

# Homogenization of chiral magnetic materials

A mathematical evidence of Dzyaloshinskii's predictions on helical structures

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**ABSTRACT.** In this paper we investigate the influence of the bulk Dzyaloshinskii-Moriya interaction on the magnetic properties of composite ferromagnetic materials with highly oscillating heterogeneities, in the framework of  $\Gamma$ -convergence and 2-scale convergence. The homogeneous energy functional resulting from our analysis provides an effective description of most of the magnetic composites of interest nowadays. Although our study covers more general scenarios than the micromagnetic one, it builds on the phenomenological considerations of Dzyaloshinskii on the existence of helicoidal textures, as the result of possible instabilities of ferromagnetic structures under small relativistic spin-lattice or spin-spin interactions. In particular we provide the first quantitative counterpart to Dzyaloshinskii's predictions on helical structures.

**Keywords:** chiral magnetic materials, micromagnetics, Dzyaloshinskii-Moriya interaction, homogenization, manifold-valued Sobolev spaces

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## 1. INTRODUCTION

Composite ferromagnetic materials are the subject of growing interest, as they often display unusual properties which turn out to be strikingly different from the corresponding ones of their constituents. For this reason, it is possible to engineer ferromagnetic composites exhibiting physical and chemical behaviors which rarely, if ever, emerge in bulk materials [30].

A systematic study of composite materials, and more generally of media with microstructures, is the primary source of inspiration for the mathematical theory of homogenization. The theory aims at a description of composite materials with highly oscillating heterogeneities, through a simplified homogeneous model whose material-dependent properties are now related to specific averages of the physical and geometrical parameters of the constituents (cf., e.g., [8, 7]). The origins of homogenization in micromagnetics date back to 1824, when Poisson, in his *Mémoire sur la théorie du magnétisme* [34], laid the foundations of the theory of induced magnetism, proposing a model in which a ferromagnet is composed of conducting spheres embedded in a nonconducting material.

The homogenization analysis performed in our paper is motivated by recent technological advances in the field of spintronics; first and foremost, by the observation, in magnetic systems lacking inversion symmetry, of chiral spin textures known as magnetic skyrmions [19], whose origin is ascribed to the *Dzyaloshinskii-Moriya interaction* (DMI) [18, 23]. More precisely, our

work builds on Dzyaloshinskii's observations in [16, 17] where, based on the Landau theory of second-order phase transitions, the emergence of helicoidal structures is predicted. According to Dzyaloshinskii, the appearance of these textures is the result of possible instabilities of the ferromagnetic structure created by relativistic spin-lattice or spin-spin forces, or by a sharp anisotropy in the exchange interaction. The results of our paper, based on the continuum theory of micromagnetics, make Dzyaloshinskii predictions quantitative (cf. Theorem 3).

From the mathematical point of view, magnetic skyrmions emerge as topological defects in the magnetization texture that carry a specific topological charge, also referred to as the *skyrmion (winding) number*. If  $\mathcal{H}$  is a compact smooth hypersurface of  $\mathbb{R}^{n+1}$  and  $m: \mathcal{H} \rightarrow \mathbb{S}^n$  is a sufficiently smooth vector field on  $\mathcal{H}$ , the skyrmion number of  $m$  is defined by the Kronecker integral [31]

$$N_{\text{sk}}(m) := \frac{1}{|\mathbb{S}^n|} \int_{\mathcal{H}} m^* \omega_n \quad (1)$$

with  $\omega_n(x) := \sum_{j=1}^n (-1)^{j-1} x_j dx_1 \wedge \dots \wedge \widehat{dx}_j \wedge \dots \wedge dx_n$  the volume form on  $\mathbb{S}^n$ , and  $m^* \omega_n$  the pull-back of  $\omega_n$  by  $m$  on  $\mathcal{H}$ . In local coordinates  $x := (x_1, \dots, x_n)$  this gives:

$$N_{\text{sk}}(m) := \frac{1}{|\mathbb{S}^n|} \int_{\mathcal{H}} \det \left( m(x), \frac{\partial}{\partial x_1} m(x), \dots, \frac{\partial}{\partial x_n} m(x) \right) dx. \quad (2)$$

According to Hadamard [33],  $N_{\text{sk}}(m)$  is always an integer number and coincides with the topological degree of  $m$ . By Hopf's theorem [29], skyrmions with different topological charges belong to different homotopy classes and, therefore, from the physical point of view, skyrmions are expected to be topologically protected against external perturbations and thermal fluctuations [12].

Since their discovery, magnetic skyrmions have been the object of intense research work in condensed matter physics. Their stability, the reduced size, and the small current densities sufficient to control them, make skyrmions extremely attractive for applications in modern spintronics [20, 21, 25].

In this paper, in the framework of  $\Gamma$ -convergence and 2-scale convergence, we investigate the influence of the bulk Dzyaloshinskii-Moriya interaction [18, 23] on the magnetic properties of composite ferromagnetic materials with highly oscillating heterogeneities. The homogeneous energy functional resulting from our analysis provides an effective description of most of the magnetic composites of interest nowadays. Indeed, although the homogenized coefficients of the limiting energy functional involve the solution of a system of PDEs, chiral multilayers are essentially one-dimensional structures, and this allows us for a complete characterization of the minimal configurations and their topological degree, at least under some simplified hypotheses on the distribution of the constituents. Precisely, we show that depending on the effective DMI constant of the homogeneous model, two Bloch-type chiral skyrmions with opposite topological charges can arise. Our results provide a solid ground to the experimental observations that ground states with a non-trivial topological degree do exist here in a stable state [40, 22, 11] (see Remark 13 below).

In order to describe our main contributions, we first collect below some preliminary notation and results.

**1.1. The micromagnetic theory of (single crystal) chiral magnets.** In the continuum theory of micromagnetism [10, 24], which dates back to the seminal work of Landau–Lifshitz [26] on fine ferromagnetic particles, the observable states of a rigid ferromagnetic body, filling a

region  $\Omega \subseteq \mathbb{R}^3$ , are described by the magnetization  $M: \Omega \rightarrow \mathbb{R}^3$ , a vector field subject to the fundamental constraint of micromagnetism: the existence of a material-dependent constant  $M_s$  such that  $|M| = M_s$  in  $\Omega$ . For single-crystal ferromagnets (cf. [1, 4]), the saturation magnetization  $M_s := M_s(T)$  depends only on the temperature  $T$  and vanishes above a critical value  $T_c$ , characteristic of each crystal type, known as the Curie temperature. When the specimen is at a fixed temperature well below  $T_c$ , the function  $M_s$  is constant in  $\Omega$  and the magnetization takes the form  $M := M_s m$ , where  $m: \Omega \rightarrow \mathbb{S}^2$  is a vector field with values in the unit sphere of  $\mathbb{R}^3$  (cf. [10, 24]).

Although the length of  $m$  is constant in space, this is, in general, not the case for its direction, and the observable states of the magnetization result as the local minimizers of the micromagnetic energy functional, which, for non-centrosymmetric (chiral) magnets, reads as

$$\mathcal{F}_\Omega(m) := \int_{\Omega} a_{\text{ex}} |\nabla m|^2 + \int_{\Omega} \kappa \operatorname{curl} m \cdot m - \frac{\mu_0}{2} \int_{\Omega} \mathbf{h}_d[M_s m \chi_\Omega] \cdot M_s m, \quad (3)$$

$\quad \quad \quad =: \mathcal{E}_\Omega(m) \quad \quad \quad =: \mathcal{K}_\Omega(m) \quad \quad \quad =: \mathcal{W}_\Omega(m)$

for every  $m \in H^1(\Omega, \mathbb{R}^3)$ ,  $m(x) \in \mathbb{S}^2$  a.e. in  $\Omega$ , where  $m \chi_\Omega$  denotes the extension by zero of  $m$  to the whole space.

The *exchange energy*  $\mathcal{E}_\Omega$  penalizes spatial variations of the magnetization. The quantity  $a_{\text{ex}} > 0$  represents a phenomenological (material-dependent) constant that summarizes the effect of short-range exchange interactions.

The second term,  $\mathcal{K}_\Omega(m)$ , represents the bulk Dzyaloshinskii-Moriya interaction (DMI), and accounts for possible lacks of inversion symmetry in the crystal structure of the magnetic material. The material-dependent constant  $\kappa \in \mathbb{R}$  is the bulk DMI constant; its sign affects the chirality of the ferromagnetic system [39, 37].

The third term,  $\mathcal{W}_\Omega$ , is the *magnetostatic self-energy*, that is, the energy due to the demagnetizing (or stray) field  $\mathbf{h}_d$  generated by  $m$ . The stray field  $-\mathbf{h}_d[M_s m \chi_\Omega]$  is characterized as the projection of  $m \chi_\Omega \in L^2(\mathbb{R}^3, \mathbb{R}^3)$  on the closed subspace of gradient vector fields

$$W_{\nabla}^1(\mathbb{R}^3, \mathbb{R}^3) := \{\nabla u : u \in \mathcal{D}'(\mathbb{R}^3), \nabla u \in L^2(\mathbb{R}^3, \mathbb{R}^3)\}. \quad (4)$$

The physical constant  $\mu_0$  denotes the vacuum permeability.

The competition among the contributions in (3) explains *most* of the striking pictures of the magnetization observable in ferromagnetic materials [24]; in particular, the emergence of chiral spin textures with a non-trivial topological degree, i.e., magnetic skyrmions [21, 22].

We note that, usually, the micromagnetic energy includes two additional energy contributions: the *magnetocrystalline anisotropy energy*  $\mathcal{A}_\Omega$  and the *Zeeman energy*  $\mathcal{Z}_\Omega$ :

$$\mathcal{A}_\Omega(m) := \int_{\Omega} \varphi_{\text{an}}(m(x)) \, dx, \quad \mathcal{Z}_\Omega(m) := -\mu_0 \int_{\Omega} \mathbf{h}_a \cdot M_s m(x) \, dx.$$

The energy density  $\varphi_{\text{an}}: \mathbb{S}^2 \rightarrow \mathbb{R}^+$  accounts for the existence of preferred directions of the magnetization: it vanishes on a finite set of directions, called easy axes, that depend on the crystallographic structure of the material. Instead,  $\mathcal{Z}_\Omega$  models the tendency of a specimen to have the magnetization aligned with the external applied field  $\mathbf{h}_a \in L^2(\Omega, \mathbb{R}^3)$ , assumed to be unaffected by variations of  $m$ . Although both  $\mathcal{A}_\Omega$  and  $\mathcal{Z}_\Omega$  are of fundamental importance in ferromagnetism, in a homogenization setting they behave like  $\Gamma$ -continuous perturbations, and their analysis has already been performed in [4]. Therefore, to shorten notation, they will be neglected in our investigation.



For what concerns the statement in *i*. we recall the following result which we state here (without proof) in the slightly more general setting of a bounded,  $C^2$  orientable hypersurface  $\mathcal{M}$  of  $\mathbb{R}^3$ .

**Proposition 1.** (Prop. 4.4 in [4]) *The family of magnetostatic self-energies  $(\mathcal{W}_\Omega^\varepsilon)_{\varepsilon \in \mathbb{R}_+}$   $\Gamma$ -continuously converges in  $\mathbb{R}_+ \times L^2(\Omega, \mathcal{M})$  to the functional*

$$\mathcal{W}_0(m) := -\frac{\mu_0}{2} \left( \int_Q M_s(y) dy \right)^2 \int_\Omega \mathbf{h}_d[m(x) \chi_\Omega(x)] \cdot m(x) + \int_{\Omega \times Q} |\nabla_y v_m(x, y)|^2 dx dy \quad (6)$$

where for almost every  $x \in \Omega$  the scalar potential  $v_m(x, \cdot)$  is the unique solution in  $H_{\sharp}^1(Q)$  of the cell problem  $\Delta_y v_m(x, y) = m(x) \cdot \nabla_y M_s(y)$ .

In particular, this guarantees that  $\mathcal{W}_\Omega^\varepsilon$  can be treated as a continuous perturbation (cf. [14, Prop. 6.20, p. 62]). Namely,  $\Gamma\text{-lim}_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon = \Gamma\text{-lim}_{\varepsilon \rightarrow 0} (\mathcal{E}_\Omega^\varepsilon + \mathcal{K}_\Omega^\varepsilon) + \mathcal{W}_0$ . For this reason, in the sequel, our analysis will be focused on the family

$$\mathcal{G}_\varepsilon := (\mathcal{E}_\Omega^\varepsilon + \mathcal{K}_\Omega^\varepsilon)_{\varepsilon \in \mathbb{R}_+}. \quad (7)$$

Regarding point *ii*. we observe that the energy densities in  $(\mathcal{E}_\Omega^\varepsilon + \mathcal{K}_\Omega^\varepsilon)_{\varepsilon \in \mathbb{R}_+}$  explicitly depend on  $m$  and cannot be expressed in the form  $f(x/\varepsilon, \nabla m)$  for some Caratheodory integrand  $f$  fitting the analysis in [5].

**1.4. Contribution of the present work.** Moving beyond [5] and departing from the observations in [4], our analysis tackles the more general setting of *periodic* chiral magnets, that is *composite* chiral magnets in which the heterogeneities are evenly distributed inside the media.

The contribution of the present work is threefold. First, we provide a characterization of the asymptotic behavior of the energy functionals  $(\mathcal{G}_\varepsilon)_{\varepsilon \in \mathbb{R}_+}$  in terms of  $\Gamma$ -convergence in the weak  $H^1(\Omega, \mathcal{M})$ -topology. Our homogenization result reads as follows (see Proposition 5 and Theorem 7).

