A VARIFOLD PERSPECTIVE ON THE p-ELASTIC ENERGY OF PLANAR SETS

MARCO POZZETTA

Abstract. Under suitable regularity assumptions, the p-elastic energy of a planar set $E \subset \mathbb{R}^2$ is defined to be

$$\mathcal{F}_p(E) = \int_{\partial E} 1 + |k_{\partial E}|^p \, d\mathcal{H}^1,$$

where $k_{\partial E}$ is the curvature of the boundary ∂E. In this work we use a varifold approach to investigate this energy, that can be well defined on varifolds with curvature. First we show new tools for the study of 1-dimensional curvature varifolds, such as existence and uniform bounds on the density of varifolds with finite elastic energy. Then we characterize a new notion of L^1-relaxation of this energy by extending the definition of regular sets by an intrinsic varifold perspective, also comparing this relaxation with the classical one of [BeMu04], [BeMu07]. Finally we discuss an application to the inpainting problem, examples and qualitative properties of sets with finite relaxed energy.

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MSC Codes: 49Q15, 49Q20, 49Q10, 53A07.
Keywords: Curvature varifolds, p-elastic energy, Relaxation.

1. Introduction

Consider $p \in [1, \infty)$. Let us denote by S^1 the interval $[0, 2\pi]$ with the identification $0 \sim 2\pi$. For an immersion $\gamma : S^1 \to \mathbb{R}^2$ such that $\gamma \in W^{2,p}(S^1)$ we can define the functional

$$\mathcal{E}_p(\gamma) = \int_0^{2\pi} |k_\gamma|^p |\gamma'| \, dt, \tag{1}$$

Date: November 26, 2019.

and the p-elastic energy

$$\mathcal{F}_p(\gamma) = L(\gamma) + \mathcal{E}_p(\gamma), \tag{2}$$

where $L(\gamma)$ denotes the length of γ.

In this work we want to study the elastic properties of the boundaries of measurable sets in $\mathbb{R}^2$. Our first purpose is to give a new definition of the sets which are enough regular for having finite p-elastic energy. We want such definition to be intrinsically dependent on the given set, using immersions of curves only as a tool for the calculation of the energy.

In order to study the functionals defined in (1) and (2) one would classically call regular set a set E with a boundary of class C^2, i.e. a set E whose ∂E is the image of closed injective immersions $\gamma : S^1 \to \mathbb{R}^2$ of class C^2. This would be a possible definition of set with finite classic p-elastic energy, and it is the definition considered in [BeDaPa93], [BeMu04] and [BeMu07] indeed. But with this classical definition it turns out that sets like the one in Fig. 1 not only have infinite energy, but they also have infinite relaxed energy (calculated with respect to the L^1-convergence of sets, see [BeMu04]).

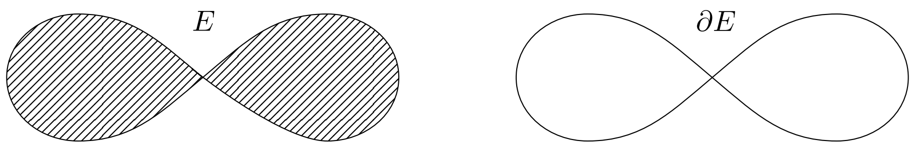

FIGURE 1. A set of finite perimeter E with boundary ∂E that can be parametrized by a smooth non-injective immersion.

However functionals (1) and (2) are very well defined on immersions which are not necessarily injective. Also for many applications one would like to consider sets like the one in Fig. 1 as regular sets, or at least as sets with finite relaxed energy (applications will also be discussed below). A good definition of regular elastic set, i.e. a definition of set with finite energy, comes intrinsically from the geometric properties of the boundary of sets of finite perimeter studied in the context of varifolds. In fact by De Giorgi's Theorem, if E is a set of finite perimeter in $\mathbb{R}^2$ then the reduced boundary $\mathcal{F}E$ is 1-rectifiable, and therefore the integer rectifiable varifold $V_E = \mathbf{v}(\mathcal{F}E, 1)$ is well defined. If a 1-rectifiable varifold $V = \mathbf{v}(\Gamma, \theta_V)$ has generalized curvature vector k_V, the analogue of the functionals (1) and (2) in the varifold context are defined by

$$\mathcal{E}_p(V) = \int_\Gamma |k_V|^p \, d\mu_V, \tag{3}$$

$$\mathcal{F}_p(V) = \mu_V(\mathbb{R}^2) + \mathcal{E}_p(V). \tag{4}$$

So such elastic energies can be calculated on the varifold V_E associated to a set of finite perimeter E, thus giving elasticity properties to the set E in a pure intrinsic way.
We can introduce the class of elastic varifolds without boundary as the integer rectifiable varifolds $V = \mathbf{v}(\Gamma, \theta_V)$ such that there exist a finite family of immersions $\gamma_i : S^1 \to \mathbb{R}^2$ such that

$$V = \sum_{i=1}^N (\gamma_i)_\sharp(\mathbf{v}(S^1, 1)), \tag{5}$$

where each $(\gamma_i)\sharp(\mathbf{v}(S^1, 1))$ is the image varifold of S^1 induced by γ_i. We shall see that a representation like (5) is not ambiguous and that the curves appearing in the formula can be used to compute the $\mathcal{F}_p$ energy (Lemma 2.8).

In this way we will eventually define that a set E is regular (in the sense that is has finite elastic energy) if

$$\mathbf{v}(\mathcal{F}E, 1) = \sum_{i=1}^{N} (\gamma_i)_\sharp(\mathbf{v}(S^1, 1)), \tag{6}$$

for some C^2 immersions $\gamma_i : S^1 \to \mathbb{R}^2$. In such a way the set in Fig. 1 is considered to be regular, and it has finite elastic energy.

We have to mention that a significant attempt in order to give a good definition of the elastic energy on sets that are not natural limits of smooth sets with bounded energy is contained in [BePa95] (see also the references therein). Here the authors consider an interesting generalization of the elastic energy functional whose relaxation is able to take into account the energy of angles and cusps.

Beside the study of the elastic properties of varifolds contained in Section 2, there are other fundamental motivations for studying this alternative notion of relaxed energy. We would like to extend this ambient perspective and strategy (at least starting from the basic definitions) to the study of the relaxation of functionals depending on the curvature of surfaces in $\mathbb{R}^3$, such as the Willmore energy. Moreover this work is the starting point for the study of the gradient flow of the elastic energy of planar sets using an intrinsic definition of the functional, not completely relying on immersions covering the boundary of the set; the characterization of the relaxed energy allows us to define the gradient flow on a huge family of sets and therefore to try to obtain a generalized flow (for example using a minimizing movements technique in the spirit of [LuSt95], and this will be the reason of some assumptions we will make in the following). Observe that in particular in a generalized flow one certainly wants to consider sets like the one in Fig. 1, hence a definition in which its energy is finite is required (see also [OkPoWh18]).

The paper is organized as follows. The first part of the work is devoted to the proof of some results about curves and varifolds with curvature from an ambient point of view. We prove a basic inequality concerning the elastic energy of immersed curves using a varifold perspective (Lemma 2.1), then we show an extension to 1-dimensional varifolds with curvature in $\mathbb{R}^n$ together with uniform bounds on the multiplicity function (Theorem 2.2) and a monotonicity formula for the case $p = 2$. This helps us to prove the main structural properties of elastic varifolds, which are contained in Lemma 2.8 and Lemma 2.9. Such results are stated for any $p \in [1, \infty)$.
In the second part we focus on the $p > 1$ case and we give a precise characterization of the L^1-relaxation of the energy $\mathcal{F}_p$ starting from our new notion (6) of regular set. The expression of the relaxed energy $\overline{\mathcal{F}_p}(E)$ takes the form of a minimization problem defined on a class $\mathcal{A}(E)$ of elastic varifolds suitably related to the set E (Theorem 3.2).
The relaxed energy $\overline{\mathcal{F}_p}$ has to be compared with the classical results contained in [BeDaPa93], [BeMu04] and [BeMu07], and in Subsection 4.1 we discuss an example of a set E with finite relaxed energy $\overline{\mathcal{F}_p}$ which is strictly less then its relaxed energy in the sense of [BeMu04] (which is still finite however). The last part of the work continues with an application to a minimization problem arising from the inpainting problem in image processing ([AmMa03], [BeCaMaSa11]). The relevance of our new definition of relaxed energy $\overline{\mathcal{F}_p}$ is particularly evident in this application. Then we conclude the work with some comments on the qualitative properties of sets having finite relaxed energy; here we prove that a set E with a boundary except that is smooth but at finitely many cusps has finite relaxed energy if and only if the number fo such cusps is even (Theorem 4.6), and we show that polygons always have infinite relaxed energy (Proposition 4.8).

2. Elastic energy of planar sets

2.1. **Notation and definitions.** In the following if γ is any parametrization of a curve, we denote by (γ) its image. The letter E will usually denote a measurable set in $\mathbb{R}^2$. We recall that E has finite perimeter in an open set $\Omega \subset \mathbb{R}^2$ if the characteristic function χ_E restricted to Ω belongs to $BV(\Omega)$, and in such case we denote by $P(E,\Omega)$ the perimeter of E in Ω. For the theory of sets of finite perimeter we refer to [AmFuPa00].
If $E \subset \mathbb{R}^2$ is measurable and $\Omega \subset \mathbb{R}^2$ is an open set such that E has finite perimeter in Ω, we denote by $D\chi_E$ the gradient measure of χ_E and by $|D\chi_E|$ the corresponding total variation measure. Then we denote by

$$\mathcal{F}E = \left\{ x \in \mathrm{supp}|D\chi_E| \cap \Omega \;\middle|\; \exists \lim_{\rho \searrow 0} \frac{D\chi_E(B_\rho(x))}{|D\chi_E(B_\rho(x))|} =: \nu_E(x),\, |\nu_E(x)| = 1 \right\},$$

and we call $\mathcal{F}E$ the reduced boundary of E, and ν_E is the generalized inner normal of E. By De Giorgi's Theorem the set $\mathcal{F}E$ is 1-rectifiable and $|D\chi_E| = \mathcal{H}^1 \llcorner \mathcal{F}E$.

Let $G(1,2)$ be the Cartesian product between $\mathbb{R}^2$ and the set of 1-dimensional subspaces in $\mathbb{R}^2$. We call $G(1,2)$ the Grassmannian of 1-dimensional spaces in $\mathbb{R}^2$. A point $(x,v) \in G(1,2)$ (where $v \in \mathbb{R}^2$ with $|v| = 1$ generates the given 1-dimensional subspace) is identified by the matrix $\pi_{x,v}$ that projects vectors in $T_x(\mathbb{R}^2)$ onto the subspace spanned by v; therefore $G(1,2)$ obtains a structure of metric space calculating the distance between two elements as the distance between the corresponding projection matrices. A 1-dimensional varifold in $\mathbb{R}^2$ is a positive finite Radon measure on $G(1,2)$. For the theory of varifolds we refer to [Si84], and in this work we will always deal with integer rectifiable varifolds.
For a 1-dimensional varifold V in $\mathbb{R}^2$ we denote by μ_V the induced measure in $\mathbb{R}^2$. We recall that a 1-dimensional rectifiable varifold $V = \mathbf{v}(\Gamma, \theta_V)$ in $\mathbb{R}^2$ has generalized curvature vector $k_V \in L^1_{loc}(\mu_V)$ and generalized boundary $\sigma_V \in \mathcal{M}^2(\mathbb{R}^2)$ if for any $X \in C^1_c(\mathbb{R}^2;\mathbb{R}^2)$ it holds

$$\int \left(\mathrm{div}_{T_x\Gamma} X\right) d\mu_V(x) = -\int \langle X, k_V\rangle \, d\mu_V + \int X \, d\sigma_V.$$

Recall that is such case the measure σ_V is singular with respect to μ_V. If $f : \mathrm{supp}V \to \mathbb{R}^2$ is Lipschitz we define the image varifold $f_\sharp(V) := \mathbf{v}(f(\Gamma), \tilde\theta)$ with $\tilde\theta(y) = \sum_{x \in f^{-1}(y)\cap\Gamma} \theta_V(x)$ for any $y \in \mathbb{R}^2$.
If E has finite perimeter in $\mathbb{R}^2$ we denote by V_E the associated varifold $V_E := \mathbf{v}(\mathcal{F}E, 1)$. If a varifold $V = \mathbf{v}(\Gamma, \theta_V)$ has generalized curvature k_V, then we define

$$\mathcal{E}_p(V) := \int |k_V|^p(x) \, d\mu_V(x),$$

while if V does not admit generalized curvature we then set $\mathcal{E}_p(V) = +\infty$.
At some point we will also use for a while some very basic facts about the theory of currents; for such definitions and results we refer to [Si84].

Here we recall with proof some basic properties of sets of finite perimeter, together with the choice of a convention and of the notation. The following observations actually work for sets of finite perimeter in any dimension.
If $E \subset \mathbb{R}^2$ is a measurable set, for any $t \in [0,1]$ we denote by E^t the subset of t-density points, that is

$$E^t := \left\{ x \in \mathbb{R}^2 \;\middle|\; \lim_{\rho \searrow 0} \frac{|E \cap B_\rho(x)|}{|B_\rho(x)|} = t \right\}. \tag{7}$$

The essential boundary $\partial^* E$ is then $\partial^* E := \mathbb{R}^2 \setminus (E^0 \cup E^1)$. Recall that for a set of perimeter E in $\mathbb{R}^3$ it holds that $\mathcal{F}E \subset E^{\frac12} \subset \partial^* E \subset \partial E$ and $\mathcal{H}^1(\partial^* E \setminus \mathcal{F}E) = 0$.
Following [LuSt95], since in BV we only consider equivalence classes of functions, for any set E with

$\chi_E \in BV(\mathbb{R}^2)$ we will always assume we have chosen the element of the class given by

$$E = \left\{ x \in \mathbb{R}^2 \;\middle|\; \int_{B_\rho(x)} \chi_E > 0 \ \forall \rho > 0 \right\}. \tag{8}$$

In this way the distance function $d(\cdot, \partial E)$ is well defined. Assuming (8) we also have

$$E = \overline{E^1}. \tag{9}$$

In fact $E^1 \subset E$ and E is closed, indeed if $x \notin E$ then there is $\rho_0 > 0$ such that $\int_{B_{\rho_0}(x)} \chi_E = 0$ and we can conclude that $B_{\frac{\rho_0}{2}}(x) \cap E = \emptyset$ (so that the complement of E is open), because if by contradiction there is $y \in B_{\frac{\rho_0}{2}}(x) \cap E$ then $0 < \int_{B_{\frac{\rho_0}{2}}(y)} \chi_E \le \int_{B_{\rho_0}(x)} \chi_E = 0$. In this way we have $\overline{E^1} \subset E$. But χ_E, χ_{E^1} are in the same equivalence class, that is $|E \Delta E^1| = 0$; if $x \in E$ there exists $x_n \in E^1$ converging to x, otherwise $|E^1 \cap B_\rho(x)| = 0$ for some $\rho > 0$ and by definition $|E \cap B_\rho(x)| > 0$, contradicting that $|E \Delta E^1| = 0$.
Moreover

$$\partial E \equiv \partial E^1 = \partial^m E := \left\{ x \in \mathbb{R}^2 \mid \forall \rho > 0 \ |B_\rho(x) \cap E| > 0, \, |B_\rho(x) \cap E^c| > 0 \right\}. \tag{10}$$

In fact if $x \in \partial^m E$ then there are sequences $x_n^1 \in E, x_n^2 \in E^c$ converging to x, thus $x \in E \cap \overline{E^c} = \partial E$. Conversely if $x \in \partial E = \partial E^c = \overline{E^c} \setminus E^c$, since $E^c = \{x : \exists r > 0 \text{ s.t. } |E \cap B_r(x)| = 0\}$, then $|E \cap B_\rho(x)| = \int_{B_\rho(x)} \chi_E > 0$ for any $\rho > 0$. But also $|E^c \cap B_\rho(x)| = \int_{B_\rho(x)} \chi_{E^c} > 0$ for any $\rho > 0$, indeed if by contradiction there is $\rho_0 > 0$ with $\int_{B_{\rho_0}(x)} \chi_{E^c} = 0$, since E^c is open we have that $E^c \cap B_{\rho_0}(x) = \emptyset$, but this is impossible because $x \in \overline{E^c}$.
Finally under assumption (8) we have that if E is a set of finite perimeter in $\mathbb{R}^2$, then the reduced boundary is dense in the boundary of E, that is

$$\overline{\mathcal{F}E} = \partial^m E = \partial E. \tag{11}$$

In fact $\mathcal{F}E \subset \partial E$ and if by contradiction there is $x \in \partial^m E \setminus \overline{\mathcal{F}E}$, then for some $\rho_0 > 0$ we have $B_{\rho_0}(x) \cap \overline{\mathcal{F}E} = \emptyset$ and $0 < |E \cap B_{\rho_0}(x)| < \pi \rho_0^2$. Hence by relative isoperimetric inequality in the ball $B_{\rho_0}(x)$ we get that $P(E, B_{\rho_0}(x)) > 0$, but since $B_{\rho_0}(x) \cap \overline{\mathcal{F}E} = \emptyset$ using De Giorgi's Theorem we also have $P(E, B_{\rho_0}(x)) = \mathcal{H}^1(\mathcal{F}E \cap B_{\rho_0}(x)) = 0$, which gives a contradiction.
Observe that it also follows that $\text{diam}\mathcal{F}E = \text{diam}\partial E$.

