Existence and Regularity of Spheres Minimising the Canham-Helfrich Energy

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Abstract

We prove existence and regularity of minimisers for the Canham-Helfrich energy in the class of weak (possibly branched and bubbled) immersions of the 2-sphere. This solves (the spherical case) of the minimisation problem proposed by Helfrich in 1973, modelling lipid bilayer membranes. On the way to prove the main results we establish the lower semicontinuity of the Canham-Helfrich energy under weak convergence of (possibly branched and bubbled) weak immersions.

1   Introduction

The basic structural and functional unit of all known living organisms is the cell. The interior material of a cell, the cytoplasm, is enclosed by biological membranes. Most of the cell membranes of living organisms are made of a lipid bilayer, which is a thin polar membrane consisting of two opposite oriented layers of lipid molecules.

In 1970, in order to explain the biconcave shape of red blood cells, Canham [Can70] proposed a bending energy density dependent on the squared mean curvature.

Three years later Helfrich proposed the following curvature elastic energy per unit area of a closed lipid bilayer [Hel73, Equation (12)]

$$\frac{1}{2}k_c(H-c_0)^2 + \bar{k}_c K, \tag{1.1}$$

where H is the mean curvature, K is the Gauss curvature, c_0 is the so-called *spontaneous curvature*, and $k_c, \bar{k}_c$ are the curvature elastic moduli. The values of the parameters can be measured experimentally (e.g. see [EF72], [DH76] for c_0, and [MH90] for k_c). The constant $\bar{k}_c$ is not important for the purpose of this paper as by the Gauss-Bonnet Theorem, the integrated Gauss curvature is a topological constant.

Lipid bilayers are very thin compared to their lateral dimensions, thus are usually modelled as surfaces. Suppose the surface and hence the membrane is represented by a

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smooth isometric embedding $\vec{\Phi} : \mathbb{S}^2 \to \mathbb{R}^3$ of the 2-sphere $\mathbb{S}^2$. We will be concerned with the following integrated version of (1.1)

$$\mathcal{H}^{c_0}(\vec{\Phi}) := \int_{\mathbb{S}^2} (H - c_0)^2 \, d\mu = \int_{\mathbb{S}^2} \left(H^2 - 2c_0 H + 1 \right) d\mu \tag{1.2}$$

where again, H is the mean curvature, c_0 is a constant, and μ is the Radon measure corresponding to the pull back of the Euclidean metric along $\vec{\Phi}$. The integral in (1.2) is known as *Canham-Helfrich energy*. It is also referred to as *Canham-Evans-Helfrich* or just *Helfrich* energy. Its most important reduction is the *Willmore energy*, where $c_0 = 0$. Due to its simplicity and fundamental nature, the Willmore energy appears in many areas of science and technology, and has been studied a lot in the past. Its first appearances were found in the works of Poisson [Poi14] in 1814 and Germain [Ger21] in 1821. It was finally brought onto physical grounds by Kirchhoff [Kir50] in 1850 as the free energy of an elastic membrane. In the early 20th century, Blaschke considered the Willmore energy in the context of differential geometry and proved its conformal invariance, see for instance [Bla55].

The difference between the Willmore energy and the Canham-Helfrich energy comes from the constant c_0, known as spontaneous curvature. According to Seifert [Sei97], it is mainly caused by asymmetry between the two layers of the membrane. Geometrically, the asymmetric area difference between the two layers is given by the total mean curvature, i.e. the integrated mean curvature. This is due to the fact that the infinitesimal variation of the area, i.e. the area difference between two nearby surfaces, is the total mean curvature. Döbereiner et al. [DSL99] observed that spontaneous curvature may also arise from differences in the chemical properties of the aqueous solution on the two sides of the lipid bilayer. Many approaches about how to derive the Canham-Helfrich energy density as the energy density of a lipid bilayer have appeared in the literature. We refer to Seifert [Sei97] for more details.

Our goal is to minimise the Canham-Helfrich energy as well as to study the regularity of minimisers (and more generally of critical points). In the language of the calculus of variations we are concerned with the following Problem 1.2 stated in Bernard, Wheeler and Wheeler [BWW17, Introduction, Problem (P1)]. Given a smooth embedding $\vec{\Phi} : \mathbb{S}^2 \to \mathbb{R}^3$ denote by $\text{Area}\,\vec{\Phi} = \int_{\mathbb{S}^2} d\mu$ the area of the surface $\vec{\Phi}(\mathbb{S}^2)$ and by $\text{Vol}\,\vec{\Phi}$ the enclosed volume.

1.1 Remark. A candidate embedding $\vec{\Phi}_0$ which achieves the global minimum is called a *minimiser*. In general it is not unique and, more dramatically, it may not exist: later in the introduction we show that for a suitable choice of parameters the minimum is achieved by a singular immersion and it cannot be achieved by a smooth one. The constraints and the functional $\mathcal{H}^{c_0}$ are invariant under reparametrisation as well as rigid motions in $\mathbb{R}^3$. Of course, in order to have a non-empty class of competitors, the constraints have to satisfy the Euclidean isoperimetric inequality $A_0^3 \geqslant 36\pi V_0^2$.

1.2 Problem. Let c_0, A_0, and V_0 be given constants. Minimise $\mathcal{H}^{c_0}(\vec{\Phi})$ in the class of smooth embeddings $\vec{\Phi} : \mathbb{S}^2 \to \mathbb{R}^3$ subject to the constraints

$$\text{Area}\,\vec{\Phi} = A_0 \quad \text{and} \quad \text{Vol}\,\vec{\Phi} = V_0. \tag{1.3}$$

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That is, find an embedding $\vec{\Phi}_0 : \mathbb{S}^2 \to \mathbb{R}^3$ such that Area $\vec{\Phi}_0 = A_0$, Vol $\vec{\Phi}_0 = V_0$, and

$$\mathcal{H}^{c_0}(\vec{\Phi}_0) \leqslant \mathcal{H}^{c_0}(\vec{\Phi})$$

for any other smooth embedding $\vec{\Phi} : \mathbb{S}^2 \to \mathbb{R}^3$ satisfying the constraints (1.3).

Problem 1.2 is the classical formulation suggested in [Hel73] and [DH76]. According to Bernard, Wheeler and Wheeler [BWW17], many issues for the Canham-Helfrich energy, including Problem 1.2, remain open and form important questions that future research should address. A similar problem in the 2-dimensional case (i.e. closed curves in the Euclidean plane) was formulated and solved by Bellettini, Dal Maso, and Paolini [BDMP93] by a relaxation procedure.

While there was essentially no work on the variational theory of the Willmore energy after Blaschke's seminal work, in 1965 Willmore [Wil65] reintroduced this Lagrangian which is now named after him. He showed that the round sphere is a minimiser of Problem 1.2 in the special case $c_0 = 0$ without constraints (1.3), see [Wil82]. Simon [Sim86] proved existence of higher genus minimisers for the Willmore energy (see also Kusner [Kus96] and Bauer-Kuwert [BK03]), using the so-called *ambient approach*, i.e. convergence of surfaces is considered in the measure-theoretic sense. Rivière [Riv08, Riv14] proved the analogous result with the so called *parametric approach*, i.e. based on PDE theory and functional analysis as opposed to geometric measure theory. In the present paper we shall adapt the parametric approach. The case $c_0 = 0$ with constraints (1.3) was solved by Schygulla [Sch12] using the ambient approach, and generalised to higher genus surfaces by Keller, the first author, and Rivière [KMR14] using the parametric approach.

From the mathematical point of view, the spontaneous curvature c_0 causes a couple of differences between the Willmore energy and the Canham-Helfrich energy. Most obviously, the Canham-Helfrich energy cannot be bounded below by a strictly positive constant, whereas the Willmore energy is bounded below by 4π, see (2.13). Secondly, while the Willmore functional is invariant under conformal transformations, the Canham-Helfrich energy is not conformally invariant. We will be concerned with yet another property that fails for the Canham-Helfrich energy due to non-negative spontaneous curvature. Namely lower semi-continuity with respect to varifold convergence: while it is well known that the Willmore functional is lower semi-continuous under varifold convergence, the Canham-Helfrich energy in general is not. Indeed, Große-Brauckmann [GB93, *Remark* (ii) on page 550] constructed a sequence of non-compact infinite genus surfaces $\Sigma_1, \Sigma_2, \ldots$ with constant mean curvature equal to 1 which converges in the varifold sense to a double plane Σ_∞. Hence, the mean curvature H_∞ of the limit Σ_∞ is zero and

$$0 = \int_{\Sigma_k} (H_k - 1)^2 \eta \, d\mathscr{H}^2 < 2 \int_{\Sigma_\infty} (H_\infty - 1)^2 \eta \, d\mathscr{H}^2 = 2 \int_{\Sigma_\infty} \eta \, d\mathscr{H}^2$$

for any continuous non-negative, non-zero function η on $\mathbb{R}^3$ of compact support, where $\mathscr{H}^2$ is the 2-dimensional Hausdorff measure. Hence, the general Canham-Helfrich energy is not lower semi-continuous under varifold convergence. However, in order to solve Problem 1.2 by the so-called *direct method of calculus of variations*, lower semi continuity is required.

According to Röger [MR208], it was an open question under which conditions/in which natural weak topology on the space of immersions one obtains lower semi continuity of the Canham-Helfrich energy. This is presumably the reason why Problem 1.2 was only partially solved for non-zero spontaneous curvature c_0. The axisymmetric case was solved in 2013 by Choksi and Veneroni [CV13], who proved the existence of a minimiser of the Canham-Helfrich energy among a suitable class of axisymmetric (possibly singular) surfaces under fixed surface area and enclosed volume constraints. Five years later, Dalphin [Dal18] showed existence of minimisers in a class of $C^{1,1}$ surfaces whose principal curvatures are bounded by a given constant $1/\varepsilon$. Though, in his setting, it is still unclear how to get compactness and lower semi-continuity as ε tends to zero.

As already alluded to, we tackle Problem 1.2 by the direct method of calculus of variations. The issue is of course to find a suitable class $\mathcal{F}_{A_0,V_0}$ of admissible maps (endowed with a suitable topology) having area A_0 and enclosed volume V_0 such that the Canham-Helfrich energy is lower semi-continuous and has (pre-)compact sub-levels. A natural choice is the class of weak (Sobolev) immersions in $W^{2,2}(\mathbb{S}^2, \mathbb{R}^3)$, already employed in the context of the Willmore energy for instance by Rivière [Riv14] or Kuwert and Li [KL12]. We will use the space of bubble trees of weak possibly branched immersions. It will shortly become clear why we have to allow branched points and multiple bubbles.

In the following, we denote by $\cdot$ (resp. $\times$) the Euclidean scalar (resp. vector) product on $\mathbb{R}^3$.

1.3 Definition. A map $\vec{\Phi} : \mathbb{S}^2 \to \mathbb{R}^3$ is called *weak (possibly branched) immersion with finite total curvature* if $\vec{\Phi} \in W^{1,\infty}(\mathbb{S}^2, \mathbb{R}^3)$ and the following holds:

1. There exists $C > 1$ such that, for a.e. $p \in \mathbb{S}^2$,

$$C^{-1}|d\vec{\Phi}|^2(p) \leq |d\vec{\Phi} \times d\vec{\Phi}|(p) \leq C|d\vec{\Phi}|^2(p), \qquad (1.4)$$

where the norms are taken with respect to the standard metric on $\mathbb{S}^2$ and with respect to the Euclidean metric of $\mathbb{R}^3$, and where $d\vec{\Phi} \times d\vec{\Phi}$ is the tensor given in local coordinates on $\mathbb{S}^2$ by

$$d\vec{\Phi} \times d\vec{\Phi} := 2\partial_{x^1}\vec{\Phi} \times \partial_{x^2}\vec{\Phi} \, dx^1 \wedge dx^2 \in \wedge^2 T^*\mathbb{S}^2 \otimes \mathbb{R}^3;$$

2. There exist a positive integer N and finitely many points $b_1, \ldots, b_N \in \mathbb{S}^2$ such that $\log|d\vec{\Phi}| \in L^\infty_{loc}(\mathbb{S}^2 \setminus \{b_1, \cdots, b_N\})$;

3. The Gauss map $\vec{n}$, defined by

$$\vec{n} := \frac{\partial_{x^1}\vec{\Phi} \times \partial_{x^2}\vec{\Phi}}{|\partial_{x^1}\vec{\Phi} \times \partial_{x^2}\vec{\Phi}|}$$

in any local chart x of $\mathbb{S}^2$, satisfies

$$\vec{n} \in W^{1,2}(\mathbb{S}^2, \mathbb{R}^3).$$

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The space of weak (possibly branched) immersions with finite total curvature is denoted by $\mathcal{F}$.

Since by assumption $\vec{\Phi}$ is a Lipschitz map, it induces an L^∞-metric g given by

$$g(X,Y) = d\vec{\Phi}(X) \cdot d\vec{\Phi}(Y)$$

for elements X, Y of the tangent bundle $T\mathbb{S}^2$. In the usual way (see for instance [Heb99, 1.2]), the L^∞-metric g induces a Radon measure μ on $\mathbb{S}^2$ which is mutually absolutely continuous to the 2-dimensional Hausdorff measure on $\mathbb{S}^2$.

Using Müller-Svěrák theory of weak isothermic charts [MŠ95] and Hélein's moving frame technique [Hél02] one can prove the following proposition (see for instance [Riv16])

1.4 Proposition. *Let $\vec{\Phi} \in \mathcal{F}$ be a weak (possibly branched) immersion of $\mathbb{S}^2$ into $\mathbb{R}^3$. Then there exists a bilipschitz homeomorphism Ψ of $\mathbb{S}^2$ such that $\vec{\Phi} \circ \Psi$ is weakly conformal: it satisfies almost everywhere on $\mathbb{S}^2$*

$$\begin{cases} |\partial_{x^1}(\vec{\Phi} \circ \Psi)|^2 = |\partial_{x^2}(\vec{\Phi} \circ \Psi)|^2 \\ \partial_{x^1}(\vec{\Phi} \circ \Psi) \cdot \partial_{x^2}(\vec{\Phi} \circ \Psi) = 0, \end{cases}$$

where x is a local arbitrary conformal chart on $\mathbb{S}^2$ for the standard metric. Moreover $\vec{\Phi} \circ \Psi$ is in $W^{1,\infty}(\mathbb{S}^2, \mathbb{R}^3) \cap W^{2,2}_{loc}(\mathbb{S}^2 \setminus \{b_1, \ldots, b_N\}, \mathbb{R}^3)$.

