THE LOCAL STRUCTURE OF THE FREE BOUNDARY IN THE FRACTIONAL OBSTACLE PROBLEM

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Abstract. Building upon the recent results in [15] we provide a thorough description of the free boundary for solutions to the fractional obstacle problem in $\mathbb{R}^{n+1}$ with obstacle function φ (suitably smooth and decaying fast at infinity) up to sets of null $\mathcal{H}^{n-1}$ measure. In particular, if φ is analytic, the problem reduces to the zero obstacle case dealt with in [15] and therefore we retrieve the same results:

(i) local finiteness of the $(n-1)$-dimensional Minkowski content of the free boundary (and thus of its Hausdorff measure),

(ii) $\mathcal{H}^{n-1}$-rectifiability of the free boundary,

(iii) classification of the frequencies and of the blow-ups up to a set of Hausdorff dimension at most $(n-2)$ in the free boundary.

Instead, if $\varphi \in C^{k+1}(\mathbb{R}^n)$, $k \geq 2$, similar results hold only for distinguished subsets of points in the free boundary where the order of contact of the solution with the obstacle function φ is less than $k+1$.

1. Introduction

Quasi-geostrophic flow models [10], anomalous diffusion in disordered media [4] and American options with jump processes [11] are some instances of constrained variational problems involving free boundaries for thin obstacle problems. In this paper we analyze the fractional obstacle problem with exponent $s \in (0,1)$, a problem that can be stated in several ways, each motivated by a different application and suited to be studied with different techniques. We follow here the variational approach: given $\varphi : \mathbb{R}^n \to \mathbb{R}$ smooth and decaying sufficiently fast at infinity, one seeks for minimizers of the H^s-seminorm

$$[v]^2_{H^s} := \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|v(x') - v(y')|^2}{|x' - y'|^{n+2s}} \mathrm{d}x' \, \mathrm{d}y'$$

$s \in (0,1)$, on the cone

$$\mathscr{A} := \Big\{ v \in \dot{H}^s(\mathbb{R}^n) : \, v(x') \geq \varphi(x') \Big\},$$

where $\dot{H}^s(\mathbb{R}^n)$ is the homogeneous space defined as the closure in the H^s seminorm of $C^\infty_c(\mathbb{R}^n)$ functions. Existence and uniqueness of a minimizer w follow for all $s \in (0,1)$ if $n \geq 2$ (the case $n = 1$ requires some care see [26] and [3]). In addition, defining the fractional laplacian as

$$(-\Delta)^s v(x') := c_{n,s} \, \mathrm{P.V.} \int_{\mathbb{R}^n} \frac{v(x') - v(y')}{|x' - y'|^{n+2s}} \mathrm{d}y',$$

for $v \in \dot{H}^s(\mathbb{R}^n)$, the Euler-Lagrange conditions characterize w as a distributional solution to the system of inequalities

$$\begin{cases} w(x') \geq \varphi(x') & \text{for } \ x' \in \mathbb{R}^n, \\ (-\Delta)^s w(x') = 0 & \text{for } \ w(x') > \varphi(x'), \\ (-\Delta)^s w(x') \geq 0 & \text{for } \ x' \in \mathbb{R}^n \,. \end{cases} \tag{1.1}$$

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The most challenging regularity issues are then that of w itself and that of its free boundary

$$\Gamma_\varphi(w) := \partial\big\{x' \in \mathbb{R}^n : w(x') = \varphi(x')\big\}.$$

To investigate the fine properties of the solution w of (1.1) the groundbreaking paper by Caffarelli and Silvestre [8] introduces an equivalent local counterpart for the fractional obstacle problem in terms of the so called a-harmonic extension argument. Indeed, it is inspired by the case $s = 1/2$, in which it is nothing but the harmonic extension problem. More precisely, setting $a = 1 - 2s$ for $s \in (0,1)$ and $\mathfrak{m} = |x_{n+1}|^a \mathcal{L}^{n+1}$, it turns out that any function w satisfying (1.1) is the trace of a function $u \in H^1(\mathbb{R}^{n+1}, \mathrm{d}\mathfrak{m})$ solving for $x = (x', x_{n+1}) \in \mathbb{R}^{n+1}$

$$\begin{cases} u(x',0) \geq \varphi & \text{for } (x',0) \in \mathbb{R}^n \times \{0\},\\ u(x',x_{n+1}) = u(x',-x_{n+1}) & \text{for all } x \in \mathbb{R}^{n+1},\\ \mathrm{div}\big(|x_{n+1}|^a \nabla u(x)\big) = 0 & \text{for } x \in \mathbb{R}^{n+1} \setminus \big\{(x',0) : u(x',0) = \varphi(x')\big\},\\ \mathrm{div}\big(|x_{n+1}|^a \nabla u(x)\big) \leq 0 & \text{in } \mathscr{D}'(\mathbb{R}^{n+1}). \end{cases} \tag{1.2}$$

In particular, note that u is unique minimizer of the Dirichlet energy

$$\int_{\mathbb{R}^{n+1}} |\nabla \widetilde{v}|^2 |x_{n+1}|^a \mathrm{d}x$$

on the class $\widetilde{\mathscr{A}} := \big\{\widetilde{v} \in H^1(\mathbb{R}^{n+1}, \mathrm{d}\mathfrak{m}) : \widetilde{v}(x',0) \geq \varphi(x')\big\}$. Viceversa, the trace $u(x',0)$ on the hyperplane $\{x_{n+1} = 0\}$ of a solution u to (1.2) is a solution w to (1.1), as for all $x' \in \mathbb{R}^n$ (cf. [8])

$$\lim_{x_{n+1}\to 0^+} |x_{n+1}|^a \partial_{n+1} u(x) = -(-\triangle)^s u(x',0)\,.$$

One then is interested into regularity issues for u and for the corresponding free boundary $\Gamma_\varphi(u)$ (with a slight abuse of notation we use the same symbol as for the analogous set for w): the topological boundary, in the relative topology of $\mathbb{R}^n$, of the coincidence set of a solution u

$$\Lambda_\varphi(u) := \big\{(x',0) \in \mathbb{R}^{n+1} : u(x',0) = \varphi(x')\big\}.$$

The locality of the operator

$$L_a(v) := \mathrm{div}\big(|x_{n+1}|^a \nabla v(x)\big) \tag{1.3}$$

in (3.1) is the main advantage of the new formulation to perform the analysis of $\Gamma_\varphi(u)$. Indeed, being $\Gamma_\varphi(u) = \Gamma_\varphi(w)$ it permits the use of monotonicity and almost monotonicity type formulas analogous to those introduced by Weiss and Monneau for the classical obstacle problem (cf. [5, 6, 27, 23]).

Optimal interior regularity for u has been established Caffarelli, Salsa and Silvestre in [9, Theorem 6.7 and Corollary 6.8] for any $s \in (0,1)$ (see also [7]). The particular case $s = 1/2$ had been previously addressed by Athanasopoulos, Caffarelli and Salsa in [1]. Instead, despite all the mentioned progresses, the current picture for free boundary regularity theory is still incomplete. In this paper we go further on in this direction and deal with the non-zero obstacle case following the recent achievements obtained in the zero-obstacle case in [15, 16]. Drawing a parallel with the theory in the zero-obstacle case, the free boundary $\Gamma_\varphi(u)$ can be split as a pairwise disjoint union of sets:

$$\Gamma_\varphi(u) = \mathrm{Reg}(u) \cup \mathrm{Sing}(u) \cup \mathrm{Other}(u), \tag{1.4}$$

termed in the existing literature as the subset of regular, singular and nonregular/nonsingular points, respectively. These sets are defined via the infinitesimal behaviour of appropriate rescalings of the solution itself. More precisely, for $x_0 \in \Gamma_\varphi(u)$ a function φ_{x_0} related to φ can be conveniently defined (cf. (3.46) and (3.48)) in a way that if

$$u_{x_0,r}(y) := \frac{r^{\frac{n+a}{2}}\big(u(x_0 + r\,y) - \varphi_{x_0}(x_0 + r\,y)\big)}{\Big(\int_{\partial B_r} (u - \varphi_{x_0})^2\, |x_{n+1}|^a d\mathcal{H}^n\Big)^{1/2}}\,,$$

then the family of functions $\{u_{x_0,r}\}_{r>0}$ is pre-compact in $H^1_{\mathrm{loc}}(\mathbb{R}^{n+1}, \mathrm{d}\mathfrak{m})$ (see [9, Section 6]). The limits are called blowups of u at x_0, they are homogeneous solutions of a fractional obstacle problem with zero obstacle. The set of all such functions is denoted by $\mathrm{BU}(x_0)$. Their homogeneity

$\lambda(x_0)$ depends only on the base point x_0 and not on the extracted subsequence, and it is called *infinitesimal homogeneity* or *frequency* of u at x_0. It is indeed the limit value, as the radius vanishes, of an Almgren's type frequency function related to u which turns out to be non decreasing in the radius. Given this, one defines

$$\mathrm{Reg}(u) := \{x \in \Gamma_\varphi(u) : \lambda(x_0) = 1+s\}, \quad \mathrm{Sing}(u) := \{x \in \Gamma_\varphi(u) : \lambda(x_0) = 2m,\, m \in \mathbb{N}\},$$
$$\mathrm{Other}(u) := \Gamma_\varphi(u) \setminus \Big(\mathrm{Reg}(u) \cup \mathrm{Sing}(u)\Big).$$

According to the regularity of φ different results are known in literature:

(i) Regular points: in [9] for $\varphi \in C^{2,1}(\mathbb{R}^n)$ optimal one-sided $C^{1,s}$ regularity of solutions is established. Moreover, $\mathrm{Reg}(u)$ is shown to be locally a $C^{1,\alpha}$ submanifolf of codimension 2 in $\mathbb{R}^{n+1}$ (non-optimal regularity of the solution had been previously established in [26]);

(ii) Singular points: for φ analytic and $a = 0$ it is proved in [17] that $\mathrm{Sing}(u)$ is $(n-1)$-rectifiable. The latter result has been very recently extended to the full range $a \in (-1, 1)$ and to $\varphi \in C^{k+1}(\mathbb{R}^n)$, $k \geq 2$, in [18]. Furthermore, fine properties of the singular set have been studied very recently by Fernández-Real and Jhaveri [13].

It is also worth mentioning the paper by Barrios, Figalli and Ros-Oton [3], in which the authors study the fractional obstacle problem (1.1) with non zero obstacle φ having compact support and satisfying suitable concavity assumptions. Under these assumptions, they are able to fully characterize the free boundary, showing that $\mathrm{Other}(u) = \emptyset$ and that at every point of $\mathrm{Sing}(u)$ the blowup is quadratic, *i.e.* the only admissible value of m is 1. In addition, they are able to show that the singular set $\mathrm{Sing}(u)$ is locally contained in a single C^1-regular submanifold (see also [7] for the case of less regular obstacles and [12, 20, 21, 22] for higher regularity results on $\mathrm{Reg}(u)$).

For ease of expositions we start with the simpler case in which the obstacle function φ is analytic, actually the slightly milder assumption (1.5) below suffices (see Section 3 for related results in the case $\varphi \in C^{k+1}(\mathbb{R}^n)$). Indeed, after a suitable transformation (see Section 2.1) such a framework reduces to the zero obstacle case since in this setting $\widetilde{\varphi}$ turns out to be exactly the a-harmonic extension of φ. Thus, in view of [15, Theorems 1.1-1.3] we may deduce the following result.

Theorem 1.1. *Let u be a solution to the fractional obstacle problem (1.2) with obstacle function $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that*

$$\{\varphi > 0\} \subset\subset \mathbb{R}^n, \quad \varphi \text{ is real analytic on } \{\varphi > 0\}. \tag{1.5}$$

Then,

(i) *the free boundary $\Gamma_\varphi(u)$ has finite $(n-1)$-dimensional Minkowski content: more precisely, there exists a constant $C > 0$ such that*
$$\mathcal{L}^{n+1}\big(\mathcal{T}_r(\Gamma_\varphi(u))\big) \leq C\, r^2 \quad \forall\, r \in (0,1), \tag{1.6}$$
where $\mathcal{T}_r(\Gamma_\varphi(u)) := \{x \in \mathbb{R}^{n+1} : \mathrm{dist}(x, \Gamma(u)) < r\}$;

(ii) *the free boundary $\Gamma_\varphi(u)$ is $(n-1)$-rectifiable, i.e. there exist at most countably many C^1-regular submanifolds $M_i \subset \mathbb{R}^n$ of dimension $n-1$ such that*
$$\mathcal{H}^{n-1}\big(\Gamma_\varphi(u) \setminus \cup_{i\in\mathbb{N}} M_i\big) = 0. \tag{1.7}$$

Moreover, there exists a subset $\Sigma(u) \subset \Gamma_\varphi(u)$ with Hausdorff dimension at most $n-2$ such that for every $x_0 \in \Gamma_\varphi(u) \setminus \Sigma(u)$ the infinitesimal homogeneity λ of u at x_0 belongs to $\{2m, 2m-1+s, 2m+2s\}_{m\in\mathbb{N}\setminus\{0\}}$.

The analysis is more involved in case φ is not analytic, since one cannot in principle avoid contact points of infinite order between the solution and the obstacle, and the free boundary can be locally an arbitrary compact set $K \subset \mathbb{R}^n$ (explicit examples are provided in [14]). In view of this, we follow the existing literature and we consider only those points in the free boundary in which u has order of contact with φ less than $k+1$: given u a solution to the fractional obstacle problem (1.2) and given a constant $\theta \in (0,1)$ we set

$$\Gamma_{\varphi,\theta}(u) := \big\{x_0 \in \Gamma_\varphi(u) : \liminf_{r\downarrow 0} r^{-(n+a+2(k+1-\theta))} H_{u_{x_0}}(r) > 0\big\}, \tag{1.8}$$

where $H_{u_{x_0}}$ is defined in (3.17) and it is related to the $L^2(\partial B_r, \mathrm{d}\mathfrak{m}')$ norm of $u_{x_0,r}$ (cf. Section 3 for more details). For this subset of points of the free boundary we can still prove some of the results stated in Theorem 1.1.

Theorem 1.2. *Let u be a solution to the fractional obstacle problem (1.2) with obstacle function $\varphi \in C^{k+1}(\mathbb{R}^n)$, $k \geq 2$, and let $\theta \in (0,1)$. Then, $\Gamma_{\varphi,\theta}(u)$ is $(n-1)$-rectifiable. Moreover, there exists a subset $\Sigma_\theta(u) \subset \Gamma_{\varphi,\theta}(u)$ with Hausdorff dimension at most $n-2$ such that for every $x_0 \in \Gamma_{\varphi,\theta}(u) \setminus \Sigma_\theta(u)$ the infinitesimal homogeneity λ of u at x_0 belongs to $\{2m, 2m-1+s, 2m+2s\}_{m\in\mathbb{N}\setminus\{0\}}$.*

This note extends the results of [15] to the case of nonconstant obstacles. It is clear by the examples of arbitrary compact sets as contact sets of suitable solutions of the problem, that in general the free boundary for nonconstant smooth obstacles does not possess any structure and that the key ingredient for the analysis of the free boundary is the *analiticity* of the obstacles as shown in Theorem 1.1. Nevertheless, for a subset of the free boundary, characterized as those points of finite order of contact, e.g. the points $\Gamma_{\varphi,\theta}$ already considered in the literature (see [18]), a partial regularity still holds even in the framework of non analytic obstacles, as proven in Theorem 1.2. The main novelty of this paper with respect to [15] consists in the analysis of the spatial dependence of the frequency for nonconstant obstacles: indeed, in this case the frequency is defined differently from point to point, by taking into account the geometry of the obstacle itself. It is not at all evident to which extent the oscillation of the frequency can be controlled. The results of Section 4 show that this kind of estimates are not completely rigid and extend to nonflat obstacles. Hence, this paper contributes to the program of broadening the results initially proven for the Signorini problem with zero obstacles to the case of the obstacle problem for the fractional Laplacian (see e.g. [1, 18]), providing a generalization of the known results on the structure of the free boundary firstly proven in [15].

2. Analytic obstacles

In this section we deal with analytic obstacles. We report first on some results related to the Caffarelli-Silvestre a-harmonic extension argument that will be instrumental to reduce the analytic type fractional obstacle problem to the lower dimensional obstacle problem. We provide then the proof of Theorem 1.1.

2.1. **Extension results.** We start off stating a lemma in which it is proved that there exists a canonical a-harmonic extension of a polynomial which is a polynomial itself (see [18, Lemma 5.2]). We denote by $\mathscr{P}_l(\mathbb{R}^n)$ the finite dimensional vector space of homogeneous polynomials of degree $l \in \mathbb{N}$.

Lemma 2.1. *For every $l \in \mathbb{N}$, there exists a unique linear extension operator $\mathscr{E}_l : \mathscr{P}_l(\mathbb{R}^n) \to \mathscr{P}_l(\mathbb{R}^{n+1})$ such that for every $p \in \mathscr{P}_l(\mathbb{R}^n)$ we have*

$$\begin{cases} -\mathrm{div}(|x_{n+1}|^a \nabla \mathscr{E}_l[p]) = 0 & \text{in } \mathscr{D}'(\mathbb{R}^{n+1}), \\ \mathscr{E}_l[p](x', 0) = p(x') & \text{for all } x' \in \mathbb{R}^n, \\ \mathscr{E}_l[p](x', -x_{n+1}) = \mathscr{E}_l[p](x', x_{n+1}) & \text{for all } x \in \mathbb{R}^{n+1}. \end{cases}$$

Proof. Let $p \in \mathscr{P}_l(\mathbb{R}^n)$ and set

$$\mathscr{E}_l[p](x', x_{n+1}) := \sum_{j=0}^{\lfloor l/2 \rfloor} p_{2j}(x')\, x_{n+1}^{2j},$$

with $p_{2j}(x') := -\frac{1}{4j(j-s)} \triangle p_{2j-2}(x')$ if $j \in \{1, \ldots, \lfloor l/2 \rfloor\}$ and $p_0 := p$. It is then easy to verify that $\mathscr{E}_l$ satisfies all the stated properties. □

Remark 2.2. In particular, $\mathscr{E}_l$ is a continuous operator, $l \in \mathbb{N}$. We will use in what follows that there exists a constant $C = C(n,l) > 0$ such that for every $p \in \mathscr{P}_l(\mathbb{R}^n)$ and for every $r > 0$

$$\|\mathscr{E}_l[p]\|_{L^\infty(B_r)} \leq C\, \|p\|_{L^\infty(B'_r)}. \tag{2.1}$$

We provide next the main result that reduces locally the analytic case to the zero obstacle case (cf. [18, Lemma 5.1]).

