Another brick in the wall

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Abstract. We study the homogenization of a linearly elastic energy defined on a periodic collection of disconnected sets with a unilateral condition on the contact region between two such sets, with the model of a brick wall in mind. Using the language of Γ-convergence we show that the limit homogenized behaviour of such an energy can be described on the space of functions with bounded deformation using the masonry-type functionals studied by Anzellotti, Giaquinta and Giusti. In this case, the limit energy density is given by the homogenization formula related to the brick-wall type energy.

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1. Introduction

The modeling of 'masonry-like materials' can be undertaken both from a 'macroscopic' and a 'microscopic' standpoint. In the first case the masonry structure is viewed as an elastic continuum sustaining compression but (little or) no tension. The translation of this approach in mathematical terms and within a linearized elasticity theory can be performed by introducing energies of the form

$$\mathcal{F}(u) = \int_\Omega f\big(P_{K^\perp}\mathcal{E}u\big)\,dx, \tag{1.1}$$

where Ω is a reference configuration, $\mathcal{E}u$ is the linearized strain of the deformation u, K is the 'cone of tensile strains' (correspondingly, $K^\perp$ is the cone of 'compressive strains' defined by duality from K), and f is a linear elastic energy density. The operator $P_{K^\perp}$ is the *projection* on the cone of compressive strains. Note that for a uniform compressive strain, when $\mathcal{E}u \equiv A \in K^\perp$ then $f\big(P_{K^\perp}\mathcal{E}u\big) = f(\mathcal{E}u)$ and the material response is linearly elastic, while for a uniform tensile strain, when $\mathcal{E}u \equiv A \in K$, we have $P_{K^\perp}A = 0$, so that $f\big(P_{K^\perp}\mathcal{E}u\big) \equiv 0$. This degeneracy corresponds to the inability of the material to sustain tension. Note that this degenerate behaviour renders the problem mathematically ill-posed, so that in order to solve some problems involving the energy $\mathcal{F}$ within the framework of the direct methods of the Calculus of Variations, it is necessary to extend the definition of $\mathcal{F}$ to the space $BD(\Omega)$ of *functions of bounded deformations* on Ω consisting of functions whose distributional strain Eu is a measure.

Conversely, a masonry structure can be described 'microscopically' as a domain with a structure of a brick wall, with (linear) elastic elements that can be detached from one another at the expense of no energy, but satisfy some unilateral condition at their common boundaries. In the simplest situation, by taking as a model the geometry in Fig. 1, we can consider a periodic 2-dimensional closed set B (in that figure, the union of the boundary of the rectangles in the reference configuration) that subdivides the reference configuration $\Omega \setminus B$ into connected sets. On each of these subsets the material is linearly elastic; i.e., upon possibly changing the norm on the space of symmetric matrices, the energy density of a deformation u is simply $\|\mathcal{E}u\|^2$. If we denote by $\nu(x)$ the normal to B at a point x (that we assume exists almost everywhere with respect to the surface measure) and by $u^{\pm}(x)$ the traces on both sides of B at x (that exist almost everywhere since automatically $u \in H^1(\Omega \setminus B)$) then a unilateral condition can be again expressed by considering a cone of matrices K_0 and requiring that

$$(u^+(x) - u^-(x)) \otimes \nu(x) \in K_0 \tag{1.2}$$

for a.e. $x \in B$. This expression includes for example the constraints $\langle u^+(x) - u^-(x), \nu(x)\rangle \geq 0$ or $u^+(x) - u^-(x) = \lambda\nu(x)$ with $\lambda \geq 0$ a.e. on B, that express a linearized condition of impenetrability.

