NONLINEAR DIFFUSION EQUATIONS WITH DEGENERATE FAST-DECAY MOBILITY BY COORDINATE TRANSFORMATION

Nadia Ansini
Department of Mathematics, Sapienza University of Roma
P.le Aldo Moro 2, 00185 Rome, Italy

Simone Fagioli
DISIM - Department of Information Engineering, Computer Science and Mathematics, University of L'Aquila
Via Vetoio 1 (Coppito), 67100 L'Aquila (AQ), Italy

Abstract. We prove an existence and uniqueness result for solutions to nonlinear diffusion equations with degenerate mobility posed on a bounded interval for a certain density u. In case of *fast-decay* mobilities, namely mobilities functions under a Osgood integrability condition, a suitable coordinate transformation is introduced and a new nonlinear diffusion equation with linear mobility is obtained. We observe that the coordinate transformation induces a mass-preserving scaling on the density and the nonlinearity, described by the original nonlinear mobility, is included in the diffusive process. We show that the rescaled density ρ is the unique weak solution to the nonlinear diffusion equation with linear mobility. Moreover, the results obtained for the density ρ allow us to motivate the aforementioned change of variable and to state the results in terms of the original density u without prescribing any boundary conditions.

1. Introduction

Spreading behaviours appear in a large class of phenomena in biology such as animal swarming, chemiotaxis and bacterial movements, but also in modelling pedestrian movements and opinon formation, and it is often in competition with other effects, such as transport driven by external forces (local potentials) and/or aggregation or repulsion induced by the presence of non-local potentials. In order to handle the aforementioned dynamics mathematical models composed by nonlinear aggregation/diffusion/transport equations were introduced [5, 18, 22, 25, 26, 29] and deeply studied in recent years adopting different techniques and investigating possible modeling extensions (see e.g.[2, 4, 7, 11, 17, 19, 24] and references therein). The presence of a nonlinear mobility term in the equation may help to improve the ability of the models to catch more sophisticated phenomena. The general form of the equation we are considering is

$$\partial_t u = \operatorname{div}\left(G(x,u)\nabla\left(\Phi(u)+W(x)\right)\right), \tag{1}$$

where u is the density population, the function Φ models the spreading effects and, in general, it is a nonlinear function of the density, W is an external potential. Non-linear mobilities functions G, depending only on the density u and degenerating for a certain value $u_{max} > 0$, are used to prevent the overcrowding effect that may produce blow-up in finite time as in classical chemotaxis models (see [2, 4, 20, 33]). The presence of such mobility induces a more *realistic* behavior since aggregation stops once u_{max} is reached and the overcrowding phenomenon is prevented, see [8, 6, 29].

In this paper we deal with a mobility function of the form, $G(x,u) = g(x)^2 u$, that is linear in u and *non homogeneous* in x. Such mobility may model the possible presence of spatial heterogeneity in the domain of u. In the sequel we call mobility the function $g(x)$; *i.e.*, the x-dependent part of G. We reduce to the one-dimensional initial value problem for nonlinear convection-diffusion equation on bounded intervals with degenerate mobility by considering the following equation

$$\partial_t u = (g(x)^2 u(\varphi'(u) + W(x))_x)_x\,, \tag{2}$$

where $u = u(x,t)$ is defined on the domain $Q_\Omega := \{(x,t) \in \Omega \times [0,+\infty)\}$ with $\Omega = (-1,1)$. We assume that the mobility function $g : \Omega \to [0,+\infty)$ (or *inverse metric coefficient*) vanishes at the edges $x = \pm 1$. We can consider as reference example $g(x) = (1-x^2)^{p/2}$, $p > 0$. The function

$\varphi : [0, +\infty) \to \mathbb{R}$ represents a *free energy density*, resulting from local repulsive effects or volume filling mechanisms, and $W : \Omega \to \mathbb{R}$ is the *external potential.* Since g vanishing at the edges $x = \pm 1$, the problem of posing suitable zero-flux boundary conditions arises in order to have a (unique) solution u with constant mass. Roughly speaking, if we consider (2) as the continuum limit equation of a many particles system and we assume that g vanishes very fast at $x = \pm 1$ then the particles slow down so fast at the boundary that no boundary condition has to be prescribed in order to preserve the total mass of u. On the other hand, if g goes to zero very slowly at $x = \pm 1$, a zero-flux boundary condition could tackle the loss of mass.

The formulation of equation (2) as gradient flows, in the sense of [1], on a *modified* Wasserstein space was first proven in [23] for a class of mobility functions $G : \mathbb{R}^n \to \mathbb{R}^n$ statisfying a uniform ellipticity assumption,

$$\lambda|\zeta|^2 \le \langle G(x)\zeta, \zeta\rangle \le \Lambda|\zeta|^2$$

for all $x, \zeta \in \mathbb{R}^n$ and for some $\lambda, \Lambda > 0$, inducing a metric coefficient $M = G^{-1}$ that satisfy a similar condition, see also [9, 10, 32]. Unfortunately this result does not apply to our case. Therefore, a new mathematical approach is needed in order to prove existence and uniqueness of solutions to equation (2). Moreover, models with mobility degenerating at the boundary are of high interest also for applications (see e.g. the modeling of the opinion formation phenomena [34]).

Our approach consists in introducing a suitable coordinate transformation with the aim of getting a Fokker-Planck type equation in a new variable ρ defined on the whole space $\mathbb{R}$ and with homogeneous mobility. Indeed, we set $\alpha : \Omega \to \mathbb{R}$ as

$$\alpha(x) := \int_0^x \frac{1}{g(z)} dz. \tag{3}$$

By definition of g we have that α is a $C^1(\Omega)$, strictly increasing function. We assume that g satisfies also the Osgood condition

$$\int_0^1 \frac{1}{g(z)} dz = +\infty\,, \tag{4}$$

that is, the mobility has a *fast-decay* behaviour. The function α is a $1:1$ map from Ω onto $\mathbb{R}$. Our reference example $g(x) = (1 - x^2)^{p/2}$ is a fast-decay mobility provided $p \ge 2$. Setting the coordinate transformation

$$y = \alpha(x) \in \mathbb{R}, \quad \forall x \in \Omega,$$

and the mass preserving scaling as follows

$$u(x, t) = \alpha'(x)\rho(\alpha(x), t)\,, \tag{5}$$

we have that, by assumption (4), ρ is defined on

$$Q_{\mathbb{R}} = \{(y, t) \in \mathbb{R} \times [0, +\infty)\}.$$

Replacing the ansatz (5) into (2) we obtain

$$\partial_t \rho = (\rho(\varphi'(a(y)\rho) + V)_y)_y\,, \tag{6}$$

where

$$a(y) := \frac{1}{g(\alpha^{-1}(y))}\,, \qquad V(y) := W(\alpha^{-1}(y))\,.$$

Therefore, we may conclude, at least formally, that if u solves (2) then ρ solves (6) and vice versa.

There are two main advantages in studying problem (6) in place of (2). First of all, as already observed, the new equation is posed on the whole real line $\mathbb{R}$, and no boundary conditions should be prescribed. Moreover, the mobility in the continuity equation is linear and no longer depending on the space variable.

If g does not satisfy (4); *i.e.*, there exists $l > 0$ such that

$$\int_0^1 \frac{1}{g(z)} dz = l < +\infty\,, \tag{7}$$

then the map α is a bi-jection from $(-1, 1)$ into $(-l, l)$ as e.g. in case of $0 \le p < 2$ for $g(x) = (1 - x^2)^{p/2}$. Condition (7) corresponds then to the *slow-decay* behaviour of the mobility. We argue that the scaling (5) can be still applied, and a new density $\rho(y, t)$ still solves (6). However,

ρ is defined on the bounded spatial domain $(-l, l)$, and a zero-flux boundary condition must be prescribed in order to preserve its total mass. In a forthcoming paper we explore in details this argument.

Another interesting case that, in our opinion, deserves to be investigated is the Cauchy problem on $\mathbb{R}$ with unbounded mobilities given by the following equation

$$\partial_t u = (\beta(x)^2 u(\varphi'(b(x)u) + W)_x)_x \,. \tag{8}$$

Here, $\beta \in C^1(\mathbb{R}; (0, +\infty)$ is the inverse metric factor bounded from below.

In [23], the solution to the Cauchy problem as in (8) was tackled by considering a variant of the theory developed in [1] and the usual Wasserstein distance is replaced by a distance constructed in the same spirit as [3]; *i.e.*,

$$d_\beta(u_1, u_2) = \inf \left\{ \int_0^1 \int_{\mathbb{R}} \frac{1}{\beta(x)^2} u(x,s) w(x,s)^2 \, dx \, ds \,,\ u(x,0) = u_1 \,,\ u(x,1) = u_2 \,,\ u_s + (uw)_x = 0 \right\}.$$

The results in [23] are valid with b smooth, uniformly bounded and uniformly positive on $\mathbb{R}$, and they holds in arbitrary space dimension. We believe that our scaling approach, introduced in Section 2.1, can be adapted in order to reduce, also in this case, (8) to an equation with homogeneous mobility.

In this paper we deal with fast-decay mobility. The equation (6) has the structure of a gradient flow with respect to the Wasserstein metric with energy functional

$$\mathcal{F}^a[\rho] = \int_{\mathbb{R}} \frac{\varphi(a(y)\rho(y))}{a(y)} dy + \int_{\mathbb{R}} V(y)\rho(y) dy, \tag{9}$$

(see e.g. [1]) . We will recall the basic notions of Wasserstein gradient flow theory in Section 2.3. It is well known by the theory developed in [1, 27, 31, 35] that (6) has a unique solution in the space of probability measures with finite second moment provided the functional $\mathcal{F}^a$ above is *displacement λ-convex* (in addition to some further technical assumptions); *i.e.*, geodesically convex on the Wasserstein space up to a quadratic perturbation. Hence, following the approach as in [16], we will collect conditions on g, φ, and W such that the corresponding functional $\mathcal{F}^a$ obtained after the scaling (5) is geodesically λ-convex. Moreover, we state the existence and uniqueness result for (6) by using the minimizing movements method and the by-now classical JKO approach [21], and we reformulate the result for the density $u = u(x,t)$ via the scaling (5). In particular, we determine the class of initial conditions for u such that a unique solution for (2) exists without imposing any boundary condition.

The paper is organized as follows. In Section 2 we first derive (6) using the coordinate transformation and the scaling (5), then we list the assumptions and we collect some useful tools and results that we will apply to prove the main result stated in Theorem 2.1. Section 3 is devoted to prove existence and uniqueness for the rescaled density ρ (Section 3.1 and Section 3.2, respectively). In Section 4 we reformulate the result obtained for ρ in terms of the density function u. Finally, in Section 5 we focus on three relevant more specific cases obtained by introducing degenerate mobility in the classical Heat equation, linear Fokker-Planck equation and Porous Medium equation.

2. Preliminaries

In this section we collect general assumptions and properties on functions a, g, V and W that are involved in the definition of the equations (2) and (6). Moreover, we derive equation (6) and we recall the notion of Wasserstein gradient flow and the extension version of the Aubin-Lions Lemma.

We use the usual notations $h'(z)$ and $\partial_z h$ to denote the first derivative of a function h depending only on one variable and the first order partial derivative for h depending on two variables; respectively. To the aim to not overburden the notations, we will use also any of the following notations h_z , $[h]_z$, $(h)_z$ to denote the first derivative or first order partial derivative. We leave the interpretation up to the reader, it will be clear anyway from the context. Similarly, for the second derivative and for the second order partial derivative.

