by $\mathcal{M}(\Omega; \mathbb{M}^{n \times n})$ the space of $\mathbb{M}^{n \times n}$ -valued Borel measures and by \mathcal{H}^{n-1} the $(n-1)$ -dimensional Hausdorff measure in \mathbb{R}^n .

For $f : \mathbb{R}^m \rightarrow [0, +\infty)$ a convex function and $\mu \in \mathcal{M}(A; \mathbb{M}^{n \times n})$ a measure, we use the notation $\int_A f(\mu) = \int_A f(h) dx + \int_A f^\infty\left(\frac{d\mu^s}{d|\mu^s|}\right) d|\mu^s|$, where $\mu = h\mathcal{L}^n + \mu^s$ is the Lebesgue decomposition of μ and f^∞ is the recession function of f (e.g., see [37]) defined by

$$f^\infty(\xi) = \lim_{t \rightarrow +\infty} \frac{f(t\xi)}{t}.$$

2.1 The space BD

The space $BD(\Omega)$ of *functions of bounded deformation* is the space of the functions $u \in L^1(\Omega; \mathbb{R}^n)$ whose symmetric distributional gradient

$$Eu = \frac{1}{2} (Du + Du^T)$$

is a measure in $\mathcal{M}(\Omega, \mathbb{M}_{\text{sym}}^{n \times n})$. For any $u \in BD(\Omega)$, we consider the Radon-Nikodym decomposition of the strain

$$Eu = \mathcal{E}u \mathcal{L}^n + E^s u,$$

where $\mathcal{E}u$ is the density of the absolutely continuous part of Eu with respect to \mathcal{L}^n and $E^s u$ is the singular part of Eu with respect to \mathcal{L}^n . We further decompose $E^s u$ into a jump part $E^j u = Eu \llcorner J_u$ and a Cantor part $E^c u = E^s u \llcorner (\Omega \setminus J_u)$. Specifically, it results that

$$E^j u = (u^+ - u^-) \odot \nu_u \mathcal{H}^{n-1} \llcorner J_u$$

and $|E^c u|(B) = 0$ on any Borel subset $B \subseteq \Omega$ which is σ -finite with respect to \mathcal{H}^{n-1} . The space $SBD(\Omega)$ of *special functions of bounded deformation* in Ω is the space of the functions $u \in BD(\Omega)$ such that $E^c u$ is the null measure; i.e., satisfying

$$Eu = \mathcal{E}u \mathcal{L}^n + (u^+ - u^-) \odot \nu_u \mathcal{H}^{n-1} \llcorner J_u.$$

A sequence (u_h) in $BD(\Omega)$ weakly converges to a function $u \in BD(\Omega)$ if $u_h \rightarrow u$ in $L^1(\Omega; \mathbb{R}^n)$ and $(|Eu_h|(\Omega))$ is bounded.

2.2 Γ -convergence

Let (X, d) be a metric space and (F_h) be a sequence of functionals from X into $\overline{\mathbb{R}}$. We say that (F_h) Γ -converges to F in X with respect to the topology induced by d if the following conditions are satisfied:

- (i) for every $x \in X$ and for every sequence (x_h) converging to x in X we have
$$F(x) \leq \liminf_h F_h(x_h);$$
- (ii) for every $x \in X$ there exists a sequence (x_h) converging to x in X such that
$$F(x) = \lim_h F_h(x_h).$$

Under appropriate coercivity conditions, Γ -convergence guarantees the convergence of the minimum values of the functionals F_h to the minimum value of

their Γ -limit.

3 Setting of the problem and main result

Let Ω be a bounded open set of \mathbb{R}^n . For $Y = (0, 1)^n$ the unit cube of \mathbb{R}^n , we consider a closed rectifiable Y -periodic $(n - 1)$ -dimensional subset of \mathbb{R}^n , i.e. such that $B + k = B$ for all $k \in \mathbb{Z}^n$. Accordingly, Ω is partitioned into a periodic collection of disconnected sets (Figure 1(a)).

We fix a positive definite symmetric linear operator $A : \mathbb{M}_{\text{sym}}^{n \times n} \rightarrow \mathbb{M}_{\text{sym}}^{n \times n}$, the scalar product on $\mathbb{M}_{\text{sym}}^{n \times n}$ given by $\langle \xi, \eta \rangle = (A\xi, \eta)$ and introduce the associated norm $\|\xi\| = \langle \xi, \xi \rangle^{1/2}$. Note that $\|\cdot\|$ is equivalent to $|\cdot|$ in $\mathbb{M}_{\text{sym}}^{n \times n}$; that is,

$$\sqrt{\alpha} |\xi| \leq \|\xi\| \leq M |\xi|, \quad (3.1)$$

for suitable constants $\alpha, M > 0$.

We assume that K_0 is a closed cone in $\mathbb{M}_{\text{sym}}^{n \times n}$ consisting of matrices of the form $a \odot b$, satisfying the following convexity assumption:

$$a \odot (b + c) \in K_0 \text{ whenever } a \odot b, a \odot c \in K_0. \quad (3.2)$$

For $\varepsilon > 0$, we define the functionals $F_\varepsilon : L^2(\Omega; \mathbb{R}^n) \rightarrow [0, +\infty]$ as

$$F_\varepsilon(u) = \begin{cases} \frac{1}{2} \int_{\Omega \setminus \varepsilon B} \|\mathcal{E}u\|^2 dx + \int_{J_u \cap \varepsilon B} |u^+ - u^-| d\mathcal{H}^{n-1}, & u \in \mathcal{U}_\varepsilon(\Omega), \\ +\infty, & \text{otherwise,} \end{cases} \quad (3.3)$$

where the set $\mathcal{U}_\varepsilon(\Omega)$ of admissible displacements is given by

$$\mathcal{U}_\varepsilon(\Omega) = \{u \in SBD(\Omega) : J_u \subseteq \varepsilon B, (u^+ - u^-) \odot \nu_u \in K_0 \mathcal{H}^{n-1}\text{-a.e.}\}, \quad (3.4)$$

i.e. consists of all special functions with bounded deformation whose jump set is contained in εB and such that the density of the singular part of the strain belongs to the cone K_0 (Figure 1(b)), which describes the admissible singular part of the strain.

Accordingly, over the set of admissible displacements, the material behaves as a linear elastic medium within brick regions $\Omega \setminus \varepsilon B$, and a surface energy over interfaces between adjacent bricks, following the Barenblatt's model of cohesive fracture [9], is considered. Possible fractures localize over such interfaces and

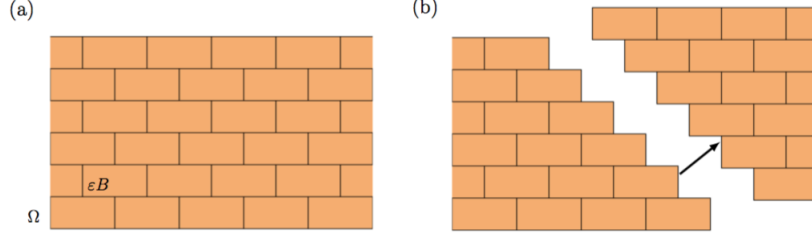


Figure 1: Setting of the problem: (a) reference configuration Ω is subdivided into a periodic collection of disconnected sets $\Omega \setminus \varepsilon B$ (blocks) by the periodic microstructure εB (interfaces) and (b) typical admissible displacement field, i.e. in $\mathcal{U}_\varepsilon(\Omega)$, yielding fracture.

produce displacement jumps obeying the unilateral constraint associated to K_0 .

Our aim is to study the asymptotic behavior of F_ε as $\varepsilon \rightarrow 0^+$ in the sense of Γ -convergence. To this end, we define the (candidate) *homogenized energy density* as given by the cell-problem

$$f_{\text{hom}}(\xi) = \inf \left\{ \frac{1}{2} \int_Y \|\mathcal{E}u\|^2 dx + \int_{J_u \cap Y} |u^+ - u^-| d\mathcal{H}^{n-1} : \right. \\ \left. u \in BD_{\text{loc}}(\mathbb{R}^n), J_u \subseteq B, (u^+ - u^-) \odot \nu_u \in K_0 \mathcal{H}^{n-1}\text{-a.e.}, u - u_\xi \text{ } Y\text{-periodic} \right\} \quad (3.5)$$

for all $\xi \in \mathbb{M}_{\text{sym}}^{n \times n}$. Moreover, we introduce the *homogenized cone* K_{hom} associated to the microgeometry B and the cone K_0 as the set

$$K_{\text{hom}} = \text{dom}(f_{\text{hom}}^\infty), \quad (3.6)$$

i.e. the domain of the recession function f_{hom}^∞ of the homogenized energy density f_{hom} . We denote by K_{hom}^\perp the cone orthogonal to K_{hom} ; i.e.,

$$K_{\text{hom}}^\perp = \left\{ \eta \in \mathbb{M}_{\text{sym}}^{n \times n} : (\xi, \eta) \leq 0 \text{ for all } \xi \in K_{\text{hom}} \right\},$$

and by $P_{K_{\text{hom}}}$ [resp., $P_{K_{\text{hom}}^\perp}$] the orthogonal projection on K_{hom} [resp., K_{hom}^\perp].