Lemma 2.3. *Let $\varphi : \Omega \to \mathbb{R}$ be analytic, $\Omega \subset \mathbb{R}^n$ open. Then for all $K \subset\subset \Omega \times \{0\}$ there exists $r > 0$ such that, for every $x_0 \in K$, there exists a function $\mathscr{E}_{x_0}[\varphi] : B_r(x_0) \to \mathbb{R}$ such that*

(i) $-\mathrm{div}(|x_{n+1}|^a \nabla \mathscr{E}_{x_0}[\varphi]) = 0$ *in* $\mathscr{D}'\big(B_r(x_0)\big)$;
(ii) $\mathscr{E}_{x_0}[\varphi](x', 0) = \varphi(x')$ $\forall$ $(x', 0) \in B_r(x_0)$;
(iii) $\mathscr{E}_{x_0}[\varphi]$ *is analytic in* $B_r(x_0)$.

Proof. For every x_0 as in the statement, we can locally expand φ in power series as $\varphi(x') = \sum_\alpha c_\alpha (x' - x_0)^\alpha$. Then, we set $\mathscr{E}_{x_0}[\varphi](x) := \sum_\alpha c_\alpha \mathscr{E}_{|\alpha|}[p_\alpha](x - x_0)$ where $p_\alpha(x') := (x')^\alpha$. From the explicit formulas in the proof of Lemma 2.1 it is easily verified that the power series defining $\mathscr{E}_{x_0}[\varphi]$ is converging in $B_r(x_0)$ and gives an analytic a-harmonic extension in $B_r(x_0)$ with $r > 0$ uniform on compact sets. □

2.2. **Proof of Theorem 1.1.** Theorem 1.1 follows straightforwardly from [15, Theorems 1.1-1.3]. As explained in the introduction $w(x') = u(x', 0)$ solves the fractional obstacle problem (1.1). By the maximum principle $u(x', 0) > 0$ for all $x' \in \mathbb{R}^n$. Therefore, $\Gamma_\varphi(u) \subset \{\varphi > 0\} \subset\subset \mathbb{R}^n$. Let $r > 0$ be the radius in Lemma 2.3 corresponding to the compact set $\Gamma_\varphi(u)$. By compactness we cover $\Gamma_\varphi(u)$ with a finite number of balls $B_r(x_i)$, with $x_i \in \mathbb{R}^n \times \{0\}$. In each ball $B_r(x_i)$ we consider the corresponding function $u - \mathscr{E}_{x_i}[\varphi]$, with $\mathscr{E}_{x_i}[\varphi]$ provided by Lemma 2.3, and note that it solves a zero lower dimensional obstacle problem (1.2). Hence, we can conclude by the quoted [15, Theorems 1.1-1.3].

3. C^{k+1} OBSTACLES

In this section we deal with the more demanding case of C^{k+1} obstacles, $k \geq 2$. It is convenient to reduce the analysis of (1.2) to that of the following localized problem

$$\begin{cases} u(x', 0) \geq \varphi(x') & \text{for } (x', 0) \in B_1', \\ u(x', x_{n+1}) = u(x', -x_{n+1}) & \text{for } x = (x', x_{n+1}) \in B_1, \\ \mathrm{div}\big(|x_{n+1}|^a \nabla u(x)\big) = 0 & \text{for } x \in B_1 \setminus \big\{(x', 0) \in B_1' : u(x', 0) = \varphi(x')\big\}, \\ \mathrm{div}\big(|x_{n+1}|^a \nabla u(x)\big) \leq 0 & \text{in } \mathscr{D}'(B_1), \end{cases} \tag{3.1}$$

for $\varphi \in C^{k+1}(B_1')$. In what follows, we shall assume that $\|\varphi\|_{C^{k+1}(B_1')} \leq 1$. This assumption can be easily matched by a simple scaling argument (cf. the proof of Theorem 1.2).

For any $x_0 \in B_1'$ we denote by $T_{k,x_0}[\varphi]$ the Taylor polynomial of φ of order k at x_0:

$$T_{k,x_0}[\varphi](x') := \sum_{|\alpha| \leq k} \frac{D^\alpha \varphi(x_0)}{\alpha!} p_\alpha(x' - x_0),$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $D^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$, $p_\alpha(x') := (x')^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $|\alpha| := \alpha_1 + \ldots + \alpha_n$ and $\alpha! := \alpha_1! \cdots \alpha_n!$. In what follows we will repeatedly use that (recall that $\|\varphi\|_{C^{k+1}(B_1')} \leq 1$)

$$|T_{k,x_0}[\varphi](x') - \varphi(x')| \leq \frac{1}{(k+1)!} |x' - x_0|^{k+1}, \tag{3.2}$$

and

$$|T_{k,x_0}[\partial_e \varphi](x') - \partial_e \varphi(x')| \leq 2|x' - x_0|^k \tag{3.3}$$

for all unit vectors $e \in \mathbb{R}^{n+1}$ such that $e \cdot e_{n+1} = 0$.

Let then $\mathscr{E}\big[T_{k,x_0}[\varphi]\big]$ be the a-harmonic extension of $T_{k,x_0}[\varphi]$, namely

$$\mathscr{E}\big[T_{k,x_0}[\varphi]\big](x) := \sum_{|\alpha| \leq k} \frac{D^\alpha \varphi(x_0)}{\alpha!} \mathscr{E}_{|\alpha|}[p_\alpha(\cdot - x_0)](x),$$

where $\mathscr{E}_l$ are the extension operators in Lemma 2.1. By the translation invariance of the operator, we point out that

$$\mathscr{E}_{|\alpha|}[p_\alpha(\cdot - x_0)](x) = \mathscr{E}_{|\alpha|}[p_\alpha](x - x_0). \tag{3.4}$$

Set

$$\varphi_{x_0}(x) := \varphi(x') - T_{k,x_0}[\varphi](x') + \mathscr{E}\big[T_{k,x_0}[\varphi]\big](x), \tag{3.5}$$

and

$$u_{x_0}(x) := u(x) - \varphi_{x_0}(x). \tag{3.6}$$

Recalling that $\mathscr{E}\big[T_{k,x_0}[\varphi]\big](x',0) = T_{k,x_0}[\varphi](x')$, then $\Lambda_\varphi(u) = \{(x',0) \in B_1' : u_{x_0}(x',0) = 0\}$, and thus in particular $\Gamma_\varphi(u) = \partial_{B_1'}\{(x',0) \in B_1' : u_{x_0}(x',0) = 0\}$, where $\partial_{B_1'}$ is the relative boundary in the hyperplane $\{x_{n+1} = 0\}$. We note that u_{x_0} is not a solution of a fractional obstacle problem as in (3.1) with null obstacle, but rather of a related obstacle problem with drift as discussed in what follows (cf. (3.14)).

First, from the regularity assumption on φ, from Lemma 2.1 and from estimate (3.2) we infer that $L_a(\varphi_{x_0})$ is a function in $L^1(B_1)$ (recall the definition of the operator L_a given in (1.3)). Moreover, estimate (3.2) gives for all $x \in B_1 \setminus B_1'$

$$\begin{aligned} |L_a(\varphi_{x_0}(x))| &= |\mathrm{div}\big(|x_{n+1}|^a \nabla(\varphi - T_{k,x_0}[\varphi])(x')\big)| \\ &= |x_{n+1}|^a |\triangle(\varphi(x') - T_{k,x_0}[\varphi](x'))| \le |x_{n+1}|^a |x' - x_0|^{k-1}. \end{aligned} \tag{3.7}$$

In turn, this yields that the distribution $L_a(u_{x_0})$ is given by the sum of a function in $L^1(B_1)$ and of a non-positive measure supported on B_1', namely,

$$L_a(u_{x_0}(x)) = \mathrm{div}\big(|x_{n+1}|^a \nabla u(x)\big) - L_a(\varphi_{x_0}(x))\mathcal{L}^n \llcorner B_1\,. \tag{3.8}$$

The following result resumes the regularity theory developed by Caffarelli, Salsa and Silvestre in [9, Proposition 4.3].

Theorem 3.1. *Let u be a solution to the fractional obstacle problem (3.1) in B_1, $x_0 \in B_r'$, $r \in (0,1)$, then $u_{x_0} \in C^{0,\min\{2s,1\}}(B_{1-r}(x_0))$, $\partial_{x_i} u_{x_0} \in C^{0,s}(B_{1-r}(x_0))$ for $i = 1,\dots,n$, and $|x_{n+1}|^a \partial_{x_{n+1}} u_{x_0} \in C^{0,\alpha}(B_{1-r}(x_0))$ for all $\alpha \in (0, 1-s)$. Moreover, there exists a constant $C_{3.1} = C_{3.1}(n,a,\alpha,r) > 0$ such that*

$$\begin{aligned} &\|u_{x_0}\|_{C^{0,\min\{2s,1\}}(B_{\frac{1-r}{2}}(x_0))} + \|\nabla' u_{x_0}\|_{C^{0,s}(B_{\frac{1-r}{2}}(x_0); R^n)} \\ &+ \|\mathrm{sign}(x_{n+1})\, |x_{n+1}|^a \partial_{x_{n+1}} u_{x_0}\|_{C^{0,\alpha}(B_{\frac{1-r}{2}}(x_0))} \le C_{3.1}\, \|u_{x_0}\|_{L^2(B_{1-r}(x_0), \mathrm{d}\mathfrak{m})}, \end{aligned} \tag{3.9}$$

where $\nabla' u_{x_0} = (\partial_{x_1} u_{x_0}, \dots, \partial_{x_n} u_{x_0})$ is the horizontal gradient.

In particular, the function u is analytic in $\{x_{n+1} > 0\}$ (see, *e.g.*, [19]) and the following boundary conditions holds:

$$\lim_{x_{n+1}\downarrow 0^+} x_{n+1}^a \partial_{n+1} u(x', x_{n+1}) = 0 \quad \text{for } \ x' \in B_1' \setminus \Lambda_\varphi(u), \tag{3.10}$$

$$\lim_{x_{n+1}\downarrow 0^+} x_{n+1}^a \partial_{n+1} u(x', x_{n+1}) \le 0 \quad \text{for } \ x' \in B_1'. \tag{3.11}$$

In particular,

$$(u(x',0) - \varphi(x')) \lim_{x_{n+1}\downarrow 0^+} x_{n+1}^a \partial_{n+1} u(x', x_{n+1}) = 0 \quad \text{for } \ x' \in B_1'. \tag{3.12}$$

Furthermore, for $B_r(x_0) \subset B_1$ and $x_0 \in B_1'$, an integration by parts implies that

$$\begin{aligned} &\int_{B_r(x_0)} |\nabla u|^2 |x_{n+1}|^a \mathrm{d}x - \int_{B_r(x_0)} |\nabla \varphi_{x_0}|^2 |x_{n+1}|^a \mathrm{d}x \\ &\quad = \int_{B_r(x_0)} |\nabla u_{x_0}|^2 |x_{n+1}|^a \mathrm{d}x + 2\int_{B_r(x_0)} \nabla u_{x_0} \cdot \nabla \varphi_{x_0} |x_{n+1}|^a \mathrm{d}x \\ &\quad = \int_{B_r(x_0)} |\nabla u_{x_0}|^2 |x_{n+1}|^a \mathrm{d}x - 2\int_{B_r(x_0)} u_{x_0} L_a(\varphi_{x_0})\, \mathrm{d}x + 2\int_{\partial B_r(x_0)} u_{x_0} \partial_\nu \varphi_{x_0} |x_{n+1}|^a \mathrm{d}x\,, \end{aligned} \tag{3.13}$$

where in the second equality we have used that $\mathscr{E}[T_{k,x_0}(\varphi)]$ is even with respect to the hyperplane $\{x_{n+1} = 0\}$ to deduce that

$$\lim_{x_{n+1}\to 0} \partial_{n+1}\varphi_{x_0}(x)|x_{n+1}|^a = 0.$$

In particular, since the last addend in (3.13) only depends on the boundary values of u_{x_0}, it follows that u_{x_0} is a minimizer of the functional

$$\int_{B_r(x_0)} |\nabla v|^2 |x_{n+1}|^a \mathrm{d}x - 2\int_{B_r(x_0)} v L_a(\varphi_{x_0})\,\mathrm{d}x \tag{3.14}$$

among all functions $v \in u_{x_0} + H^1_0(B_r(x_0), \mathrm{d}\mathfrak{m})$ and satisfying $v(x',0) \geq 0$ on $B'_r(x_0)$. Equivalently, we will say that u_{x_0} is a local minimizer of the functional in (3.14) subject to null obstacle conditions.

Remark 3.2. We record here some bounds that shall be employed extensively in what follows. By using the linearity and continuity of the extension operator $\mathscr{E}_k$ (cf. Remark 2.2), together with estimate (3.2) we get for all $z \in B_r$

$$\begin{aligned}
|u_{x_0}(z) - u_{x_1}(z)| &= |\varphi_{x_0}(z) - \varphi_{x_1}(z)| \\
&\leq \big|T_{k,x_0}[\varphi](z') - T_{k,x_1}[\varphi](z')\big| + \big|\mathscr{E}\big[T_{k,x_0}[\varphi]\big](z) - \mathscr{E}\big[T_{k,x_1}[\varphi]\big](z)\big| \\
&\leq \|T_{k,x_0}[\varphi] - T_{k,x_1}[\varphi]\|_{L^\infty(B'_r)} + \|\mathscr{E}\big[T_{k,x_0}[\varphi]\big] - \mathscr{E}\big[T_{k,x_1}[\varphi]\big]\|_{L^\infty(B_r)} \\
&\overset{(2.1)}{\leq} C\,\|T_{k,x_0}[\varphi] - T_{k,x_1}[\varphi]\|_{L^\infty(B'_r)} \leq C\big(\|\varphi - T_{k,x_0}[\varphi]\|_{L^\infty(B'_r)} + \|\varphi - T_{k,x_1}[\varphi]\|_{L^\infty(B'_r)}\big) \\
&\overset{(3.2)}{\leq} C\,\big(\max_{z\in B'_r} |z - x_0|^{k+1} + \max_{z\in B'_r} |z - x_1|^{k+1}\big)\,.
\end{aligned} \tag{3.15}$$

for some constant $C = C(n,a,k) > 0$. Since $\nabla\big(T_{k,x_i}[\varphi]\big) = T_{k-1,x_i}[\nabla\varphi]$, $i \in \{0,1\}$, arguing as above, using (3.3) rather than (3.2), we conclude that

$$\big|\nabla(u_{x_0}(z) - u_{x_1}(z))\big| = \big|\nabla(\varphi_{x_0}(z) - \varphi_{x_1}(z))\big| \leq C\,\big(\max_{z\in B'_r} |z - x_0|^k + \max_{z\in B'_r} |z - x_1|^k\big)\,, \tag{3.16}$$

for some constant $C = C(n,a,k) > 0$.

3.1. **A frequency type function.** Building upon the approach developed in [15] we consider a quantity strictly related to *Almgren's frequency function* and instrumental for developing the free boundary analysis in the subsequent sections. Let $\phi : [0,+\infty) \to [0,+\infty)$ be defined by

$$\phi(t) := \begin{cases} 1 & \text{for } 0 \leq t \leq \frac{1}{2}, \\ 2\,(1-t) & \text{for } \frac{1}{2} < t \leq 1, \\ 0 & \text{for } 1 < t, \end{cases}$$

then given the solution u to (3.1), a point $x_0 \in B'_1$ and the corresponding function u_{x_0} in (3.6), we define for all $0 < r < 1 - |x_0|$

$$I_{u_{x_0}}(r) := \frac{r\,G_{u_{x_0}}(r)}{H_{u_{x_0}}(r)}$$

where

$$G_{u_{x_0}}(r) := -\frac{1}{r}\int \dot{\phi}\Big(\tfrac{|x-x_0|}{r}\Big)\, u_{x_0}(x)\nabla u_{x_0}(x)\cdot \tfrac{x-x_0}{|x-x_0|}\,|x_{n+1}|^a\,\mathrm{d}x,$$

and

$$H_{u_{x_0}}(r) := -\int \dot{\phi}\Big(\tfrac{|x-x_0|}{r}\Big)\tfrac{u^2_{x_0}(x)}{|x-x_0|}\,|x_{n+1}|^a\,\mathrm{d}x. \tag{3.17}$$

Here $\dot{\phi}$ indicates the derivative of ϕ. Clearly, $I_{u_{x_0}}(r)$ is well-defined as long as $H_{u_{x_0}}(r) > 0$, in what follows when writing $I_{u_{x_0}}(r)$ we shall tacitly assume that the latter condition is satisfied.

For later convenience, we introduce also the notation

$$D_{u_{x_0}}(r) := \int \phi\big(\tfrac{|x-x_0|}{r}\big)\,|\nabla u_{x_0}(x)|^2\,|x_{n+1}|^a\mathrm{d}x,$$

and

$$E_{u_{x_0}}(r) := \int -\dot{\phi}\Big(\tfrac{|x-x_0|}{r}\Big)\tfrac{|x-x_0|}{r^2}\Big(\nabla u_{x_0}(x)\cdot\tfrac{x-x_0}{|x-x_0|}\Big)^2\,|x_{n+1}|^a\,\mathrm{d}x.$$

In particular, note that for all $r > 0$

$$H_{u_{x_0}}(r)\,E_{u_{x_0}}(r) - G^2_{u_{x_0}}(r) \geq 0 \tag{3.18}$$

by Cauchy-Schwarz inequality.

Remark 3.3. In case $\varphi = 0$, then $u_{x_0} = u$ for all $x_0 \in B_1'$ and $G_u = D_u$. Thus, $I_{u_{x_0}}$ boils down to the variant of Almgren's frequency function used in [15].

Remark 3.4. If u is a solution to the fractional obstacle problem (3.1), then for every $c > 0$, $x_0 \in B_1'$ and $r > 0$ such that $B_r(x_0) \subset B_1$, the function $\hat{u}(y) := c\,u(x_0 + r\,y)$ solves (3.1) on B_1 with obstacle function $\hat{\varphi}(y) := c\,\varphi(x_0 + r\,y)$. Therefore, if $x_1 = x_0 + ry_1 \in B_1'$ we have $T_{k,y_1}[\hat{\varphi}](y') = c\,T_{k,x_1}[\varphi](x_0 + ry')$ and $\hat{u}_{y_1}(y) = c\,u_{x_1}(x_0 + ry)$. Thus, $I_{\hat{u}_{y_1}}(\rho) = I_{u_{x_1}}(\rho\,r)$ for every $\rho \in (0,1)$.

In particular, this shows that the frequency function is scaling invariant, in the sequel we will use this property repeatedly.

3.2. **Almost monotonicity of $I_{u_{x_0}}$ at distinguished points.** In this subsection we prove the quais-monotonicity of $I_{u_{x_0}}$ for a suitable subset of points of the free boundary. We prove first some useful identities in a generic point x_0 of B_1'.

