

DEGENERATE NONLOCAL CAHN-HILLIARD EQUATIONS: WELL-POSEDNESS, REGULARITY AND LOCAL ASYMPTOTICS

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ABSTRACT. Existence and uniqueness of solutions for nonlocal Cahn-Hilliard equations with degenerate potential is shown. The nonlocality is described by means of a symmetric singular kernel not falling within the framework of any previous existence theory. A convection term is also taken into account. Building upon this novel existence result, we prove convergence of solutions for this class of nonlocal Cahn-Hilliard equations to their local counterparts, as the nonlocal convolution kernels approximate a Dirac delta. Eventually, we show that, under suitable assumptions on the data, the solutions to the nonlocal Cahn-Hilliard equations exhibit further regularity, and the nonlocal-to-local convergence is verified in a stronger topology.

CONTENTS

1. Introduction	1
2. Setting and main results	5
2.1. Hypotheses	5
2.2. Notation, preliminaries and comments	6
2.3. Main results	9
3. Proof of Theorem 2.1	11
3.1. Approximation	12
3.2. Uniform estimates	14
3.3. Passage to the limit as $\lambda \searrow 0$	17
3.4. Continuous dependence	19
4. Proof of Theorem 2.2	20
5. Proof of Theorems 2.3–2.4	22
Acknowledgements	24
References	24

1. INTRODUCTION

The Cahn-Hilliard equation was originally introduced in [13] in order to model the so-called “spinodal decomposition” phenomenon occurring during the phase separation processes in binary metallic alloys. Since then it has acquired fundamental importance in several diffuse-interface models in different fields, ranging from physics and engineering to biology.

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This nonlinear parabolic PDE exhibits a gradient-flow structure (in the H^{-1} -metric) in terms of the free energy functional given by, cf. [13],

$$E_{CH}(u) = \int_{\Omega} \left(\frac{\tau^2}{2} |\nabla u(x)|^2 + F(u(x)) \right) dx, \quad (1.1)$$

where Ω is the d -dimensional flat torus, F is a double-well potential, and τ is a small positive parameter related to the thickness of the transition region. The choice of the set Ω is classical in the literature, and corresponds to imposing periodic boundary conditions. The corresponding evolution problem reads as follows

$$\begin{aligned} \partial_t u + \operatorname{div} J_{CH} &= 0, \\ J_{CH} &= -m(u) \nabla \mu_{CH}, \\ \mu_{CH} &= \frac{\delta E_{CH}(u)}{\delta u} = -\tau^2 \Delta u + F'(u), \end{aligned} \quad (1.2)$$

where μ_{CH} is the chemical potential associated to the energy E_{CH} , and the symbol $\operatorname{div}(\cdot)$ denotes the divergence operator. The function $m(\cdot)$ in (1.2) is known as mobility.

The mathematical literature on the classical Cahn-Hilliard equation has been widely developed in the last decades, in terms of well-posedness of the system with possibly degenerate potentials, viscosity terms and dynamic boundary conditions, but also in the direction of regularity, long-time behaviour of solutions, and optimal control problems. Among the extensive literature, we mention the works [14, 15, 16, 18, 19, 21, 37] dealing with existence-uniqueness of solutions, [20, 27, 38] for studies on the asymptotic behaviour of solutions, and [9, 46, 51] for analyses of the system incorporating possibly nonlinear viscosity terms. As far as optimal control problems are concerned, we point out the contributions [17, 22, 23, 28, 40].

In the early 90's in [36] G. Giacomin and J. Lebowitz considered the hydrodynamic limit of a microscopic model describing a d -dimensional lattice gas evolving via a Poisson nearest-neighbor process. In this seminal paper, the authors rigorously derived a nonlocal energy functional of the form

$$E_{NL}(u) = \frac{1}{4} \int_{\Omega} \int_{\Omega} K(x, y) (u(x) - u(y))^2 dx dy + \int_{\Omega} F(u(x)) dx, \quad (1.3)$$

where $K(x, y)$ is a positive and symmetric convolution kernel, and proposed the corresponding gradient flow as a model for binary alloys undergoing phase change.

The associated evolution problem, providing a nonlocal variant of the Cahn-Hilliard PDE, is given by the following system of equations:

$$\begin{aligned} \partial_t u + \operatorname{div} J_{NL} &= 0, \\ J_{NL} &= -m(u) \nabla \mu_{NL}, \\ \mu_{NL} &= \frac{\delta E_{NL}(u)}{\delta u} = (K * 1)u - K * u + F'(u), \end{aligned} \quad (1.4)$$

where $(K * 1)(x) := \int_{\Omega} K(x, y) dy$ and $(K * u)(x) := \int_{\Omega} K(x, y) u(y) dy$, for $x \in \Omega$.

The study of such nonlocal Cahn-Hilliard equations has recently been the subject of an intense research activity (see, e.g. [1, 5, 33, 35, 39] and the references therein). All the available results in the literature dealing with nonlocal evolution of phase interfaces require

the kernel K to be symmetric and of class $W^{1,1}$. Such requirements are usually met by checking a condition in the following form

$$|K(x, y)| \leq C|x - y|^{-\alpha} \quad \text{with} \quad 0 < \alpha < \frac{3}{2} \quad (1.5)$$

(see [25, Remark 1]).

The interest in this nonlocal model is motivated by its atomistic justification and its generality. A further motivation for the study of models in the form (1.4) is the observation that, at least formally, when the interaction kernel K is of the form $K(x, y) = K(|x - y|)$ and concentrates around the origin, then the behavior of the nonlocal interface evolution problems approaches that of the standard local Cahn-Hilliard equation.

This formal argument is enforced by the rigorous theory involving the variational convergence of nonlocal energies of the form (1.3) to local integral functionals as in (1.1). Building upon the seminal papers by J. Bourgain, H. Brezis, and P. Mironescu [10, 11], and of V. Mazzy'a and T. Shaposhnikova [43, 44], a whole nonlocal-to-local framework has been developed for singular nonlocal kernels associated to fractional Sobolev spaces. This study has been complemented by the Γ -convergence analysis and Poincaré inequalities obtained by A. C. Ponce in [48, 49]. More specifically, considering the following family of convolution kernels, identified by a small positive parameter ε ,

$$K_\varepsilon(x, y) = \frac{\rho_\varepsilon(|x - y|)}{|x - y|^2}, \quad (1.6)$$

where $(\rho_\varepsilon)_\varepsilon$ is a suitable sequence of mollifiers, A. C. Ponce showed the variational convergence

$$\frac{1}{4} \int_{\Omega} \int_{\Omega} K_\varepsilon(x, y)(u(x) - u(y))^2 dx dy \rightarrow \frac{1}{2} \int_{\Omega} |\nabla u(x)|^2 dx.$$

The first positive result towards rendering the formal nonlocal-to-local convergence of the Cahn-Hilliard models rigorously has been achieved in [45], where the authors have focused on convergence of weak solutions of the nonlocal Cahn-Hilliard equation (1.4) to weak solutions of its local counterpart (1.2), as the convolution kernel K approximates a Dirac delta centered in the origin. In the aforementioned paper, the convergence is studied in the case of constant mobility, with a non-singular double-well potential satisfying a bounded-concavity assumption of the form

$$F'' \geq -B_1,$$

for a positive constant B_1 small enough, (see [45, Assumption **H3**]).

Due to the above-mentioned variational convergence result, kernels in the form (1.6) are the most natural choice in the study of nonlocal phase transition problems. However, in general it is not true that these kernels enjoy a $W^{1,1}$ regularity, so that the available existence results in the literature do *not* apply. In addition, the usual condition (1.5) is *not* satisfied by K_ε as in (1.6). This observation renders the analysis of this class of problems very delicate and several nontrivial difficulties arise. For example, the definition and regularity of the chemical potential μ_{NL} in (1.4) relies on the properties of the linear unbounded operator

$(\mathbf{B}, D(\mathbf{B}))$, defined as

$$D(\mathbf{B}) := \{v \in L^2(\Omega) : (K * 1)v - (K * v) \in L^2(\Omega)\},$$

$$\mathbf{B}(v) := (K * 1)v - (K * v), \quad \forall v \in D(\mathbf{B}),$$

whose domain $D(\mathbf{B})$ is, a priori, not explicitly characterizable and not even necessarily containing $H^1(\Omega)$ (see Subsection 2.2). Such endeavours are further enhanced when turning to the analysis of nonlocal diffusions driven by degenerate potentials.

The first contribution of this paper (see Theorem 2.1) is the development of a well-posedness theory for nonlocal Cahn-Hilliard equations having singular kernels K_ε (for $\varepsilon > 0$ being fixed) defined as in (1.6).

In our analysis, we remove the small-concavity assumption on the potential that was required in [45], and include possibly degenerate double-well potentials F defined on bounded domains. Indeed, while the classical choice for F is the fourth-order polynomial $F_{\text{pol}}(r) := \frac{1}{4}(r^2 - 1)^2$, $r \in \mathbb{R}$, with minima in ± 1 (corresponding to the pure phases), it is well-known that, in view of the physical interpretation of the model, a more realistic description is given by the logarithmic double-well potential

$$F_{\text{log}}(s) = \frac{\theta}{2}((1+s)\log(1+s) + (1-s)\log(1-s)) + \frac{\theta_c}{2} - cs^2$$

for $0 < \theta < \theta_c$ and $c > 0$, which by contrast is defined on the bounded domain $(-1, 1)$ and possesses minima within the open interval $(-1, 1)$. Another interesting example of F which is included in our treatment is the so-called double-obstacle potential (see [7, 47]), having the form

$$F_{\text{ob}}(s) = I_{[-1,1]}(s) + \frac{1}{2}(1 - s^2), \quad I_{[-1,1]}(s) := \begin{cases} 0 & \text{if } s \in [-1, 1] \\ +\infty & \text{otherwise.} \end{cases}$$

In this latter case, the derivative F'_{ob} is not defined in the usual way, and has to be interpreted as the subdifferential ∂F_{ob} in the sense of convex analysis (see [4]). Analogously the equations defining the chemical potential must be read as a differential inclusion instead.

A further extension provided by our work is to consider a nonlocal Cahn-Hilliard equation augmented by a convection term in divergence form, i.e.

$$\begin{aligned} \partial_t u + \text{div } J_{NL} + \text{div}(\beta u) &= 0, \\ J_{NL} &= -\nabla \mu_{NL}, \\ \mu_{NL} &= \frac{\delta E_{NL}(u)}{\delta u} = (K * 1)u - K * u + F'(u). \end{aligned} \tag{1.7}$$

Here, $\beta = \beta(t, x)$ denotes the velocity field, depending on time and space, which may be acting on the particular system in consideration. As a common choice in the literature, we considered constant mobility equal to one.

The interest in additional convective contributions is connected with applications in mixing and stirring of fluids, as well as in biological realizations of thin films via Langmuir-Blodgett transfer [6, 42]. We mention in this direction the contributions [8, 26, 31, 52] on the local Cahn-Hilliard with convection, [29, 30, 50] dealing with the nonlocal Cahn-Hilliard with

local convection, and [32, 41] on the nonlocal case with nonlocal convection. A nonlocal convective Cahn-Hilliard type system modelling phase-separation has been analyzed in [24, 25]. Relevant studies in coupling the Cahn-Hilliard equation with a further equation for the velocity field have been the subject of [2, 3, 12, 34].

From a mathematical viewpoint, the presence of convection terms (i.e. when $\beta \neq 0$) destroys the gradient-flow structure of the equation, causing the analysis to be even more delicate.

The proof strategy for Theorem 2.1 relies on three main ingredients: (1) a suitable approximation of the nonlinearity and an existence analysis for the approximating equations based on a fixed point argument (see Subsection 3.1); (2) the establishment of uniform estimates by ad-hoc multiplication of the equations with suitable test functions (see Subsection 3.2); (3) a passage to the limit relying on nontrivial compactness and monotonicity arguments, falling outside the framework of classical Aubin-Lions embedding results (see Lemma 4 and Subsection 3.3). A delicate point is the proof of a uniform H^1 -estimate, which strongly relies on the choice of periodic boundary conditions.

Our second contribution is established in Theorem 2.2, where we show convergence of solutions for the nonlocal convective Cahn-Hilliard equation with singular kernel to solutions of the associated local one. Our analysis extends the work in [45] to a wider class of double-well potentials, satisfying no bounded-concavity assumptions and being possibly degenerate. The nonlocal-to-local convergence in Theorem 2.2 relies in an essential way on the uniform a-priori estimates established in the proof of Theorem 2.1, and on showing the independence of the identified upper bounds from the non-locality parameter ε .

The third and fourth main results of the paper are a regularity analysis for solutions to (1.4). In particular, in Theorem 2.3 we show that, if the initial datum and the convection velocity satisfy additional integrability and differentiability assumptions, then solutions to the nonlocal Cahn-Hilliard equations exhibit further regularity. In Theorem 2.4 we prove that they also converge to their local counterparts in stronger topologies. The regularity analysis in Theorems 2.3 and 2.4 is the byproduct of a time-differentiation of the nonlocal Cahn-Hilliard equation, and of the use of higher-order-in-time test functions.

