A DYNAMIC MODEL FOR VISCOELASTIC MATERIALS WITH PRESCRIBED GROWING CRACKS

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Abstract. In this paper, we prove the existence of solutions for a class of viscoelastic dynamic systems on time–dependent cracked domains, with possibly degenerate viscosity coefficients. Under stronger regularity assumptions we also show a uniqueness result. Finally, we exhibit an example where the energy–dissipation balance is not satisfied, showing there is an additional dissipation due to the crack growth.

Keywords: linear second order hyperbolic systems, dynamic fracture mechanics, elastodynamics, viscoelasticity, cracking domains.

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1. Introduction

In the theory of Dynamic Fracture, the deformation of an elastic material evolves according to the elastodynamics system, while the evolution of the crack follows Griffith's dynamic criterion, see [13]. This principle, originally formulated in [11] for the quasi–static setting, states that there is an exact balance between the energy released during the evolution and the energy used to increase the crack, which is postulated to be proportional to the area increment of the crack itself.

For an antiplane displacement, the elastodynamics system leads to the following wave equation

$$\ddot{u}(t,x) - \Delta u(t,x) = f(t,x) \quad t \in [0,T],\, x \in \Omega \setminus \Gamma_t, \tag{1.1}$$

with some prescribed boundary and initial conditions. Here, $\Omega \subset \mathbb{R}^d$ is an open bounded set with Lipschitz boundary, which represents the cross–section of the material, the closed set $\Gamma_t \subset \overline{\Omega}$ models the crack at time t in the reference configuration, $u(t)\colon \Omega \setminus \Gamma_t \to \mathbb{R}$ is the antiplane displacement, and f is a forcing term. In this case, Griffith's dynamic criterion reads

$$\mathcal{E}(t) + \mathcal{H}^{d-1}(\Gamma_t \setminus \Gamma_0) = \mathcal{E}(0) + \text{work of external forces},$$

where $\mathcal{E}(t)$ is the total energy at time t, given by the sum of kinetic and elastic energy, and $\mathcal{H}^{d-1}$ is the $(d-1)$–dimensional Hausdorff measure.

From the mathematical point of view, a first step to study the evolution of the fracture is to solve the wave equation (1.1) when the evolution of the crack is assigned, see for example [14, 3, 7, 2, 18] (we refer also to [10, 6, 16] for the case of a 1–dimensional model). When we want to take into account the viscoelastic properties of the material, Kelvin–Voigt's model is the most common one. If no crack is present, this leads to the damped wave equation

$$\ddot{u}(t,x) - \Delta u(t,x) - \Delta \dot{u}(t,x) = f(t,x) \quad (t,x) \in (0,T) \times \Omega. \tag{1.2}$$

As it is well known, the solutions to (1.2) satisfy the energy–dissipation balance

$$\mathcal{E}(t) + \int_0^t \int_\Omega |\nabla \dot{u}|^2 \,\mathrm{d}x \,\mathrm{d}s = \mathcal{E}(0) + \text{work of external forces}. \tag{1.3}$$

When we consider a crack in a viscoelastic material, Griffith's dynamic criterion becomes

$$\mathcal{E}(t) + \mathcal{H}^{d-1}(\Gamma_t \setminus \Gamma_0) + \int_0^t \int_\Omega |\nabla \dot{u}|^2 \,\mathrm{d}x \,\mathrm{d}s = \mathcal{E}(0) + \text{work of external forces}. \tag{1.4}$$

For a prescribed crack evolution, this model was already considered by [3] in the antiplane case, and more in general by [18] for the vector–valued case. As proved in the quoted papers, the solutions to (1.2) on a domain with a prescribed time–dependent crack, i.e., with Ω replaced by $\Omega \setminus \Gamma_t$, satisfy (1.3) for every time.

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This equality implies that (1.4) cannot be satisfied unless $\Gamma_t = \Gamma_0$ for every t. This phenomenon was already well known in mechanics as the viscoelastic paradox, see for instance [17, Chapter 7].

To overcome this problem, we modify Kelvin–Voigt's model by considering a possibly degenerate viscosity term depending on t and x. More precisely, we study the following equation

$$\ddot{u}(t,x) - \Delta u(t,x) - \operatorname{div}(\Psi^2(t,x)\nabla\dot{u}(t,x)) = f(t,x) \quad t \in [0,T],\, x \in \Omega \setminus \Gamma_t. \tag{1.5}$$

On the function $\Psi\colon (0,T)\times\Omega \to \mathbb{R}$ we only require some regularity assumptions (see (2.7)); a particularly interesting case is when Ψ assumes the value zero on some points of Ω, which means that the material has no longer viscoelastic properties in such a zone.

The main result of this paper is Theorem 3.1, in which we show the existence of a weak solution to (1.5). This is done in the more general case of linear elasticity, that is when the displacement is vector–valued and the elastic energy depends only on the symmetric part of its gradient. To this aim, we first perform a time discretization in the same spirit of [3], and then we pass to the limit as the time step goes to zero by relying on energy estimates; as a byproduct, we obtain the energy–dissipation inequality (4.4). By using the change of variables method implemented in [14, 7], we also prove a uniqueness result, but only in dimension $d = 2$ and when $\Psi(t)$ vanishes on a neighborhood of the tip of Γ_t.

We complete our work by providing an example in $d = 2$ of a weak solution to (1.5) for which the fracture can grow while balancing the energy. More precisely, when the cracks Γ_t move with constant speed along the x_1–axis and $\Psi(t)$ is zero in a neighborhood of the crack tip, we construct a function u which solves (1.5) and satisfies

$$\mathcal{E}(t) + \int_0^t \int_\Omega |\Psi\nabla\dot{u}|^2\,\mathrm{d}x\,\mathrm{d}s + \mathcal{H}^1(\Gamma_t\setminus\Gamma_0) = \mathcal{E}(0) + \text{work of external forces}. \tag{1.6}$$

Notice that this is the natural extension of Griffith's dynamic criterion (1.4) to this setting.

The paper is organized as follows. In Section 2 we fix the notation adopted throughout the paper, we list the standard assumptions on the family of cracks $\{\Gamma_t\}_{t\in[0,T]}$ and on the function Ψ, and we specify the notion of weak solution to problem (1.5). In Section 3 we state our main existence result (Theorem 3.1), and we implement the time discretization method. We conclude the proof of Theorem 3.1 in Section 4, where we show the validity of the initial conditions and the energy–dissipation inequality (4.4). Section 5 deals with uniqueness: under stronger regularity assumptions on the cracks sets, in Theorem 5.5 we prove the uniqueness of a weak solution, but only when the space dimension is $d = 2$. To this aim, we assume also that the function Ψ is zero in a neighborhood of the crack tip. We conclude with Section 6, where in dimension $d = 2$ we show an example of a moving crack that satisfies Griffith's dynamic energy–dissipation balance (1.6).

2. Notation and Preliminary Results

The space of $m \times d$ matrices with real entries is denoted by $\mathbb{R}^{m\times d}$; in case $m = d$, the subspace of symmetric matrices is denoted by $\mathbb{R}^{d\times d}_{sym}$. Given two vectors $v_1, v_2 \in \mathbb{R}^d$, their Euclidean scalar product is denoted by $v_1 \cdot v_2 \in \mathbb{R}$ and their tensor product is denoted by $v_1 \otimes v_2 \in \mathbb{R}^{d\times d}$; we use $v_1 \odot v_2 \in \mathbb{R}^{d\times d}_{sym}$ to denote the symmetric part of $v_1 \otimes v_2$, namely $v_1 \odot v_2 := \frac{1}{2}(v_1\otimes v_2 + v_2 \otimes v_1)$. Given $A \in \mathbb{R}^{m\times d}$, we use A^T to denote its transpose; we use $A_1 \cdot A_2 \in \mathbb{R}$ to denote the Euclidean scalar product of two matrices $A_1, A_2 \in \mathbb{R}^{d\times d}$.

The partial derivatives with respect to the variable x_i are denoted by ∂_i. Given a function $f\colon \mathbb{R}^d \to \mathbb{R}^m$, we denote its Jacobian matrix by ∇f, whose components are $(\nabla f)_{ij} := \partial_j f_i$, $i = 1,\dots,m$, $j = 1,\dots,d$. For a tensor field $F\colon \mathbb{R}^d \to \mathbb{R}^{m\times d}$, by $\operatorname{div} F$ we mean the divergence of F with respect to rows, namely $(\operatorname{div} F)_i := \sum_{j=1}^d \partial_j F_{ij}$, for $i = 1,\dots,m$.

The d–dimensional Lebesgue measure is denoted by $\mathcal{L}^d$ and the $(d-1)$–dimensional Hausdorff measure by $\mathcal{H}^{d-1}$. We adopted standard notations for Lebesgue and Sobolev spaces on open subsets of $\mathbb{R}^d$; given an open set $\Omega \subseteq \mathbb{R}^d$ we use $\|\cdot\|_\infty$ to denote the norm of $L^\infty(\Omega;\mathbb{R}^m)$. The boundary values of a Sobolev function are always intended in the sense of traces. Given a bounded open set Ω with Lipschitz boundary, we denote by ν the outer unit normal vector to $\partial\Omega$, which is defined $\mathcal{H}^{d-1}$–a.e. on the boundary.

Given a Banach space X, its norm is denoted by $\|\cdot\|_X$; if X is an Hilbert space, we use $(\cdot,\cdot)_X$ to denote its scalar product. The dual space of X is denoted by X', and we use $\langle\cdot,\cdot\rangle_{X'}$ to denote the duality product between X' and X. Given two Banach spaces X_1 and X_2, the space of linear and continuous maps from X_1

to X_2 is denoted by $\mathscr{L}(X_1; X_2)$; given $\mathbb{A} \in \mathscr{L}(X_1; X_2)$ and $u \in X_1$, we write $\mathbb{A}u \in X_2$ to denote the image of u under $\mathbb{A}$.

Given an open interval $(a, b) \subseteq \mathbb{R}$, $L^p(a, b; X)$ is the space of L^p functions from (a, b) to X. Given $u \in L^p(a, b; X)$, we denote by $\dot{u} \in \mathcal{D}'(a, b; X)$ its distributional derivative. The set of continuous functions from $[a, b]$ to X is denoted by $C^0([a, b]; X)$. Given a reflexive Banach space X, $C^0_w([a, b]; X)$ is the set of weakly continuous functions from $[a, b]$ to X, namely

$$C^0_w([a, b]; X) := \{u\colon [a, b] \to X : t \mapsto \langle x', u(t)\rangle_{X'} \text{ is continuous from } [a, b] \text{ to } \mathbb{R} \text{ for every } x' \in X'\}.$$

Let T be a positive real number and let $\Omega \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary. Let $\partial_D\Omega$ be a (possibly empty) Borel subset of $\partial\Omega$ and let $\partial_N\Omega$ be its complement. We assume the following hypotheses on the geometry of the cracks:

(E1) $\Gamma \subset \overline{\Omega}$ is a closed set with $\mathcal{L}^d(\Gamma) = 0$ and $\mathcal{H}^{d-1}(\Gamma \cap \partial\Omega) = 0$;

(E2) for every $x \in \Gamma$ there exists an open neighborhood U of x in $\mathbb{R}^d$ such that $(U \cap \Omega) \setminus \Gamma$ is the union of two disjoint open sets U^+ and U^- with Lipschitz boundary;

(E3) $\{\Gamma_t\}_{t\in[0,T]}$ is a family of closed subsets of Γ satisfying $\Gamma_s \subset \Gamma_t$ for every $0 \le s \le t \le T$.

Thanks (E1)–(E3) the space $L^2(\Omega \setminus \Gamma_t; \mathbb{R}^m)$ coincides with $L^2(\Omega; \mathbb{R}^m)$ for every $t \in [0, T]$ and $m \in \mathbb{N}$. In particular, we can extend a function $u \in L^2(\Omega \setminus \Gamma_t; \mathbb{R}^m)$ to a function in $L^2(\Omega; \mathbb{R}^m)$ by setting $u = 0$ on Γ_t. Moreover, the trace of $u \in H^1(\Omega \setminus \Gamma)$ is well defined on $\partial\Omega$. Indeed, we may find a finite number of open sets with Lipschitz boundary $U_j \subset \Omega \setminus \Gamma$, $j = 1, \dots m$, such that $\partial\Omega \setminus (\Gamma \cap \partial\Omega) \subset \cup_{j=1}^m \partial U_j$. Since $\mathcal{H}^{d-1}(\Gamma \cap \partial\Omega) = 0$, there exists a constant $C > 0$, depending only on Ω and Γ, such that

$$\|u\|_{L^2(\partial\Omega)} \le C\|u\|_{H^1(\Omega\setminus\Gamma)} \quad \text{for every } u \in H^1(\Omega \setminus \Gamma; \mathbb{R}^d). \tag{2.1}$$

Similarly, we can find a finite number of open sets $U_j \subset \Omega \setminus \Gamma$, $j = 1, \dots m$, with Lipschitz boundary, such that $\Omega \setminus \Gamma = \cup_{j=1}^m U_j$. By using second Korn's inequality in each U_j (see, e.g., [15, Theorem 2.4]) and taking the sum over j we can find a constant C_K, depending only on Ω and Γ, such that

$$\|\nabla u\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})} \le C_K \left(\|u\|^2_{L^2(\Omega;\mathbb{R}^d)} + \|Eu\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_{sym})}\right) \quad \text{for every } u \in H^1(\Omega \setminus \Gamma; \mathbb{R}^d), \tag{2.2}$$

where Eu is the symmetric part of ∇u, i.e., $Eu := \frac{1}{2}(\nabla u + \nabla u^T)$.

For every $t \in [0, T]$ we define

$$V_t := \{u \in L^2(\Omega \setminus \Gamma_t; \mathbb{R}^d) : Eu \in L^2(\Omega \setminus \Gamma_t; \mathbb{R}^{d\times d}_{sym})\}.$$

Notice that in the definition of V_t we are considering only the distributional gradient of u in $\Omega \setminus \Gamma_t$ and not the one in Ω. The set V_t is a Hilbert space with respect to the following norm

$$\|u\|_{V_t} := (\|u\|^2_H + \|Eu\|^2_H)^{\frac{1}{2}} \quad \text{for every } u \in V_t.$$

To simplify our exposition, for every $m \in \mathbb{N}$ we set $H := L^2(\Omega; \mathbb{R}^m)$ and $H_N := L^2(\partial_N\Omega; \mathbb{R}^m)$; we always identify the dual of H by H itself and $L^2(0, T; L^2(\Omega; \mathbb{R}^m))$ by $L^2((0, T) \times \Omega; \mathbb{R}^m)$.

Thanks to (2.2), the space V_t coincides with the usual Sobolev space $H^1(\Omega \setminus \Gamma_t; \mathbb{R}^d)$. Therefore, by (2.1), it makes sense to consider for every $t \in [0, T]$ the set

$$V^D_t := \{u \in V_t : u = 0 \text{ on } \partial_D\Omega\},$$

which is a Hilbert space with respect to $\|\cdot\|_{V_t}$. Moreover, by combining (2.2) with (2.1), we derive also the existence of a constant $C_{tr} > 0$ such that

$$\|u\|_{H_N} \le C_{tr}\|u\|_{V_T} \quad \text{for every } u \in V_T. \tag{2.3}$$

Let $\mathbb{C}, \mathbb{B}\colon \Omega \to \mathscr{L}(\mathbb{R}^{d\times d}_{sym}; \mathbb{R}^{d\times d}_{sym})$ be two fourth-order tensors satisfying:

$$\mathbb{C}_{ijhk}, \mathbb{B}_{ijhk} \in L^\infty(\Omega) \quad \text{for every } i, j, h, k = 1, \dots, d, \tag{2.4}$$

$$\mathbb{C}(x)\eta_1 \cdot \eta_2 = \eta_1 \cdot \mathbb{C}(x)\eta_2, \quad \mathbb{B}(x)\eta_1 \cdot \eta_2 = \eta_1 \cdot \mathbb{B}(x)\eta_2 \quad \text{for a.e. } x \in \Omega \text{ and for every } \eta_1, \eta_2 \in \mathbb{R}^{d\times d}_{sym}, \tag{2.5}$$

$$\mathbb{C}(x)\eta \cdot \eta \ge \lambda_1|\eta|^2, \quad \mathbb{B}(x)\eta \cdot \eta \ge \lambda_2|\eta|^2 \quad \text{for a.e. } x \in \Omega \text{ and for every } \eta \in \mathbb{R}^{d\times d}_{sym}, \tag{2.6}$$

for two positive constants λ_1, λ_2 independent of x. Consider a function $\Psi\colon (0, T) \times \Omega \to \mathbb{R}$ satisfying

$$\Psi \in L^\infty((0, T) \times \Omega), \quad \nabla\Psi \in L^\infty((0, T) \times \Omega; \mathbb{R}^d). \tag{2.7}$$

Given $f \in L^2(0,T;H)$, $w \in H^2(0,T;H) \cap H^1(0,T;V_0)$, $g \in H^1(0,T;H_N)$, $u^0 \in V_0$ with $u^0 - w(0) \in V_0^D$, and $u^1 \in H$, we want to find a solution to the viscoelastic dynamic system

$$\ddot{u}(t) - \operatorname{div}(\mathbb{C}Eu(t)) - \operatorname{div}(\Psi^2(t)\mathbb{B}E\dot{u}(t)) = f(t) \quad \text{in } \Omega \setminus \Gamma_t,\, t \in (0,T), \tag{2.8}$$

satisfying the following boundary and initial conditions

$$u(t) = w(t) \qquad \text{on } \partial_D\Omega,\, t \in (0,T), \tag{2.9}$$

$$(\mathbb{C}Eu(t) + \Psi^2(t)\mathbb{B}E\dot{u}(t))\nu = g(t) \quad \text{on } \partial_N\Omega,\, t \in (0,T), \tag{2.10}$$

$$(\mathbb{C}Eu(t) + \Psi^2(t)\mathbb{B}E\dot{u}(t))\nu = 0 \qquad \text{on } \Gamma_t, \quad t \in (0,T), \tag{2.11}$$

$$u(0) = u^0, \quad \dot{u}(0) = u^1. \tag{2.12}$$

As usual, the Neumann boundary conditions are only formal, and their meaning will be specified in Definition 2.4.