Lemma 3.5. *Let u be a solution to the fractional obstacle problem (3.1) in B_1. Then, for all $x_0 \in B_1'$ and $t \in (0, 1 - |x_0|)$, it holds*

$$D_{u_{x_0}}(t) = G_{u_{x_0}}(t) - \int \phi\big(\tfrac{|x-x_0|}{t}\big) u_{x_0}(x) L_a(u_{x_0}(x))\mathrm{d}x \tag{3.19}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(H_{u_{x_0}}(t)\big) = \frac{n+a}{t}\, H_{u_{x_0}}(t) + 2\, G_{u_{x_0}}(t) \tag{3.20}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(D_{u_{x_0}}(t)\big) = \frac{n+a-1}{t}\, D_{u_{x_0}}(t) + 2\, E_{u_{x_0}}(t) - \frac{2}{t}\int \phi\big(\tfrac{|x-x_0|}{t}\big) \nabla u_{x_0}(x)\cdot(x-x_0) L_a\big(u_{x_0}(x)\big)\mathrm{d}x. \tag{3.21}$$

Remark 3.6. With an abuse of notation, the integration in the last addends in (3.19) and (3.21) is meant with respect to the reference measure $L_a(u_{x_0})$. Actually, we use this notation because from the proofs of (3.19) and (3.21) it turns out that one can consider equivalently its absolutely continuous part $L_a(\varphi_{x_0})$.

Proof. To show (3.19), (3.20) and (3.21), we assume without loss of generality that $x_0 = \underline{0}$.

For (3.19) we consider the vector field $V(x) := \phi\big(\frac{|x|}{t}\big)\, u_{\underline{0}}(x)\, \nabla u_{\underline{0}}(x)\, |x_{n+1}|^a$. Clearly, V has compact support and $V \in C^\infty(B_1 \setminus B_1', \mathbb{R}^{n+1})$. Moreover, for $x_{n+1} \neq 0$

$$V(x)\cdot e_{n+1} = \phi\big(\tfrac{|x|}{t}\big)\, u_{\underline{0}}(x)\, \partial_{n+1} u_{\underline{0}}(x)\, |x_{n+1}|^a\,,$$

so that $\lim_{y \downarrow (x',0^+)} V(y)\cdot e_{n+1} = 0$. Indeed, recalling that $\mathscr{E}[T_{k,x_0}(\varphi)]$ is even with respect to the hyperplane $\{x_{n+1} = 0\}$ (cf. Lemma 2.1): if $(x',0) \in \Lambda_\varphi(u)$ we exploit the regularity of u resumed in Theorem 3.1 to conclude; instead, if $(x',0) \notin \Lambda_\varphi(u)$ it suffices to use (3.10). Thus, the distributional divergence of V is the $L^1(B_1)$ function given by

$$\begin{aligned}\operatorname{div} V(x) &= \phi\big(\tfrac{|x|}{t}\big)|\nabla u_{\underline{0}}(x)|^2\,|x_{n+1}|^a + \dot{\phi}\big(\tfrac{|x|}{t}\big) u_{\underline{0}}(x) \nabla u_{\underline{0}}(x)\cdot \tfrac{x}{t\,|x|}\,|x_{n+1}|^a \\ &\quad + \phi\big(\tfrac{|x|}{t}\big) u_{\underline{0}}(x) L_a(u_{\underline{0}}(x)).\end{aligned}$$

Therefore, (3.19) follows from the divergence theorem by taking into account that V is compactly supported.

Next, (3.20) is a consequence of (3.19) and the direct computation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big(H_{u_{\underline{0}}}(t)\big) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(-t^{n+a}\int \dot{\phi}(|y|)\,\frac{u_{\underline{0}}^2(t\,y)}{|y|}\,|y_{n+1}|^a\,\mathrm{d}y\right) \\ &= \frac{n+a}{t}\,H_{u_{\underline{0}}}(t) - 2\,t^{n+a}\int \dot{\phi}(|y|)\,u_{\underline{0}}(t\,y)\,\nabla u_{\underline{0}}(t\,y)\cdot\frac{y}{|y|}\,|y_{n+1}|^a\,\mathrm{d}y \\ &= \frac{n+a}{t}\,H_{u_{\underline{0}}}(t) + 2\,G_{u_{\underline{0}}}(t).\end{aligned}$$

Finally, to prove (3.21) we consider the compactly supported vector field $W \in C^\infty(B_1 \setminus B_1', \mathbb{R}^{n+1})$ defined by

$$W(x) = \Big(\frac{|\nabla u_{\underline{0}}(x)|^2}{2} x - (\nabla u_{\underline{0}}(x) \cdot x) \nabla u_{\underline{0}}(x) \Big) \phi\big(\tfrac{|x|}{t}\big) |x_{n+1}|^a .$$

Moreover, conditions (3.10)-(3.12) and Lemma 2.1 imply that $\lim_{y \downarrow (x',0)} W(y) \cdot e_{n+1} = 0$. Thus, $\operatorname{div} W$ has no singular part in B_1', and we can compute pointwise the distributional divergence as follows for $x_{n+1} \neq 0$

$$\begin{aligned} \operatorname{div} W(x) =& \dot\phi\big(\tfrac{|x|}{t}\big) \tfrac{x}{t\,|x|} \cdot \Big(\tfrac{|\nabla u_{\underline{0}}(x)|^2}{2} x - (\nabla u_{\underline{0}}(x) \cdot x) \nabla u_{\underline{0}}(x) \Big) |x_{n+1}|^a \\ &+ \phi\big(\tfrac{|x|}{t}\big) \tfrac{n+a-1}{2} |\nabla u_{\underline{0}}(x)|^2 |x_{n+1}|^a - \phi\big(\tfrac{|x|}{t}\big) (\nabla u_{\underline{0}}(x) \cdot x) L_a(u_{\underline{0}}(x)). \end{aligned}$$

Therefore, we infer that

$$\begin{aligned} 0 = \int \operatorname{div} W(x)\, \mathrm{d}x = & \int \dot\phi\Big(\tfrac{|x|}{t}\Big) \tfrac{|x|}{2\,t} |\nabla u_{\underline{0}}(x)|^2 |x_{n+1}|^a \, \mathrm{d}x + t\, E_{u_{\underline{0}}}(t) + \tfrac{n+a-1}{2} D_{u_{\underline{0}}}(t) \\ & - \int \phi\big(\tfrac{|x|}{t}\big) (\nabla u_{\underline{0}}(x) \cdot x) L_a\big(u_{\underline{0}}(x)\big) \mathrm{d}x, \end{aligned}$$

and we conclude (3.21) by direct differentiation since

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(D_{u_{\underline{0}}}(t)\big) = - \int \dot\phi\Big(\tfrac{|x|}{t}\Big) \tfrac{|x|}{t^2} |\nabla u_{\underline{0}}(x)|^2 |x_{n+1}|^a \, \mathrm{d}x. \qquad \square$$

As a consequence we derive a first monotonicity formula for $H_{u_{x_0}}$ in B_1.

Corollary 3.7. *Let u be a solution to the fractional obstacle problem (3.1). Then, for all $x_0 \in B_1'$ and $0 < r_0 < r_1 < 1 - |x_0|$ such that $H_{u_{x_0}}(t) > 0$ for all $t \in (r_0, r_1)$, we have*

$$\frac{H_{u_{x_0}}(r_1)}{r_1^{n+a}} = \frac{H_{u_{x_0}}(r_0)}{r_0^{n+a}}\, e^{2\int_{r_0}^{r_1} \frac{I_{u_{x_0}}(t)}{t} \mathrm{d}t}. \tag{3.22}$$

In particular, if $A_1 \leq I_{u_{x_0}}(t) \leq A_2$ for every $t \in (r_0, r_1)$, then

$$(r_0, r_1) \ni r \mapsto \frac{H_{u_{x_0}}(r)}{r^{n+a+2A_2}} \quad \textit{is monotone decreasing,} \tag{3.23}$$

$$(r_0, r_1) \ni r \mapsto \frac{H_{u_{x_0}}(r)}{r^{n+a+2A_1}} \quad \textit{is monotone increasing.} \tag{3.24}$$

Moreover, for all $x_0 \in B_1'$ and $0 < r < 1 - |x_0|$

$$\int_{B_r(x_0)} |u_{x_0}|^2 |x_{n+1}|^a \, \mathrm{d}x \leq r\, H_{u_{x_0}}(r). \tag{3.25}$$

Proof. The proof of (3.22) (and hence of (3.23) and (3.24)) follows from the differential equation in (3.20). The proof of (3.25) is a simple consequence of a dyadic integration argument:

$$\begin{aligned} \int_{B_r(x_0)} |u_{x_0}|^2 |x_{n+1}|^a \, \mathrm{d}x &= \sum_{j \in \mathbb{N}} \int_{B_{r/2^j}(x_0) \setminus B_{r/2^{j+1}}(x_0)} |u_{x_0}|^2 |x_{n+1}|^a \, \mathrm{d}x \\ &\leq \sum_{j \in \mathbb{N}} \frac{r}{2^j} H_{u_{x_0}}\big(r/2^j\big) \leq r\, H_{u_{x_0}}(r), \end{aligned}$$

where in the last inequality we used that $H_{u_{x_0}}(s) \leq H_{u_{x_0}}(r)$ for $s \leq r$ by (3.24) (with $A_1 = 0$). $\square$

We establish next an auxiliary lemma containing useful bounds for some quantities related to the L^2-norm of u_{x_0}, for points x_0 in the contact set.

Lemma 3.8. *Let u be a solution to the fractional obstacle problem (3.1) in B_1. Then, there is a positive constant $C_{3.8} = C_{3.8}(n, a) > 0$ such that for every point $x_0 \in \Lambda_\varphi(u)$ we have for all $r \in (0, 1 - |x_0|)$*

$$H_{u_{x_0}}(r) \leq C_{3.8} \big(r\, D_{u_{x_0}}(r) + r^{n+a+2(k+1)} \big), \tag{3.26}$$

$$\int \phi\big(\tfrac{|x-x_0|}{r}\big) |u_{x_0}(x)|^2 \, |x_{n+1}|^a \mathrm{d}x \leq C_{3.8} \big(r^2\, D_{u_{x_0}}(r) + r^{n+a+1+2(k+1)} \big), \tag{3.27}$$

and

$$\int_{B_r(x_0)\setminus B_{\frac{r}{2}}(x_0)} |u_{x_0}(x)|^2\,|x_{n+1}|^a \mathrm{d}x \le C_{3.8}\big(r^2\, D_{u_{x_0}}(r) + r^{n+a+1+2(k+1)}\big). \tag{3.28}$$

Proof. By the co-area formula for Lipschitz functions we check that

$$H_{u_{x_0}}(r) = 2\int_{\frac{r}{2}}^{r} \frac{\mathrm{d}t}{t}\int_{\partial B_t(x_0)} |u_{x_0}(x)|^2\,|x_{n+1}|^a\,\mathrm{d}\mathcal{H}^n(x), \tag{3.29}$$

and

$$\begin{aligned} D_{u_{x_0}}(r) = &\int_{B_{\frac{r}{2}}(x_0)} |\nabla u_{x_0}(x)|^2 |x_{n+1}|^a\,\mathrm{d}x \\ &+ \frac{2}{r}\int_{\frac{r}{2}}^{r}\mathrm{d}t\int_{\partial B_t(x_0)} (r-t)|\nabla u_{x_0}(x)|^2\,|x_{n+1}|^a\,\mathrm{d}\mathcal{H}^n(x). \end{aligned}$$

Therefore, an integration by parts gives

$$D_{u_{x_0}}(r) = \frac{2}{r}\int_{\frac{r}{2}}^{r}\mathrm{d}t\int_{B_t(x_0)} |\nabla u_{x_0}(x)|^2\,|x_{n+1}|^a\,\mathrm{d}x. \tag{3.30}$$

By (3.8), as $x_0 \in \Lambda_\varphi(u)$, [9, Lemma 2.13] and [18, Lemma 6.3] yield the Poincaré inequality

$$\frac{1}{t}\int_{\partial B_t(x_0)} |u_{x_0}(x)|^2\,|x_{n+1}|^a\,\mathrm{d}\mathcal{H}^n(x) \le C\int_{B_t(x_0)} |\nabla u_{x_0}(x)|^2\,|x_{n+1}|^a\,\mathrm{d}x + Ct^{n+a-1+2(k+1)}, \tag{3.31}$$

with $C = C(n,a) > 0$. Integrating the latter inequality on $(r/2, r)$ we find (3.26) in view of (3.30). Instead, by first multiplying formula (3.31) by t and then integrating over $(0,r)$, we infer

$$\int_{B_r(x_0)} |u_{x_0}(x)|^2\,|x_{n+1}|^a\,\mathrm{d}x \le C\,r^2\int_{B_r(x_0)} |\nabla u_{x_0}(x)|^2\,|x_{n+1}|^a\,\mathrm{d}x + Cr^{n+a+1+2(k+1)}.$$

In conclusion, (3.27) and (3.28) follow directly. □

Next we show an explicit expression for the radial derivative of $I_{u_{x_0}}$ at all points $x_0 \in B_1'$. We follow here [15, Proposition 2.7].

Proposition 3.9. *Let u be a solution to the fractional obstacle problem (3.1) in B_1. Then, if $x_0 \in B_1'$ is such that $H_{u_{x_0}}(t) > 0$ for all $t \in [r_0, r_1]$, we have*

$$I_{u_{x_0}}(r_1) - I_{u_{x_0}}(r_0) = \int_{r_0}^{r_1}\Big(\frac{2\,t}{H^2_{u_{x_0}}(t)}\big(H_{u_{x_0}}(t)\,E_{u_{x_0}}(t) - G^2_{u_{x_0}}(t)\big) + R_{u_{x_0}}(t)\Big)\,\mathrm{d}t \tag{3.32}$$

for $0 < r_0 < r_1 < 1 - |x_0|$, with

$$|R_{u_{x_0}}(t)| \le C_{3.9}\frac{t^{n+a+2k+1}}{H_{u_{x_0}}(t)}\Big(\Big(\frac{D_{u_{x_0}}(t)}{t^{n+a+2k+1}}\Big)^{1/2} + 1\Big) \tag{3.33}$$

and $C_{3.9} = C_{3.9}(n,a) > 0$.

Proof. It is not restrictive to assume $x_0 = \underline{0}$. We use the identities in (3.19), (3.20) and (3.21) to compute (the lenghty details are left to the reader)

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\big(I_{u_{\underline{0}}}(t)\big) &= I_{u_{\underline{0}}}(t)\Big(\frac{1}{t} + \frac{\frac{\mathrm{d}}{\mathrm{d}t}\big(G_{u_{\underline{0}}}(t)\big)}{G_{u_{\underline{0}}}(t)} - \frac{\frac{\mathrm{d}}{\mathrm{d}t}\big(H_{u_{\underline{0}}}(t)\big)}{H_{u_{\underline{0}}}(t)}\Big) \\ &= 2I_{u_{\underline{0}}}(t)\Big(\frac{E_{u_{\underline{0}}}(t)}{G_{u_{\underline{0}}}(t)} - \frac{G_{u_{\underline{0}}}(t)}{H_{u_{\underline{0}}}(t)}\Big) + R_{u_{\underline{0}}}(t), \end{aligned}$$

where

$$\begin{aligned} R_{u_{\underline{0}}}(t) := &-\frac{1}{H_{u_{\underline{0}}}(t)}\int \phi\big(\tfrac{|x|}{t}\big)\Big((n+a-1)u_{\underline{0}}(x) + 2(\nabla u_{\underline{0}}(x)\cdot x)\Big)L_a(u_{\underline{0}}(x))\mathrm{d}x \\ &+ \frac{1}{t}\int \dot\phi\Big(\tfrac{|x|}{t}\Big)|x|u(x)L_a(u_{\underline{0}}(x))\mathrm{d}x\,. \end{aligned}$$

From this we conclude (3.32) straightforwardly.

For (3.33), we estimate separately each term appearing in the integral defining $R_{u_{\underline{0}}}(t)$. We start with

$$
\begin{aligned}
\left|\int \phi(\tfrac{|x|}{t}) u_{\underline{0}}(x) L_a(u_{\underline{0}}(x)) \mathrm{d}x\right| &\overset{(3.8)}{\leq} \left|\int \phi(\tfrac{|x|}{t}) |u_{\underline{0}}(x)| |x'|^{k-1} |x_{n+1}|^a \mathrm{d}x\right| \\
&\leq t^{k-1} \left(\int_{B_t} |x_{n+1}|^a \mathrm{d}x\right)^{1/2} \left(\int \phi(\tfrac{|x|}{t}) |u_{\underline{0}}(x)|^2 |x_{n+1}|^a \mathrm{d}x\right)^{1/2} \\
&\leq C t^{\frac{n+a+1}{2}+k-1} \left(\int \phi(\tfrac{|x|}{t}) |u_{\underline{0}}(x)|^2 |x_{n+1}|^a \mathrm{d}x\right)^{1/2} \\
&\overset{(3.27)}{\leq} C t^{\frac{n+a-1}{2}+k+1} \Big(D^{1/2}_{u_{\underline{0}}}(t) + t^{\frac{n+a-1}{2}+k+1}\Big),
\end{aligned} \tag{3.34}
$$

with $C_{3.34} = C_{3.34}(n,a) > 0$. Arguing similarly we infer

$$
\begin{aligned}
&\left|\int \phi(\tfrac{|x|}{t}) (\nabla u_{\underline{0}}(x) \cdot x) L_a(u_{\underline{0}}(x)) \mathrm{d}x\right| \leq t \int \phi(\tfrac{|x|}{t}) |\nabla u_{\underline{0}}(x)| |L_a(u_{\underline{0}}(x))| \mathrm{d}x \\
&\overset{(3.8)}{\leq} t^k \int \phi(\tfrac{|x|}{t}) |\nabla u_{\underline{0}}(x)| |x_{n+1}|^a \mathrm{d}x \leq C t^{\frac{n+a-1}{2}+k+1} D^{1/2}_{u_{\underline{0}}}(t),
\end{aligned} \tag{3.35}
$$

and

$$
\begin{aligned}
\left|\frac{1}{t}\int \dot{\phi}\Big(\tfrac{|x|}{t}\Big) |x| u_{\underline{0}}(x) L_a(u_{\underline{0}}(x)) \mathrm{d}x\right| &\overset{(3.8)}{\leq} t^{k-1} \int_{B_t \setminus B_{t/2}} |u_{\underline{0}}(x)| |x_{n+1}|^a \mathrm{d}x \\
&\overset{(3.28)}{\leq} C\, t^{\frac{n+a-1}{2}+k+1} \Big(D^{1/2}_{u_{\underline{0}}}(t) + t^{\frac{n+a-1}{2}+k+1}\Big),
\end{aligned} \tag{3.36}
$$

with $C = C(n,a) > 0$. Therefore, (3.33) follows at once from (3.34)-(3.36). □

Estimate (3.33) turns out to be useful to analyze the subsets of points $\Gamma_{\varphi,\theta}(u)$ of $\Gamma_\varphi(u)$, for every $\theta \in (0,1)$ (cf. (1.8)). With fixed $\theta \in (0,1)$, we then look at points of the free boundary in the subset

$$
\mathscr{Z}_{\varphi,\theta,\delta}(u) := \big\{x_0 \in \Gamma_\varphi(u) \cap B_{1/2} : \ H_{u_{x_0}}(r) \geq \delta\, r^{n+a+2(k+1-\theta)} \quad \forall\, r \in (0, 1/2)\big\}, \tag{3.37}
$$

where $\delta > 0$ is any.