1.5 Remark. In view of Proposition 1.4, a careful reader could wonder why we do not work with *conformal* $W^{2,2}$ weak, possibly branched, immersions only and why we do not impose for the membership in $\mathcal{F}$, $\vec{\Phi}$ to be conformal from the beginning. The reason why it is technically convenient not to impose conformality from the beginning is to allow general perturbations in the variational problem, which do not have to respect infinitesimally the conformal condition.

The reason why we chose the class $\mathcal{F}$ as above is the following theorem of the first author and Rivière [MR14, Theorem 1.5] (see also [CL14]).

1.6 Theorem. *Suppose $\vec{\Phi}_1, \vec{\Phi}_2, \ldots$ is a sequence in $\mathcal{F}$ of conformal weak (possibly branched) immersions such that*

$$\limsup_{k \to \infty} \int_{\mathbb{S}^2} 1 + |d\vec{n}_k|^2 \, d\mu_k < \infty, \qquad \liminf_{k \to \infty} \operatorname{diam} \vec{\Phi}_k[\mathbb{S}^2] > 0$$

where $\vec{n}_k$ are the Gauss maps, μ_k are the corresponding Radon measures, and $\operatorname{diam} \vec{\Phi}_k[\mathbb{S}^2] := \sup_{a,b \in \mathbb{S}^2} |\vec{\Phi}(a) - \vec{\Phi}(b)|$.

Then, after passing to a subsequence, there exist a family Ψ_k of bilipschitz homeomorphisms of $\mathbb{S}^2$, a positive integer N, sequences $f_k^1, \ldots, f_k^N$ of positive conformal diffeomorphisms of $\mathbb{S}^2$, $\vec{\xi}_\infty^1, \ldots, \vec{\xi}_\infty^N \in \mathcal{F}$, non-negative integers $N_1, \ldots, N_N$, and finitely many points on the sphere

$$\{b^{i,j} : j = 1, \ldots, N_i, \, i = 1, \ldots, N\} \subset \mathbb{S}^2$$

such that

$$\vec{\Phi}_k \circ \Psi_k \to \vec{f}_\infty \qquad \text{as } k \to \infty \text{ strongly in } C^0(\mathbb{S}^2, \mathbb{R}^3) \tag{1.5}$$

for some $\vec{f}_\infty \in W^{1,\infty}(\mathbb{S}^2, \mathbb{R}^3)$ *and*

$$\vec{\Phi}_k \circ f_k^i \rightharpoonup \vec{\xi}_\infty^i \qquad \text{as } k \to \infty \text{ weakly in } W_{\text{loc}}^{2,2}(\mathbb{S}^2 \setminus \{b^{i,1}, \dots, b^{i,N_i}\}, \mathbb{R}^3)$$

for $i = 1, \dots, N$. *Moreover,*

$$\sum_{i=1}^N \int_{\mathbb{S}^2} 1 \, d\mu_{\vec{\xi}_\infty^i} = \lim_{k \to \infty} \int_{\mathbb{S}^2} 1 \, d\mu_k.$$

The theorem already gives (pre-)compactness, a notion of convergence, and lower semi-continuity (actually, continuity) of the third summand in (1.2) of the Canham-Helfrich energy, i.e. of the area functional. Indeed, $\vec{T} := (\vec{f}_\infty, \vec{\xi}_\infty^1, \dots, \vec{\xi}_\infty^N)$ forms a *bubble tree*, see Definition 3.2. In particular, the limit $\vec{T}$ is not in the class $\mathcal{F}$ anymore. At an informal level, a non expert reader can think of a bubble tree $\vec{T} := (\vec{f}, \vec{\xi}^1, \dots, \vec{\xi}^N)$ as a "pearl necklace" where each "pearl" corresponds to the image of a possibly branched weak immersion $\vec{\xi}^i(\mathbb{S}^2)$ and $\vec{f}$ is a Lipschitz map from $\mathbb{S}^2$ to $\mathbb{R}^3$ "parametrising" the whole pearl necklace, in particular $\vec{f}(\mathbb{S}^2) = \bigcup_{i=1}^N \vec{\xi}^i(\mathbb{S}^2)$.

To get a better understanding of why we obtain a bubble tree in the limit, we will look at an example of Problem 1.2. Let

$$c_0 = 1, \qquad A_0 = 2 \operatorname{Area} \mathbb{S}^2, \qquad V_0 = 2 \operatorname{Vol} \mathbb{S}^2.$$

Then, the infimum in Problem 1.2 is achieved by the bubble tree $\vec{T} = (\vec{f}, \vec{\text{Id}}_{\mathbb{S}^2}, \vec{\text{Id}}_{\mathbb{S}^2})$ of twice the unit sphere. Indeed, $\mathcal{H}^{c_0}(\vec{T}) = 0$ and $\mathcal{H}^{c_0}(\vec{\Phi}) \geq 0$ for any other smooth immersion $\vec{\Phi}$ of $\mathbb{S}^2$ into $\mathbb{R}^3$, so $\vec{T}$ achieves the infimum. A minimising sequence $\vec{\Phi}_k(\mathbb{S}^2)$ of smoothly embedded spheres converging to such a bubble tree can be achieved by glueig $(1 + 1/k)\mathbb{S}^2$ to $(1 - 1/k)\mathbb{S}^2$ via a small catenoidal neck of size $2/k$.

Notice also that if $\vec{\Phi}$ satisfies $\mathcal{H}^{c_0}(\vec{\Phi}) = 0$, then the image $\vec{\Phi}[\mathbb{S}^2]$ is the unit sphere by a classical theorem of Hopf [Hop83].

Getting a bubble tree in the limit is in accordance with the earlier result on existence of minimisers by Choksi and Veneroni [CV13] in the axisymmetric case: indeed the minimiser in [CV13, Theorem 1] is made by a finite union of axisymmetric surfaces.

Moreover, the bubbling phenomenon is also known as *budding transition* in biology and has been recorded with video microscopy, see Seifert [Sei97] or Seifert, Berndl, and Lipowsky [SBL91].

In Chapter 3 we sharpen Theorem 1.6 in a way that we get lower semi-continuity for the Canham-Helfrich functional. This can be seen as a possible answer to the aforementioned open question raised by Röger [MR208]. In Chapter 4 we compute the Euler-Lagrange equation for the Canham-Helfrich energy in divergence form. Moreover, we prove that all the weak branched conformal immersions of a minimising bubble tree (actually more generally for a critical bubble tree) are smooth away from their branch points. Our proof is based on the regularity theory for Willmore surfaces developed by

Rivière [Riv08]. It relies on conservation laws discovered by Rivière [Riv08] in the context of the Willmore energy and adjusted by Bernard [Ber16] for the Canham-Helfrich energy. We get the following final result.

1.7 Theorem. *Suppose $c_0 \in \mathbb{R}$, $A_0, V_0 > 0$, and $A_0^3 \geqslant 36\pi V_0^2$.*

Then, there exist a positive integer N and weak branched conformal immersions of finite total curvature $\vec{\Phi}_1, \dots, \vec{\Phi}_N \in \mathcal{F}$ such that $\cup_{i=1}^N \vec{\Phi}_i(\mathbb{S}^2)$ is connected,

$$\inf_{\substack{\vec{\Phi} \in \mathcal{F} \\ \text{Area } \vec{\Phi} = A_0 \\ \text{Vol } \vec{\Phi} = V_0}} \mathcal{H}^{c_0}(\vec{\Phi}) = \sum_{i=1}^N \mathcal{H}^{c_0}(\vec{\Phi}_i) \tag{1.6}$$

and

$$\sum_{i=1}^N \text{Area } \vec{\Phi}_i = A_0, \qquad \sum_{i=1}^N \text{Vol } \vec{\Phi}_i = V_0.$$

Moreover, for each $i \in \{1, \dots, N\}$ there exist a non-negative integer N^i and finitely many points $b^{i,1}, \dots, b^{i,N^i} \in \mathbb{S}^2$ such that $\vec{\Phi}_i$ is a C^∞ immersion of $\mathbb{S}^2 \setminus \{b^{i,1}, \dots, b^{i,N^i}\}$ into $\mathbb{R}^3$ and $b^{i,1}, \dots, b^{i,N^i}$ are branch points for $\vec{\Phi}_i$.

Furthermore, there exists a constant $\varepsilon_{4.4}(A_0, V_0) > 0$ such that if $|c_0| < \varepsilon_{4.4}(A_0, V_0)$, then $N = 1$ and $\vec{\Phi} := \vec{\Phi}_1$ is a smooth embedding of $\mathbb{S}^2$ into $\mathbb{R}^3$.

Proof. Let $\vec{\Phi}_1, \vec{\Phi}_2, \dots$ be a minimising sequence of (1.6). There holds

$$\int_{\mathbb{S}^2} H_{\vec{\Phi}_k}^2 \, d\mu_{\vec{\Phi}_k} = \int_{\mathbb{S}^2} 2(H_{\vec{\Phi}_k} - c_0)^2 - (H_{\vec{\Phi}_k} - 2c_0)^2 + 2c_0^2 \, d\mu_{\vec{\Phi}_k} \leqslant 2\mathcal{H}^{c_0}(\vec{\Phi}_k) + 2c_0^2 A_0 \tag{1.7}$$

where $H_{\vec{\Phi}_k}$ is the mean curvature corresponding to $\vec{\Phi}_k$, see (2.3). By the Gauss-Bonnet theorem (see (2.6) for the precise statement in case of weak branched immersions and (2.8) for the estimate below),

$$\int_{\mathbb{S}^2} |d\vec{n}_{\vec{\Phi}_k}|^2 \, d\mu_{\vec{\Phi}_k} \leq 4 \int_{\mathbb{S}^2} H_{\vec{\Phi}_k}^2 \, d\mu_{\vec{\Phi}_k}$$

and thus

$$\int_{\mathbb{S}^2} 1 + |d\vec{n}_{\vec{\Phi}_k}|^2 \, d\mu_{\vec{\Phi}_k} \leqslant 8\mathcal{H}^{c_0}(\vec{\Phi}_k) + (1 + 8c_0^2)A_0$$

which means the first inequality of (3.17) is satisfied. Moreover, (2.10) implies the second inequality of (3.17). Hence, we can apply Theorem 3.3, Theorem 4.3, Lemma 4.4, (2.13) and (2.12) to conclude the proof. $\qquad\square$

1.8 Remark. The arguments in the proof of Theorem 1.7 yield also that the minimum of $\mathcal{H}^{c_0}$ is achieved in the class of bubble trees of possibly branched weak immersions, by a bubble tree of possibly branched immersions which are smooth out of the branch points.

A more general form of the Canham-Helfrich energy is given by

$$\mathcal{H}_{\alpha,\rho}^{c_0}(\vec{\Phi}) := \int_{\mathbb{S}^2} (H_{\vec{\Phi}} - c_0)^2 \, d\mu_{\vec{\Phi}} + \alpha \operatorname{Area} \vec{\Phi} + \rho \operatorname{Vol} \vec{\Phi},$$

for $\vec{\Phi} \in \mathcal{F}$ where the parameter $\alpha \geqslant 0$ is referred to as *tensile stress*, and $\rho \geqslant 0$ as *osmotic pressure*. We get the following solution of Problem (P2) from the introduction in [BWW17].

1.9 Theorem. *Suppose $c_0 \in \mathbb{R}$, $\alpha > 0$, and $\rho \geqslant 0$. Then, there holds*

$$\inf_{\vec{\Phi} \in \mathcal{F}} \mathcal{H}_{\alpha,\rho}^{c_0}(\vec{\Phi}) \leqslant 4\pi.$$

Moreover, if the inequality is strict, then there exist $\vec{\Phi}_0 \in \mathcal{F}$, a positive integer N_0, and points $b_1, \ldots, b_{N_0} \in \mathbb{S}^2$ such that

$$\inf_{\vec{\Phi} \in \mathcal{F}} \mathcal{H}_{\alpha,\rho}^{c_0}(\vec{\Phi}) = \mathcal{H}_{\alpha,\rho}^{c_0}(\vec{\Phi}_0),$$

$\vec{\Phi}_0$ is a C^∞ immersion of $\mathbb{S}^2 \setminus \{b^{i,1}, \ldots, b^{i,N^i}\}$ into $\mathbb{R}^3$ and $b^{i,1}, \ldots, b^{i,N^i}$ are branch points. Furthermore, if $|c_0| < \sqrt{\alpha}$, then $\vec{\Phi}_0$ is a smooth embedding.