**Theorem 2.** *Let  $\mathcal{M}$  be a bounded,  $C^2$  orientable hypersurface of  $\mathbb{R}^3$  that admits a tubular neighborhood of uniform thickness. Then, the family  $(\mathcal{G}_\varepsilon)_{\varepsilon \in \mathbb{R}_+}$   $\Gamma$ -converges with respect to the weak topology in  $H^1(\Omega, \mathcal{M})$ , to the energy functional*

$$\begin{aligned} \mathcal{G}_0(m) &:= \int_\Omega T_{\text{hom}}(m(x), \nabla m(x)) dx \\ &= \frac{1}{2} \int_{\Omega \times Q} a(y) \left( |\nabla^\top m(x)|^2 - |\nabla_y \phi[m(x), \nabla^\top m(x)](y)|^2 \right) dy dx \\ &\quad - \int_{\Omega \times Q} \kappa(y) \chi(m(x)) : \nabla m(x) dy dx, \end{aligned} \quad (8)$$

for every  $m \in H^1(\Omega, \mathcal{M})$ , where  $\phi[m, \nabla^\top m]$  is the unique solution of the cell-problem described in Proposition 5.

We point out that the range of surfaces included in our study is quite broad. Indeed, any compact and smooth surface is orientable and admits a tubular neighborhood (of uniform thickness), cf. [15, Prop. 1, p. 113]. In particular, our analysis covers the class of bounded surfaces that are diffeomorphic to an open subset of a compact surface (e.g., a finite cylinder,

or the graph of a  $C^2$  function). The proof strategy relies on a characterization of the two-scale asymptotic behavior of sequences in  $H^1(\Omega, \mathcal{M})$  (see Proposition 8), on an application of the theory of two-scale convergence (see [2, 32, 27]), and on a careful projection argument guaranteeing the optimality of  $\mathcal{G}_0$  as a lower bound for the energies  $\mathcal{G}_\varepsilon$ , as  $\varepsilon$  converges to zero.

Our second main result concerns the case in which  $\mathcal{M} = \mathbb{S}^2$  and  $\Omega \subseteq \mathbb{R}^3$  has a *laminated* structure (see Figure 1 below). In this micromagnetic setting of chiral multilayers we provide an explicit identification of minimizers of the functional  $\mathcal{G}_0$ . Our theorem reads as follows (see Theorem 12).

**Theorem 3.** *Assume that  $\Omega := \omega \times I_\lambda$  with  $\omega \subset \mathbb{R}^2$ ,  $|\omega| = 1$ ,  $\lambda > 0$ ,  $I_\lambda = (0, \lambda)$ , and that the material-dependent functions  $a, \kappa \in L^\infty_\#(Q, \mathbb{R})$  depend only on the third coordinate:  $a(y) = a(y \cdot e_3)$  and  $\kappa(y) = \kappa(y \cdot e_3)$ . Then, for  $\int_Q \kappa(y) \, dy = 0$ , the only energy minimizers are the helical textures*

$$m_*(x) := \cos(\theta(x \cdot e_3))e_1 + \sin(\theta(x \cdot e_3))e_2, \quad \theta(t) := \theta_0 + \langle \kappa/a \rangle_Q t \quad \text{for every } t \in \mathbb{R},$$

with  $\theta_0 \in \mathbb{R}$  arbitrary, and the minimum value of the energy is  $\mathcal{G}_0(m_*) = -\frac{\lambda}{2} \int_Q \frac{\kappa^2(y)}{a(y)} \, dy$ .

The arising of helical magnetic structures in composite alloys had been originally theorized by Dzyaloshinskii in [16, 17] (see also [36]), as the result of possible instabilities of ferromagnetic structures with respect to small relativistic spin-lattice or spin-spin interactions. A concrete realization of Dzyaloshinskii's conjecture has been shown in [6], where the authors exhibited long-period structures in MnSi and FeGe alloys stemming from the phenomenon described in [16, 17]. Despite the growing interest in chiral skyrmions (see, e.g., [35] for a review of the main properties) a rigorous theoretical justification for the presence of helical magnetic structures in alloys was, so far, still missing. To the authors' knowledge, Theorem 3 provides thus the first mathematical evidence of Dzyaloshinskii's conjectures in multilayers when long-range effects are neglected (see Remark 15).

Our third main contribution consists of an extension of the characterization in Theorem 2 to the higher-dimensional setting. To be precise, we consider the family of energy functionals

$$\begin{aligned} \mathcal{G}_{\varepsilon, v}(m) &:= \int_\Omega f_v\left(\frac{x}{\varepsilon}, m(x), \nabla m(x)\right) \, dx \\ &= \frac{1}{2} \sum_{i=1}^n \int_\Omega A_\varepsilon^i(x) (\partial_i m(x) - J_\varepsilon^i(x) m(x)) \cdot (\partial_i m(x) - J_\varepsilon^i(x) m(x)) \, dx \\ &\quad + \frac{1}{2} \sum_{i=1}^n \int_\Omega K_\varepsilon^i(x) m(x) \cdot m(x) \, dx, \end{aligned} \quad (9)$$

where for every  $i = 1, \dots, n$  the maps  $A^i, K^i \in L^\infty_\#(\mathbb{R}^n, \mathbb{R}_{\text{sym}}^{n \times n})$  are  $Q$ -periodic functions, taking values in the set  $\mathbb{R}_{\text{sym}}^{n \times n}$  of symmetric matrices, and where  $J^i \in L^\infty_\#(\mathbb{R}^n, \mathbb{R}^{n \times n})$  is a  $Q$ -periodic function, taking values in the set of  $n \times n$ -matrices. Additionally, we assume that each map  $A^i$  is *uniformly positive definite*, namely that for every  $i = 1, \dots, n$  there exists  $c_i > 0$  such that

$$A^i(y) \lambda \cdot \lambda \geq c_i |\lambda| \quad \text{for every } \lambda \in \mathbb{R}^n \text{ and for all } y \in Q. \quad (10)$$

As highlighted in Remark 16 the class of energies densities as above includes the setting in which both the exchange energy coefficient and the material-dependent DMI constant are anisotropic. In Theorem 18 we prove the following.



**Theorem 4.** *The family  $(\mathcal{G}_{\varepsilon,v})_{\varepsilon \in \mathbb{R}_+}$   $\Gamma$ -converges with respect to the weak topology in  $H^1(\Omega, \mathcal{M})$ , to the energy functional*

$$\begin{aligned} \mathcal{G}_{0,v}(m) &:= \int_{\Omega} T_{\text{hom}}^v(m(x), \nabla m(x)) \, dx \\ &= \frac{1}{2} \sum_{i=1}^n \int_{\Omega \times Q} A^i(y) (\partial_i m(x) + \partial_{y_i} \phi_v[m(x), \nabla m(x)](y) - J^i(y) m(x)) \\ &\quad \cdot (\partial_i m(x) + \partial_{y_i} \phi_v[m(x), \nabla m(x)](y) - J^i(y) m(x)) \, dx \, dy \\ &\quad + \frac{1}{2} \sum_{i=1}^n \int_{\Omega \times Q} K^i(y) m(x) \cdot m(x) \, dx \, dy \end{aligned}$$

for every  $m \in H^1(\Omega, \mathcal{M})$ , where  $\phi_v[m, \nabla m]$  solves the cell problem in Proposition 17.

The paper is organized as follows: In Section 2 we introduce the setting of the problem and prove some first preliminary results. In Section 2.1 we characterize the two-scale limits of  $H^1(\Omega, \mathcal{M})$ -maps. Sections 2.2 and 2.3 are devoted to the proof of Theorem 2. The study of chiral multilayers and the higher-dimensional setting are the subject of Section 3 and Section 4, respectively.

## 2. THE THREE-DIMENSIONAL SETTING

In what follows,  $\Omega$  will be an open bounded domain of  $\mathbb{R}^3$ . Our analysis will focus on vector-valued functions taking values on surfaces  $\mathcal{M}$  in  $\mathbb{R}^3$ . We will always assume that  $\mathcal{M}$  is a bounded,  $C^2$  orientable hypersurface of  $\mathbb{R}^3$  that admits a tubular neighborhood of uniform thickness.

The normal field associated with the choice of an orientation for  $\mathcal{M}$  will be denoted by  $n: \mathcal{M} \rightarrow \mathbb{S}^2$ . For every  $m \in \mathcal{M}$  and every  $\delta \in \mathbb{R}$  we denote by  $\ell_{\delta}(m) := \{m + tn(m) : -\delta < t < \delta\}$  the normal segment to  $\mathcal{M}$  having radius  $\delta$  and centered at  $m$ . We recall that if  $\mathcal{M}$  admits a *tubular neighborhood* (of uniform thickness) then there exists a  $\delta \in \mathbb{R}_+$  such that the following properties hold (cf. [15, p. 112]):

- i. For every  $m_1, m_2 \in \mathcal{M}$  there holds  $\ell_{\delta}(m_1) \cap \ell_{\delta}(m_2) = \emptyset$  whenever  $m_1 \neq m_2$ , and the union  $\mathcal{M}_{\delta} := \cup_{m \in \mathcal{M}} \ell_{\delta}(m)$  is an open set of  $\mathbb{R}^3$  containing  $\mathcal{M}$ .
- ii. The nearest-point projection

$$\pi_{\mathcal{M}}: \mathcal{M}_{\delta} \rightarrow \mathcal{M}, \tag{11}$$

which maps every  $p \in \mathcal{M}_{\delta}$  onto the unique  $m \in \mathcal{M}$  such that  $p \in \ell_{\delta}(m)$ , is a  $C^1$  map.

The open set  $\mathcal{M}_{\delta}$  is called the tubular neighborhood of  $\mathcal{M}$  of thickness  $\delta$ .

We denote by  $T\mathcal{M}$  the tangent bundle of  $\mathcal{M}$ , and by  $\mathbf{T}\mathcal{M} := \cup_{s \in \mathcal{M}} \{s\} \times \mathbf{T}_s\mathcal{M}$  the vector bundle, with  $\mathbf{T}_s\mathcal{M} := (T_s\mathcal{M})^3$ . We will indicate by  $\xi^{\top} := (\xi_1^{\top}, \xi_2^{\top}, \xi_3^{\top})$  a generic element of  $\mathbf{T}_s\mathcal{M}$ . The notation is motivated by the fact that if  $\xi := \nabla m$ , with  $m \in H^1(\Omega, \mathcal{M})$  and  $\nabla m$  is the transpose of the Jacobian matrix of  $m$ , the columns  $(\xi_1^{\top}, \xi_2^{\top}, \xi_3^{\top}) := (\partial_1 m(x), \partial_2 m(x), \partial_3 m(x))$  of  $(\nabla m(x))^{\top}$  are in  $T_{m(x)}\mathcal{M}$  for almost every  $x \in \Omega$ .

In what follows,  $Q$  will be the unit cube in  $\mathbb{R}^3$ . We will denote by  $H_{\#}^1(Q)$  the set of corresponding periodic  $H^1$ -maps, namely the collection of functions  $u \in H^1(\mathbb{R}^3)$  such that  $u(x + ke_i) = u(x)$  for every  $k \in \mathbb{N}$ , and for almost every  $x \in \mathbb{R}^3$ ,  $i = 1, 2, 3$ . With a slight abuse of notation, we will identify  $H_{\#}^1(Q)$  with  $H^1(Q)/\mathbb{R}$  and for  $\phi, \psi \in H_{\#}^1(Q)$  we will write  $\phi = \psi$  if

$\phi - \psi \in \mathbb{R}$ . Throughout the paper, the symbol  $\rightharpoonup$  will denote weak two-scale convergence. The symbol  $\mathcal{D}(\Omega)$  will represent the class of smooth functions having compact support in  $\Omega$ . Also, to shorten notation, for every map  $\psi \in L^1(Q)$  we will denote by  $\langle \psi \rangle_Q$  the average of  $\psi$  on  $Q$ .

We consider the energy density

$$f: (x, (s, \xi^\top)) \in \mathbb{R}^3 \times \mathbf{TM} \mapsto f(x, s, \xi) = \frac{1}{2}a(x)|\xi^\top|^2 - \sum_{i=1}^3 \kappa(x)(e_i \times s) \cdot \xi_i^\top, \quad (12)$$

where  $\kappa \in L^\infty_{\#}(\mathbb{R}^3)$  is a  $Q$ -periodic function, representing the material-dependent DMI constant,  $a \in L^\infty_{\#}(\mathbb{R}^3, \mathbb{R}_+)$  is a 1-periodic *positive* function accounting for the range of exchange interactions. For every  $\varepsilon > 0$  we set  $a_\varepsilon(x) := a(x/\varepsilon)$ , and  $\kappa_\varepsilon(x) := \kappa(x/\varepsilon)$  for almost every  $x \in \mathbb{R}^3$ . For every  $m \in H^1(\Omega, \mathcal{M})$  we define the family of energy functionals

$$\begin{aligned} \mathcal{G}_\varepsilon(m) &:= \int_{\Omega} f\left(\frac{x}{\varepsilon}, m(x), \nabla m(x)\right) dx \\ &= \frac{1}{2} \int_{\Omega} a_\varepsilon(x) |\nabla m(x)|^2 dx - \int_{\Omega} \kappa_\varepsilon(x) \chi(m(x)) : \nabla m(x) dx \end{aligned} \quad (13)$$

with  $\chi: s \in \mathcal{M} \mapsto (e_1 \times s, e_2 \times s, e_3 \times s)^\top \in \mathbb{R}^{3 \times 3}$ . We aim at identifying a homogenized functional capturing the limiting behavior of minimizers of  $\mathcal{G}_\varepsilon$  as  $\varepsilon \rightarrow 0$ , that is, as the period over which the heterogeneities are evenly distributed inside the media shrinks to zero.

Before stating our main result, we introduce the so-called *tangentially homogenized* energy density  $T_{\text{hom}}: (s, \xi^\top) \in \mathbf{TM} \rightarrow \mathbb{R}$ , defined by the minimization problem

$$T_{\text{hom}}(s, \xi^\top) := \inf_{\phi \in H^1_{\#}(Q, T_s \mathcal{M})} \int_Q \left\{ \frac{1}{2} a(y) |\xi + \nabla \phi(y)|^2 - \kappa(y) \chi(s) : (\xi + \nabla \phi(y)) \right\} dy \quad (14)$$

for every  $s \in \mathcal{M}$  and  $\xi^\top \in T_s \mathcal{M}$ . We first show an explicit characterization of solutions to (14), guaranteeing, as a by-product, the measurability of the map  $x \rightarrow T_{\text{hom}}(m(x), \nabla m(x))$  for every  $m \in H^1(\Omega, \mathcal{M})$ .