2.1. **Derivation of nonlinear convection-diffusion equation on $\mathbb{R}$ with homogeneous mobility.**

We want to derive equation (6) from equation (2) by applying the scaling (5). More precisely, we replace

$$u(x,t) = \alpha'(x)\rho(\alpha(x),t),$$

into (2) and we obtain

$$\alpha'\rho_t \circ \alpha = \left(g^2\alpha'\rho \circ \alpha[\varphi'(\alpha'\rho \circ \alpha) + W]_x\right)_x . \tag{10}$$

We define now the functions $a : \mathbb{R} \to \mathbb{R}_+$ and $V : \mathbb{R} \to \mathbb{R}$ as

$$a(y) := \frac{1}{g(\alpha^{-1}(y))}, \qquad V(y) := W(\alpha^{-1}(y)) . \tag{11}$$

Hence, by (3) we have that

$$a \circ \alpha(x) = \frac{1}{g(x)} = \alpha'(x) , \quad V \circ \alpha(x) = W(x) , \tag{12}$$

and

$$\alpha''(x) = (a' \circ \alpha)\alpha' , \quad W'(x) = \alpha'(x)V' \circ \alpha(x) . \tag{13}$$

Therefore, we have that

$$\begin{aligned} [\varphi'(\alpha'\rho \circ \alpha)]_x &= \varphi''(\alpha'\rho \circ \alpha)[\alpha''\rho \circ \alpha + (\alpha')^2\partial_y\rho \circ \alpha] \\ &= \alpha'\varphi''(a\rho \circ \alpha)[a'\rho \circ \alpha + a\partial_y\rho \circ \alpha] \\ &= \alpha'[\varphi'(a\rho)]_y \circ \alpha . \end{aligned} \tag{14}$$

By applying (12), (13), and (14) we have that the metric factor in (10) disappears and the equation (10) becomes

$$\begin{aligned} \alpha'\rho_t \circ \alpha &= \left(\rho \circ \alpha[\varphi'(a\rho) + V]_y \circ \alpha\right)_x \\ &= \alpha'\left(\rho[\varphi'(a\rho) + V]_y\right)_y \circ \alpha . \end{aligned} \tag{15}$$

Therefore, we get equation (6).

2.2. **Main assumptions and properties.**

We assume that the mobility function $g : \Omega \to [0,1]$ is a $C^2(\bar{\Omega})$ function satisfying the following conditions:

- (g1) $g(\pm 1) = 0$, g has a maximum point at $x = 0$ and $g(0) = 1$;
- (g2) the Osgood condition (4);
- (g3) there exists a constant $C_g > 0$ such that $0 \le (g')^2 - g\,g'' \le C_g$.

We collect in the following Proposition some useful properties of the function a defined in (11).

Proposition 2.1. *Let g be a function as above satisfying* (g1), (g2) *and* (g3)*. Let α and a be defined as in* (3) *and* (11)*, respectively. Then, $a : \mathbb{R} \mapsto [1,+\infty)$ is a convex function satisfying the following properties:*

- (i) *$a(y) \ge a(0) = 1$ for every $y \in \mathbb{R}$;*
- (ii) *there exists a constant C_a such that $|a'(y)/a(y)| \le C_a$ and $y\,a'(y)/a(y) \ge 0$;*
- (iii) *$a''(y)/a(y)$ is bounded for every $y \in \mathbb{R}$.*

In particular, if $g(x) = (1-x^2)^{p/2}$, with $p \ge 2$, then conditions (i) *and* (ii) *are still satisfied with $C_a = p$. Moreover, condition* (iii) *still holds for every $p \ge 2$ and $a''(y)$ is bounded for every $p \ge 4$.*

Proof. By (g1) and (12) we have that the function $a(y)$ has a global minimum at $y = 0$, that implies condition (i). By (3) and (13) we have that

$$a' \circ \alpha(x) = -\frac{g'(x)}{g(x)} , \quad \frac{a'}{a} \circ \alpha(x) = -g'(x) ; \tag{16}$$

hence, we have that $|a'(y)/a(y)|$ remains bounded. Moreover

$$a'' \circ \alpha(x) = \frac{(g'(x))^2 - g''(x)g(x)}{g(x)}, \quad \frac{a''}{a} \circ \alpha(x) = (g'(x))^2 - g''(x)g(x); \tag{17}$$

therefore by (g3) we get that the function a is convex and condition (iii) is satisfied. In particular, the convexity of a implies that $a(y) - a'(y)y \leq a(0)$; *i.e.*,

$$\frac{a'(y)}{a(y)} y \geq 0 .$$

In the particular case $g(x) = (1 - x^2)^{p/2}$, a direct computation shows

$$\frac{a'(y)}{a(y)} = p\,\alpha^{-1}(y)(1 - (\alpha^{-1}(y))^2)^{p/2-1},$$

and

$$a''(y) = p\left(1 + (\alpha^{-1}(y))^2\right)\left(1 - (\alpha^{-1}(y))^2\right)^{p/2-2};$$

hence, $|a'(y)/a(y)| \leq p$ for every $y \in \mathbb{R}$, and a'' is bounded for every $p \geq 4$. Moreover, since

$$\frac{a''(y)}{a(y)} = p\left(1 + (\alpha^{-1}(y))^2\right)\left(1 - (\alpha^{-1}(y))^2\right)^{p-2}$$

we have that the ratio a''/a remains bounded for all $p \geq 2$ and $y \in \mathbb{R}$. □

Let $\varphi : [0, +\infty) \to [0, +\infty)$ be a lower semi-continuous and convex function satisfying the following growth conditions:

(D) for $m > 1$ and $\mu \in [m, 3m)$ there exist two constants $c_m, C_\mu \geq 0$ such that

$$c_m s^{m-2} \leq \varphi''(s) \leq C_\mu s^{\mu-2},$$

for every $s \geq 0$.

Let $W : \Omega \to [0, +\infty)$ be a non-negative $C^2(\Omega)$ function. We further assume that

(gW1) there exists $\lambda \in \mathbb{R}$ such that

$$\lambda \leq g^2 W''(x) + g\,g' W' \text{ for all } x \in [-1, 1].$$

(gW2) there exists $L > 0$ such that

$$\left[g^2(x)W'(x)\right]_x \leq L, \text{ for all } x \in [-1, 1].$$

Note that

$$V'' \circ \alpha(x) = g^2(x)W''(x) + g(x)g'(x)W'(x) \tag{18}$$

$$= g^2(x)W''(x) + \frac{1}{2}[g^2(x)]_x W'(x). \tag{19}$$

Remark 2.1. *We observe that condition* (gW1) *naturally arises in the porous medium case (see Section 5.3). Indeed, condition* (gW1) *implies the λ-convexity of function V; while, condition* (gW2) *implies*

$$a(y)\left[\frac{V'(y)}{a(y)}\right]_y \leq L, \text{ for all } y \in \mathbb{R}.$$

We can now state the main result of the paper (see Section 4 for the proof of Theorem).

Theorem 2.1. *Let $g : \Omega \to [0, 1]$ be a $C^2(\bar{\Omega})$ function under assumptions* (g1)-(g3). *Let $\varphi : [0, +\infty) \to [0, +\infty)$ be a lower semi-continuous and convex function satisfying* (D) *and let $W : \Omega \to [0, +\infty)$ be a non-negative $C^2(\Omega)$ function under the assumption* (gW1)-(gW2). *Consider, for $m > 1$, the initial condition $u_0 \in L^1 \cap L^m(\Omega)$ and fix $T > 0$. Then there exists a Hölder-continuous curve $u : [0, T] \to L^m(\Omega)$ such that,*

(i) *$u \in L^\alpha([0, T] \times \Omega)$ for some $\alpha \in (1, 3m)$;*
(ii) *$g[u^{\frac{m}{2}}]_x \in L^2([0, +\infty) \times \Omega)$;*

(iii) *for almost every* $t \in [0,+\infty)$ *and for all* $\psi \in C_c^\infty(\Omega)$, *we have*

$$\frac{d}{dt}\int_\Omega \psi(x)u(x,t)dx = -\int_\Omega g^2(x)\left(\varphi'(u)+W(x)\right)_x \psi_x(x)u(x,t)dx. \tag{20}$$

Note that the result is obtained without prescribing any boundary condition on u.

2.3. **Preliminaries on Wasserstein gradient flows.** We recall some basic notions in Optimal Transport theory, see [1, 31, 35]. Let us denote with $\mathcal{P}(\mathbb{R})$ the space of all probability measures on $\mathbb{R}$ and with $\mathcal{P}_2(\mathbb{R})$ the set of all probability measures with finite second moment; *i.e.*,

$$\mathcal{P}_2(\mathbb{R}) = \{\rho \in \mathcal{P}(\mathbb{R}) : m_2(\rho) < +\infty\},$$

where

$$m_2(\rho) = \int_{\mathbb{R}^d} |x|^2 \, d\rho(x).$$

Consider now a measure $\rho \in \mathcal{P}(\mathbb{R})$ and a Borel map $T : \mathbb{R}^d \to \mathbb{R}^n$. We denote by $T_\# \rho$ the push-forward of ρ through T, defined by

$$\int_{\mathbb{R}^n} f(y)\, dT_\#\rho(y) = \int_{\mathbb{R}^d} f(T(x))\, d\rho(x) \qquad \text{for all } f \text{ Borel functions on } \mathbb{R}^n.$$

Let us recall the 2-Wasserstein distance between $\mu_1, \mu_2 \in \mathcal{P}_2(\mathbb{R})$ defined by

$$W_2^2(\mu_1,\mu_2) = \min_{\gamma\in\Gamma(\mu_1,\mu_2)} \left\{ \int_{\mathbb{R}^2} |x-y|^2 \, d\gamma(x,y) \right\}, \tag{21}$$

where $\Gamma(\mu_1,\mu_2)$ is the class of all transport plans between μ_1 and μ_2, that is the class of measures $\gamma \in \mathcal{P}_2(\mathbb{R})^2$ such that, denoting by π_i the projection operator on the i-th component of the product space, the marginality condition

$$(\pi_i)_\#\gamma = \mu_i \quad \text{for } i = 1,2,$$

is satisfied. Setting $\Gamma_0(\mu_1,\mu_2)$ as the class of optimal plans; *i.e.*, minimizers of (21), we can write the Wasserstein distance as

$$W_2^2(\mu_1,\mu_2) = \int_{\mathbb{R}^2} |x-y|^2 \, d\gamma(x,y), \qquad \gamma \in \Gamma_0(\mu_1,\mu_2).$$

For $I \subset \mathbb{R}$ we consider an absolutely continuous curve in W_2, $\rho : I \to \mathcal{P}_2(\mathbb{R})$, namely a curve such that there exists a function $g \in L^1_{loc}(I)$ such that

$$W_2(\rho(t),\rho(s)) \le \left|\int_s^t g(\tau)d\tau\right| \quad \text{for all } t,s \in I.$$

We introduce the concept of *k-flow*, which is linked to the λ-convexity along geodesics. See [13, 28] for further details.

Definition 2.1. *A semigroup* $G_\Psi : [0,+\infty] \times \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ *is a k-flow for a functional* $\Psi : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R} \cup \{+\infty\}$ *with respect to the Wasserstein distance* W_2 *if, for an arbitrary* $\rho \in \mathcal{P}_2(\mathbb{R})$, *the curve* $s \mapsto G_\Psi^s \rho$ *is absolutely continuous on* $[0,+\infty]$ *and satisfies the Evolution Variational Inequality (E.V.I.)*

$$\frac{1}{2}\frac{d^+}{d\sigma} W_2^2(G_\Psi^\sigma\rho,\tilde\rho)|_{\sigma=s} + \frac{k}{2} W_2^2(G_\Psi^s\rho,\tilde\rho) \le \Psi(\tilde\rho) - \Psi(G_\Psi^\sigma\rho), \tag{22}$$

for all $s > 0$ *and for any* $\tilde\rho \in \mathcal{P}_2(\mathbb{R})$, *such that* $\Psi(\tilde\rho) < \infty$.

Remark 2.2. *The symbol* $d^+/d\sigma$ *stands for the limit superior of the respective difference quotients and equals to the derivative if the latter exists.*

Theorem 2.2. *Asssume that a functional* $\Psi : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R} \cup \{+\infty\}$ *is* λ*-convex (along geodesics), with a modulus of convexity* $\lambda \in \mathbb{R}$, *that is, along every constant speed geodesic* $\rho : [0,1] \to \mathcal{P}_2(\mathbb{R})$

$$\Psi[\rho(t)] \le (1-t)\Psi[\rho(0)] + t\Psi[\rho(1)] - \frac{\lambda}{2} t(1-t) W_2^2(\rho(0),\rho(1))$$

holds for every $t \in [0,1]$. Then Ψ posses a uniquely determined k-flow, with some $k \geq \lambda$. Conversely, if a functional Ψ posses a k-flow, and if is monotonically non-increasing along that flow, then Ψ is λ-convex, with some $\lambda \geq k$.

