DIRECT AND INVERSE LIMITS OF NORMED MODULES

ENRICO PASQUALETTO

ABSTRACT. The aim of this note is to study existence and main properties of direct and inverse limits in the category of normed L^0-modules (in the sense of Gigli) over a metric measure space.

CONTENTS

INTRODUCTION

Recent years have witnessed a growing interest of the mathematical community towards the differential calculus on nonsmooth spaces. In this regard, an important contribution is represented by N. Gigli's paper [3], wherein a first-order differential structure for metric measure spaces has been proposed. Such theory is based upon the key notion of *normed L^0-module*, which provides a generalisation of the concept of 'space of measurable sections of a measurable Banach bundle'. The main aim of the present manuscript is to prove that direct limits always exist in the category of normed L^0-modules. Furthermore, we shall report the proof of existence of inverse limits of normed L^0-modules, which has been originally achieved in [5]. Finally, we will investigate the relation between direct/inverse limits and other natural operations that are available in this framework, such as dual and pullback.

Overview of the content. The concept of normed L^0-module that we are going to describe has been originally introduced in [3] and then further refined in [4]. We propose here an equivalent reformulation of its definition, which is tailored to our purposes.

Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a given metric measure space. Consider an algebraic module $\mathscr{M}$ over the commutative ring $L^0(\mathfrak{m})$ of all real-valued Borel functions defined on X (up to $\mathfrak{m}$-a.e. equality). By *pointwise norm* on $\mathscr{M}$ we mean a map $|\cdot|\colon \mathscr{M} \to L^0(\mathfrak{m})$ satisfying the following properties:

$$|v| \geq 0 \quad \mathfrak{m}\text{-a.e.} \quad \text{for every } v \in \mathscr{M}, \text{ with equality if and only if } v = 0,$$

$$|v + w| \leq |v| + |w| \quad \mathfrak{m}\text{-a.e.} \quad \text{for every } v, w \in \mathscr{M},$$

$$|f \cdot v| = |f||v| \quad \mathfrak{m}\text{-a.e.} \quad \text{for every } v \in \mathscr{M} \text{ and } f \in L^0(\mathfrak{m}).$$

The pointwise norm $|\cdot|$ can be naturally associated with a distance $\mathsf{d}_{\mathscr{M}}$ on $\mathscr{M}$: chosen a Borel probability measure $\mathfrak{m}'$ on X that is mutually absolutely continuous with respect to $\mathfrak{m}$, we define

$$\mathsf{d}_{\mathscr{M}}(v, w) := \int |v - w| \wedge 1 \, \mathrm{d}\mathfrak{m}' \quad \text{for every } v, w \in \mathscr{M}.$$

Then we say that the couple $\big(\mathscr{M}, |\cdot|\big)$ is a *normed $L^0(\mathfrak{m})$-module* provided the relative metric space $(\mathscr{M}, \mathsf{d}_{\mathscr{M}})$ is complete. The crucial example of normed L^0-module one should keep in mind is the space of Borel vector fields on a Riemannian manifold (with the usual pointwise operations).

Given two normed $L^0(\mathfrak{m})$-modules $\mathscr{M}$ and $\mathscr{N}$, we say that a map $\varphi\colon \mathscr{M} \to \mathscr{N}$ is a *morphism* provided it is a morphism of $L^0(\mathfrak{m})$-modules satisfying the inequality $\big|\varphi(v)\big| \leq |v|$ in the $\mathfrak{m}$-a.e. sense for every $v \in \mathscr{M}$. Consequently, we can consider the category of normed $L^0(\mathfrak{m})$-modules. The scope of these notes is to analyse direct and inverse limits in such category. More in detail:

i) We prove that any direct system in the category of normed $L^0(\mathfrak{m})$-modules admits a direct limit (cf. Theorem 2.1). Among other properties, we show (cf. Lemma 2.5) that any normed $L^0(\mathfrak{m})$-module can be written as a direct limit of finitely-generated modules (which is significant to the application b) we shall illustrate at the end of this introduction) and (cf. Theorem 2.12) that the direct limit functor commutes with the pullback functor.

ii) Existence of inverse limits in the category of normed $L^0(\mathfrak{m})$-modules has been already proven by the author, together with N. Gigli and E. Soultanis, in the paper [5]. In order to make these notes self-contained, we shall recall the proof of such fact in Theorem 3.1. We also examine several (not previously known) properties of inverse limits in this setting; for instance, we prove that 'the dual of the direct limit coincides with the inverse limit of the duals' (see Corollary 3.11). On the other hand, inverse limit functor and pullback functor do not commute (see Remark 3.12).

It is worth to underline that the category of normed $L^0(\mathfrak{m})$-modules reduces to that of Banach spaces as soon as the reference measure $\mathfrak{m}$ is a Dirac measure $\delta_{\bar{x}}$ concentrated on some point $\bar{x} \in \mathrm{X}$, whence the above-mentioned features of direct and inverse limits of normed L^0-modules might be considered as a generalisation of the corresponding ones for Banach spaces.

Motivation and related works. Besides the theoretical interest, the study of direct and inverse limits in the category of normed L^0-modules is principally motivated by the following two applications:

a) Differential of a metric-valued locally Sobolev map. Any metric measure space $(\mathrm{X}, \mathsf{d}_{\mathrm{X}}, \mathfrak{m})$ can be canonically associated with a *cotangent module* $L^0_{\mathfrak{m}}(T^*\mathrm{X})$ and a *tangent module* $L^0_{\mathfrak{m}}(T\mathrm{X})$, which are normed $L^0(\mathfrak{m})$-modules that supply an abstract notion of 'measurable 1-forms on X' and 'measurable vector fields on X', respectively; we refer the interested reader to [3, 4] for a detailed description of such objects. Moreover, there are several ways to define Sobolev maps from $(\mathrm{X}, \mathsf{d}_{\mathrm{X}}, \mathfrak{m})$ to a complete metric space $(\mathrm{Y}, \mathsf{d}_{\mathrm{Y}})$. One of possible approaches is via post-composition with Lipschitz functions (cf. [6]). Given any map $u\colon \mathrm{X} \to \mathrm{Y}$ that is locally Sobolev in the above sense, one can always select a distinguished object $|Du| \in L^2_{\mathrm{loc}}(\mathrm{X}, \mathsf{d}_{\mathrm{X}}, \mathfrak{m})$ – called *minimal weak upper gradient of u* –

which plays the role of the 'modulus of the differential of u'. The purpose of the work [5] was to build the differential $\mathrm{d}u$ associated to u, defined as a linear operator between (suitable variants of) tangent modules. More precisely, in the special case in which $|Du|$ is globally 2-integrable the differential of u is a map from $L^0_\mathfrak{m}(T\mathrm{X})$ to $\left(u^* L^0_\mu(T^*\mathrm{Y})\right)^*$, where the measure μ is defined as $\mu := u_*\big(|Du|^2\,\mathfrak{m}\big)$. The precise choice of this finite Borel measure μ on Y is due to the fact that it enjoys nice composition properties. On the other hand, if the function $|Du|$ is just locally 2-integrable, then the measure $u_*\big(|Du|^2\,\mathfrak{m}\big)$ may no longer be σ-finite (thus accordingly the cotangent module $L^0_\mu(T^*\mathrm{Y})$ is not well-defined). The strategy one can adopt to overcome such difficulty is the following: the family $\mathcal{F}(u)$ of all open subsets Ω of X satisfying $\int_\Omega |Du|^2\,\mathrm{d}\mathfrak{m} < +\infty$ is partially ordered by inclusion, whence the idea is to initially deal with the 'approximating' modules $L^0_{\mu_\Omega}(T^*\mathrm{Y})$ – where we set $\mu_\Omega := u_*\big(\chi_\Omega\,|Du|^2\,\mathfrak{m}\big)$ – and then pass to the inverse limit with respect to $\Omega \in \mathcal{F}(u)$.

b) CONCRETE REPRESENTATION OF A SEPARABLE NORMED L^0-MODULE. A significant way to build normed L^0-modules is to provide some reasonable notion of measurable Banach bundle and consider the space of its measurable sections (up to a.e. equality). Nevertheless, it is not clear whether any normed L^0-module actually admits a similar representation. In this direction, it is proven in [8] that each finitely-generated normed L^0-module can be viewed as the space of sections of some bundle. The aim of the forthcoming paper [1] is to extend this result to all separable normed L^0-modules. One of the possible approaches to achieve such goal is to realise any separable normed L^0-module $\mathscr{M}$ as the direct limit (with respect to a countable set of indices) of finitely-generated normed L^0-modules $\mathscr{M}_n$ and to apply the previously known result to each module $\mathscr{M}_n$.

1. PRELIMINARIES

1.1. **Normed $L^0(\mathfrak{m})$-modules.** For our purposes, a *metric measure space* is a triple $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$, where (X, d) is a complete and separable metric space, while $\mathfrak{m} \geq 0$ is a Radon measure on (X, d). We denote by $L^0(\mathfrak{m})$ the space of all Borel functions $f\colon \mathrm{X} \to \mathbb{R}$ considered up to $\mathfrak{m}$-a.e. equality. It is well-known that $L^0(\mathfrak{m})$ is both a topological vector space and a topological ring when equipped with the usual pointwise operations and with the topology induced by the distance

$$\mathsf{d}_{L^0(\mathfrak{m})}(f, g) := \int |f - g| \wedge 1\,\mathrm{d}\mathfrak{m}' \quad \text{for every } f, g \in L^0(\mathfrak{m}),$$

where $\mathfrak{m}'$ is any Borel probability measure on X with $\mathfrak{m} \ll \mathfrak{m}' \ll \mathfrak{m}$. Given any (not necessarily countable) family $\{f_i\}_{i \in I} \subseteq L^0(\mathfrak{m})$, we denote by $\operatorname{ess\,sup}_{i \in I} f_i \in L^0(\mathfrak{m})$ and $\operatorname{ess\,inf}_{i \in I} f_i \in L^0(\mathfrak{m})$ its essential supremum and essential infimum, respectively.