Remark 3.10. Note that $\mathscr{Z}_{\varphi,\theta,\delta}(u) \subseteq \mathscr{Z}_{\varphi,\theta,\delta'}(u)$ if $\delta' \leq \delta$. Hence, in what follows it is enough to consider the values of δ small enough.

Proposition 3.11. *For every $\delta > 0$, there exist $C_{3.11}, \varrho_{3.11} > 0$ such that for every $x_0 \in \mathscr{Z}_{\varphi,\theta,\delta}(u)$, the function $(0, \varrho_{3.11}] \ni r \mapsto e^{C_{3.11} r^\theta} I_{u_{x_0}}(r)$ is nondecreasing. In particular, the ensuing limits exist finite and are equal*

$$
\lim_{r \downarrow 0} \frac{r D_{u_{x_0}}(r)}{H_{u_{x_0}}(r)} = \lim_{r \downarrow 0} I_{u_{x_0}}(r) =: I_{u_{x_0}}(0^+). \tag{3.38}
$$

Proof. Since $x_0 \in \mathscr{Z}_{\varphi,\theta,\delta}(u)$, formula (3.26) yields for $r \in (0, 1/2)$

$$
C_{3.8} D_{u_{x_0}}(r) \geq \delta r^{n+a-1+2(k+1-\theta)} - C_{3.8} r^{n+a-1+2(k+1)},
$$

therefore, for $\varrho_{3.11}$ sufficiently small, we have for all $r \in (0, \varrho_{3.11}]$

$$
D_{u_{x_0}}(r) \geq C r^{n+a-1+2(k+1-\theta)}. \tag{3.39}
$$

In addition, from (3.19) and (3.34) we get for all $r \in (0, \varrho_{3.11}]$, if $\varrho_{3.11}$ small enough,

$$\begin{aligned}
|G_{u_{x_0}}(r) - D_{u_{x_0}}(r)| &\overset{(3.19)}{=} \left| \int \phi\big(\tfrac{|x-x_0|}{r}\big) u_{x_0}(x) L_a(u_{x_0}(x)) \mathrm{d}x \right| \\
&\overset{(3.34)}{\leq} C_{3.34} D_{u_{x_0}}(r) \Big(\frac{r^{n+a+2k+1}}{D_{u_{x_0}}(r)} + \Big(\frac{r^{n+a+2k+1}}{D_{u_{x_0}}(r)} \Big)^{1/2} \Big) \\
&\overset{(3.39)}{\leq} C D_{u_{x_0}}(r) \Big(2Cr^{2\theta} + \big(2Cr^{2\theta}\big)^{1/2} \Big) \leq Cr^{\theta} D_{u_{x_0}}(r).
\end{aligned} \tag{3.40}$$

Therefore, from (3.33), if $\varrho_{3.11}$ is sufficiently small, we get for all $r \in (0, \varrho_{3.11}]$,

$$\begin{aligned}
|R_{u_{x_0}}(r)| &\leq C_{3.9}\, I_{u_{x_0}}(r) \frac{r^{n+a+2k}}{G_{u_{x_0}}(r)} \Big(\Big(\frac{D_{u_{x_0}}(r)}{r^{n+a+2k+1}} \Big)^{1/2} + 1 \Big) \\
&\overset{(3.40)}{\leq} C\, \frac{I_{u_{x_0}}(r)}{r} \Big(\Big(\frac{r^{n+a+2k+1}}{D_{u_{x_0}}(r)} \Big)^{1/2} + \frac{r^{n+a+2k+1}}{D_{u_{x_0}}(r)} \Big) \overset{(3.39)}{\leq} C\, r^{\theta-1}\, I_{u_{x_0}}(r).
\end{aligned} \tag{3.41}$$

Hence, from (3.18), (3.32) and (3.41) we find

$$\frac{\mathrm{d}}{\mathrm{d}r}\big(I_{u_{x_0}}(r)\big) \geq -C\, r^{\theta-1} I_{u_{x_0}}(r), \tag{3.42}$$

and the monotonicity of $(0, \varrho_{3.11}] \ni r \mapsto e^{C_{3.11} r^{\theta}} I_{u_{x_0}}(r)$ follows by direct integration. In addition, we also infer (3.38), because from (3.40) for all $r \in (0, \varrho_{3.11}]$ we have

$$(1 - Cr^{\theta}) \frac{rD_{u_{x_0}}(r)}{H_{u_{x_0}}(r)} \leq I_{u_{x_0}}(r) \leq (1 + Cr^{\theta}) \frac{rD_{u_{x_0}}(r)}{H_{u_{x_0}}(r)}. \tag{3.43}$$

□

Remark 3.12. The monotonicity for the truncated Almgren's frequency function

$$r(1 + Cr^{\theta}) \frac{\mathrm{d}}{\mathrm{d}r} \log \max \big\{ H_{u_{x_0}}(r), r^{n+a+2(k+1-\theta)} \big\}$$

proved in [9] and [18] is essentially equivalent to Proposition 3.11.

We derive next an additive quasi-monotonicity formula for the frequency.

Corollary 3.13. *For every $A, \delta > 0$, there exist $C_{3.13}$, $\varrho_{3.13} > 0$ with this property: if $x_0 \in \mathscr{Z}_{\varphi,\theta,\delta}(u)$ and $I_{u_{x_0}}(\varrho_{3.13}) \leq A$, then for all $\Lambda \geq A\, C_{3.13}$ the function*

$$(0, \varrho_{3.13}] \ni r \mapsto I_{u_{x_0}}(r) + \Lambda\, r^{\theta} \quad \textit{is nondecreasing.} \tag{3.44}$$

Proof. Under the standing assumptions, the quasi-monotonicity of $I_{u_{x_0}}$ and (3.42) yield that

$$\frac{\mathrm{d}}{\mathrm{d}r}\big(I_{u_{x_0}}(r)\big) \geq -Ce^{C_{3.11}}\, A\, r^{\theta-1},$$

for r sufficiently small. Hence, we conclude (3.44) at once by integration. □

3.3. **Lower bound on the frequency and compactness.** We first show that the frequency of a solution u to (3.1) at points in $\mathscr{Z}_{\varphi,\theta,\delta}(u)$ is bounded from below by a universal constant.

Lemma 3.14. *For every $\delta > 0$ there exists $\varrho_{3.14} > 0$ such that, for all $x_0 \in \mathscr{Z}_{\varphi,\theta,\delta}(u)$ and $r \in (0, \varrho_{3.14}]$,*

$$I_{u_{x_0}}(r) \geq \frac{1}{2C_{3.8}}. \tag{3.45}$$

Proof. In view of (3.26) and since $x_0 \in \mathscr{Z}_{\varphi,\theta,\delta}(u)$, we have for all r sufficiently small,

$$\frac{1}{C_{3.8}} \leq \frac{rD_{u_{x_0}}(r)}{H_{u_{x_0}}(r)} + \frac{r^{2\theta}}{\delta}.$$

Inequality (3.45) is a straightforward consequence of estimate (3.43) and the latter estimate provided that $\varrho_{3.14}$ is sufficiently small. □

For the free boundary analysis developed in [15] it is mandatory to consider the critical set of a solution. In the current framework, the natural subsitute for the critical set is given by

$$\mathscr{N}_\varphi(u) := \Big\{(x',0) \in B_1' \,:\, u(x',0) - \varphi(x') = |\nabla'\big(u(x',0) - \varphi(x')\big)| = \lim_{y\downarrow 0^+} t^a \partial_{n+1} u(x',y) = 0\Big\}.$$

Notice that $\Gamma_\varphi(u) \subseteq \mathscr{N}_\varphi(u) \subseteq \Lambda_\varphi(u)$ (the first inclusion is a consequence of (3.10)).

We can then give the following compactness result. For $u : B_1 \to \mathbb{R}$ solution of (3.1) and $x_0 \in B_1'$ we introduce the rescalings

$$u_{x_0,r}(y) := \frac{r^{\frac{n+a}{2}}\, u_{x_0}(x_0 + ry)}{H^{1/2}_{u_{x_0}}(r)} \qquad \forall\, r \in (0, 1 - |x_0|),\ \forall\, y \in B_{\frac{1-|x_0|}{r}}. \tag{3.46}$$

Note that $u_{x_0,r}$ is a minimizer of the functional

$$\int_{B_1} |\nabla v|^2 |x_{n+1}|^a \mathrm{d}x - 2\int_{B_1} v L_a(\varphi_{x_0,r})\,\mathrm{d}x \tag{3.47}$$

with obstacle function

$$\varphi_{x_0,r}(y) := \frac{r^{\frac{n+a}{2}}\, \varphi_{x_0}(x_0 + ry)}{H^{1/2}_{u_{x_0}}(r)}\,, \tag{3.48}$$

among all functions $v \in u_{x_0,r} + H^1_0(B_1, \mathrm{d}\mathfrak{m})$ satisfying $v(x',0) \geq 0$ on B_1'.

Corollary 3.15. *Let $\delta > 0$ be given. Let $(u_l)_{l\in\mathbb{N}}$ be a sequence of solutions to the fractional obstacle problem (3.1) in B_1 with obstacle functions φ_l equi-bounded in $C^{k+1}(B_1)$, and let $x_l \in \mathscr{Z}_{\varphi_l,\theta,\delta}(u_l)$ be such that $\sup_l I_{(u_l)_{x_l}}(\varrho_l) < +\infty$, for some $\varrho_l \downarrow 0$.*

Then, there exist a subsequence $l_j \uparrow \infty$ and a solution v_∞ to the fractional obstacle problem (3.1) in B_1 with null obstacle function, such that on setting $v_j := (u_{l_j})_{x_{l_j},\varrho_{l_j}}$ we have

$$v_j \to v_\infty \quad \text{in } H^1(B_1, \mathrm{d}\mathfrak{m}), \tag{3.49}$$

$$v_j \to v_\infty \quad \text{in } C^{0,\alpha}_{\mathrm{loc}}(B_1),\ \forall\, \alpha < \min\{1, 2s\} \tag{3.50}$$

$$\nabla' v_j \to \nabla' v_\infty \quad \text{in } C^{0,\alpha}_{\mathrm{loc}}(B_1),\ \forall\, \alpha < s, \tag{3.51}$$

$$\mathrm{sign}(x_{n+1})\,|x_{n+1}|^a \partial_{x_{n+1}} v_j \to \mathrm{sign}(x_{n+1})\,|x_{n+1}|^a \partial_{x_{n+1}} v_\infty \ \text{ in } C^{0,\alpha}_{\mathrm{loc}}(B_1),\ \forall\, \alpha < 1 - s. \tag{3.52}$$

Proof. By taking into account inequality (3.43) in Proposition 3.11 we get for l large

$$\frac{\varrho_l D_{(u_l)_{x_l}}(\varrho_l)}{H_{(u_l)_{x_l}}(\varrho_l)} \leq (1 + C\|\varphi_{l_j}\|_{C^{k+1}(B_1')}\varrho_l^\theta) I_{(u_l)_{x_l}}(\varrho_l).$$

In particular, we infer that $\sup_l D_{(u_l)_{x_l,\varrho_l}}(1) < \infty$. Thus, a subsequence $v_j := (u_{l_j})_{x_{l_j},\varrho_{l_j}}$ converges weakly $H^1(B_1, \mathrm{d}\mathfrak{m})$ to some function v_∞. Moreover, v_j is a local minimizer of

$$F_j(v) := \int_{B_1} |\nabla v|^2 |y_{n+1}|^a \mathrm{d}y - 2\int_{B_1} v L_a\big((\varphi_{l_j})_{x_{l_j},\varrho_{l_j}}\big)\,\mathrm{d}y$$

among all functions $v \in v_j + H^1_0(B_1, \mathrm{d}\mathfrak{m})$ satisfying $v(x',0) \geq 0$ on B_1' (cf. (3.46)-(3.47)).

By taking into account that $x_{l_j} \in \mathscr{Z}_{\varphi_{l_j},\theta,\delta}(u_{l_j})$, inequality (3.7) implies that for all $y \in B_1 \setminus B_1'$

$$|\big(L_a(\varphi_{l_j})_{x_{l_j},\varrho_{l_j}}\big)(y)| \leq \frac{1}{\delta^{1/2}}\|\varphi_{l_j}\|_{C^{k+1}(B_1')}\varrho_{l_j}^\theta |y_{n+1}|^a. \tag{3.53}$$

Therefore, one can easily show that the sequence $(F_j)_j$ $\Gamma(L^2(B_1, \mathrm{d}\mathfrak{m}))$-converges to the functional $F_\infty : L^2(B_1, \mathrm{d}\mathfrak{m}) \to [0, +\infty]$ defined by

$$F_\infty(v) := \int_{B_1} |\nabla v|^2 |y_{n+1}|^a \mathrm{d}y$$

if $v \in v_\infty + H^1_0(B_1, \mathrm{d}\mathfrak{m})$ with $v(x',0) \geq 0$ on B_1', and $+\infty$ otherwise on $L^2(B_1, \mathrm{d}\mathfrak{m})$. In addition, being the F_j's equicoercive in $L^2(B_1, \mathrm{d}\mathfrak{m})$, $F_j(v_j) \to F_\infty(v_\infty)$, so that by (3.53) the convergence of $(v_j)_j$ to v_∞ is actually strong $H^1(B_1, \mathrm{d}\mathfrak{m})$.

Items (3.50)-(3.52) are then a straightforward consequence of Theorem 3.1 and (3.53) (cf. the arguments in [9, Lemma 6.2]). □

A sharp lower bound on the frequency then follows.

Corollary 3.16. *Let $\delta > 0$. If $x_0 \in \mathscr{Z}_{\varphi,\theta,\delta}(u)$, then*

$$I_{u_{x_0}}(0^+) \geq 1 + s\,. \tag{3.54}$$

Proof. Note that $I_{u_{x_0}}(0^+) = \lim_{r\downarrow 0} I_{u_{x_0}}(r) = \lim_{r\downarrow 0} I_{u_{x_0,r}}(1) = I_{v_\infty}(1)$, for some v_∞ homogeneous solution to the fractional obstacle problem (3.1) with null obstacle function provided by Corollary 3.15. Thus, we conclude (3.54) by [9, Proposition 5.1] (see also [15, Corollary 2.12]). □

4. Main estimates on the frequency

In this section we prove the principal estimates on the frequency that we are going to exploit in the sequel. We start with an elementary lemma. Recall that all obstacles functions φ are assumed to satisfy the normalization condition $\|\varphi\|_{C^{k+1}(B_1')} \leq 1$.

Lemma 4.1. *Let A, $\delta > 0$. Then, there exist $C_{4.1}$, $\varrho_{4.1} > 0$ such that, if u is a solution of to the fractional obstacle problem (3.1) in B_1, with $\underline{0} \in \mathscr{Z}_{\varphi,\theta,\delta}(u)$ and $I_{u_{\underline{0}}}(2\varrho) \leq A$, $\varrho \leq \varrho_{4.1}$, then for every $x \in B'_{\varrho/2}$*

$$\frac{1}{C_{4.1}} \leq \frac{H_{u_x}(\varrho)}{H_{u_{\underline{0}}}(\varrho)} \leq C_{4.1} \quad \textit{and} \quad \frac{1}{C_{4.1}} \leq \frac{D_{u_x}(\varrho)}{D_{u_{\underline{0}}}(\varrho)} \leq C_{4.1}, \tag{4.1}$$

$$\Big| I_{u_{\underline{0}}}(\varrho) - I_{u_x}(\varrho)\Big| \leq C_{4.1}\,. \tag{4.2}$$

Remark 4.2. Note that as a byproduct of the first estimate in (4.1) in Lemma 4.1 frequencies at the scale ϱ are well-defined at every point $x \in B'_{\varrho/2}$, recalling that $\underline{0} \in \mathscr{Z}_{\varphi,\theta,\delta}(u)$.

Proof. In order to prove (4.1), we argue by contradiction: we can assume that there exist A, $\delta > 0$ and solutions u_j to the fractional obstacle problem with obstacles φ_j, $\|\varphi_j\|_{C^{k+1}(B_1')} \leq 1$, with $\underline{0} \in \mathscr{Z}_{\varphi_j,\theta,\delta}(u_j)$, such that $I_{(u_j)_{\underline{0}}}(\varrho_j) \leq A$, for some $\varrho_j \downarrow 0$, and there exist points $x_j \in B'_{\varrho_j/4}$ contradicting one of the sets of inequalities in (4.1).

In particular, by almost monotonicity of the frequency function (cf. Proposition 3.11) and the lower bound on the frequency (cf. Corollary 3.16) we infer that $1+s \leq I_{(u_j)_{\underline{0}}}(t) \leq A\, e^{C_{3.11}(2\varrho_j)^\theta} \leq A\, e^{C_{3.11}} =: A'$ for all $t \in (0, 2\varrho_j]$. By Corollary 3.15, up to a subsequence, $v_j := (u_j)_{\underline{0},\varrho_j}$ converges strongly in $H^1(B_2, \mathrm{d}\mathfrak{m})$ to a function v_∞ solution of the fractional obstacle problem in B_2 with zero obstacle function. We assume in addition that $\varrho_j^{-1} x_j \to x_\infty \in \bar{B}'_{1/2}$.