We now recall an extension of the Aubin-Lions Lemma first introduced in [30]. This result will be used later to prove the existence of weak solutions to (6).

Theorem 2.3 (**Extended Aubin-Lions Lemma** [30])**.** *On a Banach space X, let be given*

- *a normal coercive integrand $\mathcal{Y} : X \to [0,\infty]$; i.e., $\mathcal{Y}$ is lower semi-continuous and its sub-levels are relatively compact in X;*
- *a pseudo-distance $\mathrm{d} : X \times X \to [0,\infty]$; i.e., d is lower semi-continuous and $\mathrm{d}(\rho,\eta) = 0$ for any $\rho, \eta \in X$ with $\mathcal{Y}(\rho), \mathcal{Y}(\eta) < \infty$ implies $\rho = \eta$.*

Let further U be a set of measurable functions $u : [0,T] \to X$, with a fixed $T > 0$. If

$$\sup_{u\in U} \int_0^T \mathcal{Y}[u(t))]dt < \infty \text{ and } \lim_{h\downarrow 0} \sup_{u\in U} \int_0^{T-h} \mathrm{d}\left(u(t+h), u(t)\right) dt = 0, \tag{23}$$

U contains an infinite sequence $\{u_n\}_{n\in\mathbb{N}}$ that converges in measure (with respect to $t \in [0,T]$) to a limit $u : [0,T] \to X$.

3. Existence and uniqueness of weak solutions to nonlinear convection-diffusion equation on $\mathbb{R}$ with homogeneous mobility

In this section we study existence and uniqueness of solutions to (6). In particular, we investigate on λ-convexity property for the related functional $\mathcal{F}^a$ introduced in (9). Let us recall that the equation (6) obtained in Section 2.1 for the scaled density ρ is

$$\rho_t = \left(\rho \left(\varphi'(a\rho) + V\right)_y\right)_y \text{ for } (t,y) \in [0,+\infty) \times \mathbb{R}. \tag{24}$$

For technical convenience we define the following functions

$$F^a(y,\eta) = \frac{1}{a(y)}\varphi(a(y)\eta)\,, \quad H(y,\eta) = \eta F^a(y,\frac{1}{\eta}), \tag{25}$$

and we reformulate equation (24) and the functional $\mathcal{F}^a$ as

$$\rho_t = (\rho(F^a_\eta(y,\rho) + V)_y)_y\,. \tag{26}$$

and

$$\mathcal{F}^a\left[\rho\right] = \int_{\mathbb{R}} F^a(a(y),\rho(y))dy + \int_{\mathbb{R}} V(y)\rho(y)\,dy\,, \tag{27}$$

respectively. At least formally, we may introduce the cumulative distribution function R of ρ, defined as

$$R(t,y) = \int_{-\infty}^{y} \rho(t,z)\,dz\,,$$

and its pseudo-inverse function

$$Y(t,\omega) = \inf\left\{y : R(t,y) > \omega\right\}, \tag{28}$$

for any $y \in \mathbb{R}$, $w \in (0,1)$ and $t \geq 0$, respectively. Note that

$$Y_\omega \rho \circ Y = 1\,. \tag{29}$$

The functions R and Y formally satisfy the equations

$$R_t = R_y\left(F^a_\eta(y,R_y) + V(y)\right)_y, \tag{30}$$

$$Y_t = -\frac{1}{Y_\omega}\Big(F^a_\eta\Big(Y,\frac{1}{Y_\omega}\Big) + V(Y)\Big)_\omega\,; \tag{31}$$

respectively. We will make use of this reformulation in Section 3.2.

Definition 3.1. *We say that a curve $\rho : [0,T] \to \mathcal{P}_2(\mathbb{R})$ is a weak solution to (26) if*

(i) *$\rho \in L^\alpha([0,T]\times\mathbb{R})$, with $\alpha \in (1,\mu)$ for all $T > 0$;*
(ii) *$[\rho^{m/2}]_y \in L^2([0,+\infty)\times\mathbb{R})$;*

(iii) *for almost every* $t \in [0,+\infty)$ *and for all* $\zeta \in C_c^\infty(\mathbb{R})$*, we have*

$$\frac{d}{dt}\int_{\mathbb{R}} \zeta(y)\rho(y,t)dy = -\int_{\mathbb{R}} \left(\varphi'(a\rho)+V(y)\right)_y \zeta_y(y)\rho(y,t)\,dy. \tag{32}$$

3.1. **The minimising movements or JKO scheme.** In this section we construct solutions to (24) by applying the so called *implicit-Euler* or *minimising movements* scheme (the last notion has been introduced by De Giorgi in [14] in the general setting of metric spaces). Here we follow the interpretation of the Fokker- Planck equation as Wasserstein gradient flow originally suggested by Jordan, Kinderlehrer, and Otto in [21].

Given $\rho \in \mathcal{P}_2(\mathbb{R})$, for a fixed time step $\tau > 0$ and for every $\eta \in \mathcal{P}_2(\mathbb{R})$ we introduce the penalization functional $\Phi_\tau\left(\rho;\eta\right)$ defined by

$$\Phi_\tau\left(\rho;\eta\right) = \frac{1}{2\tau}W_2^2\left(\rho,\eta\right) + \mathcal{F}^a\left[\rho\right]. \tag{33}$$

Let $\rho^0 \in \mathcal{P}_2(\mathbb{R})$ with $\mathcal{F}^a\left[\rho^0\right] < \infty$, the approximation scheme consists in constructing recursively the sequence of minimizers $\{\rho_\tau^n\}_{n\in\mathbb{N}}$ as

$$\rho_\tau^n = \text{argmin}_{\rho\in\mathcal{P}_2(\mathbb{R})}\Phi_\tau\left(\rho;\rho_\tau^{n-1}\right), \quad \rho_\tau^0 := \rho^0. \tag{34}$$

We define the piece-wise constant sequence as

$$\bar\rho_\tau(t) = \rho_\tau^n \qquad \text{for } t \in \left((n-1)\,\tau, n\tau\right], \tag{35}$$

for $n \geq 1$.

Lemma 3.1 (**Existence of minimzers).** *Under the assumptions* (g1) - (g3),(D),(gW1), (gW2), *we have that for any given* $\rho_\tau^{n-1} \in \mathcal{P}_2(\mathbb{R})$ *the functional* $\Phi_\tau\left(\rho;\rho_\tau^{n-1}\right)$ *admits a minimiser* $\rho_\tau^n \in \mathcal{P}_2(\mathbb{R})$*.*

Proof. The well-posedness of the scheme is an application of Direct Methods of Calculus of Variations. Indeed, since φ and V are non-negative then the functional $\Phi_\tau\left(\rho;\rho_\tau^{n-1}\right)$ satisfies the coercivity condition, that is, for any given $\eta \in \mathcal{P}_2(\mathbb{R})$ and for every constant c we have that

$$\inf_{\rho\in\mathcal{P}_2(\mathbb{R})}\left\{c\,W_2^2\left(\rho,\eta\right) + \mathcal{F}^a\left[\rho\right]\right\} > -\infty.$$

Hence, for any given $\rho_\tau^{n-1} \in \mathcal{P}_2(\mathbb{R})$ there exists a bounded minimising sequence in $\mathcal{P}_2(\mathbb{R})$ that satisfies the integral condition for tightness and therefore, it is tight in $P_2(\mathbb{R})$ (precompact with respect to the narrow convergence, see e.g. [1, Remark 5.1.5]). Moreover, by the superlinear growth condition at infinity of φ and Dunford- Pettis Theorem we have that the minimising sequence is precompact also with respect to the weak-L^1 convergence and the weak-L^1 limit $\rho_\tau^n \in K$. The lower semicontinuity of $\Phi_\tau\left(\rho;\rho_\tau^{n-1}\right)$ with respect to the L^1-weak convergence easy follows by [1]. Indeed, by [1, Lemma 5.1.7 and Lemma 7.1.4], we have that the functionals $\rho \to \int_{\mathbb{R}} \rho(y)V(y)\,dy$ and $\rho \to W_2^2\left(\rho,\rho_\tau^{n-1}\right)$ are lower semicontinuous with respect to the narrow convergence, respectively; therefore they are also L^1-weak lower semicontinuous. By classical results on the L^1- weak lower semicontinuity of integral functionals with positive, convex and lower semicontinuous integrand, we have that also $\rho \to \int_{\mathbb{R}} \varphi(a\rho)/a\,dy$ is lower semicontinuous with respect to the L^1- weak convergence, which concludes the proof. □

Lemma 3.2 (**Compactness and limit trajectory).** *The piecevise constant interpolating sequence* $\bar\rho_\tau$ *narrow converges up to (non-relabelled) sub-sequence to a Hölder continuous limit curve* $\rho : [0,\infty) \to \mathcal{P}_2(\mathbb{R})$*.*

Proof. Directly from the definition of the minimising sequence we get,

$$\frac{1}{2\tau}\sum_{n=1}^{N} W_2^2\left(\rho_\tau^{n-1},\rho_\tau^n\right) \leq \mathcal{F}^a\left[\rho^0\right] - \mathcal{F}^a\left[\rho_\tau^N\right], \tag{36}$$

which easily induces a *monotonicity* property for the functional along the sequence,

$$\mathcal{F}^a\left[\rho_\tau^n\right] \leq \mathcal{F}^a\left[\rho_\tau^0\right], \quad \forall n \geq 0.$$

Moreover, since $\mathcal{F}^a$ is non-negative we have that

$$\sum_{n=1}^{\infty} W_2^2\left(\rho_\tau^{n-1}, \rho_\tau^n\right) \leq 2\tau \mathcal{F}^a\left[\rho^0\right]. \tag{37}$$

Reasoning as in the proof of [1, Theorem 11.1.6, Steps 1-2] we get

$$W_2(\bar{\rho}_\tau(s), \bar{\rho}_\tau(t)) \leq \sqrt{2\mathcal{F}^a[\rho^0]}\max(\tau, |t-s|)^{\frac{1}{2}}, \quad s,t \geq 0. \tag{38}$$

By the refined version of Ascoli-Arzelà Theorem in [1, Proposition 3.3.1] we get the narrow convergence. □

We now show that the piece-wise constant interpolation sequence actually is strongly convergent in some L^p space, where the exponent will depend only on the growth condition (D) of φ.

Remark 3.1. *There exists a constant $C := C(\rho^0, \varphi, a, V)$, such that*

$$m_2\left[\bar{\rho}_\tau\right](T) := \int_{\mathbb{R}} |x|^2 \bar{\rho}_\tau(T, y)dy \leq C(1+T) \text{ for all } T \geq 0. \tag{39}$$

Indeed, given an optimal transport plan γ between ρ_τ^n and ρ_0 we have that

$$\begin{aligned} m_2\left[\rho_\tau^n\right] = \int_{\mathbb{R}^2} y^2 d\gamma(y,z) &\leq 2\int_{\mathbb{R}^2} z^2 d\gamma(y,z) + 2\int_{\mathbb{R}^2} |y-z|^2 d\gamma(y,z) \\ &= 2m_2\left[\rho_0\right] + W_2^2(\rho_\tau^n, \rho^0) \\ &\leq 2\, m_2\left[\rho_0\right] + \sum_{h=0}^{n} W_2^2(\rho_\tau^k, \rho^{k-1}) \\ &\leq 2\, m_2\left[\rho_0\right] + 2\tau\mathcal{F}^a[\rho^0]. \end{aligned}$$

We now prove a key tool, the so called *flow interchange lemma* (see [16] for further details).