Definition 1.1 (Pointwise norm). *Let $\mathscr{M}$ be a module over the commutative ring $L^0(\mathfrak{m})$. Then we say that a map $|\cdot|\colon \mathscr{M} \to L^0(\mathfrak{m})$ is a* pointwise seminorm *on $\mathscr{M}$ provided*

$$|v| \geq 0 \quad \text{for every } v \in \mathscr{M},$$
$$|v + w| \leq |v| + |w| \quad \text{for every } v, w \in \mathscr{M},$$
$$|f \cdot v| = |f||v| \quad \text{for every } v \in \mathscr{M} \text{ and } f \in L^0(\mathfrak{m}),$$

where all inequalities are intended in the $\mathfrak{m}$-a.e. sense. Moreover, we say that $|\cdot|$ is a pointwise norm *on $\mathscr{M}$ if in addition it holds that $|v| = 0$ $\mathfrak{m}$-a.e. if and only if $v = 0$.*

Any pointwise seminorm can be naturally associated with the following pseudometric:

$$\mathsf{d}_{\mathscr{M}}(v,w) := \int |v - w| \wedge 1 \, \mathrm{d}\mathfrak{m}' \quad \text{for every } v, w \in \mathscr{M},$$

where $\mathfrak{m}'$ is any given Borel probability measure on X such that $\mathfrak{m} \ll \mathfrak{m}' \ll \mathfrak{m}$. It holds that $\mathsf{d}_{\mathscr{M}}$ is a distance if and only if $|\cdot|$ is a pointwise norm.

With this said, we can give a definition of normed $L^0(\mathfrak{m})$-module that is fully equivalent to the one that has been proposed in [3, 4]:

Definition 1.2 (Normed $L^0(\mathfrak{m})$-module)**.** *A normed $L^0(\mathfrak{m})$-module is a module $\mathscr{M}$ over $L^0(\mathfrak{m})$ endowed with a pointwise norm $|\cdot|$ whose associated distance $\mathsf{d}_{\mathscr{M}}$ is complete.*

A *morphism* $\varphi \colon \mathscr{M} \to \mathscr{N}$ between two normed $L^0(\mathfrak{m})$-modules $\mathscr{M}$ and $\mathscr{N}$ is any $L^0(\mathfrak{m})$-module morphism – i.e. satisfying $\varphi(f \cdot v) = f \cdot \varphi(v)$ for every $f \in L^0(\mathfrak{m})$ and $v \in \mathscr{M}$ – such that

$$\big|\varphi(v)\big| \leq |v| \quad \text{holds } \mathfrak{m}\text{-a.e. for every } v \in \mathscr{M}.$$

This allows us to speak about the category of normed $L^0(\mathfrak{m})$-modules.

Example 1.3. Let us suppose that $\mathfrak{m} = \delta_{\bar{x}}$ for some point $\bar{x} \in X$. Then the ring $L^0(\delta_{\bar{x}})$ can be canonically identified with the field $\mathbb{R}$, thus accordingly the category of normed $L^0(\delta_{\bar{x}})$-modules is (equivalent to) the category of Banach spaces. ∎

Definition 1.4 (Generators)**.** *Let $\mathscr{M}$ be a normed $L^0(\mathfrak{m})$-module. Then a family $S \subseteq \mathscr{M}$ is said to* generate *$\mathscr{M}$ provided the smallest $L^0(\mathfrak{m})$-module containing S is $\mathsf{d}_{\mathscr{M}}$-dense in $\mathscr{M}$, i.e.*

$$\Big\{ \sum\nolimits_{i=1}^{n} f_i \cdot v_i \,\Big|\, n \in \mathbb{N}, \, (f_i)_{i=1}^{n} \subseteq L^0(\mathfrak{m}), \, (v_i)_{i=1}^{n} \subseteq S \Big\} \quad \text{is } \mathsf{d}_{\mathscr{M}}\text{-dense in } \mathscr{M}.$$

Lemma 1.5 (Metric identification)**.** *Let $\mathscr{M}$ be an $L^0(\mathfrak{m})$-module with a pointwise seminorm $|\cdot|$. Consider the following equivalence relation on $\mathscr{M}$: given any $v, w \in \mathscr{M}$, we declare that $v \sim w$ provided $|v - w| = 0$ holds $\mathfrak{m}$-a.e. on X. Then the quotient $\mathscr{M}/\!\sim$ inherits an $L^0(\mathfrak{m})$-module structure and the map $\big|[v]_\sim\big| := |v|$ is a pointwise norm on $\mathscr{M}/\!\sim$.*

Proof. The set $\mathscr{N} := \big\{ v \in \mathscr{M} : |v| = 0 \ \mathfrak{m}\text{-a.e.} \big\}$ is clearly a submodule of $\mathscr{M}$, thus the quotient space $\mathscr{M}/\!\sim = \mathscr{M}/\mathscr{N}$ has a canonical $L^0(\mathfrak{m})$-module structure. Given that

$$\big||v| - |w|\big| \leq |v - w| \quad \text{holds } \mathfrak{m}\text{-a.e. for every } v, w \in \mathscr{M},$$

the map $|\cdot| \colon \mathscr{M}/\!\sim \to L^0(\mathfrak{m})$ defined by $\big|[v]_\sim\big| := |v|$ is well-posed and satisfies all the pointwise norm axioms. This gives the statement. □

Lemma 1.6 (Metric completion)**.** *Let $\mathscr{M}$ be an $L^0(\mathfrak{m})$-module with a pointwise norm $|\cdot|$. Then there exists a unique (up to unique isomorphism) couple $(\mathscr{M}_0, \iota)$, where*

  i) *$\mathscr{M}_0$ is a normed $L^0(\mathfrak{m})$-module,*
  ii) *$\iota \colon \mathscr{M} \to \mathscr{M}_0$ is an $L^0(\mathfrak{m})$-linear map preserving the pointwise norm,*

such that the range $\iota(\mathscr{M})$ is dense in $\mathscr{M}_0$ with respect to the distance $\mathsf{d}_{\mathscr{M}_0}$.

Proof. Denote by $(\mathscr{M}_0, \iota)$ the completion of the metric space $(\mathscr{M}, \mathsf{d}_{\mathscr{M}})$, which is known to be unique up to unique isomorphism. The pointwise norm $|\cdot| \colon \iota(\mathscr{M}) \to L^0(\mathfrak{m})$ can be easily proved to be continuous from $\big(\iota(\mathscr{M}), \mathsf{d}_{\iota(\mathscr{M})}\big)$ to $\big(L^0(\mathfrak{m}), \mathsf{d}_{L^0(\mathfrak{m})}\big)$, whence it can be uniquely extended to a continuous map $|\cdot| \colon \mathscr{M}_0 \to L^0(\mathfrak{m})$. Arguing by approximation, we conclude that the extended map $|\cdot|$ is a pointwise norm on $\mathscr{M}_0$ and that $\mathsf{d}_{\mathscr{M}_0}(v, w) = \int |v - w| \wedge 1 \, \mathrm{d}\mathfrak{m}'$ for every $v, w \in \mathscr{M}_0$. This proves the validity of the statement. □

Given any two normed $L^0(\mathfrak{m})$-modules $\mathscr{M}$ and $\mathscr{N}$, we define the space $\mathrm{HOM}(\mathscr{M},\mathscr{N})$ as

$$\mathrm{HOM}(\mathscr{M},\mathscr{N}) \coloneqq \Big\{ T \colon \mathscr{M} \to \mathscr{N} \,\Big|\, T \text{ is } L^0(\mathfrak{m})\text{-linear and continuous} \Big\}.$$

Standard arguments show that for any $T \in \mathrm{HOM}(\mathscr{M},\mathscr{N})$ there exists $\ell \in L^0(\mathfrak{m})$ such that

$$\big|T(v)\big| \le \ell\,|v| \quad \text{ holds } \mathfrak{m}\text{-a.e. for every } v \in \mathscr{M}. \tag{1.1}$$

It turns out that the function

$$|T| \coloneqq \operatorname{ess\,sup} \Big\{ \big|T(v)\big| \,\Big|\, v \in \mathscr{M},\, |v| \le 1 \text{ holds } \mathfrak{m}\text{-a.e.} \Big\} \in L^0(\mathfrak{m}) \tag{1.2}$$

is the minimal function ℓ (in the $\mathfrak{m}$-a.e. sense) for which (1.1) is satisfied. We point out that an element $T \in \mathrm{HOM}(\mathscr{M},\mathscr{N})$ is a morphism between $\mathscr{M}$ and $\mathscr{N}$ (in the categorical sense) if and only if $|T| \le 1$ holds $\mathfrak{m}$-a.e. on X. Furthermore, the space $\mathrm{HOM}(\mathscr{M},\mathscr{N})$ inherits a natural structure of normed $L^0(\mathfrak{m})$-module if endowed with the pointwise operations

$$(T+S)(v) \coloneqq T(v) + S(v) \quad \text{ for every } T, S \in \mathrm{HOM}(\mathscr{M},\mathscr{N}),$$
$$(f \cdot T)(v) \coloneqq f \cdot T(v) \quad \text{ for every } f \in L^0(\mathfrak{m}) \text{ and } T \in \mathrm{HOM}(\mathscr{M},\mathscr{N})$$

and with the pointwise norm operator $\mathrm{HOM}(\mathscr{M},\mathscr{N}) \ni T \mapsto |T| \in L^0(\mathfrak{m})$ introduced in (1.2).

Definition 1.7 (Dual of a normed $L^0(\mathfrak{m})$-module)**.** *Let $\mathscr{M}$ be a normed $L^0(\mathfrak{m})$-module. Then we define its* dual *normed $L^0(\mathfrak{m})$-module $\mathscr{M}^*$ as*

$$\mathscr{M}^* \coloneqq \mathrm{HOM}\big(\mathscr{M}, L^0(\mathfrak{m})\big).$$

(Observe that $L^0(\mathfrak{m})$ itself can be viewed as a normed $L^0(\mathfrak{m})$-module.)

Let $\mathscr{M}$, $\mathscr{N}$ be any two normed $L^0(\mathfrak{m})$-modules and let $\varphi \colon \mathscr{M} \to \mathscr{N}$ be a given morphism. Then the *adjoint* operator $\varphi^{\mathrm{adj}} \colon \mathscr{N}^* \to \mathscr{M}^*$ is defined as

$$\varphi^{\mathrm{adj}}(\omega) \coloneqq \omega \circ \varphi \quad \text{ for every } \omega \in \mathscr{N}^*. \tag{1.3}$$

It is immediate to check that φ^{adj} is a morphism of normed $L^0(\mathfrak{m})$-modules as well.