Throughout the paper we always assume that the family $\{\Gamma_t\}_{t\in[0,T]}$ satisfies (E1)–(E3), as well as $\mathbb{C}$, $\mathbb{B}$, Ψ, f, w, g, u^0, and u^1 the previous hypotheses. Let us define the following functional spaces:

$$\mathcal{V} := \{\varphi \in L^2(0,T;V_T) : \dot{\varphi} \in L^2(0,T;H),\, \varphi(t) \in V_t \text{ for a.e. } t \in (0,T)\},$$

$$\mathcal{V}^D := \{\varphi \in \mathcal{V} : \varphi(t) \in V_t^D \text{ for a.e. } t \in (0,T)\},$$

$$\mathcal{W} := \{u \in \mathcal{V} : \Psi\dot{u} \in L^2(0,T;V_T),\, \Psi(t)\dot{u}(t) \in V_t \text{ for a.e. } t \in (0,T)\}.$$

Remark 2.1. In the classical viscoelastic case, namely when Ψ is identically equal to 1, the solution u to system (2.8) has derivative $\dot{u}(t) \in V_t$ for a.e. $t \in (0,T)$ with $E\dot{u} \in L^2(0,T;H)$. For a generic Ψ we expect to have $\Psi E\dot{u} \in L^2(0,T;H)$. Therefore $\mathcal{W}$ is the natural setting where looking for a solution to (2.8). Indeed, from a distributional point of view we have

$$\Psi(t)E\dot{u}(t) = E(\Psi(t)\dot{u}(t)) - \nabla\Psi(t) \odot \dot{u}(t) \quad \text{in } \mathcal{D}'(\Omega \setminus \Gamma_t; \mathbb{R}^{d\times d}_{sym}) \text{ for a.e. } t \in (0,T),$$

and $E(\Psi\dot{u}), \nabla\Psi \odot \dot{u} \in L^2(0,T;H)$ if $u \in \mathcal{W}$, thanks to (2.7).

Remark 2.2. The set $\mathcal{W}$ coincides with the space of functions $u \in H^1(0,T;H)$ such that $u(t) \in V_t$ and $\Psi(t)\dot{u}(t) \in V_t$ for a.e. $t \in (0,T)$, and satisfying

$$\int_0^T \|u(t)\|^2_{V_t} + \|\Psi(t)\dot{u}(t)\|^2_{V_t}\,\mathrm{d}t < \infty. \tag{2.13}$$

This is a consequence of the strong measurability of the maps $t \mapsto u(t)$ and $t \mapsto \Psi(t)\dot{u}(t)$ from $(0,T)$ into V_T, which gives that (2.13) is well defined and $u, \Psi\dot{u} \in L^2(0,T;V_T)$. To prove the strong measurability of these two maps, it is enough to observe that V_T is a separable Hilbert space and that the maps $t \mapsto \dot{u}(t)$ and $t \mapsto \Psi(t)\dot{u}(t)$ from $(0,T)$ into V_T are weakly measurable. Indeed, for every $\varphi \in C_c^\infty(\Omega \setminus \Gamma_T)$ the maps

$$t \mapsto \int_{\Omega\setminus\Gamma_T} Eu(t,x)\varphi(x)\,\mathrm{d}x = -\int_{\Omega\setminus\Gamma_T} u(t,x) \odot \nabla\varphi(x)\,\mathrm{d}x,$$

$$t \mapsto \int_{\Omega\setminus\Gamma_T} E(\Psi(t,x)\dot{u}(t,x))\varphi(x)\,\mathrm{d}x = -\int_{\Omega\setminus\Gamma_T} \Psi(t,x)\dot{u}(t,x) \odot \nabla\varphi(x)\,\mathrm{d}x$$

are measurable from $(0,T)$ into $\mathbb{R}$, and $C_c^\infty(\Omega \setminus \Gamma_T)$ is dense in $L^2(\Omega)$.

Lemma 2.3. *The spaces $\mathcal{V}$ and $\mathcal{W}$ are Hilbert spaces with respect to the following norms:*

$$\|\varphi\|_{\mathcal{V}} := (\|\varphi\|^2_{L^2(0,T;V_T)} + \|\dot{\varphi}\|^2_{L^2(0,T;H)})^{\frac{1}{2}} \quad \textit{for every } \varphi \in \mathcal{V},$$

$$\|u\|^2_{\mathcal{W}} := (\|u\|_{\mathcal{V}} + \|\Psi\dot{u}\|^2_{L^2(0,T;V_T)})^{\frac{1}{2}} \quad \textit{for every } u \in \mathcal{W}.$$

Moreover, $\mathcal{V}^D$ is a closed subspace of $\mathcal{V}$.

Proof. It is clear that $\|\cdot\|_{\mathcal{V}}$ and $\|\cdot\|_{\mathcal{W}}$ are norms on $\mathcal{V}$ and $\mathcal{W}$ induced by scalar products. We just have to check the completeness of such spaces with respect to these norms.

Let $\{\varphi_k\}_k \subset \mathcal{V}$ be a Cauchy sequence. Then, $\{\varphi_k\}_k$ and $\{\dot{\varphi}_k\}_k$ are Cauchy sequences, respectively, in $L^2(0,T;V_T)$ and $L^2(0,T;H)$, which are complete Hilbert spaces. Thus there exists $\varphi \in L^2(0,T;V_T)$ with $\dot{\varphi} \in L^2(0,T;H)$ such that $\varphi_k \to \varphi$ in $L^2(0,T;V_T)$ and $\dot{\varphi}_k \to \dot{\varphi}$ in $L^2(0,T;H)$. In particular there exists a subsequence $\{\varphi_{k_j}\}_j$ such that $\varphi_{k_j}(t) \to \varphi(t)$ in V_T for a.e. $t \in (0,T)$. Since $\varphi_{k_j}(t) \in V_t$ for a.e. $t \in (0,T)$ we deduce that $\varphi(t) \in V_t$ for a.e. $t \in (0,T)$. Hence $\varphi \in \mathcal{V}$ and $\varphi_k \to \varphi$ in $\mathcal{V}$. With a similar argument, we can prove that $\mathcal{V}^D \subset \mathcal{V}$ is a closed subspace.

Let us now consider a Cauchy sequence $\{u_k\}_k \subset \mathcal{W}$. We have that $\{u_k\}_k$ and $\{\Psi \dot{u}_k\}_k$ are Cauchy sequences, respectively, in $\mathcal{V}$ and $L^2(0,T;V_T)$, which are complete Hilbert spaces. Thus there exist two functions $u \in \mathcal{V}$ and $z \in L^2(0,T;V_T)$ such that $u_k \to u$ in $\mathcal{V}$ and $\Psi \dot{u}_k \to z$ in $L^2(0,T;V_T)$. Since $\dot{u}_k \to \dot{u}$ in $L^2(0,T;H)$ and $\Psi \in L^\infty((0,T)\times\Omega)$, we also have that $\Psi\dot{u}_k \to \Psi\dot{u}$ in $L^2(0,T;H)$, which gives that $z = \Psi\dot{u}$. Finally let us prove that $\Psi(t)\dot{u}(t) \in V_t$ for a.e. $t \in (0,T)$. By the fact that $\Psi\dot{u}_k \to \Psi\dot{u}$ in $L^2(0,T;V_T)$, there exists a subsequence $\{\Psi\dot{u}_{k_j}\}_j$ such that $\Psi(t)\dot{u}_{k_j}(t) \to \Psi(t)\dot{u}(t)$ in V_T for a.e. $t \in (0,T)$. Since $\Psi(t)\dot{u}_{k_j}(t) \in V_t$ for a.e. $t \in (0,T)$ we deduce that $\Psi(t)\dot{u}(t) \in V_t$ for a.e. $t \in (0,T)$. Hence $u \in \mathcal{W}$ and $u_k \to u$ in $\mathcal{W}$. □

We are now in position to define a weak solution to (2.8)–(2.11).

Definition 2.4 (Weak solution)**.** We say that $u \in \mathcal{W}$ is a *weak solution* to system (2.8) with boundary conditions (2.9)–(2.11) if $u - w \in \mathcal{V}^D$ and

$$\begin{aligned}&-\int_0^T (\dot{u}(t),\dot{\varphi}(t))_H\,\mathrm{d}t + \int_0^T (\mathbb{C}Eu(t), E\varphi(t))_H\,\mathrm{d}t + \int_0^T (\mathbb{B}E(\Psi(t)\dot{u}(t)), \Psi(t)E\varphi(t))_H\,\mathrm{d}t\\&-\int_0^T (\mathbb{B}\nabla\Psi(t)\odot\dot{u}(t), \Psi(t)E\varphi(t))_H\,\mathrm{d}t = \int_0^T (f(t),\varphi(t))_H\,\mathrm{d}t + \int_0^T (g(t),\varphi(t))_{H_N}\,\mathrm{d}t\end{aligned} \tag{2.14}$$

for every $\varphi \in \mathcal{V}^D$ such that $\varphi(0) = \varphi(T) = 0$.

Notice that the Neumann boundary conditions (2.10) and (2.11) can be obtained from (2.14), by using integration by parts in space, only when $u(t)$ and Γ_t are sufficiently regular.

Remark 2.5. If $\dot{u}$ is regular enough (for example $\dot{u} \in L^2(0,T;V_T)$ with $\dot{u}(t) \in V_t$ for a.e. $t \in (0,T)$), then we have $\Psi E\dot{u} = E(\Psi\dot{u}) - \nabla\Psi\odot\dot{u}$. Therefore (2.14) is coherent with the strong formulation (2.8). In particular, for a function $u \in \mathcal{W}$ we can define

$$\Psi E\dot{u} := E(\Psi\dot{u}) - \nabla\Psi\odot\dot{u} \in L^2(0,T;H), \tag{2.15}$$

so that equation (2.14) can be rephrased as

$$\begin{aligned}&-\int_0^T (\dot{u}(t),\dot{\varphi}(t))_H\,\mathrm{d}t + \int_0^T (\mathbb{C}Eu(t), E\varphi(t))_H\,\mathrm{d}t + \int_0^T (\mathbb{B}\Psi(t)E\dot{u}(t), \Psi(t)E\varphi(t))_H\,\mathrm{d}t\\&= \int_0^T (f(t),\varphi(t))_H\,\mathrm{d}t + \int_0^T (g(t),\varphi(t))_{H_N}\,\mathrm{d}t\end{aligned}$$

for every $\varphi \in \mathcal{V}^D$ such that $\varphi(0) = \varphi(T) = 0$.

Definition 2.6 (Initial conditions)**.** We say that $u \in \mathcal{W}$ satisfies the initial conditions (2.12) if

$$\lim_{h\to 0^+} \frac{1}{h}\int_0^h (\|u(t)-u^0\|^2_{V_t} + \|\dot{u}(t)-u^1\|^2_H)\,\mathrm{d}t = 0. \tag{2.16}$$

3. Existence

We now state our main existence result, whose proof will be given at the end of Section 4.

Theorem 3.1. *There exists a weak solution $u \in \mathcal{W}$ to (2.8)–(2.11) satisfying the initial conditions $u(0) = u^0$ and $\dot{u}(0) = u^1$ in the sense of (2.16). Moreover $u \in C_w([0,T];V_T)$, $\dot{u} \in C_w([0,T];H) \cap H^1(0,T;(V_0^D)')$, and*

$$\lim_{t\to 0^+} u(t) = u^0 \text{ in } V_T, \quad \lim_{t\to 0^+} \dot{u}(t) = u^1 \text{ in } H.$$

To prove the existence of a weak solution to (2.8)–(2.11), we use a time discretization scheme in the same spirit of [3]. Let us fix $n \in \mathbb{N}$ and set

$$\tau_n := \frac{T}{n}, \quad u_n^0 := u^0, \quad u_n^{-1} := u^0 - \tau_n u^1.$$

We define

$$V_n^k := V_{k\tau_n}^D, \quad g_n^k := g(k\tau_n), \quad w_n^k := w(k\tau_n) \quad \text{for } k = 0,\dots,n,$$

$$f_n^k := \frac{1}{\tau_n}\int_{(k-1)\tau_n}^{k\tau_n} f(s)\,\mathrm{d}s, \quad \Psi_n^k := \frac{1}{\tau_n}\int_{(k-1)\tau_n}^{k\tau_n} \Psi(s)\,\mathrm{d}s, \quad \delta g_n^k := \frac{g_n^k - g_n^{k-1}}{\tau_n} \quad \text{for } k = 1,\dots,n,$$
$$\delta w_n^0 := \dot w(0), \quad \delta w_n^k := \frac{w_n^k - w_n^{k-1}}{\tau_n}, \quad \delta^2 w_n^k := \frac{\delta w_n^k - \delta w_n^{k-1}}{\tau_n} \quad \text{for } k = 1,\dots,n.$$

For every $k = 1,\dots,n$ let $u_n^k \in V_T$, with $u_n^k - w_n^k \in V_n^k$, be the solution to

$$(\delta^2 u_n^k, v)_H + (\mathbb{C}Eu_n^k, Ev)_H + (\mathbb{B}\Psi_n^k E\delta u_n^k, \Psi_n^k Ev)_H = (f_n^k, v)_H + (g_n^k, v)_{H_N} \quad \text{for every } v \in V_n^k, \tag{3.1}$$

where

$$\delta u_n^k := \frac{u_n^k - u_n^{k-1}}{\tau_n} \quad \text{for } k = 0,\dots,n, \quad \delta^2 u_n^k := \frac{\delta u_n^k - \delta u_n^{k-1}}{\tau_n} \quad \text{for } k = 1,\dots,n.$$

The existence of a unique solution u_n^k to (3.1) is an easy application of Lax–Milgram's theorem.

Remark 3.2. Since $\delta u_n^k \in V_{(k-1)\tau_n}$, then $\Psi_n^k E\delta u_n^k = E(\Psi_n^k u_n^k) - \nabla\Psi_n^k \odot u_n^k$, so that the discrete equation (3.1) is coherent with the weak formulation given in (2.14).

In the next lemma, we show a uniform estimate for the family $\{u_n^k\}_{k=1}^n$ with respect to $n \in \mathbb{N}$ that will be used later to pass to the limit in the discrete equation (3.1).