**Proposition 5.** *For every  $(s, \xi^\top) \in \mathbf{TM}$ , the minimization problem (14) has a unique solution. Specifically, let  $\tau_1(s), \tau_2(s)$  be an orthonormal basis at  $s \in \mathcal{M}$ . Let  $\varphi_a, \varphi_\kappa \in H^1_{\#}(Q, \mathbb{R}^3)$  be the unique solutions to the cell equations*

$$-\mathbf{div}(a \nabla \varphi_a) = \nabla a \quad \text{in } H^{-1}_{\#}(Q, \mathbb{R}^3), \quad (15)$$

$$-\mathbf{div}(a \nabla \varphi_\kappa) = -\nabla \kappa \quad \text{in } H^{-1}_{\#}(Q, \mathbb{R}^3). \quad (16)$$

Here the operator  $\mathbf{div}$  acts on columns. Then, the unique solution  $\phi[s, \xi^\top] \in H^1_{\#}(Q, T_s \mathcal{M})$  of the minimization problem (14) is given by

$$\begin{aligned} \phi[s, \xi^\top](y) &= [\varphi_a(y) \cdot \xi \tau_1(s) + \varphi_\kappa(y) \cdot \mu_1(s)] \tau_1(s) + [\varphi_a(y) \cdot \xi \tau_2(s) + \varphi_\kappa(y) \cdot \mu_2(s)] \tau_2(s) \\ &= (\tau_1(s) | \tau_2(s)) \begin{pmatrix} \xi^\top \varphi_a(y) \cdot \tau_1(s) \\ \xi^\top \varphi_a(y) \cdot \tau_2(s) \end{pmatrix} + (\tau_1(s) | \tau_2(s)) \begin{pmatrix} \varphi_\kappa(y) \cdot \mu_1(s) \\ \varphi_\kappa(y) \cdot \mu_2(s) \end{pmatrix}, \end{aligned} \quad (17)$$

with  $\mu_j(s) := s \times \tau_j(s)$ , for every  $s \in \mathcal{M}$ , and for almost every  $y \in Q$ . Additionally,

$$T_{\text{hom}}(s, \xi^\top) = \frac{1}{2} \langle a \rangle_Q |\xi|^2 - \langle \kappa \rangle_Q \chi(s) : \xi - \int_Q a(y) |\nabla_y \phi[s, \xi^\top](y)|^2 dy. \quad (18)$$



**Proof.** We first observe that for every  $\phi \in H_{\sharp}^1(Q, T_s\mathcal{M})$  there holds  $\phi(y) = \sum_{j=1}^2 \phi^j(y) \tau_j(s)$  for almost every  $y \in Q$ . Analogously,  $\xi_i^{\top} = \sum_{j=1}^2 (\xi_i^{\top} \cdot \tau_j(s)) \tau_j(s)$ ,  $i = 1, 2, 3$ , because  $\xi^{\top} \in T_s\mathcal{M}$ . Therefore

$$\begin{aligned} |\xi + \nabla \phi(y)|^2 &= \sum_{i=1}^3 |\xi_i^{\top} + \partial_i \phi(y)|^2 \\ &= \sum_{i=1}^3 \sum_{j=1}^2 (\xi_i^{\top} \cdot \tau_j(s))^2 + (\partial_i \phi^j(y))^2 + 2(\xi_i^{\top} \cdot \tau_j(s)) \partial_i \phi^j(y) \\ &= \sum_{j=1}^2 |\xi \tau_j(s)|^2 + |\nabla \phi^j(y)|^2 + 2(\xi \tau_j(s)) \cdot \nabla \phi^j(y) \\ &= |\xi_{\tau}(s) + \nabla \phi_{\tau}(y)|^2, \end{aligned}$$

where we set  $\xi_{\tau}(s) := (\xi \tau_1(s), \xi \tau_2(s)) \in \mathbb{R}^{3 \times 2}$  and  $\phi_{\tau}(y) = (\phi^1(y), \phi^2(y))$ , for every  $s \in \mathcal{M}$  and for almost every  $y \in Q$ .

Additionally, we have

$$\begin{aligned} \sum_{i=1}^3 (e_i \times s) \cdot (\xi_i^{\top} + \partial_i \phi(y)) &= \sum_{j=1}^2 \sum_{i=1}^3 ((\xi_i^{\top} \cdot \tau_j(s)) + \partial_i \phi^j(y)) \tau_j(s) \cdot (e_i \times s) \\ &= \sum_{j=1}^2 (\xi \tau_j(s) + \nabla \phi^j(y)) \cdot (s \times \tau_j) \\ &= (\xi_{\tau}(s) + \nabla \phi_{\tau}(y)) : \mu(s), \end{aligned} \tag{19}$$

with  $\mu(s) := (\mu_1(s), \mu_2(s)) \in \mathbb{R}^{3 \times 2}$  and  $\mu_j(s) := s \times \tau_j(s)$ ,  $j = 1, 2$ , for every  $s \in \mathcal{M}$ .

Summarizing, the minimization problem in (14) can be restated under the form

$$\inf_{\phi \in H_{\sharp}^1(Q, T_s\mathcal{M})} \int_Q \left\{ \frac{1}{2} a(y) |\xi_{\tau}(s) + \nabla \phi_{\tau}(y)|^2 - \kappa(y) (\xi_{\tau}(s) + \nabla \phi_{\tau}(y)) : \mu(s) \right\} dy, \tag{20}$$

which in turn can be rephrased as the combination of two scalar infimization procedures. Indeed, for every  $\sigma, \omega \in \mathbb{R}^3$  we consider the minimization problem

$$\min_{\psi \in H_{\sharp}^1(Q, \mathbb{R})} \int_Q \left\{ \frac{1}{2} a(y) |\sigma + \nabla \psi(y)|^2 - \kappa(y) \nabla \psi(y) \cdot \omega \right\} dy. \tag{21}$$

By Lax-Milgram's lemma, for every  $\sigma, \omega \in \mathbb{R}^3$  problem (21) admits a unique solution in  $H_{\sharp}^1(Q, \mathbb{R})$ , denoted henceforth by  $\psi[\sigma, \omega]$ . In particular,  $\psi[\sigma, \omega]$  coincides with the unique solution to the Poisson equation

$$-\operatorname{div}(a \nabla \psi) = \nabla a \cdot \sigma - \nabla \kappa \cdot \omega \quad \text{in } H_{\sharp}^{-1}(Q, \mathbb{R}). \tag{22}$$

Accordingly, the unique solution of the main minimization problem (14) reads as

$$\phi[s, \xi^{\top}](y) = \phi^1[s, \xi^{\top}](y) \tau_1(s) + \phi^2[s, \xi^{\top}](y) \tau_2(s),$$

where

$$\phi^1[s, \xi^\top](y) = \psi[\xi\tau_1(s), \mu_1(s)](y), \quad (23)$$

$$\phi^2[s, \xi^\top](y) = \psi[\xi\tau_2(s), \mu_2(s)](y), \quad (24)$$

for every  $s \in \mathcal{M}$ , and almost every  $y \in Q$ . The expression of  $\phi[s, \xi^\top]$  can be further simplified. Indeed, by (15) and (16) we have  $-\operatorname{div}(a \nabla(\varphi_a \cdot \sigma)) = -\mathbf{div}(a \nabla \varphi_a) \cdot \sigma = \nabla a \cdot \sigma$  and  $-\operatorname{div}(a \nabla(\varphi_\kappa \cdot \omega)) = -\nabla \kappa \cdot \omega$ . Therefore  $\psi[\sigma, \omega] = \varphi_a \cdot \sigma + \varphi_\kappa \cdot \omega$ , and we deduce the following identities:

$$\phi^1[s, \xi^\top](y) = \psi[\xi\tau_1(s), \mu_1(s)](y) = \varphi_a(y) \cdot \xi\tau_1(s) + \varphi_\kappa(y) \cdot \mu_1(s), \quad (25)$$

$$\phi^2[s, \xi^\top](y) = \psi[\xi\tau_2(s), \mu_2(s)](y) = \varphi_a(y) \cdot \xi\tau_2(s) + \varphi_\kappa(y) \cdot \mu_2(s), \quad (26)$$

for every  $s \in \mathcal{M}$  and almost every  $y \in Q$ . This completes the proof of (17).

To prove (18) we observe that  $\phi(y) := \phi[s, \xi^\top](y) \in H_{\sharp}^1(Q, T_s \mathcal{M})$  satisfies the Euler-Lagrange equations

$$\int_Q \{a(y)(\xi + \nabla \phi(y)) : \nabla \phi(y) - \kappa(y) \chi(s) : \nabla \phi(y)\} dy = 0 \quad \forall \phi \in H_{\sharp}^1(Q, T_s \mathcal{M}).$$

In particular, choosing  $\varphi = \phi$  we get

$$\int_Q \kappa(y) \chi(s) : \nabla \phi(y) dy = \int_Q a(y)(\xi + \nabla \phi(y)) : \nabla \phi(y) dy.$$

This implies that

$$\begin{aligned} T_{\text{hom}}(s, \xi^\top) &= \int_Q \left\{ \frac{1}{2} a(y) |\xi + \nabla \phi(y)|^2 - a(y)(\xi + \nabla \phi(y)) : \nabla \phi(y) - \kappa(y) \chi(s) : \xi \right\} dy \\ &= \frac{1}{2} \int_Q a(y) (|\xi|^2 - |\nabla \phi(y)|^2) dy - \int_Q \kappa(y) \chi(s) : \xi dy, \end{aligned}$$

which yields (18). ■

**Corollary 6.** *For every  $m_0 \in H^1(\Omega, \mathcal{M})$ , the map  $x \in \Omega \mapsto T_{\text{hom}}(m_0(x), \nabla m_0(x)) \in \mathbb{R}$  is measurable.*

**Proof.** From the proof of Proposition 5 we find that the tangentially homogenized energy density in (14) can be written as

$$T_{\text{hom}}(s, \xi^\top) := \int_Q \left\{ \frac{1}{2} a(y) |\xi_\tau(s) + \nabla_y \phi_\tau[s, \xi^\top](y)|^2 - \kappa(y) (\xi_\tau(s) + \nabla_y \phi_\tau[s, \xi_\tau](y)) : \mu(s) \right\} dy, \quad (27)$$

with  $\phi_\tau[s, \xi^\top] = (\phi^1[s, \xi^\top], \phi^2[s, \xi^\top])$ , and  $\xi_\tau(s) := (\xi\tau_1(s), \xi\tau_2(s))$ , for every  $s \in \mathcal{M}$  and for almost every  $y \in Q$ . Note that, when  $s := m(x)$  and  $\xi := \nabla m(x)$  we have  $\xi_\tau(x) := \nabla_\tau m(x) = (\nabla m(x) \tau_1(m(x)), \nabla m(x) \tau_2(m(x)))$ .

The measurability of the map  $x \in \Omega \mapsto T_{\text{hom}}(m_0(x), \nabla m_0(x))$  follows from (27), the regularity of  $\mathcal{M}$ , and the explicit expressions of  $\phi_\tau[s, \xi^\top]$  in (17). ■

Our main result is to show that  $T_{\text{hom}}$  represents the effective energy density associated to our homogenization problem.

**Theorem 7.** *The family  $(\mathcal{G}_\varepsilon)_{\varepsilon \in \mathbb{R}_+}$   $\Gamma$ -converges with respect to the weak topology in  $H^1(\Omega, \mathcal{M})$ , to the energy functional*

$$\begin{aligned} \mathcal{G}_0(m) &:= \int_{\Omega} T_{\text{hom}}(m(x), \nabla m(x)) \, dx \\ &= \frac{1}{2} \int_{\Omega \times Q} a(y) \left( |\nabla^{\text{T}} m(x)|^2 - |\nabla_y \phi[m(x), \nabla^{\text{T}} m(x)](y)|^2 \right) \, dy \, dx \\ &\quad - \int_{\Omega \times Q} \kappa(y) \chi(m(x)) : \nabla m(x) \, dy \, dx, \end{aligned}$$

for every  $m \in H^1(\Omega, \mathcal{M})$ .

**Proof.** The proof of Theorem 7 is subdivided into two main steps: the compactness of sequences with equibounded energies and the liminf inequality are the subject of Theorem 10; the optimality of the upper bound follows from Theorem 11. The second equality in (7) is a direct consequence of Proposition 5.  $\blacksquare$

**2.1. Two-scale limits of fields in  $H^1(\Omega, \mathcal{M})$ .** In this section, we characterize the two-scale asymptotic behavior of sequences in  $H^1(\Omega, \mathcal{M})$ .

**Proposition 8.** *Let  $\mathcal{M}$  be a  $C^2$  orientable hypersurface in  $\mathbb{R}^3$ , and let  $(u_\varepsilon)_{\varepsilon \in \mathbb{R}_+} \subset H^1(\Omega, \mathcal{M})$ . Assume that there exist  $u_0 \in H^1(\Omega, \mathbb{R}^3)$  and  $u_1 \in L^2(\Omega, H_{\sharp}^1(Q, \mathbb{R}^3))$  such that*

$$u_\varepsilon \rightarrow u_0 \quad \text{strongly in } L^2(\Omega, \mathbb{R}^3), \quad (28)$$

$$\nabla u_\varepsilon \rightharpoonup \nabla u_0 + \nabla_y u_1 \quad \text{weakly in } L^2(\Omega \times Q, \mathbb{R}^{3 \times 3}). \quad (29)$$

Then,  $u_0 \in H^1(\Omega, \mathcal{M})$  and  $u_1(x, y) \in T_{u_0(x)} \mathcal{M}$  for almost every  $(x, y) \in \Omega \times Q$ .