Lemma 3.3 (Flow Interchange). *Let $\Psi : \mathcal{P}_2(\mathbb{R}) \to (-\infty, +\infty]$ be a lower semi-continuous functional which posses a k-flow G_Ψ. Define the dissipation of a functional $\mathcal{F}$ along G_Ψ by*

$$\mathcal{D}_\Psi \mathcal{F}^a(\rho) := \limsup_{s \downarrow 0} \frac{1}{s}\left(\mathcal{F}^a\left[\rho\right] - \mathcal{F}^a\left[G_\Psi^s \rho\right]\right),$$

for every $\rho \in \mathcal{P}_2(\mathbb{R})$. If ρ_τ^{n-1} and ρ_τ^n are two consecutive steps in the JKO scheme (34), then

$$\Psi\left[\rho_\tau^{n-1}\right] - \Psi\left[\rho_\tau^n\right] \geq \tau \mathcal{D}_\Psi \mathcal{F}^a(\rho_\tau^n) + \frac{k}{2} W_2^2(\rho_\tau^n, \rho_\tau^{n-1}). \tag{40}$$

In addition, assume that G_Ψ is such that for every $n \in \mathbb{N}$, the curve $s \mapsto G_\Psi^s \rho_\tau^n$ lies in $L^m(\mathbb{R})$, it is differentiable for $s > 0$ and continuous at $s = 0$. Let $\mathcal{R} : \mathcal{P}_2(\mathbb{R}) \to (-\infty, +\infty]$ be a functional satisfying

$$\liminf_{s \downarrow 0}\left(-\frac{d}{d\sigma}|_{\sigma=s} \mathcal{F}^a\left[G_\Psi^\sigma \rho_\tau^n\right]\right) \geq \mathcal{R}\left[\rho_\tau^n\right].$$

Then the following estimate holds: for every $n \in \mathbb{N}$,

$$\Psi\left[\rho_\tau^{n-1}\right] - \Psi\left[\rho_\tau^n\right] \geq \tau \mathcal{R}\left[\rho_\tau^n\right] + \frac{k}{2} W_2^2(\rho_\tau^n, \rho_\tau^{n-1}); \tag{41}$$

In particular, for every $N \in \mathbb{N}$,

$$\Psi\left[\rho_\tau^N\right] \leq \Psi\left[\rho^0\right] - \tau \sum_{n=1}^{N} \mathcal{R}\left[\rho_\tau^n\right] - \frac{k}{2}\sum_{n=1}^{N} W_2^2(\rho_\tau^n, \rho_\tau^{n-1}). \tag{42}$$

Proof. The proof of (40) easily follows as in [16, Lemma 4.2], after recalling Definition 2.1, the W_2- absolute continuity of the curve $s \mapsto G_\Psi^s \rho_\tau^n$ and the definition of $\rho_\tau^n, \rho_\tau^{n-1}$ as in (34). □

We now apply Lemma 3.3 with the entropy

$$H\left[\eta\right]=\int_{\mathbb{R}}\eta(y)\log\eta(y)\,dy,$$

as auxiliary functional in place of Ψ. It is well-known that H posses the *heat flow* as 0-flow, that is, $G^s_H\rho_0$ is a solution to the heat equation

$$\eta_s=\eta_{yy},\quad \eta(0,y)=\rho^0(y).$$

Lemma 3.4. *There exists a constant A depending only on ρ_0 such that the piece-wise interpolants $\bar\rho_\tau$ satisfy*

$$\|\bar\rho_\tau^{m/2}\|_{L^2(0,T;H^1(\mathbb{R}))}\le A(1+T), \tag{43}$$

for all $T>0$. In particular, $\bar\rho_\tau^{m/2}\in H^1(\mathbb{R})$ for every $t>0$.

Proof. The proof of (43) is an application of the Flow Interchange Lemma. Indeed, we first compute

$$\begin{aligned}\frac{d}{ds}\mathcal{F}^a\left[G^s_H\rho_0\right]&=\int_{\mathbb{R}}\left(\varphi'(a\eta)+V(y)\right)\eta_s\,dy\\&=\int_{\mathbb{R}}\varphi'(a\eta)\eta_{yy}\,dy+\int_{\mathbb{R}}V(y)\eta_{yy}\,dy\\&=-\int_{\mathbb{R}}\varphi''(a\eta)\partial_y(a\eta)\eta_y\,dy+\int_{\mathbb{R}}V''(y)\eta\,dy\,.\end{aligned}$$

By assumption (D),

$$\begin{aligned}\frac{d}{ds}\mathcal{F}^a\left[G^s_H\rho_0\right]&\le-c_m\int_{\mathbb{R}}(a\eta)^{m-2}(a'\eta+a\eta_y)\eta_y\,dy+\int_{\mathbb{R}}V''(y)\eta\,dy\\&=-c_m\int_{\mathbb{R}}\left(a^{m-2}a'\eta^{m-1}+a^{m-1}\eta^{m-2}\eta_y\right)\eta_y+\int_{\mathbb{R}}V''(y)\eta\,dy\\&=-\frac{c_m}{m}\int_{\mathbb{R}}a^{m-2}a'[\eta^m]_y\,dy-\frac{4c_m}{m^2}\int_{\mathbb{R}}a^{m-1}([\eta^{m/2}]_y)^2\,dy+\int_{\mathbb{R}}V''(y)\eta\,dy\\&=\frac{c_m}{m}\int_{\mathbb{R}}[a^{m-2}a']_y\eta^m\,dy-\frac{4c_m}{m^2}\int_{\mathbb{R}}a^{m-1}([\eta^{m/2}]_y)^2\,dy+\int_{\mathbb{R}}V''(y)\eta\,dy\end{aligned}$$

Note that by Proposition 2.1(ii)-(iii)

$$\partial_y\left(a^{m-2}a'\right)=a^{m-1}\left((m-2)\Big(\frac{a'}{a}\Big)^2+\frac{a''}{a}\right)\le K\,a^{m-1};$$

therefore,

$$\frac{d}{ds}\mathcal{F}^a\left[G^s_H\rho\right]\le c\int_{\mathbb{R}}a^{m-1}\eta^m\,dy-\frac{4c_m}{m^2}\int_{\mathbb{R}}([\eta^{\frac{m}{2}}]_y)^2\,dy+\int_{\mathbb{R}}V''\,\eta\,dy\,.$$

We define

$$\mathcal{R}[\eta]=-c\int_{\mathbb{R}}a^{m-1}\eta^m\,dy+\frac{4c_m}{m^2}\int_{\mathbb{R}}([\eta^{\frac{m}{2}}]_y)^2\,dy-\bar V$$

where $\bar V=\sup_{\mathbb{R}}V''<+\infty$. By applying Lemma 3.3 with $\Psi=H$, we have that

$$H\left[\rho^N_\tau\right]\le H\left[\rho^0\right]-\tau\sum_{n=1}^N\mathcal{R}\left[\rho^n_\tau\right]\,;$$

hence,

$$\tau\frac{4c_m}{m^2}\sum_{n=1}^N\int_{\mathbb{R}}([(\rho^n_\tau)^{\frac{m}{2}}]_y)^2\,dy\le H\left[\rho^0\right]-H\left[\rho^N_\tau\right]+c\tau\sum_{n=1}^N\int_{\mathbb{R}}a^{m-1}(\rho^n_\tau)^m\,dy+\bar VN\tau.$$

In particular,

$$\begin{aligned}\tau\frac{4c_m}{m^2}\sum_{n=1}^{N}\|(\rho_\tau^n)^{\frac{m}{2}}\|_{H^1}^2 &\leq \tau\frac{4c_m}{m^2}\sum_{n=1}^{N}\int_{\mathbb{R}}([(\rho_\tau^n)^{\frac{m}{2}}]_y)^2\,dy+\tau\frac{4c_m}{m^2}\sum_{n=1}^{N}\int_{\mathbb{R}}a^{m-1}(\rho_\tau^n)^m\,dy\\ &\leq H\left[\rho^0\right]-H\left[\rho_\tau^N\right]+(c+\frac{4c_m}{m^2})\tau\sum_{n=1}^{N}\int_{\mathbb{R}}a^{m-1}(\rho_\tau^n)^m\,dy+\bar{V}N\tau\,. \quad (44)\end{aligned}$$

Recalling the standard inequalities

$$-\frac{2}{e}s^{1/2}\leq s\log s\leq\frac{1}{(m-1)e}s^m$$

for all $s>0$, we have that

$$H(\eta)=\int_{\mathbb{R}}\eta\log\eta\,dy\leq\frac{1}{(m-1)e}\int_{\mathbb{R}}\eta^m\,dy\leq\frac{1}{(m-1)e}\int_{\mathbb{R}}a^{m-1}\eta^m\,dy\leq c\,\mathcal{F}^a(\eta)\,,$$

and, on the other hand,

$$H(\eta)\geq-\frac{2\sqrt{\pi}}{e}\Big(1+\int_{\mathbb{R}}x^2\eta\,dy\Big)^{1/2},$$

(see e.g. [16, Lemma 4.6]). By (44) we have that

$$\begin{aligned}&\tau\frac{4c_m}{m^2}\sum_{n=1}^{N}\|(\rho_\tau^n)^{\frac{m}{2}}\|_{H^1}^2\\ &\leq\; c\,\mathcal{F}^a(\rho^0)+\bar{c}\Big(1+\int_{\mathbb{R}}x^2\bar{\rho}_\tau(y,N\tau)\,dy\Big)^{1/2}+\tilde{c}N\tau\mathcal{F}^a(\rho_0)+\bar{V}N\tau\,. \quad (45)\end{aligned}$$

By (39) we get the thesis. □

Proposition 3.1. *The converging sub-sequence $\bar{\rho}_\tau$ in Lemma 3.2 converges to a limit function ρ in $L^\mu([0,T]\times\mathbb{R})$ for every $T>0$, with $\mu<3m$.*

Proof. We first prove the convergence in $L^m([0,T]\times\mathbb{R})$. The proof is a standard application of Theorem 2.3 and we sketch here for completeness, see also [16, Proposition 4.8] . The strategy is to check that the hypotheses of Theorem 2.3 are satisfied with $X=L^m(\mathbb{R})$ and

$$\mathcal{Y}[\rho]=\begin{cases}\int_{\mathbb{R}}\left([\rho^{\frac{m}{2}}]_y\right)^2\,dy+m_2\left[\rho\right], & \rho\in\mathcal{P}_2(\mathbb{R}),\ [\rho^{\frac{m}{2}}]_y\in L^2(\mathbb{R}),\\ +\infty & \text{otherwise,}\end{cases}$$

and

$$\mathrm{d}(\rho,\eta)=\begin{cases}W_2^2(\rho,\eta) & \rho,\eta\in\mathcal{P}_2(\mathbb{R}),\\ +\infty & \text{otherwise.}\end{cases}$$

By Frechet-Kolmogorov Theorem (see e.g. [15, Theorem IV.8.20]), it can be shown that the sub-levels of $\mathcal{Y}$, $\mathcal{Y}_c=\{\rho\in L^m(\mathbb{R})|\mathcal{Y}[\rho]\leq c\}$ for $c>0$, are relatively compact in $L^m(\mathbb{R})$. The estimates (39) and (43) imply the first condition in (23), that is

$$\sup_{u\in U}\int_0^T\mathcal{Y}[u(t))]dt<\infty$$

where $U=\{\bar{\rho}_{\tau_k}|k\in\mathbb{N}\}$. The second condition in (23) is a direct consequence of the Hölder continuity (38). The hypotheses of Theorem 2.3 are then satisfied and we can extract a sub-sequence $\bar{\rho}_{\tau_k'}$ converging in measure with respect to $t\in[0,T]$ to some limit ρ^* in $L^m(\mathbb{R})$. By Lemma 3.2 ρ^* coincides with the narrow limit ρ for every $t\in[0,T]$ and so the entire sequence $\bar{\rho}_{\tau_k}$ converges in measure to ρ. By (43) and the dominated convergence theorem we can conclude the strong convergence of $\bar{\rho}_\tau$ to ρ in $L^m(0,T;L^m(\mathbb{R}))$.