Proof. Taking $\vec{\Phi}_k = \frac{1}{k} \mathbb{S}^2$ for each integer k leads to

$$\inf_{\vec{\Phi} \in \mathcal{F}} \mathcal{H}_{\alpha,\rho}^{c_0}(\vec{\Phi}) \leqslant \liminf_{k \to \infty} \mathcal{H}_{\alpha,\rho}^{c_0}(\vec{\Phi}_k) = \int_{\mathbb{S}^2} H_{\mathbb{S}^2}^2 \, d\mu_{\mathbb{S}^2} = 4\pi,$$

which proves the first statement. Now assume $\vec{\Phi}_1, \vec{\Phi}_2, \ldots$ is a sequence in $\mathcal{F}$ such that

$$\lim_{k \to \infty} \mathcal{H}_{\alpha,\rho}^{c_0}(\vec{\Phi}_k) = \inf_{\vec{\Phi} \in \mathcal{F}} \mathcal{H}_{\alpha,\rho}^{c_0}(\vec{\Phi}) < 4\pi.$$

As $\alpha > 0$, we have $\sup_k \operatorname{Area} \vec{\Phi}_k < \infty$ and thus, using (1.7), also $\sup_k \int_{\mathbb{S}^2} H_{\vec{\Phi}_k}^2 \, d\mu_{\vec{\Phi}_k} < \infty$. A simple contradiction argument using (2.10) now leads to

$$\liminf_{k \to \infty} \operatorname{diam} \vec{\Phi}_k[\mathbb{S}^2] > 0,$$

as otherwise we had

$$\lim_{k \to \infty} \operatorname{Area} \vec{\Phi}_k = 0, \qquad \lim_{k \to \infty} \mathcal{H}_{\alpha,\rho}^{c_0}(\vec{\Phi}_k) = \lim_{k \to \infty} \int_{\mathbb{S}^2} H_{\vec{\Phi}_k}^2 \, d\mu_{\vec{\Phi}_k} \geqslant 4\pi.$$

Therefore, analogously to the proof of Theorem 1.7, we can apply Theorem 3.3 to obtain an integer N and $\vec{\Phi}_1, \ldots, \vec{\Phi}_N \in \mathcal{F}$ such that

$$\inf_{\vec{\Phi} \in \mathcal{F}} \mathcal{H}_{\alpha,\rho}^{c_0}(\vec{\Phi}) = \sum_{i=1}^{N} \mathcal{H}_{\alpha,\rho}^{c_0}(\vec{\Phi}_i).$$

Obviously,

$$\mathcal{H}_{\alpha,\rho}^{c_0}(\vec{\Phi}_1) \leqslant \sum_{i=1}^{N} \mathcal{H}_{\alpha,\rho}^{c_0}(\vec{\Phi}_i)$$

and since there are no constraints, we simply get $N = 1$. Letting $\vec{\Phi}_0 := \vec{\Phi}_1$, we infer from (1.7) that in case $|c_0| \leqslant \sqrt{\alpha}$

$$\int_{\mathbb{S}^2} H_{\vec{\Phi}_0}^2 \, d\mu_{\vec{\Phi}_0} \leqslant 2\mathcal{H}^{c_0}(\vec{\Phi}_k) + 2c_0^2 \operatorname{Area} \vec{\Phi}_0[\mathbb{S}^2] \leqslant 2\mathcal{H}_{\alpha,\rho}^{c_0}(\vec{\Phi}_0) < 8\pi.$$

The conclusion follows from Theorem 4.3, and (2.12). $\qquad\square$

2 Preliminaries

2.1 Notation

We adopt the conventions of [Riv16]. To avoid indices and to get clearly arranged equations, we will employ the following suggestive notation. For $\mathbb{R}^3$ valued maps $\vec{e}$ and $\vec{f}$ defined on the unit disk D^2, we write

$$\nabla \vec{e} := \begin{pmatrix} \partial_{x^1} \vec{e} \\ \partial_{x^2} \vec{e} \end{pmatrix}, \qquad \nabla^\perp \vec{e} := \begin{pmatrix} -\partial_{x^2} \vec{e} \\ \partial_{x^1} \vec{e} \end{pmatrix}$$

$$\langle \vec{e}, \nabla \vec{f} \rangle := \begin{pmatrix} \vec{e} \cdot \partial_{x^1} \vec{f} \\ \vec{e} \cdot \partial_{x^2} \vec{f} \end{pmatrix}, \qquad \vec{e} \times \nabla \vec{f} := \begin{pmatrix} \vec{e} \times \partial_{x^1} \vec{f} \\ \vec{e} \times \partial_{x^2} \vec{f} \end{pmatrix}$$

as well as

$$\nabla \vec{e} \times \nabla \vec{f} := \partial_{x^1} \vec{e} \times \partial_{x^1} \vec{f} + \partial_{x^2} \vec{e} \times \partial_{x^2} \vec{f},$$

$$\vec{e} \cdot \nabla \vec{f} := \vec{e} \cdot \partial_{x^1} \vec{f} + \vec{e} \cdot \partial_{x^2} \vec{f}, \qquad \nabla \vec{e} \cdot \nabla f := \partial_{x^1} \vec{e} \cdot \partial_{x^1} \vec{f} + \partial_{x^2} \vec{e} \cdot \partial_{x^2} \vec{f},$$

where $\cdot$ denotes the Euclidean inner product and $\times$ denotes the usual vector product on $\mathbb{R}^3$. Similarly, for $\lambda : D^2 \to \mathbb{R}$ we write

$$\langle \nabla \lambda, \vec{e} \rangle := \begin{pmatrix} (\partial_{x^1} \lambda) \vec{e} \\ (\partial_{x^2} \lambda) \vec{e} \end{pmatrix}, \qquad \langle \nabla \lambda, \nabla \vec{e} \rangle := \partial_{x^1} \lambda \partial_{x^1} \vec{e} + \partial_{x^2} \lambda \partial_{x^2} \vec{e}.$$

Moreover, for a vector field

$$\vec{X} = \begin{pmatrix} \vec{X}^1 \\ \vec{X}^2 \end{pmatrix}$$

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with components $\vec{X}^1, \vec{X}^2 : D^2 \to \mathbb{R}^3$, we define the divergence

$$\operatorname{div} \vec{X} := \partial_{x^1} \vec{X}^1 + \partial_{x^2} \vec{X}^2.$$

The m-dimensional Lebesgue measure is denoted by $\mathscr{L}^m$.

2.2 Weak (possibly branched) conformal immersions

We adapt the notion of weak immersions which was independently formalized by Rivière [Riv14] and Kuwert and Li [KL12].

Let (Σ, c_0) be a smooth closed Riemann surface (in the rest of the paper we will take (Σ, c_0) to be the 2-sphere endowed with the standard round metric). Without loss of generality we can assume that (Σ, c_0) is endowed with a metric g_{c_0} of constant curvature and area 4π (see for instance [Jos06]). For the definition of the Sobolev spaces $W^{k,p}(\Sigma, \mathbb{R}^3)$ on Σ see for instance Hebey [Heb99]. A map $\vec{\Phi} : \Sigma \to \mathbb{R}^3$ is called a *weak branched conformal immersion with finite total curvature* if and only if there exists a positive integer N, finitely many points $b_1, \dots, b_N \in \Sigma$ such that

$$\vec{\Phi} \in W^{1,\infty}(\Sigma, \mathbb{R}^3) \cap W^{2,2}_{\mathrm{loc}}(\Sigma \setminus \{b_1, \cdots, b_N\}, \mathbb{R}^3),$$

there holds

$$\begin{cases} |\partial_{x^1}\vec{\Phi}| = |\partial_{x^2}\vec{\Phi}| \\ \partial_{x^1}\vec{\Phi} \cdot \partial_{x^2}\vec{\Phi} = 0 \end{cases} \tag{2.1}$$

almost everywhere for any conformal chart x of Σ,

$$\log|d\vec{\Phi}| \in L^\infty_{\mathrm{loc}}(\Sigma \setminus \{b_1, \dots, b_N\}),$$

and its Gauss map $\vec{n}$ defined by

$$\vec{n} := \frac{\partial_{x^1}\vec{\Phi} \times \partial_{x^2}\vec{\Phi}}{|\partial_{x^1}\vec{\Phi} \times \partial_{x^2}\vec{\Phi}|}$$

in any local positive chart x of Σ satisfies

$$\vec{n} \in W^{1,2}(\Sigma, \mathbb{R}^3). \tag{2.2}$$

The space of weak branched conformal immersions with finite total curvature is denoted by $\mathcal{F}_\Sigma$ or just $\mathcal{F}$ in case $\Sigma = \mathbb{S}^2$. We define the L^∞-metric g pointwise for almost every $p \in \Sigma$ by

$$g_p(X, Y) := d\vec{\Phi}_p(X) \cdot d\vec{\Phi}_p(Y)$$

for elements X, Y of the tangent space $T_p\Sigma$. In the usual way, the L^∞-metric g induces a Radon measure μ_g on Σ. The conformality condition (2.1) implies that $g = e^{2\lambda}g_{c_0}$ for some $\lambda \in L^\infty_{loc}(\Sigma \setminus \{b_1, \dots, b_N\})$ called *conformal factor*. Moreover, we define the second fundamental form $\vec{\mathbb{I}}$ pointwise for almost every $p \in \Sigma$ by

$$\vec{\mathbb{I}}_p : T_p\Sigma \times T_p\Sigma \to \mathbb{R}^3, \qquad \vec{\mathbb{I}}_p(X, Y) := -[d\vec{n}_p(X) \cdot d\vec{\Phi}_p(Y)]\vec{n}.$$

The mean curvature vector $\vec{H}$ and the scalar mean curvature H are given by

$$\vec{H} := \frac{1}{2}\operatorname{trace}\vec{\mathbb{I}}, \qquad H := \vec{n}\cdot\vec{H}. \tag{2.3}$$

Note that condition (2.2) ensures

$$H \in L^2(\Sigma). \tag{2.4}$$

2.2.1 Singular points and Gauss-Bonnet Theorem of weak branched immersions

First of all let us recall the following result first proved by Müller-Svěrák [MŠ95]. For a different proof using Hélein's moving frames technique [Hél02], see [Riv14, Lemma A.5]; see also [KL12, Theorem 3.1]) and [MR14, Section 2.1].

2.1 Proposition. *Let $\vec{\Phi} : \Sigma \to \mathbb{R}^3$ be a weak branched conformal immersion with finite total curvature with singular points $b_1,\ldots,b_N \in \Sigma$. Let $\lambda \in L^\infty_{loc}(\Sigma \setminus \{b_1,\ldots,b_N\})$ be the conformal factor, i.e. $g = \vec{\Phi}^* g_{\mathbb{R}^3} = e^{2\lambda} g_{c_0}$.*
Then $\vec{\Phi} \in W^{2,2}(\Sigma)$ and the conformal factor λ is an element of $L^1(\Sigma)$.
Moreover, for each singular point $b_j, j = 1,\ldots,N$, there exists a strictly positive integer $n_j \in \mathbb{N}$ such that the following holds:

- *For every b_j there exists a local conformal chart z centred at $\{b_j\} = \{z = 0\}$ such that*
$$\lambda(z) = (n_j - 1)\log|z| + \omega(z)$$
 for some $\omega \in C^0 \cap W^{1,2}$.

- *The multiplicity of the immersion $\vec{\Phi}$ at $\vec{\Phi}(b_j)$ is n_j. Moreover, if $n_j = 1$, then $\vec{\Phi}$ is a conformal immersion of a neighbourhood of b_j.*

- *The conformal factor λ satisfies the following singular Liouville equation in distributional sense*

$$-\Delta_{g_{c_0}}\lambda = K_{\vec{\Phi}}e^{2\lambda} - K_0 - 2\pi\sum_{j=1}^{N}\left[(n_j-1)\delta_{b_j}\right], \tag{2.5}$$

 where δ_{b_j} is the Dirac delta centred at b_j, $K_{\vec{\Phi}}$ is the Gaussian curvature of $\vec{\Phi}$, and $K_0 \in \mathbb{R}$ is the (constant) curvature of (Σ, g_{c_0}).

By integrating the singular Liouville equation (2.5), we obtain the Gauss-Bonnet Theorem for weak branched immersions:

$$\int_{\Sigma} K_{\vec{\Phi}}d\mu_g = 2\pi\chi(\Sigma) + 2\pi\sum_{j=1}^{N}(n_j - 1), \tag{2.6}$$

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where $\chi(\Sigma)$ is the Euler Characteristic of Σ.

Note in particular that, once the topology of Σ is fixed, the number of branch points counted with multiplicity is bounded by the Willmore energy:

$$2\pi \sum_{j=1}^{N} (n_j - 1) = \int_\Sigma K_{\vec{\Phi}} d\mu_g - 2\pi\chi(\Sigma) \leq 2 \int_\Sigma H^2 d\mu_g - 2\pi\chi(\Sigma), \qquad (2.7)$$

Moreover, the Willmore energy controls the L^2 norm squared of the second fundamental form:

$$\int_\Sigma |\vec{\mathbb{I}}|^2 \, d\mu_g = 4 \int_\Sigma H^2 \, d\mu_g - 2 \int_\Sigma K_{\vec{\Phi}} d\mu_g \leq 4 \int_\Sigma H^2 \, d\mu_g - 4\pi\chi(\Sigma). \qquad (2.8)$$

2.2.2 Simon's monotonicity formula and Li-Yau inequality for weak branched immersions

Let $\vec{\Phi} \in \mathcal{F}_\Sigma$ be any weak branched conformal immersion with finite total curvature and branch points $\{b_1, \ldots, b_N\}$. In the usual way (by splitting the vector field in its tangential and normal parts and using integration by parts) one shows

$$\int_\Sigma \operatorname{div}_{\vec{\Phi}} \vec{X} \, d\mu_{\vec{\Phi}} = -2 \int_\Sigma \vec{X} \cdot \vec{H}_{\vec{\Phi}} \, d\mu_{\vec{\Phi}} \qquad (2.9)$$

whenever $\vec{X} \in W^{1,2}(\Sigma, \mathbb{R}^3)$ has compact support in $\Sigma \setminus \{b_1, \ldots, b_N\}$, where in a local chart x,

$$\operatorname{div}_{\vec{\Phi}} \vec{X} := g^{ij} \partial_{x^i} \vec{X} \cdot \partial_{x^j} \vec{\Phi}.$$

A simple cut-off argument together with (2.4) shows that the first variation formula (2.9) is true for all $\vec{X} \in W^{1,2}(\Sigma, \mathbb{R}^3)$. In the following we will gather a couple of facts that are well known for weak unbranched immersions and, due to (2.9), are also valid for weak branched conformal immersions with finite total curvature. Firstly, letting $\vec{X}(p) := \vec{\Phi}(p) - \vec{\Phi}(a_0)$ for $p \in \Sigma$ and some fixed $a_0 \in \Sigma$, one has $\operatorname{div}_{\vec{\Phi}} \vec{X} = 2$ and hence, see Simon [Sim86, Lemma 1.1]

$$\sqrt{\operatorname{Area} \vec{\Phi}} \leqslant \operatorname{diam} \vec{\Phi}[\Sigma] \sqrt{\int_\Sigma H_{\vec{\Phi}}^2 \, d\mu_{\vec{\Phi}}}. \qquad (2.10)$$