**Proof.** Since  $\mathcal{M}$  is a  $C^2$  orientable hypersurface, there exist an open tubular neighborhood  $U \subseteq \mathbb{R}^3$  of  $\mathcal{M}$ , and a  $C^2$  function  $\gamma: U \rightarrow \mathbb{R}$ , which has zero as a regular value, and is such that  $\mathcal{M} = \gamma^{-1}(0)$ . In view of (28), we have, up to the extraction of a (not relabeled) subsequence,  $0 = \gamma(u_\varepsilon(x)) \rightarrow \gamma(u_0(x)) = 0$  for almost every  $x \in \Omega$ ; therefore  $u_0(x) \in \mathcal{M}$  for almost every  $x \in \Omega$ . Additionally, for every  $\varepsilon > 0$  there holds,

$$0 = \nabla(\gamma \circ u_\varepsilon) = \nabla u_\varepsilon \nabla \gamma(u_\varepsilon).$$

By (28) it follows that  $\nabla \gamma(u_\varepsilon) \rightarrow \nabla \gamma(u_0)$  strongly in  $L^2(\Omega, \mathbb{R}^3)$ . Thus, by (29) we obtain, for every  $\psi \in \mathcal{D}(\Omega, C_{\sharp}^\infty(Q, \mathbb{R}^3))$ ,

$$\begin{aligned} 0 &= \int_{\Omega \times Q} \nabla u_\varepsilon \nabla \gamma(u_\varepsilon) \cdot \psi(x, y) \, dx \, dy \\ &\xrightarrow{\varepsilon \rightarrow 0} \int_{\Omega \times Q} [\nabla u_0(x) + \nabla_y u_1(x, y)] \nabla \gamma(u_0(x)) \cdot \psi(x, y) \, dx \, dy. \end{aligned} \quad (30)$$

In particular, considering a test function  $\psi$  independent of  $y$ , since  $\int_Q \nabla_y u_1(x, y) \, dy = 0$  for almost every  $x \in \Omega$ , we conclude that  $\nabla u_0 \nabla \gamma(u_0) = 0$  in  $\Omega$ . In particular, from (30) we infer that

$$\int_{\Omega \times Q} \nabla_y(u_1(x, y) \cdot \nabla \gamma(u_0(x))) \cdot \psi(x, y) \, dx \, dy = 0 \quad \forall \psi \in \mathcal{D}(\Omega, C_{\sharp}^\infty(Q, \mathbb{R}^N)).$$

Hence,  $u_1(x, y) \cdot \nabla \gamma(u_0(x)) = c(x)$  for almost every  $(x, y) \in \Omega \times Q$ , for some function  $c \in L^2(\Omega)$ . As  $\int_Q u_1(x, y) dy = 0$  for almost every  $x \in \Omega$ , it follows that  $c \equiv 0$ . Thus,  $u_1 \cdot \nabla \gamma(u_0) \equiv 0$ . The thesis follows by observing that the vector  $\nabla \gamma(u_0(x))$  is orthogonal to  $T_{u_0(x)}\mathcal{M}$  for almost every  $x \in \Omega$ .  $\blacksquare$

**Remark 9.** The proposition holds if we assume, more generally, that  $\mathcal{M}$  is the inverse image of a regular value of a  $C^2$  function  $\gamma: U \subseteq \mathbb{R}^N \rightarrow \mathbb{R}^M$  with  $M < N$ . Indeed, in this case, there exist  $M$  linearly independent normal vector fields,  $(n_j)_{j \in \mathbb{N}^M}$ , which at every point  $p \in \mathcal{M}$  span the orthogonal complement of  $T_p\mathcal{M}$ . Repeating the same argument, one then finds that  $u_1 \cdot n_j(u_0) = 0$  for every  $j \in \mathbb{N}^M$ , and therefore  $u_1(x, y) \in T_{u_0(x)}\mathcal{M}$ .

**2.2. Compactness and  $\Gamma$ -Liminf inequality in the 3d-setting.** This section is devoted to the identification of a lower bound for the limiting behavior of the energy functionals  $\mathcal{E}_\varepsilon$ . In what follows, as suggested by Lemma 8, we will denote by  $L^2(\Omega, H_{\sharp}^1(Q, T_{m_0}\mathcal{M}))$  the set of maps  $\phi \in L^2(\Omega, H_{\sharp}^1(Q, \mathbb{R}^3))$  such that  $\phi(x) \in T_{m_0(x)}\mathcal{M}$  for almost every  $x \in \Omega$ .

**Theorem 10.** *Let  $(m_\varepsilon)_\varepsilon \subset H^1(\Omega, \mathcal{M})$  be such that  $\sup_{\varepsilon > 0} \mathcal{G}_\varepsilon(m_\varepsilon) < +\infty$ . Then, there exists  $m_0 \in H^1(\Omega, \mathcal{M})$  and  $m_1 \in L^2(\Omega, H_{\sharp}^1(Q, T_{m_0}\mathcal{M}))$  such that*

$$m_\varepsilon \rightharpoonup m_0 \quad \text{weakly in } H^1(\Omega, \mathcal{M}), \quad (31)$$

$$\nabla m_\varepsilon \rightharpoonup \nabla m_0 + \nabla_y m_1 \quad \text{weakly in } L^2(\Omega \times Q, \mathbb{R}^{3 \times 3}). \quad (32)$$

Additionally,

$$\mathcal{G}_0(m_0) \leq \liminf_{\varepsilon \rightarrow 0} \mathcal{G}_\varepsilon(m_\varepsilon). \quad (33)$$

**Proof.** The compactness result is a direct consequence of Proposition 8, the assumptions on  $a$  and  $\kappa$ , and the boundedness of  $\mathcal{M}$ . First, we observe that the energy density (12) can be rearranged as follows:

$$\begin{aligned} f(x, s, \xi^\top) &= \frac{1}{2} a_\varepsilon(x) \left( |\xi|^2 - 2 \frac{\kappa_\varepsilon(x)}{a_\varepsilon(x)} \chi(s) : \xi \right) \\ &= \frac{1}{2} a_\varepsilon(x) \left( \left| \xi - \frac{\kappa_\varepsilon(x)}{a_\varepsilon(x)} \chi(s) \right|^2 - \left| \frac{\kappa_\varepsilon(x)}{a_\varepsilon(x)} \chi(s) \right|^2 \right) \\ &= \frac{1}{2} a_\varepsilon(x) \left| \xi - \frac{\kappa_\varepsilon(x)}{a_\varepsilon(x)} \chi(s) \right|^2 - \frac{1}{2} \frac{\kappa_\varepsilon^2(x)}{a_\varepsilon(x)} |s|^2, \end{aligned} \quad (34)$$

with  $\chi: s \in \mathcal{M} \mapsto (e_1 \times s, e_2 \times s, e_3 \times s)^\top \in \mathbb{R}^{3 \times 3}$ . Therefore

$$\mathcal{G}_\varepsilon(m_\varepsilon) = \mathcal{H}_\varepsilon(m_\varepsilon) + \mathcal{I}_\varepsilon(m_\varepsilon), \quad (35)$$

where

$$\mathcal{H}_\varepsilon(m_\varepsilon) := \frac{1}{2} \int_\Omega a_\varepsilon(x) \left| \nabla m_\varepsilon(x) - \frac{\kappa_\varepsilon(x)}{a_\varepsilon(x)} \chi(m_\varepsilon(x)) \right|^2 dx,$$

and

$$\mathcal{I}_\varepsilon(m_\varepsilon) := - \int_\Omega \frac{\kappa_\varepsilon^2(x)}{a_\varepsilon(x)} |m_\varepsilon(x)|^2 dx.$$

Next, we point out that for every  $\psi \in \mathcal{D}(\Omega, H_{\sharp}^1(Q, \mathbb{R}^3))$  there holds

$$\begin{aligned} \mathcal{H}_{\varepsilon}(m_{\varepsilon}) &\geq \int_{\Omega \times Q} a_{\varepsilon}(x) \left( \nabla m_{\varepsilon}(x) - \frac{\kappa_{\varepsilon}(x)}{a_{\varepsilon}(x)} \chi(m_{\varepsilon}(x)) \right) : \\ &\quad \left( \nabla m_0(x) + \nabla_y \psi(x, y) - \frac{\kappa(y)}{a(y)} \chi(m_0(x)) \right) dy dx \\ &\quad - \frac{1}{2} \int_{\Omega \times Q} a_{\varepsilon}(x) \left| \nabla m_0(x) + \nabla_y \psi(x, y) - \frac{\kappa(y)}{a(y)} \chi(m_0(x)) \right|^2 dy dx, \end{aligned}$$

because the difference of the integrand on the left-hand side with the integrand on the right-hand side is a perfect square. Thus, by standard properties of two-scale convergence, and owing to the regularity of  $\psi$ , we obtain the inequality

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \mathcal{H}_{\varepsilon}(m_{\varepsilon}) &\geq \int_{\Omega \times Q} a(y) \left( \nabla m_0(x) + \nabla_y m_1(x, y) - \frac{\kappa(y)}{a(y)} \chi(m_0(x)) \right) : \\ &\quad \left( \nabla m_0(x) + \nabla_y \psi(x, y) - \frac{\kappa(y)}{a(y)} \chi(m_0(x)) \right) dy dx \\ &\quad - \frac{1}{2} \int_{\Omega \times Q} a(y) \left| \nabla m_0(x) + \nabla_y \psi(x, y) - \frac{\kappa(y)}{a(y)} \chi(m_0(x)) \right|^2 dy dx. \end{aligned}$$

By density, there exists  $(\psi_n)_{n \in \mathbb{N}} \subset \mathcal{D}(\Omega, H_{\sharp}^1(Q, \mathbb{R}^3))$  such that  $\psi_n \rightarrow m_1$  in  $L^2(\Omega, H_{\sharp}^1(Q, \mathbb{R}^3))$ . Therefore, the previous inequality yields

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \mathcal{H}_{\varepsilon}(m_{\varepsilon}) &\geq \frac{1}{2} \int_{\Omega \times Q} a(y) \left| \nabla m_0 + \nabla_y m_1(x, y) - \frac{\kappa(y)}{a(y)} \chi(m_0(x)) \right|^2 dy dx \\ &\geq \frac{1}{2} \int_{\Omega} \inf_{\phi \in H_{\sharp}^1(Q, T_{m_0(x)} \mathcal{M})} \int_Q a(y) \left| \nabla m_0 + \nabla_y \phi(y) - \frac{\kappa(y)}{a(y)} \chi(m_0(x)) \right|^2 dy dx. \end{aligned} \quad (36)$$

Since  $\kappa_{\varepsilon}^2/a_{\varepsilon} \rightharpoonup^* \langle \kappa^2/a \rangle_Q$  weakly\* in  $L^{\infty}(Q)$ , we conclude that

$$\lim_{\varepsilon \rightarrow 0} \mathcal{I}_{\varepsilon}(m_{\varepsilon}) = - \int_{\Omega} \langle \kappa^2/a \rangle_Q |m_0(x)|^2 dx. \quad (37)$$

By combining (35), (36), and (37) we obtain (33).  $\blacksquare$

**2.3. The Limsup inequality in the 3d-setting.** In this section we show that the lower bound identified in Theorem 10 is optimal. To be precise, we prove the following result.

**Theorem 11.** *Let  $\mathcal{M}$  be a  $C^2$  orientable hypersurface of  $\mathbb{R}^N$  such that  $\mathcal{M}$  has a tubular neighborhood of uniform thickness  $\delta > 0$ . Let  $m_0 \in H^1(\Omega, \mathcal{M})$ . Then, there exists a sequence  $(m_{\varepsilon})_{\varepsilon > 0}$  in  $H^1(\Omega, \mathcal{M})$  such that, as  $\varepsilon \rightarrow 0$ ,*

$$m_{\varepsilon} \rightharpoonup m_0 \quad \text{weakly in } H^1(\Omega, \mathcal{M}), \quad (38)$$

and

$$\mathcal{G}_0(m_0) \geq \limsup_{\varepsilon \rightarrow 0} \mathcal{G}_{\varepsilon}(m_{\varepsilon}). \quad (39)$$

**Proof.** Denote by  $U_{\delta}$  the tubular neighborhood of size  $\delta$  around  $\mathcal{M}$ , and let

$$\pi_{\mathcal{M}}: U_{\delta} \rightarrow \mathcal{M}$$

be the pointwise projection operator. Note that, since  $\mathcal{M}$  is  $C^2$ , the projection satisfies  $\pi_{\mathcal{M}} \in C^1(U_{\delta}, \mathcal{M})$ . Clearly, we can choose  $\delta$  small enough so that  $\pi_{\mathcal{M}} \in C^1(\overline{U_{\delta}}, \mathcal{M})$ . For convenience of the reader we subdivide the proof into two steps.

**Step 1.** Given  $\psi \in C^\infty(\bar{\Omega}, W_{\#}^{1,\infty}(Q, \mathbb{R}^3))$ , for every  $\varepsilon > 0$  and almost every  $x \in \Omega$  we set

$$\hat{m}_\varepsilon(x) := m_0(x) + \varepsilon \psi\left(x, \frac{x}{\varepsilon}\right), \quad m_\varepsilon := \pi_{\mathcal{M}}[\hat{m}_\varepsilon].$$

We note that,

$$m_\varepsilon \rightarrow \pi_{\mathcal{M}}[m_0] \equiv m_0 \quad \text{strongly in } L^2(\Omega, \mathbb{R}^3), \quad (40)$$

$$\nabla \hat{m}_\varepsilon \rightharpoonup \nabla m_0(x) + \nabla_y \psi(x, y) \quad \text{strongly two-scale in } L^2(\Omega \times Q, \mathbb{R}^3). \quad (41)$$

Given the regularity of  $\psi$ , for  $\varepsilon$  small enough there holds  $\hat{m}_\varepsilon \in U_\delta$  for almost every  $x \in \Omega$ . By the regularity of  $\pi_{\mathcal{M}}$ , there exists a constant  $c_{\mathcal{M}} > 0$  depending only on  $\mathcal{M}$ , such that

$$|\nabla m_\varepsilon| \leq c_{\mathcal{M}} |\nabla \hat{m}_\varepsilon| \quad \text{a.e. in } \Omega. \quad (42)$$

By (40) and the boundedness of  $(\nabla \hat{m}_\varepsilon)$  in  $L^2(\Omega, \mathbb{R}^{3 \times 3})$ , we deduce that, up to the extraction of a not relabelled subsequence,

$$\hat{m}_\varepsilon \rightharpoonup m_0 \quad \text{weakly in } H^1(\Omega, \mathcal{M}), \quad (43)$$

$$m_\varepsilon = \pi_{\mathcal{M}}[\hat{m}_\varepsilon] \rightharpoonup m_0 \quad \text{weakly in } H^1(\Omega, \mathcal{M}). \quad (44)$$

Moreover, by Proposition 8, we infer that, up to the extraction of a not relabelled subsequence, there holds

$$\nabla m_\varepsilon \rightharpoonup \nabla m_0 + \nabla_y \phi \quad \text{weakly two-scale in } L^2(\Omega \times Q, \mathbb{R}^3), \quad (45)$$

for some  $\phi \in L^2(\Omega, H_{\#}^1(Q, T_{m_0}\mathcal{M}))$  with  $\int_Q \phi(x, y) dy = 0$  for almost every  $x \in \Omega$ .