Notice that, for every $T>0$

$$\int_0^T\|\bar{\rho}_\tau(t,\cdot)-\rho(t,\cdot)\|_{L^m(\mathbb{R})}^\sigma dt\to0,$$

as $\tau \to 0$, for every $\sigma > 0$. By Gagliardo-Nirenberg inequality we get

$$\begin{aligned}\int_0^T \left\| \bar{\rho}_\tau^{\frac{m}{2}} - \rho^{\frac{m}{2}} \right\|_{L^p}^p dt &\leq C \int_0^T \left\| \left[\bar{\rho}_\tau^{\frac{m}{2}} - \rho^{\frac{m}{2}} \right]_y \right\|_{L^2}^{p\theta} \left\| \bar{\rho}_\tau^{\frac{m}{2}} - \rho^{\frac{m}{2}} \right\|_{L^2}^{p(1-\theta)} dt \\ &\leq C \left(\int_0^T \left\| \left[\bar{\rho}_\tau^{\frac{m}{2}} - \rho^{\frac{m}{2}} \right]_y \right\|_{L^2}^2 dt \right)^{\frac{p\theta}{2}} \left(\int_0^T \left\| \bar{\rho}_\tau^{\frac{m}{2}} - \rho^{\frac{m}{2}} \right\|_{L^2}^{\gamma} dt \right)^{\frac{m-\mu\theta}{m}},\end{aligned}$$

with $p = 2\mu/m$, $\theta = (\mu - m)/2\mu$ and $\mu > m$. The exponent γ is given by

$$\gamma = \frac{(1-\theta)2\mu}{m - \mu\theta},$$

and it is a positive exponent provided $\mu < 3m$. □

Proposition 3.2 (Existence of weak solutions). *The approximating sequence $\bar{\rho}_\tau$ converges to a weak solution ρ to* (24) *in the sense of Definition 3.1.*

Proof. In order to not overburden the notations we denote ρ_0 and ρ two consecutive minimisers as defined in (34). For $\epsilon > 0$ and $\zeta \in C_c^\infty(\mathbb{R})$, define

$$P^\epsilon(y) = y + \epsilon\zeta_y(y), \qquad \rho^\epsilon = P^\epsilon_{\#}\rho.$$

The minimality of ρ gives

$$0 \leq \frac{1}{2\tau}\left(W_2^2(\rho^\epsilon, \rho_0) - W_2^2(\rho, \rho_0)\right) + \mathcal{F}^a\left[\rho^\epsilon\right] - \mathcal{F}^a\left[\rho\right].$$

Let T be the optimal map pushing ρ_0 to ρ, then by definition

$$\begin{aligned} W_2^2(\rho, \rho_0) &= \int_{\mathbb{R}} |y - T(y)|^2 \rho_0(y) dy, \\ W_2^2(\rho^\epsilon, \rho_0) &\leq \int_{\mathbb{R}} |y - P^\epsilon\left(T(y)\right)|^2 \rho_0(y) dy. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{1}{2\tau}\left(W_2^2(\rho^\epsilon, \rho_0) - W_2^2(\rho, \rho_0)\right) &\leq \frac{1}{2\tau}\int_{\mathbb{R}} \left(|y - P^\epsilon\left(T(y)\right)|^2 - |y - T(y)|^2\right)\rho_0(y) dy \\ &= \frac{1}{2\tau}\int_{\mathbb{R}} \left(|y - \left(T(y) + \epsilon\zeta_y(T(y))\right)|^2 - |y - T(y)|^2\right)\rho_0(y) dy \\ &= -\frac{\epsilon}{\tau}\int_{\mathbb{R}} \left(y - T(y)\right)\zeta_y(T(y))\rho_0(y) dy + o(\epsilon) := I_1. \end{aligned} \tag{46}$$

The term involving the functional can be reformulated as follows

$$\begin{aligned} \mathcal{F}^a\left[\rho^\epsilon\right] - \mathcal{F}^a\left[\rho\right] &= \int_{\mathbb{R}} \left(\frac{\varphi(a(y)\rho^\epsilon)}{a(y)} + V(y)\rho^\epsilon - \frac{\varphi(a(y)\rho)}{a(y)} - V(y)\rho \right) dy \\ &= I_2 + I_3, \end{aligned} \tag{47}$$

where I_2 and I_3 are defined by

$$I_2 = \int_{\mathbb{R}} \left(\frac{\varphi(a(y)\rho^\epsilon)}{a(y)} - \frac{\varphi(a(y)\rho)}{a(y)} \right) dy = \int_{\mathbb{R}} \left(\varphi\left(\frac{a(P^\epsilon(y))\rho}{1 + \epsilon\zeta_{yy}(y)} \right) \frac{1 + \epsilon\zeta_{yy}(y)}{a(P^\epsilon(y))} - \frac{\varphi(a(y)\rho)}{a(y)} \right) dy, \tag{48}$$

and

$$I_3 = \int_{\mathbb{R}} \left(V(P^\epsilon(y)) - V(y)\right)\rho(y) dy\,; \tag{49}$$

respectively. In order to handle with the term I_2 we introduce the following function

$$B(\chi, \eta) = \frac{1}{\chi}\varphi(\chi\eta).$$

The first-order Taylor series approximation of B about the point (χ, η) with perturbation $(\chi^\epsilon, \eta^\epsilon)$ is given by

$$B(\chi^\epsilon, \eta^\epsilon) = B(\chi, \eta) + \left(\frac{\eta}{\chi}\varphi'(\chi\eta) - \frac{1}{\chi^2}\varphi(\chi\eta)\right)(\chi^\epsilon - \chi) + \varphi'(\chi\eta)(\eta^\epsilon - \eta) + R_\epsilon(\chi, \eta)\,,$$

where $R_\epsilon(\chi, \eta)$ is the remainder term. We choose $(\chi, \eta) = (a(y), \rho)$ and $(\chi^\epsilon, \eta^\epsilon) = (a(P^\epsilon(y)), \frac{\rho}{1+\epsilon\zeta_{yy}})$, then I_2 becomes

$$\begin{aligned}&\int_{\mathbb{R}} \left[\frac{\varphi(a\rho)}{a} + \left(\frac{\rho}{a}\varphi'(a\rho) - \frac{\varphi(a\rho)}{a^2}\right)(a \circ P^\epsilon - a) + \varphi'(a\rho)\left(\frac{\epsilon\zeta_{yy}}{1+\epsilon\zeta_{yy}}\right)\rho + R_\epsilon\right](1+\epsilon\zeta_{yy}) - \frac{\varphi(a\rho)}{a}dy \\ &= \epsilon\int_{\mathbb{R}} \frac{\varphi(a\rho)}{a}\zeta_{yy} + \left(\frac{\rho}{a}\varphi'(a\rho) - \frac{\varphi(a\rho)}{a^2}\right)\frac{a\circ P^\epsilon - a}{\epsilon}(1+\epsilon\zeta_{yy}) + \rho\varphi'(a\rho)\zeta_{yy}dy + \int_{\mathbb{R}} R_\epsilon\,(1+\epsilon\zeta_{yy})\,dy.\end{aligned}$$

By dominated convergence theorem we can prove that the last term involving R_ϵ is $o(\epsilon)$. Indeed,

$$\frac{1}{\epsilon}\int_{\mathbb{R}} R_\epsilon\,(1+\epsilon\zeta_{yy})\,dy = \frac{1}{\epsilon}\int_{\mathbb{R}} \left(R^1_\epsilon + R^2_\epsilon + R^3_\epsilon\right)(1+\epsilon\zeta_{yy})\,dy, \tag{50}$$

where

$$\begin{aligned} R^1_\epsilon &= \frac{1}{2}\left(\frac{\tilde\rho^2}{\tilde a}\varphi''(\tilde a\tilde\rho) - 2\frac{\tilde\rho\varphi'(\tilde a\tilde\rho)}{\tilde a^2} + 2\frac{\varphi(\tilde a\tilde\rho)}{\tilde a^3}\right)(a\circ P^\epsilon - a)^2\,, \\ R^2_\epsilon &= \frac{1}{2}\tilde a\varphi''(\tilde a\tilde\rho)\left(\frac{\epsilon\zeta_{yy}}{1+\epsilon\zeta_{yy}}\right)^2, \\ R^3_\epsilon &= \tilde\rho\varphi''(\tilde a\tilde\rho)\left(\frac{\epsilon\zeta_{yy}}{1+\epsilon\zeta_{yy}}\right)(a\circ P^\epsilon - a)\,, \end{aligned}$$

for some $\tilde a$ between a and $a\circ P^\epsilon$ and $\tilde\rho$ between ρ and $\rho/(1+\epsilon\zeta_{yy})$. Thanks to the growth condtions (D) it is easy to see that the remainder goes to zero in view of the L^μ estimate of ρ^n_τ.

We now sum up all contributions in (46), (47), (48), (49), we divide by ϵ, and we let go ϵ to 0; hence, we have

$$\begin{aligned}&\frac{1}{\tau}\int_{\mathbb{R}} (y - T(y))\,\zeta_y(T(y))\rho_0(y)dy \\ &= \int_{\mathbb{R}} \frac{\varphi(a\rho)}{a}\zeta_{yy} + \left(\varphi'(a\rho)\frac{\rho}{a}(a'\zeta_y - a\zeta_{yy}) - \varphi(a\rho)\frac{a'}{a^2}\zeta_y\right)dy + \int_{\mathbb{R}} V'(y)\rho(y)\zeta_y(y)dy\,.\end{aligned}$$

By the Taylor series approximation of ζ about T we get that

$$\frac{1}{\tau}\int_{\mathbb{R}} (y - T(y))\,\zeta_y(T(y))\rho_0(y)dy = \frac{1}{\tau}\int_{\mathbb{R}} \zeta(y)\left[\rho_0(y) - \rho(y)\right]dy + O(\tau)\,.$$

We recall now that ρ_0 and ρ are two consecutive minimisers as in (34), so that by replacing ρ_0 with ρ^n_τ and ρ with ρ^{n+1}_τ, into the two previous formulas, we get

$$\begin{aligned}&\int_{\mathbb{R}} \zeta\left[\rho^n_\tau - \rho^{n+1}_\tau\right]dy + O(\tau) \\ &= \tau\int_{\mathbb{R}} V'\rho^{n+1}_\tau\zeta_y + \frac{\varphi(a\rho^{n+1}_\tau)}{a}\zeta_{yy} + \left(\varphi'(a\rho^{n+1}_\tau)\frac{\rho^{n+1}_\tau}{a}(a'\zeta_y - a\zeta_{yy}) - \varphi(a\rho^{n+1}_\tau)\frac{a'}{a^2}\zeta_y\right)dy. \end{aligned}\tag{51}$$

Let $0 \le t < s$ be fixed, with

$$h = \left[\frac{t}{\tau}\right] + 1 \text{ and } k = \left[\frac{s}{\tau}\right].$$

Summing (51) from h to k we get,

$$\begin{aligned}&\int_{\mathbb{R}} \zeta\left[\rho^h_\tau - \rho^{k+1}_\tau\right]dy + O(\tau) \\ &= \tau\sum_{n=h}^{k}\int_{\mathbb{R}} V'\rho^{n+1}_\tau\zeta_y + \frac{\varphi(a\rho^{n+1}_\tau)}{a}\zeta_{yy} + \left(\varphi'(a\rho^{n+1}_\tau)\frac{\rho^{n+1}_\tau}{a}(a'\zeta_y - a\zeta_{yy}) - \varphi(a\rho^{n+1}_\tau)\frac{a'}{a^2}\zeta_y\right)dy\,.\end{aligned}$$

The formula can be rewritten also in terms of the piecewise constant interpolation $\bar{\rho}_\tau$ introduced in (35); *i.e.*,

$$\begin{aligned}&\int_{\mathbb{R}} \zeta \left[\bar{\rho}_\tau(t) - \bar{\rho}_\tau(s)\right] dy + O(\tau)\\ &= \int_t^s \int_{\mathbb{R}} V'\bar{\rho}_\tau(\sigma)\zeta_y + \frac{\varphi(a\bar{\rho}_\tau(\sigma))}{a}\zeta_{yy} + \left(\varphi'(a\bar{\rho}_\tau(\sigma))\frac{\bar{\rho}_\tau(\sigma)}{a}\left(a'\zeta_y - a\zeta_{yy}\right) - \varphi(a\bar{\rho}_\tau(\sigma))\frac{a'}{a^2}\zeta_y\right) dy\, d\sigma\,.\end{aligned}$$

By Lemma 3.2, Proposition 3.1, and growth condition (D), letting $\tau \to 0$ we obtain

$$\begin{aligned}&\int_{\mathbb{R}} \zeta \left[\rho(t) - \rho(s)\right] dy\\ &= \int_t^s \int_{\mathbb{R}} V'\rho(\sigma)\zeta_y + \frac{\varphi(a\rho(\sigma))}{a}\zeta_{yy} + \left(\varphi'(a\rho(\sigma))\frac{\rho(\sigma)}{a}\left(a'\zeta_y - a\zeta_{yy}\right) - \varphi(a\rho(\sigma))\frac{a'}{a^2}\zeta_y\right) dy\, d\sigma.\end{aligned}$$

We now Integrate by parts the second and in the forth term on the right-hand side of the previous formula and we get

$$\int_{\mathbb{R}} \zeta \left[\rho(t) - \rho(s)\right] = \int_t^s \int_{\mathbb{R}} V'\rho(\sigma)\zeta_y + \rho(\sigma)(\varphi'(a\rho(\sigma)))_y\zeta_y dy\, d\sigma\,.$$

It remains to divide by $s - t$ and pass to the limit as $s \to t$, to recover Definition 3.1 of weak solutions. ☐

3.2. **λ-convexity and k-flow.** We want to study the convexity of the functional $\mathcal{F}^a$ under the assumptions in Section 2.2. In Section 5 we will show some explicit examples as the heat equation, linear Fokker-Planck equation, and Porous medium equation with degenerate mobility.