2.2. **Preliminary estimates.** Here we prove a fundamental estimate concerning curves in $\mathbb{R}^2$.

Lemma 2.1. *Let $\gamma : S^1 \to \mathbb{R}^2$ be a regular curve in $W^{2,p}$ for some $p \in [1, \infty)$. Then*

$$2\pi \le \mathcal{E}_1(\gamma) \le (\mathcal{E}_p(\gamma))^{\frac{1}{p}} (L(\gamma))^{\frac{1}{p'}}, \tag{12}$$

where $L(\gamma)$ denotes the length of the curve. Moreover for $p > 1$ equality holds if and only if γ parametrizes a circumference of radius 1, while for $p = 1$ equality holds if and only if γ parametrizes a circumference.

Proof. By approximation it is enough to prove the statement for $\gamma \in C^\infty$. Call $\sigma : [0, L] \to \mathbb{R}^2$ the arclength parametrization of the curve. We want to prove that

$$\int_0^L |k_\sigma| \ge 2\pi, \tag{13}$$

so that applying Holder inequality on the left, (13) will give the claim. Let $T : [0, L] \to S^1$ be the unit tangent vector $T = \dot{\sigma}$. The support (σ) is compact, then there are two parallel lines $\{x = \alpha\}, \{x = \beta\}$ such that $(\sigma) \subset \{\alpha \le x \le \beta\}$ and (σ) touches tangentially the two lines at points $\sigma(0) = x_0, \sigma(b) = x_b$ (up to reparametrization). Assume that $T(0) = (0, 1)$. By continuity there exists $a \in (0, b)$ such that

$T(a) = (-1,0)$ and $c \in (b,L)$ such that $T(c) = (1,0)$. Now two cases can occur: $T(b) = (0,1)$ or $T(b) = (0,-1)$. If $T(b) = (0,-1)$, then $S^1 \subset T([0,L])$; if $T(b) = (0,1)$, then $S^1 \cap \{(x,y)\,|\,y \geq 0\} \subset T([0,L])$ and for any $v \in S^1 \cap \{(x,y)\,|\,y > 0\}$ we have that $\sharp T^{-1}(v) \geq 2$. Therefore using the area formula we obtain

$$\int_0^L |k_\sigma|(s)\,ds = \int_0^L |T'|(s)\,ds = \int_{S^1} \sharp T^{-1}(v)\,d\mathcal{H}^1(v) \geq 2\pi.$$

By the above calculations equality holds if and only if σ is convex with constant curvature $|k_\sigma|$ and with $\sharp T^{-1}(v) = 1$ for $\mathcal{H}^1$ almost every $v \in S^1$. Hence this completes the proof. $\square$

We mention that inequality (13) is already present in [DaNoPl18], proved with a different method in the setting of networks, but in the following we will need the specific approach used in the proof of Lemma 2.1. Also, we are going to prove that inequality (12) is true (up to changing the constant) in an analogous sense in the setting of varifolds as stated in the inequality (15) in Subsection 2.3.

2.3. **Monotonicity.** Here we develop a monotonicity-type argument that is the direct analogue of Simon's Monotonicity Formula ([Si93]), which is fundamental in the study of the Willmore energy, that in some sense is the two-dimensional energy corresponding to the functional $\mathcal{E}_2$. This result is of independent interest and it will be sated in general in $\mathbb{R}^n$.

Throughout this Subsection consider $x_0 \in \mathbb{R}^n$, $0 < \sigma < \rho < +\infty$, $V = \mathbf{v}(\Gamma, \theta_V) \neq 0$ an integer 1-dimensional rectifiable varifold in $\mathbb{R}^n$ with curvature k_V such that $\mathcal{E}_p(V) < +\infty$ for some $p \in [1,\infty)$ (for the moment μ_V is just locally finite on $\mathbb{R}^n$). Also we are assuming that $\sigma_V = 0$.

Consider the field $X(x) = \left(\frac{1}{|x-x_0|_\sigma} - \frac{1}{\rho}\right)_+ x$, where $(\cdot)_+$ denotes the nonnegative part and $|\cdot|_\sigma = \max\{|\cdot|, \sigma\}$. For any set E let $E_\sigma = E \cap B_\sigma$, $E_\rho = E \cap B_\rho$, $E_{\rho,\sigma} = E_\rho \setminus \overline{B_\sigma}$. Then the tangential divergence of X is

$$\mathrm{div}_{T\Gamma} X(x) = \begin{cases} \frac{1}{\sigma} - \frac{1}{\rho} & \text{on } \Gamma_\sigma, \\ \frac{|(x-x_0)^\perp|^2}{|x-x_0|^3} - \frac{1}{\rho} & \text{on } \Gamma_{\rho,\sigma}. \end{cases}$$

We want to prove the following result.

Theorem 2.2. *Under the above assumptions it holds that*

$$\limsup_{\sigma \searrow 0} \frac{\mu_V(B_\sigma(x_0))}{\sigma} \leq \liminf_{\rho \nearrow \infty} \frac{\mu_V(B_\rho(x_0))}{\rho} + \int_{B_\rho(x_0)} |k_V|\,d\mu_V(x). \tag{14}$$

If also $\mu_V(\mathbb{R}^n) < +\infty$, then

$$2 \leq \mathcal{E}_1(V) \leq \mu_V(\mathbb{R}^n)^{\frac{1}{p'}} \mathcal{E}_p(V)^{\frac{1}{p}}, \tag{15}$$

and we have the following bounds on the multiplicity function:

$$p > 1 \quad \Rightarrow \quad \theta_V(x) \leq \frac{1}{2}\mathcal{E}_1(V) \qquad \forall\, x \in \mathbb{R}^n, \tag{16}$$

$$p = 1 \quad \Rightarrow \quad \theta_V(x) \leq \frac{1}{2}\mathcal{E}_1(V) \qquad \textit{for } \mathcal{H}^1\textit{-ae } x \in \mathbb{R}^n. \tag{17}$$

If also $\mu_V(\mathbb{R}^n) < +\infty$ and Γ is essentially bounded, i.e. $\mathcal{H}^1(\Gamma \setminus B_R(0)) = 0$ for R large enough, then

$$p = 1 \quad \Rightarrow \quad \exists \lim_{r \searrow 0} \frac{\mu_V(B_r(x))}{2r} = \theta_V(x) \leq \frac{1}{2}\mathcal{E}_1(V) \qquad \forall\, x \in \mathbb{R}^n. \tag{18}$$

Proof. Integrating the divergence $\mathrm{div}_{T\Gamma}X$ above with respect to μ_V and using the first variation formula we get

$$\begin{aligned}\frac{\mu_V(B_\sigma(x_0))}{\sigma}+\frac{1}{\sigma}\int_{B_\sigma(x_0)}\langle k_V,x-x_0\rangle\,d\mu_V(x)+\int_{B_\rho(x_0)\setminus B_\sigma(x_0)}\frac{|(x-x_0)^\perp|^2}{|x-x_0|^3}\,d\mu_V(x)=\\ =\frac{\mu_V(B_\rho(x_0))}{\rho}+\frac{1}{\rho}\int_{B_\rho(x_0)}\langle k_V,x-x_0\rangle\,d\mu_V(x)-\int_{B_\rho(x_0)\setminus B_\sigma(x_0)}\left\langle k_V,\frac{x-x_0}{|x-x_0|}\right\rangle d\mu_V(x).\end{aligned}\tag{19}$$

Dropping the positive term on the left we obtain

$$\begin{aligned}\frac{\mu_V(B_\sigma(x_0))}{\sigma}+\frac{1}{\sigma}\int_{B_\sigma(x_0)}\langle k_V,x-x_0\rangle\,d\mu_V(x)\leq\\ \leq\frac{\mu_V(B_\rho(x_0))}{\rho}+\int_{B_\rho(x_0)}\left\langle k_V,\frac{x-x_0}{\rho}-\frac{x-x_0}{|x-x_0|}\chi_{B_\rho(x_0)\setminus B_\sigma(x_0)}\right\rangle d\mu_V(x).\end{aligned}$$

Since

$$\left|\frac{1}{\sigma}\int_{B_\sigma(x_0)}\langle k_V,x-x_0\rangle\,d\mu_V(x)\right|\leq\left(\int_{B_\sigma(x_0)}|k_V|^p\,d\mu_V\right)^{\frac{1}{p}}\big(\mu_V(B_\sigma(x_0))^{\frac{1}{p'}}\xrightarrow[\sigma\to0]{}0,$$

and

$$\frac{x-x_0}{|x-x_0|}\chi_{B_\rho(x_0)\setminus B_\sigma(x_0)}\xrightarrow[\sigma\to0]{}\frac{x-x_0}{|x-x_0|}\chi_{B_\rho(x_0)}\qquad\text{in }L^{p'}(\mu_V),$$

letting $\sigma\searrow0$ and then $\rho\nearrow\infty$ we get the inequality

$$\begin{aligned}\limsup_{\sigma\searrow0}\frac{\mu_V(B_\sigma(x_0))}{\sigma}&\leq\liminf_{\rho\nearrow\infty}\frac{\mu_V(B_\rho(x_0))}{\rho}+\int_{B_\rho(x_0)}\left\langle k_V,\left(\frac{1}{\rho}-\frac{1}{|x-x_0|}\right)(x-x_0)\right\rangle d\mu_V(x)\leq\\ &\leq\liminf_{\rho\nearrow\infty}\frac{\mu_V(B_\rho(x_0))}{\rho}+\int_{B_\rho(x_0)}|k_V|\left|\frac{|x-x_0|}{\rho}-1\right|d\mu_V(x)\leq\\ &\leq\liminf_{\rho\nearrow\infty}\frac{\mu_V(B_\rho(x_0))}{\rho}+\int_{B_\rho(x_0)}|k_V|\,d\mu_V(x).\end{aligned}$$

that is (14).
Suppose from now on that $\mu_V(\mathbb{R}^n)<+\infty$, then (14) gives

$$\limsup_{\sigma\searrow0}\frac{\mu_V(B_\sigma(x_0))}{\sigma}\leq\mathcal{E}_1(V).\tag{20}$$

Equation (20) gives us the pointwise bounds on the multiplicity function θ_V as follows.
If $p>1$ we know that the density $\lim_{\sigma\searrow0}\frac{\mu_V(B_\sigma(p))}{2\sigma}$ exists at any p and can be used as multiplicity function θ_V for V ([Si84], page 86). So in this case (20) gives

$$\theta_V(x)=\lim_{\sigma\searrow0}\frac{\mu_V(B_\sigma(p))}{2\sigma}\leq\frac{1}{2}\mathcal{E}_1(V)\qquad\forall\,x\in\Gamma.\tag{21}$$

Instead in the $p=1$ case we can say the following. Since Γ has generalized tangent space at $\mathcal{H}^1$-ae point we have that

$$\theta_V(x)=\lim_{\sigma\searrow0}\frac{\mu_V(B_\sigma(x_0))}{2\sigma}\leq\frac{1}{2}\mathcal{E}_1(V)\qquad\text{for }\mathcal{H}^1\text{-ae }x\in\Gamma.\tag{22}$$

Therefore, since $\theta_V(x)\geq1$ at some point x, for any $p\in[1,\infty)$ we can state inequality (15).

Now assume $p=1$ and without loss of generality $\Gamma\subset B_{R_0}(0)$ is bounded, then we want to show that the limit $\lim_{\sigma\searrow 0}\frac{\mu_V(B_\sigma(x_0))}{\sigma}$ does exist for any $x_0\in\mathbb{R}^n$. In fact in Equation (19) we have

$$\begin{aligned}
&\left|\frac{1}{\sigma}\int_{B_\sigma(x_0)}\langle k_V,x-x_0\rangle\,d\mu_V(x)\right|\to 0 \qquad \text{as } \sigma\to 0,\\
&\left|\frac{1}{\rho}\int_{B_\rho(x_0)}\langle k_V,x-x_0\rangle\,d\mu_V(x)\right|\le\frac{R_0}{\rho}\int_{B_{R_0}(x_0)}|k_V|\,d\mu_V\to 0 \qquad \text{as } \rho\to\infty,\\
&\frac{x-x_0}{|x-x_0|}\chi_{B_\rho(x_0)\setminus B_\sigma(x_0)}\to\frac{x-x_0}{|x-x_0|} \qquad \text{in } L^1(\mathbb{R}^n,\mu_V),
\end{aligned}$$

where the last statement follows by Dominated Convergence. Therefore there exists the limit

$$\lim_{\sigma\searrow 0,\,\rho\nearrow\infty}\int\left\langle k_V,\frac{x-x_0}{|x-x_0|}\chi_{B_\rho(x_0)\setminus B_\sigma(x_0)}\right\rangle d\mu_V(x),$$

which is also finite. Hence (19) implies that

$$\sup_{\sigma,\rho>0}\int_{B_\rho(x_0)\setminus B_\sigma(x_0)}\frac{|(x-x_0)^\perp|^2}{|x-x_0|^3}\,d\mu_V(x)<+\infty, \tag{23}$$

thus by monotonicity the limit

$$\lim_{\sigma\searrow 0,\,\rho\nearrow\infty}\int_{B_\rho(x_0)\setminus B_\sigma(x_0)}\frac{|(x-x_0)^\perp|^2}{|x-x_0|^3}\,d\mu_V(x)$$

exists finite. Since $\lim_{\rho\to\infty}\frac{\mu_V(B_\rho(x_0))}{\rho}\to 0$ as $\rho\to 0$, Equation (19) implies that

$$\exists\lim_{\sigma\searrow 0}\frac{\mu_V(B_\sigma(x_0))}{\sigma}<+\infty \qquad \forall x_0\in\mathbb{R}^n, \tag{24}$$

which completes the proof. □

We mention that the inequality (15) is probably not sharp, but still new in the context of 1-dimensional varifolds.

We conclude with a monotonicity statement concerning the $p=2$ case.

Remark 2.3. Let $p=2$. For $r>0$ let

$$A(r)=\left(\frac12+\frac1r\right)\mu_V(B_r(x_0))+\frac1r\int_{B_r(x_0)}\langle k_V,x-x_0\rangle\,d\mu_V(x)+\frac12\int_{B_r(x_0)}|k_V|^2\,d\mu_V.$$

Then

$$A(\sigma)+\int_{B_\rho(x_0)\setminus B_\sigma(x_0)}\left(\frac{|(x-x_0)^\perp|^2}{|x-x_0|^3}+\frac12\left|k_V+\frac{x-x_0}{|x-x_0|}\right|^2\right)d\mu_V(x)=A(\rho), \tag{25}$$

in particular $r\mapsto A(r)$ is nondecreasing.
Indeed to prove (25) just insert the identity $\langle k_V,\frac{x}{|x|}\rangle=\frac12\left(\left|k_V+\frac{x}{|x|}\right|^2-|k_V|^2-1\right)$ in (19).
Moreover if we additionally require that $\mu_V(\mathbb{R}^n)<+\infty$, then

$$\begin{aligned}
&\frac1R\int_{B_R(x_0)}\langle k_V,x-x_0\rangle\,d\mu_V(x)=\frac1R\int_{B_r(x_0)}\langle k_V,x-x_0\rangle\,d\mu_V(x)+\frac1R\int_{B_R(x_0)\setminus B_r(x_0)}\langle k_V,x-x_0\rangle\,d\mu_V(x)\le\\
&\quad\le\frac1R\int_{B_r(x_0)}\langle k_V,x-x_0\rangle\,d\mu_V(x)+\left(\int_{B_R(x_0)\setminus B_r(x_0)}|k_V|^2\right)^{\frac12}\big(\mu_V(B_R(x_0)\setminus B_r(x_0))\big)^{\frac12}
\end{aligned}$$

for any $r < R$. So letting first $R \to \infty$ and then $r \to \infty$ we get that $\frac{1}{R}\int_{B_R(x_0)} \langle k_V, x - x_0\rangle \, d\mu_V(x) \to 0$ as $R \to \infty$. And thus we obtain that

$$\lim_{r\to\infty} A(r) = \frac{1}{2}\bigg(\mu_V(\mathbb{R}^n) + \mathcal{E}_2(V)\bigg), \tag{26}$$

for any choice of $x_0 \in \mathbb{R}^n$.

Remark 2.4. After the conclusion of the work the author became aware of the fact that Theorem 2.2 also follows from Corollary 4.8 in [Me16] (see also Theorem 3.5 in [MeSc18]).