The homogenization theorem for functionals in (3.3)–(3.4) takes the following form.

Theorem 3.1. *Let Ω be a bounded open set of \mathbb{R}^n , let B be a closed rectifi-*

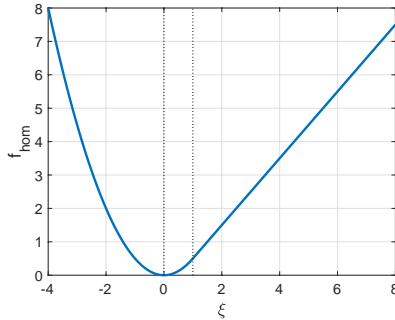


Figure 2: One-dimensional model problem: homogenized energy density.

able Y -periodic $(n - 1)$ -dimensional subset of \mathbb{R}^n . Moreover, let K_0 be a closed cone in $\mathbb{M}_{\text{sym}}^{n \times n}$ consisting of matrices of the form $a \odot b$ and satisfying condition (3.2). Then the family (F_ε) of functionals defined by (3.3)–(3.4) Γ -converges on $BD(\Omega) \cap L^2(\Omega; \mathbb{R}^n)$, with respect to the $L^2(\Omega; \mathbb{R}^n)$ -topology, to the functional $F_{\text{hom}} : L^2(\Omega; \mathbb{R}^n) \rightarrow [0, +\infty]$ given by

$$F_{\text{hom}}(u) = \begin{cases} \int_{\Omega} f_{\text{hom}}(\mathcal{E}u) \, dx + \int_{\Omega} f_{\text{hom}}^{\infty} \left(\frac{d\mathbf{E}^s u}{d|\mathbf{E}^s u|} \right) d|\mathbf{E}^s u| & \text{if } u \in \mathcal{U}_{\text{hom}}(\Omega) \\ +\infty & \text{otherwise,} \end{cases} \quad (3.7)$$

where f_{hom} is the homogenized energy density in (3.5) and the set $\mathcal{U}_{\text{hom}}(\Omega)$ of homogenized admissible displacements is

$$\mathcal{U}_{\text{hom}}(\Omega) = \left\{ u \in BD(\Omega) : P_{K_{\text{hom}}^{\perp}} \mathbf{E}^s u = 0 \right\}, \quad (3.8)$$

with K_{hom} the homogenized cone associated to B and K_0 defined by (3.6).

4 A one-dimensional model problem

In this section we consider an illustration of the homogenization result stated in Theorem 3.1 referring to a one-dimensional setting. In such a context, the only choices for the microgeometry set B and the cone K_0 involved in the unilateral constraint on the strain, are $B = \mathbb{Z}$ and $K_0 = [0, +\infty)$ respectively. Hence, the

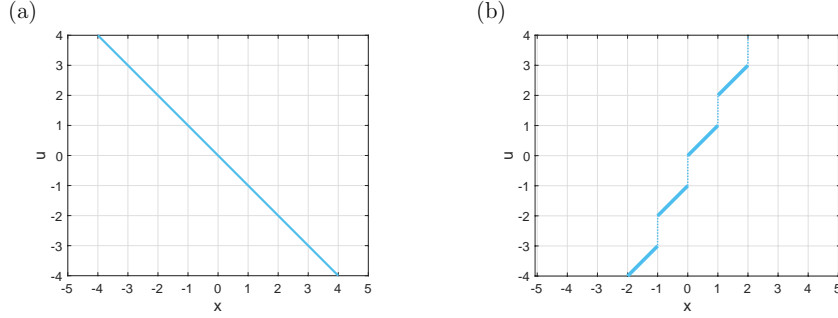


Figure 3: One-dimensional model problem: admissible displacement minimizer in the computation of the homogenized energy density. The strain is (a) $\xi \leq 1$ and (b) $\xi > 1$.

cell-problem (3.5), yielding the homogenized energy density f_{hom} , reduces to

$$f_{\text{hom}}(\xi) = \inf \left\{ \frac{1}{2} \int_{Y \setminus \mathbb{Z}} (u')^2 dx + \sum_{J_u \cap Y} (u^+ - u^-) : \right. \\ \left. u \in H^1(\mathbb{R} \setminus \mathbb{Z}), u^+ - u^- \geq 0, u - u_\xi \text{ } Y\text{-periodic} \right\} \quad (4.1)$$

for all $\xi \in \mathbb{R}$. Problem (4.1) turns out to be simplified when, instead of an admissible displacement function u , its periodic part $\tilde{u} = u - u_\xi$ is considered as unknown. Indeed, by an integration by parts, it follows that

$$f_{\text{hom}}(\xi) = \inf \left\{ \frac{1}{2} \int_{Y \setminus \mathbb{Z}} (\tilde{u}')^2 dx - (\xi - 1) \sum_{J_{\tilde{u}} \cap Y} (\tilde{u}^+ - \tilde{u}^-) + \frac{1}{2} \xi^2 : \right. \\ \left. \tilde{u} \in H^1(\mathbb{R} \setminus \mathbb{Z}), \tilde{u}^+ - \tilde{u}^- \geq 0, \tilde{u} \text{ } Y\text{-periodic} \right\}.$$

Assume first that $\xi \leq 1$. Since the second term of the energy is positive for any admissible function \tilde{u} , the minimum is attained in $H^1(\mathbb{R})$. In particular, a minimizer is given by $\tilde{u} = 0$, and we obtain the solution

$$f_{\text{hom}}(\xi) = \frac{1}{2} \xi^2, \quad u = u_\xi, \quad \text{for } \xi \leq 1.$$

Next we assume $\xi > 1$. In such a case, because of their opposite sign, there is a competition between the first and the second energy terms. We claim that the function $\tilde{u}(x) = -(\xi - 1)x$ on Y , extended by Y -periodicity over \mathbb{R} , is a minimizer. In fact, for that choice the energy is equal to $\xi - 1/2$ and coincides with the estimate from below in the next Proposition 5.1. Accordingly, we

derive the solution

$$f_{\text{hom}}(\xi) = \xi - \frac{1}{2}, \quad u(x) = x + \xi \lfloor x \rfloor, \quad \text{for } \xi > 1,$$

in which $\lfloor x \rfloor$ denotes the integer part of x . The homogenized energy density f_{hom} is depicted in Figure 2, whereas Figure 3 shows the admissible displacement minimizer in its computation for strains (a) $\xi \leq 1$ and (b) $\xi > 1$.

Upon observing that the recession function f_{hom}^∞ of the homogenized energy density f_{hom} is

$$f_{\text{hom}}^\infty(\xi) = \begin{cases} +\infty & \text{if } \xi < 0 \\ \xi & \text{if } \xi \geq 0, \end{cases}$$

from (3.6) the homogenized cone K_{hom} associated to the microgeometry B and the cone K_0 results to be

$$K_{\text{hom}} = [0, +\infty),$$

in particular coinciding with K_0 itself. Hence, for a bounded open set Ω of \mathbb{R} , the homogenized functional F_{hom} given in (3.7) is expressed as

$$F_{\text{hom}}(u) = \begin{cases} \int_{\Omega} f_{\text{hom}}(u') \, dx + E^s u(\Omega) & \text{if } u \in \mathcal{U}_{\text{hom}}(\Omega) \\ +\infty & \text{otherwise,} \end{cases}$$

with the space $\mathcal{U}_{\text{hom}}(\Omega)$ of homogenized admissible displacements given in (3.8)

$$\mathcal{U}_{\text{hom}}(\Omega) = \{u \in BV(\Omega; \mathbb{R}) : E^s u \geq 0\}.$$

It is worth noticing some properties enjoyed by the homogenized energy density f_{hom} in the present one-dimensional model problem:

- (i) it exhibits a mixed growth. In fact, it is quadratic under compression $\xi < 0$, where no cracking occurs, and linear under tension $\xi > 1$, where cracking occurs accompanied by expense of cohesive fracture energy. Interestingly, in the tensile region $0 < \xi < 1$ no fracture develops: in mechanical terms, that corresponds to the capability of the material to sustain moderate tensile stresses; in mathematical terms, the expense of fracture energy is not convenient compared to that of elastic energy under moderate tensile strain;
- (ii) it attains the upper bound value $\xi^2/2$ over K_0^\perp ;
- (iii) it is of linear growth over K_{hom} .

