ON LOCAL LIPSCHITZ REGULARITY FOR QUASILINEAR EQUATIONS IN THE HEISENBERG GROUP

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Abstract. The goal of this article is to establish local Lipschitz continuity of solutions for a class of degenerated sub-elliptic equations of divergence form, in the Heisenberg Group. The considered hypothesis for the growth and ellipticity condition, is a natural generalisation of the sub-elliptic p-Laplace equation and more general quasilinear equations with polynomial or exponential type growth.

1. Introduction

Lipschitz continuity of weak solutions for variational problems in the Heisenberg Group $\mathbb{H}^n$, has been studied in [37], where equations with growth conditions of p-Laplacian type was considered. The purpose of this paper is to reproduce the result, for a larger class of more general quasilinear equations.

In a domain $\Omega \subset \mathbb{H}^n$, for $n \geq 1$, we consider the equation

$$\operatorname{div}_H \mathcal{A}(\mathfrak{X}u) = \sum_{i=1}^{2n} X_i(\mathcal{A}_i(\mathfrak{X}u)) = 0, \tag{1.1}$$

where $X_1, \ldots, X_{2n}$ are the horizontal vector fields, $\mathfrak{X}u = (X_1u, \ldots, X_{2n}u)$ is the horizontal gradient of a function $u : \Omega \to \mathbb{R}$, the horizontal divergence div_H defined similarly and $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{2n}) : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ for given data $\mathcal{A}_i \in C^1(\mathbb{R}^{2n})$. We denote $D\mathcal{A}(z)$ as the $2n \times 2n$ Jacobian matrix $(\partial \mathcal{A}_i(z)/\partial z_j)_{ij}$ for $z \in \mathbb{R}^{2n}$. In addition, we assume that $D\mathcal{A}(z)$ is symmetric and satisfies

$$\begin{gathered} \frac{g(|z|)}{|z|}\,|\xi|^2 \;\leq\; \big\langle D\mathcal{A}(z)\,\xi, \xi\big\rangle \;\leq\; L\,\frac{g(|z|)}{|z|}\,|\xi|^2; \\ |\mathcal{A}(z)| \;\leq\; L\,g(|z|), \end{gathered} \tag{1.2}$$

for every $z, \xi \in \mathbb{R}^{2n}$, where $L \geq 1$ and $g : [0, \infty) \to [0, \infty)$ such that $g \in C^1((0, \infty))$, $g(0) = 0$ and there exists constants $g_0 \geq \delta > 0$, such that

$$\delta \leq \frac{tg'(t)}{g(t)} \leq g_0, \quad \text{for all } t > 0. \tag{1.3}$$

In the Euclidean setting, conditions (1.2) and (1.3) have been introduced by Lieberman [24], in order to produce a natural extension of the structure conditions for elliptic operators in divergence form previously considered in Ladyzhenskaya-Ural'tseva [22], which in his words is "in a sense, the best generalization". We refer to [33, 7, 34, 17, 16, 36, 12, 23] and

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references therein, for earlier works on regularity theory of elliptic equations in divergence form, including the p-Laplace equations in the setting of the Euclidean spaces.

A prominent special case appears from minimization of the scalar variational integral

$$I(u) = \int_\Omega G(|\mathfrak{X}u|)\,dx,$$

where $G(t) = \int_0^t g(s)\,ds$ with g satisfying (1.3). The corresponding Euler-Lagrange equation

$$\operatorname{div}_H\Big(g(|\mathfrak{X}u|)\,\frac{\mathfrak{X}u}{|\mathfrak{X}u|}\Big) = \sum_{i=1}^{2n} X_i\Big(g(|\mathfrak{X}u|)\,\frac{X_iu}{|\mathfrak{X}u|}\Big) = 0, \tag{1.4}$$

forms a prototype example of the equation (1.1) with $\mathcal{A}(z) = zg(|z|)/|z|$; in this case, by explicit computation and the condition (1.3), it is not difficult to show that

$$\min\{1,\delta\}\frac{g(|z|)}{|z|}\,|\xi|^2 \,\le\, \big\langle D\mathcal{A}(z)\,\xi,\xi\big\rangle \,\le\, \max\{1,g_0\}\frac{g(|z|)}{|z|}\,|\xi|^2$$

for every $z,\xi \in \mathbb{R}^{2n}$, which resembles (1.2). In particular, if $g(t) = t^{p-1}$ for $1 < p < \infty$, then g satisfies (1.3) with $\delta = p-1 = g_0$ and (1.4) becomes the sub-elliptic p-Laplace equation $\operatorname{div}_H(|\mathfrak{X}u|^{p-2}\mathfrak{X}u) = 0$. The condition (1.3) can appear naturally if one considers defining

$$\delta \,=\, \inf_{t>0}\,\frac{tg'(t)}{g(t)} \quad \text{and} \quad g_0 \,=\, \sup_{t>0}\,\frac{tg'(t)}{g(t)}. \tag{1.5}$$

However, positivity and finiteness of the constants in (1.5), are essential and the techniques of this paper do not apply to the borderline cases e.g. $\delta = 0$. Hence, more singular equations like $\operatorname{div}_H(\mathfrak{X}u/|\mathfrak{X}u|) = 0$ or $\operatorname{div}_H(\mathfrak{X}u/\sqrt{1+|\mathfrak{X}u|^2}) = 0$ are excluded from our setting.

The conditions (1.2) and (1.3) encompass quasilinear equations for a large class of structure function g. Some natural examples include functions having growth similar to that of power-like functions and there logarithmic perturbations. We enlist two particular examples:

(1) $g(t) = (e+t)^{a+b\sin(\log\log(e+t))} - e^a$ for $b > 0, a \ge 1 + b\sqrt{2}$

(2) $g(t) = t^\alpha(\log(a+t))^\beta$ for $\alpha,\beta > 0, a \ge 1$,

see [15, 27]. In addition, multiple candidates satisfying condition (1.3) can be glued together to form the function g. A suitable gluing of the monomials $t^{\alpha-\varepsilon}, t^\alpha$ and $t^{\beta+\varepsilon}$ for $\beta > \alpha > \varepsilon$ as shown in [24], can be constructed in such a way that certain non-standard growth conditions (so called (p,q)-growth condition) of Marcellini [26], can also be analyzed.

Regularity theory in the sub-elliptic setting goes back to the seminal work of Hörmander [19] in 1967, from which one can verify that sub-elliptic linear operators are hypoelliptic and hence, distributional solutions of sub-Laplace equation are smooth. Since then, regularity of quasilinear equations in the sub-elliptic setting, has been a subject of extensive investigation throughout the following decades. We refer to [2, 3, 5, 13, 10, 11, 28, 25, 9] etc. for earlier results. Local Lipschitz continuity of weak solutions for p-Laplace equation in $\mathbb{H}^n$, has been recently proved in [37]. The techniques used in there, paves the way for this paper.

The natural domain for the weak solution of (1.1) is the Horizontal Orlicz-Sobolev space $HW^{1,G}(\Omega)$ (see Section 2 for details), defined similarly as the Horizontal Sobolev space $HW^{1,p}(\Omega)$ as in [25, 29, 37]. The natural class of metrics for localizing the estimates, is the class of homogeneous metrics equivalent to the CC-metric (see Section 2).

The following theorem is the main result of this paper.

Theorem 1.1. *If* $u \in HW^{1,G}(\Omega)$ *is a weak solution of equation* (1.1) *equipped with the structure condition* (1.2), *where* $G(t) = \int_0^t g(s)\,ds$ *and* g *satisfies* (1.3) *for some* $g_0 \geq \delta > 0$, *then* $\mathfrak{X}u \in L^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n})$. *Moreover, for any CC-metric ball* $B_r \subset \Omega$, *we have the estimate*

$$\sup_{B_{\sigma r}} G(|\mathfrak{X}u|) \leq \frac{c}{(1-\sigma)^Q} \fint_{B_r} G(|\mathfrak{X}u|)\,dx \tag{1.6}$$

for any $0 < \sigma < 1$, *where* $c = c(n, \delta, g_0, L) > 0$ *is a constant.*

Although the above theorem is stated in terms of CC-metric balls, but as evident from its proof, (1.6) holds for any homogeneous metrics of $\mathbb{H}^n$ with appropriate choice of constant c.

This paper is organised as follows. We provide some essential preliminaries on Heisenberg group, Orlicz-Sobolev spaces and sub-elliptic equations in Section 2. Then we prove several Caccioppoli type inequalities of the horizontal and vertical derivatives in Section 3, followed by the proof of Theorem 1.1 in the end.

Lastly, we remark that local $C^{1,\alpha}$-regularity of weak solutions of the p-Laplace equation in $\mathbb{H}^n$, has been shown recently in [31]; the techniques can be adopted to show the same result for the equation (1.1) with structure conditions (1.2) and (1.3), as well. Furthermore, $C^{1,\alpha}$-regularity can also be shown for general quasilinear equations of the form

$$\operatorname{div}_H A(x, u, \mathfrak{X}u) + B(x, u, \mathfrak{X}u) = 0, \tag{1.7}$$

with appropriate growth and ellipticity conditions. Towards this pursuit, the estimate (1.6) is necessary for both equations (1.1) and (1.7) and to obtain uniform $C^{1,\alpha}$ estimates, further technical difficulties appear that require lengthier and more delicate analysis. Therefore, they are omitted from this paper and have been addressed in a follow up article [30].

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2. Preliminaries

In this section, we fix the notations used and introduce the Heisenberg Group $\mathbb{H}^n$. Also, we provide some essential facts on Orlicz-Sobolev spaces and sub-elliptic equations.

Throughout this paper, we shall denote a postive constant by c which may vary from line to line. But c would depend only on n, the constant g_0 of (1.3) and and L of (1.2), unless it is explicitly specified otherwise. The dependence on δ of (1.3) shall appear at the very end.

2.1. **Heisenberg Group.**

Here we provide the definition and properties of Heisenberg group that would be useful in this paper. For more details, we refer the reader to the books [1, 4].

Definition 2.1. For $n \geq 1$, the *Heisenberg Group* denoted by $\mathbb{H}^n$, is identified to the Euclidean space $\mathbb{R}^{2n+1}$ with the group operation

$$x \cdot y := \Big(x_1 + y_1,\ \ldots,\ x_{2n} + y_{2n},\ t + s + \frac{1}{2}\sum_{i=1}^n (x_i y_{n+i} - x_{n+i} y_i)\Big) \tag{2.1}$$

for every $x = (x_1, \ldots, x_{2n}, t)$, $y = (y_1, \ldots, y_{2n}, s) \in \mathbb{H}^n$.

Thus, $\mathbb{H}^n$ with the group operation (2.1) forms a non-Abelian Lie group, whose left invariant vector fields corresponding to the canonical basis of the Lie algebra, are

$$X_i = \partial_{x_i} - \frac{x_{n+i}}{2}\partial_t, \quad X_{n+i} = \partial_{x_{n+i}} + \frac{x_i}{2}\partial_t, \tag{2.2}$$

for every $1 \le i \le n$ and the only non zero commutator $T = \partial_t$. We have

$$[X_i\,, X_{n+i}] = T \quad \text{and} \quad [X_i\,, X_j] = 0 \;\; \forall\; j \neq n+i. \tag{2.3}$$

We call $X_1, \ldots, X_{2n}$ as *horizontal vector fields* and T as the *vertical vector field.* For a scalar function $f : \mathbb{H}^n \to \mathbb{R}$, we denote

$$\mathfrak{X}f := (X_1 f, \ldots, X_{2n} f) \quad \text{and} \quad \mathfrak{X}\mathfrak{X}f := (X_j X_i f)_{i,j}$$

as the *Horizontal gradient* and *Horizontal Hessian*, respectively. From (2.3), we have the following trivial but nevertheless, an important inequality

$$|Tf| \le 2|\mathfrak{X}\mathfrak{X}f|. \tag{2.4}$$

For a vector valued function $F = (f_1, \ldots, f_{2n}) : \mathbb{H}^n \to \mathbb{R}^{2n}$, the *Horizontal divergence* is defined as $\mathrm{div}_H(F) := \sum_{i=1}^{2n} X_i f_i$.

The Euclidean gradient of a function $h : \mathbb{R}^k \to \mathbb{R}$, shall be denoted by $\nabla h = (D_1 h, \ldots, D_k h)$ with $D_j = \partial_{x_j}$ and the Hessian matrix by $D^2 h = (D_i D_j h)_{i,j}$.