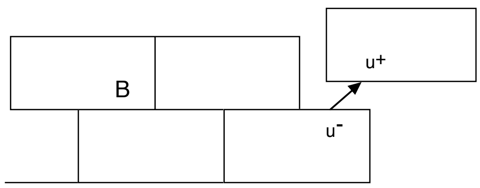

Figure 1. Admissible deformation for a brick wall

In this paper we make a connection between the two standpoints described above by showing that the first 'macroscopic' model can be obtained by *homogenization* of the second 'microscopic' one. Namely, we introduce a small parameter ε and consider the energies

$$\mathcal{F}_\varepsilon(u) = \int_{\Omega\setminus\varepsilon B} \|\mathcal{E}u\|^2\,dx \qquad (u^+ - u^-) \otimes \nu^\varepsilon \in K_0 \text{ a.e. on } \varepsilon B \tag{1.3}$$

($\nu^\varepsilon(x)$ is a fixed normal to εB in x), and we show that these energies Γ-converge as $\varepsilon \to 0^+$ to an energy $\mathcal{F}_{\rm hom}$ of the form (1.1) (more precisely, its extension to

$BD(\Omega)$) that may be written as

$$\mathcal{F}_{\rm hom}(u) = \int_\Omega f_{\rm hom}\big(P_{K^\perp_{\rm hom}} \mathcal{E}u\big)\, dx, \tag{1.4}$$

where the cone of *homogenized tensile stresses* $K_{\rm hom}$ and the *effective homogenized energy density* $f_{\rm hom}$ depend both on the 'microgeometry' described by B and on the constraint imposed by K_0, and are expressed by suitable homogenization formulas.

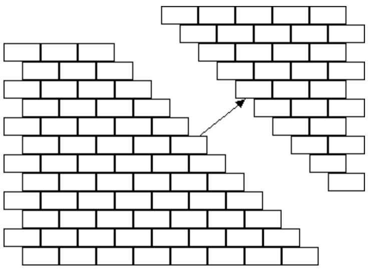

Figure 2. A limit macroscopic discontinuity

The proof relies on the localization techniques of Γ-convergence and on results of the BD-theory of masonry-like energies as in (1.1) by Anzellotti [4], Giaquinta and Giusti [16], etc. It has some strong connections with a recent result by Braides, Defranceschi and Vitali [11], where the relaxation of energies of the form

$$\mathcal{H}(u) = \int_{\Omega\setminus J_u} \|\mathcal{E}u\|^2\, dx \qquad (u^+ - u^-)\otimes\nu \in K_0 \text{ a.e. on } J_u \tag{1.5}$$

is performed, where u is constrained to be a piecewise-smooth function outside a closed set J_u (that is itself a variable of the problem). In this case the relaxed energy is again of the form (1.4) but the effective homogenized energy density depends solely on K_0. A detailed analysis when K_0 is related to a no-slip condition in contained in [10]. A mechanical insight in the subject cam be found in [15].

2. Statement of the problem and main result

We denote by $Y = [0,1)^n$ the unit cube of $\mathbb{R}^n$; a set $B \subset \mathbb{R}^n$ is Y-periodic if $B + k = B$ for all $k \in \mathbb{Z}^n$ $\mathbb{M}^{n\times n}$ denotes the space of $n\times n$ real matrices, ${}^tA$ is the transposed of the matrix A, and $A^s := \frac{1}{2}(A + {}^tA)$ its symmetric part. If $a, b \in \mathbb{R}^n$ are vectors then $a \odot b = \frac{1}{2}(a\otimes b + b\otimes a)$ is their symmetric tensor product (i.e., $a\odot b$ is the symmetric part of the tensor product $a\otimes b$) $\mathbb{M}^{n\times n}_{\rm sym}$ denotes the subspace of symmetric matrices of $\mathbb{M}^{n\times n}$ (i.e., such that $A={}^tA$). We fix a scalar product on $\mathbb{M}^{n\times n}_{\rm sym}$ that will be denoted by $\langle A, B\rangle$ and the corresponding norm $\|A\|^2 = \langle A, A\rangle$.

$\mathcal{M}(\Omega;\mathbb{M}^{n\times n})$ is the set of $\mathbb{M}^{n\times n}$-valued measures on Ω with finite total variation. We will use standard notation for Lebesgue and Sobolev spaces.