Lemma 3.3. *There exists a constant $C > 0$, independent of $n \in \mathbb{N}$, such that*

$$\max_{i=1,..,n} \|\delta u_n^i\|_H + \max_{i=1,..,n} \|Eu_n^i\|_H + \sum_{i=1}^n \tau_n \|\Psi_n^i E\delta u_n^i\|_H^2 \le C. \tag{3.2}$$

Proof. We fix $n \in \mathbb{N}$. To simplify the notation we set

$$a(u,v) := (\mathbb{C}Eu, Ev)_H, \quad b_n^k(u,v) := (\mathbb{B}\Psi_n^k Eu, \Psi_n^k Ev)_H \quad \text{for every } u, v \in V_T.$$

By taking as test function $v = \tau_n(\delta u_n^k - \delta w_n^k) \in V_n^k$ in (3.1), for $k = 1,\dots,n$ we obtain

$$\|\delta u_n^k\|_H^2 - (\delta u_n^{k-1}, \delta u_n^k)_H + a(u_n^k, u_n^k) - a(u_n^k, u_n^{k-1}) + \tau_n b_n^k(\delta u_n^k, \delta u_n^k) = \tau_n L_n^k,$$

where

$$L_n^k := (f_n^k, \delta u_n^k - \delta w_n^k)_H + (g_n^k, \delta u_n^k - \delta w_n^k)_{H_N} + (\delta^2 u_n^k, \delta w_n^k)_H + a(u_n^k, \delta w_n^k) + b_n^k(\delta u_n^k, \delta w_n^k).$$

Thanks to the following identities

$$\|\delta u_n^k\|_H^2 - (\delta u_n^{k-1}, \delta u_n^k)_H = \frac12\|\delta u_n^k\|_H^2 - \frac12\|\delta u_n^{k-1}\|_H^2 + \frac{\tau_n^2}{2}\|\delta^2 u_n^k\|_H^2,$$
$$a(u_n^k, u_n^k) - a(u_n^k, u_n^{k-1}) = \frac12 a(u_n^k, u_n^k) - \frac12 a(u_n^{k-1}, u_n^{k-1}) + \frac{\tau_n^2}{2} a(\delta u_n^k, \delta u_n^k),$$

and by omitting the terms with τ_n^2, which are non negative, we derive

$$\frac12\|\delta u_n^k\|_H^2 - \frac12\|\delta u_n^{k-1}\|_H^2 + \frac12 a(u_n^k, u_n^k) - \frac12 a(u_n^{k-1}, u_n^{k-1}) + \tau_n b_n^k(\delta u_n^k, \delta u_n^k) \le \tau_n L_n^k.$$

We fix $i \in \{1,\dots,n\}$ and sum over $k = 1,\dots,i$ to obtain the following discrete energy inequality

$$\frac12\|\delta u_n^i\|_H^2 + \frac12 a(u_n^i, u_n^i) + \sum_{k=1}^i \tau_n b_n^k(\delta u_n^k, \delta u_n^k) \le \mathcal{E}_0 + \sum_{k=1}^i \tau_n L_n^k, \tag{3.3}$$

where $\mathcal{E}_0 := \frac12\|u^1\|_H^2 + \frac12(\mathbb{C}Eu^0, Eu^0)_H$. Let us now estimate the right–hand side in (3.3) from above. By (2.3) and (2.4) we have

$$\left|\sum_{k=1}^i \tau_n (f_n^k, \delta u_n^k - \delta w_n^k)_H\right| \le \|f\|_{L^2(0,T;H)}^2 + \frac12\|\dot w\|_{L^2(0,T;H)}^2 + \frac12\sum_{k=1}^i \tau_n\|\delta u_n^k\|_H^2, \tag{3.4}$$

$$\left|\sum_{k=1}^i \tau_n a(u_n^k, \delta w_n^k)\right| \le \frac{\|\mathbb{C}\|_\infty}{2}\|\dot w\|_{L^2(0,T;V_0)}^2 + \frac{\|\mathbb{C}\|_\infty}{2}\sum_{k=1}^i \tau_n\|Eu_n^k\|_H^2, \tag{3.5}$$

$$\left|\sum_{k=1}^i \tau_n (g_n^k, \delta w_n^k)_{H_N}\right| \le \frac12\|g\|_{L^2(0,T;H_N)}^2 + \frac{C_{tr}^2}{2}\|\dot w\|_{L^2(0,T;V_0)}^2. \tag{3.6}$$

For the other term involving g_n^k, we perform the following discrete integration by parts

$$\sum_{k=1}^{i} \tau_n (g_n^k, \delta u_n^k)_{H_N} = (g_n^i, u_n^i)_{H_N} - (g(0), u^0)_{H_N} - \sum_{k=1}^{i} \tau_n (\delta g_n^k, u_n^{k-1})_{H_N}. \tag{3.7}$$

Hence for every $\epsilon \in (0,1)$, by using (2.3) and Young's inequality, we get

$$\begin{aligned}\left|\sum_{k=1}^{i} \tau_n (g_n^k, \delta u_n^k)_{H_N}\right| &\le \frac{\epsilon}{2}\|u_n^i\|_{H_N}^2 + \frac{1}{2\epsilon}\|g\|^2_{L^\infty(0,T;H_N)} + \|g(0)\|_{H_N}\|u^0\|_{H_N} + \sum_{k=1}^{i}\tau_n\|\delta g_n^k\|_{H_N}\|u_n^{k-1}\|^2_{H_N} \\ &\le C_\epsilon + \frac{\epsilon C_{tr}^2}{2}\|u_n^i\|^2_{V_T} + \frac{C_{tr}^2}{2}\sum_{k=1}^{i}\tau_n\|u_n^k\|^2_{V_T},\end{aligned} \tag{3.8}$$

where C_ϵ is a positive constant depending on ϵ. Thanks to Jensen's inequality we can write

$$\|u_n^l\|^2_{V_T} \le \|Eu_n^l\|_H^2 + \left(\|u_0\|_H + \sum_{j=1}^{l}\tau_n\|\delta u_n^j\|_H\right)^2 \le \|Eu_n^l\|_H^2 + 2\|u^0\|_H^2 + 2T\sum_{j=1}^{l}\tau_n\|\delta u_n^j\|_H^2,$$

so that (3.8) can be further estimated as

$$\begin{aligned}\left|\sum_{k=1}^{i} \tau_n (g_n^k, \delta u_n^k)_{H_N}\right| &\le C_\epsilon + \frac{\epsilon C_{tr}^2}{2}\left(\|Eu_n^i\|_H^2 + 2\|u^0\|_H^2 + 2T\sum_{j=1}^{i}\tau_n\|\delta u_n^j\|_H^2\right) \\ &\quad + \frac{C_{tr}^2}{2}\sum_{k=1}^{i}\tau_n\left(\|Eu_n^k\|_H^2 + 2\|u^0\|_H^2 + 2T\sum_{j=1}^{k}\tau_n\|\delta u_n^j\|_H^2\right) \\ &\le \tilde{C}_\epsilon + \frac{\epsilon C_{tr}^2}{2}\|Eu_n^i\|_H^2 + \tilde{C}\sum_{k=1}^{i}\tau_n\left(\|\delta u_n^k\|_H^2 + \|Eu_n^k\|_H^2\right),\end{aligned} \tag{3.9}$$

for some positive constants $\tilde{C}_\epsilon$ and $\tilde{C}$, with $\tilde{C}_\epsilon$ depending on ϵ. Similarly to (3.7), we can say

$$\sum_{k=1}^{i}\tau_n(\delta^2 u_n^k, \delta w_n^k)_H = (\delta u_n^i, \delta w_n^i)_H - (\delta u_n^0, \delta w_n^0)_H - \sum_{k=1}^{i}\tau_n(\delta u_n^{k-1}, \delta^2 w_n^k)_H, \tag{3.10}$$

from which we deduce that for every $\epsilon > 0$

$$\begin{aligned}\left|\sum_{k=1}^{i}\tau_n(\delta^2 u_n^k, \delta w_n^k)_H\right| \le& \|\delta u_n^i\|_H\|\delta w_n^i\|_H + \|u^1\|_H\|\dot{w}(0)\|_H + \sum_{k=1}^{i}\tau_n\|\delta u_n^{k-1}\|_H\|\delta^2 w_n^k\|_H \\ \le& \frac{1}{2\epsilon}\|\delta w_n^i\|_H^2 + \frac{\epsilon}{2}\|\delta u_n^i\|_H^2 + \|u^1\|_H\|\dot{w}(0)\|_H + \frac{1}{2}\sum_{k=1}^{i}\tau_n\|\delta u_n^{k-1}\|_H^2 + \frac{1}{2}\sum_{k=1}^{i}\tau_n\|\delta^2 w_n^k\|_H^2 \\ \le& \bar{C}_\epsilon + \frac{\epsilon}{2}\|\delta u_n^i\|_H^2 + \frac{1}{2}\sum_{k=1}^{i}\tau_n\|\delta u_n^k\|_H^2,\end{aligned} \tag{3.11}$$

where $\bar{C}_\epsilon$ is a positive constant depending on ϵ. We estimate from above the last term in right-hand side of (3.3) in the following way

$$\begin{aligned}\sum_{k=1}^{i}\tau_n b_n^k(\delta u_n^k, \delta w_n^k) &\le \sum_{k=1}^{i}\tau_n (b_n^k(\delta u_n^k, \delta u_n^k))^{\frac{1}{2}}(b_n^k(\delta w_n^k, \delta w_n^k))^{\frac{1}{2}} \\ &\le \frac{1}{2}\sum_{k=1}^{i}\tau_n b_n^k(\delta u_n^k, \delta u_n^k) + \frac{1}{2}\|\mathbb{B}\|_\infty\|\Psi\|_\infty^2\|\dot{w}\|^2_{L^2(0,T;V_0)}.\end{aligned} \tag{3.12}$$

By considering (3.3)–(3.12) and using (2.6) we obtain

$$\left(\frac{1-\epsilon}{2}\right)\|\delta u_n^i\|_H^2 + \frac{\lambda_1 - \epsilon C_{tr}^2}{2}\|Eu_n^i\|_H^2 + \frac{1}{2}\sum_{k=1}^{i}\tau_n b_n^k(\delta u_n^k, \delta u_n^k) \le \hat{C}_\epsilon + \hat{C}\sum_{k=1}^{i}\tau_n\left(\|\delta u_n^k\|_H^2 + \|Eu_n^k\|_H^2\right)$$

for two positive constants $\hat{C}_\epsilon$ and $\hat{C}$, with $\hat{C}_\epsilon$ depending on ϵ. We can now choose $\epsilon < \frac{1}{2}\min\left\{1, \frac{\lambda_1}{C_{tr}^2}\right\}$ to derive the following estimate

$$\frac{1}{4}\|\delta u_n^i\|_H^2 + \frac{1}{4}\|Eu_n^i\|_H^2 + \frac{1}{2}\sum_{k=1}^i \tau_n b_n^k(\delta u_n^k, \delta u_n^k) \le C_1 + C_2 \sum_{k=1}^i \tau_n \left(\|\delta u_n^k\|_H^2 + \|Eu_n^k\|_H^2\right), \tag{3.13}$$

where C_1 and C_2 are two positive constants depending only on u^0, u^1, f, g, and w. Thanks to a discrete version of Gronwall's lemma (see, e.g., [1, Lemma 3.2.4]) we deduce the existence of a constant $C_3 > 0$, independent of i and n, such that

$$\|\delta u_n^i\|_H + \|Eu_n^i\|_H \le C_3 \quad \text{for every } i = 1, \dots, n \text{ and for every } n \in \mathbb{N}.$$

By combining this last estimate with (3.13) and (2.6) we finally get (3.2) and we conclude. □

We now want to pass to the limit into the discrete equation (3.1) to obtain a weak solution to (2.8)–(2.11). We start by defining the following approximating sequences of our limit solution

$$\begin{aligned}
&u_n(t) := u_n^k + (t - k\tau_n)\delta u_n^k, && \tilde{u}_n(t) := \delta u_n^k + (t - k\tau_n)\delta^2 u_n^k && t \in [(k-1)\tau_n, k\tau_n],\ k = 1, \dots, n,\\
&u_n^+(t) := u_n^k, && \tilde{u}_n^+(t) := \delta u_n^k && t \in ((k-1)\tau_n, k\tau_n],\ k = 1, \dots, n,\\
&u_n^-(t) := u_n^{k-1}, && \tilde{u}_n^-(t) := \delta u_n^{k-1} && t \in [(k-1)\tau_n, k\tau_n),\ k = 1, \dots, n.
\end{aligned}$$

Notice that $u_n \in H^1(0,T;H)$ with $\dot{u}_n(t) = \delta u_n^k = \tilde{u}_n^+(t)$ for $t \in ((k-1)\tau_n, k\tau_n)$ and $k = 1, \dots, n$. Let us approximate Ψ and w by

$$\begin{aligned}
&\Psi_n^+(t) := \Psi_n^k, && w_n^+(t) := w_n^k && t \in ((k-1)\tau_n, k\tau_n],\ k = 1, \dots, n,\\
&\Psi_n^-(t) := \Psi_n^{k-1}, && w_n^-(t) := w_n^{k-1} && t \in [(k-1)\tau_n, k\tau_n),\ k = 1, \dots, n.
\end{aligned}$$

Lemma 3.4. *There exists a function $u \in \mathcal{W}$, with $u - w \in \mathcal{V}^D$, such that, up to a not relabeled subsequence*

$$u_n \xrightarrow[n\to\infty]{H^1(0,T;H)} u, \quad u_n^\pm \xrightarrow[n\to\infty]{L^2(0,T;V_T)} u, \quad \tilde{u}_n^\pm \xrightarrow[n\to\infty]{L^2(0,T;H)} \dot{u}, \tag{3.14}$$

$$\nabla\Psi_n^\pm \odot \tilde{u}_n^\pm \xrightarrow[n\to\infty]{L^2(0,T;H)} \nabla\Psi \odot \dot{u}, \quad E(\Psi_n^\pm \tilde{u}_n^\pm) \xrightarrow[n\to\infty]{L^2(0,T;H)} E(\Psi\dot{u}). \tag{3.15}$$

Proof. Thanks to Lemma 3.3 the sequences $\{u_n\}_n \subset H^1(0,T;H) \cap L^\infty(0,T;V_T)$, $\{u_n^\pm\}_n \subset L^\infty(0,T;V_T)$, and $\{\tilde{u}_n^\pm\}_n \subset L^\infty(0,T;H)$ are uniformly bounded. By Banach-Alaoglu's theorem there exist $u \in H^1(0,T;H)$ and $v \in L^2(0,T;V_T)$ such that, up to a not relabeled subsequence

$$u_n \xrightarrow[n\to\infty]{L^2(0,T;V_T)} u, \quad \dot{u}_n \xrightarrow[n\to\infty]{L^2(0,T;H)} \dot{u}, \quad u_n^+ \xrightarrow[n\to\infty]{L^2(0,T;V_T)} v.$$

Since there exists a constant $C > 0$ such that

$$\|u_n - u_n^+\|_{L^\infty(0,T;H)} \le C\tau_n \xrightarrow[n\to\infty]{} 0,$$

we can conclude that $u = v$. Moreover, given that $u_n^-(t) = u_n^+(t - \tau_n)$ for $t \in (\tau_n, T)$, $\tilde{u}_n^+(t) = \dot{u}_n(t)$ for a.e. $t \in (0,T)$, and $\tilde{u}_n^-(t) = \tilde{u}_n^+(t - \tau_n)$ for $t \in (\tau_n, T)$, we deduce

$$u_n^- \xrightarrow[n\to\infty]{L^2(0,T;V_T)} u, \quad \tilde{u}_n^\pm \xrightarrow[n\to\infty]{L^2(0,T;H)} \dot{u}.$$

By (3.2) we derive that the sequences $\{E(\Psi_n^+\tilde{u}_n^+)\}_n \subset L^2(0,T;H)$ and $\{\nabla\Psi_n^+ \odot \tilde{u}_n^+\}_n \subset L^2(0,T;H)$ are uniformly bounded. Indeed there exists a constant $C > 0$ independent of n such that

$$\|\nabla\Psi_n^+ \odot \tilde{u}_n^+\|_{L^2(0,T;H)}^2 = \sum_{k=1}^n \int_{(k-1)\tau_n}^{k\tau_n} \|\nabla\Psi_n^k \odot \delta u_n^k\|_H^2 \,\mathrm{d}t \le \|\nabla\Psi\|_\infty^2 \sum_{k=1}^n \tau_n \|\delta u_n^k\|_H^2 \le C,$$

$$\begin{aligned}
\|E(\Psi_n^+\tilde{u}_n^+)\|_{L^2(0,T;H)}^2 &= \sum_{k=1}^n \int_{(k-1)\tau_n}^{k\tau_n} \|E(\Psi_n^k \delta u_n^k)\|_H^2 \,\mathrm{d}t = \sum_{k=1}^n \tau_n \|\Psi_n^k E\delta u_n^k + \nabla\Psi_n^k \odot \delta u_n^k\|_H^2\\
&\le 2\sum_{k=1}^n \tau_n \|\Psi_n^k E\delta u_n^k\|_H^2 + 2\sum_{k=1}^n \tau_n \|\nabla\Psi_n^k \odot \delta u_n^k\|_H^2 \le C.
\end{aligned}$$