Remark 1.8. Define $\mathcal{L}_1 \coloneqq \mathcal{L}^1\big|_{[0,1]}$ and consider a Banach space $\mathbb{B}$. Then the space $L^0\big([0,1],\mathbb{B}\big)$ of all (strongly) Borel maps from $[0,1]$ to $\mathbb{B}$ (considered up to $\mathcal{L}_1$-a.e. equality) can be easily shown to be a normed $L^0(\mathcal{L}_1)$-module if endowed with the following operations:

$$(u+v)(t) \coloneqq u(t) + v(t),$$
$$(f \cdot u)(t) \coloneqq f(t)\,u(t), \qquad \text{for } \mathcal{L}_1\text{-a.e. } t \in [0,1],$$
$$|u|(t) \coloneqq \big\|u(t)\big\|_{\mathbb{B}}$$

for every $u, v \in L^0\big([0,1],\mathbb{B}\big)$ and $f \in L^0(\mathcal{L}_1)$. By combining the results of [3, Section 1.6] with the properties of the L^0-completion studied in [4], one can deduce that $L^0\big([0,1],\mathbb{B}'\big)$ is isometrically embedded into $L^0\big([0,1],\mathbb{B}\big)^*$ and that

$$L^0\big([0,1],\mathbb{B}\big)^* \cong L^0\big([0,1],\mathbb{B}'\big) \quad \Longleftrightarrow \quad \mathbb{B}' \text{ has the Radon-Nikodým property}, \tag{1.4}$$

where $\mathbb{B}'$ stands for the dual of $\mathbb{B}$ as a Banach space. (We refer to [2] for the definition of the Radon-Nikodým property and its main properties.) ■

Theorem 1.9 (Pullback of a normed $L^0(\mathfrak{m})$-module)**.** *Let $(\mathrm{X}, \mathsf{d}_\mathrm{X}, \mathfrak{m}_\mathrm{X})$, $(\mathrm{Y}, \mathsf{d}_\mathrm{Y}, \mathfrak{m}_\mathrm{Y})$ be metric measure spaces. Let $f \colon \mathrm{X} \to \mathrm{Y}$ be a Borel map with $f_*\mathfrak{m}_\mathrm{X} \ll \mathfrak{m}_\mathrm{Y}$. Then it holds that:*

i) *Let $\mathscr{M}$ be a given normed $L^0(\mathfrak{m}_Y)$-module. Then there exists a unique couple $(f^*\mathscr{M}, f^*)$ –
where $f^*\mathscr{M}$ is a normed $L^0(\mathfrak{m}_X)$-module and $f^*\colon \mathscr{M} \to f^*\mathscr{M}$ is a linear map – such that*

$$|f^*v| = |v| \circ f \quad \mathfrak{m}\text{-a.e.} \quad \text{for every } v \in \mathscr{M},$$
$$\{f^*v \,:\, v \in \mathscr{M}\} \quad \text{generates } f^*\mathscr{M}. \tag{1.5}$$

*Uniqueness is up to unique isomorphism: given any other couple $(\mathscr{M}_0, T)$ with the same
properties, there is a unique normed $L^0(\mathfrak{m}_X)$-module morphism $\Phi\colon f^*\mathscr{M} \to \mathscr{M}_0$ such that*

$$
\begin{array}{ccc}
\mathscr{M} & \xrightarrow{\ f^*\ } & f^*\mathscr{M} \\
& {\scriptstyle T}\searrow & \downarrow{\scriptstyle \Phi} \\
& & \mathscr{M}_0
\end{array}
$$

is a commutative diagram.

ii) *Let $\mathscr{M}$, $\mathscr{N}$ be two given normed $L^0(\mathfrak{m}_Y)$-modules. Let $\varphi\colon \mathscr{M} \to \mathscr{N}$ be a morphism of
normed $L^0(\mathfrak{m}_Y)$-modules. Then there exists a unique morphism $f^*\varphi\colon f^*\mathscr{M} \to f^*\mathscr{N}$ of
normed $L^0(\mathfrak{m}_X)$-modules such that*

$$
\begin{array}{ccc}
\mathscr{M} & \xrightarrow{\ \varphi\ } & \mathscr{N} \\
{\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\
f^*\mathscr{M} & \xrightarrow{\ f^*\varphi\ } & f^*\mathscr{N}
\end{array}
$$

is a commutative diagram.

Remark 1.10. The notion of pullback of a normed $L^0(\mathfrak{m})$-module introduced in Theorem 1.9
above fits in the framework of category theory; we refer to [3, Remark 1.6.4] for the details.   ∎

Example 1.11. Consider two metric measure spaces $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$, $(Z, \mathsf{d}_Z, \mathfrak{m}_Z)$ with $\mathfrak{m}_Z$ finite. We
endow the space $X \coloneqq Z \times Y$ with the product distance $\mathsf{d}_X = \mathsf{d}_Z \times \mathsf{d}_Y$, defined as

$$(\mathsf{d}_Z \times \mathsf{d}_Y)\big((z_1, y_1), (z_2, y_2)\big) \coloneqq \sqrt{\mathsf{d}_Z^2(z_1, z_2) + \mathsf{d}_Y^2(y_1, y_2)} \quad \text{for every } (z_1, y_1), (z_2, y_2) \in X,$$

and the product measure $\mathfrak{m}_X \coloneqq \mathfrak{m}_Z \otimes \mathfrak{m}_Y$. Moreover, we call $\pi\colon X \to Y$ the natural projection
map $(z, y) \mapsto y$, which is continuous and satisfies $\pi_*\mathfrak{m}_X = \mathfrak{m}_Z(Z)\,\mathfrak{m}_Y \ll \mathfrak{m}_Y$.

Given any normed $L^0(\mathfrak{m}_Y)$-module $\mathscr{M}$, we define the space $L^0(Z, \mathscr{M})$ as the family of all
(strongly) Borel maps $V\colon Z \to \mathscr{M}$ considered up to $\mathfrak{m}_Z$-a.e. equality. It is straightforward to check
that the space $L^0(Z, \mathscr{M})$ is a normed $L^0(\mathfrak{m}_X)$-module if equipped with the following operations:

$$(V + W)(z) \coloneqq V(z) + W(z) \in \mathscr{M} \quad \text{for } \mathfrak{m}_Z\text{-a.e. } z \in Z,$$
$$(f \cdot V)(z) \coloneqq f(z, \cdot) \cdot V(z) \in \mathscr{M} \quad \text{for } \mathfrak{m}_Z\text{-a.e. } z \in Z,$$
$$|V|(z, y) \coloneqq \big|V(z)\big|(y) \quad \text{for } \mathfrak{m}_X\text{-a.e. } (z, y) \in X,$$

for every $V, W \in L^0(Z, \mathscr{M})$ and $f \in L^0(\mathfrak{m}_X)$; this constitutes a generalisation of what has been
described in Remark 1.8. Finally, we denote by $T\colon \mathscr{M} \to L^0(Z, \mathscr{M})$ the linear operator sending
any element $v \in \mathscr{M}$ to the map $T(v)\colon Z \to \mathscr{M}$ identically equal to v. We thus claim that

$$\big(L^0(Z, \mathscr{M}), T\big) \cong (\pi^*\mathscr{M}, \pi^*). \tag{1.6}$$

In order to prove it, we need to show that the two properties in (1.5) are satisfied. For the first
one, notice that for any $v \in \mathscr{M}$ it holds that

$$\big|T(v)\big|(z, y) = \big|T(v)(z)\big|(y) = |v|(y) = |v|\big(\pi(z, y)\big) = \big(|v| \circ \pi\big)(z, y) \quad \text{for } \mathfrak{m}_X\text{-a.e. } (z, y) \in X.$$

For the second one, just observe that simple maps (i.e. Borel maps from Z to $\mathscr{M}$ whose range is
of finite cardinality) are dense in $L^0(Z, \mathscr{M})$. Therefore the claim (1.6) is proven.   ∎

1.2. **Direct and inverse limits in a category.** The purpose of this subsection is to recall the notion of direct/inverse limit in an arbitrary category; we refer, for instance, to [9] for a detailed account on this topic.

Fix a *directed (partially ordered) set* $(I, \leq)$, which is a nonempty partially ordered set such that any pair of elements admits an upper bound (i.e. for every $i, j \in I$ there exists $k \in I$ satisfying both $i \leq k$ and $j \leq k$). The directed set $(I, \leq)$ can be considered as a small category $\mathcal{I}$, whose objects are the elements of I and whose morphisms are defined as follows: given any $i, j \in I$, there is a (unique) morphism $i \to j$ if and only if $i \leq j$. Let us also fix an arbitrary category $\mathcal{C}$.

A *direct system* in $\mathcal{C}$ over I is any couple $(\{X_i\}_{i \in I}, \{\varphi_{ij}\}_{i \leq j})$, where $\{X_i : i \in I\}$ is a family of objects of $\mathcal{C}$, while $\{\varphi_{ij} : i, j \in I, i \leq j\}$ is a family of morphisms $\varphi_{ij} \colon X_i \to X_j$ satisfying the following properties:

i) φ_{ii} is the identity of X_i for every $i \in I$.
ii) $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ for every $i, j, k \in I$ with $i \leq j \leq k$.

Equivalently, a direct system in $\mathcal{C}$ over I is a covariant functor $\mathcal{I} \to \mathcal{C}$.

We can define the *direct limit* of the direct system $(\{X_i\}_{i \in I}, \{\varphi_{ij}\}_{i \leq j})$ via a universal property. We say that $(\varinjlim X_\star, \{\varphi_i\}_{i \in I})$ – where $\varinjlim X_\star$ is an object of $\mathcal{C}$ and $\{\varphi_i : i \in I\}$ is a family of morphisms $\varphi_i \colon X_i \to \varinjlim X_\star$ called *canonical morphisms* – is the direct limit of $(\{X_i\}_{i \in I}, \{\varphi_{ij}\}_{i \leq j})$ provided the following properties hold:

a) $(\varinjlim X_\star, \{\varphi_i\}_{i \in I})$ is a *target*, i.e. the diagram

$$X_i \xrightarrow{\varphi_{ij}} X_j$$
$$\varphi_i \searrow \quad \downarrow \varphi_j$$
$$\varinjlim X_\star$$

commutes for every $i, j \in I$ such that $i \leq j$.
b) Given any target $(Y, \{\psi_i\}_{i \in I})$, there exists a unique morphism $\Phi \colon \varinjlim X_\star \to Y$ such that

$$X_i \xrightarrow{\varphi_i} \varinjlim X_\star$$
$$\psi_i \searrow \quad \downarrow \Phi$$
$$Y$$

is a commutative diagram for every $i \in I$.