A piecewise smooth rectifiable curve γ is called a *horizontal curve* if its tangent vectors are contained in the *horizontal sub-bundle* $\mathcal{H} = \mathrm{span}\{X_1, \ldots, X_{2n}\}$, that is $\gamma'(t) \in \mathcal{H}_{\gamma(t)}$ for almost every t. For any $x, y \in \mathbb{H}^n$, if the set of all horizontal curves is denoted as

$$\Gamma(x,y) = \left\{\gamma : [0,1] \to \mathbb{H}^n : \gamma(0) = x, \gamma(1) = y,\; \gamma'(t) \in \mathcal{H}_{\gamma(t)}\right\},$$

then Chow's accessibility theorem (see [6]) gurantees $\Gamma(x,y) \neq \emptyset$. The *Carnot-Carathèodory metric* (CC-metric) is defined in terms of the length $\ell(\gamma)$ of horizontal curves, as

$$d(x,y) = \;\inf\left\{\ell(\gamma) : \gamma \in \Gamma(x,y)\right\}. \tag{2.5}$$

This is equivalent to the *Korànyi metric* $d_{\mathbb{H}^n}(x,y) = \|y^{-1}\cdot x\|_{\mathbb{H}^n}$, where $\|\cdot\|_{\mathbb{H}^n}$ is the Korànyi norm, see [4]. We shall require the following norm, which is equivalent to the Korànyi norm,

$$\|x\| := \Big(\sum_{i=1}^{2n} x_i^2 + |t|\Big)^{\frac{1}{2}} \quad \text{for all } \; x = (x_1, \ldots, x_{2n}, t) \in \mathbb{H}^n. \tag{2.6}$$

Throughout this article we use CC-metric balls denoted by $B_r(x) = \{y \in \mathbb{H}^n : d(x,y) < r\}$ for $r > 0$ and $x \in \mathbb{H}^n$. However, by virtue of the equivalence of the metrics, all assertions for CC-balls can be restated to Korànyi balls or metric balls defined by the norm (2.6).

Throughout this paper, the Hausdorff dimension with respect to d shall be denoted as

$$Q := 2n+2, \tag{2.7}$$

which is also the homogeneous dimension of the group $\mathbb{H}^n$. The Lebesgue measure of $\mathbb{R}^{2n+1}$ is a Haar measure of $\mathbb{H}^n$; hence for any metric ball B_r, we have that $|B_r| = c(n)r^Q$.

For $1 \le p < \infty$, the *Horizontal Sobolev space* $HW^{1,p}(\Omega)$ consists of functions $u \in L^p(\Omega)$ such that the distributional horizontal gradient $\mathfrak{X}u$ is in $L^p(\Omega\,, \mathbb{R}^{2n})$. $HW^{1,p}(\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{HW^{1,p}(\Omega)} = \;\|u\|_{L^p(\Omega)} + \|\mathfrak{X}u\|_{L^p(\Omega,\mathbb{R}^{2n})}. \tag{2.8}$$

We define $HW^{1,p}_{\text{loc}}(\Omega)$ as its local variant and $HW^{1,p}_0(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $HW^{1,p}(\Omega)$ with respect to the norm in (2.8). The Sobolev Inequality has the following version in the sub-elliptic setting (see [3, 4]).

Theorem 2.2 (Sobolev Inequality)**.** *Let $B_r \subset \mathbb{H}^n$ and $1 < q < Q$. There exists a constant $c = c(n, q) > 0$ such that for all $u \in HW^{1,q}_0(B_r)$, we have*

$$\left(\fint_{B_r} |u|^{\frac{Qq}{Q-q}}\, dx \right)^{\frac{Q-q}{Qq}} \leq cr \left(\fint_{B_r} |\mathfrak{X}u|^q\, dx \right)^{\frac{1}{q}}. \tag{2.9}$$

We remark that the Lipschitz continuity that is considered, is implied in the sense of Folland-Stein [14], i.e. the Lipschitz continuity with respect to the CC-metric. It does not make any assertion on the regularity of the vertical derivative.

2.2. **Orlicz-Sobolev Spaces.**
In this subsection, we recall some facts on Orlicz-Sobolev functions, which shall be necessary later. Further details can be found in textbooks e.g. [21, 32].

Definition 2.3 (Young function)**.** If $\psi : [0, \infty) \to [0, \infty)$ is an non-decreasing, left continuous function with $\psi(0) = 0$ and $\psi(s) > 0$ for all $s > 0$, then any function $\Psi : [0, \infty) \to [0, \infty]$ of the form

$$\Psi(t) = \int_0^t \psi(s)\, ds \tag{2.10}$$

is called a *Young function.* A continuous Young function $\Psi : [0, \infty) \to [0, \infty)$ satisfying $\Psi(t) = 0$ iff $t = 0$, $\lim_{t\to\infty} \Psi(t)/t = \infty$ and $\lim_{t\to 0} \Psi(t)/t = 0$, is called *N-function.*

There are several different definitions available in various references. However, within a slightly restricted range of functions (as in our case), all of them are equivalent. We refer to the book of Rao-Ren [32], for a more general discussion.

Definition 2.4 (Conjugate)**.** The *generalised inverse* of a montone function ψ is defined as $\psi^{-1}(t) := \inf\{s \geq 0 \mid \psi(s) > t\}$. Given any Young function $\Psi(t) = \int_0^t \psi(s)ds$, its *conjugate* function $\Psi^* : [0, \infty) \to [0, \infty]$ is defined as

$$\Psi^*(s) := \int_0^s \psi^{-1}(t)\, dt \tag{2.11}$$

and (Ψ, Ψ^*) is called a *complementary pair*, which is *normalised* if $\Psi(1) + \Psi^*(1) = 1$.

A Young function Ψ is convex, increasing, left continuous and satisfies $\Psi(0) = 0$ and $\lim_{t\to\infty} \Psi(t) = \infty$. The generalised inverse of Ψ is right continuous, increasing and coincides with the usual inverse when Ψ is continuous and strictly increasing. In general, the inequality

$$\Psi(\Psi^{-1}(t)) \leq t \leq \Psi^{-1}(\Psi(t)) \tag{2.12}$$

is satisfied for all $t \geq 0$ and equality holds when $\Psi(t)$ and $\Psi^{-1}(t) \in (0, \infty)$. It is also evident that that the conjugate function Ψ^* is also a Young function, $\Psi^{**} = \Psi$ and for any constant $c > 0$, we have $(c\,\Psi)^*(t) = c\,\Psi^*(t/c)$. Here are two standard examples of complementary pair of Young functions.

(1) $\Psi(t) = t^p/p$ and $\Psi^*(t) = t^{p^*}/p^*$ when $1 < p, p^* < \infty$ and $1/p + 1/p^* = 1$.
(2) $\Psi(t) = (1 + t)\log(1 + t) - t$ and $\Psi^*(t) = e^t - t - 1$.

Lemma 2.5. *If (Ψ, Ψ^*) is a complementary pair of N-functions, then for any $t > 0$ we have*

$$\Psi^*\left(\frac{\Psi(t)}{t}\right) \le \Psi(t). \tag{2.13}$$

Proof. Let $\Psi(t) = \int_0^t \psi(s)ds$. From mean value theorem, there exists $s_0 \in (0, t]$ such that

$$\psi(s_0) = \frac{1}{t}\int_0^t \psi(s)\, ds = \frac{\Psi(t)}{t}$$

for every $t > 0$. Using definition (2.11) and mean value theorem again, we find that there exist $r_0 \in (0, \psi(s_0))$, such that we have

$$\Psi^*\left(\frac{\Psi(t)}{t}\right) = \int_0^{\Psi(t)/t} \psi^{-1}(r)\, dr = \frac{\Psi(t)}{t}\, \psi^{-1}(r_0).$$

Since ψ and ψ^{-1} are non-decreasing functions, hence $\psi^{-1}(r_0) \le \psi^{-1}(\psi(s_0)) = s_0 \le t$. Using this on the above, one easily gets (2.13), to complete the proof. ☐

The following Young's inequality is well known. We refer to [32] for a proof.

Theorem 2.6 (Young's Inequality)**.** *Given a Young function $\Psi(t) = \int_0^t \psi(s)ds$, we have the following for all $s, t > 0$;*

$$st \le \Psi(s) + \Psi^*(t) \tag{2.14}$$

and equality holds iff $t = \psi(s)$ or $s = \psi^{-1}(t)$.

Definition 2.7 (Doubling function)**.** The Young function Ψ is called *doubling* if there exists a constant $C_2 > 0$ such that for all $t \ge 0$, we have

$$\Psi(2t) \le C_2\, \Psi(t).$$

In the growth and ellipticity condition (1.2), the structure function g satisfying (1.3), is a doubling function. Its doubling constant $C_2 = 2^{g_0}$ (see Lemma 2.12 below). Henceforth, we restrict to Orlicz spaces of doubling functions, thereby avoiding unnecessary technicalities.

Definition 2.8. Let $\Omega \subset \mathbb{R}^m$ be open and μ be a σ-finite measure on Ω. For a doubling Young function Ψ, the *Orlicz space* $L^\Psi(\Omega, \mu)$ is defined as the vector space generated by the set $\{u : \Omega \to \mathbb{R} \mid u \text{ measurable}, \int_\Omega \Psi(|u|)\, d\mu < \infty\}$. The space is equipped with the following *Luxemburg norm*

$$\|u\|_{L^\Psi(\Omega,\mu)} := \inf\left\{k > 0 : \int_\Omega \Psi\left(\frac{|u|}{k}\right)\, d\mu \le 1\right\} \tag{2.15}$$

If μ is the Lebesgue measure, the space is denoted by $L^\Psi(\Omega)$ and any $u \in L^\Psi(\Omega)$ is called a Ψ-integrable function.

The function $u \mapsto \|u\|_{L^\Psi(\Omega,\mu)}$ is lower semi continuous and $L^\Psi(\Omega, \mu)$ is a Banach space with the norm in (2.15). The following theorem is a generalised version of Hölder's inequality, which follows easily from the Young's inequality (2.14), see [32] or [35].

Theorem 2.9 (Hölder's Inequality)**.** *For every $u \in L^\Psi(\Omega, \mu)$ and $v \in L^{\Psi^*}(\Omega, \mu)$, we have*

$$\int_\Omega |uv|\, d\mu \le 2\, \|u\|_{L^\Psi(\Omega,\mu)} \|v\|_{L^{\Psi^*}(\Omega,\mu)} \tag{2.16}$$

Remark 2.10. The factor 2 on the right hand side of the above, can be dropped if (Ψ, Ψ^*) is normalised and one is replaced by $\Psi(1)$ in the definition (2.15) of Luxemburg norm.

The *Orlicz-Sobolev space* $W^{1,\Psi}(\Omega)$ can be defined similarly by L^Ψ norms of the function and its gradient, see [32], that resembles $W^{1,p}(\Omega)$ for the special case of $\Psi(t) = t^p$. But here for $\Omega \subset \mathbb{H}^n$, we require the notion of *Horizontal Orlicz-Sobolev spaces*, analoguous to the horizontal Sobolev spaces defined in the previous subsection.

Definition 2.11. We define the space $HW^{1,\Psi}(\Omega) = \{u \in L^\Psi(\Omega) \mid \mathfrak{X}u \in L^\Psi(\Omega, \mathbb{R}^{2n})\}$ for an open set $\Omega \subset \mathbb{H}^n$ and a doubling Young function Ψ, along with the norm

$$\|u\|_{HW^{1,\Psi}(\Omega)} := \|u\|_{L^\Psi(\Omega)} + \|\mathfrak{X}u\|_{L^\Psi(\Omega,\mathbb{R}^{2n})};$$

the spaces $HW^{1,\Psi}_{\mathrm{loc}}(\Omega)$, $HW^{1,\Psi}_0(\Omega)$ are defined, similarly as earlier.

We remark that, all these notions can be defined for a general metric space, equipped with a doubling measure and upper gradient. More details of these can be found in [35].

2.3. **Sub-elliptic equations.**

Here, we discuss the known results on existence and uniqueness of weak solutions of the equation (1.1). Using the notation of horizontal divergence, we rewrite (1.1) as

$$-\operatorname{div}_H(\mathcal{A}(\mathfrak{X}u)) = 0 \quad \text{in } \Omega, \tag{2.17}$$

where $\mathcal{A} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ satisfies (1.2) and the matrix $D\mathcal{A}(z)$ is symmetric. Now, we enlist monotonicity and doubling properties of the structure function g, in the following lemma.