Therefore, there exists $w_1, w_2 \in L^2(0,T;H)$ such that, up to a further not relabeled subsequence

$$\nabla\Psi_n^+ \odot \tilde u_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H)} w_1, \quad E(\Psi_n^+\tilde u_n^+) \xrightarrow[n\to\infty]{L^2(0,T;H)} w_2.$$

We want to identify the limit functions w_1 and w_2. Consider $\varphi \in L^2(0,T;H)$, then

$$\int_0^T (\nabla\Psi_n^+ \odot \tilde u_n^+, \varphi)_H \,\mathrm{d}t = \frac12 \int_0^T (\tilde u_n^+, \varphi\nabla\Psi_n^+)_H \,\mathrm{d}t + \frac12\int_0^T (\tilde u_n^+, \varphi^T\nabla\Psi_n^+)_H \,\mathrm{d}t = \int_0^T (\tilde u_n^+, \varphi^{sym}\nabla\Psi_n^+)_H \,\mathrm{d}t,$$

where $\varphi^{sym} := \frac{\varphi+\varphi^T}{2}$. Since $\tilde u_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H)} \dot u$ and $\varphi^{sym}\nabla\Psi_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H)} \varphi^{sym}\nabla\Psi$ by dominated convergence theorem, we obtain

$$\int_0^T (\nabla\Psi_n^+ \odot \tilde u_n^+, \varphi)_H \,\mathrm{d}t \xrightarrow[n\to\infty]{} \int_0^T (\dot u, \varphi^{sym}\nabla\Psi)_H \,\mathrm{d}t = \int_0^T (\nabla\Psi\odot\dot u, \varphi)_H \,\mathrm{d}t,$$

and so $w_1 = \nabla\Psi\odot\dot u$. Moreover for $\phi \in L^2(0,T;H)$ we have

$$\int_0^T (\Psi_n^+\tilde u_n^+, \phi)_H \,\mathrm{d}t = \int_0^T (\tilde u_n^+, \phi\Psi_n^+)_H \,\mathrm{d}t \xrightarrow[n\to\infty]{} \int_0^T (\dot u, \Psi\phi)_H \,\mathrm{d}t = \int_0^T (\Psi\dot u, \phi)_H \,\mathrm{d}t,$$

thanks to $\tilde u_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H)} \dot u$ and $\Psi_n^+\phi \xrightarrow[n\to\infty]{L^2(0,T;H)} \Psi\phi$, again implied by dominated convergence theorem. Therefore $\Psi_n^+\tilde u_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H)} \Psi\dot u$, from which $E(\Psi_n^+\tilde u_n^+) \xrightarrow[n\to\infty]{\mathcal D'(0,T;H)} E(\Psi\dot u)$, that gives $w_2 = E(\Psi\dot u)$. In particular we have $\Psi\dot u \in L^2(0,T;V_T)$. By arguing in a similar way we also obtain

$$\nabla\Psi_n^- \odot \tilde u_n^- \xrightarrow[n\to\infty]{L^2(0,T;H)} \nabla\Psi\odot\dot u, \quad E(\Psi_n^-\tilde u_n^-) \xrightarrow[n\to\infty]{L^2(0,T;H)} E(\Psi\dot u).$$

Let us check that $u \in \mathcal W$. To this aim, let us consider the following set

$$F := \{v \in L^2(0,T;V_T) : v(t) \in V_t \text{ for a.e. } t \in (0,T)\} \subset L^2(0,T;V_T).$$

We have that F is a (strong) closed convex subset of $L^2(0,T;V_T)$, and so by Hahn-Banach's theorem the set F is weakly closed. Notice that $\{u_n^-\}_n, \{\Psi_n^-\tilde u_n^-\}_n \subset F$, indeed

$$u_n^-(t) = u_n^{k-1} \in V_{(k-1)\tau_n} \subset V_t \quad \text{for } t \in [(k-1)\tau_n, k\tau_n),\ k = 1,\dots,n,$$
$$\Psi_n^-(t)\tilde u_n^-(t) = \Psi_n^{k-1}\delta u_n^{k-1} \in V_{(k-1)\tau_n} \subseteq V_t \quad \text{for } t \in [(k-1)\tau_n, k\tau_n),\ k = 1,\dots,n.$$

Since $u_n^- \xrightarrow[n\to\infty]{L^2(0,T;V_T)} u$ and $\Psi_n^-\tilde u_n^- \xrightarrow[n\to\infty]{L^2(0,T;V_T)} \Psi\dot u$, we conclude that $u, \Psi\dot u \in F$. Finally, to show that $u - w \in \mathcal V^D$ we observe

$$u_n^-(t) - w_n^-(t) = u_n^{k-1} - w_n^{k-1} \in V_n^{k-1} \subseteq V_t^D \qquad \text{for } t \in [(k-1)\tau_n, k\tau_n),\ k = 1,\dots,n.$$

Therefore $\{u_n^- - w_n^-\}_n \subset \{v \in L^2(0,T;V_T) : v(t) \in V_t^D \text{ for a.e. } t \in (0,T)\}$, which is a (strong) closed convex subset of $L^2(0,T;V_T)$, and so it is weakly closed. Since $u_n^- \xrightarrow[n\to\infty]{L^2(0,T;V_T)} u$ and $w_n^- \xrightarrow[n\to\infty]{L^2(0,T;V_0)} w$, we get that $u(t) - w(t) \in V_t^D$ for a.e. $t \in (0,T)$, which implies $u - w \in \mathcal V^D$. □

We now use Lemma 3.4 to pass to the limit in the discrete equation (3.1).

Lemma 3.5. *The limit function $u \in \mathcal W$ of Lemma 3.4 is a weak solution to* (2.8)–(2.11).

Proof. We only need to prove that $u \in \mathcal W$ satisfies (2.14). We fix $n \in \mathbb N$, $\varphi \in C_c^1(0,T;V_T)$ such that $\varphi(t) \in V_t^D$ for every $t \in (0,T)$, and we consider

$$\varphi_n^k := \varphi(k\tau_n) \quad \text{for } k = 0,\dots,n, \quad \delta\varphi_n^k := \frac{\varphi_n^k - \varphi_n^{k-1}}{\tau_n} \quad \text{for } k = 1,\dots,n,$$

and the approximating sequences

$$\varphi_n^+(t) := \varphi_n^k, \qquad \tilde\varphi_n^+(t) := \delta\varphi_n^k \qquad t \in ((k-1)\tau_n, k\tau_n],\ k = 1,\dots,n.$$

If we use $\tau_n\varphi_n^k \in V_n^k$ as test function in (3.1), after summing over $k = 1, ..., n$, we get

$$\sum_{k=1}^n \tau_n(\delta^2 u_n^k, \varphi_n^k)_H + \sum_{k=1}^n \tau_n(\mathbb{C}Eu_n^k, E\varphi_n^k)_H + \sum_{k=1}^n \tau_n(\mathbb{B}\Psi_n^k E\delta u_n^k, \Psi_n^k E\varphi_n^k)_H = \sum_{k=1}^n \tau_n(f_n^k, \varphi_n^k)_H + \sum_{k=1}^n \tau_n(g_n^k, \varphi_n^k)_{H_N}. \tag{3.16}$$

By these identities

$$\sum_{k=1}^n \tau_n(\delta^2 u_n^k, \varphi_n^k)_H = -\sum_{k=1}^n \tau_n(\delta u_n^{k-1}, \delta\varphi_n^k)_H = -\int_0^T (\tilde{u}_n^-(t), \tilde{\varphi}_n^+(t))_H \,\mathrm{d}t,$$

from (3.16) we deduce

$$-\int_0^T (\tilde{u}_n^-, \tilde{\varphi}_n^+)_H \,\mathrm{d}t + \int_0^T (\mathbb{C}Eu_n^+, E\varphi_n^+)_H \,\mathrm{d}t - \int_0^T (\mathbb{B}\nabla\Psi_n^+ \odot \tilde{u}_n^+, E\varphi_n^+)_H \,\mathrm{d}t + \int_0^T (\mathbb{B}E(\Psi_n^+\tilde{u}_n^+), E\varphi_n^+)_H \,\mathrm{d}t = \int_0^T (f_n^+, \varphi_n^+)_H \,\mathrm{d}t + \int_0^T (g_n^+, \varphi_n^+)_{H_N} \,\mathrm{d}t. \tag{3.17}$$

Thanks to (3.14), (3.15), and the following convergences

$$\varphi_n^+ \xrightarrow[n\to\infty]{L^2(0,T;V_T)} \varphi, \quad \tilde{\varphi}_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H)} \dot{\varphi}, \quad f_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H)} f, \quad g_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H_N)} g,$$

we can pass to the limit in (3.17), and we get that $u \in \mathcal{W}$ satisfies (2.14) for every $\varphi \in C_c^1(0, T; V_T)$ such that $\varphi(t) \in V_t^D$ for every $t \in (0, T)$. Finally, by using a density argument (see [8, Remark 2.9]), we conclude that $u \in \mathcal{W}$ is a weak solution to (2.8)–(2.11). □

4. Initial Conditions and Energy–Dissipation Inequality

To complete our existence result, it remains to prove that the function $u \in \mathcal{W}$ given by Lemma 3.5 satisfies the initial conditions (2.12) in the sense of (2.16). Let us start by showing that the second distributional derivative $\ddot{u}$ belongs to $L^2(0, T; (V_0^D)')$. If we consider the discrete equation (3.1), for every $v \in V_0^D \subseteq V_n^k$, with $\|v\|_{V_0} \le 1$, we have

$$|(\delta^2 u_n^k, v)_H| \le \|\mathbb{C}\|_\infty \|Eu_n^k\|_H + \|\mathbb{B}\|_\infty \|\Psi\|_\infty \|\Psi_n^k E\delta u_n^k\|_H + \|f_n^k\|_H + C_{tr}\|g_n^k\|_{H_N}.$$

Therefore, taking the supremum over $v \in V_0^D$ with $\|v\|_{V_0} \le 1$, we obtain the existence of a positive constant C such that

$$\|\delta^2 u_n^k\|^2_{(V_0^D)'} \le C(\|Eu_n^k\|_H^2 + \|\Psi_n^k E\delta u_n^k\|_H^2 + \|f_n^k\|_H^2 + \|g_n^k\|_{H_N}^2).$$

If we multiply this inequality by τ_n and we sum over $k = 1, \dots, n$, we get

$$\sum_{k=1}^n \tau_n \|\delta^2 u_n^k\|^2_{(V_0^D)'} \le C\left(\sum_{k=1}^n \tau_n\|Eu_n^k\|_H^2 + \sum_{k=1}^n \tau_n \|\Psi_n^k E\delta u_n^k\|_H^2 + \|f\|^2_{L^2(0,T;H)} + \|g\|^2_{L^2(0,T;H_N)}\right). \tag{4.1}$$

Thanks to (4.1) and Lemma 3.3 we conclude that $\sum_{k=1}^n \tau_n \|\delta^2 u_n^k\|^2_{(V_0^D)'} \le \tilde{C}$ for every $n \in \mathbb{N}$ for a positive constant $\tilde{C}$ independent on $n \in \mathbb{N}$. In particular the sequence $\{\tilde{u}_n\}_n \subset H^1(0, T; (V_0^D)')$ is uniformly bounded (notice that $\dot{\tilde{u}}_n(t) = \delta^2 u_n^k$ for $t \in ((k-1)\tau_n, k\tau_n)$ and $k = 1, \dots, n$). Hence, up to extract a further (not relabeled) subsequence from the one of Lemma 3.4, we get

$$\tilde{u}_n \xrightarrow[n\to\infty]{H^1(0,T;(V_0^D)')} w_3, \tag{4.2}$$

and by using the following estimate

$$\|\tilde{u}_n - \tilde{u}_n^+\|_{L^2(0,T;(V_0^D)')} \le \tau_n \|\dot{\tilde{u}}_n\|_{L^2(0,T;(V_0^D)')} \le \tilde{C}\tau_n \xrightarrow[n\to\infty]{} 0$$

we conclude that $w_3 = \dot{u}$.

Let us recall the following result, whose proof can be found for example in [9].

Lemma 4.1. *Let X, Y be two reflexive Banach spaces such that $X \hookrightarrow Y$ continuously. Then*

$$L^\infty(0, T; X) \cap C_w^0([0, T]; Y) = C_w^0([0, T]; X).$$

Since $H^1(0,T;(V_0^D)') \hookrightarrow C^0([0,T],(V_0^D)')$, by using Lemmas 3.4 and 4.1 we get that our weak solution $u \in \mathcal{W}$ satisfies

$$u \in C_w^0([0,T];V_T), \quad \dot{u} \in C_w^0([0,T];H), \quad \ddot{u} \in L^2(0,T;(V_0^D)').$$

By (3.14) and (4.2) we hence obtain

$$u_n(t) \xrightarrow[n\to\infty]{H} u(t), \quad \tilde{u}_n(t) \xrightarrow[n\to\infty]{(V_0^D)'} \dot{u}(t) \quad \text{for every } t \in [0,T], \tag{4.3}$$

so that $u(0) = u^0$ and $\dot{u}(0) = u^1$, since $u_n(0) = u^0$ and $\tilde{u}_n(0) = u^1$.

To prove that

$$\lim_{h\to 0^+} \frac{1}{h}\int_0^h \left(\|u(t)-u^0\|_{V_t}^2 + \|\dot{u}(t)-u^1\|_H^2\right) \mathrm{d}t = 0$$

we will actually show

$$\lim_{t\to 0^+} u(t) = u^0 \text{ in } V_T, \quad \lim_{t\to 0^+} \dot{u}(t) = u^1 \text{ in } H.$$

This is a consequence of following energy–dissipation inequality which holds for the weak solution $u \in \mathcal{W}$ of Lemma 3.5. Let us define the total energy as

$$\mathcal{E}(t) := \frac{1}{2}\|\dot{u}(t)\|_H^2 + \frac{1}{2}(\mathbb{C}Eu(t), Eu(t))_H \quad t \in [0,T].$$

Notice that $\mathcal{E}(t)$ is well defined for every $t \in [0,T]$ since $u \in C_w^0([0,T];V_T)$ and $\dot{u} \in C_w^0([0,T];H)$, and that $\mathcal{E}(0) = \frac{1}{2}\|u^1\|_H^2 + \frac{1}{2}(\mathbb{C}Eu^0, Eu^0)_H$.