The paper is organized as follows. Section 2 contains a description of the mathematical setting of the paper, the definition of weak solutions for the nonlocal and local convective Cahn-Hilliard equations, and the precise statements of the four main results. Sections 3 and 4 are devoted to the proof of Theorems 2.1 and 2.2, respectively. Eventually, in Section 5 we prove Theorems 2.3 and 2.4.

2. SETTING AND MAIN RESULTS

2.1. Hypotheses. Throughout the paper we will assume the following:

H1: Ω is the d -dimensional ($d = 2, 3$) flat torus and $T > 0$ is a fixed final time.

H2: The kernel $K_\varepsilon : \Omega \times \Omega \rightarrow \mathbb{R}$ is defined as in (1.6):

$$K_\varepsilon(x, y) := \frac{1}{|x - y|^2} \rho_\varepsilon(|x - y|), \quad \text{for a.e. } (x, y) \in \Omega \times \Omega$$

where $(\rho_\varepsilon)_{\varepsilon>0} \subset L^1_{\text{loc}}(0, +\infty)$ is a family of radial mollifiers on \mathbb{R} , satisfying

$$\begin{aligned} \rho_\varepsilon(r) &\geq 0 && \forall r \in \mathbb{R}, \forall \varepsilon > 0, \\ \text{supp}(\rho_\varepsilon) &\subset [0, \text{diam}(\Omega)] && \forall \varepsilon > 0, \\ \frac{1}{2} \int_0^{+\infty} \rho_\varepsilon(r) r^{d-1} dr &= 1 && \forall \varepsilon > 0, \\ \lim_{\varepsilon \searrow 0} \int_\delta^{+\infty} \rho_\varepsilon(r) r^{d-1} dr &= 0 && \forall \delta > 0. \end{aligned}$$

H3: $\gamma : \mathbb{R} \rightarrow 2^{\mathbb{R}}$ is a maximal monotone graph such that $0 \in \gamma(0)$. This implies that $\gamma = \partial \hat{\gamma}$, where $\hat{\gamma} : \mathbb{R} \rightarrow [0, +\infty]$ is a proper, convex and lower semicontinuous function. The map $\Pi : \mathbb{R} \rightarrow \mathbb{R}$ is a Lipschitz-continuous function with Lipschitz constant $C_\Pi > 0$. The double-well potential F will be represented by $\hat{\gamma} + \hat{\Pi}$, where $\hat{\Pi}(t) := \int_0^t \Pi(r) dr$ for every $t \in \mathbb{R}$. Without restriction we will assume that F is nonnegative.

H4: The velocity β depends on space and time, and satisfies $\beta \in L^2(0, T; L^\infty(\Omega; \mathbb{R}^d))$.

We point out that all assumptions collected in **H2** correspond to the requirements in [49, 48].

For every $\varepsilon > 0$, we consider the nonlocal Cahn-Hilliard equation with local convection

$$\partial_t u_\varepsilon - \Delta \mu_\varepsilon + \text{div}(\beta u_\varepsilon) = 0 \quad \text{in } (0, T) \times \Omega, \quad (2.1)$$

$$\mu_\varepsilon \in (K_\varepsilon * 1)u_\varepsilon - K_\varepsilon * u_\varepsilon + \gamma(u_\varepsilon) + \Pi(u_\varepsilon) \quad \text{in } (0, T) \times \Omega, \quad (2.2)$$

$$u_\varepsilon(0) = u_{0,\varepsilon} \quad \text{in } \Omega, \quad (2.3)$$

and its local counterpart

$$\partial_t u - \Delta \mu + \text{div}(\beta u) = 0 \quad \text{in } (0, T) \times \Omega, \quad (2.4)$$

$$\mu \in -\Delta u + \gamma(u) + \Pi(u) \quad \text{in } (0, T) \times \Omega, \quad (2.5)$$

$$u(0) = u_0 \quad \text{in } \Omega. \quad (2.6)$$

2.2. Notation, preliminaries and comments. In the sequel we will identify $L^2(\Omega)$ with its dual, so that $(H^1(\Omega), L^2(\Omega), (H^1(\Omega))^*)$ will be a classical Hilbert triplet. We will use the symbol $(v)_\Omega$ for $\frac{1}{|\Omega|} \langle v, 1 \rangle_{(H^1(\Omega))^*, H^1(\Omega)}$ for every $v \in (H^1(\Omega))^*$. Note that for $v \in L^2(\Omega)$, $(v)_\Omega$ coincides with the usual average. We recall that the operator

$$(-\Delta)^{-1} : \{v \in (H^1(\Omega))^* : (v)_\Omega = 0\} \rightarrow \{w \in H^1(\Omega) : (w)_\Omega = 0\}$$

is defined as the map assigning to every $v \in (H^1(\Omega))^*$ with null mean the unique element $w \in H^1(\Omega)$ such that

$$(w)_\Omega = 0, \quad \text{and} \quad \int_\Omega \nabla w(x) \cdot \nabla \varphi(x) dx = \langle v, \varphi \rangle_{(H^1(\Omega))^*, H^1(\Omega)} \quad \forall \varphi \in H^1(\Omega).$$

It is well known that $(-\Delta)^{-1}$ is a linear isomorphism.

In this paper C indicates a generic positive constant, possibly varying from line to line, depending only on the setting **H1–H4**. The dependence of constants on a specific parameter will be indicated explicitly through a subscript.

We collect here some useful properties of the nonlocal term. We define the operator $(\mathbf{B}_\varepsilon, D(\mathbf{B}_\varepsilon))$ on $L^2(\Omega)$ in the following way:

$$\begin{aligned} D(\mathbf{B}_\varepsilon) &:= \{v \in L^2(\Omega) : (K_\varepsilon * 1)v - (K_\varepsilon * v) \in L^2(\Omega)\}, \\ \mathbf{B}_\varepsilon(v) &:= (K_\varepsilon * 1)v - (K_\varepsilon * v), \quad \forall v \in D(\mathbf{B}_\varepsilon). \end{aligned}$$

It is clear that \mathbf{B}_ε is a linear unbounded operator in $L^2(\Omega)$, and that for every $v \in D(\mathbf{B}_\varepsilon)$ we have the representation

$$\mathbf{B}_\varepsilon(v)(x) = \int_{\Omega} \rho_\varepsilon(|x-y|) \frac{v(x) - v(y)}{|x-y|^2} dy \quad \text{for a.e. } x \in \Omega.$$

We point out that the domain $D(\mathbf{B}_\varepsilon)$ is non-trivial. More specifically, we have the following result.

Lemma 1. *For every $\varepsilon, \sigma > 0$, there holds $C^{0,\sigma}(\overline{\Omega}) \subset D(\mathbf{B}_\varepsilon)$. Additionally, there exists a constant $C_{\varepsilon,\sigma} > 0$ (only dependent on ε and σ) such that*

$$\|\mathbf{B}_\varepsilon(v)\|_{L^2(\Omega)} \leq C_{\varepsilon,\sigma} \|v\|_{C^{0,\sigma}(\overline{\Omega})} \quad \forall v \in C^{0,\sigma}(\overline{\Omega}). \quad (2.7)$$

In particular, for every $s > \frac{3}{2}$, $H^s(\Omega) \subset D(\mathbf{B}_\varepsilon)$ and there exists $C_{\varepsilon,s} > 0$ such that

$$\|\mathbf{B}_\varepsilon(v)\|_{L^2(\Omega)} \leq C_{\varepsilon,s} \|v\|_{H^s(\Omega)} \quad \forall v \in H^s(\Omega). \quad (2.8)$$

Proof. A direct computation shows that for every $v \in C^{0,\sigma}(\overline{\Omega})$ and for almost every $x \in \Omega$,

$$|\mathbf{B}_\varepsilon(v)(x)| \leq \int_{\Omega} \rho_\varepsilon(|x-y|) \frac{|v(x) - v(y)|}{|x-y|^2} dy \leq \|v\|_{C^{0,\sigma}(\overline{\Omega})} \int_{\Omega} \frac{\rho_\varepsilon(|x-y|)}{|x-y|^{2-\sigma}} dy,$$

where

$$C_{\varepsilon,\sigma} := \int_{\Omega} \frac{\rho_\varepsilon(|x-y|)}{|x-y|^{2-\sigma}} dy < +\infty$$

thanks to **H2**. The second part of the Lemma follows by the Sobolev embedding $H^s(\Omega) \hookrightarrow C^{0,\sigma}(\overline{\Omega})$ for every $s > \frac{3}{2}$ and $\sigma \in (0, s - \frac{3}{2})$. \square

The operator $(\mathbf{B}_\varepsilon, D(\mathbf{B}_\varepsilon))$ has been defined as a linear unbounded operator on $L^2(\Omega)$. Note that it is not necessarily true that $H^1(\Omega) \subset D(\mathbf{B}_\varepsilon)$. Nevertheless, we now show that actually $(\mathbf{B}_\varepsilon, D(\mathbf{B}_\varepsilon))$ can be extended, uniformly in ε , to a linear bounded operator from $H^1(\Omega)$ to its dual.

Lemma 2. *For every $\varepsilon > 0$ the operator $(D(\mathbf{B}_\varepsilon), \mathbf{B}_\varepsilon)$ can be uniquely extended to a linear continuous operator $\mathbf{B}_\varepsilon : H^1(\Omega) \rightarrow (H^1(\Omega))^*$. Additionally, there exists a positive constant C , independent of ε , such that*

$$\|\mathbf{B}_\varepsilon(v)\|_{(H^1(\Omega))^*} \leq C \|\nabla v\|_{L^2(\Omega)} \quad \forall v \in H^1(\Omega). \quad (2.9)$$

In particular, the family $(\mathbf{B}_\varepsilon)_\varepsilon$ is uniformly bounded in $\mathcal{L}(H^1(\Omega), (H^1(\Omega))^)$ and there exists $\mathbf{B} \in \mathcal{L}(H^1(\Omega), (H^1(\Omega))^*)$ and an infinitesimal sequence $(\varepsilon_n)_n$ such that*

$$\lim_{n \rightarrow \infty} \langle \mathbf{B}_{\varepsilon_n}(v), \psi \rangle_{(H^1(\Omega))^*, H^1(\Omega)} = \langle \mathbf{B}(v), \psi \rangle_{(H^1(\Omega))^*, H^1(\Omega)} \quad \forall v, \psi \in H^1(\Omega).$$

Proof. By the Hölder inequality and [10, Theorem 1], we infer that

$$\begin{aligned} \langle \mathbf{B}_\varepsilon(v), \psi \rangle_{(H^1(\Omega))^*, H^1(\Omega)} &= \frac{1}{2} \int_{\Omega} \int_{\Omega} K_\varepsilon(x, y) (v(x) - v(y)) (\psi(x) - \psi(y)) dy dx \\ &\leq \frac{1}{2} \left(\int_{\Omega} \int_{\Omega} K_\varepsilon(x, y) |v(x) - v(y)|^2 dy dx \right)^{1/2} \left(\int_{\Omega} \int_{\Omega} K_\varepsilon(x, y) |\psi(x) - \psi(y)|^2 dy dx \right)^{1/2} \\ &\leq C \|\nabla v\|_{L^2(\Omega)} \|\nabla \psi\|_{L^2(\Omega)} \end{aligned}$$

for every $v, \psi \in H^1(\Omega)$. This implies that $(D(\mathbf{B}_\varepsilon), \mathbf{B}_\varepsilon)$ can be extended uniquely as required (the uniqueness follows Lemma 1, and from the density of $C^{0,\sigma}(\overline{\Omega})$ in $D(\mathbf{B}_\varepsilon)$). The second part of the lemma follows by observing that (2.9) implies the uniform boundedness of $(\mathbf{B}_\varepsilon)_\varepsilon$ in $\mathcal{L}(H^1(\Omega), (H^1(\Omega))^*)$, and hence its precompactness in the weak operator topology of $\mathcal{L}(H^1(\Omega), (H^1(\Omega))^*)$. \square

In what follows, a crucial role is also played by the nonlocal energy contribution

$$E_\varepsilon(v) := \frac{1}{4} \int_{\Omega} \int_{\Omega} K_\varepsilon(x, y) (v(x) - v(y))^2 dy dx \quad \forall v \in H^1(\Omega).$$

Owing to [10, Theorem 1], we have that E_ε is well-defined, convex, and its differential $DE_\varepsilon : H^1(\Omega) \rightarrow (H^1(\Omega))^*$ is given by

$$DE_\varepsilon = \mathbf{B}_\varepsilon.$$

Moreover, by [10] the asymptotic behavior of E_ε as $\varepsilon \rightarrow 0^+$ can be characterized as follows

$$\lim_{\varepsilon \rightarrow 0^+} E_\varepsilon(v) = \frac{1}{2} \int_{\Omega} |\nabla v(x)|^2 dx \quad \forall v \in H^1(\Omega). \quad (2.10)$$

As a corollary, we deduce the following identification of the operator \mathbf{B} in Lemma 3.