Next, a direct computation shows that

$$\nabla m_\varepsilon(x) = \nabla \pi_{\mathcal{M}}[\hat{m}_\varepsilon(x)] \left[ \nabla m_0(x) + \varepsilon \nabla_x \psi\left(x, \frac{x}{\varepsilon}\right) + \nabla_y \psi\left(x, \frac{x}{\varepsilon}\right) \right], \quad (46)$$

for almost every  $x \in \Omega$ . By (43), and by the regularity of  $\psi$  and  $\pi_{\mathcal{M}}$ , it follows that

$$\nabla \pi_{\mathcal{M}}[\hat{m}_\varepsilon] \rightarrow \nabla \pi_{\mathcal{M}}[m_0] \quad \text{strongly in } L^2(\Omega, \mathbb{R}^{3 \times 3}), \quad (47)$$

$$\varepsilon \nabla_x \psi\left(x, \frac{x}{\varepsilon}\right) \rightarrow 0 \quad \text{strongly in } L^\infty(\Omega, \mathbb{R}^{3 \times 3}), \quad (48)$$

$$\nabla_y \psi\left(x, \frac{x}{\varepsilon}\right) \rightarrow \nabla_y \psi \quad \text{strongly two-scale in } L^2(\Omega \times Q, \mathbb{R}^{3 \times 3}). \quad (49)$$

By combining the above convergences and (46), we conclude that

$$\nabla m_\varepsilon \rightharpoonup \nabla \pi_{\mathcal{M}}[m_0] (\nabla m_0 + \nabla_y \psi) \quad \text{weakly two-scale in } L^2(\Omega \times Q, \mathbb{R}^{3 \times 3}). \quad (50)$$

Therefore, in view of (45),

$$\nabla m_0 + \nabla_y \phi = \nabla \pi_{\mathcal{M}}[m_0] (\nabla m_0 + \nabla_y \psi),$$

almost everywhere in  $\Omega \times Q$ . Since both  $\phi$  and  $\psi$  have null average in  $Q$ , this implies that  $\nabla \pi_{\mathcal{M}}[m_0] \nabla m_0 = \nabla m_0$  for almost every  $x \in \Omega$ . In particular, by (50) we infer that

$$\nabla m_\varepsilon \rightharpoonup \nabla m_0 + \nabla \pi_{\mathcal{M}}[m_0] \nabla_y \psi \quad \text{weakly two-scale in } L^2(\Omega \times Q, \mathbb{R}^{3 \times 3}). \quad (51)$$

To see that the previous convergence is actually stronger, we observe that

$$\begin{aligned} & \left| \|\nabla m_\varepsilon\|_{L^2(\Omega, \mathbb{R}^{3 \times 3})} - \|\nabla \pi_{\mathcal{M}}[m_0] (\nabla m_0 + \nabla_y \psi)\|_{L^2(\Omega \times Q)} \right| \\ & \leq \|\nabla m_\varepsilon - \nabla \pi_{\mathcal{M}}[m_0] (\nabla m_0 + \nabla_y \psi)\|_{L^2(\Omega \times Q)} \end{aligned} \quad (52)$$



and

$$\begin{aligned}
\nabla m_\varepsilon - \nabla \pi_{\mathcal{M}}[m_0](\nabla m_0 + \nabla_y \psi) &= \nabla \pi_{\mathcal{M}}[\hat{m}_\varepsilon] \nabla \hat{m}_\varepsilon - \nabla \pi_{\mathcal{M}}[m_0] \nabla m_0 - \nabla \pi_{\mathcal{M}}[m_0] \nabla_y \psi \\
&= (\nabla \pi_{\mathcal{M}}[\hat{m}_\varepsilon] - \nabla \pi_{\mathcal{M}}[m_0]) \nabla m_0 \\
&\quad + \nabla \pi_{\mathcal{M}}[m_0] (\nabla \hat{m}_\varepsilon - (\nabla m_0 + \nabla_y \psi)) \\
&\quad + (\nabla \pi_{\mathcal{M}}[\hat{m}_\varepsilon] - \nabla \pi_{\mathcal{M}}[m_0]) (\nabla \hat{m}_\varepsilon - \nabla m_0). \tag{53}
\end{aligned}$$

Now, the first term in the right-hand side of (53) converges to zero in  $L^2(\Omega, \mathbb{R}^{3 \times 3})$  owing to (47), the regularity of  $\mathcal{M}$ , and the Dominated Convergence Theorem. The second term converges to zero strongly two-scale in  $L^2(\Omega \times Q, \mathbb{R}^{3 \times 3})$  due to (41) and the regularity of  $\mathcal{M}$ . Finally, the last term in the right-hand side of (53) converges to zero strongly in  $L^2(\Omega, \mathbb{R}^{3 \times 3})$  because of (47), the boundedness of  $(\nabla \hat{m}_\varepsilon)$  in  $L^2(\Omega, \mathbb{R}^{3 \times 3})$ , and Lebesgue's dominated convergence theorem. This proves that

$$\nabla m_\varepsilon \rightarrow \nabla m_0 + \nabla \pi_{\mathcal{M}}[m_0] \nabla_y \psi \quad \text{strongly in } L^2(\Omega \times Q, \mathbb{R}^{3 \times 3}). \tag{54}$$

In view of (54) and of the regularity of  $\psi$  we directly obtain that

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0} \mathcal{G}_\varepsilon(m_\varepsilon) &= \frac{1}{2} \int_{\Omega \times Q} a(y) |\nabla m_0(x) + \nabla \pi_{\mathcal{M}}[m_0(x)] \nabla_y \psi(x, y)|^2 dy dx \\
&\quad - \int_{\Omega \times Q} \kappa(y) \chi(m_0(x)) : (\nabla m_0(x) + \nabla \pi_{\mathcal{M}}[m_0(x)] \nabla_y \psi(x, y)) dy dx. \tag{55}
\end{aligned}$$

with  $\chi: s \in \mathcal{M} \mapsto (e_1 \times s, e_2 \times s, e_3 \times s)^\top \in \mathbb{R}^{3 \times 3}$ .

**Step 2.** Let now  $m_0 \in H^1(\Omega, \mathcal{M})$ , and let

$$m_1(x, y) := \phi[m_0(x), \nabla^\top m_0(x)](y), \tag{56}$$

where  $\phi[x, \xi^\top](y)$  is the map defined in Proposition 5. We observe that  $m_1$  has the same regularity in  $y$  as the maps  $\varphi_a(y)$ ,  $\varphi_\kappa(y)$  defined in (15), (16), and the same regularity in  $x$  as  $\nabla m_0$ . Thus, in particular,  $m_1 \in L^2(\Omega, H_\#^1(Q, T_{m_0} \mathcal{M}))$ . By density, and by means of a mollification procedure in the  $y$  variable, we find a sequence  $\psi_k \in \mathcal{D}(\Omega, W_\#^{1, \infty}(\bar{Q}, \mathbb{R}^3))$  such that

$$\psi_k \rightarrow m_1 \quad \text{strongly in } L^2(\Omega, H_\#^1(Q, \mathbb{R}^3)). \tag{57}$$

Setting

$$\begin{aligned}
\mathcal{F}(m_0, \psi) &:= \frac{1}{2} \int_{\Omega \times Q} a(y) |\nabla m_0(x) + \nabla \pi_{\mathcal{M}}[m_0(x)] \nabla_y \psi(x, y)|^2 dy dx \\
&\quad - \int_{\Omega \times Q} \kappa(y) \chi(m_0(x)) : (\nabla m_0(x) + \nabla \pi_{\mathcal{M}}[m_0(x)] \nabla_y \psi(x, y)) dy dx
\end{aligned}$$

for every  $\psi \in L^2(\Omega; H_\#^1(Q, T_{m_0} \mathcal{M}))$ , we have

$$\lim_{k \rightarrow +\infty} \mathcal{F}(m_0, \psi_k) = \mathcal{F}(m_0, m_1). \tag{58}$$

Additionally, we observe that

$$\mathcal{G}_0(m_0) = \mathcal{F}(m_0, m_1). \tag{59}$$

This follows from the fact that for every  $s \in \mathcal{M}$ , and for every  $v \in T_s \mathcal{M}$ , there holds  $\nabla \pi_{\mathcal{M}}(s)v = v$ . Indeed, since  $m_1(x) \in T_{m_0(x)} \mathcal{M}$  for almost every  $x \in \Omega$ , we conclude that  $\nabla_y m_1 = \nabla \pi_{\mathcal{M}}[m_0] \nabla_y m_1$ .

In view of (58) and (59), for every  $\delta > 0$  there exists  $\psi_{k_\delta} \in \mathcal{D}(\Omega, W_{\#}^{1,\infty}(\bar{Q}, \mathbb{R}^3))$  such that

$$\mathcal{F}(m_0, \psi_{k_\delta}) \leq \mathcal{G}_0(m_0) + \delta.$$

By Step 1 and (55) there exists an associated sequence  $(m_\varepsilon^\delta)_{\varepsilon > 0}$  in  $H^1(\Omega, \mathcal{M})$  such that

$$m_\varepsilon^\delta \rightharpoonup m_0 \quad \text{weakly in } H^1(\Omega, \mathcal{M}) \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \mathcal{G}_\varepsilon(m_\varepsilon^\delta) = \mathcal{F}(m_0, \psi_{k_\delta}),$$

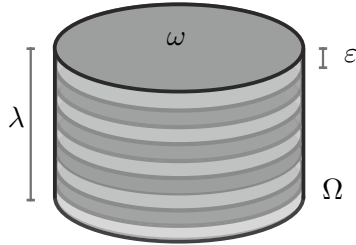
and hence

$$\lim_{\varepsilon \rightarrow 0} \mathcal{G}_\varepsilon(m_\varepsilon^\delta) \leq \mathcal{G}_0(m_0) + \delta.$$

By the arbitrariness of  $\delta$ , the limsup inequality follows then from classical properties of  $\Gamma$ -convergence (see Section 1.2 in [9]).  $\blacksquare$

### 3. THE MICROMAGNETIC SETTING: APPLICATIONS TO MULTILAYERS

In this section, we specify the characterization of the effective energy to the micromagnetic setting of chiral multilayers. In this case,  $\mathcal{M} = \mathbb{S}^2$  and  $\Omega \subseteq \mathbb{R}^3$  has a *laminated* structure as in Figure 1. We have the following result.



**Figure 1.** We consider here a chiral multilayer having a *laminated* structure, namely  $\Omega := \omega \times I_\lambda$  with  $\omega \subset \mathbb{R}^2$ ,  $|\omega| = 1$ ,  $\lambda > 0$ , and  $I_\lambda = (0, \lambda)$ . We assume that the material-dependent functions  $a_\varepsilon(y) = a(y/\varepsilon)$  and  $\kappa_\varepsilon(y) = \kappa(y/\varepsilon)$  depend only on the third coordinate:  $a(y) = a(y \cdot e_3) =: a(t)$  and  $\kappa(y) = \kappa(y \cdot e_3) =: \kappa(t)$ .

**Theorem 12.** *Assume that  $\Omega := \omega \times I_\lambda$  with  $\omega \subset \mathbb{R}^2$ ,  $|\omega| = 1$ ,  $\lambda > 0$ ,  $I_\lambda = (0, \lambda)$ , and that the material-dependent functions  $a, \kappa \in L_{\#}^{\infty}(Q, \mathbb{R})$  depend only on the third coordinate:  $a(y) = a(y \cdot e_3)$  and  $\kappa(y) = \kappa(y \cdot e_3)$ . Then, the homogenized energy functional (7) is given, for every  $m \in H^1(\Omega, \mathbb{S}^2)$ , by*

$$\begin{aligned} \mathcal{G}_0(m) &= \frac{1}{2} \int_{\Omega} (\langle a \rangle_Q |\nabla m(x)|^2 - \alpha_0 |\partial_3 m(x)|^2 - \beta_0 |e_3 \times m(x)|^2) dx \\ &\quad - \int_{\Omega} (\gamma_0 \partial_3 m(x) \cdot (e_3 \times m(x)) + \langle \kappa \rangle_Q \chi(m(x)) : \nabla m(x)) dx, \end{aligned} \quad (60)$$

where the effective parameters are defined by

$$\alpha_0 := \langle a \rangle_Q - \frac{1}{\langle a^{-1} \rangle_Q}, \quad \beta_0 := \langle \kappa^2 / a \rangle_Q - \frac{\langle \kappa / a \rangle_Q^2}{\langle a^{-1} \rangle_Q}, \quad \gamma_0 := \frac{\langle \kappa / a \rangle_Q}{\langle a^{-1} \rangle_Q} - \langle \kappa \rangle_Q. \quad (61)$$

Additionally, for  $\langle \kappa \rangle_Q = 0$  the only energy minimizers are the helical textures

$$m_*(x) := \cos(\theta(x \cdot e_3)) e_1 + \sin(\theta(x \cdot e_3)) e_2, \quad \theta(t) := \theta_0 + \langle \kappa / a \rangle_Q t \quad \text{for every } t \in \mathbb{R}, \quad (62)$$

with  $\theta_0 \in \mathbb{R}$  arbitrary, and the minimum value of the energy is  $\mathcal{G}_0(m_*) = -\frac{\lambda}{2} \langle \kappa^2/a \rangle_Q$ .