Theorem 4.2. *The weak solution $u \in \mathcal{W}$ to (2.8)–(2.11), given by Lemma 3.5, satisfies for every $t \in [0,T]$ the following energy–dissipation inequality*

$$\mathcal{E}(t) + \int_0^t (\mathbb{B}\Psi E\dot{u}, \Psi E\dot{u})_H \,\mathrm{d}s \le \mathcal{E}(0) + \mathcal{W}_{tot}(t), \tag{4.4}$$

where $\Psi E\dot{u}$ is the function defined in (2.15) and $\mathcal{W}_{tot}(t)$ is the total work on the solution u at time $t \in [0,T]$, which is given by

$$\begin{aligned}\mathcal{W}_{tot}(t) :&= \int_0^t [(f, \dot{u}-\dot{w})_H + (\mathbb{C}Eu, E\dot{w})_H + (\mathbb{B}\Psi E\dot{u}, \Psi E\dot{w})_H - (\dot{u}, \ddot{w})_H - (\dot{g}, u-w)_{H_N}] \,\mathrm{d}s \\ &\quad + (\dot{u}(t), \dot{w}(t))_H + (g(t), u(t)-w(t))_{H_N} - (u^1, \dot{w}(0))_H - (g(0), u^0 - w(0))_{H_N}.\end{aligned} \tag{4.5}$$

Remark 4.3. From the classical point of view, the total work on the solution u at time $t \in [0,T]$ is given by

$$\mathcal{W}_{tot}(t) := \mathcal{W}_{load}(t) + \mathcal{W}_{bdry}(t), \tag{4.6}$$

where $\mathcal{W}_{load}(t)$ is the work on the solution u at time $t \in [0,T]$ due to the loading term, which is defined as

$$\mathcal{W}_{load}(t) := \int_0^t (f(s), \dot{u}(s))_H \,\mathrm{d}s,$$

and $\mathcal{W}_{bdry}(t)$ is the work on the solution u at time $t \in [0,T]$ due to the varying boundary conditions, which one expects to be equal to

$$\mathcal{W}_{bdry}(t) := \int_0^t (g(s), \dot{u}(s))_{H_N} \,\mathrm{d}s + \int_0^t ((\mathbb{C}Eu(s) + \Psi^2(s)\mathbb{B}E\dot{u}(s))\nu, \dot{w}(s))_{H_D} \,\mathrm{d}s,$$

being $H_D := L^2(\partial_D\Omega;\mathbb{R}^d)$. Unfortunately, $\mathcal{W}_{bdry}(t)$ is not well defined under our assumptions on u. Notice that when $\Psi \equiv 1$ on a neighborhood U of the closure of $\partial_N\Omega$, then every weak solution u to (2.8)–(2.11) satisfies $u \in H^1(0,T;H^1((\Omega\cap U)\setminus\Gamma;\mathbb{R}^d))$, which gives that $u \in H^1(0,T;H_N)$ by our assumptions on Γ. Hence the first term of $\mathcal{W}_{bdry}(t)$ makes sense and satisfies

$$\int_0^t (g(s), \dot{u}(s))_{H_N} \,\mathrm{d}s = (g(t), u(t))_{H_N} - (g(0), u(0))_{H_N} - \int_0^t (\dot{g}(s), u(s))_{H_N} \,\mathrm{d}s.$$

The term involving the Dirichlet datum w is more difficult to handle since the trace of $(\mathbb{C}Eu + \Psi^2\mathbb{B}E\dot{u})\nu$ on $\partial_D\Omega$ is not well defined even when $\Psi \equiv 1$ on a neighborhood of the closure of $\partial_D\Omega$. If we assume that $u \in H^1(0,T;H^2(\Omega\setminus\Gamma;\mathbb{R}^d)) \cap H^2(0,T;L^2(\Omega;\mathbb{R}^d))$ and that Γ is a smooth manifold, then we can integrate by part equation (2.14) to deduce that u satisfies (2.8). In this case, $(\mathbb{C}Eu+\Psi^2\mathbb{B}E\dot{u})\nu\in L^2(0,T;H_D)$ and by using (2.8), together with the divergence theorem and the integration by parts formula, we deduce

$$\begin{aligned}
&\int_0^t ((\mathbb{C}Eu(s)+\Psi^2(s)\mathbb{B}E\dot{u}(s))\nu,\dot{w}(s))_{H_D}\,\mathrm{d}s\\
&=\int_0^t \left[(\operatorname{div}(\mathbb{C}Eu(s)+\Psi^2(s)\mathbb{B}E\dot{u}(s)),\dot{w}(s))_H+(\mathbb{C}Eu(s)+\Psi^2(s)\mathbb{B}E\dot{u}(s),E\dot{w}(s))_H-(g(s),\dot{w}(s))_{H_N}\right]\mathrm{d}s\\
&=\int_0^t \left[(\ddot{u}(s),\dot{w}(s))_H-(f(s),\dot{w}(s))_H+(\mathbb{C}Eu(s)+\Psi^2(s)\mathbb{B}E\dot{u}(s),E\dot{w}(s))_H-(g(s),\dot{w}(s))_{H_N}\right]\mathrm{d}s\\
&=\int_0^t [(\mathbb{C}Eu(s),E\dot{w}(s))_H+(\mathbb{B}\Psi(s)E\dot{u}(s),\Psi(s)E\dot{w}(s))_H-(f(s),\dot{w}(s))_H]\,\mathrm{d}s\\
&\quad+\int_0^t[(\dot{g}(s),w(s))_{H_N}-(\dot{u}(s),\ddot{w}(s))_H]\,\mathrm{d}s-(g(t),w(t))_{H_N}+(\dot{u}(t),\dot{w}(t))_H+(g(0),w(0))_{H_N}-(u^1,\dot{w}(0))_H.
\end{aligned}$$

Hence, the definition of total work given in (4.5) is coherent with the classical one (4.6). Notice that if u is the solution to (2.8)–(2.11) given by Lemma 3.5, then (4.5) is well defined for every $t\in[0,T]$, since $g\in C^0([0,T];H_N)$, $\dot{w}\in C^0([0,T];H)$, $u\in C^0_w([0,T];V_T)$, and $\dot{u}\in C^0_w([0,T];H)$. In particular, the function $t\mapsto\mathcal{W}_{tot}(t)$ from $[0,T]$ to $\mathbb{R}$ is continuous.

Proof. Fixed $t\in(0,T]$, for every $n\in\mathbb{N}$ there exists a unique $j\in\{1,\dots,n\}$ such that $t\in((j-1)\tau_n,j\tau_n]$. After setting $t_n:=j\tau_n$, we can rewrite (3.3) as

$$\frac{1}{2}\|\tilde{u}_n^+(t)\|_H^2+\frac{1}{2}(\mathbb{C}Eu_n^+(t),Eu_n^+(t))_H+\int_0^{t_n}(\mathbb{B}\Psi_n^+E\tilde{u}_n^+,\Psi_n^+E\tilde{u}_n^+)_H\,\mathrm{d}s\le\mathcal{E}(0)+\mathcal{W}_n^+(t), \tag{4.7}$$

where

$$\begin{aligned}
\mathcal{W}_n^+(t):=&\int_0^{t_n}\left[(f_n^+,\tilde{u}_n^+-\tilde{w}_n^+)_H+(\mathbb{C}Eu_n^+,E\tilde{w}_n^+)_H+(\mathbb{B}\Psi_n^+E\tilde{u}_n^+,\Psi_n^+E\tilde{w}_n^+)_H\right]\mathrm{d}s\\
&+\int_0^{t_n}\left[(\tilde{u}_n^+,\tilde{w}_n^+)_H+(g_n^+,\tilde{u}_n^+-\tilde{w}_n^+)_{H_N}\right]\mathrm{d}s.
\end{aligned}$$

Thanks to (3.2), we have

$$\begin{aligned}
&\|u_n(t)-u_n^+(t)\|_H=\|u_n^j+(t-j\tau_n)\delta u_n^j-u_n^j\|_H\le\tau_n\|\delta u_n^j\|_H\le C\tau_n\xrightarrow[n\to\infty]{}0,\\
&\|\tilde{u}_n(t)-\tilde{u}_n^+(t)\|_{(V_0^D)'}^2=\|\delta u_n^j+(t-j\tau_n)\delta^2u_n^j-\delta u_n^j\|_{(V_0^D)'}^2\le\tau_n^2\|\delta^2u_n^j\|_{(V_0^D)'}^2\le C\tau_n\xrightarrow[n\to\infty]{}0.
\end{aligned}$$

The last convergences and (4.3) imply

$$u_n^+(t)\xrightarrow[n\to\infty]{H}u(t),\qquad\tilde{u}_n^+(t)\xrightarrow[n\to\infty]{(V_0^D)'}\dot{u}(t),$$

and since $\|u_n^+(t)\|_{V_T}+\|\tilde{u}_n^+(t)\|_H\le C$ for every $n\in\mathbb{N}$, we get

$$u_n^+(t)\xrightarrow[n\to\infty]{V_T}u(t),\qquad\tilde{u}_n^+(t)\xrightarrow[n\to\infty]{H}\dot{u}(t). \tag{4.8}$$

By the lower semicontinuity properties of $v\mapsto\|v\|_H^2$ and $v\mapsto(\mathbb{C}Ev,Ev)_H$, we conclude

$$\|\dot{u}(t)\|_H^2\le\liminf_{n\to\infty}\|\tilde{u}_n^+(t)\|_H^2, \tag{4.9}$$

$$(\mathbb{C}Eu(t),Eu(t))_H\le\liminf_{n\to\infty}(\mathbb{C}Eu_n^+(t),Eu_n^+(t))_H. \tag{4.10}$$

Thanks to Lemma 3.4 and (2.15), we obtain

$$\Psi_n^+E\tilde{u}_n^+=E(\Psi_n^+\tilde{u}_n^+)-\nabla\Psi_n^+\odot\tilde{u}_n^+\xrightarrow[n\to\infty]{L^2(0,T;H)}E(\Psi\dot{u})-\nabla\Psi\odot\dot{u}=\Psi E\dot{u},$$

so that

$$\int_0^t(\mathbb{B}\Psi E\dot{u},\Psi E\dot{u})_H\,\mathrm{d}s\le\liminf_{n\to\infty}\int_0^t(\mathbb{B}\Psi_n^+E\tilde{u}_n^+,\Psi_n^+E\tilde{u}_n^+)_H\,\mathrm{d}s\le\liminf_{n\to\infty}\int_0^{t_n}(\mathbb{B}\Psi_n^+E\tilde{u}_n^+,\Psi_n^+E\tilde{u}_n^+)_H\,\mathrm{d}s, \tag{4.11}$$

since $t \le t_n$ and $v \mapsto \int_0^t (\mathbb{B}v, v)_H \, \mathrm{d}s$ is a non negative quadratic form on $L^2(0,T;H)$. Let us study the right–hand side of (4.7). Given that we have

$$\chi_{[0,t_n]} f_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H)} \chi_{[0,t]} f, \quad \tilde{u}_n^+ - \tilde{w}_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H)} \dot{u} - \dot{w},$$

we can deduce

$$\int_0^{t_n} (f_n^+, \tilde{u}_n^+ - \tilde{w}_n^+)_H \, \mathrm{d}s \xrightarrow[n\to\infty]{} \int_0^t (f, \dot{u} - \dot{w})_H \, \mathrm{d}s. \tag{4.12}$$

In a similar way, we can prove

$$\int_0^{t_n} (\mathbb{C}Eu_n^+, E\tilde{w}_n^+)_H \, \mathrm{d}s \xrightarrow[n\to\infty]{} \int_0^t (\mathbb{C}Eu, E\dot{w})_H \, \mathrm{d}s, \tag{4.13}$$

$$\int_0^{t_n} (\mathbb{B}\Psi_n^+ E\tilde{u}_n^+, \Psi_n^+ E\tilde{w}_n^+)_H \, \mathrm{d}s \xrightarrow[n\to\infty]{} \int_0^t (\mathbb{B}\Psi E\dot{u}, \Psi E\dot{w})_H \, \mathrm{d}s, \tag{4.14}$$

since the following convergences hold

$$\chi_{[0,t_n]} E\tilde{w}_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H)} \chi_{[0,t]} E\dot{w}, \qquad \mathbb{C}Eu_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H)} \mathbb{C}Eu,$$

$$\chi_{[0,t_n]} \Psi_n^+ E\tilde{w}_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H)} \chi_{[0,t]} \Psi E\dot{w}, \qquad \Psi_n^+ E\tilde{u}_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H)} \Psi E\dot{u}.$$

It remains to study the behaviour as $n \to \infty$ of the terms

$$\int_0^{t_n} (\dot{\tilde{u}}_n, \tilde{w}_n^+)_H \, \mathrm{d}s, \qquad \int_0^{t_n} (g_n^+, \tilde{u}_n^+ - \tilde{w}_n^+)_{H_N} \, \mathrm{d}s.$$

Thanks to formula (3.10) we have

$$\int_0^{t_n} (\dot{\tilde{u}}_n, \tilde{w}_n^+)_H \, \mathrm{d}s = (\tilde{u}_n^+(t), \tilde{w}_n^+(t))_H - (u^1, \dot{w}(0))_H - \int_0^{t_n} (\tilde{u}_n^-, \dot{\tilde{w}}_n)_H \, \mathrm{d}s.$$

By arguing as before we hence deduce

$$\int_0^{t_n} (\dot{\tilde{u}}_n, \tilde{w}_n^+)_H \, \mathrm{d}s \xrightarrow[n\to\infty]{} (\dot{u}(t), \dot{w}(t))_H - (u^1, \dot{w}(0))_H - \int_0^t (\dot{u}, \ddot{w})_H \, \mathrm{d}s, \tag{4.15}$$

thanks to (4.8) and by these convergences

$$\chi_{[0,t_n]} \dot{\tilde{w}}_n \xrightarrow[n\to\infty]{L^2(0,T;H)} \chi_{[0,t]} \ddot{w}, \quad \tilde{u}_n^- \xrightarrow[n\to\infty]{L^2(0,T;H)} \dot{u},$$

$$\|\tilde{w}_n^+(t) - \dot{w}(t)\|_H = \left\| \frac{w(j\tau_n) - w((j-1)\tau_n)}{\tau_n} - \dot{w}(t) \right\|_H = \left\| \fint_{(j-1)\tau_n}^{j\tau_n} (\dot{w}(s) - \dot{w}(t)) \, \mathrm{d}s \right\|_H$$

$$\le \fint_{(j-1)\tau_n}^{j\tau_n} \|\dot{w}(s) - \dot{w}(t)\|_H \, \mathrm{d}s \xrightarrow[n\to\infty]{} 0.$$

Notice that in the last convergence we used the continuity of w from $[0,T]$ in H. Similarly we have

$$\int_0^{t_n} (g_n^+, \tilde{u}_n^+ - \tilde{w}_n^+)_{H_N} \, \mathrm{d}s = (g_n^+(t), u_n^+(t) - w_n^+(t))_{H_N} - (g(0), u^0 - w(0))_{H_N} - \int_0^{t_n} (\dot{g}_n, u_n^- - w_n^-)_{H_N} \, \mathrm{d}s$$

so that we get

$$\int_0^{t_n} (g_n^+, \tilde{u}_n^+ - \tilde{w}_n^+)_{H_N} \, \mathrm{d}s \xrightarrow[n\to\infty]{} (g(t), u(t) - w(t))_{H_N} - (g(0), u^0 - w(0))_{H_N} - \int_0^t (\dot{g}, u - w)_{H_N} \, \mathrm{d}s \tag{4.16}$$

thanks to (4.8), the continuity of $s \mapsto g(s)$ in H_N, and the fact that

$$\chi_{[0,t_n]} \dot{g}_n \xrightarrow[n\to\infty]{L^2(0,T;H_N)} \chi_{[0,t]} \dot{g}, \qquad u_n^- - w_n^- \xrightarrow[n\to\infty]{L^2(0,T;H_N)} u - w.$$

By combining (4.9)–(4.16), we deduce the energy–dissipation inequality (4.4) for every $t \in (0,T]$. Finally, for $t = 0$ the inequality trivially holds since $u(0) = u^0$ and $\dot{u}(0) = u^1$. □

We now are in position to prove the validity of the initial conditions.

Lemma 4.4. *The weak solution $u \in \mathcal{W}$ to (2.8)–(2.11) of Lemma 3.5 satisfies*

$$\lim_{t\to 0^+} u(t) = u^0 \text{ in } V_T, \quad \lim_{t\to 0^+} \dot u(t) = u^1 \text{ in } H. \tag{4.17}$$

In particular u satisfies the initial conditions (2.12) in the sense of (2.16).

Proof. By sending $t \to 0^+$ into the energy–dissipation inequality (4.4) and using that $u \in C^0_w([0,T];V_T)$ and $\dot u \in C^0_w([0,T];H)$ we deduce

$$\mathcal{E}(0) \le \liminf_{t\to 0^+} \mathcal{E}(t) \le \limsup_{t\to 0^+} \mathcal{E}(t) \le \mathcal{E}(0),$$

since the right–hand side of (4.4) is continuous in t, $u(0) = u^0$, and $\dot u(0) = u^1$. Therefore there exists $\lim_{t\to 0^+} \mathcal{E}(t) = \mathcal{E}(0)$. By using the lower semicontinuity of $t \mapsto \|\dot u(t)\|^2_H$ and $t \mapsto (\mathbb{C}Eu(t), Eu(t))_H$, we derive

$$\lim_{t\to 0^+} \|\dot u(t)\|^2_H = \|u^1\|^2_H, \quad \lim_{t\to 0^+} (\mathbb{C}Eu(t), Eu(t))_H = (\mathbb{C}Eu^0, Eu^0)_H.$$

Finally, since we have

$$\dot u(t) \xrightarrow[t\to 0^+]{H} u^1, \quad Eu(t) \xrightarrow[t\to 0^+]{H} Eu^0,$$

we deduce (4.17). In particular the functions $u\colon [0,T] \to V_T$ and $\dot u\colon [0,T] \to H$ are continuous at $t = 0$, which implies (2.16). □

We can finally prove Theorem 3.1.

Proof of Theorem 3.1. It is enough to combine Lemmas 3.5 and 4.4. □

Remark 4.5. We have proved Theorem 3.1 for the d-dimensional linear elastic case, namely when the displacement u is a vector–valued function. The same result is true with identical proofs in the antiplane case, that is when the displacement u is a scalar function and satisfies (1.5).