Such properties have general validity, as will be shown in the next section.

5 Some properties of the homogenized energy density

This section is devoted to some properties of the homogenized energy density f_{hom} defined in (3.5). We first show that it satisfies a non-standard growth condition. That fact descends from the structure of the cell problem (3.5), as it consists in the sum of a quadratic elastic energy and a linear interface energy.

Proposition 5.1. *There exist constants $c_1, c_2 > 0$ such that the homogenized energy density f_{hom} satisfies*

$$c_1 (|\xi| - 1) \leq f_{\text{hom}}(\xi) \leq c_2 |\xi|^2,$$

for all $\xi \in \mathbb{M}_{\text{sym}}^{n \times n}$.

Proof. Let ξ in $\mathbb{M}_{\text{sym}}^{n \times n}$ be fixed. The estimate from above is an immediate consequence of taking u_ξ as a test function in (3.5):

$$f_{\text{hom}}(\xi) \leq \frac{1}{2} \|\xi\|^2 \leq \frac{M^2}{2} |\xi|^2.$$

As for the estimate from below, we first enlarge the space of test functions by removing the kinematical constraints in (3.5):

$$f_{\text{hom}}(\xi) \geq \inf \left\{ \frac{1}{2} \int_Y \|\mathcal{E}u\|^2 dx + \int_{J_u \cap Y} |u^+ - u^-| d\mathcal{H}^{n-1} : u \in BD_{\text{loc}}(\mathbb{R}^n), u - u_\xi \text{ } Y\text{-periodic} \right\}.$$

Then, it suffices to observe that for a function u in $BD_{\text{loc}}(\mathbb{R}^n)$ such that $u - u_\xi$ is Y -periodic, we have

$$\begin{aligned} & \frac{1}{2} \int_Y \|\mathcal{E}u\|^2 dx + \int_{J_u \cap Y} |u^+ - u^-| d\mathcal{H}^{n-1} \\ & \geq c \left(\int_Y |\mathcal{E}u| dx + \int_{J_u \cap Y} |u^+ - u^-| d\mathcal{H}^{n-1} \right) - \frac{1}{2} \\ & \geq c \left| \int_Y \mathcal{E}u dx + \int_{J_u \cap Y} (u^+ - u^-) \odot \nu_u d\mathcal{H}^{n-1} \right| - \frac{1}{2} = c |\xi| - \frac{1}{2}, \end{aligned}$$

with $c = \min\{\sqrt{\alpha}, 1\}$, where α is given by (3.1). □ □

Note that both estimates in Proposition 5.1 cannot be improved. Concerning the one from above, the following result holds.

Proposition 5.2. *The homogenized energy density f_{hom} is such that*

$$f_{\text{hom}}(\xi) = \frac{1}{2} \|\xi\|^2,$$

for all $\xi \in K_0^\perp$, where $K_0^\perp = \{\eta \in \mathbb{M}_{\text{sym}}^{n \times n} : (\xi, \eta) \leq 0 \text{ for all } \xi \in K_0\}$ is the cone orthogonal to K_0 .

Proof. Let $\xi \in K_0^\perp$ be fixed. For $u \in BD_{\text{loc}}(\mathbb{R}^n)$ a test function in (3.5), we denote by $\tilde{u} = u - u_\xi$ its Y -periodic part. We derive

$$\begin{aligned} \frac{1}{2} \int_Y \|\mathcal{E}u\|^2 dx &= \frac{1}{2} \int_Y \|\mathcal{E}\tilde{u}\|^2 dx + \frac{1}{2} \|\xi\|^2 + \int_Y \langle \mathcal{E}\tilde{u}, \xi \rangle dx \\ &\geq \frac{1}{2} \|\xi\|^2 - \int_{J_u \cap Y} \langle (\tilde{u}^+ - \tilde{u}^-) \odot \nu_u, \xi \rangle d\mathcal{H}^{n-1} \geq \frac{1}{2} \|\xi\|^2, \end{aligned}$$

where the periodicity of \tilde{u} has been used in the integration by parts and the fact that $(\tilde{u}^+ - \tilde{u}^-) \odot \nu_u \in K_0$ \mathcal{H}^{n-1} -a.e., $\xi \in K_0^\perp$. □ □

On the other hand, by definition (3.6), the energy density f_{hom} is sublinear over the homogenized cone K_{hom} . We now supply an alternative characterization of such a cone. Heuristically, suppose that the minimum in (3.5) is attained. Then, one might expect that over K_{hom} the absolutely continuous part of the strain associated to the minimizer is almost everywhere vanishing. Hence, if there were no surface energy, over the cone K_{hom} we would have a vanishing energy density. Otherwise stated, K_{hom} might be characterized as the kernel of the energy density obtained by dropping off the surface energy contribution instead as the cone where the homogenized energy density f_{hom} is sublinear.

The argument above suggests to introduce an energy density g_{hom} by the following cell-problem formula

$$\begin{aligned} g_{\text{hom}}(\xi) &= \inf \left\{ \frac{1}{2} \int_Y \|\mathcal{E}u\|^2 dx : \right. \\ &\left. u \in BD_{\text{loc}}(\mathbb{R}^n), J_u \subseteq B, (u^+ - u^-) \odot \nu_u \in K_0 \text{ } \mathcal{H}^{n-1}\text{-a.e., } u - u_\xi \text{ } Y\text{-periodic} \right\}, \end{aligned} \tag{5.1}$$

and to define the associated kernel

$$H_{\text{hom}} = \{\xi \in \mathbb{M}_{\text{sym}}^{n \times n} : g_{\text{hom}}(\xi) = 0\}. \quad (5.2)$$

Note that the function g_{hom} and the cone H_{hom} have been subject of investigation in [11], in the context of a homogenization result for the purely degenerate case of dry-masonry structures.

The major step for proving that in fact H_{hom} coincides with K_{hom} is addressed in the following proposition, where it is shown that the homogenized energy density f_{hom} is sublinear over H_{hom} . To this end, we introduce the technical assumption that the orthogonal cone K^\perp of the convex hull K of K_0 has non-empty interior.

Proposition 5.3. *Let K denote the convex hull of K_0 and assume that the orthogonal cone K^\perp has non-empty interior. Then there exists a constant $c > 0$ such that the homogenized energy density f_{hom} satisfies*

$$f_{\text{hom}}(\xi) \leq c|\xi|$$

for all $\xi \in H_{\text{hom}}$.

Proof. The proof will be carried out through several steps.

Step 1. *If ξ is an interior point of K such that $\|\xi\| = 1$, then there exists a constant $c_0 > 0$ such that $c_0 \|\eta\| \leq \langle \eta, -\xi \rangle$ for all $\eta \in K^\perp$.*

Since ξ is an interior point of K , upon setting $c_0 = \min \{\langle \eta, -\xi \rangle : \eta \in K^\perp, \|\eta\| = 1\}$, we have $c_0 > 0$ and the claim follows.