In general, a direct system in an arbitrary category might not admit a direct limit. Nevertheless, whenever the direct limit exists, it has to be unique up to unique isomorphism: given any other direct limit $(X, \{\varphi_i'\}_{i \in I})$ of $(\{X_i\}_{i \in I}, \{\varphi_{ij}\}_{i \leq j})$, there is a unique isomorphism $\mathscr{I} \colon X \to \varinjlim X_\star$ such that $\varphi_i = \mathscr{I} \circ \varphi_i'$ holds for every $i \in I$.

An *inverse system* in $\mathcal{C}$ over I is any couple $(\{X_i\}_{i \in I}, \{P_{ij}\}_{i \leq j})$, where $\{X_i : i \in I\}$ is a family of objects of $\mathcal{C}$, while $\{P_{ij} : i, j \in I, i \leq j\}$ is a family of morphisms $P_{ij} \colon X_j \to X_i$ satisfying the following properties:

i) P_{ii} is the identity of X_i for every $i \in I$.
ii) $P_{ik} = \varphi_{ij} \circ \varphi_{jk}$ for every $i, j, k \in I$ with $i \leq j \leq k$.

Equivalently, an inverse system in $\mathcal{C}$ over I is a contravariant functor $\mathcal{I} \to \mathcal{C}$.

We can define the *inverse limit* of the inverse system $(\{X_i\}_{i \in I}, \{P_{ij}\}_{i \leq j})$ via a universal property. We say that $(\varprojlim X_\star, \{P_i\}_{i \in I})$ – where $\varprojlim X_\star$ is an object of $\mathcal{C}$ and $\{P_i : i \in I\}$ is a family of

morphisms $P_i\colon \varprojlim X_\star \to X_i$ called *natural projections* – is the inverse limit of $\big(\{X_i\}_{i\in I}, \{P_{ij}\}_{i\leq j}\big)$ provided the following properties hold:

a) The diagram

$$
\begin{array}{ccc}
\varprojlim X_\star & & \\
{\scriptstyle P_j}\downarrow & \searrow^{P_i} & \\
X_j & \xrightarrow[P_{ij}]{} & X_i
\end{array}
$$

commutes for every $i, j \in I$ such that $i \leq j$.

b) Given any other such couple $\big(Y, \{Q_i\}_{i\in I}\big)$ – namely satisfying $Q_i = P_{ij} \circ Q_j$ for all $i, j \in I$ with $i \leq j$ – there exists a unique morphism $\Phi\colon Y \to \varprojlim X_\star$ such that

$$
\begin{array}{ccc}
Y & \xrightarrow{\ \Phi\ } & \varprojlim X_\star \\
& {\scriptstyle Q_i}\searrow & \downarrow^{P_i} \\
& & X_i
\end{array}
$$

is a commutative diagram for every $i \in I$.

In general, an inverse system in an arbitrary category does not necessarily admit an inverse limit. Nevertheless, whenever the inverse limit exists, it has to be unique up to unique isomorphism: given any other inverse limit $\big(X, \{P'_i\}_{i\in I}\big)$, there exists a unique isomorphism $\mathscr{I}\colon X \to \varprojlim X_\star$ such that $P'_i = P_i \circ \mathscr{I}$ holds for every $i \in I$.

1.3. **Direct and inverse limits of R-modules.** For the usefulness of the reader, we report here the construction of the direct/inverse limit of (algebraic) modules over a commutative ring. The material we are going to present can be found, e.g., in [7]. Let us fix a commutative ring R and a directed partially ordered set $(I, \leq)$.

Let $\big(\{M_i\}_{i\in I}, \{\varphi_{ij}\}_{i\leq j}\big)$ be a direct system of R-modules over I. We define an equivalence relation $\sim$ on $\bigsqcup_{i\in I} M_i$: given $v \in M_i$ and $w \in M_j$, we declare that $v \sim w$ provided there is $k \in I$ with $i, j \leq k$ such that $\varphi_{ik}(v) = \varphi_{jk}(w)$. Then we define the direct limit of $\big(\{M_i\}_{i\in I}, \{\varphi_{ij}\}_{i\leq j}\big)$ as

$$
\varinjlim M_\star := \bigsqcup_{i\in I} M_i \Big/ \sim .
$$

The R-module operations on $\varinjlim M_\star$ are defined in the following way:

- Let $\boldsymbol{v}, \boldsymbol{w} \in \varinjlim M_\star$ be fixed. Pick any $v \in \boldsymbol{v} \cap M_i$ and $w \in \boldsymbol{w} \cap M_j$. Choose some $k \in I$ such that $i, j \leq k$. Notice that $\varphi_{ik}(v) \in \boldsymbol{v} \cap M_k$ and $\varphi_{jk}(w) \in \boldsymbol{w} \cap M_k$. Then we define the sum $\boldsymbol{v} + \boldsymbol{w} \in \varinjlim M_\star$ as the equivalence class of $\varphi_{ik}(v) + \varphi_{jk}(w) \in M_k$.
- Let $\boldsymbol{v} \in \varinjlim M_\star$ and $r \in R$ be fixed. Pick any $v \in \boldsymbol{v} \cap M_i$. Then we define $r \cdot \boldsymbol{v} \in \varinjlim M_\star$ as the equivalence class of $r \cdot v \in M_i$.

It is easy to check that such operations are well-posed and the resulting structure $\big(\varinjlim M_\star, +, \cdot\big)$ satisfies the R-module axioms. The canonical morphisms $\varphi_i\colon M_i \to \varinjlim M_\star$ are obtained by sending each element to its equivalence class. We point out a fundamental property:

$$
\text{For every } \boldsymbol{v} \in \varinjlim M_\star \text{ there exist } i \in I \text{ and } v \in M_i \text{ such that } \varphi_i(v) = \boldsymbol{v}. \tag{1.7}
$$

The above claim is a direct consequence of the very definition of $\varinjlim M_\star$.

Now let us consider an inverse system $\big(\{M_i\}_{i\in I}, \{P_{ij}\}_{i\leq j}\big)$ of R-modules over I. Then we define its inverse limit as

$$
\varprojlim M_\star := \Big\{ v = \{v_i\}_{i\in I} \in \prod_{i\in I} M_i \,\Big|\, v_i = P_{ij}(v_j) \text{ for every } i, j \in I \text{ with } i \leq j \Big\}.
$$

The direct product $\prod_{i \in I} M_i$ has a natural R-module structure with respect to the element-wise operations. It can be readily shown that $\varprojlim M_\star$ is an R-submodule of $\prod_{i \in I} M_i$. Finally, the natural projections $\mathrm{P}_i \colon \varprojlim M_\star \to M_i$ are defined as

$$\mathrm{P}_i(v) \coloneqq v_i \quad \text{for every } v = \{v_i\}_{i \in I} \in \varprojlim M_\star.$$

In particular, given any family $\{v_i\}_{i \in I}$ such that $v_i \in M_i$ and $v_i = \mathrm{P}_{ij}(v_j)$ hold for every $i, j \in I$ with $i \le j$, there exists a unique element $v \in \varprojlim M_\star$ such that $\mathrm{P}_i(v) = v_i$ for every $i \in I$.

2. Direct limits of normed $L^0(\mathfrak{m})$-modules

2.1. **Definition.** Unless otherwise specified, let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a fixed metric measure space. The aim of this subsection is to prove that direct limits exist in the category of normed $L^0(\mathfrak{m})$-modules.

Theorem 2.1 (Direct limit of normed $L^0(\mathfrak{m})$-modules)**.** *Let $\big(\{\mathscr{M}_i\}_{i \in I}, \{\varphi_{ij}\}_{i \le j}\big)$ be a direct system of normed $L^0(\mathfrak{m})$-modules. Then its direct limit $\big(\varinjlim \mathscr{M}_\star, \{\varphi_i\}_{i \in I}\big)$ exists in the category of normed $L^0(\mathfrak{m})$-modules.*

Proof. Since $\big(\{\mathscr{M}_i\}_{i \in I}, \{\varphi_{ij}\}_{i \le j}\big)$ is a direct system in the category of algebraic $L^0(\mathfrak{m})$-modules, we can consider its direct limit $\big(\mathscr{M}_{\mathrm{Alg}}, \{\varphi_i'\}_{i \in I}\big)$ in such category (cf. Subsection 1.3). It can be readily checked that the following formula defines a pointwise seminorm on $\mathscr{M}_{\mathrm{Alg}}$:

$$|\boldsymbol{v}| \coloneqq \operatorname{ess\,inf}\big\{|v| \,:\, i \in I, \, v \in \mathscr{M}_i, \, \varphi_i'(v) = \boldsymbol{v}\big\} \quad \text{for every } \boldsymbol{v} \in \mathscr{M}_{\mathrm{Alg}}. \tag{2.1}$$

Clearly $|\boldsymbol{v}| \in L^0(\mathfrak{m})$ for every $\boldsymbol{v} \in \mathscr{M}_{\mathrm{Alg}}$ by (1.7). Consider the equivalence relation $\sim$ on $\mathscr{M}_{\mathrm{Alg}}$ as in Lemma 1.5 and the metric completion $\big(\varinjlim \mathscr{M}_\star, \iota\big)$ of $\mathscr{M}_{\mathrm{Alg}}/\sim$ as in Lemma 1.6. For $i \in I$ we set the map $\varphi_i \colon \mathscr{M}_i \to \varinjlim \mathscr{M}_\star$ as $\varphi_i(v) \coloneqq \iota\big[\varphi_i'(v)\big]_\sim \in \varinjlim \mathscr{M}_\star$ for all $v \in \mathscr{M}_i$. We claim that

$$\big(\varinjlim \mathscr{M}_\star, \{\varphi_i\}_{i \in I}\big) \text{ is the direct limit of } \big(\{\mathscr{M}_i\}_{i \in I}, \{\varphi_{ij}\}_{i \le j}\big) \text{ as normed } L^0(\mathfrak{m})\text{-modules.}$$