The push forward measure $\mu := \vec{\Phi}_{\#} \mu_{\vec{\Phi}}$ of $\mu_{\vec{\Phi}}$ defines a 2-dimensional integral varifold in $\mathbb{R}^3$ with multiplicity function $\theta^2(\mu, x) = \mathscr{H}^0(\vec{\Phi}^{-1}\{x\})$ (here $\mathscr{H}^0$ denotes the 0-dimensional Hausdorff measure, i.e. the counting measure) and approximate tangent space $T_x\mu = d\vec{\Phi}[T_p\Sigma]$ almost everywhere when $x = \vec{\Phi}(p)$. See Simon [Sim83, Chapter 4] for an introduction on varifolds and Kuwert and Li [KL12, Section 2.2] for the context of weak unbranched immersions. From (2.9) and the co-area formula (see for instance [Sim83, Equation 12.7]), the first variation formula for the varifold μ becomes

$$\int \operatorname{div}_\mu \phi \, d\mu = -2 \int \phi \cdot H_\mu \, d\mu \qquad \text{for } \phi \in C_c^1(\mathbb{R}^3, \mathbb{R}^3) \qquad (2.11)$$

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where the weak mean curvature is almost everywhere given by

$$H_\mu(x) = \begin{cases} \frac{1}{\theta^2(\mu,x)} \sum_{p \in \vec{\Phi}^{-1}(x)} \vec{H}_{\vec{\Phi}}(p) & \text{if } \theta^2(\mu,x) > 0 \\ 0 & \text{else.} \end{cases}$$

The first variation formula (2.11) leads to Simon's monotonicity formula [Sim83, 17.4] which implies (see for instance Rivière [Riv16, Section 5.3] or Kuwert and Schätzle [KS04, Appendix]) the Li-Yau inequality [LY82, Theorem 6]

$$\theta^2(\mu, x) \leqslant \frac{1}{4\pi} \int_\Sigma H_{\vec{\Phi}}^2 \, d\mu_{\vec{\Phi}}. \tag{2.12}$$

Consequently,

$$\inf_{\vec{\Phi} \in \mathcal{F}_\Sigma} \int_\Sigma H_{\vec{\Phi}}^2 \, d\mu_{\vec{\Phi}} \geqslant 4\pi. \tag{2.13}$$

Moreover, if $\vec{\Phi} \in \mathcal{F}_\Sigma$ with $\int_\Sigma H_{\vec{\Phi}}^2 \, d\mu_{\vec{\Phi}} < 8\pi$, then $\vec{\Phi}$ is an embedding (compare also with Proposition 2.1).

2.3 Canham-Helfrich energy

Given real numbers $c_0 \in \mathbb{R}$ and $\alpha, \rho \geqslant 0$ as well as a weak branched conformal immersion with finite total curvature $\vec{\Phi} : \Sigma \to \mathbb{R}^3$, we define the Canham-Helfrich energy $\mathcal{H}_{\alpha,\rho}^{c_0}(\vec{\Phi})$ in its most general form by

$$\mathcal{H}_{\alpha,\rho}^{c_0}(\vec{\Phi}) := \int_\Sigma (H_{\vec{\Phi}} - c_0)^2 \, d\mu_{\vec{\Phi}} + \alpha \int_\Sigma 1 \, d\mu_{\vec{\Phi}} + \rho \int_\Sigma \vec{n}_{\vec{\Phi}} \cdot \vec{\Phi} \, d\mu_{\vec{\Phi}}. \tag{2.14}$$

Note that, in case $\vec{\Phi} : \Sigma \to \mathbb{R}^3$ is a smooth (actually Lipschitz is enough) embedding, by the Divergence Theorem the last integral equals the volume enclosed by $\vec{\Phi}(\Sigma)$.
The parameter α is referred to as *tensile stress*, ρ as *osmotic pressure*. Compare this definition for instance with [Ber16, Equation (3.6)] or [BWW17].

3 Existence of minimisers

In this chapter we will prove compactness of sequences with uniformly bounded Willmore energy and area as well as lower semi-continuity of the Canham-Helfrich energy under this convergence, see Theorem 3.3. The proof of Theorem 3.3 will build on top of [MR14] and the next Lemma 3.1 which establishes the convergence of the constraints and the lower semi-continuity of the Willmore energy away from the branch points (Lemma 3.1 should be compared with [Riv16, Lemma 5.2]).

3.1 Lemma (Convergence outside the branch points)**.** *Suppose $\vec{\xi}_1, \vec{\xi}_2, \ldots \in \mathcal{F}_{\mathbb{S}^2}$ is a sequence of weak branched conformal immersions with finite total curvature of the 2-sphere $\mathbb{S}^2$ into $\mathbb{R}^3$, $\mu_1, \mu_2, \ldots$ are the corresponding Radon measures on $\mathbb{S}^2$, $\vec{n}_1, \vec{n}_2, \ldots$ are the corresponding Gauss maps,*

$$\sup_{k \in \mathbb{N}} \int_{\mathbb{S}^2} |d\vec{n}_k|^2 \, d\mu_k < \infty, \tag{3.1}$$

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there exists $\vec{\xi}_\infty \in \mathcal{F}_{\mathbb{S}^2}$, a positive integer N, and $b_1, \ldots, b_N \in \mathbb{S}^2$ such that

$$\sup_{k \in \mathbb{N}} \| \log |d\vec{\xi}_k| \|_{L^\infty_{\mathrm{loc}}(\mathbb{S}^2 \setminus \{b_1, \ldots, b_N\})} < \infty, \tag{3.2}$$

$$\vec{\xi}_k \rightharpoonup \vec{\xi}_\infty \qquad \text{as } k \to \infty \text{ weakly in } W^{2,2}_{\mathrm{loc}}(\mathbb{S}^2 \setminus \{b_1, \ldots, b_N\}, \mathbb{R}^3). \tag{3.3}$$

Then, there exists a sequence of positive numbers $s_1, s_2, \ldots$ converging to zero such that

$$\int_{\mathbb{S}^2} 1 \, d\mu_\infty = \lim_{k \to \infty} \int_{\mathbb{S}^2 \setminus \bigcup_{i=1}^N B_{s_k}(b_i)} 1 \, d\mu_k \tag{3.4}$$

$$\int_{\mathbb{S}^2} H_\infty \, d\mu_\infty = \lim_{k \to \infty} \int_{\mathbb{S}^2 \setminus \bigcup_{i=1}^N B_{s_k}(b_i)} H_k \, d\mu_k \tag{3.5}$$

$$\int_{\mathbb{S}^2} \vec{n}_\infty \cdot \vec{\xi}_\infty \, d\mu_\infty = \lim_{k \to \infty} \int_{\mathbb{S}^2 \setminus \bigcup_{i=1}^N B_{s_k}(b_i)} \vec{n}_k \cdot \vec{\xi}_k \, d\mu_k \tag{3.6}$$

where the balls are taken with respect to the geodesic distance on the standard $\mathbb{S}^2$, μ_∞ and $\vec{n}_\infty$ are the Radon measure and the Gauss map corresponding to $\vec{\xi}_\infty$, and the H_k's and H_∞ are the mean curvatures corresponding to the $\vec{\xi}_k$'s and $\vec{\xi}_\infty$. Equations (3.4)–(3.6) remain valid for s_k replaced by any sequence t_k converging to zero and satisfying $t_k \geqslant s_k$, for all $k \in \mathbb{N}$.

Moreover, for any sequence $s_1, s_2, \ldots$ of positive numbers converging to zero, there exists a sequence $t_k \geqslant s_k$ converging to zero such that

$$\int_{\mathbb{S}^2} H_\infty^2 \, d\mu_\infty \leqslant \liminf_{k \to \infty} \int_{\mathbb{S}^2 \setminus \bigcup_{i=1}^N B_{t_k}(b_i)} H_k^2 \, d\mu_k. \tag{3.7}$$

Proof. Suppose U is an open subset of $\mathbb{S}^2 \setminus \{b_1, \ldots, b_N\}$, K is a compact subset of U, and $x : U \to \mathbb{R}^2$ is a conformal chart for $\mathbb{S}^2$. Denote by

$$\lambda_k = \log |\partial_{x^1} \vec{\xi}_k|, \qquad \lambda_\infty = \log |\partial_{x^1} \vec{\xi}_\infty|$$

the conformal factors. Notice that the volume element corresponding to $\vec{\xi}_k$ is given by $e^{2\lambda_k}$. In a first step we will show that

$$e^{2\lambda_k} \to e^{2\lambda_\infty} \qquad \text{as } k \to \infty \text{ in } L^p(x[K]), \tag{3.8}$$

$$\vec{n}_k \cdot \vec{\xi}_k e^{2\lambda_k} \to \vec{n}_\infty \cdot \vec{\xi}_\infty e^{2\lambda_\infty} \qquad \text{as } k \to \infty \text{ in } L^p(x[K]) \tag{3.9}$$

for any $1 \leqslant p < \infty$, as well as

$$\int_K H_\infty \, d\mu_\infty = \lim_{k \to \infty} \int_K H_k \, d\mu_k, \tag{3.10}$$

$$\int_K H_\infty^2 \, d\mu_\infty \leqslant \liminf_{k \to \infty} \int_K H_k^2 \, d\mu_k. \tag{3.11}$$

A simple argument by contradiction shows that it is enough to prove the statement after passing to a subsequence of k. Since the $\vec{\xi}_k$'s and $\vec{\xi}_\infty$ are conformal and x is a conformal chart, we can write the mean curvature vector as

$$2\vec{H}_k = e^{-2\lambda_k} \Delta \vec{\xi}_k, \qquad 2\vec{H}_\infty = e^{-2\lambda_\infty} \Delta \vec{\xi}_\infty$$

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where Δ is the flat Laplacian with respect to x. By Hypothesis (3.3), we have that

$$\vec{H}_k e^{2\lambda_k} = \frac{1}{2}\Delta\vec{\xi}_k \rightharpoonup \frac{1}{2}\Delta\vec{\xi}_\infty = \vec{H}_\infty e^{2\lambda_\infty}$$

as $k \to \infty$ weakly in $L^2(x[K], \mathbb{R}^3)$, which implies (3.10).

By the Rellich-Kondrachov Compactness Theorem, after passing to a subsequence, there holds

$$\partial_{x^1}\vec{\xi}_k \to \partial_{x^1}\vec{\xi}_\infty \qquad \text{as } k \to \infty \text{ in } L^p_{\text{loc}}(x[U], \mathbb{R}^3) \tag{3.12}$$

for any $1 \leqslant p < \infty$. Therefore, using Hypothesis (3.2) and passing to a further subsequence, it follows

$$e^{-\lambda_k} = |\partial_{x^1}\vec{\xi}_k|^{-1} \to |\partial_{x^1}\vec{\xi}_\infty|^{-1} = e^{-\lambda_\infty} \qquad \text{as } k \to \infty \text{ in } L^2(x[K]).$$

It follows

$$\vec{H}_k\sqrt{e^{2\lambda_k}} = \frac{1}{2}e^{-\lambda_k}\Delta\vec{\xi}_k \rightharpoonup \frac{1}{2}e^{-\lambda_\infty}\Delta\vec{\xi}_\infty = \vec{H}_\infty\sqrt{e^{2\lambda_k}}$$

as $k \to \infty$ weakly in $L^2(x[K], \mathbb{R}^3)$, which implies (3.11) by lower semi-continuity of the L^2-norm under weak convergence.

Similarly, from Hypothesis (3.2) and the strong convergence (3.12), we infer (3.8).

Again by the strong convergence (3.12) and Hypothesis (3.2), we can extract a subsequence such that by dominated convergence,

$$\vec{n}_k = e^{-2\lambda_k}(\partial_{x^1}\vec{\xi}_k \times \partial_{x^2}\vec{\xi}_k) \to e^{-2\lambda_\infty}(\partial_{x^1}\vec{\xi}_\infty \times \partial_{x^2}\vec{\xi}_\infty) = \vec{n}_\infty$$

as $k \to \infty$ in $L^p(x[K], \mathbb{R}^3)$ for any $1 \leqslant p < \infty$. Using this and the fact that by the Rellich-Kondrachov Compactness Theorem

$$\vec{\xi}_k \to \vec{\xi}_\infty \qquad \text{as } k \to \infty \text{ in } L^p(x[K], \mathbb{R}^3)$$

for any $1 \leqslant p < \infty$, one verifies (3.9).

Next, let r_k be any sequence of positive numbers converging to zero and abbreviate

$$f_k = e^{2\lambda_k}, \qquad f_\infty = e^{2\lambda_\infty}, \qquad K_{r_k} = \mathbb{S}^2 \setminus \bigcup_{i=1}^{N} B_{r_k}(b_i).$$

First, notice that for any Borel function f on $\mathbb{S}^2$ with $\int_{\mathbb{S}^2}|f|\,d\mu_\infty < \infty$, there holds

$$\lim_{k\to\infty}\int_{K_{r_k}} f\,d\mu_\infty = \int_{\mathbb{S}^2} f\,d\mu_\infty \tag{3.13}$$

which is a consequence of the dominated convergence theorem and the fact that finite sets have μ_∞ measure zero. Let $n_0 = 1$. For each positive integer j, we use (3.8) to inductively choose $n_j > n_{j-1}$ such that

$$\int_{x[K_{r_j}]} |f_k - f_\infty|\,d\mathscr{L}^2 \leqslant \frac{1}{j} \qquad \text{for all } k \geqslant n_j.$$

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Moreover, define $l_k = j$ for all integers k with $n_{j-1} < k \leqslant n_j$ and define $s_k = r_{l_k}$. Then, we have that $s_k \to 0$ as $k \to \infty$ as well as

$$\int_{x[K_{s_k}]} |f_k - f_\infty| \, d\mathscr{L}^2 \to 0 \qquad \text{as } k \to \infty \tag{3.14}$$

which in particular remains valid for s_k replaced by any $t_k \geqslant s_k$. Hence, by (3.13) we can deduce (3.4). Using the convergence on compact sets (3.9)–(3.11), Equations (3.5)–(3.7) follow similarly. It only remains to show that Equation (3.5) is still valid after replacing s_k by any sequence $t_k \geqslant s_k$ converging to zero. Hence, we only have to show that

$$\int_{K_{s_k} \setminus K_{t_k}} H_k \, d\mu_k \to 0 \qquad \text{as } k \to \infty.$$

This follows as by Hölder's inequality

$$\left| \int_{K_{s_k} \setminus K_{t_k}} H_k \, d\mu_k \right| \leqslant \left(\int_{\mathbb{S}^2} H_k^2 \, d\mu_k \right)^{1/2} \left(\int_{K_{s_k} \setminus K_{t_k}} 1 \, d\mu_k \right)^{1/2}.$$

The first factor on the right hand side is bounded by (3.1). To see that the second factor goes to zero as k tends to infinity, we apply (3.14) and the fact that $\mu_\infty(\bigcup_{i=1}^N B_{t_k}(b_i)) \to 0$ as $k \to \infty$. $\qquad \square$

In the following we will define the notion of a bubble tree. The idea is that the different bubbles can be parametrised by decomposing a single 2-sphere. The bubbles can then be attached to each other by a Lipschitz map, see (3.15) and (3.16).