Step 2. *There exists a constant $c > 0$ such that $\int_{Y \cap J_u} |u^+ - u^-| d\mathcal{H}^{n-1} \leq c \int_Y \|\mathcal{E}u\| dx$ for all functions $u \in BD_{\text{loc}}(\mathbb{R}^n)$ which are Y -periodic and fulfill the unilateral condition $(u^+ - u^-) \odot \nu_u \in K_0$ \mathcal{H}^{n-1} -a.e. on the jump set J_u .*

Let ξ be an interior point of K^\perp such that $\|\xi\| = 1$. Integrating by parts and exploiting the Y -periodicity of u , we have

$$\begin{aligned} \int_Y \langle P_K \mathcal{E}u, -\xi \rangle dx &= \int_Y \langle \mathcal{E}u, -\xi \rangle dx - \int_Y \langle P_{K^\perp} \mathcal{E}u, -\xi \rangle dx \\ &\leq - \int_{Y \cap J_u} \langle (u^+ - u^-) \odot \nu_u, -\xi \rangle d\mathcal{H}^{n-1} + \int_Y \|P_{K^\perp} \mathcal{E}u\| dx. \end{aligned}$$

By Step 1, since $K^{\perp\perp} = K$, we derive

$$c_0 \int_Y \|P_K \mathcal{E}u\| dx + \int_{Y \cap J_u} \langle (u^+ - u^-) \odot \nu_u, -\xi \rangle d\mathcal{H}^{n-1} \leq \int_Y \|P_{K^\perp} \mathcal{E}u\| dx. \quad (5.3)$$

Analogously, as by assumption $(u^+ - u^-) \odot \nu_u \in K_0 \subseteq K$, we also have

$$\langle (u^+ - u^-) \odot \nu_u, -\xi \rangle \geq c_0 \|(u^+ - u^-) \odot \nu_u\| \geq c_1 |u^+ - u^-|, \quad (5.4)$$

with $c_1 > 0$ a suitable constant. Hence, by (5.3) and (5.4) we obtain

$$c_0 \int_Y \|P_K \mathcal{E}u\| dx + c_1 \int_{Y \cap J_u} |u^+ - u^-| d\mathcal{H}^{n-1} \leq \int_Y \|P_{K^\perp} \mathcal{E}u\| dx.$$

By the continuity of the projection operator, we conclude

$$\int_{Y \cap J_u} |u^+ - u^-| d\mathcal{H}^{n-1} \leq c \int_Y \|P_{K^\perp} \mathcal{E}u\| dx \leq c \int_Y \|\mathcal{E}u\| dx,$$

with $c = 1/c_1$, which is the desired result.

Step 3. There exists a constant $c > 0$ such that $f_{\text{hom}}(\xi) \leq c|\xi|$ for all $\xi \in H_{\text{hom}}$.

Let $\xi \in H_{\text{hom}}$. By definition of the masonry homogenized energy density (5.1), we can consider a minimizing sequence (u_h) in $BD_{\text{loc}}(\mathbb{R}^n)$ such that $J_{u_h} \subseteq B$, $(u_h^+ - u_h^-) \odot \nu_{u_h} \in K_0$ \mathcal{H}^{n-1} -a.e., $\tilde{u}_h = u_h - u_\xi$ is Y -periodic and

$$0 = g_{\text{hom}}(\xi) \geq \frac{1}{2} \int_Y \|\mathcal{E}u_h\|^2 dx - \frac{1}{h^2}.$$

For $t > 0$, we set $u_h^t = tu_h$. Since u_h^t is a test function for the computation of the homogenized energy density f_{hom} , we get

$$\begin{aligned} f_{\text{hom}}(t\xi) &\leq \frac{1}{2} \int_Y \|\mathcal{E}u_h^t\|^2 dx + \int_{Y \cap J_{u_h^t}} |(u_h^t)^+ - (u_h^t)^-| d\mathcal{H}^{n-1} \\ &\leq \frac{t^2}{h^2} + t \int_{Y \cap J_{u_h}} |u_h^+ - u_h^-| d\mathcal{H}^{n-1}. \end{aligned} \quad (5.5)$$

Since \tilde{u}_h is Y -periodic and $(\tilde{u}_h^+ - \tilde{u}_h^-) \odot \nu_{u_h} \in K_0$ \mathcal{H}^{n-1} -a.e., by Step 2 we derive

$$\int_{Y \cap J_{u_h}} |u_h^+ - u_h^-| d\mathcal{H}^{n-1} \leq c \int_Y \|\mathcal{E}\tilde{u}_h\| dx \leq c \int_Y \|\mathcal{E}u_h\| dx + c \|\xi\| \leq \frac{c}{h} + c \|\xi\|.$$

Finally, from (5.5) we obtain

$$f_{\text{hom}}(t\xi) \leq \liminf_h \left(\frac{t^2}{h^2} + c \frac{t}{h} + c \|t\xi\| \right) = c \|t\xi\|$$

and by the arbitrariness of $t > 0$ the proof is concluded. \square \square

We are finally in position to conclude that H_{hom} coincides with K_{hom} .

Corollary 5.4. *Let K denote the convex hull of K_0 and assume that the orthogonal cone K^\perp has non-empty interior. Moreover, let K_{hom} and H_{hom} be the two cones respectively defined in (3.6) and (5.1)–(5.2). Then, $K_{\text{hom}} = H_{\text{hom}}$.*

Proof. By Proposition 5.3, we obtain that $H_{\text{hom}} \subseteq K_{\text{hom}}$. For the opposite inclusion, we first observe that g_{hom} is homogeneous of degree 2 outside of H_{hom} . Accordingly, since $f_{\text{hom}} \geq g_{\text{hom}}$, it follows that f_{hom} has growth of order 2 outside of H_{hom} . Hence $\mathbb{M}_{\text{sym}}^{n \times n} \setminus H_{\text{hom}} \subseteq \mathbb{M}_{\text{sym}}^{n \times n} \setminus K_{\text{hom}}$, and the proof is accomplished. \square \square

6 Proof of the main theorem

The proof of Theorem 3.1 will be obtained at the end of the section, as a consequence of the following propositions, which adapt to the present case the localization methods of Γ -convergence and homogenization. From now on, Ω will be a fixed bounded open subset of \mathbb{R}^n .

6.1 A compactness result

In order to prove a compactness result for the integral functionals (3.3), we resort to the localization method of Γ -convergence [20, 10]. Accordingly, we extend the definition of the functionals F_ε explicitly highlighting the dependence on the open set of definition. Such functionals, defined on $L^2(\Omega; \mathbb{R}^n) \times \mathcal{A}(\Omega)$ and

still denoted by F_ε , are then given by

$$F_\varepsilon(u, A) = \begin{cases} \frac{1}{2} \int_{A \setminus \varepsilon B} \|\mathcal{E}u\|^2 dx + \int_{A \cap J_u} |u^+ - u^-| d\mathcal{H}^{n-1}, & u \in \mathcal{U}_\varepsilon(A), \\ +\infty, & \text{otherwise,} \end{cases}$$

with the set of admissible displacements $\mathcal{U}_\varepsilon(A)$ given by (3.4) with A in place of Ω . The crucial result we prove is the following so-called *fundamental estimate* for the family (F_ε) .

Proposition 6.1. *For every $\eta > 0$ and for every $A', A'', B \in \mathcal{A}(\Omega)$ with $A' \subset\subset A''$, there exists a constant $M > 0$ with the following property: For every $\varepsilon > 0$ and for every $u \in L^2(A''; \mathbb{R}^n)$, $v \in L^2(B; \mathbb{R}^n)$ there exists a function $\varphi \in C_0^\infty(A')$ with $\varphi = 1$ in a neighbourhood of \bar{A}' and $0 \leq \varphi \leq 1$ such that*

$$F_\varepsilon(\varphi u + (1 - \varphi)v, A' \cup B) \leq (1 + \eta) [F_\varepsilon(u, A'') + F_\varepsilon(v, B)] + M \|u - v\|_{L^2(S)}^2,$$

where $S = (A'' \setminus A') \cap B$.

Proof. Let $\eta > 0$, A' , A'' and B be fixed as in the statement. Let A_1, \dots, A_k be open sets satisfying the property $A' \subset\subset A_1 \subset\subset A_2 \subset\subset \dots \subset\subset A_{k+1} = A''$. For every $i \in \{1, \dots, k\}$, let $\varphi_i \in C_0^\infty(A_{i+1})$ with $\varphi_i = 1$ on an open neighbourhood V_i of \bar{A}_i and $0 \leq \varphi_i \leq 1$. Let $\varepsilon > 0$ and consider $u \in L^2(A''; \mathbb{R}^n)$ and $v \in L^2(B; \mathbb{R}^n)$; in particular, we can assume that $u \in \mathcal{U}_\varepsilon(A'')$ and $v \in \mathcal{U}_\varepsilon(B)$, and arbitrarily extend them respectively outside A'' and B . We set

$$w_i = \varphi_i u + (1 - \varphi_i)v$$

on $A' \cup B$ for every $i \in \{1, \dots, k\}$. We note that $w_i \in SBD(A' \cup B) \cap L^2(A' \cup B; \mathbb{R}^n)$. Moreover, since \mathcal{H}^{n-1} -a.e. $J_{w_i} \subseteq J_u \cup J_v \subseteq \varepsilon B$, and

$$(w_i^+ - w_i^-) \odot \nu_{w_i} = \varphi_i (u^+ - u^-) \odot \nu_u + (1 - \varphi_i) (v^+ - v^-) \odot \nu_v$$