Lemma 2.12. *Let $g \in C^1([0,\infty))$ be a function that satisfies* (1.3) *for some constant $g_0 > 0$ and $g(0) = 0$. If $G(t) = \int_0^t g(s)ds$, then the following holds.*

$$(1)\ \ G \in C^2([0,\infty)) \text{ is convex;} \tag{2.18}$$

$$(2)\ \ tg(t)/(1+g_0) \le G(t) \le tg(t) \quad \forall\ \ t \ge 0; \tag{2.19}$$

$$(3)\ \ g(s) \le g(t) \le (t/s)^{g_0} g(s) \quad \forall\ \ 0 \le s < t; \tag{2.20}$$

$$(4)\ \ G(t)/t \text{ is an increasing function} \quad \forall\ \ t > 0; \tag{2.21}$$

$$(5)\ \ tg(s) \le tg(t) + sg(s) \quad \forall\ \ t,s \ge 0. \tag{2.22}$$

The proof of the above lemma is trivial (see Lemma 1.1 of [24]), so we omit it. Notice that (2.20) implies that g is increasing and doubling, with $g(2t) \le 2^{g_0} g(t)$. In fact, it is easy to see that, (1.3) implies $t \mapsto g(t)/t^{g_0}$ is decreasing and $t \mapsto g(t)/t^\delta$ is increasing. Thus,

$$\min\{\alpha^\delta, \alpha^{g_0}\} g(t) \le g(\alpha t) \le \max\{\alpha^\delta, \alpha^{g_0}\} g(t) \quad \text{for all } \alpha, t \ge 0. \tag{2.23}$$

Here onwards, we fix the following notations,

$$\mathrm{F}(t) := g(t)/t \quad \text{and} \quad G(t) := \int_0^t g(s)\, ds. \tag{2.24}$$

Thus, F and G are also doubling functions and G is a Young function. Now we restate the structure condition (1.2). For every $z, \xi \in \mathbb{R}^{2n}$, we have that

$$\begin{gathered} \mathrm{F}(|z|)|\xi|^2 \le \big\langle D\mathcal{A}(z)\,\xi, \xi\big\rangle \le L\,\mathrm{F}(|z|)|\xi|^2; \\ |\mathcal{A}(z)| \le L\,|z|\mathrm{F}(|z|). \end{gathered} \tag{2.25}$$

Definition 2.13. Any $u \in HW^{1,G}(\Omega)$ is called a weak solution of the equation (2.17) if for every $\varphi \in C_0^\infty(\Omega)$, we have that

$$\int_\Omega \big\langle \mathcal{A}(\mathfrak{X}u), \mathfrak{X}\varphi \big\rangle \, dx = 0. \tag{2.26}$$

In addition, for all non-negative $\varphi \in C_0^\infty(\Omega)$, if the integral above is positive (resp. negative) then u is called a weak supersolution (resp. subsolution) of the equation (2.17).

Monotonicity of the operator $\mathcal{A}$ is required for existence of weak solutions. This follows from the structure condition (2.25). First, notice that, from (2.25)

$$\begin{aligned} \big\langle \mathcal{A}(z) - \mathcal{A}(w), z - w \big\rangle &= \int_0^1 \big\langle D\mathcal{A}\big(w + t(z-w)\big)(z-w), (z-w) \big\rangle \, dt \\ &\geq |z-w|^2 \int_0^1 \mathrm{F}(|w + t(z-w)|) \, dt, \end{aligned}$$

for any $z, w \in \mathbb{R}^{2n}$. Now, it is possible to show that

$$\begin{aligned} |z|/2 \leq |tz + (1-t)w| &\leq 3|z|/2 && \text{if } |z-w| \leq 2|z|, \ t \geq 3/4, \\ |z-w|/4 \leq |tz + (1-t)w| &\leq 3|z-w|/2 && \text{if } |z-w| > 2|z|, \ t \leq 1/4, \end{aligned}$$

with appropriate use of triangle inequality. Combining the above inequalities and using the doubling property, we have the following monotonicity inequality

$$\big\langle \mathcal{A}(z) - \mathcal{A}(w), z - w \big\rangle \geq c(g_0) \begin{cases} |z-w|^2 \, \mathrm{F}(|z|) & \text{if } |z-w| \leq 2|z| \\ |z-w|^2 \, \mathrm{F}(|z-w|) & \text{if } |z-w| > 2|z| \end{cases} \tag{2.27}$$

and therefore the following ellipticity condition

$$\big\langle \mathcal{A}(z), z \big\rangle \geq c(g_0) \, |z|^2 \mathrm{F}(|z|) \geq c(g_0) G(|z|). \tag{2.28}$$

Remark 2.14. The inequality in (2.27) is reminiscent of the monotonicity inequality for the p-laplacian operator. Precisely, when $\mathcal{A}(z) = |z|^{p-2}z$ for $1 < p < \infty$, we have

$$\big(|z|^{p-2}z - |w|^{p-2}w\big) \cdot (z-w) \geq c(p) \begin{cases} |z-w|^2(|z| + |w|)^{p-2} & \text{if } \ 1 < p < 2 \\ |z-w|^p & \text{if } \ p \geq 2 \end{cases} \tag{2.29}$$

and from this, one can also derive (2.27) for this special case.

Theorem 2.15 (Existence)**.** *If $u_0 \in HW^{1,G}(\Omega)$ is a given function and the operator $\mathcal{A}$ has the structure condition* (2.25)*, then there exists a unique weak solution $u \in HW^{1,G}(\Omega)$ for the Dirichlet problem*

$$\begin{cases} -\operatorname{div}_H(\mathcal{A}(\mathfrak{X}u)) = \ 0 \quad \text{in } \Omega; \\ \quad u - u_0 \in HW_0^{1,G}(\Omega). \end{cases} \tag{2.30}$$

The proof of this theorem is a standard variant of that for the Euclidean setting and relies on literature of variational inequalities for monotone operators by Kinderlehrer and Stampacchia [20]. Similarly as the proof of Theorem 17.1 in [18], it is possible to show that there exists $u \in \mathcal{K}$ satisfying the variational inequality

$$\int_\Omega \big\langle \mathcal{A}(\mathfrak{X}u), \mathfrak{X}w - \mathfrak{X}u \big\rangle \, dx \geq 0$$

for all $w \in \mathcal{K}$, where $\mathcal{K} = \{v \in HW^{1,G}(\Omega) \,|\, v - u_0 \in HW_0^{1,G}(\Omega)\}$. Arguing with $w = u \pm \varphi$ for any $\varphi \in C_0^\infty(\Omega)$, it is easy to see that u satisfies (2.26) and hence, is a weak solution of (2.30). The conditions for existence of u, can be established from the monotonicity (2.27).

The uniqueness, follows from the following comparison principle, which can be easily proved by choosing an appropriate test function on (2.17) and using monotonicity.

Lemma 2.16 (Comparison Principle)**.** *Given $u, v \in HW^{1,G}(\Omega)$, if u and v respectively are weak super and subsolution of the equation* (2.17) *and $u \geq v$ on $\partial\Omega$ in the trace sense, then we have $u \geq v$ a.e. in Ω.*

We would also require that, the weak solution of the Dirichlet problem (2.30) is Lipschitz with respect to CC-metric, if it has smooth boundary value in strictly convex domain. The proof of this resembles the Hilbert-Haar theory in the Euclidean setting. Actually, this is the only place where we require that $D\mathcal{A}$ is symmetric.

Consider a bounded domain $D \subset \mathbb{R}^{2n+1}$ which is convex and there exists a constant $\varepsilon_0 > 0$ such that the following holds: for every $y \in \partial D$, there exists $b(y) \in \mathbb{R}^{2n+1}$ with $|b(y)| = 1$, such that

$$b(y) \cdot (x - y) \geq \varepsilon_0 |x - y|^2 \tag{2.31}$$

for all $x \in \bar{D}$. Here $(\cdot)$ is the Euclidean inner product and $|.|$ is the Euclidean norm of $\mathbb{R}^{2n+1}$. The following theorem shows existence of Lipschitz continuous solutions of (2.17). The statement and the proof of this theorem, are the same as those of Theorem 5.1 of [37]. For sake of completeness, we provide a sketch of the proof.

Theorem 2.17. *Let $D \subset \mathbb{H}^n$ be a bounded and convex domain satisfying* (2.31) *for some $\varepsilon_0 > 0$. Given $u_0 \in C^2(\bar{D})$, if $u \in HW^{1,G}(D)$ is the weak solution of the Dirichlet problem*

$$\begin{cases} \operatorname{div}_H(\mathcal{A}(\mathfrak{X}u)) = 0 \quad \text{in } D; \\ u - u_0 \in HW_0^{1,G}(D). \end{cases} \tag{2.32}$$

then there exists a constant $M = M\big(n,\, \varepsilon_0,\, \|\nabla u_0\|_{L^\infty(\bar{D})} + \|D^2 u_0\|_{L^\infty(\bar{D})},\, \operatorname{diam}(D)\big) > 0$, such that we have

$$\|\mathfrak{X}u\|_{L^\infty(D)} \leq M.$$

Proof. This proof is the same as that of Theorem 5.1 in [37], with minor changes. Here, we provide a brief outline for the reader's convenience. It is enough to show that

$$|u(x) - u(y)| \leq M d(x, y) \quad \forall\, x, y \in \bar{D} \tag{2.33}$$

for some constant $M = M\big(n,\, \varepsilon_0,\, \|\nabla u_0\|_{L^\infty(\bar{D})} + \|D^2 u_0\|_{L^\infty(\bar{D})},\, \operatorname{diam}(D)\big) > 0$.

To this end, we fix $y \in \partial D$ and consider the barrier functions

$$L^{\pm}(x) = u_0(y) + [\nabla u_0(y) \pm K\, b(y)] \cdot (x - y), \quad \text{where } K = \frac{(2n+1)^2}{2\varepsilon_0} \|D^2 u_0\|_{L^\infty(\bar{D})}$$

Taking ξ as an appropriate point between x and y and using the Taylor's formula followed by the condition (2.31), we obtain

$$\begin{aligned} u_0(x) &= u_0(y) + \nabla u_0(y) \cdot (x - y) + \frac{1}{2} D^2 u_0(\xi)(x - y) \cdot (x - y) \\ &\leq u_0(y) + \nabla u_0(y) \cdot (x - y) + K\varepsilon_0 |x - y|^2 \leq L^+(x) \end{aligned}$$

and similarly $L^-(x) \le u_0(x)$ for all $x \in \bar{D}$. Thus, if $u \in HW^{1,G}(D)$ is the weak solution of (2.32), since u_0 is continuous on the boundary, we have

$$L^-(x) \le u(x) \le L^+(x) \quad \forall\, x \in \partial D \tag{2.34}$$

upto a continuous representative of u. Now, letting $b(y) = (b_1(y), \dots, b_{2n}(y), b_t(y)) \in \mathbb{R}^{2n} \times \mathbb{R}$ and explicit computations using (2.2), we find that $\mathfrak{X}\mathfrak{X}L^\pm$ is skew-symmetric. Precisely,

$$\mathfrak{X}\mathfrak{X}L^\pm(x) = \frac{1}{2}\big[\partial_t u_0(y) \pm K\, b_t(y)\big] \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

for every $x \in \bar{D}$. Since the matrix $D\mathcal{A}(z)$ has been assumed to be symmetric, we have $\mathrm{div}_H[\mathcal{A}(\mathfrak{X}L^\pm)] = \mathrm{Tr}\,\big(D\mathcal{A}(\mathfrak{X}L^\pm)^T\mathfrak{X}\mathfrak{X}L^\pm\big) = 0$. Thus, $L^\pm$ are solutions of the equation (2.32). This, together with (2.34) and comparison principle (Lemma 2.16), implies

$$L^-(x) \le u(x) \le L^+(x) \quad \forall\, x \in D.$$

Since $L^\pm$ are Lipschitz and $L^\pm(y) = u(y)$, it is evident that there exists $M > 0$ such that

$$-\,Md(x,y) \le u(x) - u(y) \le Md(x,y) \quad \forall\, x \in \bar{D}, y \in \partial D \tag{2.35}$$

Now, we need the fact that if u is a Lipschitz solution of (2.32), then the following holds,

$$\sup_{x,y\in\bar{D}} \left(\frac{|u(x) - u(y)|}{d(x,y)} \right) = \sup_{x\in\bar{D},\, y\in\partial D} \left(\frac{|u(x) - u(y)|}{d(x,y)} \right). \tag{2.36}$$

We refer to [37] for a proof of (2.36). From (2.35) and (2.36), we immediately get (2.33) and the proof is finished. $\square$

3. Local Boundedness of Horizontal gradient

We shall prove Theorem 1.1 in this section. In the following three subsections we prove some Caccioppoli type inequalities of the horizontal and vertical vector fields, under two supplementary assumptions (see (3.1) and (3.2) below). The proof of Theorem 1.1 is given at the end of this section, where we remove both assumptions one by one.

Throughout this section, we denote $u \in HW^{1,G}(\Omega)$ as a weak solution of (2.17). We assume the growth and ellipticity conditions (2.25), retaining the notations (2.24).