To prove the first set of inequalities in (4.1), we compute

$$\begin{aligned}
\frac{H_{(u_j)_{x_j}}(\varrho_j)}{H_{(u_j)_{\underline{0}}}(\varrho_j)} = \;& \frac{2\varrho_j^{n+a}}{H_{(u_j)_{\underline{0}}}(\varrho_j)} \int_{B_1\setminus B_{1/2}} (u_j)^2_{x_j}(x_j+\varrho_j x) \frac{|x_{n+1}|^a}{|x|}\mathrm{d}x \\
=& \frac{2\varrho_j^{n+a}}{H_{(u_j)_{\underline{0}}}(\varrho_j)} \int_{B_1\setminus B_{1/2}} \Big[u_j(x_j+\varrho_j x) - (\varphi_j)_{x_j}(x_j+\varrho_j x')\Big]^2 \frac{|x_{n+1}|^a}{|x|}\mathrm{d}x \\
=& 2\int_{B_1\setminus B_{1/2}} \Big[v_j(\varrho_j^{-1}x_j + x) + \frac{\varrho_j^{(n+a)/2}}{H^{1/2}_{(u_j)_{\underline{0}}}(\varrho)}\big((\varphi_j)_{\underline{0}}(x_j+\varrho_j x') - (\varphi_j)_{x_j}(x_j+\varrho_j x')\big)\Big]^2 \frac{|x_{n+1}|^a}{|x|}\mathrm{d}x\,.
\end{aligned} \tag{4.3}$$

Moreover, by estimate (3.15) in Remark 3.2 and since $\varrho_j^{-1}x_j \to x_\infty$ we get for all $x' \in B_1'$

$$|(\varphi_j)_{\underline{0}}(x_j+\varrho_j x') - (\varphi_j)_{x_j}(x_j+\varrho_j x')| \leq C\varrho_j^{k+1}\,. \tag{4.4}$$

Therefore, recalling that $\underline{0} \in \mathscr{Z}_{\varphi_j,\theta,\delta}(u_j)$, from (4.4) we infer

$$\frac{\varrho_j^{n+a}}{H_{(u_j)_{\underline{0}}}(\varrho_j)} \int_{B_1\setminus B_{\frac{1}{2}}} \big((\varphi_j)_{\underline{0}}(x_j+\varrho_j x') - (\varphi_j)_{x_j}(x_j+\varrho_j x')\big)^2 \frac{|x_{n+1}|^a}{|x|}\mathrm{d}x \leq \frac{C}{\delta}\varrho_j^{2\theta}. \tag{4.5}$$

Since $\varrho_j \downarrow 0$, by contradiction $\lim_j \frac{H_{(u_j)_{x_j}}(\varrho_j)}{H_{(u_j)_{\underline{0}}}(\varrho_j)} \in \{0,\infty\}$. Moreover, by (4.3) and (4.5), by the strong $L^2(B_1, \mathrm{d}\mathfrak{m})$ and local uniform convergence of $v_j \to v_\infty$ we conclude that,

$$2\int_{B_1(x_\infty)\setminus B_{1/2}(x_\infty)} v_\infty^2(y)\frac{|y_{n+1}|^a}{|y-x_\infty|}\mathrm{d}y = 2\lim_j \int_{B_1(\varrho_j^{-1}x_j)\setminus B_{1/2}(\varrho_j^{-1}x_j)} v_j^2(y)\frac{|y_{n+1}|^a}{|y-\varrho_j^{-1}x_j|}\mathrm{d}y$$
$$= 2\lim_j \int_{B_1\setminus B_{1/2}} v_j^2(\varrho_j^{-1}x_j + x)\frac{|x_{n+1}|^a}{|x|}\mathrm{d}x = \lim_j \frac{H_{(u_j)_{x_j}}(\varrho_j)}{H_{(u_j)_{\underline{0}}}(\varrho_j)}\,.$$

Being the left hand side finite, necessarily

$$2\int_{B_1(x_\infty)\setminus B_{1/2}(x_\infty)} v_\infty^2(y)\frac{|y_{n+1}|^a}{|y-x_\infty|}\mathrm{d}y = \lim_j \frac{H_{(u_j)_{x_j}}(\varrho_j)}{H_{(u_j)_{\underline{0}}}(\varrho_j)} = 0\,.$$

Hence, $v_\infty \equiv 0$ on $B_1(x_\infty)\setminus B_{1/2}(x_\infty)$, and thus $v_\infty \equiv 0$ on the whole of B_1 by analiticity. A contradiction to $H_{v_\infty}(1) = 1$ that follows from strong $L^2(B_1, \mathrm{d}\mathfrak{m})$ convergence and the equality $H_{v_j}(1) = 1$ for all j.

The second set of inequalities in (4.1) is proven by the same argument. Indeed, assuming that $\lim_j \frac{D_{(u_j)_{x_j}}(\varrho_j)}{D_{(u_j)_{\underline{0}}}(\varrho_j)} \in \{0,\infty\}$ we have

$$\frac{D_{(u_j)_{x_j}}(\varrho_j)}{D_{(u_j)_{\underline{0}}}(\varrho_j)} = \frac{\varrho_j^{n+a+1}}{D_{(u_j)_{\underline{0}}}(\varrho_j)}\int_{B_1}\phi(|x|)|\nabla\big(u_j(x_j+\varrho_j x) - (\varphi_j)_{x_j}(x_j+\varrho_j x')\big)|^2|x_{n+1}|^a\mathrm{d}x\,,$$

and sinceby (3.16) in Remark 3.2 and by (3.39)

$$\frac{\varrho_j^{n+a+1}}{D_{(u_j)_{\underline{0}}}(\varrho_j)}\int_{B_1}\phi(|x|)|\nabla\big((\varphi_j)_{\underline{0}}(x_j+\varrho_j x') - (\varphi_j)_{x_j}(x_j+\varrho_j x')\big)|^2|x_{n+1}|^a\mathrm{d}x$$
$$\leq C\frac{\varrho_j^{n+a+1+2k}}{D_{(u_j)_{\underline{0}}}(\varrho_j)} \leq \frac{C}{\delta}\varrho_j^{2\theta}\,,$$

we get (recall $\varrho_j \downarrow 0$)

$$\lim_j \frac{D_{(u_j)_{x_j}}(\varrho_j)}{D_{(u_j)_{\underline{0}}}(\varrho_j)} = \lim_j \frac{1}{4I_{(u_j)_{\underline{0}}}(\varrho_j)}\int_{B_1}\phi(|x|)|\nabla v_j(\varrho_j^{-1}x_j + x)|^2|x_{n+1}|^a\mathrm{d}x.$$

By the strong convergence of v_j to v_∞ in $H^1(B_1, \mathrm{d}\mathfrak{m})$, we infer that the left hand side is finite and then actually 0, so that

$$\int_{B_1}\phi(|x|)|\nabla v_\infty(x_\infty + x)|^2|x_{n+1}|^a\mathrm{d}x = 0.$$

Thus, by analiticity v_∞ is constant on B_1, and we may conclude that

$$\int_{B_1}\phi(|x|)|\nabla v_\infty(x)|^2|x_{n+1}|^a\mathrm{d}x = 0.$$

The latter equality contradicts

$$\int_{B_1}\phi(|x|)|\nabla v_\infty(x)|^2|x_{n+1}|^a\mathrm{d}x \in [1+s, 2A']\,,$$

that follows from strong $H^1(B_1, \mathrm{d}\mathfrak{m})$ convergence and recalling that $H_{v_j}(1) = 1$ and $1+s \leq I_{v_j}(1) \leq 2D_{v_j}(1) \leq A'$ for j big enough (cf. (3.43)).

Finally, (4.2) follows straightforwardly from (4.1) for $\varrho_{4.1}$ sufficiently small by taking into account (3.43):

$$\Big|I_{u_{\underline{0}}}(r) - I_{u_x}(r)\Big| = \Big|\frac{rG_{u_{\underline{0}}}(r)}{H_{u_x}(r)}\Big(\frac{H_{u_x}(r)}{H_{u_{\underline{0}}}(r)} - \frac{G_{u_x}(r)}{G_{u_{\underline{0}}}(r)}\Big)\Big| \overset{(4.1)}{\leq} C. \qquad \square$$

We introduce next the following notation for the radial variation of (modified) frequency at a point $x \in \mathscr{Z}_{\varphi,\theta,\delta}(u)$ of a solution u in B_1: given $0 < r_0 < r_1 < 1 - |x|$, we set

$$\Delta_{r_0}^{r_1}(x) := I_{u_x}(r_1) + \Lambda\, r_1^{\theta} - \big(I_{u_x}(r_0) + \Lambda\, r_0^{\theta}\big).$$

Note that, $\Delta_{r_0}^{r_1}(x) \geq 0$ if $x \in \mathscr{Z}_{\varphi,\theta,\delta}(u)$, if r_1 is sufficiently small and if $\Lambda \geq A\, C_{3.13}$ (cf. Corollary 3.13). We do not indicate the dependence of $\Delta_{r_0}^{r_1}$ on Λ since such a parameter will be fixed appropriately in the next result.

Lemma 4.3. *Let A, $\delta > 0$. Then, there exist $C_{4.3}$ and $\varrho_{4.3} > 0$ such that, if $x_0 \in \mathscr{Z}_{\varphi,\theta,\delta}(u)$, and $I_{u_{x_0}}(r_1) \leq A$, with $2r_1 \leq \varrho_{4.3}$, then for every $r_0 \in ({r_1}/{8}, r_1)$ we have*

$$\int_{B_{r_1/2}(x_0)\setminus B_{r_0/2}(x_0)} \Big(\nabla u_{x_0}(z)\cdot(z-x_0) - I_{u_{x_0}}({r_0}/{2})\, u_{x_0}(z)\Big)^2 \frac{|z_{n+1}|^a}{|z-x_0|}\,\mathrm{d}z \leq C_{4.3} H_{u_{x_0}}(r_1)\, \Delta_{r_0/2}^{r_1}(x_0). \tag{4.6}$$

Proof. Without loss of generality we prove the result for $x_0 = \underline{0}$. We start off with the following computation:

$$\begin{aligned} 2\int_{B_t\setminus B_{t/2}} &\Big(\nabla u_{\underline{0}}(z)\cdot z - I_{u_{\underline{0}}}(t)\, u_{\underline{0}}(z)\Big)^2 \frac{|z_{n+1}|^a}{|z|}\mathrm{d}z \\ &= \int -\dot{\phi}\big(\tfrac{|z|}{t}\big)\Big(\nabla u_{\underline{0}}(z)\cdot z - I_{u_{\underline{0}}}(t)\, u_{\underline{0}}(z)\Big)^2 \tfrac{|z_{n+1}|^a}{|z|}\mathrm{d}z \\ &= t^2 E_{u_{\underline{0}}}(t) - 2\,t\, I_{u_{\underline{0}}}(t)\, G_{u_{\underline{0}}}(t) + I_{u_{\underline{0}}}^2(t) H_{u_{\underline{0}}}(t) \\ &= \frac{t^2}{H_{u_{\underline{0}}}(t)}\big(E_{u_{\underline{0}}}(t) H_{u_{\underline{0}}}(t) - G_{u_{\underline{0}}}^2(t)\big) \overset{(3.32)}{=} \frac{t}{2} H_{u_{\underline{0}}}(t)\Big(\frac{\mathrm{d}}{\mathrm{d}t}\big(I_{u_{\underline{0}}}(t)\big) - R_{u_{\underline{0}}}(t)\Big). \end{aligned} \tag{4.7}$$

We now use the following integral estimate (whose elementary proof is left to the readers)

$$\int_{B_{\rho_1}\setminus B_{\rho_0}} f(z)\mathrm{d}z \leq \rho_0^{-1}\int_{\rho_0}^{2\rho_1}\int_{B_t\setminus B_{t/2}} f(z)\,\mathrm{d}z\,\mathrm{d}t \qquad \forall\ 0 < \rho_0 \leq \rho_1, \tag{4.8}$$

$f \geq 0$ a measurable function, in order to deduce

$$\begin{aligned} &\int_{B_{r_1/2}\setminus B_{r_0/2}} \Big(\nabla u_{\underline{0}}(z)\cdot z - I_{u_{\underline{0}}}({r_0}/{2})\, u_{\underline{0}}(z)\Big)^2 \frac{|z_{n+1}|^a}{|z|}\,\mathrm{d}z \\ &\overset{(4.8)}{\leq} \frac{2}{r_0}\int_{r_0/2}^{r_1}\int_{B_t\setminus B_{t/2}} \Big(\nabla u_{\underline{0}}(z)\cdot z - I_{u_{\underline{0}}}({r_0}/{2})\, u_{x_0}(z)\Big)^2 \frac{|z_{n+1}|^a}{|z|}\,\mathrm{d}z\,\mathrm{d}t \\ &\leq \frac{4}{r_0}\int_{r_0/2}^{r_1}\int_{B_t\setminus B_{t/2}} \Big[\big(\nabla u_{\underline{0}}(z)\cdot z - I_{u_{\underline{0}}}(t)\, u_{\underline{0}}(z)\big)^2 + \big(I_{u_{\underline{0}}}(t) - I_{u_{\underline{0}}}({r_0}/{2})\big)^2 u_{\underline{0}}^2(z)\Big]\frac{|z_{n+1}|^a}{|z|}\,\mathrm{d}z\,\mathrm{d}t \\ &\overset{(4.7),(3.44)}{\leq} \frac{2}{r_0}\int_{r_0/2}^{r_1}\frac{t}{2} H_{u_{\underline{0}}}(t)\Big(\frac{\mathrm{d}}{\mathrm{d}t}\big(I_{u_{\underline{0}}}(t)\big) - R_{u_{\underline{0}}}(t)\Big)\mathrm{d}t \\ &+ \frac{16}{r_0}\Big(\big(I_{u_{\underline{0}}}(r_1) - I_{u_{\underline{0}}}({r_0}/{2})\big)^2 + (A\,C_{3.13})^2\big(r_1^{\theta} - ({r_0}/{2})^{\theta}\big)^2\Big)\int_{r_0/2}^{r_1} H_{u_{\underline{0}}}(t)\,\mathrm{d}t \\ &\leq \frac{r_1}{r_0} H_{u_{\underline{0}}}(r_1)\int_{r_0/2}^{r_1}\Big(\frac{\mathrm{d}}{\mathrm{d}t}\big(I_{u_{\underline{0}}}(t)\big) - R_{u_{\underline{0}}}(t)\Big)\mathrm{d}t \\ &+ 16\,\frac{r_1}{r_0} H_{u_{\underline{0}}}(r_1)\Big(\big(I_{u_{\underline{0}}}(r_1) - I_{u_{\underline{0}}}({r_0}/{2})\big)^2 + (A\,C_{3.13})^2\big(r_1^{\theta} - ({r_0}/{2})^{\theta}\big)^2\Big). \end{aligned} \tag{4.9}$$

In the last inequality we have used that, if $\varrho_{4.3}$ is sufficiently small, then $H_{u_{\underline{0}}}(t) \leq H_{u_{\underline{0}}}(r_1)$ for all $t \leq r_1$ by (3.24), and that $\frac{\mathrm{d}}{\mathrm{d}t}\big(I_{u_{\underline{0}}}(t)\big) - R_{u_{\underline{0}}}(t) \geq 0$ thanks to (4.7). Moreover, estimate (3.41) in Proposition 3.11, $I_{u_{\underline{0}}}(r_1) \leq A$, the quasi-monotonicity of the frequency function and the choice $2r_1 \leq \varrho_{4.3}$ imply

$$\int_{r_0/2}^{r_1} |R_{u_{\underline{0}}}(t)|\,\mathrm{d}t \leq A\, C_{3.11} e^{C_{3.11} r_1^{\theta}}\big(r_1^{\theta} - ({r_0}/{2})^{\theta}\big).$$

Hence, from (4.9) we conclude that

$$\int_{B_{r_1/2}\setminus B_{r_0/2}} \Big(\nabla u_{\underline{0}}(z)\cdot z - I_{u_{\underline{0}}}({}^{r_0}\!/2)\, u_{\underline{0}}(z)\Big)^2 \frac{|z_{n+1}|^a}{|z|}\,\mathrm{d}z \\ \leq C\, H_{u_{\underline{0}}}(r_1)\,\big(I_{u_{\underline{0}}}(r_1) + \Lambda\, r_1^\theta - I_{u_{\underline{0}}}({}^{r_0}\!/2) - \Lambda({}^{r_0}\!/2)^\theta\big),$$

where we used that ${}^{r_1}\!/r_0 \leq 8$, and $C > 0$. □

4.1. **Oscillation estimate of the frequency.** The following lemma shows how the spatial oscillation of the frequency in two nearby points at a given scale is in turn controlled by the radial variations at comparable scales.

Proposition 4.4. *Let A, $\delta > 0$. Then there exist $C_{4.4}$, $\varrho_{4.4} > 0$ such that if $\underline{0} \in \mathscr{Z}_{\varphi,\theta,\delta}(u)$, $\tau \in (0, {}^{\varrho_{4.4}}\!/144)$ with $I_{u_{\underline{0}}}(72\tau) \leq A$, then*

$$\big|I_{u_{x_1}}(10\tau) - I_{u_{x_2}}(10\tau)\big| \leq C_{4.4}\,\Big[\big(\Delta^{24\tau}_{3\tau}(x_1)\big)^{1/2} + \big(\Delta^{24\tau}_{3\tau}(x_2)\big)^{1/2}\Big] + C_{4.4}\tau^\theta\,, \tag{4.10}$$

for every x_1, $x_2 \in B'_\tau \cap \mathscr{Z}_{\varphi,\theta,\delta}(u)$.

Proof. We start off noting that by Remark 4.2 and the choice $144\tau < \varrho_{4.4}$, if the constant $\varrho_{4.4}$ is suitably chosen, a simple scaling argument yields that $I_{u_x}(10\tau)$ is well-defined for every $x \in B'_{77\tau/4}$.

To ease the readability of the proof we divide it in several substeps.