3.2 Definition (Bubble tree of weak immersions, see [MR14, Definition 7.1]). An $N + 1$ tuple $\vec{T} = (\vec{f}, \vec{\Phi}^1, \ldots, \vec{\Phi}^N)$ is called a *bubble tree of weak immersions* if and only if N is a positive integer, $\vec{f} \in W^{1,\infty}(\mathbb{S}^2, \mathbb{R}^3)$, and $\vec{\Phi}^1, \ldots, \vec{\Phi}^N \in \mathcal{F}_{\mathbb{S}^2}$ are weak branched conformal immersions with finite total curvature such that the following holds.

There exist open geodesic balls $B^1, \ldots, B^N \subset \mathbb{S}^2$ such that

- $\overline{B^1} = \mathbb{S}^2$ and for all $i \neq i'$ either $\overline{B^i} \subset B^{i'}$ or $\overline{B^{i'}} \subset B^i$.

For all $i \in \{1, \ldots, N\}$ there exists a positive integer N^i and disjoint open geodesic balls $B^{i,1}, \ldots, B^{i,N^i} \subset \mathbb{S}^2$ whose closures are included in B^i such that

- for all $i' \neq i$ either $\overline{B^i} \subset B^{i'}$ or $\overline{B^{i'}} \subset B^{i,j}$ for some $j \in \{1, \ldots, N^i\}$.

For all $i \in \{1, \ldots, N\}$ there exist distinct points $b^{i,1}, \ldots, b^{i,N^i} \in \mathbb{S}^2$ and a Lipschitz diffeomorphism

$$\Xi^i : B^i \setminus \bigcup_{j=1}^{N^i-1} \overline{B^{i,j}} \to \mathbb{S}^2 \setminus \{b^{i,1}, \ldots, b^{i,N^i}\}$$

which extends to a Lipschitz map

$$\overline{\Xi}_i : \overline{B^i} \setminus \bigcup_{j=1}^{N^i-1} B^{i,j} \to \mathbb{S}^2$$

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such that

$$\overline{\Xi}_i[\partial B^{i,j}] = b^{i,j} \text{ whenever } j \in \{1, \dots, N^i - 1\}, \qquad \overline{\Xi}_i[\partial B^i] = b^{i,N^i}.$$

Moreover, for all $i \in \{1, \dots, N\}$,

$$\vec{f}(x) = (\vec{\Phi}^i \circ \Xi^i)(x) \qquad \text{whenever } x \in B^i \setminus \bigcup_{j=1}^{N^i-1} \overline{B^{i,j}} \tag{3.15}$$

and for all $j \in \{1, \dots, N^i\}$ there exists $p^{i,j} \in \mathbb{R}^3$ such that

$$\vec{f}(x) = p^{i,j} \qquad \text{whenever } x \in B^{i,j} \setminus \bigcup_{i' \in J^{i,j}} \overline{B^{i'}} \tag{3.16}$$

where $J^{i,j} = \{i' : \overline{B^{i'}} \subset B^{i,j}\}$.

Finally, we define

$$\mathcal{W}(\vec{T}) := \sum_{i=1}^{N} \int_{\mathbb{S}^2} H_{\vec{\Phi}^i}^2 \, d\mu_{\vec{\Phi}^i}, \qquad \mathrm{Area}(\vec{T}) := \sum_{i=1}^{N} \int_{\mathbb{S}^2} 1 \, d\mu_{\vec{\Phi}^i},$$

$$\mathrm{Vol}(\vec{T}) := \sum_{i=1}^{N} \int_{\mathbb{S}^2} \vec{n}_{\vec{\Phi}^i} \cdot \vec{\Phi}^i \, d\mu_{\vec{\Phi}^i}.$$

The next theorem establishes the weak closure of bubble trees, as well as the convergence of the constraints in the Helfrich problem and the lower semi-continuity of the Willmore energy. The proof builds on top of [MR14].

3.3 Theorem (Weak closure and lower semi-continuity of bubble trees)**.** *Suppose $\vec{T}_k = (\vec{f}_k, \vec{\Phi}_k^1, \dots, \vec{\Phi}_k^{N_k})$ is a sequence of bubble trees of weak immersions and*

$$\limsup_{k \to \infty} \sum_{i=1}^{N_k} \int_{\mathbb{S}^2} 1 + |d\vec{n}_{\vec{\Phi}_k^i}|^2 \, d\mu_{\vec{\Phi}_k^i} < \infty, \qquad \liminf_{k \to \infty} \sum_{i=1}^{N_k} \mathrm{diam}\, \vec{\Phi}_k^i[\mathbb{S}^2] > 0. \tag{3.17}$$

Then, there exists a subsequence of $\vec{T}_k$ which we again denote by $\vec{T}_k$ such that $N_k = N$ for some positive integer N and there exists a sequence of diffeomorphisms Ψ_k of $\mathbb{S}^2$ such that

$$\vec{f}_k \circ \Psi_k \to \vec{u}_\infty \qquad \text{as } k \to \infty \text{ uniformly in } C^0(\mathbb{S}^2, \mathbb{R}^3),$$
$$\mathrm{Area}\, \vec{f}_k[\mathbb{S}^2] \to \mathrm{Area}\, \vec{u}_\infty[\mathbb{S}^2] \qquad \text{as } k \to \infty$$

for some $\vec{u}_\infty \in W^{1,\infty}(\mathbb{S}^2, \mathbb{R}^3)$. Moreover, for all $i \in \{1, \dots, N\}$ there exists a positive integer Q^i and sequences $f_k^{i,1}, \dots, f_k^{i,Q^i}$ of positive conformal diffeomorphisms of $\mathbb{S}^2$ such that for each $j \in \{1, \dots, Q^i\}$ there exist finitely many points $b^{i,j,1}, \dots b^{i,j,Q^{i,j}} \in \mathbb{S}^2$ with

$$\vec{\Phi}_k^i \circ f_k^{i,j} \rightharpoonup \vec{\xi}_\infty^{i,j} \qquad \text{as } k \to \infty \text{ weakly in } W_{\mathrm{loc}}^{2,2}(\mathbb{S}^2 \setminus \{b^{i,j,1}, \dots b^{i,j,Q^{i,j}}\}, \mathbb{R}^3) \tag{3.18}$$

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for some branched Lipschitz conformal immersion $\vec{\xi}_\infty^{i,j} \in \mathcal{F}_{\mathbb{S}^2}$. Furthermore,

$$\vec{T}_\infty := \big(\vec{u}_\infty, (\vec{\xi}_\infty^{1,j})_{j=1,\ldots,Q^1}, \ldots, (\vec{\xi}_\infty^{N,j})_{j=1,\ldots,Q^N}\big)$$

is a bubble tree of weak immersions and

$$\mathcal{W}(\vec{T}_\infty) \leqslant \liminf_{k\to\infty} \mathcal{W}(\vec{T}_k), \quad \mathrm{Area}(\vec{T}_\infty) = \lim_{k\to\infty} \mathrm{Area}(\vec{T}_k), \quad \mathrm{Vol}(\vec{T}_\infty) = \lim_{k\to\infty} \mathrm{Vol}(\vec{T}_k)$$

as well as

$$\sum_{i=1}^{N} \sum_{j=1}^{Q^i} \int_{\mathbb{S}^2} H_{\vec{\xi}_\infty^{i,j}} \, d\mu_{\vec{\xi}_\infty^{i,j}} = \lim_{k\to\infty} \sum_{i=1}^{N} \int_{\mathbb{S}^2} H_{\vec{\Phi}_k^i} \, d\mu_{\vec{\Phi}_k^i}.$$

Proof. We first consider the special case where $N_k = 1$ for all positive integers k. By [MR14, Theorem 1.5], it then only remains to show the convergence properties of the Willmore energy $\mathcal{W}$, the volume, and the integral of the mean curvature. In view of Lemma 3.1, we can add Equations (3.5)–(3.7) for $\vec{\xi}_k$ replaced by $\vec{\Phi}_k \circ f_k^i$ to the conclusion of the Domain Decomposition Lemma [MR14, Theorem 6.1]. Therefore, adapting the proof of [MR14, Theorem 1.5], we get the following statement.

After passing to a subsequence and denoting $\vec{\Phi}_k := \vec{\Phi}_k^1$, there exists a positive integer N, sequences $f_k^1, \ldots, f_k^N$ of positive conformal diffeomorphisms of $\mathbb{S}^2$, and for each $i \in \{1, \ldots, N\}$ there exist points $b^{i,1}, \ldots, b^{i,N^i} \in \mathbb{S}^2$ such that (3.18) and (1.5) hold. Moreover, there exists a sequence of positive numbers s_k converging to zero such that for $i = 1, \ldots, N$ Equations (3.4)–(3.7) are satisfied for $\vec{\xi}_k$ replaced by $\vec{\Phi}_k \circ f_k^i$. Furthermore, defining

$$S_k^i := \mathbb{S}^2 \setminus \bigcup_{j=1}^{N^i} B_{s_k}(b^{i,j}),$$

and for $j = 1, \ldots, N^i$ the sets of indices

$$J^{i,j} := \{i' : \forall k \in \mathbb{N} : ((f_k^i)^{-1} \circ f_k^{i'})[S_k^{i'}] \subset B_{s_k}(b^{i,j})\},$$

and

$$\hat{J}^{i,j} := \{i' \in J^{i,j} : \forall k \in \mathbb{N} : \nexists i'' : ((f_k^i)^{-1} \circ f_k^{i'})[S_k^{i'}] \subset \mathrm{Conv}((f_k^i)^{-1} \circ f_k^{i''})[S_k^{i''}]\},$$

and the necks

$$S_k^{i,j} := B_{s_k}(b^{i,j}) \setminus \bigcup_{i' \in \hat{J}^{i,j}} ((f_k^i)^{-1} \circ f_k^{i'})[\mathbb{S}^2 \setminus B_{s_k}(b^{i',N^{i'}})],$$

there holds

$$\lim_{k\to\infty} \int_{S_k^{i,j}} 1 \, d\mu_{\vec{\Phi}_k \circ f_k^i} = 0, \qquad \lim_{k\to\infty} \mathrm{diam}(\vec{\Phi}_k \circ f_k^i)[S_k^{i,j}] = 0. \tag{3.19}$$

Finally, for any $\mu_{\vec{\Phi}_k}$ integrable Borel function φ on $\mathbb{S}^2$, we get

$$\int_{\mathbb{S}^2} \varphi \, d\mu_{\vec{\Phi}_k} = \sum_{i=1}^{N} \int_{\mathbb{S}^2 \setminus \bigcup_{j=1}^{N_i} B_{s_k}(b^{i,j})} \varphi \circ f_k^i \, d\mu_{\vec{\Phi}_k \circ f_k^i} + \sum_{i=1}^{N} \sum_{j=1}^{N^i-1} \int_{S_k^{i,j}} \varphi \circ f_k^i \, d\mu_{\vec{\Phi}_k \circ f_k^i}. \tag{3.20}$$

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We notice that by the strong convergence (1.5),

$$\sup_{k \in \mathbb{N}} \sup_{S_k^{i,j}} |\vec{\Phi}_k \circ f_k^i| \leqslant C < \infty$$

for some finite number $C > 0$. Hence, by Hölder's inequality

$$\left| \int_{S_k^{i,j}} H_{\vec{\Phi}_k \circ f_k^i} \, d\mu_{\vec{\Phi}_k \circ f_k^i} \right| \leqslant \left(\int_{S_k^{i,j}} 1 \, d\mu_{\vec{\Phi}_k \circ f_k^i} \right)^{1/2} \left(\int_{\mathbb{S}^2} H_{\vec{\Phi}_k \circ f_k^i}^2 \, d\mu_{\vec{\Phi}_k \circ f_k^i} \right)^{1/2},$$

$$\left| \int_{S_k^{i,j}} \vec{n}_{\vec{\Phi}_k \circ f_k^i} \cdot (\vec{\Phi}_k \circ f_k^i) \, d\mu_{\vec{\Phi}_k \circ f_k^i} \right| \leqslant \left(\int_{S_k^{i,j}} 1 \, d\mu_{\vec{\Phi}_k \circ f_k^i} \right)^{1/2} \left(\int_{S_k^{i,j}} C^2 \, d\mu_{\vec{\Phi}_k \circ f_k^i} \right)^{1/2}.$$

By (3.17) and (3.19), the right hand side of each line goes to zero as k tends to infinity. That means the last term of Equation (3.20) goes to zero as k tends to infinity when φ is replaced by $H_{\vec{\Phi}_k}$ as well as when φ is replaced by $\vec{n}_{\vec{\Phi}_k} \cdot \vec{\Phi}_k$. Therefore, using (3.5) and (3.6), we can conclude the convergence of the integrated mean curvature and the convergence of the volume from (3.20). Similarly, we can conclude the lower semi-continuity of the Willmore energy $\mathcal{W}$ from (3.20) by replacing φ with $H_{\vec{\Phi}_k}^2$, using super linearity of the limit inferior and by ignoring the non-negative second term in (3.20).

Now, the general case follows analogously to the proof of [MR14, Theorem 7.2]. $\square$

4 Regularity of minimisers

Throughout this section, Σ denotes a smooth, oriented, and closed 2-dimensional manifold. Moreover, c_0, α, and ρ are the parameters of the Canham-Helfrich energy, i.e. $c_0 \in \mathbb{R}$ and $\alpha, \rho \geqslant 0$, see (2.14). A (possibly branched) weak immersion $\vec{\Phi} \in \mathcal{F}_\Sigma$ with branch points $\{b_1, \ldots, b_N\}$ is called *weak Canham-Helfrich immersion* if

$$\left. \frac{d}{dt} \right|_{t=0} \mathcal{H}_{\alpha,\rho}^{c_0}(\vec{\Phi} + t\vec{\omega}) = 0 \tag{4.1}$$

for all $\vec{\omega} \in C_c^\infty(\Sigma \setminus \{b_1, \ldots, b_N\}, \mathbb{R}^3)$.