First of all, we know that $\varinjlim \mathscr{M}_\star$ is a normed $L^0(\mathfrak{m})$-module from Lemmata 1.5 and 1.6. Each φ_i is $L^0(\mathfrak{m})$-linear as composition of $L^0(\mathfrak{m})$-module morphisms, while for every $v \in \mathscr{M}_i$ it holds that

$$\big|\varphi_i(v)\big| = \big|\varphi_i'(v)\big| \overset{(2.1)}{\le} |v| \quad \text{in the } \mathfrak{m}\text{-a.e. sense,}$$

thus proving that φ_i is a normed $L^0(\mathfrak{m})$-module morphism. Given that $\varphi_j' \circ \varphi_{ij} = \varphi_i'$ for all $i, j \in I$ with $i \le j$, we immediately deduce that $\varphi_j \circ \varphi_{ij} = \varphi_i$ as well. Therefore it only remains to prove the universal property: let $\big(\mathscr{N}, \{\psi_i\}_{i \in I}\big)$ be any given target. It is a target even in the category of algebraic $L^0(\mathfrak{m})$-modules, therefore there exists a unique $L^0(\mathfrak{m})$-morphism $\Phi' \colon \mathscr{M}_{\mathrm{Alg}} \to \mathscr{N}$ such that $\Phi' \circ \varphi_i' = \psi_i$ for every $i \in I$. Then we are forced to define the map $\Phi \colon \iota(\mathscr{M}_{\mathrm{Alg}}/\sim) \to \mathscr{N}$ as

$$\Phi\big(\iota[\boldsymbol{v}]_\sim\big) \coloneqq \Phi'(\boldsymbol{v}) \quad \text{for every } \boldsymbol{v} \in \mathscr{M}_{\mathrm{Alg}}. \tag{2.2}$$

Observe that for any $\boldsymbol{v} \in \mathscr{M}_{\mathrm{Alg}}$ we have

$$\big|\Phi'(\boldsymbol{v})\big| = \big|\psi_i(v)\big| \le |v| \quad \mathfrak{m}\text{-a.e.} \quad \text{for every } i \in I \text{ and } v \in \mathscr{M}_i \text{ with } \varphi_i'(v) = \boldsymbol{v},$$

thus accordingly $\big|\Phi'(\boldsymbol{v})\big| \le |\boldsymbol{v}|$ holds $\mathfrak{m}$-a.e. in X. This grants that the operator Φ in (2.2) is well-defined and can be uniquely extended to a normed $L^0(\mathfrak{m})$-module morphism $\Phi \colon \varinjlim \mathscr{M}_\star \to \mathscr{N}$. This proves the universal property and concludes the proof of the statement. $\square$

Remark 2.2. It follows from the proof of Theorem 2.1 that $\bigcup_{i \in I} \varphi_i(\mathscr{M}_i)$ is dense in $\varinjlim \mathscr{M}_\star$. In particular, if I is countable and each $\mathscr{M}_i$ is separable, then $\varinjlim \mathscr{M}_\star$ is separable as well. $\blacksquare$

Definition 2.3 (Morphism of direct systems of normed $L^0(\mathfrak{m})$-modules). *A morphism Θ between two direct systems $\big(\{\mathscr{M}_i\}_{i\in I}, \{\varphi_{ij}\}_{i\leq j}\big)$ and $\big(\{\mathscr{N}_i\}_{i\in I}, \{\psi_{ij}\}_{i\leq j}\big)$ of normed $L^0(\mathfrak{m})$-modules is a family $\Theta = \{\theta_i\}_{i\in I}$ of normed $L^0(\mathfrak{m})$-module morphisms $\theta_i \colon \mathscr{M}_i \to \mathscr{N}_i$ such that*

$$
\begin{array}{ccc}
\mathscr{M}_i & \xrightarrow{\ \theta_i\ } & \mathscr{N}_i \\
{\scriptstyle \varphi_{ij}}\Big\downarrow & & \Big\downarrow{\scriptstyle \psi_{ij}} \\
\mathscr{M}_j & \xrightarrow[\ \theta_j\ ]{} & \mathscr{N}_j
\end{array}
$$

is a commutative diagram for every $i,j \in I$ with $i \leq j$.

With the notion of morphism just introduced, it makes sense to consider the category of direct systems of normed $L^0(\mathfrak{m})$-modules. Then the correspondence sending a direct system of normed $L^0(\mathfrak{m})$-modules to its direct limit can be made into a functor, as shown by the following result.

Theorem 2.4 (The direct limit functor $\varinjlim$). *Let $\Theta = \{\theta_i\}_{i\in I}$ be a morphism between two direct systems $\big(\{\mathscr{M}_i\}_{i\in I}, \{\varphi_{ij}\}_{i\leq j}\big)$ and $\big(\{\mathscr{N}_i\}_{i\in I}, \{\psi_{ij}\}_{i\leq j}\big)$ of normed $L^0(\mathfrak{m})$-modules, whose direct limits are denoted by $\big(\varinjlim \mathscr{M}_\star, \{\varphi_i\}_{i\in I}\big)$ and $\big(\varinjlim \mathscr{N}_\star, \{\psi_i\}_{i\in I}\big)$, respectively. Then there exists a unique normed $L^0(\mathfrak{m})$-module morphism $\varinjlim \theta_\star \colon \varinjlim \mathscr{M}_\star \to \varinjlim \mathscr{N}_\star$ such that the diagram*

$$
\begin{array}{ccc}
\mathscr{M}_i & \xrightarrow{\ \theta_i\ } & \mathscr{N}_i \\
{\scriptstyle \varphi_i}\Big\downarrow & & \Big\downarrow{\scriptstyle \psi_i} \\
\varinjlim \mathscr{M}_\star & \xrightarrow[\ \varinjlim \theta_\star\ ]{} & \varinjlim \mathscr{N}_\star
\end{array}
\tag{2.3}
$$

commutes for every $i \in I$. In particular, the correspondence $\varinjlim$ is a covariant functor from the category of direct systems of normed $L^0(\mathfrak{m})$-modules to the category of normed $L^0(\mathfrak{m})$-modules.

Proof. Let us denote by $\big(\mathscr{M}_{\mathrm{Alg}}, \{\varphi_i'\}_{i\in I}\big)$ and $\big(\mathscr{N}_{\mathrm{Alg}}, \{\psi_i'\}_{i\in I}\big)$ the direct limits (in the category of algebraic $L^0(\mathfrak{m})$-modules) of $\big(\{\mathscr{M}_i\}_{i\in I}, \{\varphi_{ij}\}_{i\leq j}\big)$ and $\big(\{\mathscr{N}_i\}_{i\in I}, \{\psi_{ij}\}_{i\leq j}\big)$, respectively. We define the map $\theta' \colon \mathscr{M}_{\mathrm{Alg}} \to \mathscr{N}_{\mathrm{Alg}}$ as follows: given any $\boldsymbol{v} \in \mathscr{M}_{\mathrm{Alg}}$, there exist $i \in I$ and $v \in \mathscr{M}_i$ such that $\varphi_i'(v) = \boldsymbol{v}$ by (1.7), thus we set $\theta'(\boldsymbol{v}) \coloneqq (\psi_i' \circ \theta_i)(v)$. It is straightforward to verify that θ' is well-defined and is the unique $L^0(\mathfrak{m})$-module morphism such that $\theta' \circ \varphi_i' = \psi_i' \circ \theta_i$ for every $i \in I$. As in the proof of Theorem 2.1, let us consider the dense-range operators $\iota \colon \mathscr{M}_{\mathrm{Alg}}/\sim\, \to \varinjlim \mathscr{M}_\star$ and $\iota \colon \mathscr{N}_{\mathrm{Alg}}/\sim\, \to \varinjlim \mathscr{N}_\star$ given by Lemmata 1.5 and 1.6. It can be readily checked that there exists a unique $L^0(\mathfrak{m})$-module morphism $\theta \colon \iota(\mathscr{M}_{\mathrm{Alg}}/\sim) \to \iota(\mathscr{N}_{\mathrm{Alg}}/\sim)$ such that

$$
\theta\big(\iota[\boldsymbol{v}]_\sim\big) = \iota\big[\theta'(\boldsymbol{v})\big]_\sim \quad \text{for every } \boldsymbol{v} \in \mathscr{M}_{\mathrm{Alg}}.
$$

Its well-posedness is granted by the $\mathfrak{m}$-a.e. inequality $\big|\iota\big[\theta'(\boldsymbol{v})\big]_\sim\big| \leq \big|\iota[\boldsymbol{v}]_\sim\big|$, which is satisfied for every $\boldsymbol{v} \in \mathscr{M}_{\mathrm{Alg}}$ as a consequence of the following observation:

$$
\big|\iota\big[\theta'(\boldsymbol{v})\big]_\sim\big| = \big|\theta'(\boldsymbol{v})\big| = \big|(\psi_i' \circ \theta_i)(v)\big| \overset{(2.1)}{\leq} \big|\theta_i(v)\big| \leq |v| \quad \text{holds } \mathfrak{m}\text{-a.e.}
$$

for every $i \in I$ and $v \in \mathscr{M}_i$ such that $\varphi_i'(v) = \boldsymbol{v}$, whence $\big|\iota\big[\theta'(\boldsymbol{v})\big]_\sim\big| \leq |\boldsymbol{v}| = \big|\iota[\boldsymbol{v}]_\sim\big|$ holds $\mathfrak{m}$-a.e. again by (2.1). This also ensures that θ can be uniquely extended to a normed $L^0(\mathfrak{m})$-module morphism $\varinjlim \theta_\star \colon \varinjlim \mathscr{M}_\star \to \varinjlim \mathscr{N}_\star$, which is the unique morphism satisfying $\varinjlim \theta_\star \circ \varphi_i = \psi_i \circ \theta_i$ for every $i \in I$. This concludes the proof of the statement. $\qquad\square$

2.2. **Main properties.** In this subsection we collect the most important properties of direct limits of normed $L^0(\mathfrak{m})$-modules.