\mathcal{H}^{n-1} -a.e. on J_{w_i} , by the convexity assumption (3.2) on K_0 , we obtain that $(w_i^+ - w_i^-) \odot \nu_{w_i} \in K_0$. Hence $w_i \in \mathcal{U}_\varepsilon(A' \cup B)$ and we have

$$F_\varepsilon(w_i, A' \cup B) = F_\varepsilon(u, (A' \cup B) \cap V_i) + F_\varepsilon(v, B \setminus \text{spt } \varphi_i) + F_\varepsilon(w_i, B \cap (A_{i+1} \setminus \bar{A}_i)). \quad (6.1)$$

Let $T_i = B \cap (A_{i+1} \setminus \bar{A}_i)$. We estimate the last term:

$$\begin{aligned}
F_\varepsilon(w_i, T_i) &= \frac{1}{2} \int_{T_i \setminus \varepsilon B} \|\varphi_i \mathcal{E}u + (1 - \varphi_i) \mathcal{E}v + \mathcal{E}\varphi_i \odot (u - v)\|^2 dx \\
&\quad + \int_{T_i \cap (J_u \setminus J_v)} |u^+ - u^-| d\mathcal{H}^{n-1} + \int_{T_i \cap (J_v \setminus J_u)} |v^+ - v^-| d\mathcal{H}^{n-1} \\
&\quad + \int_{T_i \cap (J_u \cap J_v)} |\varphi_i (u^+ - u^-) + (1 - \varphi_i) (v^+ - v^-)| d\mathcal{H}^{n-1} \\
&\leq c \int_{T_i \setminus \varepsilon B} \left(\|\mathcal{E}u\|^2 + \|\mathcal{E}v\|^2 + |\mathcal{E}\varphi_i|^2 |u - v|^2 \right) dx \\
&\quad + \int_{T_i \cap J_u} |u^+ - u^-| d\mathcal{H}^{n-1} + \int_{T_i \cap J_v} |v^+ - v^-| d\mathcal{H}^{n-1} \\
&\leq c [F_\varepsilon(u, T_i) + F_\varepsilon(v, T_i)] + cM' \|u - v\|_{L^2(T_i; \mathbb{R}^n)}^2,
\end{aligned}$$

where we have set $M' = \max_{1 \leq i \leq k} \|\mathcal{E}\varphi_i\|_{L^2(T_i; \mathbb{R}^n)}^2$.

As the sets T_i are pairwise disjoint and $\bigcup_{i=1}^k T_i \subseteq B \cap (A'' \setminus A') = S$, there exists $i_0 \in \{1, \dots, k\}$ such that:

$$F_\varepsilon(w_{i_0}, T_{i_0}) \leq \frac{1}{k} \sum_{i=1}^k F_\varepsilon(w_i, T_i) \leq \frac{c}{k} [F_\varepsilon(u, A'') + F_\varepsilon(v, B)] + M \|u - v\|_{L^2(S; \mathbb{R}^n)}^2,$$

where $M = cM'/k$. From equation (6.1), it follows that:

$$F_\varepsilon(w_{i_0}, A' \cup B) = \left(1 + \frac{c}{k}\right) [F_\varepsilon(u, A'') + F_\varepsilon(v, B)] + M \|u - v\|_{L^2(S; \mathbb{R}^n)}^2,$$

and the proof is accomplished. \square \square

Next, we derive the following compactness result on the family (F_ε) .

Proposition 6.2. *Let (ε_h) be a sequence of positive numbers converging to 0. Then there exists a subsequence $(\varepsilon_{\sigma(h)})$ of (ε_h) and a functional $F : L^2(\Omega; \mathbb{R}^n) \times \mathcal{A}(\Omega) \rightarrow [0, +\infty]$ such that*

$$F(\cdot, A) = \Gamma\text{-}\lim_h F_{\varepsilon_{\sigma(h)}}(\cdot, A)$$

for every $A \in \mathcal{A}_0(\Omega)$ with respect to the $L^2(A; \mathbb{R}^n)$ -topology. Moreover, for every $u \in L^2(\Omega; \mathbb{R}^n)$, the set function $F(u, \cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Borel measure on Ω .

Proof. Using Proposition 6.1, the proof follows from the general localization method of Γ -convergence (for an illustrative description of the method, see Chapter 16 in [10]; a detailed proof of the method is given in [20], where, relying on the fundamental estimate, it is developed through Theorem 8.5, Theorem 14.23, Theorem 15.18 and Theorem 18.5). \square

\square

6.2 Integral representation on $H^1(\Omega; \mathbb{R}^n)$

On account of Proposition 6.2, we intend to identify the Γ -limit of a convergent sequence of functionals F_ε . Therefore, we assume that a sequence (ε_h) of positive numbers converging to 0 is given, such that for every $A \in \mathcal{A}_0(\Omega)$ the limit

$$F(\cdot, A) = \Gamma\text{-}\lim_h F_{\varepsilon_h}(\cdot, A) \quad (6.2)$$

exists on $BD(A) \cap L^2(A; \mathbb{R}^n)$. Unfortunately, in investigating a representation of the limit F , we cannot directly resort to existing general results because the functionals F_ε do not fulfill standard growth conditions on the whole space $BD(A) \cap L^2(A; \mathbb{R}^n)$. To bypass this difficulty, we first restrict our attention to the behavior of F on $H^1(A; \mathbb{R}^n)$, where the growth condition of order 2 from above can be exploited. Then, we extend such representation on $BD(A) \cap L^2(A; \mathbb{R}^n)$ by convexity arguments.

A first result concerns the translation-invariance properties of the limit F .

Lemma 6.3. *Let F be defined as in (6.2). Then, for every $A \in \mathcal{A}_0(\Omega)$ and $u \in \text{dom } F(\cdot, A)$, the following properties hold:*

$$(i) \ F(u + a, A) = F(u, A), \quad (ii) \ F(\tau_y u, \tau_y A) = F(u, A),$$

for every $a \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, with $(\tau_y u)(x) = u(x - y)$ and $\tau_y A = A + y$.

Proof. These are general properties of the Γ -limit of periodic energies, which can be deduced with minor modifications e.g. as in Lemma 3.7 in [14]. \square \square

By exploiting that the limit F satisfies a growth condition of order 2 on $H^1(\Omega; \mathbb{R}^n)$, we can then prove that it admits an integral representation. In particular, the relevant density function is convex, satisfies a growth condition of order 2 and depends on the symmetric part of the gradient only.

Proposition 6.4. *There exists a unique convex function $f : \mathbb{M}^{n \times n} \in [0, +\infty)$ enjoying the following properties:*

- (i) $f(\xi) \leq c(1 + |\xi|^2)$ for every $\xi \in \mathbb{M}^{n \times n}$, with $c > 0$ a suitable constant;
- (ii) $F(u, A) = \int_A f(\mathcal{E}u) dx$ for every $A \in \mathcal{A}(\Omega)$ and $u \in H^1(A; \mathbb{R}^n)$.

Proof. The functional $F : H^1(\Omega; \mathbb{R}^n) \times \mathcal{A}(\Omega) \rightarrow [0, +\infty)$ enjoys the assumptions required in Theorem 1.1 in [18]. Namely, for every $u, v \in H^1(\Omega; \mathbb{R}^n)$ and $A \in \mathcal{A}(\Omega)$:

- (a) $F(u, A) = F(v, A)$ provided $u|_A = v|_A$;
- (b) the set function $F(u, \cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Borel measure on Ω ;
- (c) $F(u, A) = F(u + a, A)$ for every $a \in \mathbb{R}^n$;
- (d) $F(u, A) \leq c \int_A (1 + |Du|^2) dx$, with c a positive constant;
- (e) $F(\cdot, A)$ is sequentially weakly lower semicontinuous on $H^1(\Omega; \mathbb{R}^n)$.

Properties (b) and (c) follow from Proposition 6.2 and Lemma 6.3, and properties (a), (d) and (e) are consequences of the representation of $F(\cdot, A)$ as Γ -limit in (6.2).