In the following, we will first compute the Canham-Helfrich equation in divergence form, see Lemma 4.1. Then, we will prove that a weak immersion satisfying the Canham-Helfrich equation is smooth away from its branch points, see Theorem 4.3. The proof is based on the regularity theory for weak Willmore immersions developed by Rivière [Riv08, Riv16]. An important step in Riviere's regularity theory is the discovery of hidden conservation laws for weak Willmore immersions. In the framework of Canham-Helfrich immersions, the corresponding hidden conservation laws were discovered by Bernard [Ber16].

4.1 Lemma (Canham-Helfrich Euler-Lagrange equation in divergence form)**.** *Suppose $\vec{\Phi} \in \mathcal{F}_\Sigma$ is a weak Canham-Helfrich immersion with branch points $\{b_1, \ldots, b_N\}$. Then, away from its branch points, i.e. in conformal parametrisations from the open unit disk D^2 into a subset of $\Sigma \setminus \bigcup_{i=1}^N B_\varepsilon(b_i)$ for any $\varepsilon > 0$, there holds*

$$\vec{\mathcal{W}} = -\operatorname{div}\left[c_0 \nabla \vec{n} + (2c_0 H - c_0^2 - \alpha)\nabla \vec{\Phi} - \frac{\rho}{2} \vec{\Phi} \times \nabla^\perp \vec{\Phi} \right] \tag{4.2}$$

in $\mathcal{D}'(D^2, \mathbb{R}^3)$, where

$$\vec{\mathcal{W}} := \operatorname{div} \frac{1}{2} \left[2\nabla \vec{H} - 3H\nabla \vec{n} + \vec{H} \times \nabla^\perp \vec{n} \right] \tag{4.3}$$

corresponds to the first variation of the Willmore energy.

Proof. After composing with a conformal chart away from the branch points, we may assume that $\vec{\Phi}$ is a map $D^2 \to \mathbb{R}^3$. Let $\vec{\omega} \in C_c^\infty(D, \mathbb{R}^3)$ and define $\vec{\Phi}_t := \vec{\Phi} + t\vec{\omega}$ for $t \in \mathbb{R}$. The conformal factor λ is given by $2e^{2\lambda} = |\nabla \vec{\Phi}|^2$ and the metric coefficients $(g_t)_{ij}$ by $(g_t)_{ij} = \partial_i \vec{\Phi}_t \cdot \partial_j \vec{\Phi}_t$. Standard computations (see for instance [Riv16, (7.8)–(7.10)]) give

$$\frac{d}{dt}\Big|_{t=0} (g_t)^{ij} = -e^{-4\lambda} (\partial_i \vec{\omega} \cdot \partial_j \vec{\Phi} + \partial_i \vec{\Phi} \cdot \partial_j \vec{\omega})$$

$$g^{ij} \left(\partial_i \frac{d}{dt}\Big|_{t=0} \vec{n}_t \cdot \partial_j \vec{\Phi} \right) = -e^{-2\lambda} (\partial_1(\partial_1 \vec{\omega} \cdot \vec{n}) + \partial_2(\partial_2 \vec{\omega} \cdot \vec{n}))$$

$$\frac{d}{dt}\Big|_{t=0} \sqrt{\det(g_t)_{ij}} = \partial_1 \vec{\Phi} \cdot \partial_1 \vec{\omega} + \partial_2 \vec{\Phi} \cdot \partial_2 \vec{\omega}.$$

Therefore, using

$$\frac{d}{dt}\Big|_{t=0} H_t = -\frac{d}{dt}\Big|_{t=0} \frac{1}{2} (g_t)^{ij} \left(\partial_i \vec{n}_t \cdot \partial_j \vec{\Phi}_t \right)$$

$$= -\frac{1}{2} \left(\frac{d}{dt}\Big|_{t=0} (g_t)^{ij} \right) \partial_i \vec{n} \cdot \partial_j \vec{\Phi} - g^{ij} \frac{1}{2} \left(\partial_i \frac{d}{dt}\Big|_{t=0} \vec{n}_t \cdot \partial_j \vec{\Phi} + \partial_i \vec{n} \cdot \partial_j \vec{\omega} \right),$$

we obtain

$$\frac{d}{dt}\Big|_{t=0} \int_{D^2} H_t \, d\mu_t = \int_{D^2} \left(\frac{d}{dt}\Big|_{t=0} H_t \right) \sqrt{\det g_{ij}} + H \frac{d}{dt}\Big|_{t=0} \sqrt{\det(g_t)_{ij}} \, d\mathcal{L}^2$$

$$= \int_{D^2} \frac{1}{2} e^{-2\lambda} \sum_{i,j=1}^{2} (\partial_i \vec{\omega} \cdot \partial_j \vec{\Phi} + \partial_i \vec{\Phi} \cdot \partial_j \vec{\omega})(\partial_i \vec{n} \cdot \partial_j \vec{\Phi})$$

$$+ \frac{1}{2} (\partial_1(\partial_1 \vec{\omega} \cdot \vec{n}) + \partial_2(\partial_2 \vec{\omega} \cdot \vec{n})) - \frac{1}{2} g^{ij} \partial_i \vec{n} \cdot \partial_j \vec{\omega} e^{2\lambda}$$

$$+ H(\partial_1 \vec{\Phi} \cdot \partial_1 \vec{\omega} + \partial_2 \vec{\Phi} \cdot \partial_2 \vec{\omega}) \, d\mathcal{L}^2. \tag{4.4}$$

Using that $\vec{\omega}$ has compact support in D^2,

$$g^{ij} \partial_i \vec{n} \cdot \partial_j \vec{\omega} e^{2\lambda} = e^{-2\lambda} \sum_{i,j=1}^{2} (\partial_i \vec{n} \cdot \partial_j \vec{\Phi})(\partial_j \vec{\Phi} \cdot \partial_i \vec{\omega}),$$

and using the symmetry of the second fundamental form, i.e. $\partial_i \vec{n} \cdot \partial_j \vec{\Phi} = \partial_j \vec{n} \cdot \partial_i \vec{\Phi}$, we

compute further

$$\frac{d}{dt}\Big|_{t=0} \int_{D^2} H_t\, d\mu_t = \int_{D^2} -\frac{1}{2}\vec{\omega}\cdot\partial_1\Big[e^{-2\lambda}((\partial_1\vec{n}\cdot\partial_1\vec{\Phi})\partial_1\vec{\Phi} + (\partial_1\vec{n}\cdot\partial_2\vec{\Phi})\partial_2\vec{\Phi}))\Big]$$

$$-\frac{1}{2}\vec{\omega}\cdot\partial_2\Big[e^{-2\lambda}((\partial_2\vec{n}\cdot\partial_1\vec{\Phi})\partial_1\vec{\Phi} + (\partial_2\vec{n}\cdot\partial_2\vec{\Phi})\partial_2\vec{\Phi}))\Big]$$

$$-\vec{\omega}\cdot(\partial_1(H\partial_1\vec{\Phi}) + \partial_2(H\partial_2\vec{\Phi}))\,d\mathscr{L}^2$$

$$= -\int_{D^2}\vec{\omega}\cdot\partial_1\left(\frac{1}{2}\pi_T(\partial_1\vec{n}) + H\partial_1\vec{\Phi}\right)$$

$$+\vec{\omega}\cdot\partial_2\left(\frac{1}{2}\pi_T(\partial_2\vec{n}) + H\partial_2\vec{\Phi}\right)\,d\mathscr{L}^2 \tag{4.5}$$

which gives

$$\frac{d}{dt}\Big|_{t=0} \int_{D^2} H_t\, d\mu_t = -\int_{D^2}\vec{\omega}\cdot\mathrm{div}\left[\frac{1}{2}\nabla\vec{n} + H\nabla\vec{\Phi}\right]\,d\mathscr{L}^2. \tag{4.6}$$

From [Riv16, Corollary 7.3] we know

$$\frac{d}{dt}\Big|_{t=0} \int_{D^2} H_t^2\, d\mu_t = \int_{D^2}\vec{\omega}\cdot\mathrm{div}\,\frac{1}{2}\left[2\nabla\vec{H} - 3H\nabla\vec{n} + \vec{H}\times\nabla^\perp\vec{n}\right]\,d\mathscr{L}^2 \tag{4.7}$$

see also [Riv08]. Moreover (see for instance [Ber16, Chapter 3.3])

$$\frac{d}{dt}\Big|_{t=0} \int_{D^2} 1\, d\mu_t = -\int_{D^2}\vec{\omega}\cdot\mathrm{div}\,\nabla\vec{\Phi}\,d\mathscr{L}^2 \tag{4.8}$$

and

$$\frac{d}{dt}\Big|_{t=0} \int_{D^2} \vec{n}_t\cdot\vec{\Phi}_t\, d\mu_t = -\int_{D^2}\vec{\omega}\cdot\mathrm{div}\left[\frac{1}{2}\vec{\Phi}\times\nabla^\perp\vec{\Phi}\right]\,d\mathscr{L}^2. \tag{4.9}$$

Putting (4.6) – (4.9) into (4.1) yields (4.2). $\qquad\square$

4.2 Remark. Usually in the literature (see for instance [Ber16, Chapter 3.3]) one finds the expression of the first variation for $\int H d\mu_g$ written as

$$\frac{d}{dt}\Big|_{t=0} \int_{D^2} H_t\, d\mu_t = \int_{D^2}(\vec{\omega}\cdot\vec{n})\left(\frac{1}{2}\mathbb{I}_j^i\mathbb{I}_i^j - 2H^2\right)d\mu_g. \tag{4.10}$$

It is not hard to check the equivalence of (4.10) with (4.6) proved above. The advantage of the expression (4.6) is two fold: first it invokes less regularity of the immersion map $\vec{\Phi}$, second it is already in divergence form. Both advantages will be useful in establishing the regularity of weak Canham-Helfrich immersions: indeed, (4.10) would correspond to an L^1 term in the Euler-Lagrange equation (which is usually a problematic right hand side for elliptic regularity theory) while (4.6) corresponds to the divergence of an L^2 term (which is a much better right hand side in elliptic regularity).

4.3 Theorem (Smoothness of weak Canham-Helfrich immersions). *Suppose $\vec{\Phi}\in\mathcal{F}_\Sigma$ is a weak Canham-Helfrich immersion. Then $\vec{\Phi}$ is a C^∞ immersion away from the branch points.*

Proof. After composing with a conformal chart of Σ away from the branch points onto the unit disk D^2, we may assume that $\vec{\Phi}$ is a map $D^2 \to \mathbb{R}^3$ without branch points and $\vec{\Phi}$ satisfies the Canham-Helfrich equation (4.2). It is enough to show that $\vec{\Phi} \in C^\infty(B_{1/2}(0))$. The proof splits into three parts.

Step 1: Conservation laws. In view of the Canham-Helfrich equation (4.2), we define [1] $\vec{T} \in L^2(D^2, (\mathbb{R}^3)^2)$ by letting

$$\vec{T} := c_0 \nabla \vec{n} + (2c_0 H - c_0^2 - \alpha)\nabla\vec{\Phi} - \frac{\rho}{2}\vec{\Phi} \times \nabla^\perp\vec{\Phi}.$$

Then, $\operatorname{div}\vec{T} = -\vec{\mathcal{W}}$ where $\vec{\mathcal{W}}$ is as in (4.3). Hence $\vec{\mathcal{W}} \in H^{-1}(D^2, \mathbb{R}^3)$ and there exists a solution $\vec{V}$ of

$$\begin{cases} \Delta\vec{V} = -\vec{\mathcal{W}} \\ \vec{V} \in H_0^1(D^2, \mathbb{R}^3). \end{cases}$$

Therefore, we can find

$$\vec{X} \in W^{2,2}(D^2, \mathbb{R}^3), \qquad Y \in W^{2,2}(D^2, \mathbb{R}) \tag{4.11}$$

such that

$$\begin{cases} \Delta\vec{X} = \nabla\vec{V} \times \nabla\vec{\Phi} & \text{in } D^2 \\ \vec{X} = 0 & \text{on } \partial D^2 \end{cases}$$

and

$$\begin{cases} \Delta Y = \nabla\vec{V} \cdot \nabla\vec{\Phi} & \text{in } D^2 \\ Y = 0 & \text{on } \partial D^2. \end{cases}$$

After the breakthrough of Rivière [Riv08], Bernard [Ber16, Chapter 2.2] showed that by invariance of the Willmore functional under conformal transformation and the weak Poincaré Lemma, one can find potentials $\vec{L}, \vec{R} \in W^{1,2}(D^2, \mathbb{R}^3)$, and $S \in W^{1,2}(D^2, \mathbb{R})$ such that

$$\begin{cases} \nabla^\perp\vec{L} = \vec{T} - \nabla\vec{V} \\ \nabla^\perp\vec{R} = \vec{L} \times \nabla^\perp\vec{\Phi} - \vec{H} \times \nabla\vec{\Phi} - \nabla\vec{X} \\ \nabla^\perp S = \langle\vec{L}, \nabla^\perp\vec{\Phi}\rangle - \nabla Y. \end{cases}$$

Indeed, from [Ber16, Chapter 3.3] we find that $\vec{R}, S, \vec{X}, Y$, and $\vec{\Phi}$ satisfy the following system of conservation laws

$$\begin{cases} \Delta\vec{R} = \langle\nabla^\perp\vec{n}, \nabla S\rangle + \nabla^\perp\vec{n} \times \nabla\vec{R} + \operatorname{div}\left[\langle\vec{n}, \nabla Y\rangle + \frac{\rho}{4}|\vec{\Phi}|^2\nabla\vec{\Phi}\right] & (4.12) \\[2mm] \Delta S = \nabla^\perp\vec{n} \cdot \nabla\vec{R} & (4.13) \\[2mm] \Delta Y = |\nabla\vec{\Phi}|^2\left(-(c_0^2 + \alpha) + c_0 H + \frac{\rho}{2}\vec{\Phi} \cdot \vec{n}\right) & (4.14) \\[2mm] \Delta\vec{\Phi} = -\langle\nabla^\perp S, \nabla\vec{\Phi}\rangle - \nabla^\perp\vec{R} \times \nabla\vec{\Phi} + \langle\nabla\vec{\Phi}, \nabla Y\rangle + \frac{\rho}{4}|\vec{\Phi}|^2|\nabla\vec{\Phi}|^2\vec{n}. & (4.15) \end{cases}$$

---

[1] Note that here, $\vec{T}$ is not a bubble tree.