The space $BD(\Omega)$ is defined by

$$BD(\Omega)=\{u\in L^1(\Omega;\mathbb{R}^n): Eu\in\mathcal{M}(\Omega;\mathbb{M}^{n\times n})\},$$

where Eu is the *linearized strain tensor*, whose entries are defined by $E_{ij}u=\frac{1}{2}(D_iu_j+D_ju_i)$, where Du denotes the distributional gradient of u. For the measure Eu the Radon-Nikodym decomposition $Eu=\mathcal{E}u\,dx+E_su$ holds. For a function $u\in BD(\Omega)$ the symbol J_u denotes the set of essential discontinuity points for u; we will denote by $SBD(\Omega)$ (*special functions of bounded deformation*) the set of all functions $u\in BD(\Omega)$ such that $|E_su|(\Omega\setminus J_u)=0$. For such functions we have the representation

$$E_su=(u^+-u^-)\odot\nu_u\mathcal{H}^{n-1}\llcorner J_u,$$

where $\mathcal{H}^k$ is the k-dimensional Hausdorff measure, ν_u is the normal to J_u and $u^\pm$ are the traces of u on both sides of J_u (see [1], [2]).

We fix a closed rectifiable Y-periodic $(n-1)$-dimensional subset B of $\mathbb{R}^n$ and a closed cone K_0 of $\mathbb{M}^{n\times n}_{\rm sym}$ contained in $\{a\odot b: a,b\in\mathbb{R}^n\}$. We suppose that K_0 satisfies the following condition:

$$a\odot(b+c)\in K_0 \text{ whenever } a\odot b \text{ and } a\odot c\in K_0. \tag{2.1}$$

A possible choice for K_0 is the set $\{a\odot b:\ a,b\in\mathbb{R}^n,\langle a,b\rangle\geq 0\}$.

Let Ω be a bounded open subset of $\mathbb{R}^n$. For all $\varepsilon>0$ we define

$$\mathcal{U}_\varepsilon(\Omega)=\{u\in SBD(\Omega): J_u\subseteq\varepsilon B, (u^+-u^-)\odot\nu_u\in K_0\mathcal{H}^{n-1}\text{–a.e.}\},$$

the set of all special functions with bounded deformation whose discontinuity set is contained in εB and such that the density of their singular part belongs to the cone K_0.

In this paper we deal with the homogenization of integral functionals $F_\varepsilon(u): BD(\Omega)\to[0,+\infty]$ of the form

$$F_\varepsilon(u)=\begin{cases}\displaystyle\int_\Omega\|\mathcal{E}u\|^2\,dx & \text{if } u\in\mathcal{U}_\varepsilon(\Omega)\\ +\infty & \text{otherwise,}\end{cases} \tag{2.2}$$

More precisely, we will study the asymptotic behaviour of F_ε as $\varepsilon\to 0$ in the sense of Γ-convergence, with respect to the L^2 convergence on $BD(\Omega)\cap L^2(\Omega;\mathbb{R}^n)$.

We define the *homogenized energy density* as

$$\begin{aligned} f_{\rm hom}(A) \;=\; \inf\Big\{&\int_Y\|\mathcal{E}u\|^2dx: u\in BD_{\rm loc}(\mathbb{R}^n),\\ &J_u\subseteq B,(u^+-u^-)\odot\nu_u\in K_0,\ u-Ax\ Y\text{–periodic}\Big\}, \end{aligned}\tag{2.3}$$

and the corresponding kernel

$$K_{\rm hom}=\{A\in\mathbb{M}^{n\times n}: f_{\rm hom}(A)=0\}. \tag{2.4}$$

The orthogonal cone $K^\perp_{\rm hom}$ is defined by

$$K^\perp_{\rm hom}=\{B\in\mathbb{M}^{n\times n}_{\rm sym}:\langle A,B\rangle\le 0 \text{ for all } A\in K_{\rm hom}\}$$

We will prove the following result.