Hence, the Carathéodory function $f : \mathbb{R}^n \times \mathbb{M}^{n \times n} \rightarrow [0, +\infty)$ defined by

$$f(x, \xi) = \limsup_{\rho \rightarrow 0} \frac{F(u_\xi, B_\rho(x))}{|B_\rho(x)|}, \quad (6.3)$$

gives the integral representation

$$F(u, A) = \int_A f(x, Du) dx, \quad (6.4)$$

for every $A \in \mathcal{A}(\Omega)$ and $u \in H^1(A; \mathbb{R}^n)$. In particular, since $F(\cdot, A)$ is convex for every $A \in \mathcal{A}(\Omega)$, definition (6.3) implies that the function f is convex. Moreover, as a consequence of property (d), the function f satisfies the growth condition

$$f(x, \xi) \leq c(1 + |\xi|^2). \quad (6.5)$$

We now show that f is constant with respect to its first argument. Let $x, y \in \mathbb{R}^n$ be fixed. Upon observing that $\tau_{y-x}u_\xi = u_\xi - \xi(y-x)$ and $\tau_{y-x}B_\rho(x) =$

$B_\rho(y)$, from Lemma 6.3 we obtain

$$\begin{aligned} f(x, \xi) &= \limsup_{\rho \rightarrow 0} \frac{F(u_\xi, B_\rho(x))}{|B_\rho(x)|} = \limsup_{\rho \rightarrow 0} \frac{F(u_\xi - \xi(y-x), B_\rho(y))}{|B_\rho(y)|} \\ &= \limsup_{\rho \rightarrow 0} \frac{F(u_\xi, B_\rho(y))}{|B_\rho(y)|} = f(y, \xi). \end{aligned}$$

Moreover, f depends only on the symmetric part of the gradient, i.e. $f(\xi) = f(\eta)$ whenever $\xi, \eta \in \mathbb{M}^{n \times n}$ satisfy $\xi^s = \eta^s$. In fact, let (u_{ε_h}) in $\mathcal{U}_{\varepsilon_h}(A)$ be a sequence converging to u_ξ in $L^2(A; \mathbb{R}^n)$ and such that

$$\lim_h F_{\varepsilon_h}(u_{\varepsilon_h}, A) = F(u_\xi, A) = \int_A f(\xi) dx = |A| f(\xi).$$

Set $v_{\varepsilon_h} = u_{\varepsilon_h} + (\eta - \xi)x$ and note that $v_{\varepsilon_h} \in \mathcal{U}_{\varepsilon_h}(A)$, v_{ε_h} converges to u_η in $L^2(A; \mathbb{R}^n)$ and $F_{\varepsilon_h}(u_{\varepsilon_h}, A) = F_{\varepsilon_h}(v_{\varepsilon_h}, A)$. Hence:

$$|A| f(\eta) = \int_A f(\eta) dx = F(u_\eta, A) \leq \liminf_h F_{\varepsilon_h}(v_{\varepsilon_h}, A) = \lim_h F_{\varepsilon_h}(u_{\varepsilon_h}, A) = |A| f(\xi),$$

and by symmetry $f(\eta) = f(\xi)$. Finally, (i) follows from (6.5) and (ii) follows from (6.4), whereas the uniqueness of f follows from (6.3). \square \square

6.3 Characterization of the homogenized energy density

In the previous section we have proven that the Γ -limit F of a convergent sequence of functionals (F_{ε_h}) admits an integral representation on $H^1(\Omega; \mathbb{R}^n)$. We are now in a position to show that the density f indeed coincides with the homogenized energy density f_{hom} defined in (3.5). In particular, that implies that f does not depend on the sequence (ε_h) .

Proposition 6.5. $f(\xi) \leq f_{\text{hom}}(\xi)$ for every $\xi \in \mathbb{M}^{n \times n}$.

Proof. For a fixed $\xi \in \mathbb{M}^{n \times n}$, let $u \in BD_{\text{loc}}(\mathbb{R}^n)$ be a test function for the computation of f_{hom} , i.e. such that $J_u \subseteq B$, $(u^+ - u^-) \odot \nu_u \in K_0$ \mathcal{H}^{n-1} -a.e. and $\tilde{u} = u - u_\xi$ is Y -periodic. Moreover, let (ε_h) be a sequence of positive numbers converging to 0. Upon noticing that the sequence of scaled functions $u_{\varepsilon_h}(x) = \varepsilon_h u(x/\varepsilon_h)$ converges to u_ξ in $L^2(Y; \mathbb{R}^n)$, by the liminf inequality

of the Γ -convergence, it follows that

$$f(\xi) = F(u_\xi, Y) \leq \liminf_h F_{\varepsilon_h}(u_{\varepsilon_h}, Y) = \liminf_h \left(\frac{1}{2} \int_Y \|\mathcal{E}u_{\varepsilon_h}\|^2 dx + \int_Y |u_{\varepsilon_h}^+ - u_{\varepsilon_h}^-| d\mathcal{H}^{n-1} \right). \quad (6.6)$$

For the bulk energy term we obtain

$$\frac{1}{2} \int_Y \|\mathcal{E}u_{\varepsilon_h}\|^2 dx = \frac{\varepsilon_h^n}{2} \int_{Y/\varepsilon_h} \|\mathcal{E}u\|^2 dx \leq \frac{\varepsilon_h^n}{2} \left[1 + \frac{1}{\varepsilon_h} \right]^n \int_Y \|\mathcal{E}u\|^2 dx,$$

whereas for the surface energy term

$$\int_Y |u_{\varepsilon_h}^+ - u_{\varepsilon_h}^-| d\mathcal{H}^{n-1} = \varepsilon_h^n \int_{Y/\varepsilon_h} |u^+ - u^-| d\mathcal{H}^{n-1} \leq \varepsilon_h^n \left[1 + \frac{1}{\varepsilon_h} \right]^n \int_Y |u^+ - u^-| d\mathcal{H}^{n-1}.$$

Finally, from (6.6) we get

$$f(\xi) \leq \frac{1}{2} \int_Y \|\mathcal{E}u\|^2 dx + \int_Y |u^+ - u^-| d\mathcal{H}^{n-1},$$

whence, taking the supremum over functions $u \in BD_{\text{loc}}(\mathbb{R}^n)$ such that $J_u \subseteq B$, $(u^+ - u^-) \odot \nu_u \in K_0$ \mathcal{H}^{n-1} -a.e., we obtain the desired result. \square \square

Proposition 6.6. $f_{\text{hom}}(\xi) \leq f(\xi)$ for every $\xi \in \mathbb{M}^{n \times n}$.

Proof. Let $\xi \in \mathbb{M}^{n \times n}$. We can consider a sequence (u_{ε_h}) , with $u_{\varepsilon_h} \in \mathcal{U}_{\varepsilon_h}(Y)$ for all $h \in \mathbb{N}$, converging to u_ξ and such that

$$f(\xi) = \int_Y f(\xi) dx = F(u_\xi, Y) = \lim_h F_{\varepsilon_h}(u_{\varepsilon_h}, Y).$$

Let $\delta > 0$. We define $\phi(y) = \max \{ (\delta^{-1} \text{dist}(y, \partial Y)), 1 \}$ and $S_\delta = \{y \in Y : \text{dist}(y, \partial Y) < \delta\}$.

Moreover we set

$$v_{\varepsilon_h} = \phi u_{\varepsilon_h} + (1 - \phi) u_\xi,$$

and consider the extension of v_{ε_h} to a function defined on $\varepsilon_h [1 + 1/\varepsilon_h] Y$ as

$$w_{\varepsilon_h}(y) = \begin{cases} v_{\varepsilon_h}(y), & \text{if } y \in Y, \\ u_\xi(y), & \text{if } y \in \varepsilon_h [1 + 1/\varepsilon_h] Y \setminus Y. \end{cases}$$

The function w_{ε_h} is extended to all \mathbb{R}^n by requiring $\tilde{w}_{\varepsilon_h} = w_{\varepsilon_h} - u_\xi$ to be

$\varepsilon_h [1 + 1/\varepsilon_h]$ Y -periodic. By the continuity of u_ξ , it follows that $w_{\varepsilon_h} \in \mathcal{U}_{\varepsilon_h}(Y)$. We now claim that the function

$$z_{\varepsilon_h}(y) = \frac{1}{\varepsilon_h [1 + 1/\varepsilon_h]^n} \sum_{k \in \{0, \dots, [1/\varepsilon_h]\}^n} w_{\varepsilon_h}(\varepsilon_h y + \varepsilon_h k),$$

is an admissible test function for the computation of $f_{\text{hom}}(\xi)$. In fact, by the Y -periodicity of B and since $J_{w_{\varepsilon_h}} \subseteq \varepsilon_h B$, we derive

$$J_{z_{\varepsilon_h}} \subseteq \bigcup_{k \in \{0, \dots, [1/\varepsilon_h]\}^n} J_{w_{\varepsilon_h}(\varepsilon_h \cdot + \varepsilon_h k)} \subseteq \bigcup_{k \in \{0, \dots, [1/\varepsilon_h]\}^n} \left(-k + J_{w_{\varepsilon_h}/\varepsilon_h} \right) \subseteq B;$$