Lemma 2.5. *Any normed $L^0(\mathfrak{m})$-module can be written as direct limit of some direct system of finitely-generated normed $L^0(\mathfrak{m})$-modules.*

Proof. Let $\mathscr{M}$ be a normed $L^0(\mathfrak{m})$-module. Choose any set D that generates $\mathscr{M}$ (possibly $D = \mathscr{M}$). We denote by $\mathcal{P}_F(D)$ the family of all finite subsets of D. Now choose any subset I of $\mathcal{P}_F(D)$ that is a directed partially ordered set with respect to the inclusion relation $\subseteq$ and such that $\bigcup_{F \in I} F$ generates $\mathscr{M}$ (for instance, $\mathcal{P}_F(D)$ itself satisfies these properties). Then let us define

$$\mathscr{M}_F := \text{submodule of } \mathscr{M} \text{ generated by } F,$$

$$\iota_{FG} : \mathscr{M}_F \hookrightarrow \mathscr{M}_G \text{ inclusion map,}$$

for every $F, G \in I$ with $F \subseteq G$. It is then clear that $\big(\{\mathscr{M}_F\}_{F \in I}, \{\iota_{FG}\}_{F \subseteq G}\big)$ is a direct system of (finitely-generated) normed $L^0(\mathfrak{m})$-modules. We claim that

$$\mathscr{M} \cong \varinjlim \mathscr{M}_\star, \tag{2.4}$$

the canonical morphisms being given by the inclusion maps $\iota_F : \mathscr{M}_F \hookrightarrow \mathscr{M}$. First, $\big(\mathscr{M}, \{\iota_F\}_{F \in I}\big)$ is obviously a target. To prove the universal property, fix another target $\big(\mathscr{N}, \{\psi_F\}_{F \in I}\big)$. Notice that the $L^0(\mathfrak{m})$-module $\bigcup_{F \in I} \mathscr{M}_F$ is dense in $\mathscr{M}$ by construction. Therefore it can be readily checked that there is a unique morphism $\Phi : \mathscr{M} \to \mathscr{N}$ such that $\Phi(v) = \psi_F(v)$ holds for all $F \in I$ and $v \in \mathscr{M}_F$, or equivalently $\Phi \circ \iota_F = \psi_F$ for every $F \in I$. This proves the claim (2.4). $\qquad\square$

Corollary 2.6. *Let $\mathscr{M}$ be a separable normed $L^0(\mathfrak{m})$-module. Let $(v_n)_{n \in \mathbb{N}}$ be a countable dense subset of $\mathscr{M}$. Given any $n, m \in \mathbb{N}$ with $n \leq m$, let us define:*

i) *$\mathscr{M}_n$ as the module generated by $\{v_1, \ldots, v_n\}$,*
ii) *$\iota_n : \mathscr{M}_n \hookrightarrow \mathscr{M}$ and $\iota_{nm} : \mathscr{M}_n \hookrightarrow \mathscr{M}_m$ as the inclusion maps.*

Then it holds that $\big(\mathscr{M}, \{\iota_n\}_{n \in \mathbb{N}}\big)$ is the direct limit of the direct system $\big(\{\mathscr{M}_n\}_{n \in \mathbb{N}}, \{\iota_{nm}\}_{n \leq m}\big)$.

Proof. Notice that $D := (v_n)_{n \in \mathbb{N}}$ generates $\mathscr{M}$. Then the statement follows from the proof of Lemma 2.5 by choosing as I the family of all subsets of D of the form $\{v_1, \ldots, v_n\}$ with $n \in \mathbb{N}$. $\qquad\square$

The category of normed $L^0(\mathfrak{m})$-modules is a pointed category, its zero object being the trivial space $\{0\}$. Given two normed $L^0(\mathfrak{m})$-modules $\mathscr{M}, \mathscr{N}$ and a morphism $\varphi : \mathscr{M} \to \mathscr{N}$, it holds that:

i) The kernel of φ is the normed $L^0(\mathfrak{m})$-submodule $\ker(\varphi) := \big\{v \in \mathscr{M} : \varphi(v) = 0\big\}$ of $\mathscr{M}$ (together with the inclusion map $\ker(\varphi) \hookrightarrow \mathscr{M}$).
ii) The image of φ is the normed $L^0(\mathfrak{m})$-submodule $\mathrm{im}(\varphi)$ of $\mathscr{N}$ generated by the set-theoretic range $\varphi(\mathscr{M})$ of φ (together with the inclusion map $\mathrm{im}(\varphi) \hookrightarrow \mathscr{N}$). Observe that $\varphi(\mathscr{M})$ is an $L^0(\mathfrak{m})$-submodule of $\mathscr{N}$, thus $\mathrm{im}(\varphi)$ coincides with the closure of $\varphi(\mathscr{M})$ in $\mathscr{N}$.

Remark 2.7. In general, the set-theoretic range of a normed $L^0(\mathfrak{m})$-module morphism might be not complete. For instance, consider the Banach spaces ℓ^∞ and c_0, which can be regarded as normed $L^0(\mathfrak{m})$-modules provided the measure $\mathfrak{m}$ is a Dirac delta (as pointed out in Example 1.3). The linear contraction $\varphi : \ell^\infty \to c_0$, defined as

$$\varphi\big((t_k)_{k \in \mathbb{N}}\big) := (t_k/k)_{k \in \mathbb{N}} \quad \text{for every } (t_k)_{k \in \mathbb{N}} \in \ell^\infty,$$

is injective and its range $\varphi(\ell^\infty)$ is dense in c_0. The latter is granted by the following fact: the space c_{00} (i.e. the space of all real-valued sequences having finitely many non-zero terms) is dense in c_0 and is contained in $\varphi(\ell^\infty)$. On the other hand, the operator φ cannot be surjective, as c_0 is separable while ℓ^∞ is not. Therefore the normed space $\varphi(\ell^\infty)$ is not complete. $\qquad\blacksquare$

The category of direct systems of normed $L^0(\mathfrak{m})$-modules is a pointed category, whose zero object is the direct system $\big(\{\mathscr{M}_i\}_{i \in I}, \{\varphi_{ij}\}_{i \leq j}\big)$ given by $\mathscr{M}_i := \{0\}$ for all $i \in I$ and $\varphi_{ij} := 0$ for all $i, j \in I$ with $i \leq j$. Given a morphism $\Theta = \{\theta_i\}_{i \in I}$ of two direct systems $\big(\{\mathscr{M}_i\}_{i \in I}, \{\varphi_{ij}\}_{i \leq j}\big)$ and $\big(\{\mathscr{N}_i\}_{i \in I}, \{\psi_{ij}\}_{i \leq j}\big)$ of normed $L^0(\mathfrak{m})$-modules, it holds that:

a) The kernel $\ker(\Theta)$ of Θ is given by $\big(\{\ker(\theta_i)\}_{i\in I}, \{\varphi_{ij}|_{\ker(\theta_i)}\}_{i\leq j}\big)$.

b) The image $\operatorname{im}(\Theta)$ of Θ is given by $\big(\{\operatorname{im}(\theta_i)\}_{i\in I}, \{\psi_{ij}|_{\operatorname{im}(\theta_i)}\}_{i\leq j}\big)$.

Items a) and b) make sense, since $\varphi_{ij}\big(\ker(\theta_i)\big) \subseteq \ker(\theta_j)$ and $\psi_{ij}\big(\operatorname{im}(\theta_i)\big) \subseteq \operatorname{im}(\theta_j)$ whenever $i \leq j$.

Proposition 2.8. *Let $\Theta = \{\theta_i\}_{i\in I}$ be a morphism between two direct systems $\big(\{\mathscr{M}_i\}_{i\in I}, \{\varphi_{ij}\}_{i\leq j}\big)$ and $\big(\{\mathscr{N}_i\}_{i\in I}, \{\psi_{ij}\}_{i\leq j}\big)$ of normed $L^0(\mathfrak{m})$-modules such that $\operatorname{im}(\Theta) = \big(\{\mathscr{N}_i\}_{i\in I}, \{\psi_{ij}\}_{i\leq j}\big)$. Then it holds that $\operatorname{im}\big(\varinjlim \theta_\star\big) = \varinjlim \mathscr{N}_\star$.*

Proof. First of all, we know that:

- $\theta_i(\mathscr{M}_i)$ is dense in $\mathscr{N}_i$ for every $i \in I$, as $\operatorname{im}(\theta_i) = \mathscr{N}_i$ by assumption.
- $\bigcup_{i\in I} \psi_i(\mathscr{N}_i)$ is dense in $\varinjlim \mathscr{N}_\star$ by Remark 2.2.

Hence $\bigcup_{i\in I}(\theta \circ \varphi_i)(\mathscr{M}_i)$ is dense in $\varinjlim \mathscr{N}_\star$ by (2.3), where θ stands for $\varinjlim \theta_\star$. This ensures that the set $\theta\big(\varinjlim \mathscr{M}_\star\big) \supseteq \bigcup_{i\in I}(\theta \circ \varphi_i)(\mathscr{M}_i)$ is dense in $\varinjlim \mathscr{N}_\star$ as well, thus getting the statement. $\square$

Remark 2.9. The dual statement of that of Proposition 2.8 fails in general, since it is possible to build a morphism $\Theta = \{\theta_i\}_{i\in I}$ of direct systems with $\ker(\Theta) = 0$ such that $\ker\big(\varinjlim \theta_\star\big) \neq 0$.