1. With fixed x_1, $x_2 \in B'_\tau \cap \mathscr{Z}_{\varphi,\theta,\delta}(u)$, let $x_t := tx_1 + (1-t)x_2$, $t \in [0,1]$, and consider the map $t \mapsto I_{u_{x_t}}(10\tau)$. The differentiability of the functions $x \mapsto H_{u_x}(10\tau)$ and $x \mapsto D_{u_x}(10\tau)$ yields that

$$I_{u_{x_1}}(10\tau) - I_{u_{x_2}}(10\tau) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\big(I_{u_{x_t}}(10\tau)\big)\,\mathrm{d}t.$$

Set $e := x_1 - x_2$, then $e\cdot e_{n+1} = 0$; and set for all $y \in \mathbb{R}^{n+1}$

$$\delta_t(y) := \frac{\mathrm{d}}{\mathrm{d}t}\big(u_{x_t}(x_t+y)\big).$$

Recalling the very definition of u_{x_t} in (3.6), it turns out that

$$\delta_t(y) = \partial_e u(x_t+y) - \partial_e\varphi(x_t+y') + T_{k,x_t}[\partial_e\varphi](x_t+y') - \mathscr{E}\big[T_{k,x_t}[\partial_e\varphi]\big](x_t+y)\,, \tag{4.11}$$

because by linearity (the details of the elementary computations are left to the readers)

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(T_{k,x_t}[\varphi](x_t+y')\big) = T_{k,x_t}[\partial_e\varphi](x_t+y')\,, \tag{4.12}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathscr{E}\big[T_{k,x_t}[\varphi]\big](x_t+y)\big) = \mathscr{E}\big[T_{k,x_t}[\partial_e\varphi]\big](x_t+y)\,.$$

Morever, from the very definition of u_{x_t} in (3.6) it is straightforward to prove that

$$\partial_e u_{x_t}(x_t+y) = \partial_e u(x_t+y) - \partial_e\varphi(x_t+y') + T_{k-1,x_t}[\partial_e\varphi](x_t+y') - \mathscr{E}\big[T_{k-1,x_t}[\partial_e\varphi]\big](x_t+y)\,.$$

Thus, from (4.11) and the latter equality, by direct calculation it follows that

$$\delta_t(y) - \partial_e u_{x_t}(x_t+y) = \sum_{|\alpha|=k} D^\alpha(\partial_e\varphi(x_t))\frac{(y')^\alpha}{\alpha!} - \mathscr{E}\Big(\sum_{|\alpha|=k} D^\alpha(\partial_e\varphi(x_t))\frac{p_\alpha(\cdot - x_t)}{\alpha!}\Big)(y)\,,$$

and thus we may conclude that

$$|\delta_t(y) - \partial_e u_{x_t}(x_t+y)| \leq C|x_1-x_2||y|^k\,. \tag{4.13}$$

Moreover, note also that

$$\nabla\delta_t(y) = \frac{\mathrm{d}}{\mathrm{d}t}\big(\nabla u_{x_t}(x_t+y)\big)\,. \tag{4.14}$$

2. Thanks to the previous formulas, for all $\lambda \in \mathbb{R}$ we infer

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big(H_{u_{x_t}}(10\tau)\big) &= -2\int \dot\phi\big(\tfrac{|y|}{10\tau}\big)\, u_{x_t}(x_t+y)\,\delta_t(y)\,\tfrac{|y_{n+1}|^a}{|y|}\,\mathrm{d}y\\ &= -2\int \dot\phi\big(\tfrac{|y|}{10\tau}\big)\,\big(\delta_t(y)-\lambda u_{x_t}(x_t+y)\big)\,u_{x_t}(x_t+y)\,\tfrac{|y_{n+1}|^a}{|y|}\,\mathrm{d}y + 2\lambda H_{u_{x_t}}(10\tau).\end{aligned} \tag{4.15}$$

In addition, integrating by parts gives

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\big(D_{u_{x_t}}(10\tau)\big) \overset{(4.14)}{=} 2\int \phi\big(\tfrac{|y|}{10\tau}\big)\,\nabla\delta_t(y)\cdot\nabla u_{x_t}(x_t+y)\,|y_{n+1}|^a\,\mathrm{d}y\\ &= -\frac{1}{5\tau}\int \dot\phi\big(\tfrac{|y|}{10\tau}\big)\,\delta_t(y)\,\nabla u_{x_t}(x_t+y)\cdot y\,\frac{|y_{n+1}|^a}{|y|}\,\mathrm{d}y - 2\int \phi\big(\tfrac{|y|}{10\tau}\big)\,\delta_t(y)\,L_a(u_{x_t}(x_t+y))\,\mathrm{d}y\\ &= -\frac{1}{5\tau}\int \dot\phi\big(\tfrac{|y|}{10\tau}\big)\,\big(\delta_t(y)-\lambda u_{x_t}(x_t+y)\big)\nabla u_{x_t}(x_t+y)\cdot y\,\tfrac{|y_{n+1}|^a}{|y|}\,\mathrm{d}y\\ &\quad + 2\lambda G_{u_{x_t}}(10\tau) - 2\int \phi\big(\tfrac{|y|}{10\tau}\big)\,\delta_t(y)\,L_a(u_{x_t}(x_t+y))\,\mathrm{d}y\,.\end{aligned} \tag{4.16}$$

Then, by formula (3.19) together with (4.15) and (4.16), we have that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\big(I_{u_{x_t}}(10\tau)\big) = I_{u_{x_t}}(10\tau)\left(\frac{\frac{\mathrm{d}}{\mathrm{d}t}\big(G_{u_{x_t}}(10\tau)\big)}{G_{u_{x_t}}(10\tau)} - \frac{\frac{\mathrm{d}}{\mathrm{d}t}\big(H_{u_{x_t}}(10\tau)\big)}{H_{u_{x_t}}(10\tau)}\right)\\ &= -\frac{2}{H_{u_{x_t}}(10\tau)}\int \dot\phi\big(\tfrac{|y|}{10\tau}\big)\,\big(\delta_t(y)-\lambda u_{x_t}(x_t+y)\big)\big(\nabla u_{x_t}(x_t+y)\cdot y - I_{u_{x_t}}(10\tau)\,u_{x_t}(x_t+y)\big)\,\tfrac{\mathrm{d}\mathfrak{m}(y)}{|y|}\\ &+ \frac{10\tau}{H_{u_{x_t}}(10\tau)}\int \phi\big(\tfrac{|y|}{10\tau}\big)\,\Big(u_{x_t}(x_t+y)\,\tfrac{\mathrm{d}}{\mathrm{d}t}\big(L_a(u_{x_t}(x_t+y))\big) - \delta_t(y)\,L_a(u_{x_t}(x_t+y))\Big)\,\mathrm{d}y\\ &=: J_t^{(1)} + J_t^{(2)}.\end{aligned}$$

In what follows we estimate separately the two terms $J_t^{(i)}$.

3. We start off with $J_t^{(1)}$. With this aim, first note that $I_{u_{\underline{0}}}(10\tau) \le A\,e^{72^\theta C_{3.11}}$ by Proposition 3.11 since $144\tau < \varrho_{4.4}$, provided the latter is small enough. In turn, as $x_t \in B'_\tau$, by (4.2) in Lemma 4.1 we infer that $I_{u_{x_t}}(10\tau) \le C_{4.1} + A\,e^{72^\theta C_{3.11}}$.

We estimate separately the factors of the integrand defining $J_t^{(1)}$ (setting $x_t + y = z$). We start off with the first one as follows

$$\begin{aligned}\big|\delta_t(z-x_t) - \lambda\,u_{x_t}(z)\big| &\le \big|\partial_e u_{x_t}(z) - \lambda\,u_{x_t}(z)\big| + \big|\delta_t(z-x_t) - \partial_e u_{x_t}(z)\big|\\ &\overset{(4.13)}{\le} \big|\partial_e u_{x_t}(z) - \lambda\,u_{x_t}(z)\big| + C|x_1-x_2||z-x_t|^k,\end{aligned}$$

with $C = C(n,k) > 0$. Moreover, by choosing $\lambda := I_{u_{x_2}}(10\tau) - I_{u_{x_1}}(10\tau)$, we infer

$$\begin{aligned}\big|\partial_e u_{x_t}(z) - \lambda\,u_{x_t}(z)\big| &= \big|\nabla u_{x_t}(z)\cdot e - \lambda\,u_{x_t}(z)\big|\\ &\le \big|\nabla u_{x_t}(z)\cdot(z-x_1) - I_{u_{x_1}}(10\tau)\,u_{x_t}(z)\big| + \big|\nabla u_{x_t}(z)\cdot(z-x_2) - I_{u_{x_2}}(10\tau)\,u_{x_t}(z)\big|\\ &\le \big|\nabla u_{x_1}(z)\cdot(z-x_1) - I_{u_{x_1}}(10\tau)\,u_{x_1}(z)\big| + \big|\nabla u_{x_2}(z)\cdot(z-x_2) - I_{u_{x_2}}(10\tau)\,u_{x_2}(z)\big|\\ &\quad + \big|\nabla(u_{x_t}(z)-u_{x_1}(z))\cdot(z-x_1) - I_{u_{x_1}}(10\tau)\,(u_{x_t}(z)-u_{x_1}(z))\big|\\ &\quad + \big|\nabla(u_{x_t}(z)-u_{x_2}(z))\cdot(z-x_2) - I_{u_{x_2}}(10\tau)\,(u_{x_t}(z)-u_{x_2}(z))\big|\,.\end{aligned}$$

Using inequalities (3.15)-(3.16) in Remark 3.2, we estimate the last two addends as follows

$$\big|\nabla(u_{x_t}(z)-u_{x_i}(z))\cdot(z-x_i) - I_{u_{x_i}}(10\tau)\,(u_{x_t}(z)-u_{x_i}(z))\big| \le C\tau^{k+1}\,,$$

for some constant $C > 0$, for $i = 1, 2$. In the last inequality we have used that $|z - x_i| \le 12\tau$, being $z \in B_{10\tau}(x_t)$. Therefore, we have

$$\begin{aligned}\big|\delta_t(z - x_t) - \lambda\, u_{x_t}(z)\big| &\le \big|\nabla u_{x_1}(z)\cdot(z - x_1) - I_{u_{x_1}}(10\tau)\, u_{x_1}(z)\big| \\ &\quad + \big|\nabla u_{x_2}(z)\cdot(z - x_2) - I_{u_{x_2}}(10\tau)\, u_{x_2}(z)\big| + C\,\tau^{k+1} =: \psi(z). \qquad (4.17)\end{aligned}$$

For the second factor, we note that for $i = 1, 2$

$$\begin{aligned}|\nabla u_{x_t}(z)\cdot(z - x_t) - &I_{u_{x_t}}(10\tau)u_{x_t}(z)| \\ \le&|\nabla u_{x_i}(z)\cdot(z - x_t) - I_{u_{x_i}}(10\tau)u_{x_i}(z)| + |\nabla\big(u_{x_t}(z) - u_{x_i}(z)\big)\cdot(z - x_t)| \\ &+ |I_{u_{x_t}}(10\tau)||u_{x_i}(z) - u_{x_t}(z)| + |I_{u_{x_i}}(10\tau) - I_{u_{x_t}}(10\tau)||u_{x_i}(z)| \\ \le&\tau^{k+1} + C\,|u_{x_i}(z)|.\end{aligned}$$

To estimate the last three addends we have used the very definition of u_{x_t} in (3.6), formula (4.2) and inequalities (3.15)-(3.16) in Remark 3.2, taking into account that $|z - x_i| \le 12\tau$ being $z \in B_{10\tau}(x_t)$. Therefore, we get

$$|\nabla u_{x_t}(z)\cdot(z - x_t) - I_{u_{x_t}}(10\tau)u_{x_t}(z)| \le \psi(z) + C\left(|u_{x_1}(z)| + |u_{x_2}(z)|\right). \qquad (4.18)$$

By collecting (4.17) and (4.18), using Hölder inequality we conclude that there exists $C > 0$

$$\begin{aligned}J_t^{(1)} &\le \frac{C}{H_{u_{x_t}}(10\tau)}\int -\dot\phi\big(\tfrac{|z-x_t|}{10\tau}\big)\psi(z)\big(\psi(z) + |u_{x_1}(z)| + |u_{x_2}(z)|\big)\tfrac{|z_{n+1}|^a}{|z-x_t|}\,\mathrm{d}z \\ &\le \frac{C}{H_{u_{x_t}}(10\tau)}\Big(\int -\dot\phi\big(\tfrac{|z-x_t|}{10\tau}\big)\psi^2(z)\tfrac{|z_{n+1}|^a}{|z-x_t|}\,\mathrm{d}z\Big)^{1/2}\cdot \\ &\qquad\cdot\Big(\int -\dot\phi\big(\tfrac{|z-x_t|}{10\tau}\big)\big(\psi^2(z) + |u_{x_1}(z)|^2 + |u_{x_2}(z)|^2\big)\tfrac{|z_{n+1}|^a}{|z-x_t|}\,\mathrm{d}z\Big)^{1/2}. \qquad (4.19)\end{aligned}$$

Clearly, we have that

$$\begin{aligned}&\int -\dot\phi\big(\tfrac{|z-x_t|}{10\tau}\big)\psi^2(z)\tfrac{|z_{n+1}|^a}{|z-x_t|}\,\mathrm{d}z \\ &\le C\int_{B_{10\tau}(x_t)\setminus B_{5\tau}(x_t)}|\nabla u_{x_1}(z)\cdot(z - x_1) - I_{u_{x_1}}(10\tau)u_{x_1}(z)|^2\frac{|z_{n+1}|^a}{|z - x_t|}\,\mathrm{d}z \\ &+ C\int_{B_{10\tau}(x_t)\setminus B_{5\tau}(x_t)}|\nabla u_{x_2}(z)\cdot(z - x_2) - I_{u_{x_2}}(10\tau)u_{x_2}(z)|^2\frac{|z_{n+1}|^a}{|z - x_t|}\,\mathrm{d}z + C\tau^{n+a+2(k+1)} \\ &\le C\int_{B_{12\tau}(x_1)\setminus B_{3\tau}(x_1)}\big|\nabla u_{x_1}(z)\cdot(z - x_1) - I_{u_{x_1}}(10\tau)\,u_{x_1}(z)\big|^2\,\frac{|z_{n+1}|^a}{|z - x_1|}\,\mathrm{d}z \\ &+ C\int_{B_{12\tau}(x_2)\setminus B_{3\tau}(x_2)}\big|\nabla u_{x_2}(z)\cdot(z - x_2) - I_{u_{x_2}}(10\tau)\,u_{x_2}(z)\big|^2\,\frac{|z_{n+1}|^a}{|z - x_2|}\,\mathrm{d}z + C\tau^{n+a+2(k+1)} \\ &\le CH_{u_{x_1}}(24\tau)\Delta^{24\tau}_{3\tau/2}(x_1) + CH_{u_{x_2}}(24\tau)\Delta^{24\tau}_{3\tau/2}(x_2) + C\tau^{n+a+2(k+1)}.\end{aligned}$$

In the second inequality, we have used that $B_{10\tau}(x_t) \setminus B_{5\tau}(x_t) \subset B_{12\tau}(x_i) \setminus B_{3\tau}(x_i)$ for $t \in [0, 1]$, and that $|z - x_i| \le 2|z - x_t|$ as $z \in B_{10\tau}(x_t)\setminus B_{5\tau}(x_t)$, $i = 1, 2$. Moreover, in the third inequality we have applied estimate (4.6) in Lemma 4.3 to x_1, $x_2 \in B'_\tau \cap \mathscr{Z}_{\varphi,\theta,\delta}(u)$, with $r_1 = 24\tau$ and $r_0 = 3\tau$. Furthermore, thanks to Corollary 3.7, we conclude

$$\begin{aligned}&\int -\dot\phi\big(\tfrac{|z-x_t|}{10\tau}\big)\psi^2(z)\tfrac{|z_{n+1}|^a}{|z-x_t|}\,\mathrm{d}z \\ &\le C\,H_{u_{x_1}}(10\tau)\Delta^{24\tau}_{3\tau/2}(x_1) + C\,H_{u_{x_2}}(10\tau)\Delta^{24\tau}_{3\tau/2}(x_2) + C\tau^{n+a+2(k+1)}. \qquad (4.20)\end{aligned}$$

In addition, thanks to (3.25) and $|z - x_t| \ge 5\tau$ we get

$$\begin{aligned}\int -\dot\phi\big(\tfrac{|z-x_t|}{10\tau}\big)&(|u_{x_1}(z)|^2 + |u_{x_2}(z)|^2)\frac{|z_{n+1}|^a}{|z - x_t|}\,\mathrm{d}z \\ &\le \frac{2}{5\tau}\big(\|u_{x_1}\|^2_{L^2(B_{10\tau},\mathrm{dm})} + \|u_{x_2}\|^2_{L^2(B_{10\tau},\mathrm{dm})}\big) \le 4H_{u_{x_1}}(10\tau) + 4H_{u_{x_2}}(10\tau). \qquad (4.21)\end{aligned}$$

By collecting (4.19)-(4.21) we conclude that for some $C > 0$

$$
\begin{aligned}
J_t^{(1)} \leq & \frac{C}{H_{u_{x_t}}(10\tau)} \Big(H_{u_{x_1}}(10\tau)\Delta^{24\tau}_{3\tau/2}(x_1) + H_{u_{x_2}}(10\tau)\Delta^{24\tau}_{3\tau/2}(x_2) + \tau^{n+a+2(k+1)}\Big) \\
& + \frac{C}{H_{u_{x_t}}(10\tau)} \Big(H_{u_{x_1}}(10\tau) + H_{u_{x_2}}(10\tau)\Big)^{1/2} \cdot \\
& \qquad \cdot \Big(H_{u_{x_1}}(10\tau)\Delta^{24\tau}_{3\tau/2}(x_1) + H_{u_{x_2}}(10\tau)\Delta^{24\tau}_{3\tau/2}(x_2) + \tau^{n+a+2(k+1)}\Big)^{1/2} \\
\leq & C \left(\Delta^{24\tau}_{3\tau/2}(x_1) + \big(\Delta^{24\tau}_{3\tau/2}(x_1)\big)^{1/2}\right) + C \left(\Delta^{24\tau}_{3\tau/2}(x_2) + \big(\Delta^{24\tau}_{3\tau/2}(x_2)\big)^{1/2}\right) + C\,\frac{\tau^{2\theta}}{\delta} + C\,\frac{\tau^{\theta}}{\delta^{1/2}}\,,
\end{aligned}
$$

where, in the last inequality, we have used Lemma 4.1 and that $x_1,\, x_2 \in B'_\tau \cap \mathscr{Z}_{\varphi,\theta,\delta}(u)$.

Finally, in view of the very definition of the spatial oscillation of the frequency and Corollary 3.13, we deduce for some constant depending on A that

$$
J_t^{(1)} \leq C \left(\big(\Delta^{24\tau}_{3\tau/2}(x_1)\big)^{1/2} + \big(\Delta^{24\tau}_{3\tau/2}(x_2)\big)^{1/2}\right) + C\,\tau^{\theta}\,. \tag{4.22}
$$

4. We estimate next $J_t^{(2)}$. We start off noting that for all $y \in B_1 \setminus B'_1$ (cf. (3.3))

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\big(L_a(u_{x_t}(x_t+y))\big) &= |y_{n+1}|^a \triangle \Big(\frac{\mathrm{d}}{\mathrm{d}t}\big(T_{k,x_t}[\varphi](x_t+y') - \varphi(x_t+y')\big)\Big) \\
&\overset{(4.12)}{=} |y_{n+1}|^a \triangle \Big(T_{k,x_t}[\partial_e\varphi](x_t+y') - \partial_e\varphi(x_t+y')\Big) \\
&\leq C\,|x_1-x_2||y_{n+1}|^a|y'|^{k-2} \leq C\,\tau|y_{n+1}|^a|y'|^{k-2}.
\end{aligned} \tag{4.23}
$$

Then, arguing as in (4.4), thanks to estimate (3.15) in Remark 3.2, we get as $k \geq 2$

$$
\begin{aligned}
&\Big|\int \phi\big(\tfrac{|y|}{10\tau}\big)\,\big(u_{x_t}(x_t+y)\,\frac{\mathrm{d}}{\mathrm{d}t}\big(L_a(u_{x_t}(x_t+y))\big)\,\mathrm{d}y\Big| \\
&\quad \overset{(4.23)}{\leq} C\,\tau^{k-1}\int \phi\big(\tfrac{|y|}{10\tau}\big)\,\big|u_{x_t}(x_t+y)\big||y_{n+1}|^a\,\mathrm{d}y \\
&\quad = C\,\tau^{k-1}\int_{B_{10\tau}(x_t)} \phi\big(\tfrac{|z-x_t|}{10\tau}\big)\,\big|u_{x_t}(z)\big||z_{n+1}|^a\,\mathrm{d}z \\
&\quad \overset{(3.15)}{\leq} C\,\tau^{n+a+2k+1} + C\,\tau^{k-1}\int_{B_{40\tau}(x_1)} \phi\big(\tfrac{|z-x_1|}{40\tau}\big)\,\big|u_{x_1}(z)\big||z_{n+1}|^a\,\mathrm{d}z \\
&\quad \overset{(3.27)}{\leq} C\,\tau^{n+a+2k+1} + C\,\tau^{\frac{n+a+1}{2}+k}D^{1/2}_{u_{x_1}}(40\tau)\,.
\end{aligned}
$$

In addition, (3.8) and (4.13) yield

$$
\begin{aligned}
&\Big|\int \phi\big(\tfrac{|y|}{10\tau}\big)\,\delta_t(y)\,L_a(u_{x_t}(x_t+y))\,\mathrm{d}y\Big| \\
&\qquad \leq C\tau^{n+a+2k+1} + \Big|\int \phi\big(\tfrac{|y|}{10\tau}\big)\,\partial_e u_{x_t}(x_t+y)\,L_a(u_{x_t}(x_t+y))\,\mathrm{d}y\Big| \\
&\qquad \leq C\tau^{n+a+2k+1} + \tau^k\int \phi\big(\tfrac{|y|}{10\tau}\big)\,|\nabla u_{x_t}(x_t+y)|\,|y_{n+1}|^a\mathrm{d}y \\
&\qquad \leq C\tau^{n+a+2k+1} + C\tau^{\frac{n+a+1}{2}+k}D^{1/2}_{u_{x_t}}(10\tau).
\end{aligned}
$$

Therefore, by applying repeatedly Lemma 4.1, by taking into account (3.43) and by choosing $\varrho_{4.4}$ sufficiently small, we infer that

$$\begin{aligned} J_2^{(t)} \le & \frac{C}{H_{u_{x_t}}(10\tau)}\big(\tau^{n+a+2(k+1)} + \tau^{\frac{n+a+1}{2}+k+1} D^{1/2}_{u_{x_1}}(40\tau) + \tau^{\frac{n+a+1}{2}+k+1} D^{1/2}_{u_{x_t}}(10\tau)\big) \\ \overset{(3.37)}{\le} & C\Big(\frac{\tau^{2\theta}}{\delta} + \frac{\tau^{\theta}}{\delta^{1/2}}\Big(\frac{\tau D_{u_{x_1}}(40\tau)}{H_{u_{x_1}}(10\tau)}\Big)^{1/2} + \frac{\tau^{\theta}}{\delta^{1/2}} I^{1/2}_{u_{x_t}}(10\tau)\Big) \\ \overset{(3.22)}{\le} & C\Big(\tau^{2\theta} + \tau^{\theta} I^{1/2}_{u_{x_1}}(40\tau) + \tau^{\theta} I^{1/2}_{u_{x_t}}(10\tau)\Big) \le C\tau^{\theta}\,, \end{aligned} \tag{4.24}$$

since $144\tau < \varrho_{4.4}$.