Theorem 2.1. *Suppose that the function $f_{\rm hom}$ satisfies*

$$f_{\rm hom}(A)=f_{\rm hom}\big(P_{K^\perp_{\rm hom}}A\big),\tag{2.5}$$

and that it is a convex function on $K^\perp_{\rm hom}$; then the family F_ε Γ-converges to F where

$$F(u)=\begin{cases}\displaystyle\int_\Omega f_{\rm hom}(Du)dx & \textit{if } u\in\mathcal{U}_{hom}(\Omega)\\ +\infty & \textit{otherwise,}\end{cases}\tag{2.6}$$

where

$$\mathcal{U}_{\rm hom}=\{u\in BD(\Omega):P_{K^\perp_{\rm hom}}(E_su)=0\}.\tag{2.7}$$

Note that by Korn's inequality indeed all functions in $\mathcal{U}_\varepsilon(\Omega)$ belong to $H^1(\Omega'\setminus\varepsilon B)$ for all $\Omega'\subset\subset\Omega$.

Example. Our theorem applies to a number of model geometries described as follows. For the sake of simplicity we treat the two-dimensional case only.

The simplest geometry is given by taking as B the square lattice

$$B_1=\{(x,y)\in\mathbb{R}^2:x\in\mathbb{Z}\text{ or }y\in\mathbb{Z}\}.$$

The usual brick-wall structure can be parameterized by

$$B_2=\{(x,y):2y\in\mathbb{Z}\}\cup\{(x,y):[2y]\text{ even}, x\in\mathbb{Z}\}\cup\{(x,y):[2y]\text{ odd}, x+1/2\in\mathbb{Z}\}$$

(see Fig.3 (a) and (b)).

Note that in both cases we have that $\nu\in\{\pm e_1,\pm e_2\}$ $\mathcal{H}^1$-a.e. on B.

We can consider the two cones of matrices

$$K_1=\{a\odot b:a=\lambda b,\ \lambda\ge 0\}$$

and

$$K_2=\{a\odot b:\langle a,b\rangle\ge 0\}.$$

Correspondingly, we have four cases in which $f_{\rm hom}$ can be easily described.

(1) When considering the geometry given by $B=B_1$ and $K_0=K_1$ the function $f_{\rm hom}$ is given on symmetric matrices by

$$f_{\rm hom}\begin{pmatrix}a & b\\ b & c\end{pmatrix}=(a_-)^2+b^2+(c_-)^2,$$

the minimum in problem (2.3) being given by the function $u(x,y)=(-a_-x+by,-c_-y+bx)$. The corresponding kernel is

$$K_{\rm hom}=\Big\{\begin{pmatrix}a & 0\\ 0 & c\end{pmatrix}:\ a\ge 0,\ c\ge 0\Big\}.$$

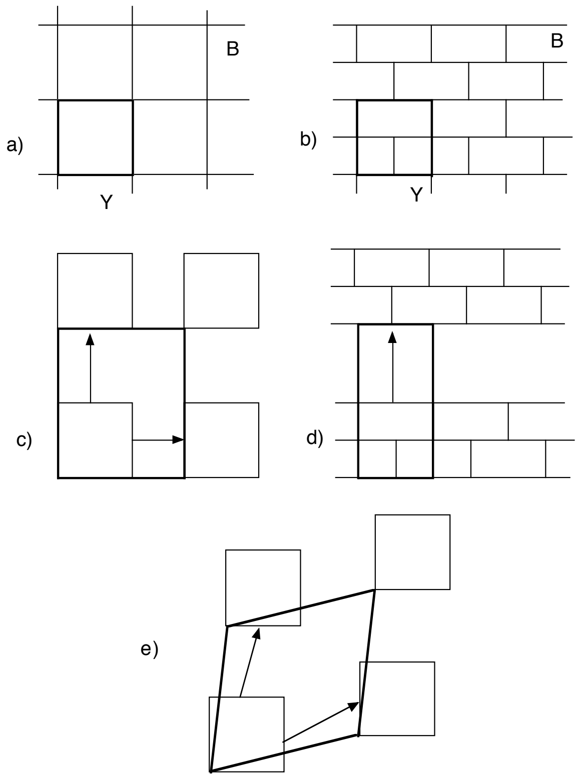

Figure 3. The geometries of the example and related zero-energy displacements

(2) If $B = B_2$ and $K_0 = K_1$ the function $f_{\rm hom}$ is given on symmetric matrices by

$$f_{\rm hom}\begin{pmatrix} a & b \\ b & c \end{pmatrix} = a^2 + b^2 + (c_-)^2,$$

a minimum in problem (2.3) being given by the function $u(x, y) = (ax+by, -c_-y+bx)$. The corresponding kernel is

$$K_{\rm hom} = \Big\{\begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} : c \geq 0\Big\}.$$