Lemma 3. *Let $(D(\mathbf{B}_\varepsilon), \mathbf{B}_\varepsilon)_\varepsilon$ and \mathbf{B} be as in Lemma 2. Then,*

$$\lim_{\varepsilon \rightarrow 0} \langle \mathbf{B}_\varepsilon(v), \psi \rangle_{(H^1(\Omega))^*, H^1(\Omega)} = \langle -\Delta v, \psi \rangle_{(H^1(\Omega))^*, H^1(\Omega)},$$

where

$$\langle -\Delta v, \psi \rangle_{(H^1(\Omega))^*, H^1(\Omega)} := \int_{\Omega} \nabla v(x) \cdot \nabla \psi(x) dx \quad \forall v, \psi \in H^1(\Omega).$$

Proof. By the characterization of the differential of E_ε , we have that

$$E_\varepsilon(v_1) + \langle \mathbf{B}_\varepsilon(v_1), v_2 - v_1 \rangle_{(H^1(\Omega))^*, H^1(\Omega)} \leq E_\varepsilon(v_2)$$

for every $v_1, v_2 \in H^1(\Omega)$. Hence, for every subsequence $(\varepsilon_n)_n$ as in Lemma 2, letting $n \rightarrow \infty$, by (2.10) we conclude that

$$\frac{1}{2} \int_{\Omega} |\nabla v_1(x)|^2 dx + \langle \mathbf{B}(v_1), v_2 - v_1 \rangle_{(H^1(\Omega))^*, H^1(\Omega)} \leq \frac{1}{2} \int_{\Omega} |\nabla v_2(x)|^2 dx,$$

from which $\mathbf{B} = -\Delta$. In particular, this implies that the convergence holds along the entire sequence ε . \square

We conclude this section with a lemma providing two fundamental compactness inequalities involving the family of operators $(\mathbf{B}_\varepsilon)_\varepsilon$. Such results are nontrivial, since they do not fall in the classical framework of the Aubin-Lions lemmas. The next lemma is a uniform counterpart to [45, Lemma 1].

Lemma 4. *For every $\delta > 0$ there exist constants $C_\delta > 0$ and $\varepsilon_\delta > 0$ with the following properties:*

(1) For every sequence $(f_\varepsilon)_\varepsilon \subset H^1(\Omega)$ there holds

$$\|f_\varepsilon\|_{H^1(\Omega)}^2 \leq \delta \int_{\Omega} \int_{\Omega} K_\varepsilon(x, y) |\nabla f_\varepsilon(x) - \nabla f_\varepsilon(y)|^2 dy dx + C_\delta \|f_\varepsilon\|_{L^2(\Omega)}^2 \quad (2.11)$$

for every $0 < \varepsilon < \varepsilon_\delta$.

(2) For every sequence $(f_\varepsilon)_\varepsilon \subset L^2(\Omega)$ there holds

$$\|f_\varepsilon\|_{L^2(\Omega)}^2 \leq \delta E_\varepsilon(f_\varepsilon) + C_\delta \|f_\varepsilon\|_{(H^1(\Omega))^*}^2 \quad (2.12)$$

for every $0 < \varepsilon < \varepsilon_\delta$.

Proof. Assume by contradiction that (2.11) is false. Then, there exists $\bar{\delta} > 0$ having the following property: for every $n \in \mathbb{N}$ we can find a sequence $(f_\varepsilon^n)_\varepsilon \subset H^1(\Omega)$ and a parameter $\varepsilon_n < \frac{1}{n}$ such that

$$\|f_{\varepsilon_n}^n\|_{H^1(\Omega)}^2 > \bar{\delta} \int_{\Omega} \int_{\Omega} K_{\varepsilon_n}(x, y) |\nabla f_{\varepsilon_n}^n(x) - \nabla f_{\varepsilon_n}^n(y)|^2 dy dx + n \|f_{\varepsilon_n}^n\|_{L^2(\Omega)}^2.$$

Noting that $\|f_{\varepsilon_n}^n\|_{H^1(\Omega)} > 0$ for every n and setting

$$g_n := \frac{f_{\varepsilon_n}^n}{\|f_{\varepsilon_n}^n\|_{H^1(\Omega)}},$$

we have

$$\bar{\delta} \int_{\Omega} \int_{\Omega} K_{\varepsilon_n}(x, y) |\nabla g_n(x) - \nabla g_n(y)|^2 dy dx + n \|g_n\|_{L^2(\Omega)}^2 < 1 \quad \forall n \in \mathbb{N}.$$

Hence, $g_n \rightarrow 0$ strongly in $L^2(\Omega)$ and the family $(\nabla g_n)_n$ is relatively strongly compact in $L^2(\Omega; \mathbb{R}^d)$ by [48, Theorem 1.2]. We deduce that $g_n \rightarrow 0$ strongly in $H^1(\Omega)$, but this is a contradiction since by definition we have $\|g_n\|_{H^1(\Omega)} = 1$ for all n . The argument for (2.12) is entirely analogous. \square

2.3. Main results. Before stating our main results, let us recall the notion of weak solutions to both the nonlocal as well as the local Cahn-Hilliard equation with local convection.

Definition 1 (Solution to the nonlocal Cahn-Hilliard equation). *Let $\varepsilon > 0$ and $T > 0$ be fixed. A solution to the nonlocal Cahn-Hilliard equation (2.1)–(2.3) on $[0, T]$, and associated with the initial datum $u_{0,\varepsilon} \in L^2(\Omega)$, is a triplet $(u_\varepsilon, \mu_\varepsilon, \xi_\varepsilon)$ with the following properties*

$$\begin{aligned} u_\varepsilon &\in H^1(0, T; (H^1(\Omega))^*) \cap L^2(0, T; H^1(\Omega)), \\ \mu_\varepsilon &\in L^2(0, T; H^1(\Omega)), \quad \xi_\varepsilon \in L^2(0, T; L^2(\Omega)), \\ \mu_\varepsilon &= \mathbf{B}_\varepsilon(u_\varepsilon) + \xi_\varepsilon + \Pi(u_\varepsilon), \quad \xi_\varepsilon \in \gamma(u_\varepsilon) \quad \text{almost everywhere in } (0, T) \times \Omega, \end{aligned}$$

satisfying $u_\varepsilon(0) = u_{0,\varepsilon}$, and such that

$$\langle \partial_t u_\varepsilon(t), \varphi \rangle_{(H^1(\Omega))^*, H^1(\Omega)} + \int_{\Omega} \nabla \mu_\varepsilon(t, x) \cdot \nabla \varphi(x) dx = \int_{\Omega} \beta(t, x) u_\varepsilon(t, x) \cdot \nabla \varphi(x) dx \quad (2.13)$$

for all $\varphi \in H^1(\Omega)$, and for almost every $t \in (0, T)$.

Definition 2 (Solution to the local Cahn-Hilliard equation). *Let $T > 0$ be fixed. A solution to the local Cahn-Hilliard equation (2.4)–(2.6) on $[0, T]$, and associated with the initial datum $u_0 \in H^1(\Omega)$, is a triplet (u, μ, ξ) with the following properties*

$$\begin{aligned} u &\in H^1(0, T; (H^1(\Omega))^*) \cap L^2(0, T; H^2(\Omega)), \\ \mu &\in L^2(0, T; H^1(\Omega)), \quad \xi \in L^2(0, T; L^2(\Omega)), \\ \mu &= -\Delta u + \xi + \Pi(u), \quad \xi \in \gamma(u) \quad \text{almost everywhere in } (0, T) \times \Omega, \end{aligned}$$

satisfying $u(0) = u_0$, and such that

$$\langle \partial_t u(t), \varphi \rangle_{(H^1(\Omega))^*, H^1(\Omega)} + \int_{\Omega} \nabla \mu(t, x) \cdot \nabla \varphi(x) \, dx = \int_{\Omega} \beta(t, x) u(t, x) \cdot \nabla \varphi(x) \, dx \quad (2.14)$$

for all $\varphi \in H^1(\Omega)$, for almost every $t \in (0, T)$.

Our first result is the well-posedness of solutions to the nonlocal Cahn-Hilliard equation.

Theorem 2.1. *Let assumptions **H1**–**H4** be satisfied, and for every $\varepsilon > 0$ let*

$$u_{0,\varepsilon} \in L^2(\Omega), \quad \hat{\gamma}(u_{0,\varepsilon}) \in L^1(\Omega), \quad E_{\varepsilon}(u_{0,\varepsilon}) < +\infty, \quad (u_{0,\varepsilon})_{\Omega} \in \text{Int } D(\gamma). \quad (2.15)$$

*Then, there exists $\varepsilon_0 > 0$ having the following property: for every $\varepsilon < \varepsilon_0$ there exists a unique solution $(u_{\varepsilon}, \mu_{\varepsilon}, \xi_{\varepsilon})$ to (2.1)–(2.3) associated with the initial datum $u_{0,\varepsilon}$, according to Definition 1. Furthermore, if $(\beta^1, u_{0,\varepsilon}^1)$ and $(\beta^2, u_{0,\varepsilon}^2)$ are two sets of data satisfying **H4** and (2.15), with $(u_{0,\varepsilon}^1)_{\Omega} = (u_{0,\varepsilon}^2)_{\Omega}$, then there exists a positive constant M_{ε} , depending only on the setting **H1**–**H3** and on the norms of the data $(\beta^1, u_{0,\varepsilon}^1)$ and $(\beta^2, u_{0,\varepsilon}^2)$ appearing in **H4** and (2.15), such that, for any respective solution $(u_{\varepsilon}^1, \mu_{\varepsilon}^1, \xi_{\varepsilon}^1)$ and $(u_{\varepsilon}^2, \mu_{\varepsilon}^2, \xi_{\varepsilon}^2)$ to the nonlocal equation (2.1)–(2.3),*

$$\begin{aligned} \|u_{\varepsilon}^1 - u_{\varepsilon}^2\|_{C^0([0,T];(H^1(\Omega))^*)}^2 + \|E_{\varepsilon}(u_{\varepsilon}^1 - u_{\varepsilon}^2)\|_{L^1(0,T)} \\ \leq M_{\varepsilon} \left(\|u_{0,\varepsilon}^1 - u_{0,\varepsilon}^2\|_{(H^1(\Omega))^*}^2 + \|\beta^1 - \beta^2\|_{L^2(0,T;L^3(\Omega))}^2 \right). \end{aligned}$$

The second result concerns nonlocal-to-local convergence.

Theorem 2.2. *Let assumptions **H1**–**H4** be satisfied. Let $u_0 \in H^1(\Omega)$, and for every $\varepsilon > 0$ let $u_{0,\varepsilon}$ satisfy (2.15) and be such that*

$$\sup_{\varepsilon \in (0, \varepsilon_0)} \left(\|u_{0,\varepsilon}\|_{L^2(\Omega)}^2 + \|\hat{\gamma}(u_{0,\varepsilon})\|_{L^1(\Omega)} + E_{\varepsilon}(u_{0,\varepsilon}) \right) < +\infty, \quad (2.16)$$

$$\exists [a_0, b_0] \subset \text{Int } D(\gamma) : \quad a_0 \leq (u_{0,\varepsilon})_{\Omega} \leq b_0 \quad \forall \varepsilon \in (0, \varepsilon_0), \quad (2.17)$$

$$u_{0,\varepsilon} \rightharpoonup u_0 \quad \text{in } L^2(\Omega) \quad \text{as } \varepsilon \rightarrow 0^+. \quad (2.18)$$

Let $(u_{\varepsilon}, \mu_{\varepsilon}, \xi_{\varepsilon})$ be the unique solution to (2.1)–(2.3) associated to $u_{0,\varepsilon}$ given by Theorem 2.1, and let (u, μ, ξ) be the unique solution to the local equation (2.4)–(2.6) associated to u_0 , according to Definition 2.

Then, as $\varepsilon \searrow 0$,

$$\begin{aligned} u_{\varepsilon} &\rightarrow u && \text{strongly in } C^0([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \\ \partial_t u_{\varepsilon} &\rightharpoonup \partial_t u && \text{weakly* in } L^2(0, T; (H^1(\Omega))^*), \\ \mu_{\varepsilon} &\rightharpoonup \mu && \text{weakly in } L^2(0, T; H^1(\Omega)), \\ \xi_{\varepsilon} &\rightharpoonup \xi && \text{weakly in } L^2(0, T; L^2(\Omega)). \end{aligned}$$

The last two results that we present deal with regularity of solutions to the nonlocal equation. In particular, we show that if the data are more regular, then the solution to the nonlocal equation inherits a further regularity, and the convergences to the local equation are obtained in stronger topologies.