**Remark 13.** It is interesting to note that the DMI layers contribute to an increase in the crystal anisotropy of the magnetic system. When  $\langle \kappa \rangle_Q = 0$ , the minimizers of  $\mathcal{G}_0$  are always planar and constant on each layer. Moreover, depending on the effective DMI constant  $\gamma_0$  of the homogeneous model, two Bloch-type chiral skyrmions with opposite (and possibly non-integral) topological charges can arise. Indeed, the sign of  $\gamma_0$ , equivalently the sign of  $\langle \kappa/a \rangle_Q$ , controls the chirality of the system: the minimizers describe right-handed helices when  $\gamma_0 > 0$  and left-handed helices when  $\gamma_0 < 0$  (cf. Figure 2). Additionally, identifying  $m_*$  with the curve

$$\tilde{m}_*(t) := e^{i\theta(t)} \quad \text{for every } t \in I_\lambda,$$

in the complex plane, for

$$\langle \kappa/a \rangle_Q \neq 0 \quad \text{and} \quad \lambda = \frac{2\pi n}{|\langle \kappa/a \rangle_Q|}, \quad n \in \mathbb{N}, \quad (63)$$

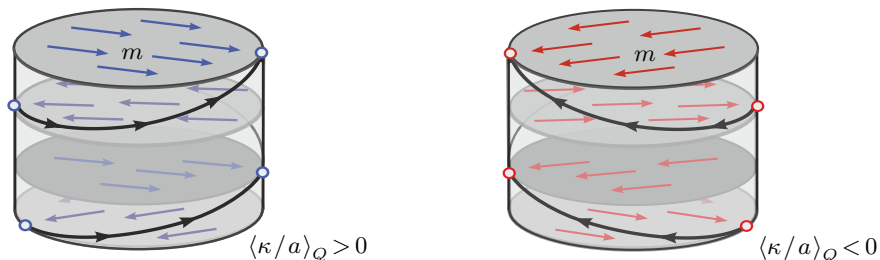
one can interpret  $\tilde{m}_*$  as a map from  $\mathbb{S}^1$  to  $\mathbb{S}^1$ , whose skyrmion number (cf. (2)) coincides with the *winding number* of  $\tilde{m}_*$  around the origin:

$$N_{\text{sk}}(\tilde{m}_*) = \text{Ind}_{\tilde{m}_*}(0) := \frac{1}{2\pi i} \int_0^\lambda \frac{\tilde{m}'_*(t)}{\tilde{m}_*(t)} dt = \frac{\theta(\lambda) - \theta_0}{2\pi} = n \text{sign}(\langle \kappa/a \rangle_Q) = \frac{\lambda \langle \kappa/a \rangle_Q}{2\pi}.$$

In other words, for the values of  $\lambda$  specified in (63), the sign of  $\langle \kappa/a \rangle_Q$  determines that of the topological degree of  $m_*$ .

**Remark 14.** Note that, the shape of the minimizers does not depend on the height  $\lambda$  of the multilayer. Instead, the minimum value of the energy scales linearly in  $\lambda$ .

**Remark 15.** Our analysis in Theorem 12 does not take into account long-range effects such as the ones originating from magnetostatic interaction, as well as magnetocrystalline effects. As already pointed out these contributions can be superimposed to our energy functional because, from the variational point of view, they play the role of a continuous perturbation. For example, to include magnetostatic interaction, one has to consider the augmented energy functional (cf. Proposition 1)  $\mathcal{G}_0 + \mathcal{W}_0$ , with  $\mathcal{W}_0$  given by (6). In this case, although we expect similar qualitative considerations, an explicit characterization of the minimizing profiles of  $\mathcal{G}_0 + \mathcal{W}_0$  will be hardly achievable.



**Figure 2.** A schematic representation of the minimizers of  $\mathcal{G}_\perp$ . A posteriori, they completely characterize the profiles of the minimizers of  $\mathcal{G}_0$ . The minimizers of  $\mathcal{G}_0$  are always planar and constant on every layer. According to the sign of  $\gamma_0$ , equivalently on the sign of  $\langle \kappa/a \rangle_Q$ , they describe right-handed (left) or left-handed helices (right).

**Proof.** We denote by  $t$  the third coordinate in  $Q$ , we write  $Q = Q' \times I$ , and we consider the setting in which  $a(y) = a(y \cdot e_3) =: a(t)$  and  $\kappa(y) = \kappa(y \cdot e_3) =: \kappa(t)$  for almost every  $y \in Q$ . In this framework, the two functions  $\varphi_\kappa$  and  $\varphi_a$  can be computed explicitly in terms of  $\kappa$  and  $a$ . Indeed, consider the equation

$$\int_Q a \nabla \varphi_\kappa : \nabla \psi = \int_Q \kappa \operatorname{div} \psi. \quad (64)$$

Testing (64) against vector fields  $\psi \in H_{\#}^1(Q, \mathbb{R}^3)$  that do not depend on  $t$  we get that, for a.e.  $t \in I_\lambda$ , the distribution  $\varphi_\kappa(\cdot, t)$  is harmonic:

$$\Delta_\omega \varphi_\kappa(\cdot, t) = 0 \quad \text{in } \mathcal{D}'_{\#}(Q', \mathbb{R}^3), \quad (65)$$

where we set  $\Delta_\omega \varphi_\kappa(\cdot, t) := \partial_{y_1}^2 \varphi_\kappa(\cdot, t) + \partial_{y_2}^2 \varphi_\kappa(\cdot, t)$ . Hence, by Weyl's lemma,  $\varphi_\kappa(\cdot, t) \in C_{\#}^\infty(Q', \mathbb{R}^3)$ . Also, for a.e.  $t \in I_\lambda$ ,  $\varphi_\kappa(\cdot, t) \equiv 0$  is the unique solution of (65) in  $H_{\#}^1(Q', \mathbb{R}^3)/\mathbb{R} \supseteq C_{\#}^\infty(Q', \mathbb{R}^3)/\mathbb{R}$ . We conclude that  $\varphi_\kappa(y)$  depends only on the  $t$ -variable. Therefore, we set  $\varphi_\kappa(y) = \varphi_\kappa(y \cdot e_3) =: \varphi_\kappa(t)$ .

In view of (64) it follows that  $\varphi_\kappa(t)$  solves the ordinary differential equation

$$(a \varphi'_\kappa)' = \kappa' e_3 \quad \text{in } H_{\#}^{-1}(I, \mathbb{R}^3).$$

Integrating in  $[0, t]$  yields the equation

$$\varphi'_\kappa(t) = \frac{a_0}{a(t)} \varphi'_\kappa(0) + \left( \frac{\kappa(t) - \kappa_0}{a(t)} \right) e_3 \quad \text{for every } t \in I,$$

where we set  $a_0 := a(0)$ ,  $\kappa_0 := \kappa(0)$ . Integrating again in  $I$  and imposing periodicity we deduce that

$$a_0 \langle a^{-1} \rangle_Q \varphi'_\kappa(0) + (\langle \kappa/a \rangle_Q - \kappa_0 \langle a^{-1} \rangle_Q) e_3 = 0.$$

The previous relation implies that  $\varphi'_\kappa(0) \cdot e_1 = 0$ ,  $\varphi'_\kappa(0) \cdot e_2 = 0$ , and

$$\varphi'_\kappa(0) \cdot e_3 = \frac{\kappa_0}{a_0} - \frac{\langle \kappa/a \rangle_Q}{a_0 \langle a^{-1} \rangle_Q}.$$

Therefore, for every  $t \in I$

$$\mathfrak{K}(t) := \varphi'_\kappa(t) \cdot e_3 = \frac{\kappa(t)}{a(t)} - \frac{\langle \kappa/a \rangle_Q}{a(t) \langle a^{-1} \rangle_Q}. \quad (66)$$

Noting that  $\varphi_a$  solves an analogous differential equation as  $\varphi_\kappa$ , in which  $\kappa$  is replaced by  $-a$ , we conclude that  $\varphi'_a(0) \cdot e_1 = 0$ ,  $\varphi'_a(0) \cdot e_2 = 0$ , and  $\varphi'_a(0) \cdot e_3 = -1 + (a_0 \langle a^{-1} \rangle_I)^{-1}$ . Thus, for every  $t \in I$ ,

$$\mathfrak{a}(t) := \varphi'_a(t) \cdot e_3 = -1 + \frac{1}{a(t) \langle a^{-1} \rangle_Q}. \quad (67)$$

We stress that  $\mathfrak{K}(t)$  and  $\mathfrak{a}(t)$  in (66) and (67) have been introduced because they are the only quantity of interest for the tangentially homogenized energy density  $T_{\text{hom}}$ .

We recall in fact that  $T_{\text{hom}}$  reads as (cf. (18))

$$T_{\text{hom}}(s, \xi^{\text{T}}) = \frac{1}{2} \int_Q a(y) \left( |\xi|^2 - |\nabla_y \phi[s, \xi^{\text{T}}](y)|^2 \right) dy - \int_Q \kappa(y) \chi(s) : \xi dy,$$

where, for every  $s \in \mathcal{M}$  and  $\xi^\top \in T_s \mathcal{M}$ , the map  $\phi(y) := \phi[s, \xi^\top](y)$  is given by

$$\phi(y) = (\tau_1(s) | \tau_2(s)) \begin{pmatrix} \xi^\top \varphi_a(y) \cdot \tau_1(s) \\ \xi^\top \varphi_a(y) \cdot \tau_2(s) \end{pmatrix} + (\tau_1(s) | \tau_2(s)) \begin{pmatrix} \varphi_\kappa(y) \cdot \mu_1(s) \\ \varphi_\kappa(y) \cdot \mu_2(s) \end{pmatrix}.$$

In the micromagnetic setting, in which the vector field  $m$  take values in  $\mathcal{M} := \mathbb{S}^2$ , an explicit identification of  $|\nabla \phi|^2$  is available. Indeed, we find that  $\mu_1(s) = \tau_2(s)$  and  $\mu_2(s) = -\tau_1(s)$  for every  $s \in \mathbb{S}^2$ . Hence, recalling that  $\varphi_a$  and  $\varphi_\kappa$  depend only on the  $t$ -variable, for almost every  $y \in Q$ ,  $s \in \mathcal{M}$  and  $\xi^\top \in T_s \mathcal{M}$ , we deduce

$$\partial_{y_1} \phi[s, \xi^\top](y) = 0, \quad \partial_{y_2} \phi[s, \xi^\top](y) = 0,$$

$$\partial_{y_3} \phi[s, \xi^\top](y) := \mathbf{a}(t)(\tau_1(s) | \tau_2(s)) \begin{pmatrix} \xi^\top e_3 \cdot \tau_1(s) \\ \xi^\top e_3 \cdot \tau_2(s) \end{pmatrix} + \mathfrak{K}(t)(\tau_1(s) | \tau_2(s)) \begin{pmatrix} e_3 \cdot \tau_2(s) \\ -e_3 \cdot \tau_1(s) \end{pmatrix},$$

with  $\mathfrak{K}(t)$  and  $\mathbf{a}(t)$  given by (66) and (67). A direct computation shows that

$$\begin{aligned} \partial_{y_3} \phi[s, \xi^\top](y) &= \mathbf{a}(t) \xi^\top e_3 + \mathfrak{K}(t)[e_3 \cdot \tau_2(s)] \tau_1(s) - \mathfrak{K}(t)[e_3 \cdot \tau_1(s)] \tau_2(s) \\ &= \mathbf{a}(t) \xi^\top e_3 + \mathfrak{K}(t) e_3 \times (\tau_1(s) \times \tau_2(s)) \\ &= \mathbf{a}(t) \xi^\top e_3 + \mathfrak{K}(t) e_3 \times s. \end{aligned}$$

Thus, for almost every  $y \in Q$ ,  $s \in \mathcal{M}$ , and  $\xi^\top \in T_s \mathcal{M}$ , we have

$$|\nabla \phi(y)|^2 = |\mathbf{a}(y) \xi^\top e_3 + \mathfrak{K}(y) e_3 \times s|^2.$$

Substituting the previous expression in (18) gives

$$\begin{aligned} T_{\text{hom}}(s, \xi^\top) &= \frac{1}{2} \langle a \rangle_Q |\xi^\top|^2 - \langle \kappa \rangle_Q \chi^\top(s) : \xi^\top - \frac{1}{2} \int_Q a(y) |\nabla_y \phi[s, \xi^\top](y)|^2 dy \\ &= \frac{1}{2} \langle a \rangle_Q |\xi^\top|^2 - \langle \kappa \rangle_Q \chi^\top(s) : \xi^\top - \frac{1}{2} \int_Q a(y) |\mathbf{a}(y) \xi^\top e_3 + \mathfrak{K}(y) e_3 \times s|^2 dy \\ &= \frac{1}{2} \langle a \rangle_Q |\xi^\top|^2 - \langle \kappa \rangle_Q \chi^\top(s) : \xi^\top - \frac{1}{2} \alpha_0 |\xi^\top e_3|^2 - \frac{1}{2} \beta_0 |e_3 \times s|^2 - \gamma_0 (\xi^\top e_3) \cdot (e_3 \times s), \end{aligned}$$

with  $\alpha_0 := \langle a \mathbf{a}^2 \rangle_Q$ ,  $\beta_0 := \langle a \mathfrak{K}^2 \rangle_Q$ , and  $\gamma_0 := \langle a \mathbf{a} \mathfrak{K} \rangle_Q$  explicitly given by

$$\alpha_0 := \langle a \mathbf{a}^2 \rangle_Q = \langle a \rangle_Q - \frac{1}{\langle a^{-1} \rangle_Q}, \quad \beta_0 := \langle a \mathfrak{K}^2 \rangle_Q = \langle \kappa^2 / a \rangle_Q - \frac{\langle \kappa / a \rangle_Q^2}{\langle a^{-1} \rangle_Q} \quad (68)$$

and

$$\gamma_0 := \langle a \mathbf{a} \mathfrak{K} \rangle_Q = \frac{\langle \kappa / a \rangle_Q}{\langle a^{-1} \rangle_Q} - \langle \kappa \rangle_Q. \quad (69)$$

This yields (60). To complete the proof of the theorem it remains to show that when  $\langle \kappa \rangle_Q = 0$  the energy minimizers depend on  $y \cdot e_3$  only and can be fully characterized. We proceed in two steps:

1. We assume that any minimizer  $m_*$  of  $\mathcal{G}_0$  is of the form  $m_*(x) = u(x \cdot e_3)$  for some planar one-dimensional profile  $u: I_\lambda \rightarrow \mathbb{S}^1 \times \{0\}$ , and we characterize the minimizers in this class.
2. We prove that every minimizer of  $\mathcal{G}_0$  satisfies the assumptions in step 1.