(3) In the two remaining cases with $K_0 = K_2$ the function $f_{\rm hom}$ is given on symmetric matrices by

$$f_{\rm hom}\begin{pmatrix} a & b \\ b & c \end{pmatrix} = (a_-)^2 + (c_-)^2.$$

The corresponding kernel is

$$K_{\rm hom} = \left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} : \ a \geq 0, \ c \geq 0 \right\}.$$

In the case $B = B_1$ the minimum in problem (2.3) is given by the function $u(x, y) = (-a_- x, -c_- y)$, and similarly in the case $B = B_2$.

In Fig. 3 (c)–(e) we have pictured three displacement fields with zero energy, related to case (1)–(3), respectively. The area with a thick contour represents the image of Y through $I + A$.

3. Proof of the result

The result can be proven in part following the usual localization methods of Γ-convergence, and in part using recent (and less recent) results on energies defined on BD with constraints on the strain. The proof can be divided into three steps: 1) existence and representation of the Γ-limit on $H^1(\Omega; \mathbb{R}^n)$; 2) Γ-liminf inequality by a convolution argument and translation invariance; 3) Γ-limsup inequality by density.

Step 1. As customary, we localize the energies on open subsets U of Ω by setting

$$F_\varepsilon(u, U) = \begin{cases} \displaystyle\int_{U \setminus \varepsilon B} \|\mathcal{E}u\|^2 \, dx & \text{if } u \in \mathcal{U}_\varepsilon(\Omega) \\ +\infty & \text{otherwise.} \end{cases} \tag{3.1}$$

By a density argument we may assume that the Γ-limit $F(u, U)$ exists for all u and for U in a dense class $\mathcal{D}$ of open sets (e.g. all polyrectangles with rational vertices). The extension of such a set function $F(u, \cdot)$ can be proved to be a measure (the crucial point is to prove the subadditivity with respect to the set variable; this can be done as in [11] Step 5 in the proof of Theorem 5.1). By comparing (the extension of) F with the pointwise limit on $H^1(\Omega; \mathbb{R}^n)$ we have that

$$F(u, U) \leq \int_U \|\mathcal{E}u\|^2 \, dx \leq c \int_U |Du|^2 \, dx;$$

hence we may apply the usual integral representation theorems on $H^1(\Omega; \mathbb{R}^n)$ (see [9] Section 9,[12]) to conclude that there exists a function f such that

$$F(u, U) = \int_U f(x, Du) \, dx.$$

It is easily seen that indeed $f(x, Du)$ does not depend on x (see e.g. [9] Proposition 14.3). Moreover, f depends only on the symmetric part of the gradient; i.e., $f(A) = f(B)$ whenever $A^s = B^s$. In fact, If $u_\varepsilon \to Ax$ is a sequence in $\mathcal{U}_\varepsilon(\Omega)$ such that

$$|\Omega| f(A) = \lim_{\varepsilon \to 0^+} F_\varepsilon(u_\varepsilon),$$

then we may set $v_\varepsilon = u_\varepsilon + (B - A)x$ and note that $v_\varepsilon \in \mathcal{U}_\varepsilon(\Omega)$, $v_\varepsilon \to Bx$ and $Ev_\varepsilon = Eu_\varepsilon$, so that

$$|\Omega| f(B) \leq \liminf_{\varepsilon\to 0^+} F_\varepsilon(v_\varepsilon) = \lim_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon) = |\Omega| f(A).$$

Hence, we have $f(B) \leq f(A)$ and by symmetry $f(B) = f(A)$.