For instance, suppose that $\mathfrak{m} = \delta_{\bar{x}}$ for some $\bar{x} \in X$, so that we are dealing with Banach spaces (as observed in Example 1.3). Consider the sequence space ℓ^2 and the morphism $T\colon \ell^2 \to \ell^2$ defined as $T(\lambda_1, \lambda_2, \lambda_3, \ldots) := (0, \lambda_2, \lambda_3, \ldots)$. Moreover, let us define the sequence $(a_n)_{n\in\mathbb{N}} \subseteq \ell^2$ as follows: $a_1 := (1/k)_{k\in\mathbb{N}}$ and $a_n := (\delta_{kn})_{k\in\mathbb{N}}$ for all $n \geq 2$. Then we set $\mathscr{M}_n := \operatorname{span}\{a_1, \ldots, a_n\}$ and $\mathscr{N}_n := \ell^2$ for every $n \in \mathbb{N}$, while for every $n \leq m$ we define the morphisms $\varphi_{nm}\colon \mathscr{M}_n \to \mathscr{M}_m$ and $\psi_{nm}\colon \mathscr{N}_n \to \mathscr{N}_m$ as the inclusion map and the identity map, respectively. Finally, let us define the morphism $\theta_n\colon \mathscr{M}_n \to \mathscr{N}_n$ as $\theta_n := T_{|\mathscr{M}_n}$ for every $n \in \mathbb{N}$. Therefore it is immediate to check that $\big(\{\mathscr{M}_n\}_{n\in\mathbb{N}}, \{\varphi_{nm}\}_{n\leq m}\big)$, $\big(\{\mathscr{N}_n\}_{n\in\mathbb{N}}, \{\psi_{nm}\}_{n\leq m}\big)$ are direct systems of Banach spaces and that $\Theta := \{\theta_n\}_{n\in\mathbb{N}}$ is a morphism between them satisfying $\ker(\Theta) = 0$. Obviously $\varinjlim \mathscr{N}_\star = \ell^2$, but also $\varinjlim \mathscr{M}_\star = \ell^2$ by Corollary 2.6 and by density of the sequence $(a_n)_{n\in\mathbb{N}}$ in ℓ^2. It also turns out that $\varinjlim \theta_\star = T$. This yields the desired counterexample, as the map T is not injective. $\blacksquare$

Lemma 2.10. *Suppose that the directed set $(I, \leq)$ admits a greatest element $m \in I$. Then for any direct system $\big(\{\mathscr{M}_i\}_{i\in I}, \{\varphi_{ij}\}_{i\leq j}\big)$ of normed $L^0(\mathfrak{m})$-modules it holds that*

$$\big(\mathscr{M}_m, \{\varphi_{im}\}_{i\in I}\big) \text{ is the direct limit of } \big(\{\mathscr{M}_i\}_{i\in I}, \{\varphi_{ij}\}_{i\leq j}\big). \tag{2.5}$$

In particular, given any morphism $\Theta = \{\theta_i\}_{i\in I}$ between two direct systems $\big(\{\mathscr{M}_i\}_{i\in I}, \{\varphi_{ij}\}_{i\leq j}\big)$ and $\big(\{\mathscr{N}_i\}_{i\in I}, \{\psi_{ij}\}_{i\leq j}\big)$ of normed $L^0(\mathfrak{m})$-modules, it holds that $\varinjlim \theta_\star = \theta_m$.

Proof. It easily follows from the fact that m is the greatest element of $(I, \leq)$ that $\big(\mathscr{M}_m, \{\varphi_{im}\}_{i\in I}\big)$ is a target. To prove the universal property, fix another target $\big(\mathscr{N}, \{\psi_i\}_{i\in I}\big)$. Then ψ_m is the unique normed $L^0(\mathfrak{m})$-module morphism between $\mathscr{M}_m$ and $\mathscr{N}$ such that $\psi_m \circ \varphi_{im} = \psi_i$ holds for every $i \in I$, which shows the validity of the universal property and accordingly the claim (2.5). $\square$

Remark 2.11. The direct limit functor $\varinjlim$ is neither faithul nor full, as we are going to prove.

Suppose that $I = \{0, 1\}$ and that $\mathfrak{m} = \delta_{\bar{x}}$ for some $\bar{x} \in X$. Set $\mathscr{M}_0 = \mathscr{M}_1 = \mathscr{N}_0 = \mathscr{N}_1 := \mathbb{R}^2$, viewed as Banach spaces with the usual Euclidean norm (recall Example 1.3). We also define the maps $\varphi_{01}\colon \mathscr{M}_0 \to \mathscr{M}_1$ and $\psi_{01}\colon \mathscr{N}_0 \to \mathscr{N}_1$ as $\varphi_{01}(x, y) := (x, y)$ and $\psi_{01}(x, y) := (x, 0)$, respectively. We have that $\varinjlim \mathscr{M}_\star = \mathscr{M}_1$ and $\varinjlim \mathscr{N}_\star = \mathscr{N}_1$ by Lemma 2.10.

i) Let us consider the morphisms $\Theta = \{\theta_0, \theta_1\}$ and $H = \{\eta_0, \eta_1\}$ between the two direct systems $\big(\{\mathscr{M}_0, \mathscr{M}_1\}, \{\varphi_{00}, \varphi_{01}, \varphi_{11}\}\big)$ and $\big(\{\mathscr{N}_0, \mathscr{N}_1\}, \{\psi_{00}, \psi_{01}, \psi_{11}\}\big)$ defined as follows:

$$\theta_0(x, y) := (x, y),$$

$$\eta_0(x, y) := (x, -y),$$

$$\theta_1(x, y) = \eta_1(x, y) := (x, 0).$$

Then $\Theta \neq H$, but $\varinjlim \theta_\star = \varinjlim \eta_\star$ by Lemma 2.10.

ii) Consider the morphism $\theta_1 \colon \mathscr{M}_1 \to \mathscr{N}_1$ given by $\theta_1(x, y) := (x, y)$. Then there cannot exist a normed $L^0(\mathfrak{m})$-module morphism $\theta_0 \colon \mathscr{M}_0 \to \mathscr{N}_0$ such that $\theta_1 \circ \varphi_{01} = \psi_{01} \circ \theta_0$, the reason being that the map $\theta_1 \circ \varphi_{01}$ is surjective while ψ_{01} is not. This means that we cannot write the morphism $\theta_1 \colon \varinjlim \mathscr{M}_i \to \varinjlim \mathscr{N}_i$ as $\varinjlim \theta_\star$ for some morphism $\{\theta_0, \theta_1\}$ between the direct systems $\big(\{\mathscr{M}_0, \mathscr{M}_1\}, \{\varphi_{00}, \varphi_{01}, \varphi_{11}\}\big)$ and $\big(\{\mathscr{N}_0, \mathscr{N}_1\}, \{\psi_{00}, \psi_{01}, \psi_{11}\}\big)$.

Items i) and ii) above show that the functor $\varinjlim$ is neither faithul nor full, respectively. ∎

Theorem 2.12 (Pullback and direct limit commute). *Let $(X, d_X, \mathfrak{m}_X)$, $(Y, d_Y, \mathfrak{m}_Y)$ be metric measure spaces. Let $f \colon X \to Y$ be a Borel map with $f_* \mathfrak{m}_X \ll \mathfrak{m}_Y$. Let $\big(\{\mathscr{M}_i\}_{i \in I}, \{\varphi_{ij}\}_{i \leq j}\big)$ be a direct system of normed $L^0(\mathfrak{m}_Y)$-modules, whose direct limit is denoted by $\big(\varinjlim \mathscr{M}_\star, \{\varphi_i\}_{i \in I}\big)$. Then $\big(\{f^* \mathscr{M}_i\}_{i \in I}, \{f^* \varphi_{ij}\}_{i \leq j}\big)$ is a direct system of normed $L^0(\mathfrak{m}_X)$-modules. Its direct limit is*

$$\varinjlim f^* \mathscr{M}_\star \cong f^* \varinjlim \mathscr{M}_\star \tag{2.6}$$

together with the canonical morphisms $\{f^* \varphi_i\}_{i \in I}$.

Proof. It follows from Theorem 1.9 that the diagram

$$
\begin{array}{ccccccc}
\mathscr{M}_i & \xrightarrow{\varphi_{ij}} & \mathscr{M}_j & \xrightarrow{\varphi_{jk}} & \mathscr{M}_k & \xrightarrow{\varphi_k} & \varinjlim \mathscr{M}_\star \\
{\scriptstyle f^*} \downarrow & & {\scriptstyle f^*} \downarrow & & {\scriptstyle f^*} \downarrow & & {\scriptstyle f^*} \downarrow \\
f^* \mathscr{M}_i & \xrightarrow[f^* \varphi_{ij}]{} & f^* \mathscr{M}_j & \xrightarrow[f^* \varphi_{jk}]{} & f^* \mathscr{M}_k & \xrightarrow[f^* \varphi_k]{} & f^* \varinjlim \mathscr{M}_\star
\end{array}
$$

commutes for every $i, j, k \in I$ with $i \leq j \leq k$, whence accordingly $\big(\{f^* \mathscr{M}_i\}_{i \in I}, \{f^* \varphi_{ij}\}_{i \leq j}\big)$ is a direct system of normed $L^0(\mathfrak{m}_X)$-modules having $\big(f^* \varinjlim \mathscr{M}_\star, \{f^* \varphi_i\}_{i \in I}\big)$ as a target. Given any other target $\big(\mathscr{N}, \{\psi_i\}_{i \in I}\big)$, there exists a unique morphism $\Phi \colon f^* \varinjlim \mathscr{M}_\star \to \mathscr{N}$ such that

$$\Phi\big(f^*\big(\varphi_i(v)\big)\big) = \psi_i(f^* v) \quad \text{for every } i \in I \text{ and } v \in \mathscr{M}_i, \tag{2.7}$$

as we are going to show. Since $\bigcup_{i \in I} \varphi_i(\mathscr{M}_i)$ is a dense submodule of $\varinjlim \mathscr{M}_\star$ (cf. Remark 2.2), we know that $\bigcup_{i \in I} \big\{ f^*\big(\varphi_i(v)\big) \, : \, i \in I, \, v \in \mathscr{M}_i \big\}$ generates $f^* \varinjlim \mathscr{M}_\star$ by Theorem 1.9, whence uniqueness follows. To prove (well-posedness and) existence, we need to show that if $\boldsymbol{v} \in \varinjlim \mathscr{M}_\star$ can be written as $\boldsymbol{v} = \varphi_i(v) = \varphi_j(v')$ for some $v \in \mathscr{M}_i$ and $v' \in \mathscr{M}_j$, then $\psi_i(f^* v) = \psi_j(f^* v')$ and the inequality $\big| \psi_i(f^* v) \big| \leq |f^* \boldsymbol{v}|$ holds in the $\mathfrak{m}_X$-a.e. sense. Indeed, given any $k \in I$ with $i, j \leq k$ such that $\varphi_{ik}(v) = \varphi_{jk}(v')$, we deduce from the fact that the diagram

$$
\begin{array}{ccccc}
\mathscr{M}_i & \xrightarrow{\varphi_{ik}} & \mathscr{M}_k & \xleftarrow{\varphi_{jk}} & \mathscr{M}_j \\
{\scriptstyle f^*} \downarrow & & {\scriptstyle f^*} \downarrow & & {\scriptstyle f^*} \downarrow \\
f^* \mathscr{M}_i & \xrightarrow{f^* \varphi_{ik}} & f^* \mathscr{M}_k & \xleftarrow{f^* \varphi_{jk}} & f^* \mathscr{M}_j \\
& {\scriptstyle \psi_i} \searrow & {\scriptstyle \psi_k} \downarrow & \swarrow {\scriptstyle \psi_j} & \\
& & \mathscr{N} & &
\end{array}
$$

commutes that $\psi_i(f^* v)$ and $\psi_j(f^* v')$ coincide, thus it makes sense to define $\Phi(f^* \boldsymbol{v}) := \psi_i(f^* v)$. Since $\big| \Phi(f^* \boldsymbol{v}) \big| \leq |f^* v| = |v| \circ f$ holds $\mathfrak{m}$-a.e. for every $i \in I$ and $v \in \mathscr{M}_i$ with $\varphi_i(v) = \boldsymbol{v}$, we infer

that $\big|\Phi(f^*\boldsymbol{v})\big| \leq |\boldsymbol{v}| \circ f = |f^*\boldsymbol{v}|$. Therefore there exists a (unique) morphism $\Phi\colon f^* \varinjlim \mathscr{M}_\star \to \mathscr{N}$ satisfying (2.7), thus also $\Phi \circ (f^*\varphi_i) = \psi_i$ for all $i \in I$. This proves the universal property and accordingly that $\big(f^* \varinjlim \mathscr{M}_\star, \{f^*\varphi_i\}_{i \in I}\big)$ is the direct limit of $\big(\{f^*\mathscr{M}_i\}_{i \in I}, \{f^*\varphi_{ij}\}_{i \leq j}\big)$. $\qquad\square$