2.4. **Elastic varifolds.** Here we prove some important remarks about varifolds defined through immersions of elastic curves. The next definition comes from [BeMu04].

Definition 2.5. Given a family of regular C^1 curves $\alpha_i : (-a_i, a_i) \to \mathbb{R}^2$ for $i = 1, ..., N$ and a point $p \in \mathbb{R}^2$ such that $\alpha_i(t_i) = p$ for some times t_i and the curves $\{\alpha_i\}$ are tangent at p. Let $v \in S^1$ such that $\alpha_i'(t_i)$ and v are parallel for any i. We say that $R_v(p)$ is a *nice rectangle at p for the curves $\{\alpha_i\}$ with side parameters $a, b > 0$* if

$$R_v(p) = \{z \in \mathbb{R}^2 \;:\; |\langle z - p, v\rangle| < a,\, |\langle z - p, v^\perp\rangle| < b\},$$

and

$$R_v(p) \cap \bigg(\bigcup_{i=1}^{N} (\alpha_i)\bigg) = \bigcup_{i=1}^{M} graph(f_i),$$

for distinct C^1 functions $f_i : [-a, a] \to (-b, b)$, where $graph(f_i)$ denotes the graph of f_i constructed on the lower side of the rectangle.

We also give the following definition.

Definition 2.6. Let $V = \mathbf{v}(\cup_{i\in I}(\gamma_i), \theta_V)$ be a varifold defined by the $C^1 \cap W^{2,p}$ immersions $\gamma_i : S^1 \to \mathbb{R}^2$, and assume that $\mathcal{F}_p(V) < +\infty$, $\theta_V \le C < +\infty$.
For any $p \in \cup_{i\in I}(\gamma_i)$ and any $v \in S^1$ denote by $g_1, ..., g_r : [-\varepsilon, \varepsilon] \hookrightarrow \mathbb{R}^2$ arclength parametrized injective arcs such that: $g_i(0) = p$, $\dot{g}_i(0) = v$, $g_i([-\varepsilon, 0]) \ne g_j([-\varepsilon, 0])$ or $g_i([0, \varepsilon]) \ne g_j([0, \varepsilon])$ for $i \ne j$, and $\cup_{i=1}^r (g_i) \cap \overline{B}_\rho(p) = \cup_{i\in I}(\gamma_i) \cap \overline{B_\rho(p)}$. Observe that for any such p, v and ρ small enough, the arcs g_i are well defined.
We say that V verifies the *flux property* if: $\forall\, p \in \cup_{i\in I}(\gamma_i)$, $\forall\, v \in S^1$, and ρ small enough there exists a nice rectangle $R_v(p) \subset B_\rho(p)$ for the family of arcs $\{g_i\}$ such that it holds that

$$\forall |c| < a : \qquad \sum_{z \in \cup_{i=1}^r (g_i) \cap \{y \,|\, \langle y - p, v\rangle = c\}} \theta_V(z) = M,$$

for a constant $M \in \mathbb{N}$ with $M \le \theta_V(p)$.

Roughly speaking, Definition 2.6 requires that the "incoming" total amount of multiplicity at p in direction v equals the "outcoming" total amount of multiplicity at p in direction v.
Observe that if $V = \sum_{i\in I}(\gamma_i)_\sharp(\mathbf{v}(S^1, 1))$ with $\gamma_i \in C^1 \cap W^{2,p}$ immersions and $\mathcal{F}_p(V) < +\infty$, $\theta_V \le C < +\infty$, then V verifies the flux property.

Remark 2.7. Let E be a set of finite perimeter in $\mathbb{R}^2$, let $\Gamma = \cup_{i=1}^N(\gamma_i)$ with $\gamma_i \in C^1(S^1; \mathbb{R}^2)$ and regular for any i. Assume that $V_E := \mathbf{v}(\mathcal{F}E, 1) = \sum_{i=1}^N (\gamma_i)_\sharp(\mathbf{v}(S^1, 1))$. Then $\mathcal{H}^1(\partial E \setminus \mathcal{F}E) = 0$, and we can equivalently write $V_E = \mathbf{v}(\partial E, 1)$.
In fact by assumption $\mathcal{H}^1$-almost every point $p \in \Gamma$ is contained in $\mathcal{F}E$, supp$V_E = \Gamma$, and $\Gamma =$

$\mathrm{supp}V_E = \mathrm{supp}(\mathcal{H}^1 \llcorner \mathcal{F}E) = \partial E$. Therefore $0 = \mathcal{H}^1(\Gamma \setminus \mathcal{F}E) = \mathcal{H}^1(\partial E \setminus \mathcal{F}E)$.

Lemma 2.8. *Assume $p > 1$. If an integer rectifiable varifold $V = \mathbf{v}(\Gamma, \theta_V)$ is such that $V = \sum_{i=1}^N (\gamma_i)_\sharp(\mathbf{v}(S^1, 1))$ for some regular curves $\gamma_i \in W^{2,p}(S^1; \mathbb{R}^2)$ and $\mathcal{F}_p(V) < +\infty$, then V has generalized curvature*

$$k_V(p) = \frac{1}{\theta_V(p)} \sum_{i=1}^N \sum_{t \in \gamma_i^{-1}(p)} k_{\gamma_i}(t) \qquad \text{at } \mathcal{H}^1\text{-ae } p \in \Gamma, \tag{27}$$

the generalized boundary $\sigma_V = 0$, and

$$\mathcal{E}_p(V) = \sum_{i=1}^N \mathcal{E}_p(\gamma_i). \tag{28}$$

In particular, since k_V is unique, the value $\mathcal{E}_p(V)$ is independent of the choice of the family of curves $\{\gamma_i\}$ defining V.

Proof. In fact suppose first that $N = 1$, and then call $\gamma_1 = \gamma$. Up to rescaling, assume without loss of generality that γ is an arclength parametrization. By assumption $\gamma \in C^{1,\alpha}$ for $\alpha \le \frac{1}{p'}$, and clearly $\Gamma = (\gamma)$ and $\mu_V = \theta_V \mathcal{H}^1 \llcorner (\gamma) = \gamma_\sharp(\mathcal{H}^1 \llcorner S^1)$ (by the arclength parametrization assumption). If $X \in C^1_c(\mathbb{R}^2; \mathbb{R}^2)$ is a vector field, using the area formula and the fact that $\theta_V \ge 1$ $\mathcal{H}^1$-ae on Γ, we have

$$\begin{aligned}
\int \mathrm{div}_{T_p\Gamma} X \, d\mu_V(p) &= \int \mathrm{div}_{T_{p}(\gamma)} X \, d\gamma_\sharp(\mathcal{H}^1 \llcorner S^1)(p) = \int_{S^1} \langle \gamma'(t), (\nabla X)_{\gamma(t)}(\gamma'(t)) \rangle \, dt = \\
&= \int_{S^1} \langle \gamma', (X \circ \gamma)' \rangle \, dt = -\int_{S^1} \langle \gamma''(t), X(\gamma(t)) \rangle \, dt = \\
&= -\int \int_{\gamma^{-1}(p)} \langle \gamma''(t), X(\gamma(t)) \rangle \, d\mathcal{H}^0 \, d\mathcal{H}^1 \llcorner \Gamma(p) = \\
&= -\int \left\langle X(p), \int_{\gamma^{-1}(p)} k_\gamma(t) \, d\mathcal{H}^0 \right\rangle \frac{\theta_V(p)}{\theta_V(p)} \, d\mathcal{H}^1 \llcorner \Gamma(p) = \\
&= -\int \left\langle X(p), \frac{1}{\theta_V(p)} \sum_{t \in \gamma^{-1}(p)} k_\gamma(t) \right\rangle d\mu_V(p).
\end{aligned}$$

If now $N > 1$, by linearity of the first variation we get

$$\begin{aligned}
\int \mathrm{div}_{T_p\Gamma} X \, d\mu_V(p) &= -\sum_{i=1}^N \int \left\langle X(p), \sum_{t \in \gamma_i^{-1}(p)} k_{\gamma_i}(t) \right\rangle d\mathcal{H}^1 \llcorner (\gamma_i)(p) = \\
&= -\int \left\langle X(p), \frac{1}{\theta_V(p)} \sum_{i=1}^N \sum_{t \in \gamma_i^{-1}(p)} k_{\gamma_i}(t) \right\rangle \theta_V(p) \, d\mathcal{H}^1 \llcorner \left(\cup_{i=1}^n (\gamma_i) \right) = \\
&= -\int \left\langle X(p), \frac{1}{\theta_V(p)} \sum_{i=1}^N \sum_{t \in \gamma_i^{-1}(p)} k_{\gamma_i}(t) \right\rangle d\mu_V.
\end{aligned}$$

Now we want to prove (28). Let us consider the set $W = \{p \in \Gamma \,|\, \theta_V(p) > 1\}$. Up to redefining some γ_i on another circumference, we can suppose from now on that γ_i is an arclength parametrization. We can write $W = T \cup X \cup Y \cup Z$, with

$$\begin{aligned} T &= \big\{p \in W \,|\, \exists i,j,t,\tau :\ \gamma_i(t)=\gamma_j(\tau)=p,\, \gamma_i'(t) \neq \alpha\gamma_j'(\tau)\ \forall\alpha\in\mathbb{R}\big\},\\ X &= \big\{p \in W\setminus T \,|\, \exists i,t :\ \gamma_i(t)=p,\, t \text{ is not a Lebesgue point of } \gamma_i''\big\},\\ Y &= \big\{p \in W\setminus(T\cup X) \,|\, \forall i,j,t,\tau :\ \gamma_i(t)=\gamma_j(\tau)=p \Rightarrow \gamma_i''(t)=\gamma_j''(\tau)\big\},\\ Z &= \big\{p \in W\setminus(T\cup X) \,|\, \exists i,j,t,\tau :\ \gamma_i(t)=\gamma_j(\tau)=p,\, \gamma_i''(t)\neq\gamma_j''(\tau)\big\}. \end{aligned}$$

We are going to prove that T, Z are at most countable, then since $\mathcal{H}^1(X)=0$ we will get that $\mathcal{H}^1(W)=\mathcal{H}^1(Y)$. Hence by (27) one immediately gets (28).
Let $p\in\Gamma$ and $C\in\mathbb{N}$ such that $\theta_V\le C$. Let $v_1(p),...,v_k(p)\in S^1$ with $k=k(p)\le C$ such that if $\gamma_i(t)=p$ then $\gamma_i'(t)$ is proportional to some v_j. For any $i=1,...,k$ let $R_{v_i}(p)$ be a nice rectangle at p for the curves $\{\alpha_j\}_{j\in J(i)}$ which are suitable restrictions of the curves $\{\gamma_i\}$. Then let $f_1^i,...,f_l^i$ with $l=l(i)$ be C^1 functions $f_s^i:[-a_i,a_i]\to(-b_i,b_i)$ given by the definition of nice rectangle.
Let $q\in\cup_{s=1}^l graph(f_s^i)$, and assume $q\in T$. If a_i is chosen sufficiently small, the fact that q belongs to T means that the transversal intersection happens between some of the curves $\{\alpha_j\}_{j\in J(i)}$. This means that there is some $\delta_q>0, x_q\in(-a_i,a_i), r,s\in\{1,...,l\}$ such that

$$f_r^i(x_q)=f_s^i(x_q),\quad (x_q,f_r^i(x_q))=q,\quad graph\big(f_r^i|_{(x_q-\delta_q,x_q+\delta_q)}\big)\cap graph\big(f_s^i|_{(x_q-\delta_q,x_q+\delta_q)}\big)=\{q\}.$$

Letting $A_i=\{x\in(-a_i,a_i)\,|\,f_r^i\neq f_s^i\}$, which is open, we see that x_q belongs to the boundary of some connected component of A_i. This implies that $T\cap\big(\cup_{s=1}^l graph(f_s^i)\big)$ is countable, and this is true for any $i=1,...,k(p)$.
For any $p\in\Gamma$ take a ball $B_{r(p)}(p)\subset\cap_{i=1}^{k(p)}R_{v_i(p)}(p)$ for suitable rectangles $R_{v_i(p)}(p)$ as above. Then $T\cap B_{r(p)}(p)$ is countable. Since Γ is compact, taking a finite cover of such balls $B_{r(p_1)}(p_1),...,B_{r(p_L)}(p_L)$, we conclude that T is countable.

Consider now $q\in\cup_{s=1}^l graph(f_s^i)$, and assume $q\in Z$. If a_i is chosen sufficiently small, the fact that q belongs to Z means that the tangential intersection happens between some of the curves $\{\alpha_j\}_{j\in J(i)}$. Hence at some $x_q\in(-a_i,a_i)$ for some $r,s\in\{1,...,l\}$ we find that x_q is a Lebesgue point for $(f_r^i)''$ and $(f_s^i)''$, and

$$f_r^i(x_q)=f_s^i(x_q),\quad (x_q,f_r^i(x_q))=q,\quad (f_r^i)''(x_q)\neq(f_s^i)''(x_q).$$

This implies that there exists $\varepsilon>0$ such that for any $0<|t|<\varepsilon$ we have $(f_r^i)'(x_q+t)\neq(f_s^i)'(x_q+t)$. By continuity of the first derivative we have that, for example, $(f_r^i)'(x_q+t)>(f_s^i)'(x_q+t)$ for any $0<|t|<\varepsilon$, and therefore $f_r^i(x_q+t)>f_s^i(x_q+t)$ for any $0<|t|<\varepsilon$. So we find that x_q belongs to the boundary of a connected component of an A_i defined as above as in the case of the set T. Arguing as before we eventually get that Z is countable.

□

Lemma 2.9. *Let $\gamma_1,...,\gamma_N:S^1\to\mathbb{R}^2$ be Lipschitz curves and let $V=\mathbf{v}(\Gamma,\theta_V)=\sum_{i=1}^N(\gamma_i)_\sharp(\mathbf{v}(S^1,1))$. Assume that $\mathcal{H}^1(\{x\,|\,\theta_V(x)>1\})=0$, and define*

$$E=\left\{p\in\mathbb{R}^2\setminus\Gamma\ :\ \left|\sum_{i=1}^N \mathrm{Ind}_{\gamma_i}(p)\right| \text{ is odd}\right\}\cup\Gamma. \tag{29}$$

Then $V=V_E:=\mathbf{v}(\mathcal{F}E,1)$, and E is uniquely determined by V, i.e. if $V=\sum_{i=1}^N(\gamma_i)_\sharp(\mathbf{v}(S^1,1))=\sum_{i=1}^M(\sigma_i)_\sharp(\mathbf{v}(S^1,1))$ then the corresponding set E defined using (29) with the family $\{\gamma_i\}$ is the same set defined using (29) with the family $\{\sigma_i\}$.