It remains then to check that f is given by the homogenization formula (2.3). To this end, choose $U = (0,1)^n$, and a sequence $u_\varepsilon \to Ax$ in $\mathcal{U}_\varepsilon((0,1)^n)$ such that

$$f(A) = \lim_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon). \tag{3.2}$$

Fix $\delta > 0$ and define $\varphi(y) = \Big(\frac{1}{\delta}\mathrm{dist}(y, \partial Y)\Big) \wedge 1$ and $S_\delta = \{y \in Y : \mathrm{dist}(y, \partial Y)\Big) < \delta\}$. Set

$$v_\varepsilon = \varphi u_\varepsilon + (1 - \varphi) Ax,$$

and note that

$$(v_\varepsilon^+ - v_\varepsilon^-) \odot \nu_{v_\varepsilon} = \varphi(u_\varepsilon^+ - u_\varepsilon^-) \odot \nu_{u_\varepsilon}$$

on εB, so that $v_\varepsilon \in \mathcal{U}_\varepsilon((0,1)^n)$. We extend v_ε to a function defined on $\varepsilon[\frac{1}{\varepsilon} + 1]Y$ ($[t]$ is the integer part of t) by setting

$$w_\varepsilon(y) = \begin{cases} v_\varepsilon(y) & \text{if } y \in Y \\ Ay & \text{if } y \in \varepsilon[\frac{1}{\varepsilon} + 1]Y \setminus Y. \end{cases}$$

The function w_ε is then extended to all $\mathbb{R}^n$ by requiring that $w_\varepsilon(y) - Ay$ be $\varepsilon[\frac{1}{\varepsilon} + 1]Y$periodic. If we set

$$z_\varepsilon(y) = \frac{1}{\varepsilon} \sum_{k \in \{0,\dots,[\frac{1}{\varepsilon}]\}^n} w_\varepsilon(\varepsilon y + \varepsilon k),$$

then z_ε is Y-periodic and ad admissible test function for the computation of $f_{\mathrm{hom}}(A)$. By the periodicity of w_ε and Jensen's inequality we have

$$\begin{aligned} f_{\mathrm{hom}}(A) &\leq \int_Y \|\mathcal{E} z_\varepsilon\|^2\, dy \\ &= \frac{1}{\varepsilon^n[\frac{1}{\varepsilon} + 1]^n} \int_{\varepsilon[\frac{1}{\varepsilon}+1]Y} \|\mathcal{E} z_\varepsilon\|^2\, dy \leq \frac{1}{\varepsilon^n[\frac{1}{\varepsilon} + 1]^n} \int_{\varepsilon[\frac{1}{\varepsilon}+1]Y} \|\mathcal{E} w_\varepsilon\|^2\, dy \\ &\leq \int_Y \|\mathcal{E} v_\varepsilon\|^2 + \|A^s\| \Big(\varepsilon^n[\frac{1}{\varepsilon} + 1]^n - 1\Big). \end{aligned} \tag{3.3}$$

We then have to estimate this last term. Let $\eta > 0$; we have (for a suitable constant c_η)

$$\begin{aligned}\int_{(0,1)^n} \|\mathcal{E}v_\varepsilon\|^2\,dy &= \int_{(0,1)^n\setminus S_\delta} \|\mathcal{E}u_\varepsilon\|^2\,dy \\ &\quad + \int_{S_\delta} \|\varphi\mathcal{E}u_\varepsilon + (1-\varphi)A^s + D\varphi\odot(u_\varepsilon - Ay)\|^2\,dy \\ &\le (1+\eta)\int_{(0,1)^n} \|\mathcal{E}u_\varepsilon\|^2\,dy \\ &\quad + c_\eta |S_\delta| \|A^s\|^2 + c_\eta \frac{1}{\delta^2}\int_{(0,1)^n} |u_\varepsilon - Ay|^2\,dy.\end{aligned}$$