5. Uniqueness

In this section we investigate the uniqueness properties of system (2.8) with boundary and initial conditions (2.9)–(2.12). To this aim, we need to assume stronger regularity assumptions on the crack sets $\{\Gamma_t\}_{t\in[0,T]}$ and on the function Ψ. Moreover, we have to restrict our problem to the dimensional case $d = 2$, since in our proof we need to construct a suitable family of diffeomorphisms which maps the time–dependent crack Γ_t into a fixed set, and this can be explicitly done only for $d = 2$ (see [7, Example 2.14]).

We proceed in two steps; first, in Lemma 5.2 we prove a uniqueness result in every dimension d, but when the cracks are not increasing, that is $\Gamma_T = \Gamma_0$. Next, in Theorem 5.5 we combine Lemma 5.2 with the finite speed of propagation theorem of [5] and the uniqueness result of [8] to derive the uniqueness of a weak solution to (2.8)–(2.12) in the case $d = 2$.

Let us start with the following lemma, whose proof is similar to that one of [8, Proposition 2.10].

Lemma 5.1. *Let $u \in \mathcal{W}$ be a weak solution to (2.8)–(2.11) satisfying the initial condition $\dot u(0) = 0$ in the following sense*

$$\lim_{h\to 0^+} \frac{1}{h}\int_0^h \|\dot u(t)\|^2_H = 0.$$

Then u satisfies

$$\begin{aligned} &-\int_0^T (\dot u(t), \dot\varphi(t))_H \,\mathrm{d}t + \int_0^T (\mathbb{C}Eu(t), E\varphi(t))_H \,\mathrm{d}t + \int_0^T (\mathbb{B}\Psi(t)E\dot u(t), \Psi(t)E\varphi(t))_H \,\mathrm{d}t \\ &= \int_0^T (f(t), \varphi(t))_H \,\mathrm{d}t + \int_0^T (g(t), \varphi(t))_{H_N} \,\mathrm{d}t \end{aligned}$$

for every $\varphi \in \mathcal{V}^D$ such that $\varphi(T) = 0$, where $\Psi E\dot u$ is the function defined in (2.15).

Proof. We fix $\varphi \in \mathcal{V}^D$ with $\varphi(T) = 0$ and for every $\epsilon > 0$ we define the following function

$$\varphi_\epsilon(t) := \begin{cases} \frac{t}{\epsilon}\varphi(t) & t \in [0,\epsilon], \\ \varphi(t) & t \in [\epsilon, T]. \end{cases}$$

We have that $\varphi_\epsilon \in \mathcal{V}^D$ and $\varphi_\epsilon(0) = \varphi_\epsilon(T) = 0$, so we can use φ_ϵ as test function in (2.14). By proceeding as in [8, Proposition 2.10] we obtain

$$\lim_{\epsilon\to 0^+} \int_0^T (\dot u(t), \dot\varphi_\epsilon(t))_H \,\mathrm{d}t = \int_0^T (\dot u(t), \dot\varphi(t))_H \,\mathrm{d}t,$$
$$\lim_{\epsilon\to 0^+} \int_0^T (\mathbb{C}Eu(t), E\varphi_\epsilon(t))_H \,\mathrm{d}t = \int_0^T (\mathbb{C}Eu(t), E\varphi(t))_H \,\mathrm{d}t,$$
$$\lim_{\epsilon\to 0^+} \int_0^T (f(t), \varphi_\epsilon(t))_H \,\mathrm{d}t = \int_0^T (f(t), \varphi(t))_H \,\mathrm{d}t.$$

It remains to consider the terms involving $\mathbb{B}$ and g. We have

$$\int_0^T (\mathbb{B}\Psi(t)E\dot u(t), \Psi(t)E\varphi_\epsilon(t))_H \,\mathrm{d}t = \int_0^\epsilon (\mathbb{B}\Psi(t)E\dot u(t), \frac{t}{\epsilon}\Psi(t)E\varphi(t))_H \,\mathrm{d}t + \int_\epsilon^T (\mathbb{B}\Psi(t)E\dot u(t), \Psi(t)E\varphi(t))_H \,\mathrm{d}t,$$
$$\int_0^T (g(t), \varphi_\epsilon(t))_{H_N} \,\mathrm{d}t = \int_0^\epsilon (g(t), \frac{t}{\epsilon}\varphi(t))_{H_N} \,\mathrm{d}t + \int_\epsilon^T (g(t), \varphi(t))_{H_N} \,\mathrm{d}t,$$

hence by the dominated convergence theorem we get

$$\int_\epsilon^T (\mathbb{B}\Psi(t)E\dot u(t), \Psi(t)E\varphi(t))_H \,\mathrm{d}t \xrightarrow[\epsilon\to 0^+]{} \int_0^T (\mathbb{B}\Psi(t)E\dot u(t), \Psi(t)E\varphi(t))_H \,\mathrm{d}t,$$
$$\left|\int_0^\epsilon (\mathbb{B}\Psi(t)E\dot u(t), \frac{t}{\epsilon}\Psi(t)E\varphi(t))_H \,\mathrm{d}t\right| \le \|\mathbb{B}\|_\infty \|\Psi\|_\infty \int_0^\epsilon \|\Psi(t)E\dot u(t)\|_H \|E\varphi(t)\|_H \,\mathrm{d}t \xrightarrow[\epsilon\to 0^+]{} 0,$$
$$\int_\epsilon^T (g(t), \varphi(t))_{H_N} \,\mathrm{d}t \xrightarrow[\epsilon\to 0^+]{} \int_0^T (g(t), \varphi(t))_{H_N} \,\mathrm{d}t,$$
$$\left|\int_0^\epsilon (g(t), \frac{t}{\epsilon}\varphi(t))_{H_N} \,\mathrm{d}t\right| \le \int_0^\epsilon \|g(t)\|_{H_N} \|\varphi(t)\|_{H_N} \,\mathrm{d}t \xrightarrow[\epsilon\to 0^+]{} 0.$$

By combining together all the previous convergences we get the thesis. □

We now state the uniqueness result in the case of a fixed domain, that is $\Gamma_T = \Gamma_0$. We follow the same ideas of [12], and we need to assume

$$\Psi \in \mathrm{Lip}([0,T] \times \overline{\Omega}), \quad \nabla\dot\Psi \in L^\infty((0,T) \times \Omega; \mathbb{R}^d), \tag{5.1}$$

while on Γ_0 we do not require any further hypotheses.

Lemma 5.2 (Uniqueness in a fixed domain)**.** *Assume* (5.1) *and* $\Gamma_T = \Gamma_0$. *Then the viscoelastic dynamic system* (2.8) *with boundary and initial conditions* (2.9)–(2.12) (*the latter in the sense of* (2.16)) *has a unique weak solution.*

Proof. Let $u_1, u_2 \in \mathcal{W}$ be two weak solutions to (2.8)–(2.11) with initial conditions (2.12). The function $u := u_1 - u_2$ satisfies

$$\frac{1}{h}\int_0^h (\|u(t)\|^2_{V_t} + \|\dot u(t)\|^2_H) \,\mathrm{d}t \xrightarrow[h\to 0^+]{} 0, \tag{5.2}$$

hence by Lemma 5.1 it solves

$$-\int_0^T (\dot u(t), \dot\varphi(t))_H \,\mathrm{d}t + \int_0^T (\mathbb{C}Eu(t), E\varphi(t))_H \,\mathrm{d}t + \int_0^T (\mathbb{B}\Psi(t)E\dot u(t), \Psi(t)E\varphi(t))_H \,\mathrm{d}t = 0 \tag{5.3}$$

for every $\varphi \in \mathcal{V}^D$ such that $\varphi(T) = 0$. We fix $s \in (0,T]$ and consider the function

$$\varphi_s(t) := \begin{cases} -\int_t^s u(\tau)\mathrm{d}\tau & t \in [0,s], \\ 0 & t \in [s,T]. \end{cases}$$

Since $\varphi_s \in \mathcal{V}^D$ and $\varphi_s(T) = 0$, we can use it as test function in (5.3) to obtain

$$-\int_0^s (\dot u(t), u(t))_H \,\mathrm{d}t + \int_0^s (\mathbb{C}E\dot\varphi_s(t), E\varphi_s(t))_H \,\mathrm{d}t + \int_0^s (\mathbb{B}\Psi(t)E\dot u(t), \Psi(t)E\varphi_s(t))_H \,\mathrm{d}t = 0.$$

In particular we deduce

$$-\frac{1}{2}\int_0^s \frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_H^2\,\mathrm{d}t+\frac{1}{2}\int_0^s \frac{\mathrm{d}}{\mathrm{d}t}(\mathbb{C}E\varphi_s(t),E\varphi_s(t))_H\,\mathrm{d}t+\int_0^s(\mathbb{B}\Psi(t)E\dot{u}(t),\Psi(t)E\varphi_s(t))_H\,\mathrm{d}t=0,$$

which implies

$$\frac{1}{2}\|u(s)\|_H^2+\frac{1}{2}(\mathbb{C}E\varphi_s(0),E\varphi_s(0))_H=\int_0^s(\mathbb{B}\Psi(t)E\dot{u}(t),\Psi(t)E\varphi_s(t))_H\,\mathrm{d}t, \tag{5.4}$$

since $u(0)=0=\varphi_s(s)$. From the distributional point of view the following equality holds

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Psi Eu)=\dot{\Psi}Eu+\Psi E\dot{u}\in L^2(0,T;H), \tag{5.5}$$

indeed, for all $v\in C_c^\infty(0,T;H)$ we have

$$\begin{aligned}
&\int_0^T(\frac{\mathrm{d}}{\mathrm{d}t}(\Psi(t)Eu(t)),v(t))_H\mathrm{d}t=-\int_0^T(\Psi(t)Eu(t),\dot{v}(t))_H\,\mathrm{d}t\\
&=-\int_0^T(E(\Psi(t)u(t))-\nabla\Psi(t)\odot u(t),\dot{v}(t))_H\,\mathrm{d}t\\
&=\int_0^T(E(\dot{\Psi}(t)u(t))+E(\Psi(t)\dot{u}(t)),v(t))_H\,\mathrm{d}t-\int_0^T(\nabla\dot{\Psi}(t)\odot u(t)+\nabla\Psi(t)\odot\dot{u}(t),v(t))_H\,\mathrm{d}t\\
&=\int_0^T(\dot{\Psi}(t)Eu(t),v(t))_H\,\mathrm{d}t+\int_0^T(\Psi(t)E\dot{u}(t),v(t))_H\,\mathrm{d}t.
\end{aligned}$$

In particular $\Psi Eu\in H^1(0,T;H)\subset C^0([0,T],H)$, so that by (5.2)

$$\|\Psi(0)Eu(0)\|_H^2=\lim_{h\to 0}\frac{1}{h}\int_0^h\|\Psi(t)Eu(t)\|_H^2\mathrm{d}t\le C\lim_{h\to 0}\frac{1}{h}\int_0^h\|u(t)\|_{V_t}^2\mathrm{d}t=0$$

which yields $\Psi(0)Eu(0)=0$. Thanks to (5.5) and to property $\Psi u\in H^1(0,T;H)$, we deduce

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mathbb{B}\Psi Eu,\Psi E\varphi_s)_H&=(\mathbb{B}\dot{\Psi}Eu,\Psi E\varphi_s)_H+(\mathbb{B}\Psi E\dot{u},\Psi E\varphi_s)_H+(\mathbb{B}\Psi Eu,\dot{\Psi}E\varphi_s)_H+(\mathbb{B}\Psi Eu,\Psi E\dot{\varphi}_s)_H\\
&=2(\mathbb{B}\Psi Eu,\dot{\Psi}E\varphi_s)_H+(\mathbb{B}\Psi E\dot{u},\Psi E\varphi_s)_H+(\mathbb{B}\Psi Eu,\Psi E\dot{\varphi}_s)_H,
\end{aligned}$$

and by integrating on $[0,s]$ we get

$$\begin{aligned}
&\int_0^s(\mathbb{B}\Psi(t)E\dot{u}(t),\Psi(t)E\varphi_s(t))_H\,\mathrm{d}t\\
&=\int_0^s\Big[\frac{\mathrm{d}}{\mathrm{d}t}(\mathbb{B}\Psi(t)Eu(t),\Psi(t)E\varphi_s(t))_H-2(\mathbb{B}\Psi(t)Eu(t),\dot{\Psi}(t)E\varphi_s(t))_H-(\mathbb{B}\Psi(t)E\dot{\varphi}_s(t),\Psi(t)E\dot{\varphi}_s(t))_H\Big]\,\mathrm{d}t\\
&\le(\mathbb{B}\Psi(s)Eu(s),\Psi(s)E\varphi_s(s))_H-(\mathbb{B}\Psi(0)Eu(0),\Psi(0)E\varphi_s(0))_H\\
&\quad+\int_0^s\Big[2(\mathbb{B}\Psi(t)Eu(t),\Psi(t)Eu(t))_H^{\frac{1}{2}}(\mathbb{B}\dot{\Psi}(t)E\varphi_s(t),\dot{\Psi}(t)E\varphi_s(t))_H^{\frac{1}{2}}-(\mathbb{B}\Psi(t)E\dot{\varphi}_s(t),\Psi(t)E\dot{\varphi}_s(t))_H\Big]\,\mathrm{d}t\\
&\le\int_0^s\Big[(\mathbb{B}\Psi(t)Eu(t),\Psi(t)Eu(t))_H+(\mathbb{B}\dot{\Psi}(t)E\varphi_s(t),\dot{\Psi}(t)E\varphi_s(t))_H-(\mathbb{B}\Psi(t)E\dot{\varphi}_s(t),\Psi(t)E\dot{\varphi}_s(t))_H\Big]\,\mathrm{d}t\\
&\le\|\mathbb{B}\|_\infty\|\dot{\Psi}\|_\infty^2\int_0^s\|E\varphi_s(t)\|_H^2\mathrm{d}t,
\end{aligned}$$

since $E\varphi_s(s)=0=\Psi(0)Eu(0)$ and $E\dot{\varphi}_s=Eu$ in $(0,s)$. By combining the previous inequality with (5.4) and using the coercivity of the tensor $\mathbb{C}$, we derive

$$\frac{\lambda_1}{2}\|E\varphi_s(0)\|_H^2+\frac{1}{2}\|u(s)\|_H^2\le\frac{1}{2}(\mathbb{C}E\varphi_s(0),E\varphi_s(0))_H+\frac{1}{2}\|u(s)\|_H^2\le\|\mathbb{B}\|_\infty\|\dot{\Psi}\|_\infty^2\int_0^s\|E\varphi_s(t)\|_H^2\mathrm{d}t.$$

Let us set $\xi(t):=\int_0^t u(\tau)\mathrm{d}\tau$, then

$$\|E\varphi_s(0)\|_H^2=\|E\xi(s)\|_H^2,\quad \|E\varphi_s(t)\|_H^2=\|E\xi(t)-E\xi(s)\|_H^2\le 2\|E\xi(t)\|_H^2+2\|E\xi(s)\|_H^2,$$

from which we deduce

$$\frac{\lambda_1}{2}\|E\xi(s)\|_H^2+\frac{1}{2}\|u(s)\|_H^2\le C\int_0^s \|E\xi(t)\|_H^2\mathrm{d}t+Cs\|E\xi(s)\|_H^2, \tag{5.6}$$

where $C:=2\|\mathbb{B}\|_\infty\|\dot\Psi\|_\infty^2$. Therefore, if we set $s_0:=\frac{\lambda_1}{4C}$, for all $s\le s_0$ we obtain

$$\frac{\lambda_1}{4}\|E\xi(s)\|_H^2\le\left(\frac{\lambda_1}{2}-Cs\right)\|E\xi(s)\|_H^2\le C\int_0^s \|E\xi(t)\|_H^2\mathrm{d}t.$$

By Gronwall's lemma the last inequality implies $E\xi(s)=0$ for all $s\le s_0$. Hence, thanks to (5.6) we get $\|u(s)\|_H^2\le 0$ for all $s\le s_0$, which yields $u(s)=0$ for all $s\le s_0$. Since s_0 depends only on $\mathbb{C}$, $\mathbb{B}$, and Ψ, we can repeat this argument starting from s_0, and with a finite number of steps we obtain $u\equiv 0$ on $[0,T]$. ☐

In order to prove our uniqueness result in the case of a moving crack we need two auxiliary results, which are [4, Theorem 6.1] and [8, Theorem 4.3]. For the sake of the readers, we rewrite below the statements without proof.