The conclusion in (4.10) follows at once from estimates (4.22) and (4.24). □

5. Proof of the main result

5.1. **Mean-flatness.** Here we show a control of the Jones' β-number by the oscillation of the frequency. Given a Radon measure μ in $\mathbb{R}^{n+1}$, for every $x_0 \in \mathbb{R}^n$ and for every $r > 0$, we set

$$\beta_\mu(x_0, r) := \inf_{\mathcal{L}} \Big(r^{-n-1} \int_{B_r(x_0)} \operatorname{dist}^2(y, \mathcal{L}) \mathrm{d}\mu(y)\Big)^{1/2}, \tag{5.1}$$

where the infimum is taken among all affine $(n-1)$-dimensional planes $\mathcal{L} \subset \mathbb{R}^{n+1}$.

If $x_0 \in \mathbb{R}^{n+1}$ and $r > 0$ is such that $\mu(B_r(x_0)) > 0$, set $\bar{x}_{x_0,r}$ the barycenter of μ in $B_r(x_0)$, *i.e.*

$$\bar{x}_{x_0,r} := \frac{1}{\mu(B_r(x_0))} \int_{B_r(x_0)} x\, \mathrm{d}\mu(x)\,,$$

and

$$\mathbf{B}_{x_0}(v, w) := \int_{B_r(x_0)} \big((x - \bar{x}_{x_0,r}) \cdot v\big)\, \big((x - \bar{x}_{x_0,r}) \cdot w\big)\, \mathrm{d}\mu(x) \quad \forall\, v,\, w \in \mathbb{R}^{n+1}.$$

Then

$$\beta_\mu(x_0, r) = \Big(r^{-n-1}\big(\lambda_n + \lambda_{n+1}\big)\Big)^{\frac{1}{2}}, \tag{5.2}$$

where $0 \le \lambda_{n+1} \le \lambda_n \le \cdots \le \lambda_1$ are the eigenvalues of the positive semidefinite bilinear form $\mathbf{B}_{x_0}$.

Proposition 5.1. *Let $A, \delta > 0$. Then there exist constants $C_{5.1}$, $\varrho_{5.1} > 0$ with this property. Let $122r \le \varrho_{5.1}$, $\underline{0} \in \mathscr{Z}_{\varphi,\theta,\delta}(u)$ and $I_{u_{\underline{0}}}(66r) \le A$. Let μ be a finite Borel measure with* spt $(\mu) \subseteq \mathscr{Z}_{\varphi,\theta,\delta}(u)$*. Then, for all points $p \in B'_r \cap \mathscr{Z}_{\varphi,\theta,\delta}(u)$, we have*

$$\beta^2_\mu(p, r) \le \frac{C_{5.1}}{r^{n-1}} \left(\int_{B_r(p)} \Delta^{24\,r}_{5/2r}(x)\, \mathrm{d}\mu(x) + r^{2\theta} \mu(B_r(p))\right). \tag{5.3}$$

Proof. The proof is a variant of the [15, Proposition 4.2], which in turn follows closely the original arguments by Naber and Valtorta in [24, 25], therefore we only highlight the main differences.

Without loss of generality assume that $p \in B'_r \cap \Gamma_\varphi(u) \cap \mathscr{Z}_{\varphi,\theta,\delta}(u)$ is such that $\mu(B_r(p)) > 0$ (otherwise, there is nothing to prove). Let $\{v_1, \ldots, v_{n+1}\}$ be any diagonalizing basis for the bilinear form $\mathbf{B}_p$ introduced in § 5.1, with corresponding eigenvalues $0 \le \lambda_{n+1} \le \lambda_n \le \cdots \le \lambda_1$.

Since spt $(\mu) \subset \Gamma_\varphi(u) \subset \mathbb{R}^n \times \{0\}$, we may assume that $v_{n+1} = e_{n+1}$, $\lambda_{n+1} = 0$, so that $\beta_\mu(p, r) = (r^{-n-1}\lambda_n)^{1/2}$ by (5.2). Clearly, we may also assume that $\lambda_n > 0$.

From the very definitions of $\mathbf{B}_p$ and of its barycenter we deduce

$$\begin{aligned} r^{n+1}\beta^2_\mu(p, r) & \int_{B_{11r}(p)\setminus B_{10r}(p)} |\nabla' u_p(z)|^2 |z_{n+1}|^a \mathrm{d}z \\ & \le n \int_{B_r(p)} \int_{B_{12r}(x)\setminus B_{9r}(x)} \big((z - x) \cdot \nabla u_p(z) - \alpha\, u_p(z)\big)^2 |z_{n+1}|^a \mathrm{d}z\, \mathrm{d}\mu(x), \end{aligned} \tag{5.4}$$

where

$$\alpha := \frac{1}{\mu(B_r)} \int_{B_r(p)} I_{u_x}(9r) \mathrm{d}\mu(x).$$

Next we estimate the two sides of (5.4).

For estimating the left hand side of (5.4), we can show by compactness that

$$D_{u_p}(12r) \le C \int_{B_{11r}(p)\setminus B_{10r}(p)} |\nabla' u_p(z)|^2 |z_{n+1}|^a \mathrm{d}z. \tag{5.5}$$

Here we use the same contradiction argument in [15, Proposition 4.2] using the compactness given by Corollary 3.15.

For what concerns the right hand side of (5.4) we proceed as follows. By the triangular inequality we have that

$$\begin{aligned}
&\text{r.h.s. of (5.4)}\\
&\le 4n \int_{B_r(p)} \int_{B_{12r}(x)\setminus B_{9r}(x)} \big((z-x)\cdot\nabla u_x(z) - I_{u_x}(9r)\, u_x(z)\big)^2 |z_{n+1}|^a \,\mathrm{d}z\, \mathrm{d}\mu(x)\\
&+ 4n \int_{B_r(p)} \int_{B_{12r}(x)\setminus B_{9r}(x)} \big((z-x)\cdot\nabla (u_x-u_p)(z) - \alpha\, (u_x-u_p)(z)\big)^2 |z_{n+1}|^a \,\mathrm{d}z\, \mathrm{d}\mu(x)\\
&+ 4n \int_{B_r(p)} \int_{B_{12r}(x)\setminus B_{9r}(x)} \Big(I_{u_x}(9r) - \alpha\Big)^2 u_x^2(z) |z_{n+1}|^a \,\mathrm{d}z\, \mathrm{d}\mu(x) =: J_1 + J_2 + J_3.
\end{aligned} \tag{5.6}$$

The addends J_1 and J_3 can be treated as in [15, Proposition 4.2]. Indeed, for J_1 we use Lemma 4.1 and Lemma 4.3 for a suitable choice of the constants to get

$$J_1 \le C\, r \int_{B_r(p)} H_{u_x}\big(24r\big) \Delta_{9r}^{24r}(x)\, \mathrm{d}\mu(x) \le C\, r\, H_{u_p}(12r) \int_{B_r(p)} \Delta_{9r}^{24r}(x)\, \mathrm{d}\mu(x)\,. \tag{5.7}$$

For J_3 we use Jensen's inequality, Proposition 4.4, Fubini's Theorem, inequality (3.25) and (4.1) in Lemma 4.1 to get

$$J_3 \le C\, r\, H_{u_p}(12r) \Big(\int_{B_r(p)} \Delta_{5/2r}^{22r}(x)\, \mathrm{d}\mu(x) + r^{2\theta} \mu\big(B_r(p)\big)\Big). \tag{5.8}$$

Note that the extra term with respect to [15, Proposition 4.2] arises as a consequence of the additional error term in Proposition 4.4.

To estimate J_2 in (5.6) we first note that $\nabla\big(T_{k,x}[\varphi]\big) = T_{k-1,x}[\nabla\varphi]$. Then, we use estimates (3.15) and (3.16) in Remark 3.2 to deduce that for all $x \in B_r(p)$ and $z \in B_{12r}(x)\setminus B_{9r}(x)$ we have

$$\begin{aligned}
&\big((z-x)\cdot\nabla (u_x-u_p)(z) - \alpha\, (u_x-u_p)(z)\big)^2\\
&\le C \Big(r^2 |\nabla\big(T_{k,x}[\varphi](z) - T_{k,p}[\varphi](z)\big)|^2 + \alpha^2 |T_{k,x}[\varphi](z) - T_{k,p}[\varphi]|^2(z)\Big) \le C\, r^{2(k+1)}\,.
\end{aligned}$$

Therefore, integrating the last estimate we conclude that

$$J_2 \le C\, r^{n+a+2k+3} \mu(B_r(p))\,. \tag{5.9}$$

We can now collect the estimates (5.5)–(5.9) and use Corollary 3.13 to get

$$\begin{aligned}
&r^{n+1} \beta_\mu^2(p,r) D_{u_p}(12r)\\
&\qquad \le C\, r\, H_{u_p}(12r) \int_{B_r(p)} \big(\Delta_{9r}^{24r}(x) + \Delta_{5/2r}^{22r}(x)\big)\, \mathrm{d}\mu(x)\\
&\qquad + C\, r^{1+2\theta} \mu(B_r(p)) H_{u_p}(12r) + C\, r^{n+a+2k+3} \mu(B_r(p))\\
&\qquad \le C\, r\, H_{u_p}(12r) \Big(\int_{B_r(p)} \Delta_{5/2r}^{24r}(x)\, \mathrm{d}\mu(x) + \big(r^{2\theta} + \frac{r^{n+a+2(k+1)}}{H_{u_p}(12r)}\big) \mu(B_r(p))\Big).
\end{aligned}$$

Finally, by assumption $p \in \mathscr{Z}_{\varphi,\theta,\delta}$, then $e^{C_{3.11}\|\varphi\|(12r)^\theta} I_{u_p}(12r) \ge 1 + s$ (cf. Proposition 3.11, Corollary 3.16 and the choice $122r \le 1$), so that the upper inequality in (3.43) yields (5.3). □

5.2. **Rigidity of homogeneous solutions.** In this section we extend the results on the rigidity of almost homogeneous solutions established in [15].

We denote by $\mathcal{H}_\lambda$ the space of all non-zero λ-homogeneous solutions to the thin obstacle problem (3.1) with zero obstacle,

$$\mathcal{H}_\lambda := \Big\{u \in H^1_{\mathrm{loc}}(\mathbb{R}^{n+1}, \mathrm{d}\mathfrak{m}) \setminus \{0\} : \, u(x) = |x|^\lambda \, u\big(x/|x|\big), \ u|_{B_1} \text{ solves (3.1) with } \varphi \equiv 0\Big\},$$

and set $\mathcal{H} := \bigcup_{\lambda \geq 1+s} \mathcal{H}_\lambda$. The spine $S(u)$ of $u \in \mathcal{H}$ is the maximal subspace of invariance of u,

$$S(u) := \Big\{y \in \mathbb{R}^n \times \{0\} \ : \ u(x+y) = u(x) \quad \forall\, x \in \mathbb{R}^{n+1}\Big\}.$$

As observed in [15], the maximal dimension of the spine of a function in $\mathcal{H}$ is at most $n-1$ and we set $u \in \mathcal{H}^{\mathrm{top}}$ if $u \in \mathcal{H}$ and $\dim S(u) = n-1$, and $\mathcal{H}^{\mathrm{low}} := \mathcal{H} \setminus \mathcal{H}^{\mathrm{top}}$. All functions in $\mathcal{H}^{\mathrm{top}}$ are classified in [15, Lemma 5.3]. Note also that by Caffarelli, Salsa and Silvestre [9]

$$\mathcal{H}_{1+s} \subseteq \mathcal{H}^{top}. \tag{5.10}$$

We next introduce the notion of almost homogeneous solutions. Given $\delta > 0$ and $x_0 \in \mathscr{Z}_{\varphi,\theta,\delta}$ we set

$$J_{u_{x_0}}(t) := e^{C_{3.11} t^\theta} I_{u_{x_0}}(t) \quad \forall\, t \in (0, \varrho_{3.11}] \,.$$

Definition 5.2. Let $\eta > 0$ and let $u : B_1 \to \mathbb{R}$ be a solution to thin obstacle problem (3.1) with obstacle φ (as usual $\|\varphi\|_{C^{k+1}(B_1')} \leq 1$). Assume that $0 \in \mathscr{Z}_{\varphi,\theta,\delta}$ and $\varrho \leq \varrho_{3.11}$, u is called *η-almost homogeneous in B_ϱ* if

$$J_{u_{\underline{0}}}(\varrho/2) - J_{u_{\underline{0}}}(\varrho/4) \leq \eta.$$

The following lemma justifies this terminology and it is the analog of [15, Lemma 5.5].

Lemma 5.3. *Let ε, $A > 0$. There exists $\eta_{5.3} > 0$ with the following property: for every $\delta > 0$ there exists $\varrho_{5.3}$ such that, if u is a $\eta_{5.3}$-almost homogeneous solution of (3.1) in B_ϱ with $\varrho \leq \varrho_{5.3}$ and obstacle φ, $\underline{0} \in \mathscr{Z}_{\varphi,\theta,\delta}(u)$ and $I_{u_{\underline{0}}}(\varrho_{5.3}) \leq A$, then*

$$\left\|u_{\underline{0},\varrho} - w\right\|_{H^1(B_{1/4}, \mathrm{d}\mathfrak{m})} \leq \varepsilon, \tag{5.11}$$

for some homogeneous solution $w \in \mathcal{H}$.

Proof. The proof follows by a contradiction argument similar to [15, Lemma 5.5]. Assume that for some $\varepsilon, A > 0$ we could find sequences of numbers δ_l, ϱ_l and of $1/l$-almost homogeneous solutions u_l of (3.1) in B_{ϱ_l}, with ϱ_l arbitrarily small, such that $\underline{0} \in \mathscr{Z}_{\varphi,\theta,\delta}(u_l)$ and

$$\inf_l \inf_{w \in \mathcal{H}} \left\|(u_l)_{\underline{0},\varrho_l} - w\right\|_{H^1(B_{1/4}, \mathrm{d}\mathfrak{m})} \geq \varepsilon\,, \tag{5.12}$$

and satisfying the bounds $I_{(u_l)_{\underline{0}}}(\varrho_l) \leq A$.

Consider $v_l := (u_l)_{\underline{0},\varrho_l}$, then by Corollary 3.15 applied to v_l there would be a subsequence, not relabeled, converging in $H^1(B_1, \mathrm{d}\mathfrak{m})$ to a solution v_∞ of the thin obstacle problem with zero obstacle. By Proposition 3.11 there is some A' independent of l such that $I_{(u_l)_{\underline{0}}}(t) \leq A'$ for all $t \in (0, \varrho_l]$, then from (3.23) in Corollary 3.7 we would infer that

$$-\int \dot\phi\big(\tfrac{2|x|}{t}\big) \tfrac{|v_\infty|^2}{|x|} |x_{n+1}|^a \mathrm{d}x = -\lim_l \int \dot\phi\big(\tfrac{2|x|}{t}\big) \tfrac{|v_l|^2}{|x|} |x_{n+1}|^a \mathrm{d}x = \lim_l \frac{H_{(u_l)_{\underline{0}}}(\varrho_l/2)}{H_{(u_l)_{\underline{0}}}(\varrho_l)} \geq 2^{-(n+a+2A')},$$

in turn implying that v_∞ is not zero. On the other hand, we would also get

$$I_{(v_\infty)_{\underline{0}}}(1/2) - I_{(v_\infty)_{\underline{0}}}(1/4) = \lim_l \big(J_{(v_l)_{\underline{0}}}(1/2) - J_{(v_l)_{\underline{0}}}(1/4)\big) = \lim_l \big(J_{(u_l)_{\underline{0}}}(\varrho_l/2) - J_{(u_l)_{\underline{0}}}(\varrho_l/4)\big) = 0,$$

and thus we would conclude that $v_\infty \in \mathcal{H}$ being a solution to the lower dimensional obstacle problem with constant frequency (see for instance [15, Proposition 2.7]). We have thus contradicted (5.12). □

A rigidity property of the type shown in [15, Proposition 5.6] holds for the non-zero obstacle problem.