Lemma 3.5. *Let F^a and H be defined as in* (25)*, and let us assume that the matrix*

$$\mathcal{H}(y,\eta) = \begin{pmatrix} H_{yy}(y,\eta) + V''(y) - k & H_{\eta y}(y,\eta) \\ H_{\eta y}(y,\eta) & H_{\eta\eta}(y,\eta) \end{pmatrix},$$

is positive semi-definite in $\mathbb{R} \times \mathbb{R}_+$; that is,

$$H(y,\eta) + V(y) - \frac{k}{2}y^2,$$

is jointly convex on $\mathbb{R} \times \mathbb{R}_+$. Then the solution to (26) *is a k-flow for the functional $\mathcal{F}^a[\rho]$.*

Proof. We adapt the regularisation procedure used in [16] to our case. We first troncate the function F^a, introduced in (25), as follows

$$\tilde{F}^N(y,\eta) = \begin{cases} F^a(y,\eta) & \text{if } |y| \le N, \\ F^a(N,\eta) & \text{if } y > N, \\ F^a(-N,\eta) & \text{if } y < -N, \end{cases} \tag{52}$$

and we denote F^N as the C^∞ mollification of $\tilde{F}^N(y,\eta)$ such that $F^N_y = 0$ for $|y| \ge N + 1/2$ and

$$\eta F^N_{\eta\eta} \ge c > 0. \tag{53}$$

According to (25) we can define H^N and the functional $\mathcal{F}^a_N$, by replacing F^a with F^N. The following initial-boundary value problem

$$\begin{cases} \partial_t\rho_N = (\rho_N(\left(F^N_{y\eta}(y,\rho_N) + [\rho_N]_y F^N_{\eta\eta}(y,\rho_N)\right) + V'))_y \\ \partial_y\rho_N(t,N) = \partial_y\rho_N(t,-N) = 0 \\ \rho_N(0,y) = \rho_{N,0}\,. \end{cases} \tag{54}$$

is then uniformly parabolic, thanks to (53). We consider an initial datum $\rho_{N,0}$ such that the following inequality

$$\int_{\mathbb{R}} \frac{1}{a(y)}\varphi(a(y)\rho_{N,0}(y))\, dy \le \int_{\mathbb{R}} \frac{1}{a(y)}\varphi(a(y)\rho_0(y))\, dy\,, \tag{55}$$

is satisfied. Moreover, the solution ρ_N is supported in $[-N,N]$ and strictly positive. Hence, we define the corresponding cumulative distribution function R^N and its pseudo-inverse Y^N that obeys to

$$Y_t^N = (H_\eta^N(Y^N,Y_z^N))_z - H_Y^N(Y^N,Y_z^N) - \frac{1}{Y_z^N}(V(Y^N))_z\,.$$

Indeed, the first term in the (31) right-hand side can be rewritten in term of H^N as follows

$$\begin{aligned}
&-\frac{1}{Y_z^N}(F_\eta^N(Y^N,\frac{1}{\partial_z Y^N}))_z = -(F_\eta^N(y,\rho_N))_y\\
&=\frac{1}{\rho_N}(H_\eta^N(y,\frac{1}{\rho_N}))_y - H_y^N(y,\frac{1}{\rho_N})\\
&=Y_z^N(H_\eta^N(Y^N,Y_z^N))_y - H_y^N(Y^N,Y_z^N).
\end{aligned}$$

We now prove that the solution to (54) is a k-flow, showing that the E.V.I. (22) is satisfied. By the change of variable $y=Y^N(t,z)$ we get

$$\begin{aligned}
\mathcal{F}_N^a[\rho_N] &= \int_{-N}^N F^N(y,\rho_N)\,dx + \int_{-N}^N V(y)\rho_N(y)\,dy\\
&= \int_0^1 Y_z^N(t,z)F^N\left(Y^N(t,z),\frac{1}{Y_z^N(t,z)}\right)dz + \int_0^1 V(Y^N(t,z))\,dz\\
&= \int_0^1 H^N\left(Y^N(t,z),Y_z^N(t,z)\right)dz + \int_0^1 V(Y^N(t,z))\,dz.
\end{aligned}$$

Since the Wasserstein distance can be rephrased in terms of pseudo-inverse as

$$W_2^2(\rho_1,\rho_2) = \int_0^1 (Y_1-Y_2)^2\,dz,$$

for any ρ_1 and ρ_2 in $\mathcal{P}_2(\mathbb{R})$; then, for a fixed $\tilde\rho_N$ we have

$$\begin{aligned}
&\frac{1}{2}\frac{d^+}{dt}W_2^2(\rho_N(t),\tilde\rho_N) + \frac{k}{2}W_2^2(\rho_N(t),\tilde\rho_N)\\
&=\frac{1}{2}\frac{d^+}{dt}\int_0^1 (Y^N-\tilde Y^N)^2\,dz + \frac{k}{2}\int_0^1 (Y^N-\tilde Y^N)^2\,dz\\
&=\int_0^1 Y_t^N(Y^N-\tilde Y^N)\,dz + \frac{k}{2}\int_0^1 (Y^N-\tilde Y^N)^2\,dz\\
&=\int_0^1 [H_\eta^N(Y^N,Y_z^N]_z - H_Y^N(Y^N,Y_z^N) - \frac{1}{Y_z^N}[V(Y^N)]_z(Y^N-\tilde Y^N)\,dz\\
&\quad+\frac{k}{2}\int_0^1 (Y^N-\tilde Y^N)^2\,dz\,.
\end{aligned}$$

We now integrate by parts, by convexity we get

$$\begin{aligned}
&\frac{1}{2}\frac{d^+}{dt}W_2^2(\rho_N(t),\tilde\rho_N) + \frac{k}{2}W_2^2(\rho_N(t),\tilde\rho_N)\\
&=\int_0^1 H_\eta^N(Y^N,Y_z^N)(\tilde Y_z^N - Y_z^N)\,dz + \int_0^1 H_Y^N(Y^N,Y_z^N)(\tilde Y^N - Y^N)\,dz\\
&\quad+\int_0^1 V'(Y^N)(\tilde Y^N - Y^N)\,dz + \frac{k}{2}\int_0^1 (Y^N-\tilde Y^N)^2\,dz\\
&\le \mathcal{F}_N^a\left[\tilde\rho^N\right] - \mathcal{F}_N^a\left[\rho_N\right]\,.
\end{aligned}$$

In order to conclude the proof we need to pass to the limit as $N\to\infty$ in the inequality after proving that the sequence ρ_N converges to a certain limit function that is a solution to (24). To this end we first calculate the following derivative

$$\begin{aligned}\frac{1}{m}\frac{d}{dt}\int_{-N}^{N} a^{m-1}(y)\rho_N^m(y,t)\,dy &= -\Big(\frac{m-1}{m}\Big)\int_{-N}^{N}\left[(a\rho_N)^m\right]_y \frac{1}{a}\left(F^N_{y\eta}(y,\rho_N)+[\rho_N]_y F^N_{\eta\eta}(y,\rho_N)\right)\,dy\\ &\quad +\frac{m-1}{m}\int_{-N}^{N}(a\rho_N)^m\Big[\frac{V'}{a}\Big]_y\,dy\\ &= -\Big(\frac{m-1}{m}\Big)\int_{-N}^{N}\left[(a\rho_N)^m\right]_y \frac{[a\rho_N]_y}{a}\varphi''(a\rho_N)\,dy\\ &\quad +\frac{m-1}{m}\int_{-N}^{N}(a\rho_N)^m\left[\frac{V'}{a}\right]_y\,dy\,.\\ &= -(m-1)\int_{-N}^{N}\frac{(a\rho_N)^{m-1}}{a}[a\rho_N]_y^2\varphi''(a\rho_N)\,dy\\ &\quad +\frac{m-1}{m}\int_{-N}^{N}(a\rho_N)^m\left[\frac{V'}{a}\right]_y\,dy\,.\end{aligned}$$

By the growth condition from below (D) and (gW2) the last equality becomes

$$\frac{d}{dt}\int_{-N}^{N} a^{m-1}\rho_N^m\,dy \le -\frac{4m(m-1)}{(2m-1)^2}c_m\int_{-N}^{N}\frac{1}{a}\left(\left[(a\rho_N)^{m-\frac{1}{2}}\right]_y\right)^2 dy + L(m-1)\int_{-N}^{N} a^{m-1}\rho_N^m\,dy\,. \tag{56}$$

By applying the Gronwall's inequality in $(0,T)$ we deduce an L^m-estimates on ρ_N; that is,

$$\int_{-N}^{N}\rho_N^m\,dy \le \int_{-N}^{N} a^{m-1}\rho_N^m\,dy \le e^{(m-1)LT}\int_{-N}^{N} a^{m-1}\rho_{N,0}^m\,dy \le c\,e^{(m-1)LT}\mathcal{F}^a\left[\rho_0\right]\,. \tag{57}$$

This actually induce an L^∞-estimate in space on both ρ_N and $a\rho_N$. Indeed, the L^∞-estimate of ρ_N is a straightforward consequence of (57) since $\|\rho_N\|_\infty = \lim_{m\to\infty}\|\rho_N\|_m$. In order to derive the L^∞-estimate for $a\rho_N$ we consider the change of variable $x=\alpha^{-1}(y)$ that maps $[-N,N]$ to $[-1+\delta_N, 1-\delta_N]$ for some $\delta_N>0$. Hence, we define the scaling $v_N(x,t)=a(\alpha(x))\rho_N(\alpha(x),t)$ for $x\in[-1+\delta_N,1-\delta_N]$ and zero otherwise. We apply the aforementioned change of variable to (56) and we get

$$\frac{d}{dt}\int_{-1+\delta_N}^{1-\delta_N} v_N^m\,dx \le L(m-1)\int_{-1+\delta_N}^{1-\delta_N} v_N^m\,dx\,.$$

Reasoning as above we can conclude that $\|v_N\|_m \le e^{\frac{L(m-1)t}{m}}\|v_0\|_m$. Letting $m\to\infty$ and changing again variable we get the L^∞-estimate for $a\rho_N$.