Now we make two supplementary assumptions:

$$\text{(1) there exists } m_1, m_2 > 0 \text{ such that } \lim_{t\to 0} \mathrm{F}(t) = m_1 \text{ and } \lim_{t\to\infty} \mathrm{F}(t) = m_2; \tag{3.1}$$

$$\text{(2) there exists } M > 0 \text{ such that } \|\mathfrak{X}u\|_{L^\infty(\Omega)} \le M. \tag{3.2}$$

The purpose of the assumptions, is to ensure weak-differentiability of weak solutions of the equation (2.17). Since $\mathrm{F}(t) = g(t)/t$ and g is monotonic, F has possible singularities at $t \to 0$ or $t \to \infty$ (or both). The assmption (3.1) avoids this and consequently, the structure condition (2.25) together with (3.1) and (3.2), imply

$$\begin{aligned} \nu^{-1}|\xi|^2 \le \big\langle D\mathcal{A}(\mathfrak{X}u)\,\xi, \xi\big\rangle \le \nu\,|\xi|^2; \\ |\mathcal{A}(\mathfrak{X}u)| \le \nu\,|\mathfrak{X}u|, \end{aligned} \tag{3.3}$$

for some $\nu = \nu(g_0, L, M, m_1, m_2) > 0$. Thus, the equation (2.17) with (3.3), satisfies the conditions considered by Capogna [2]. From Theorem 1.1 and Theorem 3.1 of [2], we get

$$\mathfrak{X}u \in HW^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n}) \cap C^{0,\alpha}_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n}), \quad Tu \in HW^{1,2}_{\mathrm{loc}}(\Omega) \cap C^{0,\alpha}_{\mathrm{loc}}(\Omega). \tag{3.4}$$

However, every apriori estimates that follow in this section, are independent of the constants M, m_1, m_2. This enables us to remove both the assumptions (3.1) and (3.2), in the end.

3.1. **Caccioppoli type inequalities.**
By virtue of (3.4), we can weakly differentiate the equation (2.17) and obtain the equations satisfied by $X_l u$ and Tu in the weak sense. This is shown in the following two lemmas.

Lemma 3.1. *If $u \in HW^{1,G}(\Omega)$ is a weak solution of* (2.17)*, then Tu is a weak solution of*

$$\sum_{i,j=1}^{2n} X_i(D_j\mathcal{A}_i(\mathfrak{X}u)X_j(Tu)) = 0. \tag{3.5}$$

The proof of the above lemma is quite easy and similar to Lemma 3.2 in [37]. So, we omit the proof. The following lemma is similar to Lemma 3.1 in [37].

Lemma 3.2. *If $u \in HW^{1,G}(\Omega)$ is a weak solution of* (2.17)*, then for any $l \in \{1, \dots, n\}$, we have that $X_l u$ is weak solution of*

$$\sum_{i,j=1}^{2n} X_i(D_j\mathcal{A}_i(\mathfrak{X}u)X_jX_l u) + \sum_{i=1}^{2n} X_i(D_i\mathcal{A}_{n+l}(\mathfrak{X}u)Tu) + T(\mathcal{A}_{n+l}(\mathfrak{X}u)) = 0 \tag{3.6}$$

and similarly, $X_{n+l}u$ is weak solution of

$$\sum_{i,j=1}^{2n} X_i(D_j\mathcal{A}_i(\mathfrak{X}u)X_jX_{n+l} u) - \sum_{i=1}^{2n} X_i(D_i\mathcal{A}_{l}(\mathfrak{X}u)Tu) - T(\mathcal{A}_{l}(\mathfrak{X}u)) = 0. \tag{3.7}$$

Proof. We only prove (3.6), the proof of (3.7) is similar. Let $l \in \{1, 2, \dots, n\}$ and $\varphi \in C_0^\infty(\Omega)$ be fixed. We choose test function $X_l\varphi$ in (2.17) to get

$$\int_\Omega \sum_{i=1}^{2n} \mathcal{A}_i(\mathfrak{X}u)X_iX_l\varphi\, dx = 0.$$

Recalling the commutation relation (2.3) and using integral by parts, we obtain

$$\begin{aligned} 0 &= \int_\Omega \sum_{i=1}^{2n} \mathcal{A}_i(\mathfrak{X}u)X_lX_i\varphi\, dx - \int_\Omega \mathcal{A}_{n+l}(\mathfrak{X}u)T\varphi\, dx \\ &= -\int_\Omega \sum_{i=1}^{2n} X_l(\mathcal{A}_i(\mathfrak{X}u)X_i\varphi)dx + \int_\Omega T(\mathcal{A}_{n+l}(\mathfrak{X}u))\varphi\, dx. \end{aligned} \tag{3.8}$$

From (2.3) again, notice that for every $i \in \{1, 2, \dots, 2n\}$,

$$X_l(\mathcal{A}_i(\mathfrak{X}u)) = \sum_{j=1}^{2n} D_j\mathcal{A}_i(\mathfrak{X}u)X_jX_l u + D_i\mathcal{A}_{n+l}(\mathfrak{X}u)Tu. \tag{3.9}$$

Thus, (3.8) and (3.9) together completes the proof. □

The following Caccioppoli type inequality for Tu is quite standard. We provide a proof for the reader's convenience.

Lemma 3.3. *For any $\gamma \geq 0$ and $\eta \in C_0^\infty(\Omega)$, there exists $c = c(n, g_0, L) > 0$ such that*

$$\int_\Omega \eta^2 G(|Tu|)^{\gamma+1} \mathrm{F}(|\mathfrak{X}u|)|\mathfrak{X}(Tu)|^2\, dx \leq \frac{c}{(\gamma+1)^2} \int_\Omega G(|Tu|)^{\gamma+1}\mathrm{F}(|\mathfrak{X}u|)|Tu|^2|\mathfrak{X}\eta|^2 dx.$$

Proof. For some fixed $\eta \in C_0^\infty(\Omega)$ and $\gamma \geq 0$, we choose test funcion

$$\varphi = \eta^2 G(|Tu|)^{\gamma+1} Tu$$

in the equation (3.5) to get

$$\begin{aligned}
&\sum_{i,j=1}^{2n} \int_\Omega \eta^2 G(|Tu|)^{\gamma+1} D_j \mathcal{A}_i(\mathfrak{X}u) X_j(Tu) X_i(Tu)\, dx \\
&+ (\gamma+1) \sum_{i,j=1}^{2n} \int_\Omega \eta^2 G(|Tu|)^{\gamma} g(|Tu|)|Tu| D_j \mathcal{A}_i(\mathfrak{X}u) X_j(Tu) X_i(Tu)\, dx \\
&\qquad = -2 \sum_{i,j=1}^{2n} \int_\Omega \eta\, G(|Tu|)^{\gamma+1} Tu D_j \mathcal{A}_i(\mathfrak{X}u) X_j(Tu) X_i \eta\, dx.
\end{aligned}$$

We use the condition (2.19) on the first term and then use the structure condition (2.25), to estimate both sides of the above equality. We obtain

$$\begin{aligned}
&\int_\Omega \eta^2 G(|Tu|)^{\gamma+1} \mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}(Tu)|^2\, dx \\
&\qquad \leq \frac{c}{(\gamma+1)} \int_\Omega |\eta| G(|Tu|)^{\gamma+1}|Tu| \mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}(Tu)||\mathfrak{X}\eta|\, dx \\
&\qquad \leq\ c\tau \int_\Omega \eta^2 G(|Tu|)^{\gamma+1} \mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}(Tu)|^2\, dx \\
&\qquad\qquad + \frac{c}{\tau(\gamma+1)^2} \int_\Omega G(|Tu|)^{\gamma+1} \mathrm{F}(|\mathfrak{X}u|)|Tu|^2|\mathfrak{X}\eta|^2\, dx,
\end{aligned}$$

where we have used Young's inequality to obtain the latter inequality of the above. With the choice of a small enough $\tau > 0$, the proof is finished. □

The following Caccioppoli type inequality for the horizontal vector fields is more involved than the above, due to the non-commutativity (2.3). For the case of p-Laplace equations, similar inequalities have been proved before, using difference quotients for $2 \leq p \leq 4$ in [29] and directly, for $1 < p < \infty$ in [37].

Lemma 3.4. *For any $\gamma \geq 0$ and $\eta \in C_0^\infty(\Omega)$, there exists $c = c(n, g_0, L) > 0$ such that*

$$\begin{aligned}
\int_\Omega \eta^2 G(|\mathfrak{X}u|)^{\gamma+1} \mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2\, dx \leq c \int_\Omega G(|\mathfrak{X}u|)^{\gamma+1}|\mathfrak{X}u|^2 \mathrm{F}\left(|\mathfrak{X}u|\right)\left(|\mathfrak{X}\eta|^2 + |\eta T\eta|\right) dx \\
+ c\,(\gamma+1)^4 \int_\Omega \eta^2 G(|\mathfrak{X}u|)^{\gamma+1} \mathrm{F}\left(|\mathfrak{X}u|\right)|Tu|^2\, dx.
\end{aligned}$$

Proof. We fix $l \in \{1, \ldots, n\}$ and $\eta \in C_0^\infty(\Omega)$. Now, we choose $\varphi_l = \eta^2 G(|\mathfrak{X}u|)^{\gamma+1} X_l u$ as a test function in (3.6) and obtain the following,

$$
\begin{aligned}
&\sum_{i,j=1}^{2n} \int_\Omega \eta^2 G(|\mathfrak{X}u|)^{\gamma+1} D_j \mathcal{A}_i(\mathfrak{X}u) X_j X_l u X_i X_l u \, dx \\
&\quad + \ (\gamma+1) \sum_{i,j=1}^{2n} \int_\Omega \eta^2 G(|\mathfrak{X}u|)^{\gamma} X_l u D_j \mathcal{A}_i(\mathfrak{X}u) X_j X_l u X_i (G(|\mathfrak{X}u|)) \, dx \\
&\quad = -2 \sum_{i,j=1}^{2n} \int_\Omega \eta \, G(|\mathfrak{X}u|)^{\gamma+1} X_l u D_j \mathcal{A}_i(\mathfrak{X}u) X_j X_l u X_i \eta \, dx \\
&\qquad - \sum_{i=1}^{2n} \int_\Omega D_i \mathcal{A}_{n+l}(\mathfrak{X}u) X_i \varphi_l T u \ dx \\
&\qquad + \int_\Omega T(\mathcal{A}_{n+l}(\mathfrak{X}u)) \, \varphi_l \, dx \\
&\quad = J_{1,l} + J_{2,l} + J_{3,l}.
\end{aligned} \tag{3.10}
$$

Similarly, we choose $\varphi_{n+l} = \eta^2 G(|\mathfrak{X}u|)^{\gamma+1} X_{n+l} u$ in (3.7) to get

$$
\begin{aligned}
&\sum_{i,j=1}^{2n} \int_\Omega \eta^2 G(|\mathfrak{X}u|)^{\gamma+1} D_j \mathcal{A}_i(\mathfrak{X}u) X_j X_{n+l} u X_i X_{n+l} u \, dx \\
&\quad + \ (\gamma+1) \sum_{i,j=1}^{2n} \int_\Omega \eta^2 G(|\mathfrak{X}u|)^{\gamma} X_{n+l} u D_j \mathcal{A}_i(\mathfrak{X}u) X_j X_{n+l} u X_i (G(|\mathfrak{X}u|)) \, dx \\
&\quad = -2 \sum_{i,j=1}^{2n} \int_\Omega \eta \, G(|\mathfrak{X}u|)^{\gamma+1} X_{n+l} u D_j \mathcal{A}_i(\mathfrak{X}u) X_j X_{n+l} u X_i \eta \, dx \\
&\qquad + \sum_{i=1}^{2n} \int_\Omega D_i \mathcal{A}_{l}(\mathfrak{X}u) X_i \varphi_{n+l} T u \ dx \\
&\qquad - \int_\Omega T(\mathcal{A}_{l}(\mathfrak{X}u)) \, \varphi_{n+l} \, dx \\
&\quad = J_{1,n+l} + J_{2,n+l} + J_{3,n+l}.
\end{aligned} \tag{3.11}
$$

We shall add (3.10) and (3.11) and estimate both sides. First, notice that

$$
X_i(G(|\mathfrak{X}u|)) = \frac{g(|\mathfrak{X}u|)}{|\mathfrak{X}u|} \sum_{k=1}^{2n} X_k u X_i X_k u.
$$

We shall use the above along with (2.19). Adding (3.10) and (3.11) and using the structure condition (2.25), we obtain that

$$
\sum_{l=1}^{2n} (J_{1,l} + J_{2,l} + J_{3,l}) \ \geq \int_\Omega \eta^2 G(|\mathfrak{X}u|)^{\gamma+1} \mathrm{F}\,(|\mathfrak{X}u|)\, |\mathfrak{X}\mathfrak{X}u|^2 \, dx \tag{3.12}
$$