By letting first $\varepsilon \to 0^+$, $\delta \to 0^+$ and $\eta \to 0^+$ we get

$$\limsup_{\varepsilon\to 0^+} \int_{(0,1)^n} \|\mathcal{E}v_\varepsilon\|^2\,dy \le \limsup_{\varepsilon\to 0^+} \int_{(0,1)^n} \|\mathcal{E}u_\varepsilon\|^2\,dy, \tag{3.4}$$

so that, by (3.3), (3.4) and (3.2),

$$f_{\rm hom}(A) \le \liminf_{\varepsilon\to 0^+} \int_Y \|\mathcal{E}z_\varepsilon\|^2\,dy \le \lim_{\varepsilon\to 0^+} \int_Y \|\mathcal{E}u_\varepsilon\|^2\,dy \le f(A).$$

The converse inequality is obtained by estimating $f(A)$ using the liminf inequality of Γ-convergence upon choosing $u_\varepsilon \to Ay$ of the form $u_\varepsilon(x) = \varepsilon u(x/\varepsilon)$, where u is an admissible test function for (2.3).

Step 2. To prove the lower-bound inequality we use a convexity method through convolutions (see [14], [9] Section 14.3.2). We first remark that, setting $u^y(x) = u(x-y)$, we have that if the Γ-limit $F(u,U)$ exists then we have

$$F(u^y, y+U) = F(u,U).$$

Next, we choose a sequence U_k converging increasingly to Ω with $U_k \subset\subset \Omega$, and we suppose (upon subsequences) that

$$F(u,U) = \Gamma\text{-}\lim_j F_{\varepsilon_j}(u,U)$$

for all $U = U_k$ and for $U = \Omega$ (this is not restrictive upon enlarging the class $\mathcal{D}$ above). By Step 1 we have that

$$F(u,U) = \int_U f_{\rm hom}(\mathcal{E}u)\,dx$$

on such U. By hypothesis (2.5) we may also write

$$F(u,U) = \int_U f_{\rm hom}\Big(P_{K_{\rm hom}^\perp}\mathcal{E}u\Big)\,dx.$$

Let ρ_j be a sequence of mollifiers with supports in $B_{1/j}(0)$. We then have

$$\begin{aligned} F_\#(\rho_j * u, U_k) &= F(\rho_j * u, U_k) \\ &\leq \int_{B_{1/j}(0)} \rho_j(y) F(u^y, U_k)\, dy \\ &\leq \int_{B_{1/j}(0)} \rho_j(y) F(u, \Omega)\, dy = F(u, \Omega). \end{aligned}$$

By [4, 11] the functional

$$F_\#(u,U) = \begin{cases} \displaystyle\int_U f_{\rm hom}\Big(P_{K^\perp_{\rm hom}} \mathcal{E}u\Big)\, dx & \text{if } P_{K^\perp_{\rm hom}} \mathcal{E}_s u = 0 \\ +\infty & \text{otherwise} \end{cases}$$

is weakly lower semicontinuous on BD. We can then pass to the limit as $j \to +\infty$ to obtain

$$F_\#(u, U_k) \leq \liminf_j F_\#(\rho_j * u, U_k) \leq F(u, \Omega).$$

We may then take the supremum in k to get

$$F_\#(u, \Omega) \leq F(u, \Omega).$$

By the arbitrariness of the sequence (ε_j) we obtain

$$F_\#(u, \Omega) \leq \Gamma\text{-}\liminf_{\varepsilon \to 0^+} F_\varepsilon(u, \Omega).$$

Note in particular that $F(u, \Omega) = +\infty$ if $u \notin \mathcal{U}(\Omega)$.

Step 3. To prove the upper bound we use an approximation result by Anzellotti (see [4] Theorem 10.2) that states that for all $u \in \mathcal{U}(\Omega)$ there exists a sequence $u_k \in H^1(\Omega; \mathbb{R}^n)$ converging weakly* to u such that

$$P_{K^\perp_{\rm hom}} \mathcal{E}u_k \to P_{K^\perp_{\rm hom}} \mathcal{E}u$$

strongly in L^2, and hence in particular

$$F_\#(u, \Omega) = \liminf_k F_\#(u_k, \Omega).$$

By the lower semicontinuity of the Γ-lim sup we then obtain

$$\begin{aligned} \Gamma\text{-}\limsup_{\varepsilon \to 0^+} F_\varepsilon(u, \Omega) &\leq \liminf_k \Big(\Gamma\text{-}\limsup_{\varepsilon \to 0^+} F_\varepsilon(u_k, \Omega)\Big) \\ &= \lim_k F_\#(u_k, \Omega) = F_\#(u, \Omega) \end{aligned}$$