We now integrate (56) with respect to $t\in(0,T)$, by (57) we get

$$\frac{4(m-1)}{(2m-1)^2}c\int_0^T\int_{-N}^{N}\frac{1}{a(y)}\left(\left[(a(y)\rho_N(y,t))^{m-\frac{1}{2}}\right]_y\right)^2 dy\,dt \le C(T,m)\mathcal{F}^a\left[\rho_0\right]. \tag{58}$$

Note that

$$\begin{aligned}\left(\left[(a(y)\rho_N(y,t))^{m-\frac{1}{2}}\right]_y\right)^2 &= \left(\frac{2m-1}{2}\right)^2\left(\frac{a'}{a}\right)^2 a^{2m-1}(\rho_N)^{2m-1}\\ &\quad + a^{2m-1}\left([(\rho_N)^{m-\frac{1}{2}}]_y\right)^2\\ &\quad + (2m-1)\left(\frac{a'}{a}\right)a^{2m-1}(\rho_N)^{m-\frac{1}{2}}\,[(\rho_N)^{m-\frac{1}{2}}]_y\,;\end{aligned}$$

hence,

$$
\begin{aligned}
\int_{-N}^{N} \frac{1}{a(y)} \left(\left[(a(y)\rho_N(y,t))^{m-\frac{1}{2}} \right]_y \right)^2 dy &= \left(\frac{2m-1}{2}\right)^2 \int_{-N}^{N} \left(\frac{a'}{a}\right)^2 a^{2m-2} (\rho_N)^{2m-1}\, dy \\
&\quad + \int_{-N}^{N} a^{2m-2} \left([(\rho_N)^{m-\frac{1}{2}}]_y \right)^2 dy \\
&\quad + (2m-1) \int_{-N}^{N} \left(\frac{a'}{a}\right) a^{2m-2} (\rho_N)^{m-\frac{1}{2}}\, [(\rho_N)^{m-\frac{1}{2}}]_y\, dy \\
&= \left(\frac{2m-1}{2}\right)^2 \int_{-N}^{N} \left(\frac{a'}{a}\right)^2 a^{2m-2} (\rho_N)^{2m-1}\, dy \qquad (59) \\
&\quad + \int_{-N}^{N} a^{2m-2} \left([(\rho_N)^{m-\frac{1}{2}}]_y \right)^2 dy \\
&\quad + (\frac{2m-1}{2}) \int_{-N}^{N} \left(\frac{a'}{a}\right) a^{2m-2} [(\rho_N)^{2m-1}]_y\, dy\,.
\end{aligned}
$$

Since (59) is positive, $a \geq 1$, and $m > 1$ then we can minimise

$$
\begin{aligned}
&\int_{-N}^{N} \frac{1}{a(y)} \left(\left[(a(y)\rho_N(y,t))^{m-\frac{1}{2}} \right]_y \right)^2 dy \\
\geq\ & \int_{-N}^{N} \left([(\rho_N)^{m-\frac{1}{2}}]_y \right)^2 dy - (\frac{2m-1}{2}) \int_{-N}^{N} \left(\frac{a''}{a} + (2m-3) \left(\frac{a'}{a}\right)^2 \right) a^{2m-2} (\rho_N)^{2m-1}\, dy\,.
\end{aligned}
$$

Therefore, by (58) we get that

$$
\int_{-N}^{N} \left([(\rho_N)^{m-\frac{1}{2}}]_y \right)^2 dy \leq C(t,m) \mathcal{F}^a[\rho_0] + (\frac{2m-1}{2}) \int_{-N}^{N} \left(\frac{a''}{a} + (2m-3) \left(\frac{a'}{a}\right)^2 \right) a^{2m-2} \rho_N^{2m-1}\, dy\,.
$$

We recall that, by Proposition 2.1, a''/a and $|a'(y)/a(y)|$ are bounded for every $y \in \mathbb{R}$. Hence, if we denote $K := \sup_{\mathbb{R}} \left(\frac{a''}{a} + (2m-3) \left(\frac{a'}{a}\right)^2 \right)$ we can conclude that

$$
\int_{-N}^{N} \left([(\rho_N)^{m-\frac{1}{2}}]_y \right)^2 dy \leq C(t,m) \mathcal{F}^a[\rho_0] + K \int_{-N}^{N} a^{2m-2} \rho_N^{2m-1}\, dy\,.
$$

Thanks to the L^∞-estimate of $a\rho_N$ we get

$$
\begin{aligned}
\int_{-N}^{N} a^{2m-2} \rho_N^{2m-1}\, dy &= \int_{-N}^{N} a^{m-1} \rho_N^{m} \left(a^{m-1} \rho_N^{m-1} \right) dy \\
&\leq C\, \mathcal{F}^a[\rho_N] \leq C\, \mathcal{F}^a[\rho_0]\ .
\end{aligned}
$$

Since $\rho_N > 0$,the L^2-estimate of $[(\rho_N)^{m-\frac{1}{2}}]_y$ easily implies an L^2-estimate of $[(\rho_N)^{\frac{m}{2}}]_y$ and, therefore, the $L^2\left([0,T], H^1(\mathbb{R})\right)$- estimate of $\rho_N^{\frac{m}{2}}$ uniformly in N.

We now prove the H^{-1}-estimate of $[\rho_N^{m/2}]_t$. Let θ be a bounded function such that $\theta_{yy} = [\rho_N^{m/2}]_t$. We get that

$$\begin{aligned}
\int_{-N}^{N} (\theta_y)^2\,dy &= -\int_{-N}^{N} \theta\theta_{yy}\,dy = -\frac{m}{2}\int_{-N}^{N} \theta\rho_N^{\frac{m}{2}-1}[\rho_N]_t\,dy\\
&= -\frac{m}{2}\int_{-N}^{N} \theta\rho_N^{\frac{m}{2}-1}\left[\rho_N\left(F^N_{\eta y}(y,\rho_N) + F^N_{\eta\eta}(y,\rho_N)\partial_y\rho_N + V'\right)\right]_y\,dy\\
&= \frac{m}{2}\int_{-N}^{N} [\theta\rho_N^{\frac{m}{2}-1}]_y\rho_N\left(F^N_{\eta y}(y,\rho_N) + F^N_{\eta\eta}(y,\rho_N)\partial_y\rho_N + V'\right)\,dy\\
&= \frac{m}{2}\int_{-N}^{N} [\theta\rho_N^{\frac{m}{2}-1}]_y\rho_N\left((a'\rho_N + a\,[\rho_N]_y)\,\varphi''(a\rho_N) + V'\right)\,dy\\
&= \frac{m}{2}\int_{-N}^{N} \theta_y\rho_N^{\frac{m}{2}}\left((a'\rho_N + a\,[\rho_N]_y)\,\varphi''(a\rho_N) + V'\right)\,dy\\
&\quad+\frac{m}{2}\int_{-N}^{N} \theta[\rho_N^{\frac{m}{2}-1}]_y\rho_N\left((a'\rho_N + a\,[\rho_N]_y)\,\varphi''(a\rho_N) + V'\right)\,dy\\
&= I_1 + I_2.
\end{aligned}$$

Applying the weighted Cauchy inequality to I_1 we have that

$$I_1 \le \frac{1}{2}\int_{-N}^{N}\theta_y^2\,dy + \frac{m^2}{4}3C(V,a)^2\int_{-N}^{N}\rho_N^m\,dy + \frac{m^2}{4}\frac{3}{2}C^2\int_{-N}^{N}[\rho_N^{\frac{m}{2}}]_y^2\,dy. \tag{60}$$

Concerning I_2 its easy to see that we can get the following estimate

$$|I_2| \le \|\theta\|_\infty C(m,\|a\rho_N\|_\infty)\int_{-N}^{N}[\rho_N^{\frac{m}{2}}]_y^2\,dy.$$

Hence, we can conclude that

$$\frac{1}{2}\int_{-N}^{N}\theta_y^2\,dy \le C(m,\|a\rho_N\|_\infty,V)\left(\int_{-N}^{N}\rho_N^m\,dy + \int_{-N}^{N}[\rho_N^{\frac{m}{2}}]_y^2\,dy\right). \tag{61}$$

Thanks to the previous L^2-estimate of $\rho_N^{m/2}$ we can conclude that $[\rho_N^{m/2}]_t$ is N-uniformly bounded in $L^2(0,T;H^{-1}(\mathbb{R}))$. Invoking Aubin-Lions Lemma we have that $\rho_N^{m/2}$ converges to a certain limit η in $L^2_{loc}(\mathbb{R}_+\times\mathbb{R})$. The estimates above allow us to pass to the limit in the weak formulation of the regularised problem in order to recover weak solutions to (24). The passage to the limit in the E.V.I. can easily be deduced reasoning as in [16]. □

In Proposition 3.2 we proved the existence of weak solution ρ to equation (24). We are now ready to prove the uniqueness.

Theorem 3.1 (Uniqueness of weak solution ρ). *Let φ, g and W as in Section 2.2. In addition, we assume that also the assumption in Lemma 3.5 is full-filled for some $k\in\mathbb{R}$. Then, there is at most one solution to (24) with initial condition ρ_0.*

Proof. Let ρ and η be two solutions to (24) with initial data ρ_0 and η_0 respectively. Under the assumption of Lemma 3.5, the E.V.I. holds for both solutions, namely for any $\tilde\rho\in\mathcal{P}_2(\mathbb{R})$

$$\frac{1}{2}\frac{d}{dt}W_2^2(\rho,\tilde\rho) + \frac{k}{2}W_2^2(\rho,\tilde\rho) \le \mathcal{F}^a(\tilde\rho) - \mathcal{F}^a(\rho), \tag{62}$$

and

$$\frac{1}{2}\frac{d}{dt}W_2^2(\eta,\tilde\rho) + \frac{k}{2}W_2^2(\eta,\tilde\rho) \le \mathcal{F}^a(\tilde\rho) - \mathcal{F}^a(\eta), \tag{63}$$

are satisfied. Choosing $\tilde\rho = \eta$ in (62), $\tilde\rho = \rho$ in (63) and summing up the two inequalities, we get, by Gronwall's Lemma, a contraction; *i.e.*,

$$W_2^2(\rho(t),\eta(t)) \le e^{-kt}W_2^2(\rho_0,\eta_0), \tag{64}$$

that yields uniqueness provided $\rho_0 = \eta_0$. □

4. EXISTENCE AND UNIQUENESS OF WEAK SOLUTIONS TO NONLINEAR CONVECTION-DIFFUSION EQUATION ON BOUNDED INTERVALS WITH DEGENERATE MOBILITY: FAST-DECAY CASE

In this section we reformulate the results obtained in Section 3 in terms of existence and uniqueness of weak solutions to equation (2).

Theorem 4.1. *Let $g : \Omega \to [0,1]$ be a $C^2(\bar{\Omega})$ function under assumptions* (g1)-(g3)*. Let $\varphi : [0,+\infty) \to [0,+\infty)$ be a lower semi-continuous and convex function satisfying* (D) *and let $W : \Omega \to [0,+\infty)$ be a non-negative $C^2(\Omega)$ function under the assumption* (gW1)-(gW2)*. Consider, for $m > 1$, the initial condition $u_0 \in L^1 \cap L^m(\Omega)$ and fix $T > 0$. Then there exists a Hölder-continuous curve $u : [0,T] \to L^m(\Omega)$ such that,*

- (i) *$u \in L^\alpha([0,T] \times \Omega)$ for some $\alpha \in (1, 3m)$;*
- (ii) *$g[u^{m/2}]_x \in L^2([0,+\infty) \times \Omega)$;*
- (iii) *for almost every $t \in [0,+\infty)$ and for all $\psi \in C_c^\infty(\Omega)$, we have*

$$\frac{d}{dt}\int_\Omega \psi(x)u(x,t)dx = -\int_\Omega g^2(x)\left(\varphi'(u) + W(x)\right)_x \psi_x(x)u(x,t)dx. \tag{65}$$