Theorem 2.3. *Let assumptions **H1**–**H4** be satisfied, and suppose also that*

$$\beta \in H^1(0, T; L^3(\Omega; \mathbb{R}^d)). \quad (2.19)$$

For every $0 < \varepsilon < \varepsilon_0$ let $u_{0,\varepsilon}$ satisfy (2.15) and

$$u_{0,\varepsilon} \in L^6(\Omega), \quad \mathbf{B}_\varepsilon(u_{0,\varepsilon}) + \xi_{0,\varepsilon} + \Pi(u_{0,\varepsilon}) \in H^1(\Omega) \quad \forall \xi_{0,\varepsilon} \in \gamma(u_{0,\varepsilon}). \quad (2.20)$$

Then the unique solution $(u_\varepsilon, \mu_\varepsilon, \xi_\varepsilon)$ to the nonlocal equation (2.1)–(2.3) with respect to the initial datum $u_{0,\varepsilon}$ also satisfies

$$u_\varepsilon \in W^{1,\infty}(0, T; (H^1(\Omega))^*) \cap H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)).$$

If also

$$\beta \in L^\infty(0, T; L^\infty(\Omega; \mathbb{R}^d)), \quad (2.21)$$

then in addition

$$\mu_\varepsilon \in L^\infty(0, T; H^1(\Omega)), \quad \xi_\varepsilon \in L^\infty(0, T; L^2(\Omega)).$$

If also

$$\operatorname{div} \beta \in L^\infty(0, T; L^3(\Omega)), \quad (2.22)$$

then in addition

$$\mu_\varepsilon \in L^2(0, T; H^2(\Omega)).$$

Theorem 2.4. *Let assumptions **H1**–**H4** be satisfied. Let $u_0 \in H^1(\Omega)$, and for every $\varepsilon > 0$ let $u_{0,\varepsilon}$ satisfy (2.15), (2.16)–(2.18), (2.20) and*

$$\sup_{\varepsilon \in (0, \varepsilon_0), \xi_{0,\varepsilon} \in \gamma(u_{0,\varepsilon})} (\|u_{0,\varepsilon}\|_{L^6(\Omega)} + \|\mathbf{B}_\varepsilon(u_{0,\varepsilon}) + \xi_{0,\varepsilon} + \Pi(u_{0,\varepsilon})\|_{H^1(\Omega)}) < +\infty. \quad (2.23)$$

Denoting by (u, μ, ξ) the unique solution to the local equation (2.4)–(2.6), if (2.19) holds then

$$\begin{aligned} u_\varepsilon &\rightarrow u && \text{strongly in } C^0([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \\ u_\varepsilon &\rightharpoonup u && \text{weakly* in } W^{1,\infty}(0, T; (H^1(\Omega))^*) \cap H^1(0, T; L^2(\Omega)). \end{aligned}$$

If also (2.21) holds, then

$$\begin{aligned} \mu_\varepsilon &\rightharpoonup \mu && \text{weakly* in } L^\infty(0, T; H^1(\Omega)), \\ \xi_\varepsilon &\rightharpoonup \xi && \text{weakly* in } L^\infty(0, T; L^2(\Omega)). \end{aligned}$$

If also (2.22) holds, then

$$\mu_\varepsilon \rightharpoonup \mu \quad \text{weakly in } L^2(0, T; H^2(\Omega)).$$

3. PROOF OF THEOREM 2.1

This section contains the proof of existence of a solution $(u_\varepsilon, \mu_\varepsilon, \xi_\varepsilon)$ to the nonlocal convective Cahn-Hilliard equation. We subdivide it in different steps. In this section, $\varepsilon > 0$ is fixed.

3.1. Approximation. For every $\lambda > 0$, let $\gamma_\lambda : \mathbb{R} \rightarrow \mathbb{R}$ be the Yosida approximation of γ , having Lipschitz constant $1/\lambda$, and set $\hat{\gamma}_\lambda(s) := \int_0^s \gamma_\lambda(r) dr$ for every $s \in \mathbb{R}$. We consider the approximated problem

$$\partial_t u_\varepsilon^\lambda - \Delta \mu_\varepsilon^\lambda + \operatorname{div}(\beta_\lambda u_\varepsilon^\lambda) = 0 \quad \text{in } (0, T) \times \Omega, \quad (3.1)$$

$$\mu_\varepsilon^\lambda = -\lambda \Delta u_\varepsilon^\lambda + \mathbf{B}_\varepsilon(u_\varepsilon^\lambda) + \gamma_\lambda(u_\varepsilon^\lambda) + \Pi(u_\varepsilon^\lambda) \quad \text{in } (0, T) \times \Omega, \quad (3.2)$$

$$u_\varepsilon^\lambda(0) = u_{0,\varepsilon}^\lambda \quad \text{in } \Omega, \quad (3.3)$$

where $\beta_\lambda := P_\lambda \beta$, $P_\lambda : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is the projection on the closed ball of radius $\frac{1}{\lambda}$, and the initial datum $u_{0,\varepsilon}^\lambda$ satisfies

$$u_{0,\varepsilon}^\lambda \in H^1(\Omega), \quad u_{0,\varepsilon}^\lambda \rightarrow u_{0,\varepsilon} \quad \text{in } L^2(\Omega), \quad (3.4)$$

$$\sup_{\lambda \in (0, \lambda_0)} \left(\lambda \|u_{0,\varepsilon}^\lambda\|_{H^1(\Omega)}^2 + \|\hat{\gamma}(u_{0,\varepsilon}^\lambda)\|_{L^1(\Omega)} + E_\varepsilon(u_{0,\varepsilon}^\lambda) \right) < +\infty \quad (3.5)$$

for a certain $\lambda_0 > 0$ (possibly depending on ε). The existence of an approximating sequence $(u_{0,\varepsilon}^\lambda)_\lambda$ satisfying (3.4)–(3.5) is guaranteed by (2.15): for example, one can check that the classical elliptic regularization given by the unique solution to the problem

$$u_{0,\varepsilon}^\lambda - \lambda \Delta u_{0,\varepsilon}^\lambda = u_{0,\varepsilon}$$

is a possible choice. In this subsection, we show existence of an approximated solution $(u_\varepsilon^\lambda, \mu_\varepsilon^\lambda)$ for every $\lambda > 0$ fixed. The proof strategy relies on the use of a fixed-point argument.

For every $w \in L^2(0, T; H^s(\Omega))$ with $s \in (\frac{3}{2}, 2)$, Lemma 1 ensures that

$$\mathbf{B}_\varepsilon(w) \in L^2(0, T; L^2(\Omega)),$$

so that we can study the auxiliary problem

$$\partial_t v - \Delta \mu_v + \operatorname{div}(\beta_\lambda v) = 0 \quad \text{in } (0, T) \times \Omega, \quad (3.6)$$

$$\mu_v = -\lambda \Delta v + \mathbf{B}_\varepsilon(w) + \gamma_\lambda(v) + \Pi(v) \quad \text{in } (0, T) \times \Omega, \quad (3.7)$$

$$v(0) = u_{0,\varepsilon}^\lambda \quad \text{in } \Omega, \quad (3.8)$$

which can be seen as a local convective Cahn-Hilliard equation with an additional source term in the definition of the chemical potential. It is well-known (see [26] for example) that such problem admits a unique weak solution (v, μ_v) with

$$v \in H^1(0, T; (H^2(\Omega))^*) \cap L^2(0, T; H^2(\Omega)), \quad \text{and} \quad \mu_v \in L^2(0, T; L^2(\Omega)),$$

satisfying (3.6)–(3.8) for example in the sense of distributions. Hence, the map

$$\Gamma_\varepsilon^\lambda : L^2(0, T; H^s(\Omega)) \rightarrow H^1(0, T; (H^2(\Omega))^*) \cap L^2(0, T; H^2(\Omega))$$

associating to every $w \in L^2(0, T; H^s(\Omega))$ the solution v to (3.6)–(3.8) is well-defined. We proceed by showing that $\Gamma_\varepsilon^\lambda$ has also some continuity properties. For $i = 1, 2$ let $w_i \in L^2(0, T; H^s(\Omega))$, and set $v_i := \Gamma_\varepsilon^\lambda(w_i)$. Then taking the difference of the corresponding

equations (3.6) and (3.7) for $i = 1, 2$, we obtain

$$\partial_t(v_1 - v_2) - \Delta(\mu_{v_1} - \mu_{v_2}) + \operatorname{div}(\beta_\lambda(v_1 - v_2)) = 0 \quad \text{in } (0, T) \times \Omega, \quad (3.9)$$

$$\begin{aligned} \mu_{v_1} - \mu_{v_2} = & -\lambda\Delta(v_1 - v_2) + \mathbf{B}_\varepsilon(w_1 - w_2) \\ & + \gamma_\lambda(v_1) - \gamma_\lambda(v_2) + \Pi(v_1) - \Pi(v_2) \end{aligned} \quad \text{in } (0, T) \times \Omega, \quad (3.10)$$

$$v_1(0) - v_2(0) = 0 \quad \text{in } \Omega. \quad (3.11)$$

Noting that $(v_1 - v_2)_\Omega = 0$ by integrating (3.9), testing (3.9) by $(-\Delta)^{-1}(v_1 - v_2)$, equation (3.10) by $v_1 - v_2$, and taking the difference, estimate (2.9) and assumption **H4** yield

$$\begin{aligned} & \|v_1 - v_2\|_{C^0([0,t];(H^1(\Omega))^* \cap L^2(0,T;H^1(\Omega)))}^2 \\ & \leq C_{\varepsilon,\lambda} \left\{ \int_0^t \|w_1(s, \cdot) - w_2(s, \cdot)\|_{H^1(\Omega)}^2 ds + \int_0^t \|\gamma_\lambda(v_1(s, \cdot)) - \gamma_\lambda(v_2(s, \cdot))\|_{L^2(\Omega)}^2 ds \right. \\ & \left. + \int_0^t \|\Pi(v_1(s, \cdot)) - \Pi(v_2(s, \cdot))\|_{L^2(\Omega)}^2 ds + \int_0^t \|v_1(s, \cdot) - v_2(s, \cdot)\|_{(H^1(\Omega))^*}^2 ds \right\}, \end{aligned}$$

for every $t \in [0, T]$.

Testing (3.9) by $v_1 - v_2$, equation (3.10) by $-\Delta(v_1 - v_2)$, taking the difference, and using Lemma 1, a similar argument yields

$$\begin{aligned} & \|v_1 - v_2\|_{C^0([0,t];L^2(\Omega))}^2 + \|\Delta(v_1 - v_2)\|_{L^2(0,t;L^2(\Omega))}^2 \\ & \leq C_{\varepsilon,\lambda} \left\{ \int_0^t \|w_1(s, \cdot) - w_2(s, \cdot)\|_{H^s(\Omega)}^2 ds \right. \\ & \quad + \int_0^t \|\gamma_\lambda(v_1(s, \cdot)) - \gamma_\lambda(v_2(s, \cdot)) + \Pi(v_1(s, \cdot)) - \Pi(v_2(s, \cdot))\|_{L^2(\Omega)}^2 ds \\ & \quad \left. + \int_0^t \|v_1(s, \cdot) - v_2(s, \cdot)\|_{H^1(\Omega)}^2 ds \right\}. \end{aligned}$$

Hence, summing the two inequalities, using the Lipschitz-continuity of γ_λ and Π , together with Gronwall's Lemma, we deduce that there exists a positive constant $C_{\varepsilon,\lambda}$ such that

$$\|v_1 - v_2\|_{C^0([0,T];L^2(\Omega)) \cap L^2(0,T;H^2(\Omega))} \leq C_{\varepsilon,\lambda} \|w_1 - w_2\|_{L^2(0,T;H^s(\Omega))}. \quad (3.12)$$

In particular, $\Gamma_\varepsilon^\lambda$ is continuous from $L^2(0, T; H^s(\Omega))$ to $L^2(0, T; H^s(\Omega))$.

Fix $T_0 > 0$. By repeating the argument leading to (3.12) we deduce the estimate

$$\|v_1 - v_2\|_{C^0([0,T_0];L^2(\Omega)) \cap L^2(0,T_0;H^2(\Omega))} \leq C_{\varepsilon,\lambda} \|w_1 - w_2\|_{L^2(0,T_0;H^s(\Omega))},$$

for every $w \in L^2(0, T_0; H^s(\Omega))$, and $v = \Gamma_\varepsilon^\lambda(w)$. Now, since $s \in (\frac{3}{2}, 2)$, if $\vartheta \in (0, 1)$ is such that

$$s = (1 - \vartheta) \cdot 0 + \vartheta \cdot 2, \quad \text{i.e.} \quad \vartheta := \frac{s}{2} \in \left(\frac{3}{4}, 1\right),$$

by interpolation we get that

$$\|v_1(t, \cdot) - v_2(t, \cdot)\|_{H^s(\Omega)} \leq \|v_1(t, \cdot) - v_2(t, \cdot)\|_{H^2(\Omega)}^{s/2} \|v_1(t, \cdot) - v_2(t, \cdot)\|_{L^2(\Omega)}^{1-s/2} \quad \text{a.e. in } (0, T),$$

which in turn yields that

$$\begin{aligned} \|v_1 - v_2\|_{L^{4/s}(0, T_0; H^s(\Omega))} &\leq \|v_1 - v_2\|_{L^2(0, T_0; H^2(\Omega))}^{s/2} \|v_1 - v_2\|_{L^\infty(0, T_0; L^2(\Omega))}^{1-s/2} \\ &\leq \frac{s}{2} \|v_1 - v_2\|_{L^2(0, T_0; H^2(\Omega))} + \left(1 - \frac{s}{2}\right) \|v_1 - v_2\|_{L^\infty(0, T_0; L^2(\Omega))} \\ &\leq C_s \|v_1 - v_2\|_{C^0([0, T_0]; L^2(\Omega)) \cap L^2(0, T_0; H^2(\Omega))}. \end{aligned}$$

Consequently, we have that

$$\|v_1 - v_2\|_{L^{4/s}(0, T_0; H^s(\Omega))} \leq C_{\varepsilon, \lambda, s} \|w_1 - w_2\|_{L^2(0, T_0; H^s(\Omega))},$$

where $\frac{4}{s} > 2$ since $s < 2$. Hence, we infer that

$$\|v_1 - v_2\|_{L^2(0, T_0; H^s(\Omega))} \leq T_0^{\frac{1}{2} - \frac{s}{4}} \|v_1 - v_2\|_{L^{4/s}(0, T_0; H^s(\Omega))},$$

and we can choose T_0 sufficiently small such that $T_0^{\frac{1}{2} - \frac{s}{4}} C_{\varepsilon, \lambda, s} < 1$. Thus,

$$\|v_1 - v_2\|_{L^2(0, T_0; H^s(\Omega))} \leq T_0^{\frac{1}{2} - \frac{s}{4}} C_{\varepsilon, \lambda, s} \|w_1 - w_2\|_{L^2(0, T_0; H^s(\Omega))}.$$

Banach fixed point theorem ensures the existence of a unique weak solution $(u_\varepsilon^\lambda, \mu_\varepsilon^\lambda)$ to the approximated problem (3.1)-(3.3) in $(0, T_0) \times \Omega$, with

$$u_\varepsilon^\lambda \in H^1(0, T_0; (H^2(\Omega))^*) \cap L^2(0, T_0; H^2(\Omega)), \quad \text{and} \quad \mu_\varepsilon^\lambda \in L^2(0, T_0; L^2(\Omega)).$$

Since the choice of T_0 is independent of the initial time, using a standard patching argument in time allows to extend the solution to the whole interval $(0, T)$.