moreover, since $(w_{\varepsilon_h}^+ - w_{\varepsilon_h}^-) \odot \nu_{w_{\varepsilon_h}} \in K_0 \mathcal{H}^{n-1}$ -a.e., it follows that $(z_{\varepsilon_h}^+ - z_{\varepsilon_h}^-) \odot \nu_{z_{\varepsilon_h}} \in K_0 \mathcal{H}^{n-1}$ -a.e. In order to show that $\tilde{z}_{\varepsilon_h} = z_{\varepsilon_h} - u_\xi$ is Y -periodic, it suffices to notice that, for $r \in \{0, \dots, [1/\varepsilon_h]\}$ and e_l denoting the l -th vector of the canonical base of \mathbb{R}^n , we have

$$\begin{aligned} \tilde{z}_{\varepsilon_h}(y + re_j) &= \frac{1}{\varepsilon_h [1 + 1/\varepsilon_h]^n} \sum_{k \in \{0, \dots, [1/\varepsilon_h]\}^n} \{ \tilde{w}_{\varepsilon_h}(\varepsilon_h y + \varepsilon_h(k + re_j)) + \varepsilon_h \xi k \} \\ &= \frac{1}{\varepsilon_h [1 + 1/\varepsilon_h]^n} \sum_{k \in \{0, \dots, [1/\varepsilon_h]\}^n} \{ \tilde{w}_{\varepsilon_h}(\varepsilon_h y + \varepsilon_h k) + \varepsilon_h \xi k \} = \tilde{z}_{\varepsilon_h}(y), \end{aligned}$$

where the $\varepsilon_h [1 + 1/\varepsilon_h]$ Y -periodicity of $\tilde{w}_{\varepsilon_h}$ has been exploited.

Using z_{ε_h} as a test function in the computation of $f_{\text{hom}}(\xi)$, it follows that

$$f_{\text{hom}}(\xi) \leq \frac{1}{2} \int_Y \|\mathcal{E}z_{\varepsilon_h}\|^2 dy + \int_{Y \cap J_{z_{\varepsilon_h}}} |z_{\varepsilon_h}^+ - z_{\varepsilon_h}^-| d\mathcal{H}^{n-1}, \quad (6.7)$$

and we have to estimate the two terms. For the bulk energy term, by the Y -periodicity of $\tilde{z}_{\varepsilon_h}$ and by the $\varepsilon_h [1 + 1/\varepsilon_h]$ Y -periodicity of $\tilde{w}_{\varepsilon_h}$ we derive

$$\begin{aligned} \int_Y \|\mathcal{E}z_{\varepsilon_h}\|^2 dy &= \frac{1}{[1 + 1/\varepsilon_h]^n} \int_{[1 + \frac{1}{\varepsilon_h}]Y} \|\mathcal{E}z_{\varepsilon_h}\|^2 dy \\ &\leq \frac{1}{[1 + 1/\varepsilon_h]^n} \int_{\varepsilon_h [1 + \frac{1}{\varepsilon_h}]Y} \|\mathcal{E}w_{\varepsilon_h}\|^2 dy \\ &\leq \int_Y \|\mathcal{E}v_{\varepsilon_h}\|^2 dy + \|\xi^s\|^2 \left(\varepsilon_h^n \left[1 + \frac{1}{\varepsilon_h} \right]^n - 1 \right). \end{aligned}$$

In particular, for $\eta > 0$ and $c_\eta > 0$ a suitable constant, the first term can be estimated by

$$\begin{aligned} \int_Y \|\mathcal{E}v_{\varepsilon_h}\|^2 dy &= \int_{Y \setminus S_\delta} \|\mathcal{E}u_{\varepsilon_h}\|^2 dy + \int_{S_\delta} \|\phi \mathcal{E}u_{\varepsilon_h} + (1 - \phi) \xi^s + \nabla \phi \odot (u_{\varepsilon_h} - u_\xi)\|^2 dy \\ &\leq (1 + \eta) \int_Y \|\mathcal{E}u_{\varepsilon_h}\|^2 dy + c_\eta |S_\delta| \|\xi^s\|^2 + \frac{c_\eta}{\delta^2} \int_{S_\delta} |u_{\varepsilon_h} - u_\xi|^2 dy. \end{aligned}$$

Analogously, for the surface energy term the Y -periodicity of $\tilde{z}_{\varepsilon_h}$ implies

$$\begin{aligned} \int_{Y \cap J_{z_{\varepsilon_h}}} |z_{\varepsilon_h}^+ - z_{\varepsilon_h}^-| d\mathcal{H}^{n-1} &= \frac{1}{[1 + 1/\varepsilon_h]^n} \int_{[1 + \frac{1}{\varepsilon_h}]_Y} |z_{\varepsilon_h}^+ - z_{\varepsilon_h}^-| d\mathcal{H}^{n-1} \\ &= \frac{1}{[1 + 1/\varepsilon_h]^n} \int_{\varepsilon_h [1 + \frac{1}{\varepsilon_h}]_Y} |w_{\varepsilon_h}^+ - w_{\varepsilon_h}^-| d\mathcal{H}^{n-1} \\ &\leq \int_Y |u_{\varepsilon_h}^+ - u_{\varepsilon_h}^-| d\mathcal{H}^{n-1}. \end{aligned}$$

Accordingly, from (6.7) we obtain

$$\begin{aligned} f_{\text{hom}}(\xi) &\leq \frac{1}{2} (1 + \eta) \int_Y \|\mathcal{E}u_{\varepsilon_h}\|^2 dy + \frac{c_\eta}{2} |S_\delta| \|\xi^s\|^2 + \frac{c_\eta}{2\delta^2} \int_{S_\delta} |u_{\varepsilon_h} - u_\xi|^2 dy \\ &\quad + \|\xi^s\|^2 \left(\varepsilon_h^n \left[1 + \frac{1}{\varepsilon_h} \right]^n - 1 \right) + \int_Y |u_{\varepsilon_h}^+ - u_{\varepsilon_h}^-| d\mathcal{H}^{n-1}. \end{aligned}$$

As u_{ε_h} converges to u_ξ in $L^2(Y; \mathbb{R}^n)$ and $|S_\delta| = 1 - (1 - 2\delta)^2 \rightarrow 0$ for $\delta \rightarrow 0^+$, by letting first $h \rightarrow +\infty$, then $\delta \rightarrow 0^+$ and finally $\eta \rightarrow 0^+$, we get

$$f_{\text{hom}}(\xi) \leq \lim_h \left(\frac{1}{2} \int_Y \|\mathcal{E}u_{\varepsilon_h}\|^2 dy + \int_Y |u_{\varepsilon_h}^+ - u_{\varepsilon_h}^-| \right) = \lim_h F_{\varepsilon_h}(u_{\varepsilon_h}, Y) = f(\xi),$$

and the proof is concluded. \square \square

We conclude this section with a corollary immediately descending from Propositions 6.4, 6.5 and 6.6, which characterizes the Γ -limit F of a convergent sequence of functionals (F_{ε_h}) on Sobolev functions.

Corollary 6.7 ($(\Gamma$ -convergence on H^1).). *For every $A \in \mathcal{A}_0(\Omega)$, the family of functionals $(F_{\varepsilon_h}(\cdot, A))$ Γ -converges on $H^1(A; \mathbb{R}^n)$, with respect to the $L^2(\Omega; \mathbb{R}^n)$ -topology, to the functional $F_{\text{hom}}(\cdot, A)$.*

6.4 Characterization of the homogenized functional

In this section we conclude the proof of Theorem 3.1 by showing that $F_{\text{hom}}(\cdot, \Omega)$ is the Γ -limit of $F_\varepsilon(\cdot, \Omega)$ on $BD(\Omega) \cap L^2(\Omega; \mathbb{R}^n)$.

As a preliminary step, we prove a lower-semicontinuity result on the functional F_{hom} .