Proof. Fix $T > 0$ and consider the initial datum $u_0 \in L^1 \cap L^m(\Omega)$. We define

$$\rho_0(y) = g(\alpha^{-1}(y))u_0(\alpha^{-1}(y)),\ y \in \mathbb{R},$$

with α as in (3). The function ρ_0 is an admissible initial condition for (24) in the sense of Definition 3.1 and by Theorem 3.1 there exists a unique solution ρ, corresponding to this initial datum, with $\rho \in L^m([0,T] \times \mathbb{R})$ and $[\rho^{m/2}]_y \in L^2([0,+\infty) \times \mathbb{R})$. Therefore, by the usual change of variable, we can define in a unique way

$$u(x,t) = a(\alpha(x))\rho(\alpha(x),t), \quad x \in \Omega,$$

and, we get that

$$\int_\Omega u^m(x,t)dx = \int_{\mathbb{R}} a^{m-1}(y)\rho^m(y,t)\,dy.$$

By performing a similar computation as in proof of Lemma 3.5 we can show that

$$\begin{aligned}
\frac{1}{m}\frac{d}{dt}\int_\Omega u^m(x,t)\,dx &= \frac{1}{m}\frac{d}{dt}\int_{\mathbb{R}} a^{m-1}(y)\rho^m(y,t)dy \\
&\le -\frac{4(m-1)}{(2m-1)^2}c_m \int_{\mathbb{R}} \frac{1}{a}\left(\left[(a\rho)^{m-\frac{1}{2}}\right]_y\right)^2 dy + L\frac{m-1}{m}\int_{\mathbb{R}} a^{m-1}\rho^m\,dy \\
&\le -\frac{4(m-1)}{(2m-1)^2}c_m \int_\Omega g^2\left([u^{m-\frac{1}{2}}]_x\right)^2 dx + L\frac{m-1}{m}\int_\Omega u^m\,dx\,.
\end{aligned}$$

Since $u_0 \in L^m(\Omega)$ we get that $u \in L^m(\Omega)$. An L^2-estimate of $g[u^{m/2}]_x$ can be easily derived from the L^2-estimate of $[\rho^{\frac{m}{2}}]_y$ and assumption (gW2). Changing variable in (32) we get

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \psi(x)u(x,t)dx &= \frac{d}{dt}\int_{\mathbb{R}} \zeta(y)\rho(y,t)dy \\
&= -\int_{\mathbb{R}} \partial_y\left(\varphi'(a\rho) + V(y)\right)\partial_y\zeta(y)\rho(y,t)dy \\
&= -\int_\Omega g^2(x)\partial_x\left(\varphi'(u) + W(x)\right)\partial_x\psi(x)u(x,t)dx
\end{aligned}$$

where $\zeta(y) = \psi(\alpha^{-1}(y))$, for $y \in \mathbb{R}$. □

5. SPECIAL CASES WITH DEGENERATE FAST-DECAY MOBILITY

In this section we consider three examples of well-known classical equations slightly modified by adding the degenerate mobility $g(x) = (1-x^2)^{p/2}$. In order to apply the results obtained in the general case, we need to check the λ-convexity of the associated energy functionals. We recall that in Section 3.2 we associated to the equation

$$\rho_t = \left(\rho\left(\varphi'(a\rho) + V\right)_y\right)_y$$

the functional $\mathcal{F}^a\left[\rho\right]$ as in (27). By using (28) and (29), the functional can be rewritten in the following form

$$\begin{aligned} \mathcal{F}^a\left[\rho\right] &= \int_{\Omega'} \frac{\varphi(a(y)\rho(y))}{a(y)}\,\mathrm{d}y + \int_{\Omega'} V(y)\rho(y)\,\mathrm{d}y \\ &= \int_0^1 \frac{\varphi(a\circ Y\,\rho\circ Y)}{a\circ Y} Y_\omega\,\mathrm{d}\omega + \int_0^1 V\circ Y\,\rho\circ Y\,Y_\omega\,\mathrm{d}\omega \\ &= \int_0^1 \psi\Big(\frac{a\circ Y}{Y_\omega}\Big)\,\mathrm{d}\omega + \int_0^1 V\circ Y\,\mathrm{d}\omega =: \tilde{\mathcal{F}}^a\left[Y\right], \end{aligned} \tag{66}$$

where $\psi(s) = \varphi(s)/s$. We recall that λ-convexity of $\mathcal{F}^a$, with respect to the Wasserstein distance, is equivalent to the λ-convexity of $\tilde{\mathcal{F}}^a$ in L^2. The latter is implied by the convexity of $f : \Omega'\times\mathbb{R}_+ \to \mathbb{R}$ defined as

$$f(p,q) = \psi\Big(\frac{a(p)}{q}\Big) + V(p) - \frac{\lambda}{2}p^2\,. \tag{67}$$

The Hessian of f is given by

$$\mathcal{H}_f(p,q) = \begin{pmatrix} (a')^2q^{-2}\,\psi''(z) + a''q^{-1}\,\psi'(z) + V'' - \lambda\,, & -aa'q^{-3}\,\psi''(z) - a'q^{-2}\,\psi'(z) \\ \\ -aa'q^{-3}\,\psi''(z) - a'q^{-2}\,\psi'(z)\,, & a^2q^{-4}\,\psi''(z) + 2aq^{-3}\,\psi'(z) \end{pmatrix}.$$

where $z := a(p)/q$ and we have omitted the dependence of a from the variable p to not overburden the notations.

We now study the λ-convexity of $\mathcal{F}^a$ in the three relevant cases with $g(x) = (1-x^2)^{p/2}$.

5.1. **Heat equation.** We consider the following linear heat equation with degenerate mobility,

$$u_t = (g^2u_x)_x = \big(g^2u[\log u]_x\big)_x\,.$$

By Section 2.1 we get the corresponding equation in $\rho(y,t)$

$$\rho_t \;= (\rho[\log(a\rho)]_y)_y = \rho_{yy} + (\rho[\log(a)]_y)_y \tag{68}$$

and the associated functional as in (27) with $\varphi(\rho) = \rho\log\rho$ and $V(y) = \log a$. Therefore, by (16), (17), and $\psi(z) = \log z$, we have that the hessian reduces to

$$\mathcal{H}_f(p,q) = \begin{pmatrix} -2gg_{xx} - \lambda & 0 \\ 0 & q^{-2} \end{pmatrix}.$$

By (11), we have that he equation (68) is a λ-convex gradient flow, with

$$\lambda := \inf_{y\in\mathbb{R}}[\log a]_{yy} = \inf_{y\in\mathbb{R}} \frac{a''a-(a')^2}{a^2}\circ\alpha = -\sup_{x\in\Omega}(gg'')\,.$$

Since $g(x) = (1-x^2)^{p/2}$ for some $p>0$ then we get that

$$\begin{aligned} -g''(x)g(x) &= p\big(x(1-x^2)^{p/2-1}\big)_x(1-x^2)^{p/2} \\ &= p(1-x^2)^{p-2}\big(1-x^2-(p-2)x^2\big) \\ &= p(1-x^2)^{p-2}(1-(p-1)x^2); \end{aligned}$$

which implies,

$$\lambda = p\inf_{|x|<1}(1-x^2)^{p-2}(1-(p-1)x^2).$$

for $p>2$.

5.2. **Linear Fokker-Plank equation.** We now consider a linear Fokker-Planck equation with degenerate mobility g; *i.e.*,

$$u_t = \left(g^2 u[\log u + W]_x\right)_x.$$

Recalling (16) and (17), the hessian reduces

$$\mathcal{H}_f(p,q) = \begin{pmatrix} -gg'' + g^2W'' + gg'W' - \lambda & 0 \\ 0 & q^{-2} \end{pmatrix}.$$

In this case we can consider a λ that balances the diffusive and the potential part, separately; namely, we assume that there exist λ_d and λ_W such that

(i) $-gg'' \geq \lambda_d$,
(ii) $g^2W'' + gg'W' \geq \lambda_W$.

5.3. **Porous Medium.** We consider the following porous medium equation

$$u_t = \left(g^2\left[(u^m)_x + uW'\right]\right)_x.$$

Then $\varphi(s) = s^m/(m-1)$, and accordingly

$$\psi(z) = \frac{z^{m-1}}{m-1}, \quad \psi'(z) = z^{m-2}, \quad \psi''(z) = (m-2)z^{m-3}.$$

The component $(\mathcal{H}_f)_{11}$ is thus given by

$$(\mathcal{H}_f)_{11} = \frac{1}{g^{m-1}q^{m-1}}\Big((m-1)(g')^2 - g''g\Big) + (g^2W'' + gg'W' - \lambda), \tag{69}$$

and the determinant becomes

$$\det \mathcal{H}_f = \frac{1}{g^{2(m-1)}q^{2m}}\Big((m-1)(g')^2 - m\,g''g\Big) + \frac{m}{g^{m-1}q^{m+1}}(g^2W'' + gg'W' - \lambda). \tag{70}$$

Unfortunately, only in the linear case $m = 1$ both the component 11 and the determinant are homogeneous with respect to q and g, and their terms can be combined to balance each other. As soon as $m \neq 1$, all terms need to be non-negative individually. In the case $m = 2$, $(g')^2 - g''g = 2(1 + x^2)$ that is always positive, so the first entrance in the hessian is positive as soon as

$$g^2W'' + gg'W' - \lambda \geq 0.$$

In order to preserve this condition we need to impose that

$$\frac{1}{2}(g')^2 - g''g = -2g^{\frac{3}{2}}\left(g^{\frac{1}{2}}\right)_{xx} \geq 0;$$

namely, $g^{\frac{1}{2}}$ concave, that is true only for $p < 4$.

For general $m \neq 2$, we can rewrite

$$\begin{aligned}\frac{(m-1)}{m}(g')^2 - g''g &= -g^{2-\frac{1}{m}}(g^{\frac{1}{m}-1}g')_x \\ &= -mg^{2-\frac{1}{m}}(g^{\frac{1}{m}})_{xx}\,.\end{aligned}$$

Similarly, we can prove that

$$\begin{aligned}(m-1)(g')^2 - g''g &= -g^m(g^{1-m}g')_x \\ &= \frac{1}{m-2}g^m(g^{2-m})_{xx}\,.\end{aligned}$$

Therefore, the formulas in (69) and (70) become

$$(\mathcal{H}_f)_{11} = \frac{g}{q^{m-1}}\frac{(g^{2-m})_{xx}}{m-2} + (g^2W'' + gg'W' - \lambda), \tag{71}$$

and

$$\det \mathcal{H}_f = -m^2\frac{g^{2-\frac{1}{m}}}{g^{2(m-1)}q^{2m}}(g^{\frac{1}{m}})_{xx} + \frac{m}{g^{m-1}q^{m+1}}(g^2W'' + gg'W' - \lambda). \tag{72}$$

We first compute

$$(g^\alpha(x))_x = -\alpha p\, x\,(1-x^2)^{\alpha p/2-1},$$

and,

$$(g^\alpha(x))_{xx} = -\alpha p\,(1-x^2)^{\alpha\frac{p}{2}-2}[1-(\alpha p-1)x^2]\,.$$

The term $[1-(\alpha p-1)x^2]$ is always positive if $\alpha = 2-m < 0$; hence, g^{2-m} is always convex. While, g^{2-m} is concave if and only if $\alpha = 2-m > 0$ and $p < 2/(2-m)$. On the other side, if $\alpha = 1/m$ then, $[1-(\alpha p-1)x^2]$ is positive if and only if $p \le 2m$ and therefore $g^{1/m}$ is concave. Note that, if $p \le 2m$ and $m < 2$ then p satisfies also the condition $p < 2/(2-m)$. Hence, summarizing,

- if $m > 2$ then g^{2-m} is convex;
- if $m < 2$ and $p < 2/(2-m)$ then g^{2-m} is concave. Note that, if $m < 2$ and $p > 2/(2-m)$ then $\alpha p > 2$ therefore $[1-(\alpha p-1)x^2] < 0$ if $(1/(\alpha p-1)) < |x| < 1$ and $[1-(\alpha p-1)x^2] > 0$ if $|x| < 1/(\alpha p-1)$;
- if $p \le 2m$ then $g^{1/m}$ is concave;
- if $m < 2$ and $p \le 2m$ then g^{2-m} is concave and $g^{1/m}$ concave. Indeed, if $m < 2$ then $2m = \min\{2m, 2/(2-m)\}$; hence, $p \le 2m$ implies $p < 2/(2-m)$;
- if $m < 2$ and $2m < p < 2/(2-m)$: $g^{1/m}$ is convex and g^{2-m} is concave.

We can conclude that $(\mathcal{H}_f)_{11}$ and $\det \mathcal{H}_f$ are both positive as soon $g^2W'' + gg'W' - \lambda \ge 0$, $m > 2$ and $p \le 2m$.

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Email address: `ansini@mat.uniroma1.it`
Email address: `simone.fagioli@univaq.it`