**Step 1.** We start noting that under the symmetry assumptions in **1.** the minimization problem for the micromagnetic energy functional reduces to the minimization in  $H^1(I_\lambda; \mathbb{S}^2)$  of the functional

$$\mathcal{G}_\perp(u) := \frac{1}{2} \int_{I_\lambda} (\tilde{\alpha}_0 |\dot{u}(t)|^2 - \beta_0 - 2\gamma_0 \dot{u}(t) \cdot (e_3 \times u(t))) dt, \quad (70)$$

where  $\tilde{\alpha}_0 := \langle a \rangle_Q - \alpha_0 = \langle a^{-1} \rangle^{-1} > 0$ . Since  $u(t) \in \mathbb{S}^1 \times \{0\}$  for almost every  $t \in I$ , we have that  $u(t) \cdot \dot{u}(t) = 0$  and therefore, for almost every  $t \in I_\lambda$  there holds

$$\dot{u}(t) \cdot (e_3 \times u(t)) = e_3 \cdot (u(t) \times \dot{u}(t)) = |\dot{u}(t)| \sigma(u(t), \dot{u}(t)),$$

with  $\sigma(u(t), \dot{u}(t)) = 1$  if the couple  $(u, \dot{u})$  induces a positively-oriented basis of  $\mathbb{R}^2$ , and  $\sigma(u(t), \dot{u}(t)) = -1$  otherwise. In particular, if  $u^+$  is a critical point of the energy for  $\gamma_0 > 0$ , then for every  $R \in \text{SO}(3)$ , the profile  $Ru^+$  is again a critical point. By contrast, if  $R_- \in \text{O}(3)$  and  $\det R_- = -1$ , then  $u^- := R_- u^+$  is a critical point of the energy for  $\gamma_0 = -|\gamma_0|$ . Therefore, without loss of generality, we can assume that  $\gamma_0 > 0$ .

The Euler-Lagrange equations associated with (70) read as

$$u \times (\tilde{\alpha}_0 \ddot{u}(t) - 2\gamma_0 e_3 \times \dot{u}) = 0 \quad \text{in } H^{-1}(I_\lambda, \mathbb{S}^1), \quad (71)$$

with  $\tilde{\alpha}_0 := \langle a \rangle_Q - \alpha_0 = \langle a^{-1} \rangle^{-1} > 0$ . In particular, parameterizing  $u$  in polar coordinates, we obtain that the general solution of (71) is given by

$$u(t) = \cos \theta(t) e_1 + \sin \theta(t) e_2, \quad \theta(t) := \theta_0 + \frac{\theta_\lambda - \theta_0}{\lambda} t \quad \text{for every } t \in I_\lambda, \quad (72)$$

with  $\theta_0 := \theta(0)$ ,  $\theta_\lambda := \theta(\lambda)$  arbitrary real numbers. The corresponding values of the energy depends on  $\theta_\lambda - \theta_0$ , and on the height  $\lambda$  of the multilayer. Precisely, evaluating  $\mathcal{G}_\perp$  on the family (72) we find that

$$\begin{aligned} \mathcal{G}_\perp(u) &= \frac{1}{2} \left[ \int_{I_\lambda} \tilde{\alpha}_0 |\dot{\theta}(t)|^2 dt - \beta_0 \lambda - 2\gamma_0 (\theta_\lambda - \theta_0) \right] \\ &= \frac{1}{2} \left[ \tilde{\alpha}_0 \frac{(\theta_\lambda - \theta_0)^2}{\lambda} - \beta_0 \lambda - 2\gamma_0 (\theta_\lambda - \theta_0) \right]. \end{aligned}$$

Minimizing with respect to  $(\theta_\lambda - \theta_0)$ , and taking into account (68) and (69), we deduce that the corresponding minimum value of  $\mathcal{G}_\perp$  is achieved when  $(\theta_\lambda - \theta_0) = \lambda \gamma_0 / \tilde{\alpha}_0 = \lambda \langle \kappa / a \rangle_Q$ , it is strictly negative, and it is given by

$$\mathcal{G}_\perp(u) = -\frac{\lambda}{2} \left( \beta_0 + \frac{\gamma_0^2}{\tilde{\alpha}_0} \right) = -\frac{\lambda}{2} \langle \kappa^2 / a \rangle_Q.$$

Summarizing, the energy  $\mathcal{G}_\perp$  is minimized by all profiles of the form

$$u(t) = \cos \theta(t) e_1 + \sin \theta(t) e_2, \quad \theta(t) := \theta_0 + \langle \kappa / a \rangle_Q t. \quad (73)$$

As we were expecting, depending on the sign of  $\gamma_0$ , equivalently on the sign of  $\langle \kappa / a \rangle_Q$ , the rotation is clockwise or counter-clockwise (cf. Figure 2).

Note that the structure of optimal profiles does not depend on the height  $\lambda$  of the multilayer. The length  $\lambda$  affects only the minimal energy value, which is a decreasing function of  $\lambda$ .



**Step 2.** We decompose each element  $m \in H^1(\Omega, \mathbb{S}^2)$  as  $m = u + m_3 e_3$ , with  $u: \Omega \rightarrow \mathbb{R}^3$  being the projection of  $m$  on  $\mathbb{R}^2 \times \{0\}$ . We expand the energy  $\mathcal{G}_0$  as

$$\begin{aligned} \mathcal{G}_0(m) &= \frac{1}{2} \int_{\Omega} (\langle a \rangle_Q |\nabla u|^2 + \langle a \rangle_Q |\nabla m_3|^2 - \alpha_0 |\partial_3 u|^2 - \alpha_0 |\partial_3 m_3|^2 - \beta_0 |e_3 \times u|^2) dx \\ &\quad - \gamma_0 \int_{\Omega} \partial_3 u \cdot (e_3 \times u) dx \\ &= \frac{1}{2} \int_{\Omega} (\tilde{\alpha}_0 |\partial_3 u|^2 + \langle a \rangle_Q (|\partial_1 u|^2 + |\partial_2 u|^2) - \beta_0 |u|^2) dx - \gamma_0 \int_{\Omega} \partial_3 u \cdot (e_3 \times u) dx \\ &\quad + \frac{1}{2} \int_{\Omega} (\tilde{\alpha}_0 |\partial_3 m_3|^2 + \langle a \rangle_Q (|\partial_1 m_3|^2 + |\partial_2 m_3|^2)) dx, \end{aligned} \quad (74)$$

and we set  $\mathcal{I}_1(u) := \frac{1}{2} \int_{\Omega} (\tilde{\alpha}_0 |\partial_3 u|^2 + \langle a \rangle_Q (|\partial_1 u|^2 + |\partial_2 u|^2) - \beta_0 |u|^2) dx - \gamma_0 \int_{\Omega} \partial_3 u \cdot (e_3 \times u) dx$  and  $\mathcal{I}_2(m_3) := \frac{1}{2} \int_{\Omega} (\tilde{\alpha}_0 |\partial_3 m_3|^2 + \langle a \rangle_Q (|\partial_1 m_3|^2 + |\partial_2 m_3|^2)) dx$ .

Note that  $\mathcal{I}_2(m_3) \geq 0$  for every  $m \in H^1(\Omega, \mathbb{S}^2)$ . Additionally, for every  $\delta > 0$  there holds

$$-\gamma_0 \partial_3 u \cdot (e_3 \times u) = \frac{1}{2} \left| \delta \partial_3 u - \frac{\gamma_0}{\delta} e_3 \times u \right|^2 - \frac{1}{2} \delta^2 |\partial_3 u|^2 - \frac{1}{2} \frac{\gamma_0^2}{\delta^2} |u|^2.$$

Therefore, neglecting the contribution of  $\frac{1}{2} \left| \delta \partial_3 u - \frac{\gamma_0}{\delta} e_3 \times u \right|^2$  leads to the lower bound

$$\mathcal{I}_1(u) \geq \frac{1}{2} \int_{\Omega} \langle a \rangle_Q (|\partial_1 u|^2 + |\partial_2 u|^2) dx + \frac{1}{2} \int_{\Omega} (\tilde{\alpha}_0 - \delta^2) |\partial_3 u|^2 - \frac{1}{2} \int_{\Omega} \left( \beta_0 + \frac{\gamma_0^2}{\delta^2} \right) |u|^2 dx \quad (75)$$

for every  $m \in H^1(\Omega, \mathbb{S}^2)$  and for every  $\delta > 0$ . Choosing  $\delta^2 := \tilde{\alpha}_0$ , and neglecting the contribution of  $\langle a \rangle_Q (|\partial_1 u|^2 + |\partial_2 u|^2)$  we obtain

$$\mathcal{I}_1(u) \geq -\frac{1}{2} \int_{\Omega} \left( \beta_0 + \frac{\gamma_0^2}{\tilde{\alpha}_0} \right) |u|^2 dx$$

for every  $m \in H^1(\Omega, \mathbb{S}^2)$ . Recall that (cf. (68) and (69))  $\beta_0 + \gamma_0^2 / \tilde{\alpha}_0 = \langle \kappa^2 / a \rangle_Q > 0$ . Hence, since  $|u| \leq 1$  and  $|\Omega| = \lambda |\omega| = \lambda$ , we deduce the estimate

$$\mathcal{G}_0(m) \geq -\frac{\lambda}{2} \langle \kappa^2 / a \rangle_Q \quad (76)$$

for every  $m \in H^1(\Omega; \mathbb{S}^2)$ , which in turn implies that

$$\inf_{m \in H^1(\Omega, \mathbb{S}^2)} \mathcal{G}_0(m) \geq -\frac{\lambda}{2} \langle \kappa^2 / a \rangle_Q.$$

From step 1 we know that there exist configurations  $m_*: \Omega \rightarrow \mathbb{S}^2$  for which the equality holds in (76). Indeed, it is sufficient to set  $m_*(x) = u(x \cdot e_3)$  with  $u$  given by (73):

$$m_*(x) := \cos(\theta(x \cdot e_3)) e_1 + \sin(\theta(x \cdot e_3)) e_2, \quad \theta(t) := \theta_0 + \langle \kappa / a \rangle_Q t \quad (77)$$

and  $\theta_0 \in \mathbb{R}$  arbitrary. Hence

$$\inf_{m \in H^1(\Omega, \mathbb{S}^2)} \mathcal{G}_0(m) = \min_{m \in H^1(\Omega, \mathbb{S}^2)} \mathcal{G}_0(m) = \mathcal{G}_0(m_*) = -\frac{\lambda}{2} \langle \kappa^2 / a \rangle_Q.$$

To conclude the proof, we observe that the right-hand side of (76) is attained if and only if

$$\mathcal{I}_2(m_3) = 0, \quad |u| \equiv 1, \quad \int_{\Omega} |\partial_1 u|^2 + |\partial_2 u|^2 dx = 0.$$

Hence, if  $m$  is a minimizer of  $\mathcal{G}_0$  then necessarily  $m_3 \equiv 0$ , and  $\partial_1 u = \partial_2 u = 0$ . Therefore, every minimizer  $m_*$  of  $\mathcal{G}_0$  is of the form  $m_*(x) = u(x \cdot e_3)$  for some one-dimensional profile  $u: I_\lambda \rightarrow \mathbb{S}^2$ , with  $u \cdot e_3 \equiv 0$ , i.e., due to step 1,  $u$  is of the form described in (77).  $\blacksquare$

#### 4. THE HIGHER-DIMENSIONAL CASE

This section is devoted to a higher dimensional counterpart of the results presented so far in  $\mathbb{R}^3$ . Since many arguments follow along the same lines as in the  $3d$ -setting we only highlight here the main changes. Fix  $n \in \mathbb{N}$ . In this section,  $\Omega$  will be an open bounded domain of  $\mathbb{R}^n$ , and  $\mathcal{M}$  will be a bounded,  $C^2$  orientable  $n-1$  dimensional surface of  $\mathbb{R}^n$  that admits a tubular neighborhood of uniform thickness. We consider the energy density

$$f_v: (x, (s, \xi^\top)) \in \mathbb{R}^n \times \mathcal{TM} \rightarrow \frac{1}{2} \sum_{i=1}^n A^i(x) (\xi_i^\top - J^i(x)s) \cdot (\xi_i^\top - J^i(x)s) + K^i(x)s \cdot s \quad (78)$$

where for every  $i = 1, \dots, n$  the maps  $A^i, K^i \in L_\sharp^\infty(\mathbb{R}^n, \mathbb{R}_{\text{sym}}^{n \times n})$  are  $Q$ -periodic functions, taking values in the set  $\mathbb{R}_{\text{sym}}^{n \times n}$  of symmetric matrices, and where  $J^i \in L_\sharp^\infty(\mathbb{R}^n, \mathbb{R}^{n \times n})$  is a  $Q$ -periodic function, taking values in the set of  $n \times n$ - matrices. Additionally, we will assume that each map  $A^i$  is *uniformly positive definite*, namely that for every  $i = 1, \dots, n$  there exists  $c_i > 0$  such that

$$A^i(y)\lambda \cdot \lambda \geq c_i |\lambda| \quad \text{for every } \lambda \in \mathbb{R}^n \text{ and for all } y \in Q.$$