Now we claim the following, which combined with (3.12) concludes the proof of the lemma.
Claim : For every $k \in \{1,2,3\}, l \in \{1,\dots,2n\}$ and some $c = c(n, g_0, L) > 0$, we have

$$
\begin{aligned}
|J_{k,l}| \leq \frac{1}{12n} & \int_\Omega \eta^2 G(|\mathfrak{X}u|)^{\gamma+1} \mathrm{F}\left(|\mathfrak{X}u|\right) |\mathfrak{X}\mathfrak{X}u|^2 \, dx \\
& + c \int_\Omega G(|\mathfrak{X}u|)^{\gamma+1} |\mathfrak{X}u|^2 \mathrm{F}\left(|\mathfrak{X}u|\right) \left(|\mathfrak{X}\eta|^2 + |\eta T\eta|\right) \, dx \\
& + c\,(\gamma+1)^4 \int_\Omega \eta^2 G(|\mathfrak{X}u|)^{\gamma+1} \mathrm{F}\left(|\mathfrak{X}u|\right) |Tu|^2 \, dx.
\end{aligned} \tag{3.13}
$$

We prove the claim by estimating each $J_{k,l}$ in (3.10) and (3.11), using (2.25).
For the first term, we obtain

$$
|J_{1,l}| \leq c \int_\Omega |\eta| G(|\mathfrak{X}u|)^{\gamma+1} |\mathfrak{X}u| \mathrm{F}(|\mathfrak{X}u|) |\mathfrak{X}\mathfrak{X}u| |\mathfrak{X}\eta| \, dx
$$

and the claim (3.13) for $J_{1,l}$, follows from Young's inequality.
We calculate $\mathfrak{X}\varphi_l$ and similary estimate the second term using (2.25), to get

$$
\begin{aligned}
|J_{2,l}| \leq c & \int_\Omega \eta^2 G(|\mathfrak{X}u|)^{\gamma+1} \mathrm{F}\left(|\mathfrak{X}u|\right) |Tu| |\mathfrak{X}\mathfrak{X}u| \, dx \\
& + c\,(\gamma+1) \int_\Omega \eta^2 G(|\mathfrak{X}u|)^{\gamma} g(|\mathfrak{X}u|) |\mathfrak{X}u| \mathrm{F}(|\mathfrak{X}u|) |Tu| |\mathfrak{X}\mathfrak{X}u| \, dx \\
& + c \int_\Omega |\eta| G(|\mathfrak{X}u|)^{\gamma+1} |\mathfrak{X}u| \mathrm{F}(|\mathfrak{X}u|) |\mathfrak{X}\eta| |Tu| \, dx.
\end{aligned} \tag{3.14}
$$

Recalling $tg(t) \leq (1+g_0)G(t)$ from (2.19), note that the second term of the right hand side of (3.14) can be replaced by the first term. Then the claim (3.13) for $J_{2,l}$, follows by applying Young's inequality on each terms of the above.
For the third term, we show the estimate only for (3.10) i.e. for $l \in \{1,\dots,n\}$, since the estimate for the other case is the same. We first use integral by parts, then we calculate $T\varphi_l$ and obtain the following;

$$
\begin{aligned}
J_{3,l} = - & \int_\Omega \eta^2 G(|\mathfrak{X}u|)^{\gamma+1} \mathcal{A}_{n+l}(\mathfrak{X}u) X_l(Tu) \, dx \\
& - (\gamma+1) \int_\Omega \eta^2 G(|\mathfrak{X}u|)^{\gamma} X_l u \, \mathcal{A}_{n+l}(\mathfrak{X}u) T(G(|\mathfrak{X}u|)) \, dx \\
& - \ 2 \int_\Omega \eta \, G(|\mathfrak{X}u|)^{\gamma+1} X_l u \, \mathcal{A}_{n+l}(\mathfrak{X}u) T\eta \, dx.
\end{aligned}
$$

Now, notice that

$$
T(G(|\mathfrak{X}u|)) = \frac{g(|\mathfrak{X}u|)}{|\mathfrak{X}u|} \sum_{k=1}^{2n} X_k u X_k(Tu) = \mathrm{F}\left(|\mathfrak{X}u|\right) \sum_{k=1}^{2n} X_k u X_k(Tu).
$$

Using this, we carry out integral by parts again, for the first two terms of $J_{3,l}$ and obtain

$$\begin{aligned} J_{3,l} = & \int_\Omega X_l\Big(\eta^2 G(|\mathfrak{X}u|)^{\gamma+1}\mathcal{A}_{n+l}(\mathfrak{X}u)\Big)Tu\,dx \\ & -(\gamma+1)\int_\Omega\sum_{k=1}^{2n}X_k\Big(\eta^2G(|\mathfrak{X}u|)^\gamma \mathrm{F}(|\mathfrak{X}u|)X_lu\,\mathcal{A}_{n+l}(\mathfrak{X}u)X_ku\Big)Tu\,dx \\ & -2\int_\Omega \eta\, G(|\mathfrak{X}u|)^{\gamma+1}X_lu\,\mathcal{A}_{n+l}(\mathfrak{X}u)T\eta\,dx. \end{aligned}$$

From standard calculations and structure condition (2.25), we get

$$\begin{aligned} |J_{3,l}| \le & \; c\,(\gamma+1)^2\int_\Omega \eta^2 G(|\mathfrak{X}u|)^{\gamma+1}\mathrm{F}\,(|\mathfrak{X}u|)\,|Tu||\mathfrak{X}\mathfrak{X}u|\,dx \\ & + c\,(\gamma+1)^2\int_\Omega \eta^2G(|\mathfrak{X}u|)^\gamma g(|\mathfrak{X}u|)|\mathfrak{X}u|\mathrm{F}(|\mathfrak{X}u|)|Tu||\mathfrak{X}\mathfrak{X}u|\,dx \\ & + c\,(\gamma+1)\int_\Omega |\eta|G(|\mathfrak{X}u|)^{\gamma+1}|\mathfrak{X}u|\mathrm{F}(|\mathfrak{X}u|)|\mathfrak{X}\eta||Tu|\,dx \\ & + c\int_\Omega |\eta|G(|\mathfrak{X}u|)^{\gamma+1}|\mathfrak{X}u|^2\mathrm{F}\,(|\mathfrak{X}u|)\,|T\eta|\,dx. \end{aligned} \tag{3.15}$$

Similarly as the estimate of $J_{2,l}$ in (3.14), we use (2.19) to combine the first two terms of the right hand side of (3.15). Then, by applying Young's inequality on all terms except the last one, the claim (3.13) for $J_{3,l}$ follows. Thus, the proof is finished. □

3.2. **A Reverse type inequality.**

We follow the technique of Zhong [37] and obtain a reverse type inequality for Tu in the following lemma. This shall be crucial for obtaining estimates for horizontal and vertical derivatives, later. The following lemma is reminiscent to Lemma 3.5 in [37].

Lemma 3.5. *For any $\gamma \ge 1$ and all non-negative $\eta \in C_0^\infty(\Omega)$, we have*

$$\begin{aligned} & \int_\Omega \eta^2 G(\eta|Tu|)^{\gamma+1}\mathrm{F}\,(|\mathfrak{X}u|)\,|\mathfrak{X}\mathfrak{X}u|^2\,dx \\ & \quad \le c\,(\gamma+1)^2\|\mathfrak{X}\eta\|^2_{L^\infty}\int_\Omega G(\eta|Tu|)^{\gamma+1}|Tu|^{-2}|\mathfrak{X}u|^2\mathrm{F}\,(|\mathfrak{X}u|)\,|\mathfrak{X}\mathfrak{X}u|^2\,dx \end{aligned} \tag{3.16}$$

for some $c = c(n, g_0, L) > 0$.

Proof. First, notice that from (2.19), we have $G(\eta|Tu|)^{\gamma+1}|Tu|^{-2} \le \eta^2 G(\eta|Tu|)^{\gamma-1}g(\eta|Tu|)^2$ for every $\gamma \ge 1$. In other words, the integral in right hand side of (3.16), is not singular.

To prove the lemma, we fix $l \in \{1,\dots,n\}$ and invoke (3.8), i.e. for any $\varphi \in C_0^\infty(\Omega)$

$$\int_\Omega \sum_{i=1}^{2n} X_l(\mathcal{A}_i(\mathfrak{X}u)X_i\varphi)dx = \int_\Omega T(\mathcal{A}_{n+l}(\mathfrak{X}u))\varphi\,dx.$$

We choose the test function $\varphi = \eta^2 G(\eta|Tu|)^{\gamma+1}X_lu$ in the above. Notice that

$$\begin{aligned} X_i\varphi = & \; \eta^2G(\eta|Tu|)^{\gamma+1}X_iX_lu + (\gamma+1)\eta^3G(\eta|Tu|)^\gamma g(\eta|Tu|)X_lu\,X_i(|Tu|) \\ & + \Big(2\eta G(\eta|Tu|)^{\gamma+1} + (\gamma+1)\eta^2G(\eta|Tu|)^\gamma g(\eta|Tu|)|Tu|\Big)X_luX_i\eta \end{aligned}$$

and from (2.3), recall that $X_{n+l}X_l = X_lX_{n+l} - T$. Using these, we obtain

$$
\begin{aligned}
&\sum_{i=1}^{2n} \int_\Omega \eta^2 G(\eta|Tu|)^{\gamma+1} X_l(\mathcal{A}_i(\mathfrak{X}u)) X_l X_i u \, dx \\
&\qquad = \int_\Omega \eta^2 G(\eta|Tu|)^{\gamma+1} X_l(\mathcal{A}_{n+l}(\mathfrak{X}u)) Tu \, dx \\
&\qquad\quad - (\gamma+1) \sum_{i=1}^{2n} \int_\Omega \eta^3 G(\eta|Tu|)^{\gamma} g(\eta|Tu|) X_l u X_l(\mathcal{A}_i(\mathfrak{X}u)) X_i(|Tu|) \, dx \\
&\qquad\quad - \sum_{i=1}^{2n} \int_\Omega \Big(2\eta G(\eta|Tu|) + (\gamma+1)\eta^2 g(\eta|Tu|)|Tu| \Big) G(\eta|Tu|)^{\gamma} X_l u X_l(\mathcal{A}_i(\mathfrak{X}u)) X_i \eta \, dx \\
&\qquad\quad + \int_\Omega \eta^2 G(\eta|Tu|)^{\gamma+1} X_l u \, T(\mathcal{A}_{n+l}(\mathfrak{X}u)) \, dx \\
&\qquad = I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

We shall estimate both sides of the above. To estimate the left hand side, we use the structure condition (2.25), to obtain

$$
\sum_{i=1}^{2n} \int_\Omega \eta^2 G(\eta|Tu|)^{\gamma+1} X_l(\mathcal{A}_i(\mathfrak{X}u)) X_l X_i u \, dx \geq \int_\Omega \eta^2 G(\eta|Tu|)^{\gamma+1} \mathrm{F}\left(|\mathfrak{X}u|\right) |X_l(\mathfrak{X}u)|^2 \, dx.
$$

For the right hand side, we claim the following for every $k \in \{1,2,3,4\}$,

$$
\begin{aligned}
|I_k| &\leq c\tau \int_\Omega \eta^2 G(\eta|Tu|)^{\gamma+1} \mathrm{F}\left(|\mathfrak{X}u|\right) |\mathfrak{X}\mathfrak{X}u|^2 \, dx \\
&\qquad + \frac{c}{\tau}(\gamma+1)^2 \|\mathfrak{X}\eta\|_{L^\infty}^2 \int_\Omega G(\eta|Tu|)^{\gamma+1} |Tu|^{-2} |\mathfrak{X}u|^2 \mathrm{F}\left(|\mathfrak{X}u|\right) |\mathfrak{X}\mathfrak{X}u|^2 \, dx
\end{aligned}
\tag{3.17}
$$

for some $c = c(n, g_0, L) > 0$, where $\tau > 0$ is any arbitrary constant. Assuming the claim and combining it with the previous estimate, we end up with

$$
\begin{aligned}
\int_\Omega \eta^2 G(\eta|Tu|)^{\gamma+1} \mathrm{F}\left(|\mathfrak{X}u|\right) |X_l(\mathfrak{X}u)|^2 \, dx &\leq \tau \int_\Omega \eta^2 G(\eta|Tu|)^{\gamma+1} \mathrm{F}\left(|\mathfrak{X}u|\right) |\mathfrak{X}\mathfrak{X}u|^2 \, dx \\
&\quad + \frac{c}{\tau}(\gamma+1)^2 \|\mathfrak{X}\eta\|_{L^\infty}^2 \int_\Omega G(\eta|Tu|)^{\gamma+1} |Tu|^{-2} |\mathfrak{X}u|^2 \mathrm{F}\left(|\mathfrak{X}u|\right) |\mathfrak{X}\mathfrak{X}u|^2 \, dx
\end{aligned}
$$

for some $c = c(n, g_0, L) > 0$ and every $l \in \{1, \dots, n\}$. Similarly, the above inequality can also be obtained when $l \in \{n, \dots, 2n\}$. Then, by summing over the two inequalities and choosing $\tau > 0$ small enough, it is easy to obtain (3.16), as required to complete the proof.