Proof. The set E is closed and bounded, with $\mathring{E}=\{p\in\mathbb{R}^2\setminus\Gamma\ :\ |\sum_{i=1}^N\mathrm{Ind}_{\gamma_i}(p)| \text{ is odd}\}$ and $\partial E=\Gamma$, hence E is a set of finite perimeter.
Let us first check that if $V=\sum_{i=1}^N(\gamma_i)_\sharp(\mathbf{v}(S^1,1))=\sum_{i=1}^M(\sigma_i)_\sharp(\mathbf{v}(S^1,1))$, then the definition of E by

(29) is independent of the choice of the family of curves. The fact that a point $p\in\mathbb{R}^2\setminus\Gamma$ belongs to E depends on the residue class

$$\left(\sum_{i=1}^N \mathrm{Ind}_{\gamma_i}(p)\right) \mod 2, \qquad\qquad \text{or} \qquad \left(\sum_{i=1}^M \mathrm{Ind}_{\sigma_i}(p)\right) \mod 2.$$

Without loss of generality we think that $p=0$. Since the curves $\{\gamma_i\},\{\sigma_i\}$ define the same varifold, for $\mathcal{H}^1$-ae point $q\in\{(x,y)\in\mathbb{R}^2\,|\,x^2+y^2=1\}$ we have that

$$\sum_{i=1}^N \sharp\left(\frac{\gamma_i}{|\gamma_i|}\right)^{-1}(q)=\sum_{i=1}^M \sharp\left(\frac{\sigma_i}{|\sigma_i|}\right)^{-1}(q). \tag{30}$$

In the following we denote by $deg(f,y)$ the degree of a map f at y and by $deg_2(f,y)$ the degree mod 2 of f at y (we refer to [Mi65]). Since the curves are Lipschitz almost every point $q\in\{(x,y)\in\mathbb{R}^2\,|\,x^2+y^2=1\}$ is a regular value for $\frac{\gamma_i}{|\gamma_i|},\frac{\sigma_i}{|\sigma_i|}$ and we can perform the calculation

$$\begin{aligned}\left(\sum_{i=1}^N \mathrm{Ind}_{\gamma_i}(p)\right) \mod 2 &= \left(\sum_{i=1}^N deg\left(\frac{\gamma_i}{|\gamma_i|},q\right)\right) \mod 2 = \left(\sum_{i=1}^N deg_2\left(\frac{\gamma_i}{|\gamma_i|},q\right)\right) \mod 2 =\\ &= \left(\sum_{i=1}^N \sharp\left(\frac{\gamma_i}{|\gamma_i|}\right)^{-1}(q) \mod 2\right) \mod 2,\end{aligned}$$

that together with the same expression using the curves σ_i, implies that E is uniquely defined by (30). Now we prove that $V=V_E$. Let

$$X=\{p\in\Gamma\,|\,\theta_V(p)=1,\ \gamma_i(t)=p\ \Rightarrow\ \gamma_i \text{ is differentiable at } t\}.$$

We want to prove that

$$\mathcal{H}^1\big(\mathcal{F}E\Delta X\big)=0, \tag{31}$$

which implies that $V=V_E$.
If $\gamma_i(t)=p\in X$, then there is $\varepsilon>0$ such that $\gamma_i\big((t-\varepsilon,t+\varepsilon)\big)\subset\{\theta_V=1\}\subset\Gamma=\partial E$. By Rademacher we therefore have that $\mathcal{H}^1(X\cap\gamma_i\big((t-\varepsilon,t+\varepsilon)\big)\setminus\mathcal{F}E)=0$. Hence $\mathcal{H}^1(X\setminus\mathcal{F}E)=0$.
Now let $p\in\mathcal{F}E$, we want to prove that $\mathcal{H}^1(\mathcal{F}E\setminus X)=0$, and this will complete the claim (31). If $\theta_V(p)=1$ only a curve passes (once) trough p, say $\gamma_1(t_1)=p$, and since $p\in\mathcal{F}E$ such curve has to be differentiable at t_1. Conversely if $p=\gamma_i(t_i)$ for some $\{i,t_i\}$'s, assuming that each γ_i is differentiable at t_i, we want to prove that $\theta_V(p)=1$. Suppose by contradiction that $\theta_V(p)>1$, then there are $\alpha,\beta:(-\varepsilon,\varepsilon)\to\Gamma$ Lipschitz different arcs such that $\alpha(0)=\beta(0)=p$ and α,β are differentiable at time 0; moreover the hypothesis $\mathcal{H}^1(\{x\,|\,\theta_V(x)>1\})=0$ implies that $\mathcal{H}^1\big((\alpha)\cap(\beta)\big)=0$. Therefore $\mathcal{H}^1$-ae point $p\in(\alpha)\cup(\beta)$ is contained in X, and thus $\mathcal{H}^1$-ae point $p\in(\alpha)\cup(\beta)$ is contained in $\mathcal{F}E$, since we already know that $\mathcal{H}^1(X\setminus\mathcal{F}E)=0$. So for any $\varepsilon>0$ there is $r>0$ such that

$$\mathcal{H}^1\big([(\alpha)\cup(\beta)]\cap B_r(p)\big)\ge(2-\varepsilon)2r,$$

and thus

$$\mathcal{H}^1(\mathcal{F}E\cap B_r(p))\ge\mathcal{H}^1\big([(\alpha)\cup(\beta)]\cap B_r(p)\big)\ge(2-\varepsilon)2r,$$

which is a contradiction with the fact that any point in $\mathcal{F}E$ has one dimensional density equal to 1.
So we have proved that a point $p\in\mathcal{F}E$ verifies that: if $\theta_V(p)=1$ then $p\in X$, and if any curve passing through p at some time is differentiable at that time then $p\in X$. In any case we conclude that $\mathcal{H}^1$-almost every point in $\mathcal{F}E$ belongs to X, and then $\mathcal{H}^1(\mathcal{F}E\setminus X)=0$. $\square$

3. Relaxation

3.1. **Setting and results.** From now on and for the rest of Section 3 let $p > 1$ be fixed and for any set of finite perimeter E assume (8). For any measurable set $E \subset \mathbb{R}^2$ we define the energy

$$\mathcal{F}_p(E) = \begin{cases} \mu_{V_E}(\mathbb{R}^2) + \mathcal{E}_p(V_E) & \text{if } V_E = \sum_{i\in I} (\gamma_i)_\sharp(\mathbf{v}(S^1,1)), \quad \gamma_i : S^1 \to \mathbb{R}^2 \ C^2\text{-immersion}, \\ & \quad \sharp I < +\infty, \\ +\infty & \text{otherwise.} \end{cases} \tag{32}$$

We write $\mathcal{F}_p(E)$ understanding that $\mathcal{F}_p$ is defined on the set of equivalence classes of characteristic functions endowed with L^1 norm. We want to calculate the relaxed functional $\overline{\mathcal{F}_p}$ with respect to the L^1 sense of convergence of characteristic functions.
By Remark 2.7 and Lemma 2.8, if $\mathcal{F}_p(E) < \infty$, we have that

$$\mathcal{H}^1(\partial E \setminus \mathcal{F}E) = 0, \qquad \mathcal{F}_p(V_E) = \sum_{i\in I} \mathcal{F}_p(\gamma_i),$$

if $V_E = \sum_{i\in I} (\gamma_i)_\sharp(\mathbf{v}(S^1,1))$. Also up to renaming $\mathring{E}^c$ into E, we can suppose that E is bounded.

If $E \subset \mathbb{R}^2$ is measurable, we define

$$\mathcal{A}(E) = \left\{ V = \mathbf{v}(\Gamma,\theta_V) = \sum_{i\in I} (\gamma_i)_\sharp(\mathbf{v}(S^1,1)) \;\middle|\; \begin{array}{l} \gamma_i : S^1 \to \mathbb{R}^2 \ C^1 \cap W^{2,p}\text{-immersion}, \ \sharp I < +\infty, \\ \sum_{i\in I} \mathcal{F}_p(\gamma_i) < +\infty, \\ \partial E \subset \Gamma, \ V_E \le V, \\ \mathcal{F}E \subset \{x \in \mathbb{R}^2 \,|\, \theta_V(x) \text{ is odd}\}, \\ \mathcal{H}^1(\{x \,|\, \theta_V(x) \text{ is odd}\} \setminus \mathcal{F}E) = 0 \end{array} \right\},$$

Remark 3.1. Observe that if $V \in \mathcal{A}(E)$, then $\mathcal{F}_p(V) < +\infty$, and then $\theta_V(x) = \lim_{\rho \searrow 0} \frac{\mu_V(B_\rho(x))}{2\rho}$ exists and it is uniformly bounded on Γ. Moreover the condition $\partial E \subset \Gamma$ and the bound on the energy of the curves imply that $\mathcal{H}^1(\partial E) < \infty$, and then E is a set of finite perimeter.

The main result of the section is the following.

Theorem 3.2. *For any measurable set $E \subset \mathbb{R}^2$ we have that*

$$\overline{\mathcal{F}_p}(E) = \begin{cases} +\infty & \text{if } \mathcal{A}(E) = \emptyset \ \text{ or } \ E \text{ is ess. unbounded}, \\ \min\left\{\mathcal{F}_p(V) \mid V \in \mathcal{A}(E)\right\} & \text{otherwise}, \end{cases} \tag{33}$$

where we say that a set E is essentially unbounded if $|E \setminus B_r(0)| > 0$ for any $r > 0$.

The proof of Theorem 3.2 will be completed in Subsection 3.3.

Remark 3.3. Choosing for a measurable set E the L^1 representative defined in (8), then the set E is essentially unbounded if and only if it is unbounded. So in the statement of Theorem 3.2 one can actually write unbounded in place of essentially unbounded.

Remark 3.4. The characterization of $\overline{\mathcal{F}_p}$ given by Theorem 3.2 immediately implies the stability property that

$$\mathcal{F}_p(E) < +\infty \qquad \Rightarrow \qquad \overline{\mathcal{F}_p}(E) = \mathcal{F}_p(E) < +\infty. \tag{34}$$

In fact if $\mathcal{F}_p(E) < +\infty$, then $V_E \in \mathcal{A}(E)$. Consider any $W = \mathbf{v}(\Gamma, \theta_W) \in \mathcal{A}(E) \setminus \{V_E\}$, then by definition we have that $V_E \leq V$ in the sense of measures and $\mathcal{F}E \subset \{x \,|\, \theta_W(x) \text{ is odd}\}$, and this implies that $\mathcal{H}^1(\mathcal{F}E \setminus \Gamma) = 0$. Therefore $\mu_W(\mathbb{R}^2) \geq \mathcal{H}^1(\mathcal{F}E) = \mu_{V_E}(\mathbb{R}^2)$, and also $\mathcal{E}_p(W) \geq \mathcal{E}_p(V_E)$ by locality of the generalized curvature ([LeMa09]).

We conclude this part showing some properties of varifolds $V \in \mathcal{A}(E)$ (in the following remember the assumption (8)).

Lemma 3.5. *Let $E \subset \mathbb{R}^2$ be a bounded set of finite perimeter. Let $V = \mathbf{v}(\Gamma, \theta_V) = \sum_{i=1}^N (\gamma_i)_\sharp(\mathbf{v}(S^1, 1))$ with $\gamma_1, ..., \gamma_N : S^1 \to \mathbb{R}^2$ Lipschitz curves. Suppose that $\mathcal{F}E \subset \Gamma$ and*

$$\mathcal{H}^1(\mathcal{F}E \,\Delta\, \{x \mid \theta_V(x) \text{ is odd}\}) = 0.$$

Then

$$E = \left\{ p \in \mathbb{R}^2 \setminus \Gamma \,:\, \left| \sum_{i=1}^N \mathrm{Ind}_{\gamma_i}(p) \right| \text{ is odd} \right\} \cup \Gamma.$$

Proof. Fix $p \in \mathbb{R}^2 \setminus \Gamma$. In the following we suppose without loss of generality that $p = 0$. By hypotheses and by the calculations in the proof of Lemma 2.9, there exists a vector $v \in \mathbb{R}^2 \setminus \{0\}$ such that the ray $L = \{p + tv \,|\, t \in [0, \infty)\}$ verifies the properties:
i) L intersects Γ at points y such that for any $i = 1, ..., N$ if $\gamma_i(t) = y$ then γ_i is differentiable at t,
ii) L intersects $\mathcal{F}E$ a finite number $M \in \mathbb{N}$ of times at points z in $\mathcal{F}E \cap \{x \mid \theta_V(x) \text{ is odd}\}$ where $\nu_E(z), v$ are independent,
iii) L intersects $\Gamma \setminus \mathcal{F}E$ a finite number of times at points w in $\{x \mid \theta_V(x) \text{ is even}\}$ where $\gamma_i'(t), v$ are independent whenever $\gamma_i(t) = w$,
iv)

$$\begin{aligned} \left(\sum_{i=1}^N \mathrm{Ind}_{\gamma_i}(p) \right) \mod 2 &= \left(\sum_{i=1}^N \sum_{y \in L \cap (\gamma_i)} \sharp \left(\frac{\gamma_i}{|\gamma_i|} \right)^{-1} \left(\frac{y}{|y|} \right) \mod 2 \right) \mod 2 = \\ &= \left(\sum_{i=1}^N \sum_{y \in L \cap (\gamma_i) \cap \mathcal{F}E} \sharp \left(\frac{\gamma_i}{|\gamma_i|} \right)^{-1} \left(\frac{y}{|y|} \right) \mod 2 \right) \mod 2, \end{aligned}$$

where in iv) the second inequality follows from ii) and iii).
Now if $p \in \mathring{E}$, since E is bounded the number M has to be odd, and then $\left(\sum_{i=1}^N \mathrm{Ind}_{\gamma_i}(p) \right) \mod 2 = 1$. Conversely if $p \in E^c$, then M is even, and then $\left(\sum_{i=1}^N \mathrm{Ind}_{\gamma_i}(p) \right) \mod 2 = 0$. □

Remark 3.6. We observe that Lemma 3.5 applies to couples E, V with $V \in \mathcal{A}(E)$.

Lemma 3.7. *Let $V = \mathbf{v}(\Gamma, \theta_V) \in \mathcal{A}(E)$ for some measurable set E. Letting $\Sigma := \overline{\Gamma \setminus \partial E}$, it holds that if $\Sigma \neq \emptyset$ then for any $x \in \Sigma \cap \partial E$ at least one of the following holds:*

$$\begin{aligned} &i)\ \exists y \in \Sigma \cap \partial E,\ \exists f : [0, T] \to \mathbb{R}^2\ C^1 \cap W^{2,p},\ T > 0, \\ &\quad f \text{ regular curve from } x \text{ to } y \text{ with } (f) \subset \Gamma, \\ &ii)\ x \text{ is not isolated in } \Sigma \cap \partial E. \end{aligned} \tag{35}$$

The alternative above is not exclusive.

Proof. Write $V = \sum_{i=1}^{N} (\sigma_i)_\sharp(\mathbf{v}(S^1, 1))$. Assume $\Sigma \neq \emptyset$, that is equivalent to $\Gamma \setminus \partial E =: S \neq \emptyset$. Suppose $x \in \Sigma \cap \partial E$ is isolated in $\Sigma \cap \partial E$, then we want to prove that condition $i)$ in (35) holds true. There exists $r_0 > 0$ such that $B_r(x) \cap \Sigma \cap \partial E = \{x\}$ for any $r \leq r_0$. Up to reparametrization we can say that $\sigma_1|_{(-\varepsilon,\varepsilon)} : (-\varepsilon, \varepsilon) \to B_{r_0}(x)$ passes through x at time 0. Up to reparametrize $\sigma_1(t)$ into $\sigma_1(-t)$, we can say that there exists a time $T > 0$ such that $\sigma_1|_{(0,T)} \subset S$ and $y := \sigma_1(T) \in \partial S = \Sigma \cap \partial E$, looking at S as topological subspace of Σ; in fact otherwise x would not be isolated in $\partial S = \Sigma \cap \partial E$. Defining $f(t) = \sigma_1(t)$ for $t \in [0, T]$ gives alternative $i)$ in (35).

□

3.2. **Necessary conditions.** Here we prove that a set $E \subset \mathbb{R}^2$ with $\overline{\mathcal{F}_p}(E) < +\infty$ has the necessary properties that inspire formula (33).

Let E_n be any sequence of sets such that $\mathcal{F}_p(E_n) \leq C$ and $\chi_{E_n} \to \chi_E$ in $L^1(\mathbb{R}^2)$. Let us adopt the notation $V_{E_n} = \sum_{i \in I_n} (\gamma_{i,n})_\sharp(\mathbf{v}(S^1, 1)) = \mathbf{v}(\Gamma_n, \theta_{V_{E_n}})$, so that $\mathcal{F}_p(E_n) = \sum_{i \in I_n} \mathcal{H}^1(\gamma_{i,n}) + \mathcal{E}_p(\gamma_{i,n})$. Using also (12) we have that $0 < c \leq \mathcal{H}^1(\gamma_{i,n}) \leq C < \infty$ for any i, n. Also $\mathcal{E}_p(\gamma_{i,n}) \geq c > 0$ for any i, n, thus $\sharp I_n < +\infty$ for large n and then we can suppose that $I_n = I$ for any n. Also we can choose E_n bounded and by L^1 convergence we have that $|E| < +\infty$.

Moreover we observe that in order to calculate the relaxation of $\mathcal{F}_p$ we can suppose that the sequence E_n is actually uniformly bounded, hence getting that E is bounded.
Indeed if (up to subsequence) we have that for example $\gamma_{1,n} \cap B_n(0)^c \neq \emptyset$, then by boundedness of the length we have $\gamma_{1,n} \subset (B_{n-c}(0))^c$ for any n for some c. Let Λ_n be the connected component of $\cup_{i \in I}(\gamma_i)$ containing (γ_1). The component Λ_n is equal to some union $\cup_{j \in J_n}(\gamma_{j,n})$. Up to relabeling we can suppose that $J_n = J$ for any n. Since the length of each curve is uniformly bounded, then there exist open sets U_n such that $\Lambda_n \subset U_n$, $U_n \cap \left(\cup_{i \in I \setminus J} (\gamma_{i,n}) \right) = \emptyset$, and $U_n \cap B_{R_n}(0) = \emptyset$ for some sequence $R_n \to \infty$. Therefore the set $E'_n := E_n \setminus U_n$ still converges to E in $L^1(\mathbb{R}^2)$, and $\mathcal{F}_p(E'_n) < \mathcal{F}_p(E_n)$.

Under the above notation we have the following result.