3.2. Uniform estimates. In this subsection we show that there exists $\varepsilon_0 > 0$ independent of λ , and such that for $\varepsilon < \varepsilon_0$ the approximated solutions fulfill some uniform estimates independently of λ and ε . In what follows we will always assume that $\lambda \in [0, 1]$.

Step 1. We start by fixing $t \in [0, T]$, testing (3.1) with μ_ε^λ , (3.2) with $\partial_t u_\varepsilon^\lambda$, taking the difference, and integrating the resulting equation on $(0, t)$. We obtain

$$\begin{aligned} &\int_0^t \int_\Omega |\nabla \mu_\varepsilon^\lambda(s, x)|^2 dx ds + \frac{\lambda}{2} \int_\Omega |\nabla u_\varepsilon^\lambda(t, x)|^2 dx + E_\varepsilon(u_\varepsilon^\lambda(t, \cdot)) + \int_\Omega (\hat{\gamma}_\lambda + \hat{\Pi})(u_\varepsilon^\lambda(t, x)) dx \\ &\leq \int_0^t \int_\Omega \beta_\lambda(t, x) u_\varepsilon^\lambda(t, x) \cdot \nabla \mu_\varepsilon^\lambda(t, x) dx dt + \frac{\lambda}{2} \int_\Omega |\nabla u_{0, \varepsilon}^\lambda(x)|^2 dx \\ &\quad + E_\varepsilon(u_{0, \varepsilon}^\lambda) + \int_\Omega (\hat{\gamma}_\lambda + \hat{\Pi})(u_{0, \varepsilon}^\lambda(x)) dx. \end{aligned}$$

Note that such estimate is formal, since the regularity of the solutions would not allow us to perform the above computation. However, the estimate can be proved in a classical way, by further approximating the problem at ε fixed. For example, one can apply the regularizing operator $(I - \delta \Delta)^{-1}$ to the equations (3.1)-(3.2) for every $\delta > 0$, noting that $-\Delta$ and \mathbf{B}_ε commute, obtaining thus the required regularity for the further approximated solutions for every $\delta > 0$. For such regularized solutions $u_{\varepsilon, \delta}^\lambda$ the estimate is then rigorous, and one concludes by passing to the limit as $\delta \searrow 0$. From the fact that

$$\int_\Omega \hat{\gamma}_\lambda(u_{0, \varepsilon}^\lambda(x)) dx \leq \int_\Omega \hat{\gamma}(u_{0, \varepsilon}^\lambda(x)) dx \quad \text{for every } \lambda > 0,$$

using assumption **H3**, the uniform bound (3.5) and as well as Young's inequality, we get

$$\begin{aligned} & \int_0^t \int_{\Omega} |\nabla \mu_{\varepsilon}^{\lambda}(s, x)|^2 dx ds + E_{\varepsilon}(u_{\varepsilon}^{\lambda}(t, \cdot)) + \frac{\lambda}{2} \int_{\Omega} |\nabla u_{\varepsilon}^{\lambda}(t, x)|^2 dx \\ & \leq C_{\varepsilon} + \frac{1}{2} \int_0^T \int_{\Omega} |\nabla \mu_{\varepsilon}^{\lambda}(t, x)|^2 dx dt + \frac{1}{2} \int_0^T \int_{\Omega} |\beta_{\lambda}(t, x) u_{\varepsilon}^{\lambda}(t, x)|^2 dx dt \end{aligned} \quad (3.13)$$

for every $t \in [0, T]$.

We point out that, due to the periodic boundary conditions, and the fact that Ω is the d -dimensional torus, we formally have

$$\int_{\Omega} \nabla \mathbf{B}_{\varepsilon}(u_{\varepsilon}^{\lambda}(s, x)) \cdot \nabla u_{\varepsilon}^{\lambda}(s, x) dx = \int_{\Omega} \int_{\Omega} K_{\varepsilon}(x, y) |\nabla u_{\varepsilon}^{\lambda}(s, x) - \nabla u_{\varepsilon}^{\lambda}(s, y)|^2 dx dy$$

for almost every $s \in [0, T]$. Testing (3.1) with $u_{\varepsilon}^{\lambda}$ and (3.2) with $-\Delta u_{\varepsilon}^{\lambda}$, by considering the difference between the two resulting equation and by integrating in the time interval $(0, t)$, from **H3** we deduce the estimate

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} |u_{\varepsilon}^{\lambda}(t, x)|^2 dx + \lambda \int_0^t \int_{\Omega} |\Delta u_{\varepsilon}^{\lambda}(s, x)|^2 dx ds + \int_0^t \int_{\Omega} \gamma'_{\lambda}(u_{\varepsilon}^{\lambda}(s, x)) |\nabla u_{\varepsilon}^{\lambda}(s, x)|^2 dx ds \\ & + \int_0^t \int_{\Omega} \int_{\Omega} K_{\varepsilon}(x, y) |\nabla u_{\varepsilon}^{\lambda}(s, x) - \nabla u_{\varepsilon}^{\lambda}(s, y)|^2 dx dy ds \\ & \leq \frac{1}{2} \int_{\Omega} |u_{0, \varepsilon}^{\lambda}(x)|^2 dx + \frac{1}{2} \int_0^t \int_{\Omega} |\beta_{\lambda}(s, x) u_{\varepsilon}^{\lambda}(s, x)|^2 dx ds + \left(C_{\Pi} + \frac{1}{2}\right) \int_0^t \int_{\Omega} |\nabla u_{\varepsilon}^{\lambda}(s, x)|^2 dx ds \\ & \leq \frac{1}{2} \int_{\Omega} |u_{0, \varepsilon}^{\lambda}(x)|^2 dx + \frac{1}{2} \int_0^t \int_{\Omega} |\beta_{\lambda}(s, x) u_{\varepsilon}^{\lambda}(s, x)|^2 dx ds \\ & + \frac{1}{4} \int_0^t \int_{\Omega} \int_{\Omega} K_{\varepsilon}(x, y) |\nabla u_{\varepsilon}^{\lambda}(s, x) - \nabla u_{\varepsilon}^{\lambda}(s, y)|^2 dx dy ds + C \int_0^t \|u_{\varepsilon}^{\lambda}(s, \cdot)\|_{L^2(\Omega)}^2 ds, \end{aligned}$$

where the latter inequality holds for ε smaller than a suitable constant ε_0 in view of Lemma 4. Noticing that the third term in the left-hand side of the above estimate is positive owing to the monotonicity of γ_{λ} , by [48, Theorem 1.1] we infer the bound

$$\begin{aligned} & \|u_{\varepsilon}^{\lambda}(t, \cdot)\|_{L^2(\Omega)}^2 + \|u_{\varepsilon}^{\lambda}\|_{L^2(0, t; H^1(\Omega))}^2 + \int_0^t \int_{\Omega} \int_{\Omega} K_{\varepsilon}(x, y) |\nabla u_{\varepsilon}^{\lambda}(s, x) - \nabla u_{\varepsilon}^{\lambda}(s, y)|^2 dx dy ds \\ & \leq C \left(\int_{\Omega} |u_{0, \varepsilon}^{\lambda}(x)|^2 dx + \int_0^t \int_{\Omega} |\beta_{\lambda}(s, x) u_{\varepsilon}^{\lambda}(s, x)|^2 dx ds + \int_0^t \|u_{\varepsilon}^{\lambda}(s, \cdot)\|_{L^2(\Omega)}^2 ds \right). \end{aligned} \quad (3.14)$$

By the continuous embedding of $H^1(\Omega)$ into $L^6(\Omega)$, we deduce the estimate

$$\int_0^t \int_{\Omega} |\beta_{\lambda}(s, x) u_{\varepsilon}^{\lambda}(s, x)|^2 dx ds \leq \int_0^t \|\beta_{\lambda}(s, \cdot)\|_{L^{\infty}(\Omega)}^2 \|u_{\varepsilon}^{\lambda}(s, \cdot)\|_{L^2(\Omega)}^2 ds. \quad (3.15)$$

Thus, summing (3.13), (3.14), and (3.15), recalling **H4** we obtain

$$\begin{aligned} & \|u_{\varepsilon}^{\lambda}(t, \cdot)\|_{L^2(\Omega)}^2 + \|u_{\varepsilon}^{\lambda}\|_{L^2(0, t; H^1(\Omega))}^2 + \int_0^t \int_{\Omega} |\nabla \mu_{\varepsilon}^{\lambda}(s, x)|^2 dx ds \\ & + E_{\varepsilon}(u_{\varepsilon}^{\lambda}(t, \cdot)) + \int_0^t \int_{\Omega} \int_{\Omega} K_{\varepsilon}(x, y) |\nabla u_{\varepsilon}^{\lambda}(s, x) - \nabla u_{\varepsilon}^{\lambda}(s, y)|^2 dx dy ds \\ & \leq C_{\varepsilon} + C \|u_{\varepsilon}^{\lambda}\|_{L^2(0, t; L^2(\Omega))}^2 + C \|u_{\varepsilon}^{\lambda}\|_{L^2(0, t; H^1(\Omega))}^2 \|\beta\|_{L^2(0, T; L^{\infty}(\Omega; \mathbb{R}^d))}^2. \end{aligned} \quad (3.16)$$

Recalling assumption **H4** and applying Gronwall's lemma, from the arbitrariness of $t \in [0, T]$ we deduce that there exists a constant C_ε such that

$$\|\nabla \mu_\varepsilon^\lambda\|_{L^2(0,T;L^2(\Omega))} \leq C_\varepsilon, \quad (3.17)$$

$$\|u_\varepsilon^\lambda\|_{L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))} + \lambda^{1/2} \|u_\varepsilon^\lambda\|_{L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))} \leq C_\varepsilon, \quad (3.18)$$

$$\|E_\varepsilon(u_\varepsilon^\lambda)\|_{L^\infty(0,T)} + \left\| \int_\Omega \int_\Omega K_\varepsilon(x,y) |\nabla u_\varepsilon^\lambda(\cdot, x) - \nabla u_\varepsilon^\lambda(\cdot, y)|^2 dx dy \right\|_{L^1(0,T)} \leq C_\varepsilon. \quad (3.19)$$

Testing equation (3.1) with a function $\varphi \in L^2(0, T; H^1(\Omega))$, integrating in time, and using (3.17)–(3.19) gives

$$\|\partial_t u_\varepsilon^\lambda\|_{L^2(0,T;(H^1(\Omega))^*)} \leq C_\varepsilon. \quad (3.20)$$

Step 2. In order to obtain an $L^2(0, T; H^1(\Omega))$ -estimate on the chemical potential μ_ε^λ , we need a bound on the $L^2(0, T)$ -norm of the spatial mean of μ_ε^λ . Thanks to the symmetry of the kernel K , the mean of the convolution terms vanishes, i.e.

$$(\mathbf{B}_\varepsilon(u_\varepsilon^\lambda))_\Omega = 0.$$

Since also $(\Delta u_\varepsilon^\lambda)_\Omega = 0$, owing to (3.18) and the Lipschitz continuity of Π , we get

$$(\mu_\varepsilon^\lambda)_\Omega = (\gamma_\lambda(u_\varepsilon^\lambda) + \Pi(u_\varepsilon^\lambda))_\Omega \leq C_\varepsilon + \frac{1}{|\Omega|} \|\gamma_\lambda(u_\varepsilon^\lambda)\|_{L^1(\Omega)}. \quad (3.21)$$