Thus, we are left with proving the claim (3.17), which we accomplish by estimating each I_k, one by one. For I_1, first we use integral by parts to get

$$
\begin{aligned}
I_1 = & -\int_\Omega X_l\Big(\eta^2 G(\eta|Tu|)^{\gamma+1}Tu\Big)\mathcal{A}_{n+l}(\mathfrak{X}u)\,dx \\
= & -\int_\Omega \eta^2 G(\eta|Tu|)^{\gamma}\Big[G(\eta|Tu|)+(\gamma+1)\eta|Tu|g(\eta|Tu|)\Big]\mathcal{A}_{n+l}(\mathfrak{X}u)X_l(Tu)\,dx \\
& -\int_\Omega \eta\, G(\eta|Tu|)^{\gamma}\Big[2G(\eta|Tu|)+(\gamma+1)\eta|Tu|g(\eta|Tu|)\Big]Tu\,\mathcal{A}_{n+l}(\mathfrak{X}u)X_l\eta\,dx \\
= & \; I_{11}+I_{12}.
\end{aligned}
$$

Recall that $tg(t)\leq(1+g_0)G(t)$ for all $t>0$ from (2.19). Using this along with the structure condition (2.25), we will show that the claim (3.17) holds for both I_{11} and I_{12}.

For I_{11}, using (2.19),(2.25) and Young's inequality, we obtain

$$
\begin{aligned}
|I_{11}| &\leq c\,(\gamma+1)\int_\Omega \eta^2 G(\eta|Tu|)^{\gamma+1}|\mathfrak{X}u|\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}(Tu)|\,dx \\
&\leq \frac{\tau}{\|\mathfrak{X}\eta\|_{L^\infty}^2}\int_\Omega \eta^4 G(\eta|Tu|)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}(Tu)|^2\,dx \\
&\quad+\frac{c}{\tau}(\gamma+1)^2\|\mathfrak{X}\eta\|_{L^\infty}^2\int_\Omega G(\eta|Tu|)^{\gamma+1}|\mathfrak{X}u|^2\mathrm{F}\left(|\mathfrak{X}u|\right)\,dx
\end{aligned}
\tag{3.18}
$$

Now, the following inequality can be proved in a way similar to that of the Caccioppoli type inequality of Tu in Lemma 3.3, with minor modifications,

$$
\int_\Omega \eta^4 G(\eta|Tu|)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}(Tu)|^2\,dx\leq c\int_\Omega \eta^2 G(\eta|Tu|)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|Tu|^2|\mathfrak{X}\eta|^2\,dx,
$$

for some $c=c(n,g_0,L)>0$. After using the above inequality for the first term of (3.18) and then using $|Tu|\leq 2|\mathfrak{X}\mathfrak{X}u|$ for both terms, it is easy to see that (3.17) holds for I_{11}.

For I_{12}, using structure condition (2.25) and (2.19) again, we get

$$
|I_{12}|\leq c\,(\gamma+1)\int_\Omega |\eta|G(\eta|Tu|)^{\gamma+1}|\mathfrak{X}u|\mathrm{F}\left(|\mathfrak{X}u|\right)|Tu||\mathfrak{X}\eta|\,dx
\tag{3.19}
$$

from which, (3.17) follows easily from Young's inequality and $|Tu|\leq 2|\mathfrak{X}\mathfrak{X}u|$. Thus, combining the estimates (3.18) and (3.19), we conclude that the claim (3.17), holds for I_1.

The estimate of I_2 is similar. We use (2.25), (2.19) and Young's inequality, to get

$$
\begin{aligned}
|I_2| &\leq c\,(\gamma+1)\int_\Omega \eta^2 G(\eta|Tu|)^{\gamma+1}|Tu|^{-1}|\mathfrak{X}u|\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u||\mathfrak{X}(Tu)|\,dx \\
&\leq \frac{\tau}{\|\mathfrak{X}\eta\|_{L^\infty}^2}\int_\Omega \eta^4 G(\eta|Tu|)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}(Tu)|^2\,dx \\
&\quad+\frac{c}{\tau}(\gamma+1)^2\|\mathfrak{X}\eta\|_{L^\infty}^2\int_\Omega G(\eta|Tu|)^{\gamma+1}|Tu|^{-2}|\mathfrak{X}u|^2\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2\,dx.
\end{aligned}
$$

Notice that, the first term on the right hand side of the latter inequality of the above, is identical to that of (3.18). Hence, the claim (3.17) for I_2, follows similarly.

For I_3, using (2.19) and structure condition (2.25) again, we obtain

$$|I_3| \le c\,(\gamma+1)\int_\Omega |\eta|\, G(\eta|Tu|)^{\gamma+1}|\mathfrak{X}u|\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u||\mathfrak{X}\eta|\,dx$$

which together with Young's inequality, is enough for claim (3.17). Finally, the fourth term has the following estimate.

$$\begin{aligned}|I_4| &= \Big|\int_\Omega \eta^2 G(\eta|Tu|)^{\gamma+1}X_l u\sum_{i=1}^{2n} D_i\mathcal{A}_{n+l}(\mathfrak{X}u)X_i(Tu)\,dx\Big|\\ &\le \int_\Omega \eta^2 G(\eta|Tu|)^{\gamma+1}|\mathfrak{X}u|\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}(Tu)|\,dx,\end{aligned}$$

which is identical to the upper bound of I_{11} in (3.18). Hence, the claim (3.17) holds for I_4 as well and the proof is complete. $\square$

The inequality (3.16) of the above lemma yields the following intermediate inequality, which shall be essential for proving the final estimate for the horizontal gradient.

Corollary 3.6. *For any $\gamma \ge 1$ and all non-negative $\eta \in C_0^\infty(\Omega)$, we have*

$$\begin{aligned}&\int_\Omega \eta^2 G(\eta|Tu|)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2\,dx\\ &\quad\le c^{\frac{\gamma+1}{2}}(\gamma+1)^{(\gamma+1)(1+g_0)}\int_\Omega \eta^2 G\big(\|\mathfrak{X}\eta\|_{L^\infty}|\mathfrak{X}u|\big)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2\,dx\end{aligned}\tag{3.20}$$

where $c = c(n, g_0, L) > 0$.

Proof. Let us denote $\Psi(s) = \tau G(\sqrt{s})^{\gamma+1}$, where $\tau > 0$ is an arbitrary constant. Notice that Ψ is a N-function if $\gamma \ge 1$. Now we restate the inequality (3.16) of Lemma 3.5, as

$$\begin{aligned}&\int_\Omega \eta^2 G(\eta|Tu|)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2\,dx\\ &\quad\le \frac{c}{\tau}(\gamma+1)^2\|\mathfrak{X}\eta\|_{L^\infty}^2\int_\Omega \frac{\Psi(\eta^2|Tu|^2)}{|Tu|^2}|\mathfrak{X}u|^2\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2\,dx.\end{aligned}\tag{3.21}$$

Taking Ψ^* as the conjugate function of Ψ, we apply the Young's inequality (2.14) on the right hand side of the above to get

$$\begin{aligned}&\frac{c}{\tau}(\gamma+1)^2\|\mathfrak{X}\eta\|_{L^\infty}^2\int_\Omega \frac{\Psi(\eta^2|Tu|^2)}{|Tu|^2}|\mathfrak{X}u|^2\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2\,dx\\ &\quad\le \int_\Omega \eta^2\,\Psi^*\Big(\frac{\Psi(\eta^2|Tu|^2)}{\eta^2|Tu|^2}\Big)\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2\,dx\\ &\qquad+\int_\Omega \eta^2\,\Psi\Big(\frac{c}{\tau}(\gamma+1)^2\|\mathfrak{X}\eta\|_{L^\infty}^2|\mathfrak{X}u|^2\Big)\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2\,dx.\end{aligned}\tag{3.22}$$

Recalling (2.13), notice that

$$\Psi^*\Big(\frac{\Psi(\eta^2|Tu|^2)}{\eta^2|Tu|^2}\Big) \le \Psi(\eta^2|Tu|^2) = \tau G(\eta|Tu|)^{\gamma+1}$$

and using this together with (3.21) and (3.22), we end up with

$$
\begin{aligned}
&\int_\Omega \eta^2 G(\eta|Tu|)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2\,dx \\
&\qquad\qquad \le \tau\int_\Omega \eta^2\, G(\eta|Tu|)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2\,dx \\
&\qquad\qquad\qquad + \int_\Omega \eta^2\,\tau G\Big(\sqrt{c/\tau}\,(\gamma+1)\|\mathfrak{X}\eta\|_{L^\infty}|\mathfrak{X}u|\Big)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2\,dx.
\end{aligned}
$$

Thus, with a small enough $\tau>0$ and the doubling property of G, the proof is finished. $\square$

The inequality (3.20) is required in a slightly different form, which we state here in the following corollary. It is an easy consequence of Corollary 3.6, above.

Corollary 3.7. *For any $\gamma,\omega\ge 1$ and all non-negative $\eta\in C_0^\infty(\Omega)$, we have*

$$
\begin{aligned}
&\int_\Omega \eta^2 G\Big(\frac{\eta|Tu|}{\sqrt{\omega K_\eta}}\Big)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2\,dx \\
&\qquad \le \frac{c^{\frac{\gamma+1}{2}}(\gamma+1)^{(\gamma+1)(1+g_0)}}{\omega^{\frac{\gamma+1}{2}}}\int_\Omega \eta^2 G\big(|\mathfrak{X}u|\big)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2\,dx
\end{aligned}
\tag{3.23}
$$

where $K_\eta=\|\mathfrak{X}\eta\|^2_{L^\infty(\Omega)}+\|\eta T\eta\|_{L^\infty(\Omega)}$ and $c=c(n,g_0,L)>0$ is a constant.

Proof. Given any $\omega\ge 1$, note that from Lemma 2.12,

$$
G\Big(\frac{t}{\sqrt{\omega}}\Big)\le \frac{t}{\sqrt{\omega}}g\Big(\frac{t}{\sqrt{\omega}}\Big)\le \frac{1+g_0}{\sqrt{\omega}}G(t).
\tag{3.24}
$$

Taking $K_\eta=\|\mathfrak{X}\eta\|^2_{L^\infty(\Omega)}+\|\eta T\eta\|_{L^\infty(\Omega)}$, we use $\eta/\sqrt{\omega K_\eta}$ in place of η in (3.20), to get that

$$
\begin{aligned}
&\int_\Omega \frac{\eta^2}{\omega K_\eta} G\Big(\frac{\eta|Tu|}{\sqrt{\omega K_\eta}}\Big)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2\,dx \\
&\qquad \le c^{\frac{\gamma+1}{2}}(\gamma+1)^{(\gamma+1)(1+g_0)}\int_\Omega \frac{\eta^2}{\omega K_\eta} G\Big(\frac{\|\mathfrak{X}\eta\|_{L^\infty}|\mathfrak{X}u|}{\sqrt{\omega K_\eta}}\Big)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2\,dx \\
&\qquad \le \frac{c^{\frac{\gamma+1}{2}}(1+g_0)^{\gamma+1}(\gamma+1)^{(\gamma+1)(1+g_0)}}{\omega^{\frac{\gamma+1}{2}}}\int_\Omega \frac{\eta^2}{\omega K_\eta} G(|\mathfrak{X}u|)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2\,dx.
\end{aligned}
$$

In the latter inequality of the above, we have used $\|\mathfrak{X}\eta\|_{L^\infty}\le\sqrt{K_\eta}$, monotonicity of G and the inequality (3.24). After removing the factor $1/\omega K_\eta$ from both sides of the above, we end up with (3.23) for some $c=c(n,g_0,L)>0$, to complete the proof. $\square$

3.3. **Horizontal and Vertical estimates.**

We first show that, the Caccioppoli type inequality of Lemma 3.4, can be improved using Corollary 3.7. This would be essential for the proof of Theorem 1.1.