Lemma 3.8. *Suppose $E \subset \mathbb{R}^2$ verifies that $\overline{\mathcal{F}_p}(E) < +\infty$. Let $E_n \subset \mathbb{R}^2$ be uniformly bounded such that $\chi_{E_n} \to \chi_E$ in $L^1(\mathbb{R}^2)$ with $\mathcal{F}_p(E_n) \leq C$. Suppose that for any n the set $\{p \,|\, \theta_{V_{E_n}}(p) > 1\}$ is finite, then any subsequence of V_{E_n} converging in the sense of varifolds converge to an element of $\mathcal{A}(E)$.*

Proof. The arclength parametrizations $\sigma_{i,n}$ corresponding to $\gamma_{i,n}$ are uniformly bounded in $W^{2,p}$ for any $i \in I_n = I$ and for any n. Therefore, since the sequence is uniformly bounded in $\mathbb{R}^2$, up to subsequence $\sigma_{i,n} \to \sigma_i$ strongly in $C^{1,\alpha}$ for some $\alpha \leq \frac{1}{p'}$ and weakly in $W^{2,p}(\mathbb{R}^2)$ for any $i \in I$. Each σ_i is then a closed curve parametrized by arclength, and we call γ_i the parametrization on S^1 with constant velocity.
Hence the varifolds V_{E_n} converge to some limit integer rectifiable varifold $V = \mathbf{v}(\Gamma, \theta_V)$ in the sense of varifolds, and $V = \sum_{i \in I} (\gamma_i)_\sharp(\mathbf{v}(S^1, 1))$. The multiplicity function θ_V is upper semicontinuous and pointwise bounded by the discussion in Subsection 2.3. Also the sets E_n converge to E weakly* in $BV(\mathbb{R}^2)$, that is $\chi_{E_n} \to \chi_E$ and $D\chi_{E_n} \overset{\star}{\rightharpoonup} D\chi_E$, thus E is a set of finite perimeter. Observe that $|D\chi_{E_n}| = \mu_{V_{E_n}} \overset{\star}{\rightharpoonup} \mu_V$.

From now on we call $\Gamma = \cup_{i \in I}(\sigma_i)$, $\Sigma = \overline{\Gamma \setminus \partial E}$, $S = \Gamma \setminus \partial E$.

Let $x \in \partial E$, so that for any $\rho > 0$ we have

$$\lim_n \int_{B_\rho(x)} \chi_{E_n} > 0, \qquad \lim_n \int_{B_\rho(x)} \chi_{E_n^c} > 0. \tag{36}$$

Then for $\rho > 0$ there is $n(\rho)$ such that there exist $\xi_n \in E_n \cap B_\rho(x)$, $\eta_n \in E_n^c \cap B_\rho(x)$ for any $n \geq n(\rho)$ and thus there exists $w_n \in \partial E_n \cap B_\rho(x)$ for any $n \geq n(\rho)$. Taking some sequence $\rho_k \searrow 0$, we find a sequence w_n converging to x. Therefore, also by density (11), we have proved that $\mathcal{F}E \subset \partial E \subset \big\{y \,|\, y = \lim_n y_n,\, y_n \in \mathcal{F}E_n\big\} = \Gamma$. In particular ∂E is 1-rectifiable.

Now we prove that $\mathcal{F}E \subset \{x \mid \theta_V(x) \text{ is odd}\}$.
So let $p \in \mathcal{F}E$, and let $\{\gamma_k^j \,|\, j = 1, ..., N, i = 1, ..., n_j\}$ be distinct curves which are suitable disjoint restrictions of the γ_i's such that $(\gamma_k^j) \subset (\gamma_j)$ for any k (up to relabeling the γ_i's) and

$$\Gamma \cap B_{r_0}(p) = \bigcup_{j,k} (\gamma_k^j).$$

Without loss of generality we write $\gamma_k^j(t_k^j) = p$. We want to prove that $\sum_{j=1}^N n_j = \theta_V(p)$ is odd. Since $p \in \mathcal{F}E$ there is $q \in \mathring{E} \cap B_{r_0}(p)$ such that the segment

$$s(t) = q + \frac{p-q}{|p-q|} t \qquad t \in [0, 2|p-q|]$$

is such that

$$\left| \left\langle \frac{p-q}{|p-q|}, (\gamma_k^j)'(t_k^j) \right\rangle \right| > 0, \tag{37}$$

and $s|_{[0,|p-q|]} \subset E$, $s|_{(|p-q|,2|p-q|]} \subset E^c$. Also since $\gamma_{i,n} \to \gamma_i$ strongly in $C^{1,\alpha}$, by (37) we get that s intersects transversely $\gamma_{i,n}$ for any i for n big enough, and the number of such intersections is $\theta_V(p)$. Also denote $b := s(2|p-q|)$. Moreover we can write that $B_{r_q}(q) \subset \mathring{E}_n$ and $B_{r_b}(b) \subset E_n^c$ for n sufficiently big.
We know that for any $\varepsilon > 0$ there is $a_\varepsilon \in E_n^{c*}$, where $(\cdot)^*$ will always denote the unbounded connected component of $(\cdot)$, such that

$$\left| \frac{p-q}{|p-q|} - \frac{a_\varepsilon - b}{|a_\varepsilon - b|} \right| < \varepsilon,$$

$$\sum_{i\in I} \mathrm{Ind}_{\gamma_{i,n}}(b) \mod 2 = \sum_{i\in I} \sharp \left(\frac{\gamma_{i,n}}{|\gamma_{i,n}|} \right)^{-1} \left(\frac{a_\varepsilon - b}{|a_\varepsilon - b|} \right) \mod 2.$$

Hence up to a small C^∞ deformation which is different from the identity only on $\big\{x + t\frac{p-q}{|p-q|} \,|\, x \in B_{r_b}(b), t \in \mathbb{R}_{\geq 0}\big\} \setminus B_{r_b}(b)$ we can suppose that for $M > 0$ sufficiently big it holds that

$$a_0 := b + M\frac{p-q}{|p-q|} \in E_n^{c*},$$

$$\left\{ b + \mathbb{R}_{\geq 0}\left(\frac{p-q}{|p-q|} \right) \right\} \cap \left(\bigcup_{i\in I} (\gamma_{i,n}) \right) \subset \mathcal{F}E_n,$$

$$\sum_{i\in I} \mathrm{Ind}_{\gamma_{i,n}}(b) \mod 2 = \sum_{i\in I} \sharp \left(\frac{\gamma_{i,n}}{|\gamma_{i,n}|} \right)^{-1} \left(\frac{a_0 - b}{|a_0 - b|} \right) \mod 2. \tag{38}$$

Taking into account Lemma 2.9, by construction we have that the quantity in (38) is 0 mod 2. Moreover we have that

$$1 \mod 2 = \sum_{i\in I} \mathrm{Ind}_{\gamma_{i,n}}(q) \mod 2 = \left(\theta_V(p) + \sum_{i\in I} \mathrm{Ind}_{\gamma_{i,n}}(b) \right) \mod 2,$$

and then $\theta_V(p)$ is odd.

It remains to prove that $\mathcal{H}^1(\{x \mid \theta_V(x) \text{ odd}\} \setminus \mathcal{F}E) = 0$.
We observe that in the sense of currents we have the convergence $[|E_n|] \to [|E|]$ and thus

$$\partial[|E_n|] = \boldsymbol{\tau}\left(\bigcup_{i\in I}(\sigma_{i,n}), 1, \xi_0\right) \to \partial[|E|]$$

in the sense of currents where ξ_0 is the positive orientation of the boundaries with respect to $\mathbb{R}^2$. We can write $\partial[|E_n|] = \sum_{i=0}^{\infty}(\alpha_{i,n})_\sharp([|S^1|])$ for countably many Lipschitz parametrizations $\alpha_{i,n}$ ordered so that $L(\alpha_{i+1,n}) \le L(\alpha_{i,n})$ for any i, n. Such immersions positively orient the boundary ∂E^i_n of E^i_n, where E^i_n is one of the open connected components of $\mathring{E}_n$, which are at most countable. The length of each $\alpha_{i,n}$ is uniformly bounded, then we can assume that the parametrizations $\alpha_{i,n}$ are L-Lipschitz with constant L independent of i, n. Since the parametrizations $\sigma_{i,n}$ converge strongly in C^1, the immersions $\alpha_{i,n}$ uniformly converge to L-Lipschitz curves $\alpha_i : S^1 \to \mathbb{R}^2$ as $n \to \infty$. We can also reparametrize each α_i by constant velocity almost everywhere (in the sense of metric derivatives). In the sense of currents we have that

$$\sum_{i=0}^{\infty}(\alpha_{i,n})_\sharp([|S^1|]) = \partial[|E_n|] \to \partial[|E|] = \boldsymbol{\tau}\big(\mathcal{F}E, 1, \xi_0\big).$$

Let us define

$$T := \sum_{i=0}^{\infty}(\alpha_i)_\sharp([|S^1|]).$$

Since each $(\alpha_{i,n})$ is contained in some $(\sigma_{i_0,n})$ we have that $d_{\mathcal{H}}(\alpha_{i,n}, \alpha_i) \le N \max_{i=1,\dots,N} \|\sigma_{i,n} - \sigma_i\|_\infty \le \varepsilon$ for any $n \ge n_\varepsilon$. Since an equivalent definition of Hausdorff distance is $d_{\mathcal{H}}(A, B) = \inf\big\{\varepsilon > 0 \mid A \subset \mathcal{N}_\varepsilon(B),\, B \subset \mathcal{N}_\varepsilon(A)\big\}$ where $\mathcal{N}_\varepsilon(X) = \{x \mid d(x, X) \le \varepsilon\}$, we have that

$$\forall \varepsilon > 0\, \exists n_\varepsilon : \qquad d_{\mathcal{H}}\big(\cup_i (\alpha_{i,n}), \cup_i(\alpha_i)\big) < \varepsilon \qquad n \ge n_\varepsilon. \tag{39}$$

Thus $\cup_i(\alpha_{i,n})$ converges in Hausdorff distance to the set $\cup_i(\alpha_i)$. By hypothesis the set $\{p \mid \theta_{V_{E_n}} > 1\}$ is finite, hence $\cup_i(\alpha_{i,n}) = \partial E_n$ is closed, bounded, and has finitely many connected components for any n. The connected components of ∂E_n are at most $\sharp I < +\infty$, thus applying Golab Lower Semicontinuity Theorem on each component, we have that $\cup_i(\alpha_i)$ is closed as well. Moreover

$$\Gamma = \cup_i(\alpha_i). \tag{40}$$

In fact it is easy to see that $\Gamma \supset \cup_i(\alpha_i)$, and if by contradiction $\Gamma \supsetneq \cup_i(\alpha_i) = \overline{\cup_i(\alpha_i)}$, then by (39) there is $r > 0$ such that $\emptyset = B_r(p) \cap \big(\cup_i(\alpha_{i,n})\big) = B_r(p) \cap \big(\cup_{i\in I}(\gamma_{i,n})\big)$ for any n big enough, that is impossible.

Let $x \in \mathbb{R}^2 \setminus \Gamma$. By (39) and (40) we have that there is $\rho > 0$ such that $B_\rho(x) \cap \big(\cup_i (\alpha_i) \cup \cup_i(\alpha_{i,n})\big) = \emptyset$ for any n large. Then there exists n_x such that for any i the index $\mathrm{Ind}_{\alpha_{i,n}}(x)$ is the same for any $n \ge n_x$. In fact suppose by contradiction for any n there is $i_n, N_1, N_2 \ge n$ with $1 = \mathrm{Ind}_{\alpha_{i_n,N_1}}(x) \neq \mathrm{Ind}_{\alpha_{i_n,N_2}}(x) = 0$ and $i_n \to \infty$ without loss of generality. Then $L(\alpha_{i_n,N_1}) \ge C(\rho)$ for a constant $C(\rho) > 0$ depending only on ρ by isoperimetric inequality. Since $L(\alpha_{i+1,n}) \le L(\alpha_{i,n})$ for any i, n and $i_n \to \infty$, this implies $P(E_n)$ is arbitrarily big that for n large enough.

Now let $x \in \mathbb{R}^2 \setminus \Gamma$ such that there exists $\lim_n \chi_{E_n}(x)$. Since $\chi_{E_n}(x) = \sum_i \mathrm{Ind}_{\alpha_{i,n}}(x)$ for n big such that $B_\rho(x) \cap \big(\cup_i (\alpha_i) \cup \cup_i(\alpha_{i,n})\big) = \emptyset$ for some $\rho > 0$, from the above discussion we have that

$$\begin{aligned}
\lim_n \sum_i \mathrm{Ind}_{\alpha_{i,n}}(x) = 1 \quad &\Leftrightarrow \quad \forall n \ge n_0\, \exists i : \quad \mathrm{Ind}_{\alpha_{i,n}}(x) = 1 \\
&\Leftrightarrow \quad \exists i(x)\, \forall n \ge n_0 \quad \mathrm{Ind}_{\alpha_{i(x),n}}(x) = 1 \\
&\Leftrightarrow \quad \exists i(x) : \quad \mathrm{Ind}_{\alpha_{i(x)}}(x) = 1.
\end{aligned}$$

Hence

$$x \in E \quad \Leftrightarrow \quad \lim_n \sum_i \mathrm{Ind}_{\alpha_{i,n}}(x) = 1 \quad \Leftrightarrow \quad \exists i(x) : \quad \mathrm{Ind}_{\alpha_{i(x)}}(x) = 1 \quad \Leftrightarrow \quad \sum_i \mathrm{Ind}_{\alpha_i}(x) = 1.$$

In particular

$$E = \big\{x \in \mathbb{R}^2 \setminus \Gamma \mid \sum_{i=0}^{\infty} \mathrm{Ind}_{\alpha_i}(x) = 1\big\} = \left\{x \in \mathbb{R}^2 \setminus \Gamma \ : \ \left|\sum_{i\in I} \mathrm{Ind}_{\sigma_i}(x) = 1\right| \text{ is odd}\right\}, \tag{41}$$

up to $\mathcal{L}^2$-negligible sets, where the second equality follows by the uniform convergence of the finitely many curves $\sigma_i i, n$. Also for any $i \neq j$ it holds that $|\{x \in \mathbb{R}^2 \setminus (\alpha_i) \mid \mathrm{Ind}_{\alpha_i}(x) = 1\} \cap \{x \in \mathbb{R}^2 \setminus (\alpha_j) \mid \mathrm{Ind}_{\alpha_j}(x) = 1\}| = 0$, because the inequality holds for any n for $\alpha_{i,n}, \alpha_{j,n}$. Therefore we have

$$\begin{aligned}\sum_i \int_{(\alpha_{i,n})} \langle \omega, \tau_{i,n}\rangle = \int_{E_n} d\omega \underset{n}{\to} \int_{\left\{\sum_{i=0}^{\infty} \mathrm{Ind}_{\alpha_i}(x)=1\right\}} d\omega = \\ = \sum_i \int_{\left\{\mathrm{Ind}_{\alpha_i}(x)=1\right\}} d\omega = \sum_i \int_{(\alpha_i)} \langle \omega, \tau_i\rangle,\end{aligned}$$

for any 1-form ω on $\mathbb{R}^2$. This means that

$$\sum_{i=0}^{\infty} (\alpha_{i,n})_\sharp([|S^1|]) = \partial[|E_n|] \to T = \sum_{i=0}^{\infty} (\alpha_i)_\sharp([|S^1|]) = \partial[|E|] = \boldsymbol{\tau}\big(\mathcal{F}E, 1, \xi_0\big), \tag{42}$$

in the sense of currents. In particular we can write the multiplicity function of the current $\partial[|E|]$ as

$$m(x) = \sum_{i=0}^{\infty} \sum_{y\in\alpha_i^{-1}(x)} S(y), \tag{43}$$

for $\mathcal{H}^1$-ae $x \in \mathbb{R}^2$, where $S(y) = +1$ if $d(\alpha_i)_y$ preserves the orientation and the opposite in the -1 case. Note that since θ_V is bounded, $\Gamma = \cup_i(\alpha_i)$, and $\theta_V(p) \geq \sum_i \sharp\alpha_i^{-1}(p)$, then the series in (43) is actually a finite sum.
Also observe that since E is a set of finite perimeter, by Gauss-Green formula the multiplicity function m is equal to 1 $\mathcal{H}^1$-ae on $\mathcal{F}E$, $\mathcal{H}^1(\{x \,|\, m(x) \geq 1\} \setminus \mathcal{F}E) = 0$, and $m = 0$ $\mathcal{H}^1$-ae on $\mathbb{R}^2 \setminus \mathcal{F}E$.