The first one ([4, Theorem 6.1]) is a generalization of the well–known result of finite speed of propagation for the wave equation. Given an open bounded set $U\subset\mathbb{R}^d$, we define by $\partial_L U$ the Lipschitz part of the boundary ∂U, which is the collection of points $x\in\partial U$ for which there exist an orthogonal coordinate system $y_1,\dots,y_d$, a neighborhood V of x of the form $A\times I$, with A open in $\mathbb{R}^{d-1}$ and I open interval in $\mathbb{R}$, and a Lipschitz function $g\colon A\to I$, such that $V\cap U:=\{(y_1,\dots,y_d)\in V: y_d<g(y_1,\dots,y_{d-1})\}$. Moreover, given a Borel set $S\subseteq\partial_L U$, we define

$$H_S(U;\mathbb{R}^d):=\{u\in H^1(U;\mathbb{R}^d): u=0 \text{ on } S\}.$$

Notice that $H_S(U;\mathbb{R}^d)$ is a Hilbert space, and we denote its dual by $H_S^{-1}(U;\mathbb{R}^d)$.

Theorem 5.3 (Finite speed of propagation)**.** *Let $U\subset\mathbb{R}^d$ be an open bounded set and let $\partial_L U$ be the Lipschitz part of ∂U. Let S_0 and S_1 be two Borel sets with $S_0\subseteq S_1\subseteq\partial_L U$, and let $\mathbb{C}\colon U\to\mathscr{L}(\mathbb{R}^{d\times d}_{sym};\mathbb{R}^{d\times d}_{sym})$ be a fourth-order tensor satisfying (2.4)–(2.6). Let*

$$u\in L^2(0,T;H^1_{S_0}(U;\mathbb{R}^d))\cap H^1(0,T;L^2(U;\mathbb{R}^d))\cap H^2(0,T;H^{-1}_{S_1}(U;\mathbb{R}^d))$$

be a solution to

$$\langle\ddot u(t),\psi\rangle_{H^{-1}_{S_1}(U;\mathbb{R}^d)}+(\mathbb{C}Eu(t),E\psi)_{L^2(U;\mathbb{R}^{d\times d}_{sym})}=0\quad\text{for every } \psi\in H^1_{S_1}(U;\mathbb{R}^d),$$

with initial conditions $u(0)=0$ and $\dot u(0)=0$ in the sense of $L^2(U;\mathbb{R}^d)$ and $H^{-1}_{S_1}(U;\mathbb{R}^d)$, respectively. Then

$$u(t)=0\quad\text{a.e. in } U_t:=\{x\in U:\operatorname{dist}(x,S_1\setminus S_0)>t\sqrt{\|\mathbb{C}\|_\infty}\}$$

for every $t\in[0,T]$.

Proof. See [4, Theorem 6.1]. ☐

The second one ([8, Theorem 4.3]) is a uniqueness result for the weak solutions of the wave equation in a moving domain. Let $\hat H$ be a separable Hilbert space, and let $\{\hat V_t\}_{t\in[0,T]}$ be a family of separable Hilbert spaces with the following properties:

- (i) for every $t\in[0,T]$ the space $\hat V_t$ is contained and dense in $\hat H$ with continuous embedding;
- (ii) for every $s,t\in[0,T]$, with $s<t$, $\hat V_s\subset\hat V_t$ and the Hilbert space structure on $\hat V_s$ is the one induced by $\hat V_t$.

Let $a\colon\hat V_T\times\hat V_T\to\mathbb{R}$ be a bilinear symmetric form satisfying the following conditions:

- (iii) there exists M_0 such that
$$|a(u,v)|\le M_0\|u\|_{\hat V_T}\|v\|_{\hat V_T}\quad\text{for every } u,v\in\hat V_T;$$
- (iv) there exist $\lambda_0>0$ and $\nu_0\in\mathbb{R}$ such that
$$a(u,u)\ge\lambda_0\|u\|^2_{\hat V_T}-\nu_0\|u\|^2_{\hat H}\quad\text{for every } u\in\hat V_T.$$

Assume that

- (U1) for every $t\in[0,T]$ there exists a continuous and linear bijective operator $Q_t\colon\hat V_t\to\hat V_0$, with continuous inverse $R_t\colon\hat V_0\to\hat V_t$;

(U2) Q_0 and R_0 are the identity maps on $\hat{V}_0$;

(U3) there exists a constant M_1 independent of t such that

$$\|Q_t u\|_{\hat{H}} \le M_1 \|u\|_{\hat{H}} \quad \text{for every } u \in \hat{V}_t, \qquad \|R_t u\|_{\hat{H}} \le M_1 \|u\|_{\hat{H}} \quad \text{for every } u \in \hat{V}_0,$$
$$\|Q_t u\|_{\hat{V}_0} \le M_1 \|u\|_{\hat{V}_t} \quad \text{for every } u \in \hat{V}_t, \qquad \|R_t u\|_{\hat{V}_t} \le M_1 \|u\|_{\hat{V}_0} \quad \text{for every } u \in \hat{V}_0.$$

Since $\hat{V}_t$ is dense in $\hat{H}$, (U3) implies that R_t and Q_t can be extended to continuous linear operators from $\hat{H}$ into itself, still denoted by Q_t and R_t. We also require

(U4) for every $v \in \hat{V}_0$ the function $t \mapsto R_t v$ from $[0, T]$ into $\hat{H}$ has a derivative, denoted by $\dot{R}_t v$;

(U5) there exists $\eta \in (0, 1)$ such that

$$\|\dot{R}_t Q_t v\|^2_{\hat{H}} \le \lambda_0 (1 - \eta) \|v\|^2_{\hat{V}_t} \quad \text{for every } v \in \hat{V}_t;$$

(U6) there exists a constant M_2 such that

$$\|Q_t v - Q_s v\|_{\hat{H}} \le M_2 \|v\|_{\hat{V}_s} (t - s) \quad \text{for every } 0 \le s < t \le T \text{ and every } v \in \hat{V}_s;$$

(U7) for very $t \in [0, T)$ and for every $v \in \hat{V}_t$ there exists an element of $\hat{H}$, denoted by $\dot{Q}_t v$, such that

$$\lim_{h \to 0^+} \frac{Q_{t+h} v - Q_t v}{h} = \dot{Q}_t v \text{ in } \hat{H}.$$

For every $t \in [0, T]$, define

$$\alpha(t) \colon \hat{V}_0 \times \hat{V}_0 \to \mathbb{R} \quad \text{as } \alpha(t)(u, v) := a(R_t u, R_t v) \text{ for } u, v \in \hat{V}_0,$$
$$\beta(t) \colon \hat{V}_0 \times \hat{V}_0 \to \mathbb{R} \quad \text{as } \beta(t)(u, v) := (\dot{R}_t u, \dot{R}_t v) \text{ for } u, v \in \hat{V}_0,$$
$$\gamma(t) \colon \hat{V}_0 \times \hat{H} \to \mathbb{R} \quad \text{as } \gamma(t)(u, v) := (\dot{R}_t u, R_t v) \text{ for } u \in \hat{V}_0 \text{ and } v \in \hat{H},$$
$$\delta(t) \colon \hat{H} \times \hat{H} \to \mathbb{R} \quad \text{as } \delta(t)(u, v) := (R_t u, R_t v) - (u, v) \text{ for } u, v \in \hat{H}.$$

We assume that there exists a constant M_3 such that

(U8) the maps $t \mapsto \alpha(t)(u, v)$, $t \mapsto \beta(t)(u, v)$, $t \mapsto \gamma(t)(u, v)$, and $t \mapsto \delta(t)(u, v)$ are Lipschitz continuous and for a.e. $t \in (0, T)$ their derivatives satisfy

$$|\dot{\alpha}(t)(u, v)| \le M_3 \|u\|_{\hat{V}_0} \|v\|_{\hat{V}_0} \quad \text{for } u, v \in \hat{V}_0,$$
$$|\dot{\beta}(t)(u, v)| \le M_3 \|u\|_{\hat{V}_0} \|v\|_{\hat{V}_0} \quad \text{for } u, v \in \hat{V}_0,$$
$$|\dot{\gamma}(t)(u, v)| \le M_3 \|u\|_{\hat{V}_0} \|v\|_{\hat{H}} \quad \text{for } u \in \hat{V}_0 \text{ and } v \in \hat{H},$$
$$|\dot{\delta}(t)(u, v)| \le M_3 \|u\|_{\hat{H}} \|v\|_{\hat{H}} \quad \text{for } u, v \in \hat{H}.$$

Theorem 5.4 (Uniqueness for the wave equation)**.** *Assume that $\hat{H}$, $\{\hat{V}_t\}_{t \in [0,T]}$, and a satisfy (i)–(iv) and that (U1)–(U8) hold. Given $u^0 \in \hat{V}_0$, $u^1 \in \hat{H}$, and $f \in L^2(0, T; \hat{H})$, there exists a unique solution*

$$u \in \hat{\mathcal{V}} := \{\varphi \in L^2(0, T; \hat{V}_T) : \dot{u} \in L^2(0, T; \hat{H}),\ u(t) \in \hat{V}_t \text{ for a.e. } t \in (0, T)\}$$

to the wave equation

$$-\int_0^T (\dot{u}(t), \dot{\varphi}(t))_{\hat{H}} \, \mathrm{d}t + \int_0^T a(u(t), \varphi(t)) \, \mathrm{d}t = \int_0^T (f(t), \varphi(t))_{\hat{H}} \, \mathrm{d}t \quad \text{for every } \varphi \in \hat{\mathcal{V}},$$

satisfying the initial conditions $u(0) = u^0$ and $\dot{u}(0) = u^1$ in the sense that

$$\lim_{h \to 0^+} \frac{1}{h} \int_0^h \left(\|u(t) - u^0\|^2_{\hat{V}_t} + \|\dot{u}(t) - u^1\|^2_{\hat{H}} \right) \mathrm{d}t = 0.$$

Proof. See [8, Theorem 4.3]. □

We now are in position to prove the uniqueness theorem in the case of a moving domain. We consider the dimensional case $d = 2$, and we require the following assumptions:

(H1) there exists a $C^{2,1}$ simple curve $\Gamma \subset \overline{\Omega} \subset \mathbb{R}^2$, parametrized by arc–length $\gamma \colon [0, \ell] \to \overline{\Omega}$, such that $\Gamma \cap \partial\Omega = \gamma(0) \cup \gamma(\ell)$ and $\Omega \setminus \Gamma$ is the union of two disjoint open sets with Lipschitz boundary;

(H2) there exists a non decreasing function $s \colon [0, T] \to (0, \ell)$ of class $C^{1,1}$ such that $\Gamma_t = \gamma([0, s(t)])$;

(H3) $|\dot s(t)|^2 < \frac{\lambda_1}{C_K}$, where λ_1 is the ellipticity constant of $\mathbb{C}$ and C_K is the constant that appears in Korn's inequality in (2.2).

Notice that hypotheses (H1) and (H2) imply (E1)–(E3). We also assume that Ψ satisfies (5.1) and there exists a constant $\epsilon > 0$ such that for every $t \in [0,T]$

$$\Psi(t,x) = 0 \quad \text{for every } x \in \{y \in \overline{\Omega} : |y - \gamma(s(t))| < \epsilon\}. \tag{5.7}$$

Theorem 5.5. *Assume $d = 2$ and (H1)–(H3), (5.1), and (5.7). Then the system (2.8) with boundary conditions (2.9)–(2.11) has a unique weak solution $u \in \mathcal{W}$ which satisfies $u(0) = u^0$ and $\dot u(0) = u^1$ in the sense of (2.16).*

Proof. As before let $u_1, u_2 \in \mathcal{W}$ be two weak solutions to (2.8)–(2.11) with initial conditions (2.12). Then $u := u_1 - u_2$ satisfies (5.2) and (5.3) for every $\varphi \in \mathcal{V}^D$ such that $\varphi(T) = 0$. Let us define

$$t_0 := \sup\{t \in [0,T] : u(s) = 0 \text{ for every } s \in [0,t]\},$$

and assume by contradiction that $t_0 < T$. Consider first the case in which $t_0 > 0$. By (H1), (H2), (5.1), and (5.7) we can find two open sets A_1 and A_2, with $A_1 \subset\subset A_2 \subset\subset \Omega$, and a number $\delta > 0$ such that for every $t \in [t_0 - \delta, t_0 + \delta]$ we have $\gamma(s(t)) \in A_1$, $\Psi(t,x) = 0$ for every $x \in \overline{A}_2$, and $(A_2 \setminus A_1) \setminus \Gamma$ is the union of two disjoint open sets with Lipschitz boundary. Let us define

$$\hat V^1 := \{u \in H^1((A_2 \setminus A_1) \setminus \Gamma_{t_0-\delta}; \mathbb{R}^2) : u = 0 \text{ on } \partial A_1 \cup \partial A_2\}, \quad \hat H^1 := L^2(A_2 \setminus A_1; \mathbb{R}^2).$$

Since every function in $\hat V^1$ can be extended to a function in $V^D_{t_0-\delta}$, by classical results for linear hyperbolic equations (se, e.g., [9]), we deduce $\ddot u \in L^2(t_0 - \delta, t_0 + \delta; (\hat V^1)')$ and that u satisfies for a.e. $t \in (t_0 - \delta, t_0 + \delta)$

$$\langle \ddot u(t), \phi\rangle_{(\hat V^1)'} + (\mathbb{C}Eu(t), E\phi)_{\hat H^1} = 0 \quad \text{for every } \phi \in \hat V^1.$$

Moreover, we have $u(t_0) = 0$ as element of $\hat H^1$ and $\dot u(t_0) = 0$ as element of $(\hat V^1)'$, since $u(t) \equiv 0$ in $[t_0 - \delta, t_0)$, $u \in C^0([t_0 - \delta, t_0]; \hat H^1)$, and $\dot u \in C^0([t_0 - \delta, t_0]; (\hat V^1)')$. We are now in position to apply the result of finite speed of propagation of Theorem 5.3. This theorem ensures the existence of a third open set A_3, with $A_1 \subset\subset A_3 \subset\subset A_2$, such that, up to choose a smaller δ, we have $u(t) = 0$ on ∂A_3 for every $t \in [t_0, t_0 + \delta]$, and both $(\Omega \setminus A_3) \setminus \Gamma$ and $A_3 \setminus \Gamma$ are union of two disjoint open sets with Lipschitz boundary.

In $\Omega \setminus A_3$ the function u solves

$$\begin{aligned} -\int_{t_0-\delta}^{t_0+\delta}\int_{\Omega\setminus A_3} \dot u(t,x)\cdot\dot\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t + \int_{t_0-\delta}^{t_0+\delta}\int_{\Omega\setminus A_3} \mathbb{C}(x)Eu(t,x)\cdot E\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t \\ + \int_{t_0-\delta}^{t_0+\delta}\int_{\Omega\setminus A_3} \mathbb{B}(x)\Psi(t,x)E\dot u(t,x)\cdot\Psi(t,x)E\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t = 0 \end{aligned}$$

for every $\varphi \in L^2(t_0 - \delta, t_0 + \delta; \hat V^2) \cap H^1(t_0 - \delta, t_0 + \delta; \hat H^2)$ such that $\varphi(t_0 - \delta) = \varphi(t_0 + \delta) = 0$, where

$$\hat V^2 := \{u \in H^1((\Omega \setminus A_3) \setminus \Gamma_{t_0-\delta}; \mathbb{R}^2) : u = 0 \text{ on } \partial_D\Omega \cup \partial A_3\}, \quad \hat H^2 := L^2(\Omega \setminus A_3; \mathbb{R}^2).$$

Since $u(t) = 0$ on $\partial_D\Omega \cup \partial A_3$ for every $t \in [t_0 - \delta, t_0 + \delta]$ and $u(t_0 - \delta) = \dot u(t_0 - \delta) = 0$ in the sense of (2.16) (recall that $u \equiv 0$ in $[t_0 - \delta, t_0)$), we can apply Lemma 5.2 to deduce $u(t) = 0$ in $\Omega \setminus A_3$ for every $t \in [t_0 - \delta, t_0 + \delta]$.