Proposition 3.8. *If $u\in HW^{1,G}(\Omega)$ is a weak solution of equation (2.17), then for any $\gamma\ge 1$ and all non-negative $\eta\in C_0^\infty(\Omega)$, we have the following estimate*

$$
\int_\Omega \eta^2 G(|\mathfrak{X}u|)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2\,dx \le c(\gamma+1)^{10(1+g_0)}K_\eta\int_{\mathrm{supp}(\eta)} G(|\mathfrak{X}u|)^{\gamma+1}|\mathfrak{X}u|^2\mathrm{F}\left(|\mathfrak{X}u|\right)\,dx,
$$

where $K_\eta = \|\mathfrak{X}\eta\|^2_{L^\infty(\Omega)} + \|\eta T\eta\|_{L^\infty(\Omega)}$ *and* $c = c(n, g_0, L) > 0$ *is a constant.*

Proof. First, we recall the Caccioppoli type estimate of Lemma 3.4,

$$\begin{aligned}\int_\Omega \eta^2 G(|\mathfrak{X}u|)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2\,dx &\le cK_\eta \int_\Omega G(|\mathfrak{X}u|)^{\gamma+1}|\mathfrak{X}u|^2\mathrm{F}\left(|\mathfrak{X}u|\right)dx\\ &\quad + c\,(\gamma+1)^4\int_\Omega \eta^2 G(|\mathfrak{X}u|)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|Tu|^2\,dx,\end{aligned}\tag{3.25}$$

where $K_\eta = \|\mathfrak{X}\eta\|^2_{L^\infty(\Omega)} + \|\eta T\eta\|_{L^\infty(\Omega)}$ and $c = c(n, g_0, L) > 0$. Thus, to complete the proof, we require an estimate of the second integral of the right hand side of the above.

To this end, let us denote

$$\Phi(s) = \omega K_\eta\, s\, G(\sqrt{s})^{\gamma+1}\tag{3.26}$$

where $\omega \ge 1$ is a constant which shall be specified later. Let Φ^* be the conjugate of Φ. We estimate the last integral of (3.25) using the Young's inequality (2.14), as follows;

$$\begin{aligned}c(\gamma+1)^4 &\int_\Omega \eta^2 G(|\mathfrak{X}u|)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|Tu|^2\,dx\\ &\le \int_\Omega \Phi\Big(c(\gamma+1)^4\frac{\eta^2|Tu|^2}{\omega K_\eta}\Big)\mathrm{F}\left(|\mathfrak{X}u|\right)\,dx + \int_\Omega \Phi^*\Big(\omega K_\eta G(|\mathfrak{X}u|)^{\gamma+1}\Big)\mathrm{F}\left(|\mathfrak{X}u|\right)\,dx\\ &= Z_1 + Z_2\end{aligned}$$

where Z_1 and Z_2 are the respective terms of the right hand side. Now, we estimate Z_1 and Z_2, one by one. First, using (3.26), doubling property for G and $|Tu| \le 2|\mathfrak{X}\mathfrak{X}u|$, notice that

$$\begin{aligned}Z_1 &= c(\gamma+1)^4\int_\Omega \eta^2|Tu|^2 G\bigg(\sqrt{c}(\gamma+1)^2\frac{\eta|Tu|}{\sqrt{\omega K_\eta}}\bigg)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)\,dx\\ &\le c^{\frac{\gamma+1}{2}}(\gamma+1)^{4+2(\gamma+1)(1+g_0)}\int_\Omega \eta^2 G\bigg(\frac{\eta|Tu|}{\sqrt{\omega K_\eta}}\bigg)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2\,dx\end{aligned}\tag{3.27}$$

for some $c = c(n, g_0, L) > 0$. Now, we apply the estimate (3.23) from Corrollary 3.7 on the last term of (3.27), to get that

$$\begin{aligned}Z_1 &\le \frac{c^{\gamma+1}(\gamma+1)^{4+3(\gamma+1)(1+g_0)}}{\omega^{\frac{\gamma+1}{2}}}\int_\Omega \eta^2 G\big(|\mathfrak{X}u|\big)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2\,dx\\ &= \frac{1}{2}\int_\Omega \eta^2 G\big(|\mathfrak{X}u|\big)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2\,dx,\end{aligned}\tag{3.28}$$

where ω is chosen as

$$\omega = 2^{\frac{2}{\gamma+1}}c^2(\gamma+1)^{6(1+g_0)+\frac{8}{\gamma+1}},\tag{3.29}$$

where c is the constant $c = c(n, g_0, L) > 0$ in the first step of (3.28).

To estimate Z_2, first notice that, from the inequality (2.13) and the definition (3.26)

$$\Phi^*\Big(\omega K_\eta G(|\mathfrak{X}u|)^{\gamma+1}\Big) = \Phi^*\bigg(\frac{\Phi(|\mathfrak{X}u|^2)}{|\mathfrak{X}u|^2}\bigg) \le \Phi(|\mathfrak{X}u|^2) = \omega K_\eta|\mathfrak{X}u|^2\, G(|\mathfrak{X}u|)^{\gamma+1}.\tag{3.30}$$

Using the above, we immediately have that

$$Z_2 \le \omega K_\eta \int_\Omega G(|\mathfrak{X}u|)^{\gamma+1}|\mathfrak{X}u|^2\mathrm{F}\left(|\mathfrak{X}u|\right)dx.\tag{3.31}$$

Combining (3.28) and (3.31) with ω as in (3.29), we finally end up with

$$
\begin{aligned}
c(\gamma+1)^4 \int_\Omega \eta^2 G(|\mathfrak{X}u|)^{\gamma+1} \mathrm{F}\left(|\mathfrak{X}u|\right) |Tu|^2\, dx &\leq \frac{1}{2} \int_\Omega \eta^2 G\big(|\mathfrak{X}u|\big)^{\gamma+1} \mathrm{F}\left(|\mathfrak{X}u|\right) |\mathfrak{X}\mathfrak{X}u|^2\, dx \\
&\quad + c(\gamma+1)^{6(1+g_0)+\frac{8}{\gamma+1}} K_\eta \int_\Omega G(|\mathfrak{X}u|)^{\gamma+1} |\mathfrak{X}u|^2 \mathrm{F}\left(|\mathfrak{X}u|\right) dx
\end{aligned}
$$

for some $c = c(n, g_0, L) > 0$. This, together with (3.25), is enough to conclude the proof. □

The following local estimate for the vertical derivative is an immediate consequence of the horizontal estimate of Proposition 3.8 and Corrollary 3.7, with the use of $|Tu| \leq 2|\mathfrak{X}\mathfrak{X}u|$.

Corollary 3.9. *If $u \in HW^{1,G}(\Omega)$ is a weak solution of equation* (2.17)*, then for any $\gamma \geq 1$ and all non-negative $\eta \in C_0^\infty(\Omega)$, we have the following estimate.*

$$
\int_\Omega \eta^2 G\left(\frac{\eta|Tu|}{\sqrt{K_\eta}}\right)^{\gamma+1} \mathrm{F}\left(|\mathfrak{X}u|\right) |Tu|^2\, dx \leq c(\gamma) K_\eta \int_{\mathrm{supp}(\eta)} G(|\mathfrak{X}u|)^{\gamma+1} |\mathfrak{X}u|^2 \mathrm{F}\left(|\mathfrak{X}u|\right)\, dx
$$

where $K_\eta = \|\mathfrak{X}\eta\|^2_{L^\infty(\Omega)} + \|\eta T\eta\|_{L^\infty(\Omega)}$ and $c(\gamma) = c(n, g_0, L, \gamma) > 0$ is a constant.

3.4. **Proof of Theorem 1.1.**
We recall that all the estimates above, rely on the apriori assumptions (3.1) and (3.2). We prove Theorem 1.1 here in three steps; first by assuming both (3.1) and (3.2), then by removing them one by one.

Proof of Theorem 1.1. First note that, it is enough to establish the estimate (1.6) to finish the proof. If (1.6) holds apriori for a weak solution $u \in HW^{1,G}(\Omega)$ of (2.17), then monotonicity of g immediately implies $|\mathfrak{X}u| \in L^\infty(B_{\sigma r})$ along with the estimate

$$
\sup_{B_{\sigma r}} |\mathfrak{X}u| \leq \max\left\{ 1\,,\ \frac{c(n, g_0, \delta, L)}{g(1)(1-\sigma)^Q} \fint_{B_r} G(|\mathfrak{X}u|)\, dx \right\}.
$$

Step 1 : We assume both (3.1) *and* (3.2).

The estimate (1.6) follows from Proposition 3.8 by standard Moser's iteration. Here, we provide a brief outline. Letting $w = G(|\mathfrak{X}u|)$, note that from (2.19)

$$
|\mathfrak{X}w|^2 \leq |\mathfrak{X}u|^2 \mathrm{F}\left(|\mathfrak{X}u|\right)^2 |\mathfrak{X}\mathfrak{X}u|^2 \leq (1+g_0) w\, \mathrm{F}\left(|\mathfrak{X}u|\right) |\mathfrak{X}\mathfrak{X}u|^2,
$$

and hence, from Proposition 3.8 we obtain

$$
\int_\Omega \eta^2 w^\gamma |\mathfrak{X}w|^2\, dx \leq c\,(\gamma+1)^{10(1+g_0)}\, K_\eta \int_{\mathrm{supp}(\eta)} w^{\gamma+2}\, dx \tag{3.32}
$$

for some $c = c(n, g_0, L) > 0$ and $K_\eta = \|\mathfrak{X}\eta\|^2_{L^\infty(\Omega)} + \|\eta T\eta\|_{L^\infty(\Omega)}$. Now we use a standard choice of test function $\eta \in C_0^\infty(B_r)$ such that $0 \leq \eta \leq 1$ and $\eta \equiv 1$ in $B_{r'}$ for $0 < r' < r$,

$$
|\mathfrak{X}\eta| \leq 4/(r-r') \quad \text{and} \quad |\mathfrak{X}\mathfrak{X}\eta| \leq 16n/(r-r')^2.
$$

Letting $\kappa = Q/(Q-2)$ and using Sobolev's inequality (2.9) for $q = 2$ on (3.32), we get that

$$
\left(\fint_{B_{r'}} w^{(\gamma+2)\kappa}\, dx\right)^{\frac{1}{\kappa}} \leq \frac{c(\gamma+2)^{12(1+g_0)}}{(r-r')^2} \fint_{B_r} w^{\gamma+2}\, dx
$$

for every $\gamma \geq 1$. Iterating this with $\gamma_i = 3\kappa^i - 2$ and $r_i = \sigma r + (1-\sigma)r/2^i$, we get

$$\sup_{B_{\sigma r}} w \leq \frac{c}{(1-\sigma)^{Q/3}} \left(\fint_{B_r} w^3 \, dx \right)^{\frac{1}{3}}$$

for $c = c(n, g_0, L) > 0$ and this holds for every $B_r \subset \Omega$ and every $0 < \sigma < 1$. Then, a standard interpolation argument (see [8], p. 299–300) leads to

$$\sup_{B_{\sigma r}} w \leq \frac{c(q)}{(1-\sigma)^{Q/q}} \left(\fint_{B_r} w^q \, dx \right)^{\frac{1}{q}}$$

for every $q > 0$ and some $c(q) = c(n, g_0, L, q) > 0$. Taking $q = 1$, we get the estimate (1.6).

Step 2 : We assume (3.1) *but remove* (3.2).

Let $B_r = B_r(x_0) \subset \Omega$ be a fixed CC-ball. Given the weak solution $u \in HW^{1,G}(\Omega)$, there exists a smooth approximation $\phi_m \in C^\infty(B_r)$ such that $\phi_m \to u$ in $HW^{1,G}(B_r)$ as $m \to \infty$. By virtue of equivalence with the Korànyi metric, it is possible to find a concentric ball $K_{\theta r} \subset\subset B_r$ with respect to the norm (2.6), for some constant $\theta = \theta(n) > 0$.