Step 2: Morrey decrease. We will show that for some number $\alpha > 0$, there holds

$$\sup_{r < 1/4,\, a \in B_{1/2}(0)} r^{-\alpha} \int_{B_r(a)} |\nabla \vec{R}|^2 + |\nabla S|^2 \, d\mathcal{L}^2 < \infty. \tag{4.16}$$

We let $\varepsilon_0 > 0$ and fix its value later. Choose $0 < r_0 < 1/4$ such that

$$\sup_{a \in B_{1/2}(0)} \int_{B_{r_0}(a)} |\nabla \vec{n}|^2 \, d\mathcal{L}^2 < \varepsilon_0. \tag{4.17}$$

Let a be any point in $B_{1/2}(0)$. Denote by $\vec{R}_0$ the solution of

$$\begin{cases} \Delta \vec{R}_0 = \mathrm{div}\Big[\langle \vec{n}, \nabla Y \rangle + \dfrac{\rho}{4} |\vec{\Phi}|^2 \nabla \vec{\Phi} \Big] & \text{in } D^2 \\ \vec{R}_0 = 0 & \text{on } \partial D^2. \end{cases}$$

Then, from (4.11) we obtain $\vec{R}_0 \in W^{2,2}(B_1(0), \mathbb{R}^3)$ and hence $\nabla \vec{R}_0 \in L^p(B_1(0), (\mathbb{R}^3)^2)$ for any $1 \leqslant p < \infty$. Therefore, by Hölder's inequality

$$\int_{B_r(a)} |\nabla \vec{R}_0|^2 \, d\mathcal{L}^2 \leqslant r \boldsymbol{\alpha}(2)^{1/2} \Big(\int_{B_1(0)} |\nabla \vec{R}_0|^4 \, d\mathcal{L}^2 \Big)^{1/2} =: r C_1 \tag{4.18}$$

whenever $0 < r \leqslant r_0$ where $\boldsymbol{\alpha}(2)$ is the area of the unit disk. Let $0 < r \leqslant r_0$ and let $\vec{\Psi}_{\vec{R}}$ and Ψ_S be the solutions of

$$\begin{cases} \Delta \vec{\Psi}_{\vec{R}} = \langle \nabla^\perp \vec{n}, \nabla S \rangle + \nabla^\perp \vec{n} \times \nabla \vec{R} & \text{in } B_r(a) \\ \vec{\Psi}_{\vec{R}} = 0 & \text{on } \partial B_r(a) \end{cases}$$

and

$$\begin{cases} \Delta \Psi_S = \nabla^\perp \vec{n} \cdot \nabla \vec{R} & \text{in } B_r(a) \\ \Psi_S = 0 & \text{on } \partial B_r(a). \end{cases}$$

Then, the maps

$$\vec{\nu}_{\vec{R}} := \vec{R} - \vec{R}_0 - \vec{\Psi}_{\vec{R}}, \qquad \nu_S := S - \Psi_S$$

are harmonic and satisfy

$$\vec{\nu}_{\vec{R}} = \vec{R} - \vec{R}_0, \quad \vec{\nu}_S = S \qquad \text{on } \partial B_r(a).$$

Therefore, by monotonicity (see for instance [Riv16, Lemma 7.10]), the Dirichlet principle, and (4.18)

$$\begin{aligned} \int_{B_{r/3}(a)} |\nabla \vec{\nu}_{\vec{R}}|^2 + |\nabla \nu_S|^2 \, d\mathcal{L}^2 &\leqslant \frac{1}{9} \int_{B_r(a)} |\nabla(\vec{R} - \vec{R}_0)|^2 + |\nabla S|^2 \, d\mathcal{L}^2 \\ &\leqslant \frac{2}{9} \int_{B_r(a)} |\nabla \vec{R}|^2 + |\nabla S|^2 \, d\mathcal{L}^2 + \frac{2}{9} r C_1. \end{aligned} \tag{4.19}$$

By Wente's theorem (see for instance [Riv16, Theorem 3.7]) and the definition of r_0 (4.17) we find

$$
\int_{B_r(a)} |\nabla \vec{\Psi}_{\vec{R}}|^2 + |\nabla \Psi_S|^2 \, d\mathscr{L}^2
$$

$$
\leqslant C_2 \int_{B_{r_0}(a)} |\nabla \vec{n}|^2 \, d\mathscr{L}^2 \int_{B_r(a)} |\nabla \vec{R}|^2 + |\nabla S|^2 \, d\mathscr{L}^2 \tag{4.20}
$$

$$
\leqslant C_2 \varepsilon_0 \int_{B_r(a)} |\nabla \vec{R}|^2 + |\nabla S|^2 \, d\mathscr{L}^2
$$

for some constant $0 < C_2 < \infty$ independent of $r, r_0, \vec{\Psi}_{\vec{R}}, \Psi_S$. Using the inequalities (4.18)–(4.20) we compute

$$
\int_{B_{r/3}(a)} |\nabla \vec{R}|^2 + |\nabla S|^2 \, d\mathscr{L}^2
$$

$$
\leqslant 3 \int_{B_{r/3}(a)} |\nabla \vec{\Psi}_{\vec{R}}|^2 + |\nabla \Psi_S|^2 \, d\mathscr{L}^2
$$

$$
+ 3 \int_{B_{r/3}(a)} |\nabla \vec{\nu}_{\vec{R}}|^2 + |\nabla \nu_S|^2 \, d\mathscr{L}^2 + 3 \int_{B_{r/3}(a)} |\nabla \vec{R}_0|^2 \, d\mathscr{L}^2
$$

$$
\leqslant (3 C_2 \varepsilon_0 + 6/9) \int_{B_r(a)} |\nabla \vec{R}|^2 + |\nabla S|^2 \, d\mathscr{L}^2 + \frac{6}{9} r C_1 + r C_1 .
$$

Therefore, taking $\varepsilon_0 = (3 C_2 9)^{-1}$ yields

$$
\int_{B_{r/3}(a)} |\nabla \vec{R}|^2 + |\nabla S|^2 \, d\mathscr{L}^2 \leqslant \frac{7}{9} \int_{B_r(a)} |\nabla \vec{R}|^2 + |\nabla S|^2 \, d\mathscr{L}^2 + r 2 C_1 \tag{4.21}
$$

for all $0 < r \leqslant r_0$. We next show by induction that

$$
\int_{B_{3^{-n} r_0}(a)} |\nabla \vec{R}|^2 + |\nabla S|^2 \, d\mathscr{L}^2
$$

$$
\leqslant \left(\frac{7}{9}\right)^n \int_{B_{r_0}(a)} |\nabla \vec{R}|^2 + |\nabla S|^2 \, d\mathscr{L}^2 + r_0 2 C_1 \sum_{i=1}^n 3^{-i+1} \left(\frac{7}{9}\right)^{n-i} \tag{4.22}
$$

for all $n \in \mathbb{N}$. Indeed, letting

$$
A(s) := \int_{B_s(a)} |\nabla \vec{R}|^2 + |\nabla S|^2 \, d\mathscr{L}^2 \qquad \text{for } 0 < s \leqslant r_0
$$

we have from (4.21) that $A(r_0/3^1) \leqslant \left(\frac{7}{9}\right)^1 A(r_0) + r_0 2 C_1 \sum_{i=1}^1 3^{-i+1} \left(\frac{7}{9}\right)^{1-i}$. Assuming (4.22) to be true for some integer n, we get from (4.21) that

$$
A(r_0/3^{n+1}) \leqslant \frac{7}{9} A(r_0/3^n) + r_0 3^{-n} 2 C_1
$$

$$
\leqslant \frac{7}{9} \left[\left(\frac{7}{9}\right)^n A(r_0) + r_0 2 C_1 \sum_{i=1}^n 3^{-i+1} \left(\frac{7}{9}\right)^{n-i} \right] + r_0 3^{-n} 2 C_1
$$

$$
\leqslant \left(\frac{7}{9}\right)^{n+1} A(r_0) + r_0 2 C_1 \sum_{i=1}^{n+1} 3^{-i+1} \left(\frac{7}{9}\right)^{n+1-i} .
$$

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Thus, by induction, (4.22) holds true for all $n \in \mathbb{N}$. Since

$$2C_1 \sum_{i=1}^{n} 3^{-i+1} \left(\frac{7}{9}\right)^{n-i} \leqslant \left(\frac{7}{9}\right)^n 2C_1 3 \sum_{i=1}^{n} \frac{9^i}{3^i 7^i} \leqslant \left(\frac{7}{9}\right)^n 12 C_1,$$

it follows that

$$\fint_{B_{3^{-n} r_0}(a)} |\nabla \vec{R}|^2 + |\nabla S|^2 \, d\mathscr{L}^2 \leqslant \left(\frac{r_0}{3^n}\right)^\alpha C_0$$

for $\alpha = \log_3(9/7)$ and

$$C_0 = r_0^{-\alpha} \left(\fint_{B_1(0)} |\nabla \vec{R}|^2 + |\nabla S|^2 \, d\mathscr{L}^2 + 12 C_1 \right)$$

which implies (4.16) as C_0 and α are independent of a. From (4.12), (4.13), the definition of $\vec{R}_0$, and Hölder's inequality it follows

$$\sup_{r<1/4,\, a\in B_{1/2}(0)} r^{-\alpha/2} \int_{B_r(a)} |\Delta(\vec{R} - \vec{R}_0)| + |\Delta S| \, d\mathscr{L}^2 < \infty$$

and hence, by a classical estimate on Riesz potentials [Ada75],

$$\nabla(\vec{R} - \vec{R}_0) \in L^p_{\mathrm{loc}}(B_{1/2}(0), (\mathbb{R}^3)^2), \qquad \nabla S \in L^p_{\mathrm{loc}}(B_{1/2}(0), \mathbb{R}^2)$$

for some $p > 2$. Since $\nabla \vec{R}_0 \in L^q(B_{1/2}(0), (\mathbb{R}^3)^2)$ for all $1 \leqslant q < \infty$, we obtain

$$\nabla \vec{R} \in L^p_{\mathrm{loc}}(B_{1/2}(0), (\mathbb{R}^3)^2), \qquad \nabla S \in L^p_{\mathrm{loc}}(B_{1/2}(0), \mathbb{R}^2). \tag{4.23}$$

Step 3: Bootstrapping. Putting (4.11) and (4.23) into (4.15), we infer

$$\nabla \vec{n} \in L^p_{\mathrm{loc}}(B_{1/2}(0), (\mathbb{R}^3)^2),$$

for some $p > 2$ given in the previous step. By Hölder's inequality and (4.12), (4.13) we first get

$$|\Delta(\vec{R} - \vec{R}_0)| \in L^q_{\mathrm{loc}}(B_{1/2}(0)), \qquad |\Delta S| \in L^q_{\mathrm{loc}}(B_{1/2}(0))$$

for $q := p/2 > 1$ and then, by Sobolev embedding,

$$\nabla(\vec{R} - \vec{R}_0) \in L^{q^*}_{\mathrm{loc}}(B_{1/2}(0), (\mathbb{R}^3)^2), \qquad \nabla S \in L^{q^*}_{\mathrm{loc}}(B_{1/2}(0), \mathbb{R}^2)$$

where $q^* := 2q/(2-q) = 2p/(4-p)$ satisfies $q^* > 2q = p$ as $p > 2$.
Since $\nabla \vec{R}_0 \in L^q_{\mathrm{loc}}(B_{1/2}(0)$ for all $1 \leqslant q < \infty$, we infer

$$\nabla \vec{R} \in L^{q^*}_{\mathrm{loc}}(B_{1/2}(0), (\mathbb{R}^3)^2), \qquad \nabla S \in L^{q^*}_{\mathrm{loc}}(B_{1/2}(0), \mathbb{R}^2).$$

Notice that q^* as above induces a recursively defined sequence of real numbers. Given a starting point $q_0 > 1$, this sequence is unbounded as $q^* > 2q$. Hence, we can repeat this procedure to obtain

$$\nabla \vec{R} \in L^q_{\mathrm{loc}}(B_{1/2}(0), (\mathbb{R}^3)^2), \quad \nabla S \in L^q_{\mathrm{loc}}(B_{1/2}(0), \mathbb{R}^2) \qquad \text{for all } 1 \leqslant q < \infty.$$

Therefore, from the system of conservation laws (4.12)–(4.15) we get step by step for all $1 \leqslant q < \infty$

$$\vec{\Phi} \in W^{2,q}_{\text{loc}}(B_{1/2}(0), \mathbb{R}^3), \qquad \nabla \vec{n} \in L^q_{\text{loc}}(B_{1/2}(0), (\mathbb{R}^3)^2), \qquad Y \in W^{2,q}_{\text{loc}}(B_{1/2}(0), \mathbb{R}),$$
$$\vec{R} \in W^{2,q}_{\text{loc}}(B_{1/2}(0), \mathbb{R}^3), \qquad S \in W^{2,q}_{\text{loc}}(B_{1/2}(0), \mathbb{R}).$$

Iteration gives

$$\vec{\Phi} \in W^{k,p}_{\text{loc}}(B_{1/2}(0), \mathbb{R}^3) \qquad \text{for all } k \in \mathbb{N},\ 1 \leqslant p < \infty$$

and hence,

$$\vec{\Phi} \in C^\infty(B_{1/2}(0))$$

which finishes the proof. □

Let $A_0, V_0 > 0$ satisfy the isoperimetric inequality: $A_0^3 \geqslant 36\pi V_0^2$ and let

$$\mathcal{F}_{A_0, V_0} := \mathcal{F} \cap \{\vec{\Phi} : \text{Area } \vec{\Phi} = A_0,\ \text{Vol } \vec{\Phi} = V_0\}$$

be the family of weak (possibly branched) immersions with area A_0 and enclosed volume V_0. Using the scaling invariance of the Willmore energy (which implies the equivalence between a scale invariant isoperimetric-ratio constraint versus a double constraint on enclosed volume and area), from [Sch12, Lemma 2.1] we get that

$$\inf_{\vec{\Phi} \in \mathcal{F}_{A_0, V_0}} \int_{\mathbb{S}^2} H^2 \, d\mu_{\vec{\Phi}} < 8\pi.$$

Define

$$\varepsilon_{4.4}(A_0, V_0) := \frac{\sqrt{8\pi} - \sqrt{\inf_{\vec{\Phi} \in \mathcal{F}_{A_0, V_0}} \int_{\mathbb{S}^2} H^2 \, d\mu_{\vec{\Phi}}}}{2\sqrt{A_0}} \in \left(0, (\sqrt{2} - 1)\sqrt{\pi}/\sqrt{A_0}\right], \qquad (4.24)$$

where the upper bound is given by the Willmore Theorem [Wil93, Theorem 7.2.2], see also (2.13). Notice that $\varepsilon_{4.4}(A_0, V_0)$ depends continuously on A_0 and V_0. Indeed, from [Sch12, Theorem 1.1], $\inf_{\vec{\Phi} \in \mathcal{F}_{A_0, V_0}} \int_{\mathbb{S}^2} H^2 \, d\mu_{\vec{\Phi}}$ is a continuous function of the isoperimetric ratio.