Hence $\{(\mu_\varepsilon^\lambda)_\Omega\}_\varepsilon$ is uniformly bounded in $L^2(0, T)$ if $\{\gamma_\lambda(u_\varepsilon^\lambda)\}_\varepsilon$ is uniformly bounded in $L^2(0, T; L^1(\Omega))$. We test (3.1) by $(-\Delta)^{-1}(u_\varepsilon^\lambda - (u_{0,\varepsilon}^\lambda)_\Omega)$ and (3.2) by $u_\varepsilon - (u_{0,\varepsilon}^\lambda)_\Omega$, obtaining

$$\begin{aligned} & \underbrace{\langle \partial_t u_\varepsilon^\lambda(t), (-\Delta)^{-1}(u_\varepsilon^\lambda(t, \cdot) - (u_{0,\varepsilon}^\lambda)_\Omega) \rangle_{(H^1(\Omega))^*, H^1(\Omega)} + \lambda \int_\Omega |\nabla u_\varepsilon^\lambda(t, x)|^2 dx}_{=: I_1} \\ & + \underbrace{\int_\Omega \mathbf{B}_\varepsilon(u_\varepsilon^\lambda)(t, x) (u_\varepsilon^\lambda(t, x) - (u_{0,\varepsilon}^\lambda)_\Omega) dx}_{=: I_2} \\ & + \underbrace{\int_\Omega (\gamma_\lambda + \Pi)(u_\varepsilon^\lambda(t, x)) (u_\varepsilon^\lambda(t, x) - (u_{0,\varepsilon}^\lambda)_\Omega) dx}_{=: I_3} \\ & - \underbrace{\int_\Omega \beta_\lambda(t, x) u_\varepsilon^\lambda(t, x) \cdot \nabla (-\Delta)^{-1}(u_\varepsilon^\lambda(t, x) - (u_{0,\varepsilon}^\lambda)_\Omega) dx}_{=: I_4} = 0. \end{aligned}$$

We proceed by estimating each integral in the left-hand side of the above equation separately. It is readily seen that I_1 is uniformly bounded in $L^2(0, T)$ due to (3.18), (3.20) and (3.5). Regarding I_2 , since $(\mathbf{B}_\varepsilon(u_\varepsilon^\lambda))_\Omega = 0$ we have that

$$I_2 = E_\varepsilon(u_\varepsilon^\lambda),$$

which is clearly bounded in $L^2(0, T)$ by (3.19).

To estimate I_3 we observe that in view of (2.15) and (3.4) there exist constants $M_1, M_2 > 0$ depending only on the position of $(u_{0,\varepsilon}^\lambda)_\Omega$ in $\text{Int } D(\gamma)$, such that

$$\gamma_\lambda(u_\varepsilon^\lambda)(u_\varepsilon^\lambda - (u_{0,\varepsilon}^\lambda)_\Omega) \geq M_1 |\gamma_\lambda(u_\varepsilon^\lambda)| - M_2,$$

cf. for example [21, p. 984] and the references within, while

$$\int_{\Omega} \Pi(u_{\varepsilon}(x))(u_{\varepsilon}(x) - (u_{0,\varepsilon}^{\lambda})_{\Omega}) dx$$

is bounded in $L^{\infty}(0, T)$ thanks to (3.18).

Eventually, I_4 can be estimated as follows

$$\|\beta_{\lambda} u_{\varepsilon}^{\lambda} \cdot \nabla(-\Delta)^{-1}(u_{\varepsilon}^{\lambda} - (u_{0,\varepsilon}^{\lambda})_{\Omega})\|_{L^2(0,T;L^1(\Omega))}^2 \leq \int_0^T \|\beta_{\lambda} u_{\varepsilon}^{\lambda}\|_{L^2(\Omega)}^2 \|(u_{\varepsilon}^{\lambda} - (u_{0,\varepsilon}^{\lambda})_{\Omega})\|_{(H^1(\Omega))^*}^2 dt,$$

where the right-hand side is bounded due to **H4** and (3.18).

Combining this information, we conclude by difference that $\{\gamma_{\lambda}(u_{\varepsilon}^{\lambda})\}$ is uniformly bounded in $L^2(0, T; L^1(\Omega))$. Thus, from (3.17) and (3.21) we infer that

$$\|\mu_{\varepsilon}^{\lambda}\|_{L^2(0,T;H^1(\Omega))} \leq C_{\varepsilon}. \quad (3.22)$$

Step 3. We proceed by proving that $\{\gamma_{\lambda}(u_{\varepsilon}^{\lambda})\}$ is uniformly bounded in $L^2(0, T; L^2(\Omega))$.

We test (3.2) with $\gamma_{\lambda}(u_{\varepsilon}^{\lambda})$. This gives

$$\begin{aligned} & \int_0^T \int_{\Omega} |\gamma_{\lambda}(u_{\varepsilon}^{\lambda}(t, x))|^2 dx dt + \lambda \int_0^T \int_{\Omega} |\nabla u_{\varepsilon}^{\lambda}(t, x)|^2 \gamma_{\lambda}'(u_{\varepsilon}^{\lambda}(t, x)) dx dt \\ & \quad + \int_0^T \int_{\Omega} \mathbf{B}_{\varepsilon}(u_{\varepsilon}^{\lambda})(t, x) \gamma_{\lambda}(u_{\varepsilon}^{\lambda}(t, x)) dx dt \\ & = \int_0^T \int_{\Omega} (\mu_{\varepsilon}^{\lambda}(t, x) - \Pi(u_{\varepsilon}^{\lambda}(t, x))) \gamma_{\lambda}(u_{\varepsilon}^{\lambda}(t, x)) dx dt. \end{aligned}$$

We observe that the second term on the left-hand side is nonnegative owing to the monotonicity of γ_{λ} . Analogously, the third term on the left-hand side can be rewritten as

$$\int_0^T \int_{\Omega} \int_{\Omega} K_{\varepsilon}(x, y) (u_{\varepsilon}^{\lambda}(t, x) - u_{\varepsilon}^{\lambda}(t, y)) (\gamma_{\lambda}(u_{\varepsilon}^{\lambda}(t, x)) - \gamma_{\lambda}(u_{\varepsilon}^{\lambda}(t, y))) dx dy dt,$$

which is also nonnegative due to the monotonicity of γ_{λ} . Applying Young's inequality we deduce the bound

$$\begin{aligned} & \int_0^T \int_{\Omega} (\mu_{\varepsilon}^{\lambda}(t, x) - \Pi(u_{\varepsilon}^{\lambda}(t, x))) \gamma_{\lambda}(u_{\varepsilon}^{\lambda}(t, x)) dx dt \\ & \leq \int_0^T \int_{\Omega} \left[\frac{|\mu_{\varepsilon}^{\lambda}(t, x) - \Pi(u_{\varepsilon}^{\lambda}(t, x))|^2}{2} + \frac{|\gamma_{\lambda}(u_{\varepsilon}^{\lambda}(t, x))|^2}{2} \right] dx dt, \end{aligned}$$

which, together with **H3**, (3.18) and (3.22), implies the following estimate

$$\|\gamma_{\lambda}(u_{\varepsilon}^{\lambda})\|_{L^2(0,T;L^2(\Omega))} \leq C_{\varepsilon}. \quad (3.23)$$

3.3. Passage to the limit as $\lambda \searrow 0$. We perform here the passage to the limit as $\lambda \searrow 0$, with $0 < \varepsilon < \varepsilon_0$ still fixed. In view of the uniform bounds identified in Section 3.2 and the Aubin-Lions lemma, up to the extraction of (not relabeled) subsequences we have the

following convergences:

$$u_\varepsilon^\lambda \rightarrow u_\varepsilon \quad \text{strongly in } L^2(0, T; L^2(\Omega)) \cap C^0([0, T]; (H^1(\Omega))^*), \quad (3.24)$$

$$u_\varepsilon^\lambda \rightharpoonup u_\varepsilon \quad \text{weakly* in } L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \quad (3.25)$$

$$\lambda u_\varepsilon^\lambda \rightarrow 0 \quad \text{strongly in } L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)), \quad (3.26)$$

$$\partial_t u_\varepsilon^\lambda \rightharpoonup \partial_t u_\varepsilon \quad \text{weakly* in } L^2(0, T; (H^1(\Omega))^*), \quad (3.27)$$

$$\mu_\varepsilon^\lambda \rightharpoonup \mu_\varepsilon \quad \text{weakly in } L^2(0, T; H^1(\Omega)), \quad (3.28)$$

$$\gamma_\lambda(u_\varepsilon^\lambda) \rightharpoonup \xi_\varepsilon \quad \text{weakly in } L^2(0, T; L^2(\Omega)), \quad (3.29)$$

for some

$$u_\varepsilon \in H^1(0, T; (H^1(\Omega))^*) \cap L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \\ \mu_\varepsilon \in L^2(0, T; H^1(\Omega)), \quad \xi_\varepsilon \in L^2(0, T; L^2(\Omega)).$$

The strong convergence (3.24), the weak convergence (3.29) and the strong-weak closure of the maximal monotone graph γ readily implies that $\xi_\varepsilon \in \gamma(u_\varepsilon)$ almost everywhere in $(0, T) \times \Omega$. The Lipschitz continuity of Π yields also

$$\Pi(u_\varepsilon^\lambda) \rightarrow \Pi(u_\varepsilon) \quad \text{strongly in } L^2(0, T; L^2(\Omega)). \quad (3.30)$$

Furthermore, for every $\varphi \in L^2(0, T; H^2(\Omega))$ by the triangle inequality we have that

$$\left| \int_0^T \int_\Omega \beta_\lambda(t, x) u_\varepsilon^\lambda(t, x) \cdot \nabla \varphi(t, x) \, dx \, dt - \int_0^T \int_\Omega \beta(t, x) u_\varepsilon(t, x) \cdot \nabla \varphi(t, x) \, dx \, dt \right| \\ \leq \int_0^T \int_\Omega |\beta_\lambda(t, x) - \beta(t, x)| |u_\varepsilon^\lambda(t, x)| |\nabla \varphi(t, x)| \, dx \, dt \\ + \int_0^T \int_\Omega \beta(t, x) (u_\varepsilon^\lambda(t, x) - u_\varepsilon(t, x)) \cdot \nabla \varphi(t, x) \, dx \, dt.$$

By the Hölder inequality, the fact that $\beta_\lambda \rightarrow \beta$ strongly in $L^2(0, T; L^3(\Omega))$ and the embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, for the first term on the right-hand side we have

$$\int_0^T \int_\Omega |\beta_\lambda(t, x) - \beta(t, x)| |u_\varepsilon^\lambda(t, x)| |\nabla \varphi(t, x)| \, dx \, dt \\ \leq \|u_\varepsilon^\lambda\|_{L^\infty(0, T; L^2(\Omega))} \|\varphi\|_{L^2(0, T; L^6(\Omega))} \|\beta_\lambda - \beta\|_{L^2(0, T; L^3(\Omega))} \rightarrow 0.$$

For the second term on the right-hand side note that $\beta \cdot \nabla \varphi \in L^1(0, T; L^2(\Omega))$ thanks to assumption **H4**, the fact that $\varphi \in L^2(0, T; H^2(\Omega))$ and the inclusion $H^1(\Omega) \hookrightarrow L^6(\Omega)$, so that from (3.25)

$$\int_0^T \int_\Omega \beta(t, x) (u_\varepsilon^\lambda(t, x) - u_\varepsilon(t, x)) \cdot \nabla \varphi(t, x) \, dx \, dt \rightarrow 0.$$

Hence, we conclude that

$$-\operatorname{div} \beta_\lambda u_\varepsilon^\lambda \rightharpoonup -\operatorname{div} \beta u_\varepsilon \quad \text{weakly* in } L^2(0, T; (H^2(\Omega))^*).$$

From (3.25) and the fact that $\mathbf{B}_\varepsilon \in \mathcal{L}(H^1(\Omega), (H^1(\Omega))^*)$, it is readily seen that

$$\mathbf{B}_\varepsilon(u_\varepsilon^\lambda) \rightharpoonup \mathbf{B}_\varepsilon(u_\varepsilon) \quad \text{weakly* in } L^2(0, T; (H^1(\Omega))^*).$$

By (3.28)–(3.29) and (3.30), by comparison it follows that the sequence $(\mathbf{B}_\varepsilon(u_\varepsilon^\lambda))_\lambda$ is bounded in $L^2(0, T; L^2(\Omega))$, hence we also conclude that $\mathbf{B}_\varepsilon(u_\varepsilon) \in L^2(0, T; L^2(\Omega))$

$$\mathbf{B}_\varepsilon(u_\varepsilon^\lambda) \rightharpoonup \mathbf{B}_\varepsilon(u_\varepsilon) \quad \text{weakly in } L^2(0, T; L^2(\Omega)).$$

Now, passing to the limit in (3.1)–(3.2) as $\lambda \searrow 0$, we obtain, in the sense of distributions,

$$\partial_t u_\varepsilon - \Delta \mu_\varepsilon = -\operatorname{div}(\beta u_\varepsilon)$$

and

$$\mu_\varepsilon = \mathbf{B}_\varepsilon(u_\varepsilon) + \xi_\varepsilon + \Pi(u_\varepsilon).$$

Finally, the strong convergence (3.24) implies also that $u_\varepsilon(0) = u_{0,\varepsilon}$, so that $(u_\varepsilon, \mu_\varepsilon, \xi_\varepsilon)$ is a solution to the nonlocal Cahn-Hilliard equation (1.7) according to Definition 1. This completes the proof of the first assertion of Theorem 2.1.