Now, let u_m be the weak solution of the following Dirichlet problem,

$$\begin{cases} \operatorname{div}_H(\mathcal{A}(\mathfrak{X} u_m)) = 0 & \text{in } K_{\theta r} \\ u_m - \phi_m \in HW_0^{1,G}(K_{\theta r}). \end{cases} \tag{3.33}$$

The choice of test function $u_m - \phi_m$ on (3.33), yields

$$\int_{K_{\theta r}} \big\langle \mathcal{A}(\mathfrak{X} u_m), \mathfrak{X} u_m \big\rangle \, dx = \int_{K_{\theta r}} \big\langle \mathcal{A}(\mathfrak{X} u_m), \mathfrak{X} \phi_m \big\rangle \, dx \tag{3.34}$$

Now, there exists $k = c(g_0, L) > 1$ such that combining ellipticity (2.28) and structure condition (2.25), one has $\big\langle \mathcal{A}(z), z \big\rangle \geq (2/k)\, |z| |\mathcal{A}(z)|$. Using this along with (2.25) and doubling property of g, we estimate the right hand side of (3.34), as

$$\begin{aligned} \int_{K_{\theta r}} \big\langle \mathcal{A}(\mathfrak{X} u_m), \mathfrak{X} \phi_m \big\rangle \, dx &= \int_{|\mathfrak{X} u_m| \geq k |\mathfrak{X} \phi_m|} \big\langle \mathcal{A}(\mathfrak{X} u_m), \mathfrak{X} \phi_m \big\rangle \, dx + \int_{|\mathfrak{X} u_m| < k |\mathfrak{X} \phi_m|} \big\langle \mathcal{A}(\mathfrak{X} u_m), \mathfrak{X} \phi_m \big\rangle \, dx \\ &\leq \frac{1}{k} \int_{K_{\theta r}} |\mathcal{A}(\mathfrak{X} u_m)| |\mathfrak{X} u_m| \, dx \, + \int_{|\mathfrak{X} u_m| < k |\mathfrak{X} \phi_m|} L \, g(|\mathfrak{X} u_m|) \, |\mathfrak{X} \phi_m| \, dx \\ &\leq \frac{1}{2} \int_{K_{\theta r}} \big\langle \mathcal{A}(\mathfrak{X} u_m), \mathfrak{X} u_m \big\rangle \, dx \, + \, k^{g_0} L \int_{K_{\theta r}} g(|\mathfrak{X} \phi_m|) \, |\mathfrak{X} \phi_m| \, dx. \end{aligned}$$

Combining the above with (3.34) and using (2.28), we get

$$\int_{K_{\theta r}} G(|\mathfrak{X} u_m|) \, dx \leq c \int_{K_{\theta r}} G(|\mathfrak{X} \phi_m|) \, dx \leq c \int_{K_{\theta r}} G(|\mathfrak{X} u|) \, dx + o(1/m) \tag{3.35}$$

for $c = c(n, g_0, L) > 0$ and $o(1/m) \to 0$ as $m \to \infty$. Now, since ϕ_m is smooth and $K_{\theta r}$ (defined by norm (2.6)) satisfies the strong convexity condition (2.31), the equation (3.33) is an example of the Dirichlet problem (2.32). From Proposition 2.17, we have that

$$\|\mathfrak{X} u_m\|_{L^\infty(K_{\theta r})} \leq M$$

which is the assumption (3.2) for u_m. Now we can apply Step 1 and conclude

$$\sup_{B_{\sigma\tau r}} G(|\mathfrak{X}u_m|) \leq \frac{c}{(1-\sigma)^Q} \fint_{B_{\tau r}} G(|\mathfrak{X}u_m|)\, dx \tag{3.36}$$

for some $c = c(n, g_0, L) > 0$, $\sigma \in (0,1)$ and $\tau = \tau(n) > 0$ chosen such that $B_{\tau r} \subset K_{\theta r}$. This is followed up with standard argument, since (3.35) ensures that there exists $\tilde{u} \in HW^{1,G}(K_{\theta r})$ such that upto a subsequence $u_m \rightharpoonup \tilde{u}$. Since, $u_m - \phi_m \in HW_0^{1,G}(K_{\theta r})$, hence we have $\tilde{u} - u \in HW_0^{1,G}(K_{\theta r})$ and combined with the monotonicity (2.27), one can show $\tilde{u}$ is a weak solution of (2.17). From uniqueness, $\tilde{u} = u$. Taking $m \to \infty$ in (3.36) and (3.35), we conclude

$$\sup_{B_{\sigma\tau r}} G(|\mathfrak{X}u|) \leq \frac{c}{(1-\sigma)^Q} \fint_{B_r} G(|\mathfrak{X}u|)\, dx$$

and (1.6) follows from a simple covering argument.

Step 3: We remove both (3.2) *and* (3.1).

The assumption (3.1) is removed by a standard approximation argument. We use the regularization constructed in Lemma 5.2 of [24]. Here, we give a brief outline.

For any fixed $0 < \varepsilon < 1$ and some $\eta_\varepsilon \in C^{0,1}([0,\infty))$, we define

$$\mathrm{F}_\varepsilon(t) = \mathrm{F}\Big(\min\{\, t + \varepsilon\, ,\, 1/\varepsilon\,\}\Big) \quad \text{and} \quad \mathcal{A}_\varepsilon(z) = \eta_\varepsilon(|z|)\mathrm{F}_\varepsilon(|z|)\, z + \Big(1 - \eta_\varepsilon(|z|)\Big)\mathcal{A}(z) \tag{3.37}$$

where $\mathcal{A}$ is given and $\mathrm{F}(t) = g(t)/t$. Thus, F_ε satisfies the assumption (3.1) with $m_1 = \mathrm{F}(\varepsilon)$ and $m_2 = \mathrm{F}(1/\varepsilon)$. Also, with the choice of η_ε as in [24](p. 343), it is possible to show that

$$\begin{aligned} \frac{1}{\tilde{L}}\mathrm{F}_\varepsilon(|z|)|\xi|^2 \leq \big\langle D\mathcal{A}_\varepsilon(z)\,\xi, \xi\big\rangle \leq \tilde{L}\,\mathrm{F}_\varepsilon(|z|)|\xi|^2; \\ |\mathcal{A}_\varepsilon(z)| \leq \tilde{L}|z|\mathrm{F}_\varepsilon(|z|), \end{aligned} \tag{3.38}$$

for some $\tilde{L} = \tilde{L}(\delta, g_0, L) > 0$. Reducing to a subsequence if necessary, it is easy to see that $\mathcal{A}_\varepsilon \to \mathcal{A}$ uniformly and $\mathrm{F}_\varepsilon \to \mathrm{F}$ uniformly on compact subsets of $(0,\infty)$, as $\varepsilon \to 0$.

Given weak solution $u \in HW^{1,G}(\Omega)$ of (2.17), we consider u_ε as the weak solution of the following regularized equation

$$\begin{cases} -\operatorname{div}_H(\mathcal{A}_\varepsilon(\mathfrak{X}u_\varepsilon)) = 0 \;\text{ in } \Omega'; \\ \quad u_\varepsilon - u \in HW_0^{1,G}(\Omega'), \end{cases} \tag{3.39}$$

for any $\Omega' \subset\subset \Omega$. Now, we are able to apply Step 2, to obtain uniform estimates for u_ε. Taking limit $\varepsilon \to 0$, we can obtain (1.6). This concludes the proof. $\square$

References

[1] A. Bonfiglioli, E. Lanconelli, and F. Uguzzoni. *Stratified Lie groups and potential theory for their sub-Laplacians*. Springer Monographs in Mathematics. Springer, Berlin, 2007.

[2] Luca Capogna. Regularity of quasi-linear equations in the Heisenberg group. *Comm. Pure Appl. Math.*, 50(9):867–889, 1997.

[3] Luca Capogna, Donatella Danielli, and Nicola Garofalo. An embedding theorem and the Harnack inequality for nonlinear subelliptic equations. *Comm. Partial Differential Equations*, 18(9-10):1765–1794, 1993.

[4] Luca Capogna, Donatella Danielli, Scott D. Pauls, and Jeremy T. Tyson. *An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem*, volume 259 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2007.

[5] Luca Capogna and Nicola Garofalo. Regularity of minimizers of the calculus of variations in Carnot groups via hypoellipticity of systems of Hörmander type. *J. Eur. Math. Soc. (JEMS)*, 5(1):1–40, 2003.

[6] Wei-Liang Chow. Über Systeme von linearen partiellen Differentialgleichungen erster Ordnung. *Math. Ann.*, 117:98–105, 1939.

[7] E. DiBenedetto. $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations. *Nonlinear Anal.*, 7(8):827–850, 1983.

[8] E. DiBenedetto and Neil S. Trudinger. Harnack inequalities for quasiminima of variational integrals. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 1(4):295–308, 1984.

[9] András Domokos. Differentiability of solutions for the non-degenerate p-Laplacian in the Heisenberg group. *J. Differential Equations*, 204(2):439–470, 2004.

[10] András Domokos and Juan J. Manfredi. $C^{1,\alpha}$-regularity for p-harmonic functions in the Heisenberg group for p near 2. 370:17–23, 2005.

[11] András Domokos and Juan J. Manfredi. Subelliptic Cordes estimates. *Proc. Amer. Math. Soc.*, 133(4):1047–1056 (electronic), 2005.

[12] Lawrence C. Evans. A new proof of local $C^{1,\alpha}$ regularity for solutions of certain degenerate elliptic p.d.e. *J. Differential Equations*, 45(3):356–373, 1982.

[13] Anna Föglein. Partial regularity results for subelliptic systems in the Heisenberg group. *Calc. Var. Partial Differential Equations*, 32(1):25–51, 2008.

[14] G. B. Folland and Elias M. Stein. *Hardy spaces on homogeneous groups*, volume 28 of *Mathematical Notes*. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1982.

[15] Nicola Fusco and Carlo Sbordone. Higher integrability of the gradient of minimizers of functionals with nonstandard growth conditions. *Comm. Pure Appl. Math.*, 43(5):673–683, 1990.

[16] M. Giaquinta and E. Giusti. Global $C^{1,\alpha}$-regularity for second order quasilinear elliptic equations in divergence form. *J. Reine Angew. Math.*, 351:55–65, 1984.

[17] Mariano Giaquinta and Enrico Giusti. On the regularity of the minima of variational integrals. *Acta Math.*, 148:31–46, 1982.

[18] Juha Heinonen, Tero Kilpeläinen, and Oll-i Martio. *Nonlinear potential theory of degenerate elliptic equations*. Dover Publications, Inc., Mineola, NY, 2006. Unabridged republication of the 1993 original.

[19] Lars Hörmander. Hypoelliptic second order differential equations. *Acta Math.*, 119:147–171, 1967.

[20] David Kinderlehrer and Guido Stampacchia. *An introduction to variational inequalities and their applications*, volume 31 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000. Reprint of the 1980 original.

[21] Alois Kufner, Oldřich John, and Svatopluk Fučík. *Function spaces*. Noordhoff International Publishing, Leyden; Academia, Prague, 1977. Monographs and Textbooks on Mechanics of Solids and Fluids; Mechanics: Analysis.

[22] Olga A. Ladyzhenskaya and Nina N. Ural′tseva. *Linear and quasilinear elliptic equations*. Translated from the Russian by Scripta Technica, Inc. Translation editor: Leon Ehrenpreis. Academic Press, New York-London, 1968.

[23] John L. Lewis. Regularity of the derivatives of solutions to certain degenerate elliptic equations. *Indiana Univ. Math. J.*, 32(6):849–858, 1983.

[24] Gary M. Lieberman. The natural generalization of the natural conditions of Ladyzhenskaya and Ural′tseva for elliptic equations. *Comm. Partial Differential Equations*, 16(2-3):311–361, 1991.

[25] Juan J. Manfredi and Giuseppe Mingione. Regularity results for quasilinear elliptic equations in the Heisenberg group. *Math. Ann.*, 339(3):485–544, 2007.

[26] Paolo Marcellini. Regularity and existence of solutions of elliptic equations with p, q-growth conditions. *J. Differential Equations*, 90(1):1–30, 1991.

[27] Paolo Marcellini. Regularity for elliptic equations with general growth conditions. *J. Differential Equations*, 105(2):296–333, 1993.

[28] Silvana Marchi. $C^{1,\alpha}$ local regularity for the solutions of the p-Laplacian on the Heisenberg group. The case $1 + \frac{1}{\sqrt{5}} < p \leq 2$. *Comment. Math. Univ. Carolin.*, 44(1):33–56, 2003.

[29] Giuseppe Mingione, Anna Zatorska-Goldstein, and Xiao Zhong. Gradient regularity for elliptic equations in the Heisenberg group. *Adv. Math.*, 222(1):62–129, 2009.
[30] Shirsho Mukherjee. $C^{1,\alpha}$-Regularity of Quasilinear equations on the Heisenberg Group. *https://arxiv.org/abs/1805.03748*, 2018.
[31] Shirsho Mukherjee and Xiao Zhong. $C^{1,\alpha}$-Regularity for variational problems in the Heisenberg Group. *https://arxiv.org/abs/1711.04671*, 2017.
[32] M. M. Rao and Z. D. Ren. *Theory of Orlicz spaces*, volume 146 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 1991.
[33] Leon Simon. Interior gradient bounds for non-uniformly elliptic equations. *Indiana Univ. Math. J.*, 25(9):821–855, 1976.
[34] Peter Tolksdorf. Regularity for a more general class of quasilinear elliptic equations. *J. Differential Equations*, 51(1):126–150, 1984.
[35] Heli Tuominen. Orlicz-Sobolev spaces on metric measure spaces. *Ann. Acad. Sci. Fenn. Math. Diss.*, 135:86, 2004. Dissertation, University of Jyväskylä, Jyväskylä, 2004.
[36] K. Uhlenbeck. Regularity for a class of non-linear elliptic systems. *Acta Math.*, 138(3-4):219–240, 1977.
[37] Xiao Zhong. Regularity for variational problems in the Heisenberg Group. *https://arxiv.org/abs/1711.03284*.

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