Now since $\Gamma = \cup_i(\alpha_i)$ and $\mathcal{H}^1(\alpha_i(\{t \,:\, \nexists \alpha_i'(t)\})) = 0$, then

$$\mathcal{H}^1\big(\{p \in \Gamma \mid \exists\, t, i \,:\, \alpha_i(t) = p,\, \nexists \alpha_i'(t)\}\big) = 0. \tag{44}$$

So let $p \in \Gamma$ be such that if $\alpha_i(t) = p$ then $\exists \alpha_i'(t)$. We want to check that $\theta_V(p)$ and $\sum_i \sharp\alpha_i^{-1}(p)$ have the same parity. In fact if without loss of generality $\theta_V(p) > \sum_i \sharp\alpha_i^{-1}(p)$, taking into account (41), following a segment s intersecting Γ only at p and transversally (as in the first part of the proof) we have that:
i) s passes from E to E^c if and only if $\theta_V(p)$ is odd, or equivalently if and only if $\sum_i \sharp\alpha_i^{-1}(p)$ is odd;
ii) s passes from E to E if and only if $\theta_V(p)$ is even, or equivalently if and only if $\sum_i \sharp\alpha_i^{-1}(p)$ is even.
Hence by (43) we conclude that $\theta(p)$ is odd if and only if alternative i) above holds, if and only if the summands in (43) are odd, if and only if $m(p)$ is odd.
By (44) this holds for $\mathcal{H}^1$-ae point in Γ. Therefore $\mathcal{H}^1(\{x \,|\, m(x) \text{ is odd}\}\Delta\{x \,|\, \theta_V(x) \text{ is odd}\}) = 0$. So finally since $\mathcal{H}^1(\{x \,|\, m(x) = 1\} \setminus \mathcal{F}E) = 0$, then

$$\begin{aligned}0 = \mathcal{H}^1(\{x \,|\, m(x) \text{ odd}\} \setminus \{x \,|\, m(x) = 1\}) = \mathcal{H}^1(\{x \,|\, \theta_V(x) \text{ odd}\} \setminus \{x \,|\, m(x) = 1\}) = \\ = \mathcal{H}^1(\{x \,|\, \theta_V(x) \text{ is odd}\} \setminus \mathcal{F}E),\end{aligned}$$

which completes the proof.

$\square$

3.3. **Proof of Theorem 3.2.** First we want to prove the following approximation result.

Proposition 3.9. *Let $E \subset \mathbb{R}^2$ be measurable and bounded with $\mathcal{A}(E) \neq \emptyset$. Then for any $V \in \mathcal{A}(E)$ there exists a sequence E_n of uniformly bounded sets such that*

$$\mathcal{F}_p(E_n) < +\infty, \quad \chi_{E_n} \to \chi_E \ \text{ in } L^1(\mathbb{R}^2), \quad V_{E_n} \to V \text{ as varifolds}, \quad \lim_n \mathcal{F}_p(E_n) = \mathcal{F}_p(V). \tag{45}$$

Moreover for any n we have that $V_{E_n} = \sum_{i=1}^N (\gamma_i)_\sharp(\mathbf{v}(S^1,1)) = \mathbf{v}(\Gamma_n, \theta_{V_{E_n}})$ and $\{p \,|\, \theta_{V_{E_n}}(p) > 1\}$ is finite.

Proof. Let $V = \sum_{i=1}^N (\gamma_i)_\sharp(\mathbf{v}(S^1,1)) \in \mathcal{A}(E)$ with $\gamma_i \in W^{2,p}$ regular. For any i let $\{\gamma_{i,n}\}_{n\in\mathbb{N}}$ be a sequence of analytic regular immersions such that $\gamma_{i,n} \to \gamma_i$ in $W^{2,p}$ as $n \to \infty$. Hence the set

$$\{x \in \mathbb{R}^2 \,|\, \exists i, j, t \neq \tau : \ \gamma_i(t) = \gamma_j(\tau)\} \tag{46}$$

is finite. Let $V_n = \sum_{i=1}^N (\gamma_{i,n})_\sharp(\mathbf{v}(S^1,1))$. By (46) we can define E_n as in Lemma 2.9, so that $V_n = V_{E_n}$. Moreover we have that

$$\mathcal{F}_p(E_n) < +\infty, \qquad \lim_{n\to\infty} \mathcal{F}_p(V_{E_n}) = \lim_{n\to\infty} \mathcal{F}_p(E_n) = \mathcal{F}_p(V), \qquad V_{E_n} \to V.$$

By uniform convergence of $\gamma_{i,n}$ we get that for any $\varepsilon > 0$ there is n_ε such that

$$\bigcup_{i=1}^N (\gamma_{i,n}) \subset I_{\frac{\varepsilon}{2}}\left(\bigcup_{i=1}^N (\gamma_i)\right) \qquad \forall n \geq n_\varepsilon,$$

where $I_{\frac{\varepsilon}{2}}$ denotes the $\frac{\varepsilon}{2}$ open tubolar neighborhood. Hence up to passing to a subsequence by Riesz-Fréchet-Kolmogorov we have that χ_{E_n} converges strongly in $L^2(\mathbb{R}^2)$, and then in $L^1(\mathbb{R}^2)$ and pointwise almost everywhere to the characteristic function of a closed set F. Using the definition of E_n and Lemma 3.5 together with Remark 3.6 we have that $F = E$, and the proof is completed.

□

Corollary 3.10. *Let $E \subset \mathbb{R}^2$ be measurable and bounded with $\mathcal{A}(E) \neq \emptyset$. Then*

$$\exists \min \big\{\mathcal{F}_p(V) \mid V \in \mathcal{A}(E)\big\}.$$

Proof. Let V_k be a minimizing sequence in $\mathcal{A}(E)$. Up to subsequence we can assume that $V_k \to V$ in the sense of varifolds and the supports $\mathrm{supp} V_k$ are uniformly bounded. By Proposition 3.9 using a diagonal argument we find a sequence of uniformly bounded sets E_k such that

$$\begin{aligned} &\chi_{E_k} \to \chi_E \quad \text{in } L^1(\mathbb{R}^2), && \mathcal{F}_p(E_k) \leq C < +\infty, \\ &V_{E_k} \to V \quad \text{as varifolds}, && \lim_k \mathcal{F}_p(E_k) = \lim_k \mathcal{F}_p(V_k) = \inf_{\mathcal{A}(E)} \mathcal{F}_p \geq \mathcal{F}_p(V), \end{aligned}$$

and $\{p \,|\, \theta_{V_{E_k}}(p) > 1\}$ is finite. Hence E_k is a possible approximating sequence of E by regular sets, i.e. a competitor in the calculation of the relaxation $\overline{\mathcal{F}_p}(E)$. Then by Lemma 3.8 we get that $V \in \mathcal{A}(E)$, and therefore V minimizes $\mathcal{F}_p$ on $\mathcal{A}(E)$.

□

Now Proposition 3.9 together with Corollary 3.10 readily imply Theorem 3.2.

3.4. **Comment on the $p = 1$ case.** The characterization of the relaxed energy given by Theorem 3.2 fails in the $p = 1$ case. As stated in Section 1, many estimates used in the $p > 1$ case have an analogous formulation in case $p = 1$. However, if $I \subset \mathbb{R}$ is a bounded interval, functions $u \in W^{2,1}(I)$ do not have good compactness properties. In fact even if $u \in W^{2,1}(I)$ implies that $u' \in W^{1,1}(I) = AC(\bar{I})$ and hence $u \in C^1$, the immersion $W^{2,1}(I) \hookrightarrow C^1(\bar{I})$ is not continuous.
Since $W^{2,1}(I) \hookrightarrow W^{1,p}(I)$ for any $p \in [1, \infty)$, we have that $W^{2,1}(I)$ compactly embeds only in $C^{0,\alpha}(\bar{I})$ for any $\alpha \in (0,1)$. This implies that the convergence of the curves defining the boundary of sets E_n

with $\mathcal{F}_1(E_n) \le C$ is much weaker than in the $p > 1$ case.

One of the main differences is the following. As we will show in Subsection 4.3 the $\mathcal{F}_p$ energy of polygons is infinite if $p > 1$. Instead if E is a regular polygon, i.e. a set $E \subset \mathbb{R}^2$ whose boundary is the image of an injective piecewise C^2 closed curve, it can happen that $\overline{\mathcal{F}_1}(E) < +\infty$. For instance, consider a square Q in the plane: in small neighborhoods of the four vertices the boundary ∂Q can be approximated by a piece of circumference of radius converging to 0 with finite bounded energy converging to $\frac{\pi}{2}$. This is ultimately due to the invariance of the energy $\mathcal{F}_1$ under rescaling, a property that is absent if $p > 1$. This implies that a possible limit varifold does not verify the flux property (because of the arguments in the proof of Proposition 4.8).

We believe that the presence of vertices in the boundary of the limit set is the main difference with the $p > 1$ case and that sets E with $\overline{\mathcal{F}_1}(E) \le C$ have at most countably many vertices, each of them giving an additional contribution to the energy equal to the angle described by the vertex.

4. Remarks and applications

4.1. **Comparison with** [BeMu04]**,** [BeMu07]**.** In these works Bellettini and Mugnai develop a characterization of the following relaxed functional. For simplicity we reduce ourselves to the case $p = 2$. Let $E \subset \mathbb{R}^2$ be measurable and define the energy

$$G(E) = \begin{cases} \int_{\partial E} 1 + |k_{\partial E}|^2 \, d\mathcal{H}^1 & E \text{ is of class } C^2, \\ +\infty & \text{otherwise.} \end{cases} \tag{47}$$

Then the functional $\overline{G}$ is the L^1-relaxation of G. Clearly

$$G(E) < +\infty \quad \Rightarrow \quad \mathcal{F}_2(E) = G(E),$$

$$\overline{\mathcal{F}_2}(E) \le \overline{G}(E) \qquad \forall E.$$

The precise characterization of $\overline{G}$ is discussed in [BeMu04] and [BeMu07]; here we just want to point out that

$$\exists E : \qquad \overline{\mathcal{F}_2}(E) < \overline{G}(E) < +\infty.$$

In fact an example is the set E_0 in Fig. 2 described in the Example 4.4 in [BeMu07]. Let γ_1, γ_2 be as in Fig. 2. In [BeMu07] it is proved that

$$\overline{G}(E) > \mathcal{F}_2(\gamma_1) + \mathcal{F}_2(\gamma_2).$$

Here we want to prove that

$$\overline{\mathcal{F}_2}(E) = \mathcal{F}_2(\gamma_1) + \mathcal{F}_2(\gamma_2). \tag{48}$$

Observe that γ_1, γ_2 carry inside $B_1(0)$ a $\mathcal{F}_2$ energy equal to 8.
Since $\overline{\mathcal{F}_2}(E_0) < +\infty$ there exists a varifold $V = \sum_{i=1}^N (\gamma_1)_\sharp(\mathbf{v}(S^1,1)) \in \mathcal{A}(E)$. Up to renaming and reparametrization assume $\gamma_1(0) = (1,0)$, $\gamma_1'(0) = -(1,0)$, and $\gamma_1|[-T,0]$ joins $(0,1)$ and $(0,1)$ having support contained in $\mathcal{F}E_0 \setminus \overline{B_1(0)}$. Since γ_1 is C^1 and closed, by the above discussion there exists a first time $\tau > 0$ such that $\gamma_1(\tau) \in \{(1,0),(0,1),(-1,0),(0,-1)\}$. We divide two cases.
1) If $\gamma_1(\tau) \in \{(0,1),(0,-1)\}$, arguing like in the proof of inequality (12) one has

$$\frac{\pi}{2} \le \left[L(\gamma_1|_{(0,\tau)})\right]^{\frac12} \left[\mathcal{E}_2(\gamma_1|_{(0,\tau)})\right]^{\frac12} \le \frac{1}{2}\big(L(\gamma_1|_{(0,\tau)}) + \mathcal{E}_2(\gamma_1|_{(0,\tau)})\big),$$

then $\mathcal{F}_2(\gamma_1|_{(0,\tau)}) \ge \pi > 2$.
2)If $\gamma_1(\tau) = (1,0)$ by an analogous argument one gets $\mathcal{F}_2(\gamma_1|_{(0,\tau)}) \ge 2\pi > 2$.
Hence in any case it is convenient for the curve γ to pass first through the point $(-1,0)$. By the

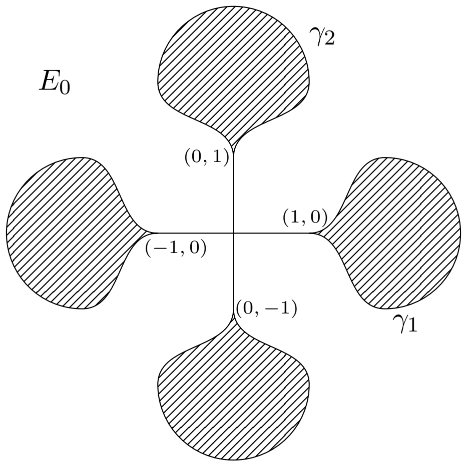

FIGURE 2. Picture of the set E_0 in Example 4.4 of [BeMu07]. The curve γ_1 parametrizes the left and the right components, while γ_2 parametrizes the upper and lower components. The varifold $(\gamma_1)_\sharp(\mathbf{v}(S^1,1)) + (\gamma_1)_\sharp(\mathbf{v}(S^1,1))$ belongs to $\mathcal{A}(E_0)$, and it has multiplicity equal to 1 on ∂E_0 and equal to 2 on the cross in the middle of the picture.

characterization of Theorem 3.2 equality (48) follows.

In this sense we can look at the relaxation $\overline{\mathcal{F}_p}$ as a generalization of the energy $\overline{G}$, in the sense that $\mathcal{F}_p$ admits a wider class of regular objects, i.e. sets E with $\mathcal{F}_p(E) < +\infty$, and this implies that the relaxed energy $\overline{\mathcal{F}_p}$ is naturally strictly less than $\overline{G}$ on some sets.

4.2. **Inpainting.** Here we describe a simple but significant application of the relaxed functional $\overline{\mathcal{F}_p}$ given by Theorem 3.2. Such application arises from the inpainting problem that roughly speaking consists in the reconstruction of a part of an image, knowing how the remaining part of the picture looks like. This problem as stated is quite involved ([BeCaMaSa11]). Assuming the only two colours of the image are black and white, as already pointed out for example in [AmMa03], one can think that the black shape contained in lost part of the image is consistent with the shape minimizing a suitable functional depending on length and curvature of its boundary. In such a setting the known part of the image plays the role of the boundary conditions. On different scales one can ask for the optimal unknown shape to minimize a weighted functional like (50), where one can give more importance to the length or to the curvature term.
Now we formalize the problem and we give a variational result.

Fix $p \in (1,\infty)$. In $\mathbb{R}^2$ consider the set H defined as follows. Let Q_1, Q_2 be the squares $Q_1 = \{(x,y) : 0 \le x \le 10, 0 \le y \le 10\}, Q_2 = \{(x,y) : -10 \le x \le 0, -10 \le y \le 0\}$, modify the squares in small neighborhoods of the vertices into convex sets $\tilde{Q}_1, \tilde{Q}_2$ with smooth boundary. Finally let

$$H = \big(\tilde{Q}_1 \cup \tilde{Q}_2\big) \setminus B_1(0). \tag{49}$$

Let $\lambda \in (0,\infty)$ and $\mathcal{F}_{\lambda,p}$ be the functional

$$\mathcal{F}_{\lambda,p}(E) = \begin{cases} \lambda \mu_{V_E}(\mathbb{R}^2) + \mathcal{E}_p(V_E) & \text{if } V_E = \sum_{i\in I} (\gamma_i)_\sharp(\mathbf{v}(S^1,1)), \quad \gamma_i : S^1 \to \mathbb{R}^2 \ C^2\text{-immersion}, \\ & \quad \sharp I < +\infty, \\ +\infty & \text{otherwise.} \end{cases} \tag{50}$$

Analogously to the functional $\mathcal{F}_p$, we have a well defined characterization of the L^1-relaxed functional $\overline{\mathcal{F}_{\lambda,p}}$.

We want to solve the minimization problem

$$\mathfrak{P} = \min\left\{\overline{\mathcal{F}_{\lambda,p}}(E) \;\middle|\; E \subset \mathbb{R}^2 \text{ measurable s.t. } E \setminus B_1(0) = H\right\}, \tag{51}$$

under the hypothesis of λ suitably small. The heuristic idea is that a good candidate minimizer is given by the set

$$E_0 = \left[(Q_1 \cup Q_2) \cap \overline{B_1(0)}\right] \cup H, \tag{52}$$

which has finite $\mathcal{F}_p$ energy. For a qualitative picture see Fig. 3.

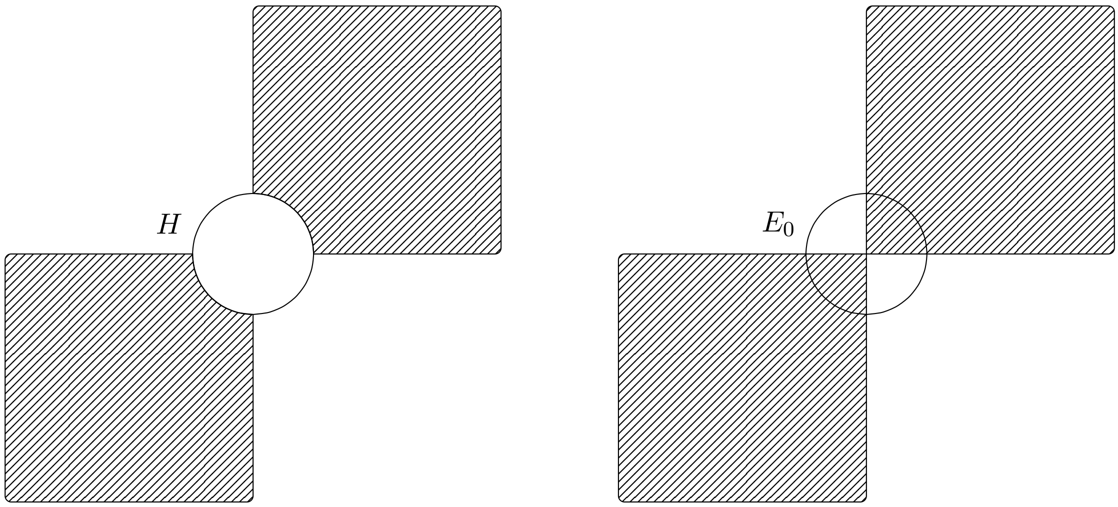

FIGURE 3. Qualitative pictures of the datum H and the minimizer E_0.