Proposition 5.4. *Let A, $\tau > 0$. There exists $\eta_{5.4} > 0$ with this property. For every $\delta > 0$ there exists $\varrho_{5.4}$ such that, if u is a $\eta_{5.4}$-almost homogeneous solution of (3.1) in B_ϱ with $\varrho \le \varrho_{5.4}$ and obstacle φ, $\underline{0} \in \mathscr{Z}_{\varphi,\theta,\delta}(u)$ and $I_{u_{\underline{0}}}(\varrho_{5.4}) \le A$, then the following dichotomy holds:*

(i) *either for every point $x \in B'_{\varrho/2} \cap \mathscr{Z}_{\varphi,\theta,\delta}(u)$ we have*

$$\left|J_{u_x}(\varrho/2) - J_{u_{\underline{0}}}(\varrho/2)\right| \le \tau, \tag{5.13}$$

(ii) *or there exists a linear subspace $V \subset \mathbb{R}^n \times \{0\}$ of dimension $n-2$ such that*

$$\begin{cases} y \in B'_{\varrho/2} \cap \mathscr{Z}_{\varphi,\theta,\delta}(u), \\ J_{u_y}(\varrho/8) - J_{u_y}(\varrho/16) \le \eta_{5.4} \end{cases} \implies \quad \mathrm{dist}(y, V) < \tau\varrho. \tag{5.14}$$

Proof. The proof proceeds by contradiction and follows the strategy developed in [15, Proposition 5.6]. Let A, $\tau > 0$ be given constants and assume that there exist δ_l, ϱ_l and a sequence $(u_l)_{l\in\mathbb{N}}$ of $1/l$-almost homogeneous solutions in B_{ϱ_l} such that $\underline{0} \in \mathscr{Z}_{\varphi_l,\theta,\delta_l}(u_l)$, $I_{(u_l)_{\underline{0}}}(\varrho_l) \le A$ and such that:

(i) there exists $x_l \in B'_{\varrho_l/4} \cap \mathscr{Z}_{\varphi_l,\theta,\delta_l}(u_l)$ for which

$$\left|J_{(u_l)_{x_l}}(\varrho_l/2) - J_{(u_l)_{\underline{0}}}(\varrho_l/2)\right| > \tau, \tag{5.15}$$

(ii) for every linear subspace $V \in \mathbb{R}^n \times \{0\}$ of dimension $n-2$ there exists $y_l \in B'_{\varrho_l/4} \cap \mathscr{Z}_{\varphi_l,\theta,\delta_l}(u_l)$ (a priori depending on V) such that

$$J_{(u_l)_{y_l}}(\varrho_l/8) - J_{(u_l)_{y_l}}(\varrho_l/16) \le 1/l \quad \text{and} \quad \mathrm{dist}\big(y_l, V\big) \ge \tau\varrho_l. \tag{5.16}$$

We consider the rescaled functions $v_l := (u_l)_{\underline{0},\varrho_l} : B_2 \to \mathbb{R}$. By the compactness result in Corollary 3.15 we deduce that, up to passing to a subsequence (not relabeled), there exists a nonzero function v_∞ solution to the thin obstacle problem (3.1) in B_1 with null obstacle such that $v_l \to v_\infty$ in $H^1(B_1, \mathrm{d}\mathfrak{m})$. Moreover, $v_\infty \in \mathcal{H}$ thanks to Lemma 5.3.

If $v_\infty \in \mathcal{H}^{\mathrm{top}}$, then (5.15) is contradicted. Indeed, up to choosing a further subsequence, we can assume that $z_l := \varrho_l^{-1} x_l \to z_\infty \in \bar{B}_{1/2}$. Note that the points $z_l \in \mathcal{N}(v_l)$, as $x_l \in \Gamma_{\varphi_l}(u_l)$, so that

$$v_l(z_l) = (u_l)_{\underline{0},\varrho_l}(z_l) = \frac{\varrho_l^{\frac{n+a}{2}}}{H^{1/2}_{(u_l)_{\underline{0}}}(\varrho_l)}(u_l)_{\underline{0}}(x_l) = 0\,.$$

In addition, by (3.10) and being $\mathscr{E}\big[T_{k,\underline{0}}[\varphi_l]\big]$ even with respect to $\{x_{n+1} = 0\}$ (cf. Lemma 2.1), for all l we infer that

$$\lim_{t\downarrow 0} t^a \partial_{n+1} v_l(z_l', t) = \frac{\varrho_l^{\frac{n+a}{2}}}{H^{1/2}_{(u_l)_{\underline{0}}}(\varrho_l)} \lim_{t\downarrow 0} t^a \partial_{n+1}\Big(u_l(x_l', t) - \mathscr{E}\big[T_{k,\underline{0}}[\varphi_l]\big](x_l', t)\Big) = 0\,.$$

Hence, we conclude that $z_\infty \in \mathcal{N}(v_\infty)$ in view of (3.52). Moreover, by taking into account the very definition of v_l and Remark 3.4 we get by scaling

$$\begin{aligned}
&\left|I_{(v_\infty)_{z_\infty}}(1/2) - I_{(v_\infty)_{\underline{0}}}(1/2)\right| = \left|\frac{1/2\int \phi(2|x-z_\infty|)|\nabla v_\infty|^2|x_{n+1}|^a \mathrm{d}x}{-\int \dot\phi(2|x-z_\infty|)\frac{|v_\infty|^2}{|x-z_\infty|}|x_{n+1}|^a\mathrm{d}x} - \frac{1/2\int \phi(2|x|)|\nabla v_\infty|^2|x_{n+1}|^a\mathrm{d}x}{-\int \dot\phi(2|x|)\frac{|v_\infty|^2}{|x|}|x_{n+1}|^a\mathrm{d}x}\right| \\
&= \lim_{l\to+\infty}\left|\frac{1/2\int \phi(2|x-z_l|)|\nabla v_l|^2|x_{n+1}|^a\mathrm{d}x}{-\int \dot\phi(2|x-z_l|)\frac{|v_l|^2}{|x-z_l|}|x_{n+1}|^a\mathrm{d}x} - \frac{1/2\int\phi(2|x|)|\nabla v_l|^2|x_{n+1}|^a\mathrm{d}x}{-\int\dot\phi(2|x|)\frac{|v_l|^2}{|x|}|x_{n+1}|^a\mathrm{d}x}\right| \\
&= \lim_{l\to+\infty}\left|\frac{\varrho_l/2\int\phi(\frac{|z-x_l|}{\varrho_l/2})|\nabla(u_l)_{\underline{0}}|^2|z_{n+1}|^a\mathrm{d}z}{-\int\dot\phi(\frac{|z-x_l|}{\varrho_l/2})\frac{|(u_l)_{\underline{0}}|^2}{|z-x_l|}|z_{n+1}|^a\mathrm{d}z} - \frac{\varrho_l/2 D_{(u_l)_{\underline{0}}}(\varrho_l/2)}{H_{(u_l)_{\underline{0}}}(\varrho_l/2)}\right| \\
&= \lim_{l\to+\infty}\left|\frac{\varrho_l/2 D_{(u_l)_{x_l}}(\varrho_l/2)}{H_{(u_l)_{x_l}}(\varrho_l/2)} - \frac{\varrho_l/2 D_{(u_l)_{\underline{0}}}(\varrho_l/2)}{H_{(u_l)_{\underline{0}}}(\varrho_l/2)}\right| \overset{(3.43)}{=} \lim_{l\to+\infty}\left|I_{(u_l)_{x_l}}(\varrho_l/2) - I_{(u_l)_{\underline{0}}}(\varrho_l/2)\right| \\
&= \lim_{l\to+\infty}\left|J_{(u_l)_{x_l}}(\varrho_l/2) - J_{(u_l)_{\underline{0}}}(\varrho_l/2)\right| \ge \tau,
\end{aligned}$$

which is a contradiction to the constancy of the frequency at critical points of the homogeneous solution $v_\infty \in \mathcal{H}^{\rm top}$ (see [15, Lemma 5.3]). The fourth equality is justifed by taking into account that $x_l \in \mathscr{Z}_{\varphi_l,\theta,\delta_l}(u_l)$ (cf. (3.37) and (3.39)), and in view of estimates (3.15) and (3.16) in Remark 3.2, in turn implying for all $z \in B_{\varrho_l/2}(x_l)$ (recall that $\varrho_l^{-1}x_l \to z_\infty$)

$$|(u_l)_{\underline{0}}(z) - (u_l)_{x_l}(z)| \le C\varrho_l^{k+1}, \qquad |\nabla\big((u_l)_{\underline{0}}(z) - (u_l)_{x_l}(z)\big)| \le C\varrho_l^{k}\,.$$

Moreover, (3.43) can be employed in the last two equalities as $x_l \in B'_{\varrho_l/4} \cap \mathscr{Z}_{\varphi_l,\theta,\delta_l}(u_l)$.

Instead, if $v_\infty \in \mathcal{H}^{\rm low}$, we show a contradiction to (5.16) with V any $(n-2)$-dimensional subspace containing $S(v_\infty)$. Indeed, let y_l be as in (5.16) for such a choice of V. By compactness, up to passing to a subsequence (not relabeled), $z_l := \varrho_l^{-1}y_l \to z_\infty$ for some $z_\infty \in \bar{B}_{1/2}$ with $\mathrm{dist}(z_\infty, V) \ge \tau$. In addition, arguing as before

$$\begin{aligned}
&\big|I_{(v_\infty)_{z_\infty}}(1/8) - I_{(v_\infty)_{z_\infty}}(1/16)\big| \\
&= \lim_{l\to+\infty}\left|\frac{\varrho_l/8\, D_{(u_l)_{y_l}}(\varrho_l/8)}{H_{(u_l)_{y_l}}(\varrho_l/8)} - \frac{\varrho_l/16\, D_{(u_l)_{y_l}}(\varrho_l/16)}{H_{(u_l)_{y_l}}(\varrho_l/16)}\right| \overset{(3.43)}{=} \lim_{l\to+\infty}\left|I_{(u_l)_{y_l}}(\varrho_l/8) - I_{(u_l)_{y_l}}(\varrho_l/16)\right| \\
&= \lim_{l\to+\infty}\left|J_{(u_l)_{y_l}}(\varrho_l/8) - J_{(u_l)_{y_l}}(\varrho_l/16)\right| = 0\,.
\end{aligned}$$

Again, note that (3.43) can be employed since $y_l \in B'_{\varrho_l/2} \cap \mathscr{Z}_{\varphi_l,\theta,\delta_l}(u_l)$. By [15, Proposition 2.7, Lemma 5.2] it follows that $z_\infty \in S(v_\infty)$, thus contradicting $S(v_\infty) \subseteq V$ and $\mathrm{dist}(z_\infty, V) \ge \tau$. □

5.3. **Proof of Theorem 1.2.** We start off noting that it suffices to prove that $\Gamma_{\varphi,\theta}(u) \cap \bar{B}_1(x_0)$ satisfies all the conclusions for all $x_0 \in \Gamma_{\varphi,\theta}(u)$. For all $R \in (0,1)$, we can find a finite number of balls $B_{R/2}(x_i)$, $x_i \in \Gamma_{\varphi,\theta}(u)$ for $i \in \{1,\ldots,M\}$, whose union cover $\Gamma_{\varphi,\theta}(u) \cap \bar{B}_1(x_0)$. We shall choose appropriately R in what follows. Moreover, with fixed $i \in \{1,\ldots,M\}$, by horizontal translation we may reduce to $x_i = \underline{0} \in \Gamma_{\varphi,\theta}(u)$ without loss of generality.

Then, recalling the definition of $\Gamma_{\varphi,\theta}(u)$ in (1.8) we have that

$$\Gamma_{\varphi,\theta}(u) \cap B'_{R/2} = \cup_{j\in\mathbb{N}} \mathscr{Z}^R_{\varphi,\theta,1/j}(u)\,,$$

where

$$\mathscr{Z}^R_{\varphi,\theta,1/j}(u) := \big\{x_0 \in \Gamma_\varphi(u) \cap B'_{R/2} :\, H_{u_{x_0}}(r) \ge r^{n+a+2(k+1-\theta)}/j \quad \forall\, r \in (0, R/2)\big\}$$

Hence, we may establish the result for $\mathscr{Z}^R_{\varphi,\theta,1/j}(u)$ with $j \in \mathbb{N}$ fixed.

Next, note that as $\underline{0} \in \Gamma(u)$, the function

$$\widetilde{u}(y) := u(Ry) - u(\underline{0})$$

solves the fractional obstacle problem (3.1) in B_1 with obstacle function $\widetilde{\varphi}(\cdot) := \varphi(R\cdot) - \varphi(\underline{0})$. Moreover, $\Gamma_{\widetilde{\varphi},\theta}(\widetilde{u}) \cap B'_{1/2} = \frac{1}{R}\big(\Gamma_{\varphi,\theta}(u) \cap B'_{R/2}\big)$, with $\widetilde{u}_{z/R}(\cdot) = u_z(R\cdot)$ if $z \in \Gamma_{\varphi,\theta}(u) \cap B'_{R/2}$, being $T_{k,z/R}[\widetilde{\varphi}](\cdot) = T_{k,z}[\varphi](R\cdot)$. Thus, we get that $z \in \mathscr{Z}^R_{\varphi,\theta,1/j}(u)$ if and only if $z/R \in B'_{1/2} \cap \mathscr{Z}_{\widetilde{\varphi},\theta,R^{2(k+1-\theta)}/j}(\widetilde{u})$. In addition, it is easy to check that

$$\|\widetilde{\varphi}\|_{C^{k+1}(B'_1)} \le R\|\nabla\varphi\|_{C^k(B'_R,\mathbb{R}^n)}\,.$$

We choose $R > 0$ sufficiently small so that $\|\widetilde{\varphi}\|_{C^{k+1}(B'_R)} \le 1$ and the smallness conditions on the radii in all the statements of Sections 3-5 are satisfied.

In such a case the proof, of the main results can be obtained by following verbatim [15, Sections 6–8]. Indeed, [15, Proposition 6.1], that leads both to the local finiteness of the Minkowskii content of $\mathscr{Z}_{\widetilde{\varphi},\theta,\delta}(\widetilde{u})$ and to its $(n-1)$-rectifiability, is based on a covering argument that exploits the lower bound on the frequency in Corollary 3.16, the control of the mean oscillation via the frequency in Proposition 5.1, the rigidity of almost homogeneous solutions in Proposition 5.4, the discrete Reifenberg theorem by Naber and Valtorta [24, Theorem 3.4, Remark 3.9], and the rectifiability criterion either by Azzam and Tolsa [2] or by Naber and Valtorta [24, 25]. Therefore, the only extra-care needed in the current setting is to start the covering argument from a scale which is small enough to validate the conclusions of the lemmas and propositions of the previous sections.

Finally, the classification of blow-up limits is exactly that stated in [15, Theorem 1.3], and proved in [15, Section 8], in view of Lemma 3.14 and Corollary 3.15.

References

[1] I. Athanasopoulos, L. A. Caffarelli, S. Salsa. The structure of the free boundary for lower dimensional obstacle problems. *Amer. J. Math.* 130 (2008), no. 2, 485–498.

[2] J. Azzam, X. Tolsa. Characterization of n-rectifiability in terms of Jones' square function: Part II. *Geom. Funct. Anal.* 25 (2015), no. 5, 1371–1412.

[3] B. Barrios, A. Figalli, X. Ros-Oton. Global regularity for the free boundary in the obstacle problem for the fractional Laplacian. *Amer. J. Math.* 140 (2018), 415–447.

[4] J.P. Bouchaud, A. Georges. Anomalous diffusion in disordered media: Statistical mechanics, models and physical applications. *Phys. Rep.* 195 (1990), 127–293.

[5] L. A. Caffarelli. The regularity of free boundaries in higher dimensions. *Acta Math.* 139 (1977), no. 3-4, 155–184.

[6] L. A. Caffarelli. The obstacle problem revisited. *J. Fourier Anal. Appl.* 4 (1998), no. 4-5, 383–402.

[7] L. Caffarelli, D. De Silva, O. Savin. The two membranes problem for different operators. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 34 (2017), no. 4, 899–932.

[8] L. A. Caffarelli, L. Silvestre. An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations* 32 (2007), no. 7-9, 1245–1260.

[9] L. A. Caffarelli, S. Salsa, L. Silvestre. Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian. *Invent. Math.* 171 (2008), no. 2, 425–461.

[10] L. A. Caffarelli, S. Vasseur. Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation. *Ann. of Math.* (2) 171 (2010), no. 3, 1903–1930.

[11] R. Cont, P. Tankov. Financial Modelling with Jump Processes. Chapman Hall/CRC Financ. Math. Ser. Chapman & Hall/CRC, Boca Raton, FL (2004)

[12] D. De Silva, O. Savin. C^∞ regularity of certain thin free boundaries. *Indiana Univ. Math. J.* 64 (2015), no. 5, 1575–1608.

[13] X. Fernández-Real, Y. Jhaveri. On the singular set in the thin obstacle problem: higher order blow-ups and the very thin obstacle problem Preprint arXiv:1812.01515

[14] X. Fernández-Real, X. Ros-Oton. Free boundary regularity for almost every solution to the Signorini problem. In preparation.

[15] M. Focardi, E. Spadaro. On the measure and the structure of the free boundary of the lower dimensional obstacle problem. *Arch. Rational Mech. Anal.*, (2018) 230: 125. https://doi.org/10.1007/s00205-018-1242-4

[16] M. Focardi, E. Spadaro. Correction to: On the Measure and the Structure of the Free Boundary of the Lower Dimensional Obstacle Problem. *Arch. Rational Mech. Anal.*, (2018) 230: 783. https://doi.org/10.1007/s00205-018-1273-x

[17] N. Garofalo, A. Petrosyan. Some new monotonicity formulas and the singular set in the lower dimensional obstacle problem. *Invent. Math.*, 177 (2009), no. 2, 415–461.

[18] N. Garofalo, X. Ros-Oton. Structure and regularity of the singular set in the obstacle problem for the fractional Laplacian. *Rev. Mat. Iberoam.* (2018), in press. Preprint arXiv:1704.00097.

[19] E. Hopf. Über den funktionalen, insbesondere den analytischen Charakter der Lösungen elliptischer Differentialgleichungen zweiter Ordnung. (German) Math. Z. 34 (1932), no. 1, 194–233.

[20] H. Koch, A. Petrosyan, W. Shi. Higher regularity of the free boundary in the elliptic Signorini problem. *Nonlinear Anal.* 126 (2015), 3–44.

[21] H. Koch, A. Rüland, W. Shi. The variable coefficient thin obstacle problem: higher regularity. *Adv. Differential Equations* 22 (2017), no. 11-12, 793–866.

[22] Y. Jhaveri, R. Neumayer. Higher regularity of the free boundary in the obstacle problem for the fractional Laplacian. *Adv. Math.* 311 (2017), 748–795.

[23] R. Monneau. On the number of singularities for the obstacle problem in two dimensions. *J. Geom. Anal.* 13 (2003), no. 2, 359–389.

[24] A. Naber, D. Valtorta. Rectifiable-Reifenberg and the regularity of stationary and minimizing harmonic maps. *Ann. of Math. (2)* 185 (2017), no. 1, 131–227.

[25] A. Naber, D. Valtorta. The Singular Structure and Regularity of Stationary and Minimizing Varifolds. *preprint* arXiv:1505.03428

[26] L. Silvestre. Regularity of the obstacle problem for a fractional power of the Laplace operator. *Comm. Pure Appl. Math.* 60 (2007), no. 1, 67–112.

[27] G. S. Weiss. A homogeneity improvement approach to the obstacle problem. *Invent. Math.*, 138 (1999), no. 1, 23–50.

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