3.4. Continuous dependence. Let $(\beta^1, u_{0,\varepsilon}^1)$ and $(\beta^2, u_{0,\varepsilon}^2)$ satisfy **H4** and (2.15), with $(u_{0,\varepsilon}^1)_\Omega = (u_{0,\varepsilon}^2)_\Omega$, and let $(u_\varepsilon^1, \mu_\varepsilon^1, \xi_\varepsilon^1)$ and $(u_\varepsilon^2, \mu_\varepsilon^2, \xi_\varepsilon^2)$ be any corresponding solutions to the nonlocal equation (2.1)–(2.3). Then we have

$$\begin{aligned} \partial_t(u_\varepsilon^1 - u_\varepsilon^2) - \Delta(\mu_\varepsilon^1 - \mu_\varepsilon^2) &= -\operatorname{div}(\beta^1 u_\varepsilon^1 - \beta^2 u_\varepsilon^2), \\ \mu_\varepsilon^1 - \mu_\varepsilon^2 &= \mathbf{B}_\varepsilon(u_\varepsilon^1 - u_\varepsilon^2) + \xi_\varepsilon^1 - \xi_\varepsilon^2 + \Pi(u_\varepsilon^1) - \Pi(u_\varepsilon^2). \end{aligned}$$

Noting that $(u_\varepsilon^1 - u_\varepsilon^2)_\Omega = 0$ by the assumption on the initial data, we test the first equation by $(-\Delta)^{-1}(u_\varepsilon^1 - u_\varepsilon^2)$, the second by $u_\varepsilon^1 - u_\varepsilon^2$, and take the difference: by performing classical computations we get

$$\begin{aligned} &\frac{1}{2} \|(u_\varepsilon^1 - u_\varepsilon^2)(t)\|_{(H^1(\Omega))^*}^2 + \int_0^t E_\varepsilon(u_\varepsilon^1 - u_\varepsilon^2)(s) \, ds + \int_0^t \int_\Omega (\xi_\varepsilon^1 - \xi_\varepsilon^2)(s, x)(u_\varepsilon^1 - u_\varepsilon^2)(s, x) \, dx \, ds \\ &= \frac{1}{2} \|(u_{0,\varepsilon}^1 - u_{0,\varepsilon}^2)\|_{(H^1(\Omega))^*}^2 - \int_0^t \int_\Omega (\Pi(u_\varepsilon^1) - \Pi(u_\varepsilon^2))(s, x)(u_\varepsilon^1 - u_\varepsilon^2)(s, x) \, dx \, ds \\ &\quad + \int_0^t \int_\Omega \beta^1(s, x)(u_\varepsilon^1 - u_\varepsilon^2)(s, x) \cdot \nabla(-\Delta)^{-1}(u_\varepsilon^1 - u_\varepsilon^2)(s, x) \, dx \, ds \\ &\quad + \int_0^t \int_\Omega (\beta^1 - \beta^2)(s, x)u_\varepsilon^2(s, x) \cdot \nabla(-\Delta)^{-1}(u_\varepsilon^1 - u_\varepsilon^2)(s, x) \, dx \, ds. \end{aligned}$$

By the Lipschitz-continuity of Π we have

$$\int_0^t \int_\Omega (\Pi(u_\varepsilon^1) - \Pi(u_\varepsilon^2))(s, x)(u_\varepsilon^1 - u_\varepsilon^2)(s, x) \, dx \, ds \leq C \|u_\varepsilon^1 - u_\varepsilon^2\|_{L^2(0,t;L^2(\Omega))}^2,$$

while the Hölder and Young inequalities yield

$$\begin{aligned} &\int_0^t \int_\Omega \beta^1(s, x)(u_\varepsilon^1 - u_\varepsilon^2)(s, x) \cdot \nabla(-\Delta)^{-1}(u_\varepsilon^1 - u_\varepsilon^2)(s, x) \, dx \, ds \\ &\leq \|u_\varepsilon^1 - u_\varepsilon^2\|_{L^2(0,t;L^2(\Omega))}^2 + \int_0^t \|\beta^1(s, x)\|_{L^\infty(\Omega)}^2 \|(u_\varepsilon^1 - u_\varepsilon^2)(s)\|_{(H^1(\Omega))^*}^2 \, ds \end{aligned}$$

and

$$\begin{aligned} &\int_0^t \int_\Omega (\beta^1 - \beta^2)(s, x)u_\varepsilon^2(s, x) \cdot \nabla(-\Delta)^{-1}(u_\varepsilon^1 - u_\varepsilon^2)(s, x) \, dx \, ds \\ &\leq \|\beta^1 - \beta^2\|_{L^2(0,T;L^3(\Omega))}^2 + \int_0^t \|u_\varepsilon^2(s, \cdot)\|_{L^6(\Omega)}^2 \|(u_\varepsilon^1 - u_\varepsilon^2)(s)\|_{(H^1(\Omega))^*}^2 \, ds. \end{aligned}$$

The continuous-dependence property stated in Theorem 2.1 follows from Lemma 4 and the Gronwall lemma.

4. PROOF OF THEOREM 2.2

In this section we perform the limit as $\varepsilon \searrow 0$.

First of all, going back to the arguments performed in the previous section to obtain estimates (3.17)–(3.23), we observe that assumptions (2.16)–(2.17) guarantee that the sequence of constants $(C_\varepsilon)_\varepsilon$ is uniformly bounded for every $\varepsilon \in (0, \varepsilon_0)$. Consequently, we deduce that there exists $C > 0$ such that

$$\begin{aligned} \|u_\varepsilon\|_{H^1(0,T;(H^1(\Omega))^*) \cap L^2(0,T;H^1(\Omega))} &\leq C, \\ \|E_\varepsilon(u_\varepsilon)\|_{L^\infty(0,T)} + \left\| \int_\Omega \int_\Omega K_\varepsilon(x,y) |\nabla u_\varepsilon(x) - \nabla u_\varepsilon(y)|^2 dx dy \right\|_{L^1(0,T)} &\leq C, \quad (4.1) \\ \|\mu_\varepsilon\|_{L^2(0,T;H^1(\Omega))} &\leq C, \\ \|\xi_\varepsilon\|_{L^2(0,T;L^2(\Omega))} &\leq C. \end{aligned}$$

Hence, by comparison

$$\|\mathbf{B}_\varepsilon(u_\varepsilon)\|_{L^2(0,T;L^2(\Omega))} \leq C.$$

By Aubin-Lions compactness results we infer that, up to the extraction of (not relabeled) subsequences,

$$u_\varepsilon \rightarrow u \quad \text{strongly in } L^2(0,T;L^2(\Omega)) \cap C^0([0,T];(H^1(\Omega))^*), \quad (4.2)$$

$$u_\varepsilon \rightharpoonup u \quad \text{weakly* in } L^\infty(0,T;L^2(\Omega)), \quad (4.3)$$

$$\partial_t u_\varepsilon \rightharpoonup \partial_t u \quad \text{weakly* in } L^2(0,T;(H^1(\Omega))^*), \quad (4.4)$$

$$\mathbf{B}_\varepsilon(u_\varepsilon) \rightharpoonup \eta \quad \text{weakly in } L^2(0,T;L^2(\Omega)), \quad (4.5)$$

$$\mu_\varepsilon \rightharpoonup \mu \quad \text{weakly in } L^2(0,T;H^1(\Omega)), \quad (4.6)$$

$$\xi_\varepsilon \rightharpoonup \xi \quad \text{weakly in } L^2(0,T;L^2(\Omega)) \quad (4.7)$$

for some

$$\begin{aligned} u &\in H^1(0,T;(H^1(\Omega))^*) \cap L^\infty(0,T;L^2(\Omega)), \\ \mu &\in L^2(0,T;H^1(\Omega)), \quad \xi, \eta \in L^2(0,T;L^2(\Omega)). \end{aligned}$$

We proceed by showing in addition that

$$u_\varepsilon^\lambda \rightarrow u_\varepsilon \quad \text{strongly in } C^0([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)). \quad (4.8)$$

Indeed, Lemma 4 implies that for every $\delta > 0$, there exist $C_\delta > 0$ and $\varepsilon_\delta > 0$ such that

$$\begin{aligned} &\|u_\varepsilon - u\|_{L^2(0,T;H^1(\Omega))}^2 \\ &\leq \delta \int_0^T \int_\Omega \int_\Omega K_\varepsilon(x,y) |\nabla(u_\varepsilon - u)(t,x) - \nabla(u_\varepsilon - u)(t,y)|^2 dx dy dt + C_\delta \|u_\varepsilon - u\|_{L^2(0,T;L^2(\Omega))}^2 \end{aligned}$$

for every $0 < \varepsilon < \varepsilon_\delta$. Thanks to (4.1), we infer that

$$\|u_\varepsilon - u\|_{L^2(0,T;H^1(\Omega))}^2 \leq C\delta + C_\delta \|u_\varepsilon - u\|_{L^2(0,T;L^2(\Omega))}^2$$

for a constant $C > 0$. Similarly, using the second inequality in Lemma 4 and (4.1), the same argument ensures also that

$$\begin{aligned} \|u_\varepsilon - u\|_{L^\infty(0,T;L^2(\Omega))}^2 &\leq \delta \|E_\varepsilon(u_\varepsilon - u)\|_{L^\infty(0,T)} + C_\delta \|u_\varepsilon - u\|_{C^0([0,T];(H^1(\Omega))^*)}^2 \\ &\leq C\delta + C_\delta \|u_\varepsilon - u\|_{C^0([0,T];(H^1(\Omega))^*)}^2. \end{aligned}$$

The strong convergence (4.8) follows then from the arbitrariness of δ , and from (4.2).

From the strong convergence of $(u_\varepsilon)_\varepsilon$ and the strong-weak closure of maximal monotone graphs it is readily seen that $\xi \in \gamma(u)$ and that

$$\Pi(u_\varepsilon) \rightarrow \Pi(u) \quad \text{strongly in } L^2(0, T; L^2(\Omega)).$$

Let us now identify the limit η as $-\Delta u$. As $DE_\varepsilon = \mathbf{B}_\varepsilon$, we have that

$$E_\varepsilon(z_1) + \langle B_\varepsilon(z_1), z_2 - z_1 \rangle_{(H^1(\Omega))^*, H^1(\Omega)} \leq E_\varepsilon(z_2), \quad (4.9)$$

for all $z_1, z_2 \in H^1(\Omega)$. Hence, for all $z \in L^2(0, T; H^1(\Omega))$ we deduce that

$$\int_0^T E_\varepsilon(u_\varepsilon(t, \cdot)) dt + \int_0^T \int_\Omega \mathbf{B}_\varepsilon(u_\varepsilon(t, x))(z(t, x) - u_\varepsilon(t, x)) dx dt \leq \int_0^T E_\varepsilon(z(t, \cdot)) dt. \quad (4.10)$$

The results in [10] and the dominated convergence theorem yield

$$\int_0^T E_\varepsilon(z(t, \cdot)) dt \rightarrow \frac{1}{2} \int_0^T \int_\Omega |\nabla z(x, t)|^2 dx dt.$$

Owing to the convergences (4.8) and (4.5), we have that

$$\int_0^T \int_\Omega B_\varepsilon(u_\varepsilon(t, x))(z(t, x) - u_\varepsilon(t, x)) dx dt \rightarrow \int_0^T \int_\Omega \eta(t, x)(z(t, x) - u(t, x)) dx dt.$$

Finally, following the exact same steps as in [45], there holds

$$\int_0^T E_\varepsilon(u_\varepsilon(t, \cdot)) dt \rightarrow \frac{1}{2} \int_0^T \int_\Omega |\nabla u(t, x)|^2 dx dt.$$

Hence, letting $\varepsilon \rightarrow 0$ in (4.10), we obtain the inequality

$$\frac{1}{2} \int_0^T \int_\Omega |\nabla u(t, x)|^2 dx dt + \int_0^T \int_\Omega \eta(t, x)(z(t, x) - u(t, x)) dx dt \leq \frac{1}{2} \int_0^T \int_\Omega |\nabla z(t, x)|^2 dx dt$$

for every $z \in L^2(0, T; H^1(\Omega))$, so that $\eta = -\Delta u \in L^2(0, T; L^2(\Omega))$. By elliptic regularity we infer that $u \in L^2(0, T; H^2(\Omega))$.