Remark 4.1. Observe that if $\overline{G}$ is the relaxed functional defined in [BeMu07] recalled in Subsection 4.1, then $\overline{G}(E_0) = +\infty$, and hence E_0 will never be detected by a minimization problem (51) analogously defined with the functional $\overline{G}$.

We have the following result.

Proposition 4.2. *There exists $\lambda_0 \in \left(0, \frac{\pi}{2}\right)$ such that for any $\lambda \in (0, \lambda_0)$ the set E_0 is the unique minimizer of problem $\mathfrak{P}$.*

Proof. Let E_n be a minimizing sequence of problem $\mathfrak{P}$. By Theorem 3.2 and Lemma 2.8 we can write $\overline{\mathcal{F}_{\lambda,p}}(E_n) = \sum_{i\in I_n} \mathcal{F}_{\lambda,p}(\gamma_{i,n})$ for some curves $\gamma_{i,n}$. Up to subsequence $I_n = I$ and the curves converge strongly in C^1 and weakly in $W^{2,p}$ to curves γ_i. In particular $E_n \to E$ in the L^1 sense, and

$$\overline{\mathcal{F}_{\lambda,p}}(E) \le \inf \mathfrak{P} \le \overline{\mathcal{F}_{\lambda,p}}(E_0) = \mathcal{F}_{\lambda,p}(E_0). \tag{53}$$

by lower semicontinuity. Moreover by C^1 strong convergence we have that

$$\forall i \forall p \in \big((\gamma_i) \cap \partial B_1(0)\big) \setminus \{(1,0),(0,1),(-1,0),(0,-1)\} \quad \Rightarrow \quad (\gamma_i) \text{ is tangent to } \partial B_1(0) \text{ at } p.$$

Observe that E_0 carries inside $B_1(0)$ a $\mathcal{F}_{\lambda,p}$ energy equal to 8λ.

Arguing as in Subsection 4.1, since $\overline{\mathcal{F}_{\lambda,p}}(E) < +\infty$ there exists a varifold $V = \sum_{i=1}^N (\gamma_1)_\sharp(\mathbf{v}(S^1,1)) \in \mathcal{A}(E)$. Up to renaming and reparametrization assume $\gamma_1(0) = (1,0)$, $\gamma_1'(0) = -(1,0)$, and $\gamma_1|[-T,0]$ joins $(0,1)$ and $(1,0)$ having support contained in $\mathcal{F}H \setminus \overline{B_1(0)}$. Since γ_1 is C^1 and closed, by the above discussion there exists a first time $\tau > 0$ such that γ_1 intersects transversally $\partial B_1(0)$. Also such

transversal intersection can take place only at one of the points in $\{(1,0),(0,1),(-1,0),(0,-1)\}$. We divide two cases.
1) If $\gamma_1(\tau)\in\{(-1,0),(0,-1)\}$, observing that there is $\overline{C}>0$ depending only on the problem $\mathfrak{P}$ such that $L(\gamma_i)\le\overline{C}$ for any i, then arguing like in (12) we get

$$\mathcal{F}_{\lambda,p}(\gamma_1|_{(0,\tau)})\ge\lambda\sqrt{2}+\frac{\pi}{2}\frac{1}{L(\gamma_1|_{(0,\tau)})^{\frac{p}{p'}}}\ge\lambda\sqrt{2}+\frac{\pi}{2}\frac{1}{\overline{\overline{C}}}>2\lambda,$$

where the last inequality holds choosing λ_0 small enough.
2) If $\gamma_1(\tau)=(1,0)$, then by the same argument leading to (12) one has

$$\pi\le\frac{L(\gamma_1|_{(0,\tau)})}{p'}+\frac{\mathcal{E}_p(\gamma_1|_{(0,\tau)})}{p}. \tag{54}$$

If $\lambda p'\ge1$, then $\pi\le\mathcal{F}_{\lambda,p}(\gamma_1|_{(0,\tau)})$. If instead $\lambda p'<1$, then also $\frac{\lambda p'}{p}<1$, and multiplying (54) by $\lambda p'$ one has $\lambda p'\pi\le\mathcal{F}_{\lambda,p}(\gamma_1|_{(0,\tau)})$. So we can write that $\mathcal{F}_{\lambda,p}(\gamma_1|_{(0,\tau)})\ge\min\{1,\lambda p'\}\pi$. Choosing $\lambda_0<\frac{\pi}{2}$ then $\pi>2\lambda$, and since $p'>1>\frac{2}{\pi}$ then $\lambda p'\pi>2\lambda$; hence in any case

$$\mathcal{F}_{\lambda,p}(\gamma_1|_{(0,\tau)})>2\lambda.$$

By inequality (53) we conclude that $\gamma_1(\tau)=(-1,0)$ and $\partial E_0\subset\cup_{i=1}^N(\gamma_i)$. Hence again by the same inequality we have that $E=E_0$, and thus $\mathfrak{P}$ has a unique minimizer, that is E_0.

□

4.3. **Examples and qualitative properties.** In this subsection we fix $p\in(1,\infty)$ and we collect some remarks about the qualitative properties of sets E having $\overline{\mathcal{F}_p}(E)<+\infty$.

First we want to prove a result that is completely analogous to the Theorem 6.5 in [BeDaPa93]. To this aim we need some definitions.

Definition 4.3. Let $E\subset\mathbb{R}^2$ be closed measurable. A point $p\in\partial E$ is called (simple) cusp if there is $r>0$ such that up to rotation and translation the set $B_r(p)\cap\partial E$ is the union of the graphs of two functions $f_1,f_2:[0,a]\to\mathbb{R}$ of class $C^1\cap W^{2,p}$ with $f_i(0)=f_i'(0)=0$, $f_1(x)\le f_2(x)$, and $f_1(x)=f_2(x)$ if and only if $x=0$.

Also, we shall need the following definitions in the context of planar graphs.

Definition 4.4. Let $G\subset\mathbb{R}^2$ be a planar finite graph, i.e. a set given by the union of finitely many embeddings of $[0,1]$ of class $C^1\cap W^{2,p}$, called edges of G, possibly meeting only at the endpoints, called vertices of G. The symbols E_G,V_G respectively denote the set of edges of G and the set of vertices of G. Together with the topology of a graph G, it is assigned a multiplicity function $m:E_G\to\mathbb{N}$.
For any vertex $v\in V_G$ there is $r_v>0$ such that for $0<r<r_v$ the set $H:=G\cap B_r(v)$ is a finite connected graph whose edges only meet at v and with multiplicity inherited from G. In this notation, the local density of G at v is the number $\rho_G(v)=\sum_{e\in E_H}m(e)$.
Now assume also that for any $v\in V_G$ and $0<r<r_v$, if f_i are regular parametrizations of the edges e_i of the graph $H=G\cap B_r(v)$ with $f_i(1)=v$, then for any i there is j such that the arclength derivatives $\dot f_i,\dot f_j$ satisfy $\dot f_i(1)=-\dot f_j(1)$. Under this assumption, we denote by $w_1(v),...,w_{N_v}(v)$ unit norm vectors identifying the possible tangent directions given by $\{\dot f_i(1)\}_i$. Hence $w_i(v)^\perp$ is the counterclockwise rotation of $w_i(v)$ of an angle equal to $\pi/2$. Finally we define

$$I^+(w_i(v)):=\big\{e_i\in E_H\mid\dot f_i(1)=\pm w_i(v),\ \ (\dot f_i(1),w_i(v))\text{ is a negative basis of }\mathbb{R}^2\big\},$$

$$I^-(w_i(v)):=\big\{e_i\in E_H\mid\dot f_i(1)=\pm w_i(v),\ \ (\dot f_i(1),w_i(v))\text{ is a positive basis of }\mathbb{R}^2\big\},$$

and

$$\rho_G^+(v, w_i(v)) = \sum_{e_i \in I^+(w_i(v))} m(e_i),$$
$$\rho_G^-(v, w_i(v)) = \sum_{e_i \in I^-(w_i(v))} m(e_i).$$

The graph G is said to be regular if for any $v \in V_G$ and for any $w_i(v)$ it holds that $\rho_G^+(v, w_i(v)) = \rho_G^-(v, w_i(v))$.

Remark 4.5. Let $V = \sum_{i=1}^N (\gamma_i)_\sharp(\mathbf{v}(S^1, 1))$ be a varifold in $\mathcal{A}(E)$ for some set E. Suppose that $\Gamma = \cup(\gamma_i)$ is a finite planar graph G_Γ. To any edge e of G_Γ we assign the multiplicity function $m_\Gamma(e) = \theta_V(p)$ for any $p \in e$ that is not a vertex. By the flux property (Definition 2.6), the graph G_Γ with the multiplicity m_Γ is regular.

We are ready to prove the following result about the energy of sets which are smooth out of finitely many cusp points. The strategy follows ideas from [BeDaPa93], but it is different in the technical parts.

Theorem 4.6. *Let $E \subset \mathbb{R}^2$ be a closed set whose boundary is $C^1 \cap W^{2,p}$ smooth at every point but at finitely many ones which are simple cusps $q_1, ..., q_k$. Then*

$$\overline{\mathcal{F}_p}(E) < +\infty \qquad \Leftrightarrow \qquad k \text{ is even.} \tag{55}$$

Proof. If k is even, Theorem 6.4 in [BeDaPa93] implies that the relaxed energy $\overline{G}(E)$ studied in [BeDaPa93] is finite. Since $\overline{\mathcal{F}_p} \le \overline{G}$, we have one implication.
Now suppose that $\overline{\mathcal{F}_p}(E)$ is finite, i.e. $\mathcal{A}(E) \neq \emptyset$. Let $V = \mathbf{v}(\Gamma, \theta_V) = \sum_{i=1}^N (\gamma_i)_\sharp(\mathbf{v}(S^1, 1)) \in \mathcal{A}(E)$. We are going to construct a set $\tilde{E}$ satisfying the hypotheses of the theorem and having the same unknown number of cusps of E, together with a varifold $\tilde{V} = \mathbf{v}(\tilde{\Gamma}, \tilde{\theta}_{\tilde{V}}) \in \mathcal{A}(\tilde{E})$ with the additional property that $\tilde{\Gamma}$ is a finite graph $G_{\tilde{\Gamma}}$ with multiplicity as given in Remark 4.5. Once the support of a varifold in $\mathcal{A}(\tilde{E})$ is a finite graph, we can prove that the number of cusps is even.

Step 1. Now we construct $\tilde{E}$ and $\tilde{\Gamma}$ as claimed. Let $\mathcal{C}(\Gamma)$ be the set of points $p \in \Gamma$ such that in any neighborhood of p it is impossible to write Γ as a single graph, i.e. p is a crossing or a branching point of two pieces of some curves γ_i, γ_j. Call K the set of accumulation points of $\mathcal{C}(\Gamma)$. Observe that K is compact.
Also, observe that if a sequence $p_n \in \mathcal{C}(\Gamma)$ converges to $\bar{p}$, $p_n = \gamma_i(t_n) = \gamma_j(\tau_n)$ with $t_n \to t^-, \tau_n \to \tau^-$ or $t_n \to t^-, \tau_n \to \tau^+$ and $t \neq \tau$, then $\dot{\gamma}_i(t) = \pm\dot{\gamma}_j(\tau)$.
Now fix $\varepsilon << 1$ and let $q \in K$. Let $v_1(q), ..., v_{N_q}(q)$ be unit vectors identifying the tangent directions at q of the curves passing through q. For $j = 1, ..., N_q$ let $\sigma_1^j, ..., \sigma_{M_{q,j}}^j$ be suitable restrictions of the curves $\{\gamma_i\}$ on disjoint intervals $I_i^j = \text{domain}(\sigma_i^j)$ such that each σ_i^j passes through q with tangent parallel to $v_j(q)$. Also, for $i = 1, ..., N_q$ let $R_i(q)$ be open rectangles with two sides parallel to $v_i(q)$. Up to restriction we assume that each σ_i^j is contained in $\overline{R_j(q)}$ with endpoints on the boundary of the rectangle. We can assume the following properties:
i) each rectangle contains at most one cusp and cusps do not lie on the boundary of any rectangle. Also if $q \in \Gamma \setminus \partial E$, then $\overline{R_i(q)} \cap \partial E = \emptyset$;
ii) $R_i(q) \cap \partial E$ is homeomorphic to a closed segment such that: if no cusps lie in $R_i(q)$ then $R_i(q) \cap \partial E$ is the graph of a $C^1 \cap W^{2,p}$ function, if a cusp lies in $R_i(q)$ then $R_i(q) \cap \partial E$ is the union of the graphs of two $C^1 \cap W^{2,p}$ functions as in the definition of simple cusp;
iii) each σ_i^j can be parametrized as graph inside $R_j(q)$, and $|\dot{\sigma}_i^j(\cdot) + v_j(q)| \le \varepsilon$ or $|\dot{\sigma}_i^j(\cdot) - v_j(q)| \le \varepsilon$;
iv) σ_i^j intersects $\partial R_j(q)$ only on the sides perpendicular to $v_j(q)$ and transversely, and σ_i^j intersects σ_k^l only in the open set $R_j(q) \cup R_l(q) \setminus (\partial R_j(q) \cup \partial R_k(q))$;

v) if $a \in \partial I_i^j, b \in \partial I_k^j$ and $\sigma_i^j(a) = \sigma_i^k(b)$, then $\dot\sigma_i^j(a) = \pm\dot\sigma_k^j(b)$;
vi) if $\sigma_i^j(a) \in \mathcal{F}E$, then $\theta_V(\sigma_i^j(a)) = \sharp\{k \mid \sigma_k^j$ passes through $\sigma_i^j(a)\}$ is odd; if $\sigma_i^j(a) \in \Gamma \setminus \partial E$, then $\theta_V(\sigma_i^j(a)) = \sharp\{k \mid \sigma_k^j$ passes through $\sigma_i^j(a)\}$ is even.
Property v) follows by the fact that transverse crossings of two curves are at most countable (as proved in Lemma 2.8), and property vi) follows from the fact that $V \in \mathcal{A}(E)$ and thus θ_V is odd (resp. even) at $\mathcal{H}^1$-ae point of $\mathcal{F}E$ (resp. $\Gamma \setminus \partial E$).
Since the set K is compact, we can extract a finite covering of rectangles corresponding to points $q_1, ..., q_L$. By Theorem 2.2 the numbers N_{q_i} of the rectangles of q_i are uniformly bounded in terms of the energy, which is finite. Hence we can add to the cover the possibly remaining rectangles corresponding to each q_i, yielding a covering that is still finite. For any $j = 1, ..., L$ and $i = 1, ..., N_{q_j}$ we are going to modify the curves σ_i^j in a finite number of steps. We start from the family $\{\sigma_i^1\}_{i=1}^{M_{q_1,1}}$ corresponding to $R_1(q_1)$, then one modifies the curves corresponding to $R_2(q_1)$ and so on up to $R_{N_{q_1}}(q_1)$, then one changes the curves of the families corresponding to q_2 and so on up to q_L. Since the procedure is the same at any step, let us describe only the case of the family $\{\sigma_i^1\}_{i=1}^{M_{q_1,1}}$ corresponding to $R_1(q_1)$. In the end we will end up with the desired $\tilde E, \tilde\Gamma$.
We modify a σ_i^1 as follows, depending on the cases $q_1 \in \Gamma \setminus \partial E$, or $q_1 \in \mathcal{F}E$, or q_1 is a cusp.
1) Suppose $q_1 \in \Gamma \setminus \partial E$. Fix σ_i^1 and split it into the two pieces divided by q_1. Let us say that one such piece of σ_i^1 is parametrized as graph by $f : [0, \alpha] \to \mathbb{R}$ with $f(0) = f'(0) = 0$ corresponding to q_1. Let u_i^1 be the solution of

$$\begin{cases} u(x) = \lambda x^3 + \mu x^2 + \nu x + \omega, \\ u(0) = u'(0) = 0, \\ u(\alpha) = f(\alpha), \;\; u'(\alpha) = f'(\alpha), \end{cases} \tag{56}$$

for the suitable constants $\lambda, \mu, \nu, \omega$. Doing the same with the other piece of σ_i^1, we substitute each σ_i^1 with the graphs of the obtained functions u_i^1 (such modification is then a change in one of the original curves γ_i's). Observe that by properties v), vi) one obtains a new varifold still in the class $\mathcal{A}(E)$, in fact graphs of finitely many polynomials meet in at most finitely many points.
2) Suppose now that $q \in \mathcal{F}E$. By construction, for example $R_1(q_1)$ contains some curves with endpoints on $\mathcal{F}E \cap \partial R_1(q_1)$. In this case we modify the curves exactly as before following the system (56); moreover we declare that the boundary $\mathcal{F}E$ is modified inside $R_1(q_1)$ following the new modified curves having endpoints on $\mathcal{F}E \cap \partial R_1(q_1)$. This leads to a new set which we already call $\tilde E$ satisfying the hypotheses of the theorem and having the same number of cusps of E, together with a new varifold already called $\tilde V$ in the class $\mathcal{A}(\tilde E)$ (as before by properties v), vi), together with the fact that the new curves are graphs of polynomials).
3) Finally suppose q_1 is a cusp of ∂E. In this case we modify the curves (and the set E) exactly in the same way of the case 2). This preserves the cusp in the new set $\tilde E$.
After performing these modifications in any $R_i(q_j)$ we end up with a varifold $\tilde V$ given by curves $\tilde\gamma_i$ such that the set $\mathcal{C}(\tilde\Gamma)$ of the points $p \in \tilde\Gamma$ such that in any neighborhood of p it is impossible to write $\tilde\Gamma$ as a single graph is finite. In fact the points of this type belonging to the union of the closure of the rectangles $R_i(q_j)$ are finite. So, if by contradiction there are points of $\mathcal{C}(\tilde\Gamma)$ accumulating to some limit point q, this would be outside the union of the rectangles $R_i(q_j)$, and q would be a limit of a sequence in $\mathcal{C}(\Gamma)$. Hence q would be in K, and thus in the interior of some rectangle $R_i(q_j)$, that is a contradiction.