4.4 Lemma. *Let $A_0 > 0, V_0 > 0$ satisfy the isoperimetric inequality: $A_0^3 \geqslant 36\pi V_0^2$ and let $\varepsilon_{4.4} = \varepsilon_{4.4}(A_0, V_0)$ be defined as in (4.24). Then, for any $c_0 \in (-\varepsilon_{4.4}, \varepsilon_{4.4})$, the following holds.*
Any minimizing sequence $\vec{\Phi}_k$ of $\inf_{\vec{\Phi} \in \mathcal{F}_{A_0, V_0}} \int_{\mathbb{S}^2} (H - c_0)^2 \, d\mu_{\vec{\Phi}}$ satisfies

$$\limsup_{k \to +\infty} \int_{\mathbb{S}^2} H^2_{\vec{\Phi}_k} \, d\mu_{\vec{\Phi}_k} < 8\pi.$$

Proof. From the Cauchy-Schwartz inequality, we have $|\int_{\mathbb{S}^2} H \, d\mu_{\vec{\Phi}}| \leqslant \sqrt{\int_{\mathbb{S}^2} H^2 \, d\mu_{\vec{\Phi}}} \sqrt{\text{Area}(\vec{\Phi})}$. Thus:

$$\left(\sqrt{\int_{\mathbb{S}^2} H^2 \, d\mu_{\vec{\Phi}}} - |c_0| \sqrt{\text{Area}(\vec{\Phi})} \right)^2 \leqslant \int_{\mathbb{S}^2} (H - c_0)^2 \, d\mu_{\vec{\Phi}} \leqslant \left(\sqrt{\int_{\mathbb{S}^2} H^2 \, d\mu_{\vec{\Phi}}} + |c_0| \sqrt{\text{Area}(\vec{\Phi})} \right)^2$$

which yields

$$\left| \sqrt{\int_{\mathbb{S}^2} H^2 \, d\mu_{\vec{\Phi}}} - \sqrt{\int_{\mathbb{S}^2} (H - c_0)^2 \, d\mu_{\vec{\Phi}}} \right| \leqslant |c_0| \sqrt{\text{Area}(\vec{\Phi})}. \tag{4.25}$$

In particular, we deduce that

$$\left| \inf_{\vec{\Phi} \in \mathcal{F}_{A_0, V_0}} \sqrt{\int_{\mathbb{S}^2} H^2 \, d\mu_{\vec{\Phi}}} - \inf_{\vec{\Phi} \in \mathcal{F}_{A_0, V_0}} \sqrt{\int_{\mathbb{S}^2} (H - c_0)^2 \, d\mu_{\vec{\Phi}}} \right| \leqslant |c_0| \sqrt{A_0}. \tag{4.26}$$

Let $c_0 \in (-\varepsilon_{4.4}(A_0, V_0), \varepsilon_{4.4}(A_0, V_0))$ and let $\vec{\Phi}_k$ be a minimizing sequence of $\inf_{\vec{\Phi} \in \mathcal{F}_{A_0, V_0}} \int_{\mathbb{S}^2} (H - c_0)^2 \, d\mu_{\vec{\Phi}}$.
For k large enough it holds

$$\sqrt{\int_{\mathbb{S}^2} (H_k - c_0)^2 \, d\mu_{\vec{\Phi}_k}} \leqslant \sqrt{\inf_{\vec{\Phi} \in \mathcal{F}_{A_0, V_0}} \int_{\mathbb{S}^2} (H - c_0)^2 \, d\mu_{\vec{\Phi}}} + (\varepsilon_{4.4}(A_0, V_0) - |c_0|) \sqrt{A_0}. \tag{4.27}$$

Combining (4.25), (4.27), and (4.26), we get

$$\begin{aligned}
\sqrt{\int_{\mathbb{S}^2} H_k^2 \, d\mu_{\vec{\Phi}_k}} &\leqslant \sqrt{\int_{\mathbb{S}^2} (H_k - c_0)^2 \, d\mu_{\vec{\Phi}_k}} + |c_0| \sqrt{A_0} \\
&\leqslant \sqrt{\inf_{\vec{\Phi} \in \mathcal{F}_{A_0, V_0}} \int_{\mathbb{S}^2} (H - c_0)^2 \, d\mu_{\vec{\Phi}}} + (\varepsilon_{4.4}(A_0, V_0) - |c_0|) \sqrt{A_0} + |c_0| \sqrt{A_0} \\
&< \sqrt{\inf_{\vec{\Phi} \in \mathcal{F}_{A_0, V_0}} \int_{\vec{\Phi}_k} H^2 \, d\mu_{\vec{\Phi}}} + 2\varepsilon_{4.4}(A_0, V_0) \sqrt{A_0} = \sqrt{8\pi},
\end{aligned}$$

where in the last identity we plugged in the definition of $\varepsilon_{4.4}(A_0, V_0)$ as in (4.24). $\qquad\square$

References

[Ada75]   D. R. Adams. A note on Riesz potentials. *Duke Math. J.*, 42(4):765–778, 1975.

[BDMP93] G. Bellettini, G. Dal Maso, and M. Paolini. Semicontinuity and relaxation properties of a curvature depending functional in 2D. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 20(2):247–297, 1993.

[Ber16]   Y. Bernard. Noether's theorem and the Willmore functional. *Adv. Calc. Var.*, 9(3):217–234, 2016.

[BK03]   M. Bauer and E. Kuwert. Existence of minimizing willmore surfaces of prescribed genus. *Int. Math. Res. Not.*, 2003(10):553–576, 2003.

[Bla55]   W. Blaschke. *Vorlesungen über Integralgeometrie*. Deutscher Verlag der Wissenschaften, Berlin, 1955. 3te Aufl.

[BWW17]   Y. Bernard, G. Wheeler, and V. M. Wheeler. Rigidity and stability of spheres in the Helfrich model. *Interfaces Free Bound.*, 19(4):495–523, 2017.

[Can70]   P. B. Canham. The minimum energy of bending as a possible explanation of the biconcave shape of the human red blood cell. *Journal of theoretical biology*, 26(1):61–81, 1970.

[CL14]   J. Chen and Y. Li. Bubble tree of branched conformal immersions and applications to the willmore functional. *Amer. J. Math.*, 136(4):1107–1154, 2014.

[CV13]   R. Choksi and M. Veneroni. Global minimizers for the doubly-constrained Helfrich energy: the axisymmetric case. *Calc. Var. Partial Differential Equations*, 48(3-4):337–366, 2013.

[Dal18]   J. Dalphin. Uniform ball property and existence of optimal shapes for a wide class of geometric functionals. *Interfaces Free Bound.*, 20(2):211–260, 2018.

[DH76]   H. J. Deuling and W. Helfrich. Red blood cell shapes as explained on the basis of curvature elasticity. *Biophysical journal*, 16(8):861–868, 1976.

[DSL99]   H. G. Döbereiner, O. Selchow, and R. Lipowsky. Spontaneous curvature of fluid vesicles induced by trans-bilayer sugar asymmetry. *European Biophysics journal*, 28(2):174–178, 1999.

[EF72]   E. Evans and Y. C. Fung. Improved measurements of the erythrocyte geometry. *Microvascular research*, 4(4):335–347, 1972.

[GB93]   K. Große-Brauckmann. New surfaces of constant mean curvature. *Math. Z.*, 214(4):527–565, 1993.

[Ger21]   S. Germain. *Recherches sur la théorie des surfaces élastiques*. V. Courcier, 1821.

[Heb99]   E. Hebey. *Nonlinear analysis on manifolds: Sobolev spaces and inequalities*, volume 5 of *Courant Lecture Notes in Mathematics*. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1999.

[Hel73]   W. Helfrich. Elastic properties of lipid bilayers: theory and possible experiments. *Zeitschrift für Naturforschung C*, 28(11-12):693–703, 1973.

[Hél02]   F. Hélein. *Harmonic maps, conservation laws and moving frames*, volume 150. Tracts in Mathematics, Cambridge, 2002.

[Hop83]   H. Hopf. *Differential geometry in the large*, volume 1000 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1983. Notes taken by Peter Lax and John Gray, With a preface by S. S. Chern.

[Jos06]   J. Jost. *Compact Riemann surfaces. An introduction to contemporary mathematics. Third edition.* Springer-Verlag, Berlin, 2006.

[Kir50]   G. Kirchhoff. Über das Gleichgewicht und die Bewegung einer elastischen Scheibe. *J. Reine Angew. Math.*, 40:51–88, 1850.

[KL12]   E. Kuwert and Y. Li. $W^{2,2}$-conformal immersions of a closed Riemann surface into $\mathbb{R}^n$. *Comm. Anal. Geom.*, 20(2):313–340, 2012.

[KMR14]   L. G. A. Keller, A. Mondino, and T. Rivière. Embedded surfaces of arbitrary genus minimizing the Willmore energy under isoperimetric constraint. *Arch. Ration. Mech. Anal.*, 212(2):645–682, 2014.

[KS04]   E. Kuwert and R. Schätzle. Removability of point singularities of Willmore surfaces. *Ann. of Math. (2)*, 160(1):315–357, 2004.

[Kus96]   R. Kusner. Estimates for the biharmonic energy on unbounded planar domains, and the existence of surfaces of every genus that minimize the squared-mean-curvature integral. *Elliptic and parabolic methods in geometry (Minneapolis, MN, 1994)*, pages 67–72, 1996.

[LY82]   P. Li and S. T. Yau. A new conformal invariant and its applications to the Willmore conjecture and the first eigenvalue of compact surfaces. *Invent. Math.*, 69(2):269–291, 1982.

[MH90]   M. Mutz and W. Helfrich. Bending rigidities of some biological model membranes as obtained from the fourier analysis of contour sections. *Journal de Physique*, 51(10):991–1001, 1990.

[MR14]   A. Mondino and T. Rivière. Immersed spheres of finite total curvature into manifolds. *Adv. Calc. Var.*, 7(4):493–538, 2014.

[MR208]   Mini-Workshop: Mathematics of Biological Membranes. *Oberwolfach Rep.*, 5(3):2293–2336, 2008. Abstracts from the mini-workshop held August 31–September 6, 2008, Organized by Harald Garcke, Barbara Niethammer, Mark A. Peletier and Matthias Röger, Oberwolfach Reports. Vol. 5, no. 3.

[MŠ95]   S. Müller and V. Šverák. On surfaces of finite total curvature. *Journal of Differential Geometry*, 42(5):229–258, 1995.

[Poi14]   S. D. Poisson. Mémoire sur les surfaces élastiques. *Mémoires de la classe des sciences mathématiques et physiques de l'Institut National de France*, pages 167–225, 1814.

[Riv08]   T. Rivière. Analysis aspects of Willmore surfaces. *Invent. Math.*, 174(1):1–45, 2008.

[Riv14]   T. Rivière. Variational principles for immersed surfaces with L^2-bounded second fundamental form. *J. Reine Angew. Math.*, 695:41–98, 2014.

[Riv16]   T. Rivière. Weak immersions of surfaces with L^2-bounded second fundamental form. In *Geometric analysis*, volume 22 of *IAS/Park City Math. Ser.*, pages 303–384. Amer. Math. Soc., Providence, RI, 2016.

[SBL91]   U. Seifert, K. Berndl, and R. Lipowsky. Shape transformations of vesicles: Phase diagram for spontaneous-curvature and bilayer-coupling models. *Physical Review A*, 44(2):1182, 1991.

[Sch12]   J. Schygulla. Willmore minimizers with prescribed isoperimetric ratio. *Arch. Ration. Mech. Anal.*, 203(3):901–941, 2012.

[Sei97]   U. Seifert. Configurations of fluid membranes and vesicles. *Advances in physics*, 46(1):13–137, 1997.

[Sim83]   L. Simon. *Lectures on geometric measure theory*, volume 3 of *Proceedings of the Centre for Mathematical Analysis, Australian National University*. Australian National University Centre for Mathematical Analysis, Canberra, 1983.

[Sim86]   L. Simon. Existence of Willmore surfaces. In *Miniconference on geometry and partial differential equations (Canberra, 1985)*, volume 10 of *Proc. Centre Math. Anal. Austral. Nat. Univ.*, pages 187–216. Austral. Nat. Univ., Canberra, 1986.

[Wil65]   T. J. Willmore. Note on embedded surfaces. *An. Şti. Univ. "Al. I. Cuza" Iaşi Secţ. I a Mat. (N.S.)*, 11B:493–496, 1965.

[Wil82]   T. J. Willmore. *Total curvature in Riemannian geometry*. Ellis Horwood Series: Mathematics and its Applications. Ellis Horwood Ltd., Chichester; Halsted Press [John Wiley & Sons, Inc.], New York, 1982.

[Wil93]   T. J. Willmore. *Riemannian geometry*. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1993.