as desired. □

4. Perspectives

The results presented here lead to various additional questions. One is whether assumption (2.5) on the effective energy density $f_{\rm hom}$ can be altogether dropped, or some general assumptions on the set B and the cone K_0 can be found that ensure its validity. Another direction of investigation may be adding some energy on the discontinuity set B, and consider energies of the form

$$F_\varepsilon(u) = \int_\Omega \|\mathcal{E}u\|^2\,dx + \int_{\Omega\cap\varepsilon B} \varphi_\varepsilon(u^+ - u^-)\,d\mathcal{H}^{n-1}.$$

Referring to the sets K_1 and K_2 in the example the functions φ_ε may satisfy different growth conditions when $u^+ - u^-$ points in the direction of ν, mimicking a plastic or an elastic behaviour, and when it is orthogonal to ν (to mimic, e.g., friction). In this case the results of the present paper should be integrated with those in [3] (see also [9] Section 18). Furthermore, most of these problems can be rephrased in a nonlinearly elastic framework, where some of the analog of the results in [10, 11] are still to be proved. Part of these questions will be addressed in [8].

References

[1] L. Ambrosio, A. Coscia and G. Dal Maso. Fine properties of functions with bounded deformation. *Arch. Ration. Mech. Anal.* **139** (1997), 201–238.

[2] L. Ambrosio, N. Fusco and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford University Press, Oxford, 2000.

[3] N. Ansini, A. Braides and V. Chiadò Piat. Homogenization of periodic multi-dimensional structures. *Boll. Un. Mat. Ital.* **2-B** (1999), 735–758.

[4] G. Anzellotti. A class of convex non-coercive functionals and masonry-like materials. *Ann. Inst. H. Poincaré, Anal. Nonlin.* **2** (1985), 261–307

[5] G. Anzellotti and M. Giaquinta, Existence of the displacements field for an elasto-plastic body subject to Hencky's law and von Mises yield condition, *Manuscripta Math.* **32** (1980), 101–136.

[6] G. Anzellotti and M. Giaquinta, On the existence of the fields of stresses and displacements for an elasto-perfectly plastic body in static equilibrium, *J. Math. Pures Appl.* **61** (1982), 219–244.

[7] A. Braides, Γ-*convergence for Beginners*, Oxford University Press, Oxford, 2002.

[8] A. Braides and V. Chiadò Piat, Another brick in the wall, Part II, In preparation.

[9] A. Braides and A. Defranceschi, *Homogenization of Multiple Integrals*, Oxford University Press, Oxford, 1998.

[10] A. Braides, A. Defranceschi and E. Vitali. A relaxation approach to Hencky's plasticity. *Appl. Math. Optim.* **35** (1997), 45–68

[11] A. Braides, A. Defranceschi and E. Vitali. Relaxation of elastic energies with free discontinuities and constraints on the strain. *Ann Scuola Norm Sup. Pisa (V)* **1** (2002), 275–317.

[12] G. Buttazzo *Semicontinuity, Relaxation and Integral Representation in the Calculus of Variations*, Pitman Res. Notes Math. Ser. **207**, Longman, Harlow 1989.

[13] G. Buttazzo and G. Dal Maso, Integral representation and relaxation of local functionals, *Nonlinear Anal.* **9** (1985), pp. 515–532.

[14] G. Dal Maso, *An Introduction to* Γ*-convergence*, Birkhäuser, Boston, 1993.

[15] G. Del Piero, Limit analysis and no-tension materials *Int. J. Plasticity* **14** (1998), 259-271.

[16] M. Giaquinta and E. Giusti. Researches on the static of masonry structures *Arch. Ration. Mech. Anal.* **88** (1985), 359–392

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