Proposition 6.8. *For every $A \in \mathcal{A}_0(\Omega)$, the functional $F_{\text{hom}}(\cdot, A)$ is lower semicontinuous on $BD(A) \cap L^2(A; \mathbb{R}^n)$ with respect to the weak convergence in $BD(A)$.*

Proof. Let $A \in \mathcal{A}_0(\Omega)$. We consider a sequence (u_h) in $BD(A) \cap L^2(A; \mathbb{R}^n)$ weakly converging to a function u in $BD(A)$. Accordingly, there exists a subsequence (u_{h_j}) such that Eu_{h_j} weakly* converges to Eu in $\mathcal{M}(A; \mathbb{M}^{n \times n})$. For $\lambda > 0$, let

$$\varphi_\lambda(\xi) = \inf_{\eta \in \mathbb{M}^{n \times n}} \{f_{\text{hom}}(\eta) + \lambda |\xi - \eta|\}$$

denote the infimal convolution of f_{hom} and $\lambda|\cdot|$. Then φ_λ enjoys the following properties: (i) $\varphi_\lambda \leq \lambda|\cdot|$, (ii) φ_λ is convex and (iii) φ_λ increasingly converges to f_{hom} for $\lambda \uparrow \infty$. Using a classical theorem by Reshetnyak on convex functional of measures [36], properties (i) and (ii) imply that

$$\int_A \varphi_\lambda(\text{Eu}) \leq \liminf_h \int_A \varphi_\lambda(\text{Eu}_h) \leq \liminf_h F_{\text{hom}}(u_h, A),$$

whence by Fatou's lemma

$$F_{\text{hom}}(u, A) \leq \liminf_h F_{\text{hom}}(u_h, A),$$

which concludes the proof. □ □

We are now in position to prove the Γ -liminf inequality. A convexity method through convolution is employed to exploit the integral representation of the Γ -limit on Sobolev functions.

Proposition 6.9. *Let (ε_h) be a sequence of positive numbers converging to 0. Then there exists a subsequence $(\varepsilon_{\sigma(h)})$ of (ε_h) such that $F_{\text{hom}}(u, \Omega) \leq \Gamma\text{-lim inf}_h F_{\varepsilon_{\sigma(h)}}(u, \Omega)$ for every $u \in BD(\Omega) \cap L^2(\Omega; \mathbb{R}^n)$.*

Proof. Let (A_j) be a sequence in $\mathcal{A}_0(\Omega)$, with $A_j \subset\subset \Omega$ for all $j \in \mathbb{N}$, converging increasingly to Ω . By Proposition 6.2, there exists a subsequence $(\varepsilon_{\sigma(h)})$ of (ε_h)

such that

$$F(\cdot, A) = \Gamma\text{-}\lim_h F_{\varepsilon_{\sigma(h)}}(\cdot, A)$$

for all $A = A_j$ with $j \in \mathbb{N}$, and for $A = \Omega$. Now, fix $u \in BD(\Omega) \cap L^2(\Omega; \mathbb{R}^n)$ and let (ρ_h) be a sequence of mollifiers such that ρ_h has support in the open ball in \mathbb{R}^n of center 0 and radius $1/h$, denoted B_h . Setting $u_h = \rho_h * u$, we notice that $u_h \in C_0^\infty(\Omega; \mathbb{R}^n)$, u_h converges to u in $L^2(\Omega; \mathbb{R}^n)$ and Eu_h weakly* converges to Eu in $\mathcal{M}(\Omega; \mathbb{M}^{n \times n})$, i.e. u_h weakly converges to u in $BD(\Omega)$. By Corollary 6.7, we derive

$$F_{\text{hom}}(u_h, A_j) = \int_{A_j} f_{\text{hom}}(\mathcal{E}u_h) dx = F(u_h, A_j). \quad (6.8)$$

Next, we observe that the functional $F(\cdot, A)$ is lower semicontinuous and convex on $L^2(\Omega; \mathbb{R}^n)$ for every $A \in \mathcal{A}_0(\Omega)$. For $h \in \mathbb{N}$ such that $1/h < \text{dist}(A_j, \partial\Omega)$, Jensen's inequality implies that

$$F(u_h, A_j) = F\left(\int_{B_h} \rho_h(y) \tau_y u dy, A_j\right) \leq \int_{B_h} \rho_h(y) F(\tau_y u, A_j) dy.$$

Since F is translation invariant (Lemma 6.3), we obtain

$$F(u_h, A_j) \leq \int_{B_h} \rho_h(y) F(u, \tau_{-y} A_j) dy.$$

In particular, because $\tau_{-y} A_j \subseteq \Omega$ for every $y \in B_h$, we get

$$F(u_h, A_j) \leq \int_{B_h} \rho_h(y) F(u, \Omega) dy = F(u, \Omega). \quad (6.9)$$

By Proposition 6.8, the functional F_{hom} is lower semicontinuous on $BD(A_j) \cap L^2(A_j; \mathbb{R}^n)$ with respect to the weak convergence in $BD(A_j)$. Then, from (6.8) and (6.9) we derive

$$F_{\text{hom}}(u, A_j) \leq \liminf_h F_{\text{hom}}(u_h, A_j) \leq F(u, \Omega)$$

for all $j \in \mathbb{N}$. Taking the supremum with respect to $j \in \mathbb{N}$, we conclude

$$F_{\text{hom}}(u, \Omega) \leq F(u, \Omega) = \Gamma\text{-}\liminf_h F_{\varepsilon_{\sigma(h)}}(u, \Omega),$$

and the proof is accomplished. \square \square

Next we prove the Γ -limsup inequality.

Proposition 6.10. *Let (ε_h) be a sequence of positive numbers converging to 0. Then there exists a subsequence $(\varepsilon_{\sigma(h)})$ of (ε_h) such that $\Gamma\text{-lim sup}_h F_{\varepsilon_{\sigma(h)}}(u, \Omega) \leq F_{\text{hom}}(u, \Omega)$ for every $u \in BD(\Omega) \cap L^2(\Omega; \mathbb{R}^n)$.*

Proof. By Proposition 6.2, there exists a subsequence $(\varepsilon_{\sigma(h)})$ of (ε_h) such that

$$F(\cdot, A) = \Gamma\text{-lim}_h F_{\varepsilon_{\sigma(h)}}(\cdot, A)$$

for all $A \in \mathcal{A}_0(\Omega)$ on $L^2(A; \mathbb{R}^n)$. If $u \in L^2(\Omega; \mathbb{R}^n) \setminus \mathcal{U}_{\text{hom}}(\Omega)$, the claim is accomplished because $F_{\text{hom}}(u, \Omega) = +\infty$; then, fix $u \in \mathcal{U}_{\text{hom}}(\Omega)$. As in the previous proof, let (ρ_j) be a sequence of mollifiers such that ρ_j has support in B_j and set $u_j = \rho_j * u$. Then $u_j \in C_0^\infty(\Omega; \mathbb{R}^n)$ and u_j converges to u in $L^2(\Omega; \mathbb{R}^n)$. By the lower semicontinuity of the Γ -limsup and Corollary 6.7, we derive

$$\Gamma\text{-lim sup}_h F_{\varepsilon_{\sigma(h)}}(u, \Omega) \leq \liminf_j \left(\Gamma\text{-lim sup}_h F_{\varepsilon_{\sigma(h)}}(u_j, \Omega) \right) = \liminf_j F_{\text{hom}}(u_j, \Omega).$$

Then, Lemma 5.2 in [42] implies

$$F_{\text{hom}}(u_j, \Omega) = \int_{\Omega} f_{\text{hom}}(\mathbb{E}u_j) \, dx = \int_{\Omega} f_{\text{hom}}(\rho_j * \mathbb{E}u) \, dx \leq \int_{\Omega} f_{\text{hom}}(\mathbb{E}u) = F_{\text{hom}}(\Omega).$$

Finally we obtain

$$\Gamma\text{-lim sup}_h F_{\varepsilon_{\sigma(h)}}(u, \Omega) \leq F_{\text{hom}}(u, \Omega),$$

which is the desired result. \square \square

7 Conclusions and perspectives

In the present work we have discussed a homogenization result dealing with masonry structures constituted by an assemblage of blocks with interposed mortar joints. Departing point of the derivation has been regarding such structures as a periodic collection of disconnected sets (blocks) interacting through their boundaries (interfaces). Accordingly, we have considered a sequence of energy functionals (scaling with the size of the microgeometry) on the space of special functions of bounded deformation comprising (i) a linear elastic behavior in the blocks, (ii) a Barenblatt's cohesive contribution at interfaces and (iii) a unilateral condition on the strain across interfaces. Exploiting the notion of

Γ -convergence, we have analyzed the asymptotic behavior of such energy functionals, thus obtaining a simple homogenization formula for the limit energy. We have investigated the behavior of the limit energy, highlighting its mathematical and mechanical main properties. Among them, we have in particular focused on the non-standard growth conditions under tension or compression.

Various additional questions and several perspectives arise from the presented results. One is to generalize this approach to the case of a more complicated energy contribution on interfaces, for instance mimicking a plastic behavior or friction. Furthermore, most of these problems can be rephrased in a nonlinearly elastic framework. Finally, yet requiring to abandon many of the techniques exploiting the convexity assumption here adopted, an interesting direction of investigation might be the extension to the Griffith theory of brittle fracture.

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