Finally, Hölder's inequality, the Sobolev embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, and the strong convergence (4.8) yield

$$\begin{aligned} \|\beta u_\varepsilon - \beta u\|_{L^2(0,T;L^2(\Omega))} &\leq \int_0^T \|\beta(t, \cdot)\|_{L^\infty(\Omega)} \|u_\varepsilon - u\|_{L^2(\Omega)} dt \\ &\leq \|\beta\|_{L^2(0,T;L^\infty(\Omega))} \|u_\varepsilon - u\|_{L^\infty(0,T;L^2(\Omega))} \rightarrow 0. \end{aligned}$$

Thus, letting $\varepsilon \searrow 0$ in Definition 1 (of solution for the nonlocal Cahn-Hilliard) we obtain

$$\partial_t u - \Delta \mu = -\operatorname{div}(\beta u)$$

in the sense of distributions, as well as

$$\mu = -\Delta u + \xi + \Pi(u).$$

This implies that u is a solution to the local Cahn-Hilliard equation (2.4)–(2.6), and concludes the proof of Theorem 2.2.

5. PROOF OF THEOREMS 2.3–2.4

We show first that under the additional assumption (2.20), the solution $(u_\varepsilon, \mu_\varepsilon, \xi_\varepsilon)$ to the nonlocal equation is more regular. Note that here $\varepsilon \in (0, \varepsilon_0)$ is fixed.

The idea is to argue in a classical way, performing some additional estimates on the approximate solutions $(u_\varepsilon^\lambda, \mu_\varepsilon^\lambda)$ constructed in Section 3. To this end, note that by (2.20), the approximating sequence $(u_{0,\varepsilon}^\lambda)_\lambda$ of initial data satisfying (3.4)–(3.5) can be chosen with the additional property

$$\sup_{\lambda \in (0, \lambda_0)} \left\{ \|u_{0,\varepsilon}^\lambda\|_{L^6(\Omega)} + \|\lambda \Delta u_{0,\varepsilon}^\lambda + \mathbf{B}_\varepsilon(u_{0,\varepsilon}^\lambda) + \gamma_\lambda(u_{0,\varepsilon}^\lambda) + \Pi(u_{0,\varepsilon}^\lambda)\|_{H^1(\Omega)} \right\} < +\infty. \quad (5.1)$$

First of all we need some preparatory work. Note that the elliptic problem corresponding to (3.1)–(3.3) at time 0, i.e.

$$\begin{cases} u'_{0,\varepsilon,\lambda} - \Delta \mu_{0,\varepsilon,\lambda} = -\operatorname{div}(\beta(0)u_{0,\varepsilon}^\lambda), \\ \mu_{0,\varepsilon,\lambda} = -\lambda \Delta u_{0,\varepsilon}^\lambda + \mathbf{B}_\varepsilon(u_{0,\varepsilon}^\lambda) + \gamma_\lambda(u_{0,\varepsilon}^\lambda) + \Pi(u_{0,\varepsilon}^\lambda), \end{cases}$$

admits a unique solution $(u'_{0,\varepsilon,\lambda}, \mu_{0,\varepsilon,\lambda}) \in (H^1(\Omega))^* \times H^1(\Omega)$. Testing the first equation by $\mu_{0,\varepsilon,\lambda}$, the second by $u'_{0,\varepsilon,\lambda}$ and taking the difference yields

$$\begin{aligned} \int_{\Omega} |\nabla \mu_{0,\varepsilon,\lambda}(x)|^2 dx &= -\langle u'_{0,\varepsilon,\lambda}, -\lambda \Delta u_{0,\varepsilon}^\lambda + \mathbf{B}_\varepsilon(u_{0,\varepsilon}^\lambda) + \gamma_\lambda(u_{0,\varepsilon}^\lambda) + \Pi(u_{0,\varepsilon}^\lambda) \rangle_{(H^1(\Omega))^*, H^1(\Omega)} \\ &\quad + \int_{\Omega} \beta(0, x) u_{0,\varepsilon}^\lambda(x) \cdot \nabla \mu_{0,\varepsilon,\lambda}(x) dx. \end{aligned}$$

From the first equation it is readily seen that

$$\|u'_{0,\varepsilon,\lambda}\|_{(H^1(\Omega))^*} \leq \|\nabla \mu_{0,\varepsilon,\lambda}\|_{L^2(\Omega)} + \|\beta(0)u_{0,\varepsilon}^\lambda\|_{L^2(\Omega)}$$

with

$$\|\beta(0)u_{0,\varepsilon}^\lambda\|_{L^2(\Omega)} \leq \|\beta(0)\|_{L^3(\Omega)} \|u_{0,\varepsilon}^\lambda\|_{L^6(\Omega)} \leq C \|\beta\|_{H^1(0,T;L^3(\Omega))} \|u_{0,\varepsilon}^\lambda\|_{L^6(\Omega)}.$$

Hence the Young inequality, (2.19), (3.4)–(3.5), and (5.1) imply that

$$\|u'_{0,\varepsilon,\lambda}\|_{(H^1(\Omega))^*} + \|\nabla \mu_{0,\varepsilon,\lambda}\|_{L^2(\Omega)} \leq C_\varepsilon. \quad (5.2)$$

We are now ready to perform the additional estimate on the approximated solutions. Again, we proceed formally in order to avoid heavy notations and since everything can be proved rigorously through a further regularization on the problem. The idea is to (formally) test the time derivative of (3.1) by $(-\Delta)^{-1}(\partial_t u_\varepsilon^\lambda)$, the time derivative of (3.2) by $\partial_t u_\varepsilon^\lambda$ and then

to take the difference: the resulting inequality is

$$\begin{aligned}
& \frac{1}{2} \|\partial_t u_\varepsilon^\lambda(t)\|_{(H^1(\Omega))^*}^2 + \lambda \int_0^t \int_\Omega |\nabla \partial_t u_\varepsilon^\lambda(s, x)|^2 dx ds + \int_0^t E_\varepsilon(\partial_t u_\varepsilon^\lambda(s, \cdot)) ds \\
& \quad + \int_0^t \int_\Omega \gamma'_\lambda(u_\varepsilon^\lambda(s, x)) |\partial_t u_\varepsilon^\lambda(s, x)|^2 dx ds + \int_0^t \int_\Omega \Pi'(u_\varepsilon^\lambda(s, x)) |\partial_t u_\varepsilon^\lambda(s, x)|^2 dx ds \\
& = \frac{1}{2} \|u'_{0,\varepsilon,\lambda}\|_{(H^1(\Omega))^*}^2 + \int_0^t \int_\Omega \partial_t u_\varepsilon^\lambda(s, x) \beta(s, x) \cdot \nabla (-\Delta)^{-1}(\partial_t u_\varepsilon^\lambda)(s, x) dx ds \\
& \quad + \int_0^t \int_\Omega u_\varepsilon^\lambda(s, x) \partial_t \beta(s, x) \cdot \nabla (-\Delta)^{-1}(\partial_t u_\varepsilon^\lambda)(s, x) dx ds.
\end{aligned}$$

Now, note that by Hölder's inequality and (2.19) we have

$$\begin{aligned}
& \int_0^t \int_\Omega \partial_t u_\varepsilon^\lambda(s, x) \beta(s, x) \cdot \nabla (-\Delta)^{-1}(\partial_t u_\varepsilon^\lambda)(s, x) dx ds \\
& \leq \int_0^t \|\partial_t u_\varepsilon^\lambda(s, \cdot)\|_{L^2(\Omega)} \|\beta(s, \cdot)\|_{L^\infty(\Omega)} \|\nabla (-\Delta)^{-1}(\partial_t u_\varepsilon^\lambda)(s, \cdot)\|_{L^2(\Omega)} ds \\
& \leq \frac{1}{2} \|\partial_t u_\varepsilon^\lambda\|_{L^2(0,t;L^2(\Omega))}^2 + \frac{1}{2} \int_0^t \|\beta(s, \cdot)\|_{L^\infty(\Omega)}^2 \|\partial_t u_\varepsilon^\lambda(s)\|_{(H^1(\Omega))^*}^2 ds
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^t \int_\Omega u_\varepsilon^\lambda(s, x) \partial_t \beta(s, x) \cdot \nabla (-\Delta)^{-1}(\partial_t u_\varepsilon^\lambda)(s, x) dx ds \\
& \leq \int_0^t \|u_\varepsilon^\lambda(s, \cdot)\|_{L^6(\Omega)} \|\partial_t \beta(s, \cdot)\|_{L^3(\Omega)} \|\nabla (-\Delta)^{-1}(\partial_t u_\varepsilon^\lambda)\|_{L^2(\Omega)} ds \\
& \leq \frac{1}{2} \|u_\varepsilon^\lambda\|_{L^2(0,t;H^1(\Omega))}^2 + \frac{1}{2} \int_0^t \|\partial_t \beta(s, \cdot)\|_{L^3(\Omega)}^2 \|\partial_t u_\varepsilon^\lambda(s)\|_{(H^1(\Omega))^*}^2 ds.
\end{aligned}$$

Thanks to Lemma 4 there holds

$$\|\partial_t u_\varepsilon^\lambda(s)\|_{(H^1(\Omega))^*}^2 \leq \delta \int_0^t E_\varepsilon(\partial_t u_\varepsilon^\lambda)(s) ds + C_\delta \|\partial_t u_\varepsilon^\lambda\|_{L^2(0,T;(H^1(\Omega))^*)}^2$$

for δ sufficiently small. Hence, putting this information together, using the Lipschitz-continuity of Π , the monotonicity of γ_λ , condition (5.2) and the already proved estimates (3.18) and (3.20), we are left with

$$\begin{aligned}
& \|\partial_t u_\varepsilon^\lambda(t)\|_{(H^1(\Omega))^*}^2 + \int_0^t E_\varepsilon(\partial_t u_\varepsilon^\lambda(s, \cdot)) ds \\
& \leq C_\varepsilon + \int_0^t \left(\|\beta(s, \cdot)\|_{L^\infty(\Omega)}^2 + \|\partial_t \beta(s, \cdot)\|_{L^3(\Omega)}^2 \right) \|\partial_t u_\varepsilon^\lambda(s)\|_{(H^1(\Omega))^*}^2 ds.
\end{aligned}$$

Since $s \mapsto \|\beta(s, \cdot)\|_{L^\infty(\Omega)}^2$ and $s \mapsto \|\partial_t \beta(s, \cdot)\|_{L^3(\Omega)}^2$ belong to $L^1(0, T)$ due to (2.19) and **H4**, using the Gronwall lemma and recalling [49, Theorem 1.1] we infer that

$$\|\partial_t u_\varepsilon^\lambda\|_{L^\infty(0,T;(H^1(\Omega))^*) \cap L^2(0,T;L^2(\Omega))} \leq C_\varepsilon. \tag{5.3}$$

Now, if (2.21) holds, we also have

$$\|\operatorname{div}(\beta u_\varepsilon^\lambda)\|_{L^\infty(0,T;(H^1(\Omega))^*)} \leq \|\beta u_\varepsilon^\lambda\|_{L^\infty(0,T;L^2(\Omega))} \leq \|\beta\|_{L^\infty(0,T;L^\infty(\Omega))} \|u_\varepsilon^\lambda\|_{L^\infty(0,T;L^2(\Omega))},$$

yielding by (3.18) and by comparison in (3.1),

$$\|\nabla\mu_\varepsilon^\lambda\|_{L^\infty(0,T;L^2(\Omega))} \leq C_\varepsilon. \quad (5.4)$$

At this point, going back to the proof of Theorem 2.1, we repeat exactly the same arguments of *Step 2* and *Step 3*: using the additional estimates (5.3)–(5.4), we deduce

$$\|\mu_\varepsilon^\lambda\|_{L^\infty(0,T;H^1(\Omega))} + \|\gamma_\lambda(u_\lambda)\|_{L^\infty(0,T;L^2(\Omega))} \leq C_\varepsilon. \quad (5.5)$$

Furthermore, if also (2.22) holds we have

$$\begin{aligned} \|\operatorname{div}(\beta u_\varepsilon^\lambda)\|_{L^2(0,T;L^2(\Omega))} &\leq \|\operatorname{div}(\beta)u_\varepsilon^\lambda\|_{L^2(0,T;L^2(\Omega))} + \|\beta \cdot \nabla u\|_{L^2(0,T;L^2(\Omega))} \\ &\leq \|\operatorname{div} \beta\|_{L^\infty(0,T;L^3(\Omega))} \|u_\varepsilon^\lambda\|_{L^2(0,T;L^6(\Omega))} \\ &\quad + \|\beta\|_{L^\infty(0,T;L^\infty(\Omega))} \|\nabla u_\varepsilon^\lambda\|_{L^2(0,T;L^2(\Omega))}, \end{aligned}$$

so that from (3.20) and by comparison in (3.1) we infer that

$$\|\Delta\mu_\varepsilon^\lambda\|_{L^2(0,T;L^2(\Omega))} \leq C_\varepsilon. \quad (5.6)$$

Hence, (5.3)–(5.6) ensure that the limit solution $(u_\varepsilon, \mu_\varepsilon, \xi_\varepsilon)$ inherits the additional regularity stated in Theorem 2.3.

The proof of Theorem 2.4 follows now as in Section 4, noting that the assumption (2.23) implies that the family $(C_\varepsilon)_\varepsilon$ appearing in (5.3)–(5.6) is uniformly bounded in ε .

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