Step 2. Now we show that, in general, if a set E is as in the hypotheses of the theorem and if $V = \mathbf{v}(\Gamma, \theta_V) \in \mathcal{A}(E)$ is such that Γ is a finite graph, then the number of cusps of E is even. Together with Step 1, this gives the conclusion. Here we essentially generalize the strategy of [BeDaPa93].
Call G_Γ the finite graph given by Γ with multiplicity m_Γ as described in Remark 4.5 (recall that G_Γ is regular). Let us construct a new graph G with multiplicity m as follows. If $e \in E_{G_\Gamma}$, then define

the multiplicity

$$m(e) := \begin{cases} \frac{m_\Gamma(e)}{2} & \text{if } m_\Gamma(e) \text{ even,} \\ \frac{m_\Gamma(e)-1}{2} & \text{if } m_\Gamma(e) \text{ odd,} \end{cases}$$

with the convention that if $m(e) = 0$, then the edge e does not appear in G. Now let $y \in V_G$. We want to evaluate the parity of $\rho_G(y)$ dividing some cases.
a) Suppose $y \notin \partial E$. Then any edge e of G_Γ with endpoint at y has $\rho^+_{G_\Gamma}(y, w_i(y)) = \rho^-_{G_\Gamma}(y, w_i(y))$ even for any $w_i(y)$. Hence by definition we have that $\rho_G(y)$ is even.
b) Suppose $y \in \mathcal{F}E$. Then exactly two edges e_1, e_2 of G_Γ having an endpoint at y have odd multiplicity: $m_\Gamma(e_i) = 2k_i + 1$ for $i = 1, 2$. Up to relabeling suppose that $e_1 \in I^+(w_1(y))$ and $e_2 \in I^-(w_1(y))$. Every other edge of G_Γ having an endpoint at y has even multiplicity. Since G_Γ is regular we have that

$$2k_1 + 1 + 2a_1^+ = \rho^+_{G_\Gamma}(y, w_1(y)) = \rho^-_{G_\Gamma}(y, w_1(y)) = 2k_2 + 1 + 2a_1^-,$$

and similarly

$$2a_i^+ = \rho^+_{G_\Gamma}(y, w_i(y)) = \rho^-_{G_\Gamma}(y, w_i(y)) = 2a_i^-,$$

for any possible $i \geq 2$. Then

$$\rho_G(y) = k_1 + a_1^+ + k_2 + a_1^- + \sum_{i\geq 2} a_i^+ + a_i^- = 2\left(k_1 + a_1^+ + \sum_{i\geq 2} a_i^+\right)$$

is even.
c) Finally suppose that y is a cusp of ∂E. Then exactly two edges e_1, e_2 of G_Γ having an endpoint at y have odd multiplicity: $m_\Gamma(e_i) = 2k_i + 1$ for $i = 1, 2$. Up to relabeling suppose that $e_1, e_2 \in I^+(w_1(y))$. Every other edge of G_Γ having an endpoint at y has even multiplicity. Since G_Γ is regular we have that

$$2k_1 + 1 + 2k_2 + 1 + 2a_1^+ = \rho^+_{G_\Gamma}(y, w_1(y)) = \rho^-_{G_\Gamma}(y, w_1(y)) = 2a_1^-,$$

and similarly

$$2a_i^+ = \rho^+_{G_\Gamma}(y, w_i(y)) = \rho^-_{G_\Gamma}(y, w_i(y)) = 2a_i^-,$$

for any possible $i \geq 2$. Then

$$\rho_G(y) = k_1 + k_2 + a_1^+ + a_1^- + \sum_{i\geq 2} a_i^+ + a_i^- = 2(k_1 + k_2 + a_1^+) + 1 + 2\sum_{i\geq 2} a_i^+,$$

that is odd.
It follows that the cusps of ∂E coincides with the vertices y of G having odd local density $\rho_G(y)$. By Theorem 1.2.1 in [Or62], the vertices of a finite graph with odd local density are even. Hence the cusps are even and the proof is completed. $\square$

Now we turn our attention to another class of sets. Let us give the following definition.

Definition 4.7. A closed measurable set $E \subset \mathbb{R}^2$ is a p-polygon if $\partial E = (\gamma)$ for a curve $\gamma : [0, 2\pi]/_\sim \simeq S^1 \to \mathbb{R}^2$ such that:
i) γ is injective,
ii) there exist finitely many times $t_1 < t_2 < ... < t_K$ such that $\gamma|_{(t_i, t_{i+1})} \in W^{2,p}$ for $i = 1, ..., K$ (with $t_{K+1} = t_1$), and $\gamma'(t_i^-), \gamma'(t_i^+)$ are linearly independent for $i = 1, ..., K$.

Proposition 4.8. *Let E be a p-polygon, then $\overline{\mathcal{F}_p}(E) = +\infty$.*

Proof. Let γ be as in the definition of p-polygon. Without loss of generality we can assume that $0 = \gamma(0)$ is such that $\gamma'(0^-)$ and $\gamma'(0^+)$ are linearly independent. Suppose by contradiction that there is a varifold $V = \mathbf{v}(\Gamma, \theta_V) = \sum_{i=1}^N (\gamma_i)_\sharp(\mathbf{v}(S^1, 1)) \in \mathcal{A}(E)$. Let $v = \gamma'(0^-)$, then since V verifies the flux property we find a nice rectangle $R_v(p)$ at p with side parameters a, b for the curves $\{g_j\}_{j=1}^r$ given by the definition of flux property. We can suppose that $g_1|_{[-\varepsilon,0)} \subset \mathcal{F}E$, $g_1|_{(0,\varepsilon]} \subset \Gamma \setminus \partial E$, and that $(g_i) \cap \partial E = \{0\}$ for $i = 2, ..., r$. Hence

$$g_1|_{[-\varepsilon,0)} \subset \{\theta_V \text{ odd}\},$$

$$\mathcal{H}^1\left(\left(\bigcup_{i=2}^{r}(g_i) \cup g_1\big((0,\varepsilon]\big)\right) \setminus \{\theta_V \text{ even}\}\right) = 0.$$

Then there exists $c_1 \in (-a, 0)$ such that

$$\sum_{z \in \cup_{j=1}^r (g_i) \cap \{y \,|\, \langle y-p, v\rangle = c_1\}} \theta_V(z) = M_1$$

with M_1 odd, and there exists $c_2 \in (0, a)$ such that

$$\sum_{z \in \cup_{j=1}^r (g_i) \cap \{y \,|\, \langle y-p, v\rangle = c_2\}} \theta_V(z) = M_2$$

with M_2 even. But by the flux property M_1 and M_2 should be equal, thus we have a contradiction. □

Remark 4.9. More generally it follows from the proof of Proposition 4.8 that roughly speaking $\overline{\mathcal{F}_p}(E) = +\infty$ whenever the boundary ∂E has an angle (in the same sense of the definition of polygon).

With the strategy in the proof of Proposition 4.8 we can construct an example of a set $E \subset \mathbb{R}^2$ such that E is a set of finite perimeter such that the associated varifold V_E verifies that

$$\sigma_{V_E} = 0, \qquad k_{V_E} \in L^2(\mu_{V_E}), \qquad \text{but } \overline{\mathcal{F}_p}(E) = +\infty.$$

Such set is discussed in the next example.

Example 4.10. Consider a positive angle $\theta > 0$ which will be taken very small and the vectors in the plane identified by the complex numbers

$$e^{-i\theta}, \qquad e^{-i2\theta}, \qquad e^{i(-\pi+\theta)}, \qquad e^{i(-\pi+2\theta)}. \tag{57}$$

The sum of such vectors gives the point $(0, -2(\sin(\theta) + \sin(2\theta)))$. Now let $\varphi > 0$ be another positive angle and consider the vectors

$$e^{i\varphi}, \qquad e^{i(\pi-\varphi)}, \tag{58}$$

so that the sum of these last vectors gives the point $(0, 2\sin(\varphi))$. Then for $\theta \to 0$, since $\sin(\theta) + \sin(2\theta) = 3\theta + o(\theta^2)$ there exists $\varphi = 3\theta + o(\theta^2)$ such that the sum of the vectors in (57) and (58) is zero.

Given these vectors we can define a set E as in Fig. 4 whose boundary is the image of three smooth closed immersions σ_i of the interval $[0, 1]$ having $\sigma_i(0) = \sigma_i(1) = 0$ with derivative $\sigma_i'(0), \sigma_i'(1)$ proportional to the vectors in (57), (58). In such a way the varifold V_E clearly verifies that $\sigma_{V_E} = 0$ and $k_{V_E} \in L^2(\mu_{V_E})$. However arguing as in the proof of Proposition 4.8 and assuming $\overline{\mathcal{F}_p}(E) < +\infty$, one immediately gets a contradiction. Hence $\overline{\mathcal{F}_p}(E) = +\infty$.

Finally we construct a simple example showing that there are sets E with $\overline{\mathcal{F}_p}(E) < \infty$, but such that $\mathcal{H}^1(\partial E \setminus \mathcal{F}E) > 0$ and ∂E is the support of a C^∞ immersion σ.

Example 4.11. Let us construct a set E such that $\partial E = (\gamma)$ for a C^∞ immersion $\gamma : S^1 \to \mathbb{R}^2$, $\mathcal{H}^1(\partial E \setminus \mathcal{F}E) > 0$, and $\overline{\mathcal{F}_p}(E) < +\infty$.

Let $\{q_n\}_{n\geq 1} = \mathbb{Q} \cap [0, 1]$ be an enumeration of the rationals in $[0, 1]$, and define $K = [0, 1] \setminus \cup_{n\geq 1}(q_n -$

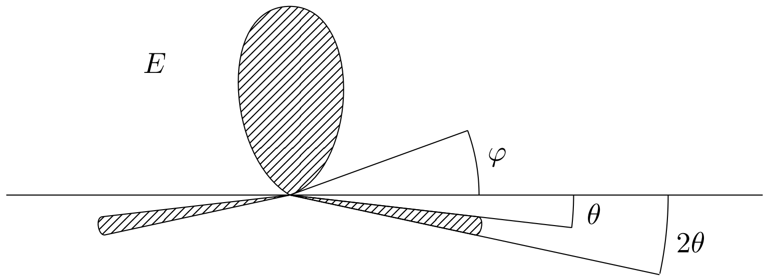

FIGURE 4. Picture describing the set E of Example 4.10. The set is symmetric with respect to the reflection about the vertical axis.

$2^{-n-2}, q_n - 2^{-n-2})$. The set K is compact and $\mathcal{L}^1(K) \geq 1 - \sum_{n=1}^{\infty} 2^{-n-1} = \frac{1}{2}$. Consider a C^∞ nonincreasing function $\varphi : [0,\infty) \to [0,1]$ such that $\varphi(0) = 1$, $\varphi(t) = 0$ for $t \geq 1$ and let

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \varphi\left(\frac{(x-q_n)^2}{(2^{n+2})^2} \right) \qquad \forall x \in [0,1].$$

By construction we have that $K = f^{-1}(0)$. Moreover $f \in C^\infty([0,1])$, in fact $\varphi \leq 1$ and $|\varphi^{(k)}| \leq c_k$ for any $k \geq 1$ for some $c_k > 0$, so that both the series f and the series of the derivatives totally converge. Then we can define a C^∞ parametrization $\sigma : [0,4] \to \mathbb{R}^2$ such that $\sigma(t) = (t, f(t))$ for $t \in [0,1]$, $\sigma(t) = (3-t, -f(t))$ for $t \in [2,3]$, while $\sigma|_{[1,2]}$ and $\sigma|_{[3,4]}$ parametrize two drops with vertices respectively at $(1,0)$ and $(0,0)$. Therefore σ parametrizes the boundary of a bounded set E which is the planar surface enclosed by the two drops and lying between the graphs of f and $-f$.
By construction $\partial E = (\sigma)$ and $\mathcal{F}E = (\sigma) \setminus K$, hence $\mathcal{H}^1(\partial E \setminus \mathcal{F}E) \geq \frac{1}{2}$. However approximating f with $f_n(x) = f(x) + \frac{1}{n}\psi(x)$, where $\psi \in C^\infty([0,1];[0,1])$ is such that $\psi(0) = \psi(1) = 0$, $\psi|_{(0,1)} > 0$, and defining σ_n in analogy with σ, we conclude that $\overline{\mathcal{F}_p}(E) < +\infty$.

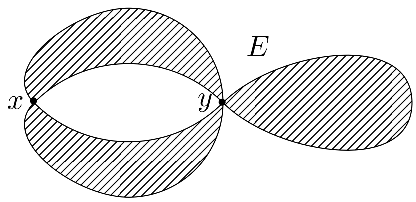

FIGURE 5. An example of a set of finite perimeter E such that $\mathcal{F}_p(E) = \mathcal{F}_p(V) < +\infty$ for any $p \in [1,\infty)$, where $V \in \mathcal{A}(E)$ is the varifold induced by a smooth immersion γ parametrizing ∂E. Here $\partial E = \mathcal{F}E \sqcup \{x, y\}$ and the strict inclusions $\mathcal{F}E \subsetneq \{x \mid \theta_V(x) \text{ is odd}\} = \mathcal{F}E \sqcup \{y\} \subsetneq \partial E$ occur.

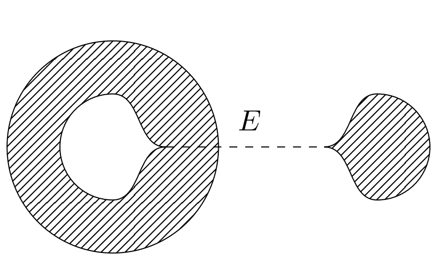

FIGURE 7. An example of a set E with finite relaxed energy such that, by Lemma 3.7, the multiplicity θ_V is not locally constant on connected components of $\mathcal{F}E$.

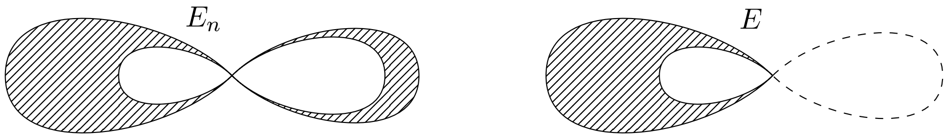

FIGURE 6. An example of a set E with finite relaxed energy such that $\partial E \setminus \mathcal{F}E$ is a singleton. A sequence of sets E_n converging to E with uniformly bounded energy is for example made of sets like in the one on the left in the picture; the dashed line represents the corresponding ghost line given by the collapsing of the right part of the sets E_n.

Acknowledgments. I warmly thank Matteo Novaga for having proposed me to study this problem and for many useful conversations. I also thank the referee for suggesting an improvement on the work.

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Dipartimento di Matematica, Università di Pisa, Largo Bruno Pontecorvo 5, 56127 Pisa, Italy
Email address: pozzetta@mail.dm.unipi.it