On the other hand in A_3, by setting

$$\hat V^3_t := \{u \in H^1(A_3 \setminus \Gamma_t; \mathbb{R}^2) : u = 0 \text{ on } \partial A_3\}, \quad \hat H^3 := L^2(A_3; \mathbb{R}^2),$$

we get that the function u solves

$$-\int_{t_0-\delta}^{t_0+\delta}\int_{A_3} \dot u(t,x)\cdot\dot\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t + \int_{t_0-\delta}^{t_0+\delta}\int_{A_3} \mathbb{C}(x)Eu(t,x)\cdot E\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t = 0$$

for every $\varphi \in L^2(t_0 - \delta, t_0 + \delta; \hat V^3_{t_0+\delta}) \cap H^1(t_0 - \delta, t_0 + \delta; \hat H^3)$ such that $\varphi(t) \in \hat V^3_t$ for a.e. $t \in (t_0 - \delta, t_0 + \delta)$ and $\varphi(t_0 - \delta) = \varphi(t_0 + \delta) = 0$. Here we would like to apply the uniqueness result of Theorem 5.4 for the spaces $\{\hat V^3_t\}_{t\in[t_0-\delta,t_0+\delta]}$ and $\hat H^3$, endowed with the usual norms, and for the bilinear form

$$a(u,v) := \int_{A_3} \mathbb{C}(x)Eu(x)\cdot Ev(x)\mathrm{d}x \quad \text{for every } u, v \in \hat V^3_{t_0+\delta}.$$

As show in [7, Example 2.14] we can construct two maps $\Phi, \Lambda \in C^{1,1}([t_0-\delta, t_0+\delta] \times \overline{A}_3; \mathbb{R}^2)$ such that for every $t \in [0,T]$ the function $\Phi(t,\cdot)\colon \overline{A}_3 \to \overline{A}_3$ is a diffeomorfism of A_3 in itself with inverse $\Lambda(t,\cdot)\colon \overline{A}_3 \to \overline{A}_3$. Moreover, $\Phi(0,y) = y$ for every $y \in \overline{A}_3$, $\Phi(t, \Gamma \cap \overline{A}_3) = \Gamma \cap \overline{A}_3$ and $\Phi(t, \Gamma_{t_0-\delta} \cap \overline{A}_3) = \Gamma_t \cap \overline{A}_3$ for every $t \in [t_0-\delta, t_0+\delta]$. For every $t \in [t_0-\delta, t_0+\delta]$, the maps $(Q_t u)(y) := u(\Phi(t,y))$, $u \in \hat{V}^3_t$ and $y \in A_3$, and $(R_t v)(x) := v(\Lambda(t,x))$, $v \in \hat{V}^3_{t_0-\delta}$ and $x \in A_3$, provide a family of linear and continuous operators which satisfy the assumptions (U1)–(U8) of Theorem 5.4 (see [8, Example 4.2]). The only condition to check is (U5). The bilinear form a satisfies the following ellipticity condition

$$a(u,u) \geq \lambda_1 \|Eu\|^2_{L^2(A_3;\mathbb{R}^{2\times 2}_{sym})} \geq \frac{\lambda_1}{\hat{C}_k} \|u\|^2_{\hat{V}^3_{t_0+\delta}} - \lambda_1 \|u\|^2_{\hat{H}^3} \quad \text{for every } u \in \hat{V}^3_{t_0+\delta}, \tag{5.8}$$

where $\hat{C}_K$ is the constant in Korn's inequality in $\hat{V}^3_{t_0+\delta}$, namely

$$\|\nabla u\|^2_{L^2(A_3;\mathbb{R}^{2\times 2})} \leq \hat{C}_K (\|u\|^2_{L^2(A_3;\mathbb{R}^2)} + \|Eu\|^2_{L^2(A_3;\mathbb{R}^{2\times 2}_{sym})}) \quad \text{for every } u \in \hat{V}^3_{t_0+\delta}.$$

Notice that for $t \in [t_0-\delta, t_0+\delta]$

$$(\dot{R}_t v)(x) = \nabla v(\Lambda(t,x)) \dot{\Lambda}(t,x) \quad \text{for a.e. } x \in A_3,$$

from which we obtain

$$\|\dot{R}_t Q_t u\|^2_{\hat{H}^3} \leq \int_{A_3} |\nabla u(x)|^2 |\dot{\Phi}(t, \Lambda(t,x))|^2 \,\mathrm{d}x.$$

Hence, have to show the property

$$|\dot{\Phi}(t,y)|^2 < \frac{\lambda_1}{\hat{C}_K} \quad \text{for every } t \in [t_0-\delta, t_0+\delta] \text{ and } y \in \overline{A}_3.$$

This is ensured by (H3). Indeed, as explained in [7, Example 3.1], we can construct the maps Φ and Λ in such a way that

$$|\dot{\Phi}(t,y)|^2 < \frac{\lambda_1}{C_K},$$

since $|\dot{s}(t)|^2 < \frac{\lambda_1}{C_K}$. Moreover, every function in $\hat{V}^3_{t_0+\delta}$ can be extended to a function in $H^1(\Omega \setminus \Gamma; \mathbb{R}^d)$. Hence, for Korn's inequality in $\hat{V}^3_{t_0+\delta}$, we can use the same constant C_K of $H^1(\Omega \setminus \Gamma; \mathbb{R}^d)$. This allows us to apply Theorem 5.4, which implies $u(t) = 0$ in A_3 for every $t \in [t_0, t_0+\delta]$. In the case $t_0 = 0$, it is enough to argue as before in $[0,\delta]$, by exploiting (5.2). Therefore $u(t) = 0$ in Ω for every $t \in [t_0, t_0+\delta]$, which contradicts the maximality of t_0. Hence $t_0 = T$, that yields $u(t) = 0$ in Ω for every $t \in [0,T]$. ☐

Remark 5.6. Also Theorem 5.5 is true in the antiplane case, with essentially the same proof. Notice that, when the displacement is scalar, we do not need to use Korn's inequality in (5.8) to get the coercivity in $\hat{V}^3_{t_0+\delta}$ of the bilinear form a defined before. Therefore, in this case in (H3) it is enough to assume $|\dot{s}(t)|^2 < \lambda_1$.

6. A Moving Crack Satisfying Griffith's Dynamic Energy–Dissipation Balance

We conclude this paper with an example of a moving crack $\{\Gamma_t\}_{t\in[0,T]}$ and weak solution to (2.8)–(2.12) which satisfy the energy–dissipation balance of Griffith's dynamic criterion, as happens in [4] for the purely elastic case. In dimension $d = 2$ we consider an antiplane evolution, which means that the displacement u is scalar, and we take $\Omega := \{x \in \mathbb{R}^2 : |x| < R\}$, with $R > 0$. We fix a constant $0 < c < 1$ such that $cT < R$, and we set

$$\Gamma_t := \{(\sigma, 0) \in \overline{\Omega} : \sigma \leq ct\}.$$

Let us define the following function

$$S(x_1,x_2) := Im(\sqrt{x_1 + ix_2}) = \frac{1}{\sqrt{2}} \frac{x_2}{\sqrt{|x| + x_1}} \quad x \in \mathbb{R}^2 \setminus \{(\sigma,0) : \sigma \leq 0\},$$

where Im denotes the imaginary part of a complex number. Notice that $S \in H^1(\Omega \setminus \Gamma_0) \setminus H^2(\Omega \setminus \Gamma_0)$, and it is a weak solution to

$$\begin{cases} \Delta S = 0 & \text{in } \Omega \setminus \Gamma_0, \\ \nabla S \cdot \nu = \partial_2 S = 0 & \text{on } \Gamma_0. \end{cases}$$

Let us consider the function

$$u(t,x) := \frac{2}{\sqrt{\pi}} S\left(\frac{x_1 - ct}{\sqrt{1-c^2}}, x_2\right) \quad t \in [0,T],\ x \in \Omega \setminus \Gamma_t$$

and let $w(t)$ be its restriction to $\partial\Omega$. Since $u(t)$ has a singularity only at the crack tip $(ct, 0)$, the function $w(t)$ can be seen as the trace on $\partial\Omega$ of a function belonging to $H^2(0,T;L^2(\Omega)) \cap H^1(0,T;H^1(\Omega\setminus\Gamma_0))$, still denoted by $w(t)$. It is easy to see that u solves the wave equation

$$\ddot{u}(t) - \Delta u(t) = 0 \quad \text{in } \Omega\setminus\Gamma_t,\ t \in (0,T),$$

with boundary conditions

$$\begin{aligned} &u(t) = w(t) && \text{on } \partial\Omega,\ t \in (0,T),\\ &\frac{\partial u}{\partial \nu}(t) = \nabla u(t)\cdot\nu = 0 && \text{on } \Gamma_t, \quad t \in (0,T), \end{aligned}$$

and initial data

$$\begin{aligned} u^0(x_1,x_2) &:= \frac{2}{\sqrt{\pi}} S\left(\frac{x_1}{\sqrt{1-c^2}}, x_2\right) \in H^1(\Omega\setminus\Gamma_0),\\ u^1(x_1,x_2) &:= -\frac{2}{\sqrt{\pi}}\frac{c}{\sqrt{1-c^2}}\partial_1 S\left(\frac{x_1}{\sqrt{1-c^2}}, x_2\right) \in L^2(\Omega). \end{aligned}$$

Let us consider a function Ψ which satisfies the regularity assumptions (5.1) and condition (5.7), namely

$$\Psi(t) = 0 \quad \text{on } B_\epsilon(t) := \{x \in \mathbb{R}^2 : |x - (ct,0)| < \epsilon\} \text{ for every } t \in [0,T],$$

with $0 < \epsilon < R - cT$. In this case u is a weak solution, in the sense of Definition 2.4, to the damped wave equation

$$\ddot{u}(t) - \Delta u(t) - \operatorname{div}(\Psi^2(t)\nabla\dot{u}(t)) = f(t) \quad \text{in } \in \Omega\setminus\Gamma_t,\ t\in(0,T),$$

with forcing term f given by

$$f := -\operatorname{div}(\Psi^2\nabla\dot{u}) = -\nabla\Psi\cdot 2\Psi\nabla\dot{u} - \Psi^2\Delta\dot{u} \in L^2(0,T;L^2(\Omega)),$$

and boundary and initial conditions

$$\begin{aligned} &u(t) = w(t) && \text{on } \partial\Omega,\ t\in(0,T),\\ &\frac{\partial u}{\partial\nu}(t) + \Psi^2(t)\frac{\partial\dot{u}}{\partial\nu}(t) = 0 && \text{on } \Gamma_t,\ \ t\in(0,T),\\ &u(0) = u^0, \quad \dot{u}(0) = u^1. \end{aligned}$$

Notice that for the homogeneous Neumann boundary conditions on Γ_t we used $\frac{\partial\dot{u}}{\partial\nu}(t) = \nabla\dot{u}(t)\cdot\nu = \partial_2\dot{u}(t) = 0$ on Γ_t. By the uniqueness result proved in the previous section, the function u coincides with that one found in Theorem 3.1. Thanks to the computations done in [4, Section 4], we know that u satisfies for every $t\in[0,T]$ the following energy–dissipation balance for the undamped equation, where ct coincides with the length of $\Gamma_t\setminus\Gamma_0$

$$\frac{1}{2}\|\dot{u}(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\Omega;\mathbb{R}^2)} + ct = \frac{1}{2}\|\dot{u}(0)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u(0)\|^2_{L^2(\Omega;\mathbb{R}^2)} + \int_0^t (\frac{\partial u}{\partial\nu}(s), \dot{w}(s))_{L^2(\partial\Omega)}\,\mathrm{d}s. \tag{6.1}$$

Moreover, we have

$$\begin{aligned} \int_0^t (\frac{\partial u}{\partial\nu}(s),\dot{w}(s))_{L^2(\partial\Omega)}\,\mathrm{d}s = &\int_0^t (\nabla u(s),\nabla\dot{w}(s))_{L^2(\Omega;\mathbb{R}^2)}\,\mathrm{d}s - \int_0^t(\dot{u}(s),\ddot{w}(s))_{L^2(\Omega)}\,\mathrm{d}s\\ &+ (\dot{u}(t),\dot{w}(t))_{L^2(\Omega)} - (\dot{u}(0),\dot{w}(0))_{L^2(\Omega)}. \end{aligned} \tag{6.2}$$

For every $t\in[0,T]$ we compute

$$\begin{aligned} (f(t),\dot{u}(t)-\dot{w}(t))_{L^2(\Omega)} &= -\int_{(\Omega\setminus B_\epsilon(t))\setminus\Gamma_t} \operatorname{div}[\Psi^2(t,x)\nabla\dot{u}(t,x)](\dot{u}(t,x)-\dot{w}(t,x))\,\mathrm{d}x\\ &= -\int_{(\Omega\setminus B_\epsilon(t))\setminus\Gamma_t}\operatorname{div}[\Psi^2(t,x)\nabla\dot{u}(t,x)(\dot{u}(t,x)-\dot{w}(t,x))]\,\mathrm{d}x \end{aligned}$$

$$+\int_{(\Omega\setminus B_\epsilon(t))\setminus\Gamma_t}\Psi^2(t,x)\nabla\dot u(t,x)\cdot(\nabla\dot u(t,x)-\nabla\dot w(t,x))\,\mathrm{d}x.$$

If we denote by $\dot u^\oplus(t)$ and $\dot w^\oplus(t)$ the traces of $\dot u(t)$ and $\dot w(t)$ on Γ_t from above and by $\dot u^\ominus(t)$ and $\dot w^\ominus(t)$ the trace from below, thanks to the divergence theorem we have

$$\begin{aligned}&\int_{(\Omega\setminus B_\epsilon(t))\setminus\Gamma_t}\mathrm{div}[\Psi^2(t,x)\nabla\dot u(t,x)(\dot u(t,x)-\dot w(t,x))]\,\mathrm{d}x\\&=\int_{\partial\Omega}\Psi^2(t,x)\frac{\partial\dot u}{\partial\nu}(t,x)(\dot u(t,x)-\dot w(t,x))\,\mathrm{d}x+\int_{\partial B_\epsilon(t)}\Psi^2(t,x)\frac{\partial\dot u}{\partial\nu}(t,x)(\dot u(t,x)-\dot w(t,x))\,\mathrm{d}x\\&\quad-\int_{(\Omega\setminus B_\epsilon(t))\cap\Gamma_t}\Psi^2(t,x)\partial_2\dot u^\oplus(t,x)(\dot u^\oplus(t,x)-\dot w^\oplus(t,x))\,\mathrm{d}\mathcal{H}^1(x)\\&\quad+\int_{(\Omega\setminus B_\epsilon(t))\cap\Gamma_t}\Psi^2(t,x)\partial_2\dot u^\ominus(t,x)(\dot u^\ominus(t,x)-\dot w^\ominus(t,x))\,\mathrm{d}\mathcal{H}^1(x)=0,\end{aligned}$$

since $u(t)=w(t)$ on $\partial\Omega$, $\Psi(t)=0$ on $\partial B_\epsilon(t)$, and $\partial_2\dot u(t)=0$ on Γ_t. Therefore for every $t\in[0,T]$ we get

$$(f(t),\dot u(t)-\dot w(t))_{L^2(\Omega)}=\|\Psi(t)\nabla\dot u(t)\|^2_{L^2(\Omega;\mathbb{R}^2)}-(\Psi(t)\nabla\dot u(t),\Psi(t)\nabla\dot w(t))_{L^2(\Omega;\mathbb{R}^2)}.\tag{6.3}$$

By combining (6.1)–(6.3) we deduce that u satisfies for every $t\in[0,T]$ the following Griffith's energy–dissipation balance for the viscoelastic dynamic equation

$$\begin{aligned}\frac12\|\dot u(t)\|^2_{L^2(\Omega)}+\frac12\|\nabla u(t)\|^2_{L^2(\Omega;\mathbb{R}^2)}+\int_0^t\|\Psi(s)\nabla\dot u(s)\|^2_{L^2(\Omega;\mathbb{R}^2)}\,\mathrm{d}s+ct\\=\frac12\|\dot u(0)\|^2_{L^2(\Omega)}+\frac12\|\nabla u(0)\|^2_{L^2(\Omega;\mathbb{R}^2)}+\mathcal{W}_{tot}(t),\end{aligned}\tag{6.4}$$

where in this case the total work takes the form

$$\begin{aligned}\mathcal{W}_{tot}(t):=\int_0^t\big[(f(s),\dot u(s)-\dot w(s))_{L^2(\Omega)}+(\nabla u(s),\nabla\dot w(s))_{L^2(\Omega;\mathbb{R}^2)}+(\Psi(s)\nabla\dot u(s),\Psi(s)\nabla\dot w(s))_{L^2(\Omega;\mathbb{R}^2)}\big]\,\mathrm{d}s\\-\int_0^t(\dot u(s),\ddot w(s))_{L^2(\Omega)}\,\mathrm{d}s+(\dot u(t),\dot w(t))_{L^2(\Omega)}-(\dot u(0),\dot w(0))_{L^2(\Omega)}.\end{aligned}$$

Notice that equality (6.4) gives (1.6). This show that in this model Griffith's dynamic energy–dissipation balance can be satisfied by a moving crack, in contrast with the case $\Psi=1$, which always leads to (1.3).

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