**Remark 16.** The motivation for taking into account the class of energy densities having the structure in (78) is the observation that, in the case in which the exchange energy coefficient and the material-dependent DMI constant are anisotropic, then the natural generalization of the energy density in (12) to an  $n$ -dimensional setting would be the following:

$$\tilde{f}_v: (x, (s, \xi^\top)) \in \mathbb{R}^n \times \mathcal{TM} \rightarrow \frac{1}{2} \sum_{i=1}^n (\mathcal{A}_i(x) \xi_i^\top \cdot \xi_i^\top - 2\mathcal{K}_i(x)s \cdot \xi_i^\top), \quad (79)$$

with  $\mathcal{A}_i \in L_\sharp^\infty(\mathbb{R}^n, \mathbb{R}_{\text{sym}}^{n \times n})$  and  $\mathcal{K}_i \in L_\sharp^\infty(\mathbb{R}^n, \mathbb{R}_{\text{skew}}^{n \times n})$ , and where  $\mathbb{R}_{\text{skew}}^{n \times n}$  denotes the set of skew-symmetric matrices,  $\mathbb{R}_{\text{skew}}^{n \times n} := \{M \in \mathbb{R}^{n \times n}: M^\top = -M\}$ . An algebraic manipulation yields the identity:

$$\tilde{f}_v(x, (s, \xi^\top)) = \frac{1}{2} \sum_{i=1}^n \mathcal{A}_i(x) (\xi_i^\top - \mathcal{A}_i(x)^{-1} \mathcal{K}_i(x)s) \cdot \xi_i^\top - \mathcal{K}_i(x)s \cdot \xi_i^\top \quad (80)$$

$$\begin{aligned} &= \frac{1}{2} \sum_{i=1}^n \mathcal{A}_i(x) (\xi_i^\top - \mathcal{A}_i(x)^{-1} \mathcal{K}_i(x)s) \cdot (\xi_i^\top - \mathcal{A}_i(x)^{-1} \mathcal{K}_i(x)s) \\ &\quad + \frac{1}{2} \sum_{i=1}^n \mathcal{A}_i(x) (\xi_i^\top - \mathcal{A}_i(x)^{-1} \mathcal{K}_i(x)s) \cdot \mathcal{A}_i(x)^{-1} \mathcal{K}_i(x)s - \mathcal{K}_i(x)s \cdot \xi_i^\top \end{aligned} \quad (81)$$

$$\begin{aligned} &= \frac{1}{2} \sum_{i=1}^n \mathcal{A}_i(x) (\xi_i^\top - \mathcal{A}_i(x)^{-1} \mathcal{K}_i(x)s) \cdot (\xi_i^\top - \mathcal{A}_i(x)^{-1} \mathcal{K}_i(x)s) \\ &\quad + \frac{1}{2} \sum_{i=1}^n \mathcal{K}_i(x) \mathcal{A}_i(x)^{-1} \mathcal{K}_i(x)s \cdot s, \end{aligned} \quad (82)$$

for almost every  $x \in \mathbb{R}^n$ , and for all  $(s, \xi^\top) \in \mathbf{TM}$ , where we have used the symmetry of  $\mathcal{A}_i$  and the skew-symmetry of  $\mathcal{K}_i$ . We point out that

$$(\mathcal{K}_i(x)\mathcal{A}_i(x)^{-1}\mathcal{K}_i(x))^\top = \mathcal{K}_i(x)^\top\mathcal{A}_i(x)^{-1}\mathcal{K}_i(x)^\top = \mathcal{K}_i(x)\mathcal{A}_i(x)^{-1}\mathcal{K}_i(x)$$

for almost every  $x \in \Omega$ . For this reason, the analysis of energy densities  $\tilde{f}_v$  as in (79) is naturally encompassed by the study of functions  $f_v$  as in (78).

For every  $\varepsilon > 0$  we set  $A_\varepsilon^i(x) := A^i(x/\varepsilon)$ ,  $J_\varepsilon^i(x) := J^i(x/\varepsilon)$ , and  $K_\varepsilon^i(x) := K^i(x/\varepsilon)$  for almost every  $x \in \mathbb{R}^n$ . For every  $m \in H^1(\Omega, \mathcal{M})$  we define the family of energy functionals

$$\begin{aligned} \mathcal{G}_{\varepsilon,v}(m) &:= \int_{\Omega} f_v\left(\frac{x}{\varepsilon}, m(x), \nabla m(x)\right) dx \\ &= \frac{1}{2} \sum_{i=1}^n \int_{\Omega} A_\varepsilon^i(x) (\partial_i m(x) - J_\varepsilon^i(x)m(x)) \cdot (\partial_i m(x) - J_\varepsilon^i(x)m(x)) dx \\ &\quad + \frac{1}{2} \sum_{i=1}^n \int_{\Omega} K_\varepsilon^i(x) m(x) \cdot m(x) dx. \end{aligned} \quad (83)$$

The main result of this section is the proof that the effective functional

$$\mathcal{G}_{0,v}(m) := \int_{\Omega} T_{\text{hom}}^v(m(x), \nabla m(x)) dx,$$

encodes the asymptotic behavior of  $\mathcal{G}_{\varepsilon,v}$  as the periodicity scale converges to zero. In the expression above  $T_{\text{hom}}^v: (s, \xi^\top) \in \mathbf{TM} \rightarrow \mathbb{R}$  is the *tangentially homogenized  $n$ -dimensional energy density*, defined as

$$\begin{aligned} T_{\text{hom}}^v(s, \xi^\top) &:= \inf_{\phi \in H_{\#}^1(Q, T_s \mathcal{M})} \left\{ \frac{1}{2} \sum_{i=1}^n \int_Q A^i(y) (\xi_i^\top + \partial_{y_i} \phi(y) - J^i(y)s) \right. \\ &\quad \left. \cdot (\xi_i^\top + \partial_{y_i} \phi(y) - J^i(y)s) dy \right. \\ &\quad \left. + \frac{1}{2} \sum_{i=1}^n \int_Q K^i(y) s \cdot s \right\} \end{aligned} \quad (84)$$

for every  $s \in \mathcal{M}$  and  $\xi^\top \in T_s \mathcal{M}$ .

We first provide the counterpart to Proposition 5 in the higher dimensional setting.

**Proposition 17.** *For every  $(s, \xi^\top) \in \mathbf{TM}$ , the minimization problem (84) has a unique solution. Specifically, let  $\varphi_J, \varphi_A^\ell \in H_{\#}^{-1}(Q, \mathbb{R}^{n \times n})$ ,  $\ell = 1, \dots, n$  be the unique solutions to the cell elliptic systems*

$$\sum_{i=1}^n \partial_{y_i} (A^i(y) \partial_{y_i} \varphi_J(y)) = \sum_{i=1}^n \partial_{y_i} (A^i(y) J^i(y)), \quad (85)$$

and

$$\sum_{i=1}^n \partial_{y_i} (A^i(y) \partial_{y_i} \varphi_A^\ell(y)) = \partial_{y_\ell} (A^\ell(y)), \quad \ell = 1, \dots, n. \quad (86)$$

Then, the unique solution  $\phi_v[s, \xi^\top] \in H_{\sharp}^1(Q, T_s\mathcal{M})$  of the minimization problem (84) is given by

$$\phi_v[s, \xi^\top](y) = \varphi_J(y)s + \sum_{\ell=1}^n \varphi_A^\ell(y)\xi_\ell^\top \quad (87)$$

for every  $s \in \mathcal{M}$ , and for almost every  $y \in Q$ .

**Proof.** The characterization of  $\phi_v[s, \xi^\top]$  follows by computing the Euler-Lagrange equations associated to the minimum problem in (84), by the fact that  $A^i$  is uniformly positive definite for every  $i = 1, \dots, n$ , and by the linearity of (84) with respect to  $s$  and  $\xi$ .  $\blacksquare$

We are now in a position to state our main result.

**Theorem 18.** *The family  $(\mathcal{G}_{\varepsilon, v})_{\varepsilon \in \mathbb{R}_+}$   $\Gamma$ -converges with respect to the weak topology in  $H^1(\Omega, \mathcal{M})$ , to the energy functional*

$$\begin{aligned} \mathcal{G}_{0, v}(m) &:= \int_{\Omega} T_{\text{hom}}^v(m(x), \nabla m(x)) \, dx \\ &= \frac{1}{2} \sum_{i=1}^n \int_{\Omega \times Q} A^i(y) (\partial_i m(x) + \partial_{y_i} \phi_v[m(x), \nabla m(x)](y) - J^i(y)m(x)) \\ &\quad \cdot (\partial_i m(x) + \partial_{y_i} \phi_v[m(x), \nabla m(x)](y) - J^i(y)m(x)) \, dx \, dy \\ &\quad + \frac{1}{2} \sum_{i=1}^n \int_{\Omega \times Q} K^i(y) m(x) \cdot m(x) \, dx \, dy \end{aligned}$$

for every  $m \in H^1(\Omega, \mathcal{M})$ .

**Proof.** We provide here a sketch of proof for the liminf inequality. Let  $(m_\varepsilon)_\varepsilon$  in  $H^1(\Omega, \mathcal{M})$  be such that

$$\sup_{\varepsilon > 0} \mathcal{G}_{\varepsilon, v}(m_\varepsilon) < +\infty. \quad (88)$$

Then, by the uniform positive definiteness of the maps  $A^i$  for every  $i = 1, \dots, n$  and by the boundedness of  $\mathcal{M}$ , the sequence  $(m_\varepsilon)_\varepsilon$  is uniformly bounded in  $H^1(\Omega, \mathcal{M})$ . By standard properties of two-scale convergence and by Proposition 8 there exists  $m_0 \in H^1(\Omega, \mathcal{M})$  and  $m_1 \in L^2(\Omega, H_{\sharp}^1(Q, T_{m_0}\mathcal{M}))$  such that

$$\begin{aligned} m_\varepsilon &\rightharpoonup m_0 && \text{weakly in } H^1(\Omega, \mathcal{M}), \\ \nabla m_\varepsilon &\rightharpoonup \nabla m_0 + \nabla_y m_1 && \text{weakly two-scale in } L^2(\Omega \times Q, \mathbb{R}^{n \times n}). \end{aligned}$$

We proceed by showing that,

$$\mathcal{G}_{0, v}(m_0) \leq \liminf_{\varepsilon \rightarrow 0} \mathcal{G}_{\varepsilon, v}(m_\varepsilon). \quad (89)$$

Arguing as in the scalar case, we decompose  $\mathcal{G}_{\varepsilon, v}(m_\varepsilon)$  as

$$\mathcal{G}_{\varepsilon, v}(m_\varepsilon) = \mathcal{H}_{\varepsilon, v}(m_\varepsilon) + \mathcal{I}_{\varepsilon, v}(m_\varepsilon),$$

where

$$\mathcal{H}_{\varepsilon, v}(m_\varepsilon) := \frac{1}{2} \sum_{i=1}^n \int_{\Omega} A_\varepsilon^i(x) (\partial_i m_\varepsilon(x) - J_\varepsilon^i(x)m_\varepsilon(x)) \cdot (\partial_i m_\varepsilon(x) - J_\varepsilon^i(x)m_\varepsilon(x)) \, dx$$

and

$$\mathcal{I}_{\varepsilon, v}(m_\varepsilon) := \frac{1}{2} \sum_{i=1}^n \int_{\Omega} K_\varepsilon^i(x) m_\varepsilon(x) \cdot m_\varepsilon(x) dx.$$

For every  $\psi \in \mathcal{D}(\Omega, H_{\sharp}^1(Q; \mathbb{R}^n))$  there holds

$$\begin{aligned} \mathcal{H}_\varepsilon(m_\varepsilon) &\geq \sum_{i=1}^n \int_{\Omega \times Q} A_\varepsilon^i(x) (\partial_i m_\varepsilon(x) - J_\varepsilon^i(x) m_\varepsilon(x)) : (\partial_i m_0(x) + \partial_{y_i} \psi(x, y) - J^i(y) m_0(x)) dy dx \\ &\quad - \frac{1}{2} \sum_{i=1}^n \int_{\Omega \times Q} A^i(y) (\partial_i m_0(x) + \partial_{y_i} \psi(x, y) - J^i(y) m_0(x)) \\ &\quad \quad \quad : (\partial_i m_0(x) + \partial_{y_i} \psi(x, y) - J^i(y) m_0(x)) dy dx, \end{aligned} \quad (90)$$

owing to the fact that the difference of the integrand on the left-hand side with the integrand on the right-hand side is a perfect square. Thus, by standard properties of two-scale convergence, and owing to the regularity of  $\psi$ , we deduce

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \mathcal{H}_\varepsilon(m_\varepsilon) &\geq \sum_{i=1}^n \int_{\Omega \times Q} A^i(y) (\partial_i m_0(x) + \partial_{y_i} m_1(x, y) - J^i(y) m_0(x)) \\ &\quad \quad \quad : (\partial_i m_0(x) + \partial_{y_i} \psi(x, y) - J^i(y) m_0(x)) dy dx \\ &\quad - \frac{1}{2} \sum_{i=1}^n \int_{\Omega \times Q} A^i(y) (\partial_i m_0(x) + \partial_{y_i} \psi(x, y) - J^i(y) m_0(x)) \\ &\quad \quad \quad : (\partial_i m_0(x) + \partial_{y_i} \psi(x, y) - J^i(y) m_0(x)) dy dx. \end{aligned} \quad (91)$$

By density, there exists  $(\psi_n)_{n \in \mathbb{N}}$  such that  $\psi_n \rightarrow m_1$  in  $L^2(\Omega, H_{\sharp}^1(Q, T_{m_0} \mathcal{M}))$ . Therefore, the previous inequality yields

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \mathcal{H}_\varepsilon(m_\varepsilon) &\geq \int_{\Omega} \inf_{\phi \in H_{\sharp}^1(Q, T_s \mathcal{M})} \int_Q \frac{1}{2} \sum_{i=1}^n A^i(y) (\xi_i^\top + \partial_{y_i} \phi(y) - J^i(y) s) \\ &\quad \quad \quad \cdot (\xi_i^\top + \partial_{y_i} \phi(y) - J^i(y) s) dy. \end{aligned} \quad (92)$$

Finally, since  $K_\varepsilon^i \rightharpoonup \int_Q K^i(y) dy$  weakly\* in  $L^\infty(Q, \mathbb{R}^{n \times n})$ , we get

$$\lim_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon(m_\varepsilon) = \frac{1}{2} \sum_{i=1}^n \int_{\Omega \times Q} K^i(y) m_0(x) \cdot m_0(x) dx dy. \quad (93)$$

Combining (92) and (93) we deduce (89). The optimality of the lower bound is a straightforward adaptation of the arguments in Theorem 11.  $\blacksquare$

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