3. Inverse limits of normed $L^0(\mathfrak{m})$-modules

3.1. **Definition.** Let us fix a metric measure space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$. As we are going to see in this subsection, inverse limits exist in the category of normed $L^0(\mathfrak{m})$-modules. This has been already proved in [5]; for the sake of completeness, we report here the full proof of such fact.

Theorem 3.1 (Inverse limit of normed $L^0(\mathfrak{m})$-modules)**.** *Let $\big(\{\mathscr{M}_i\}_{i \in I}, \{\mathrm{P}_{ij}\}_{i \leq j}\big)$ be an inverse system of normed $L^0(\mathfrak{m})$-modules. Then its inverse limit $\big(\varprojlim \mathscr{M}_\star, \{\mathrm{P}_i\}_{i \in I}\big)$ exists in the category of normed $L^0(\mathfrak{m})$-modules.*

Proof. Since $\big(\{\mathscr{M}_i\}_{i \in I}, \{\mathrm{P}_{ij}\}_{i \leq j}\big)$ is an inverse system in the category of algebraic $L^0(\mathfrak{m})$-modules, we can consider its inverse limit $\big(\mathscr{M}_{\mathrm{Alg}}, \{\mathrm{P}'_i\}_{i \in I}\big)$ in such category (cf. Subsection 1.3). Given any element $v \in \mathscr{M}_{\mathrm{Alg}}$, we define (up to $\mathfrak{m}$-a.e. equality) the Borel function $|v|\colon \mathrm{X} \to [0, +\infty]$ as

$$|v| := \operatorname*{ess\,sup}_{i \in I} \big|\mathrm{P}'_i(v)\big| \tag{3.1}$$

Then we define the $L^0(\mathfrak{m})$-submodule $\varprojlim \mathscr{M}_\star$ of $\mathscr{M}_{\mathrm{Alg}}$ as

$$\varprojlim \mathscr{M}_\star := \big\{v \in \mathscr{M}_{\mathrm{Alg}} \,:\, |v| \in L^0(\mathfrak{m})\big\} = \big\{v \in \mathscr{M}_{\mathrm{Alg}} \,:\, |v| < +\infty \ \mathfrak{m}\text{-a.e.}\big\},$$

while the natural projections $\mathrm{P}_i\colon \varprojlim \mathscr{M}_\star \to \mathscr{M}_i$ are given by $\mathrm{P}_i := \mathrm{P}'_i|_{\varprojlim \mathscr{M}_\star}$. We claim that

$$\big(\varprojlim \mathscr{M}_\star, \{\mathrm{P}_i\}_{i \in I}\big) \text{ is the inverse limit of } \big(\{\mathscr{M}_i\}_{i \in I}, \{\mathrm{P}_{ij}\}_{i \leq j}\big) \text{ as normed } L^0(\mathfrak{m})\text{-modules.} \tag{3.2}$$

First of all, we need to show that $\varprojlim \mathscr{M}_\star$ is a normed $L^0(\mathfrak{m})$-module. The only non-trivial fact to check is its completeness: fix a Cauchy sequence $(v^n)_{n \in \mathbb{N}}$ in $\varprojlim \mathscr{M}_\star$. Given that $\big|\mathrm{P}_i(v^n)\big| \leq |v^n|$ holds $\mathfrak{m}$-a.e. for all $i \in I$ and $n \in \mathbb{N}$ by (3.1), we deduce that the sequence $\big(\mathrm{P}_i(v^n)\big)_{n \in \mathbb{N}}$ is Cauchy in the complete space $\mathscr{M}_i$ for every $i \in I$, whence it admits a limit $v_i \in \mathscr{M}_i$. Since the maps P_{ij} are continuous for all $i, j \in I$ with $i \leq j$, we can pass to the limit as $n \to \infty$ in $\mathrm{P}_i(v^n) = \mathrm{P}_{ij}\big(\mathrm{P}_j(v^n)\big)$ and obtain that $v_i = \mathrm{P}_{ij}(v_j)$, which means that $v := \{v_i\}_{i \in I} \in \mathscr{M}_{\mathrm{Alg}}$. Moreover, it can be readily checked that the map $|\cdot|$ is a pointwise norm on $\varprojlim \mathscr{M}_\star$, thus the inequality $\big||v^n| - |v^m|\big| \leq |v^n - v^m|$ holds $\mathfrak{m}$-a.e. for every $n, m \in \mathbb{N}$ and accordingly $\big(|v^n|\big)_{n \in \mathbb{N}}$ is a Cauchy sequence in the space $L^0(\mathfrak{m})$. Calling $f \in L^0(\mathfrak{m})$ its limit, we infer from (3.1) that

$$|v_i| = \lim_{n \to \infty} \big|\mathrm{P}_i(v^n)\big| \leq \lim_{n \to \infty} |v^n| = f \quad \mathfrak{m}\text{-a.e.} \quad \text{for every } i \in I.$$

This grants that $|v| \leq f < +\infty$ holds $\mathfrak{m}$-a.e. in X, therefore $v \in \varprojlim \mathscr{M}_\star$. It also holds that

$$|v - v^n| \overset{(3.1)}{=} \operatorname*{ess\,sup}_{i \in I} \big|v_i - \mathrm{P}_i(v^n)\big| = \operatorname*{ess\,sup}_{i \in I} \lim_{m \to \infty} \big|\mathrm{P}_i(v^m) - \mathrm{P}_i(v^n)\big| \overset{(3.1)}{\leq} \lim_{m \to \infty} |v^m - v^n|$$

in the $\mathfrak{m}$-a.e. sense. Then by letting $n \to \infty$ we conclude that $|v - v^n| \to 0$ in $L^0(\mathfrak{m})$, or equivalently that $v_n \to v$ in $\varprojlim \mathscr{M}_\star$, which proves the completeness of $\varprojlim \mathscr{M}_\star$.

Furthermore, it is immediate from the construction that each map P_i is a normed $L^0(\mathfrak{m})$-module morphism and that $\mathrm{P}_i = \mathrm{P}_{ij} \circ \mathrm{P}_j$ holds whenever $i, j \in I$ satisfy $i \leq j$, thus in order to get the claim (3.2) it just remains to prove the universal property. To this aim, fix any couple $\big(\mathscr{N}, \{\mathrm{Q}_i\}_{i \in I}\big)$ such that $\mathrm{Q}_i(w) = (\mathrm{P}_{ij} \circ \mathrm{Q}_j)(w)$ holds for all $i, j \in I$ with $i \leq j$ and $w \in \mathscr{N}$. Then for any $w \in \mathscr{N}$ there exists a unique element $\Phi(w) \in \mathscr{M}_{\mathrm{Alg}}$ satisfying $\mathrm{P}'_i(\Phi(w)) = \mathrm{Q}_i(w)$ for every $i \in I$. Given that $\big|\mathrm{Q}_i(w)\big| \leq |w|$ holds $\mathfrak{m}$-a.e. for every $i \in I$, we deduce that

$$\big|\Phi(w)\big| \overset{(3.1)}{=} \operatorname*{ess\,sup}_{i \in I} \big|\mathrm{P}'_i\big(\Phi(w)\big)\big| = \operatorname*{ess\,sup}_{i \in I} \big|\mathrm{Q}_i(w)\big| \leq |w| < +\infty \quad \text{in the } \mathfrak{m}\text{-a.e. sense,}$$

whence $\Phi(w) \in \varprojlim \mathscr{M}_\star$. Therefore $\Phi \colon \mathscr{N} \to \varprojlim \mathscr{M}_\star$ is the unique morphism such that $Q_i = P_i \circ \Phi$ for all $i \in I$. This proves the universal property and accordingly (3.2), thus concluding the proof of the statement. $\qquad\square$

Remark 3.2. The following fact stems from the proof of Theorem 3.1: if $\{v_i\}_{i\in I}$ is a family of elements $v_i \in \mathscr{M}_i$ satisfying $v_i = P_{ij}(v_j)$ for all $i,j \in I$ with $i \leq j$ and $\operatorname{ess\,sup}_{i\in I}|v_i| < +\infty$ in the $\mathfrak{m}$-a.e. sense, then there exists a unique element $v \in \varprojlim \mathscr{M}_\star$ such that $v_i = P_i(v)$ for every $i \in I$. Moreover, it holds that $|v| = \operatorname{ess\,sup}_{i\in I}|v_i|$. $\qquad\blacksquare$

Definition 3.3 (Morphism of inverse systems of normed $L^0(\mathfrak{m})$-modules). *A morphism Θ between two inverse systems $\big(\{\mathscr{M}_i\}_{i\in I}, \{P_{ij}\}_{i\leq j}\big)$ and $\big(\{\mathscr{N}_i\}_{i\in I}, \{Q_{ij}\}_{i\leq j}\big)$ of normed $L^0(\mathfrak{m})$-modules is a family $\Theta = \{\theta_i\}_{i\in I}$ of normed $L^0(\mathfrak{m})$-module morphisms $\theta_i \colon \mathscr{M}_i \to \mathscr{N}_i$ such that*

$$
\begin{array}{ccc}
\mathscr{M}_j & \xrightarrow{\ \theta_j\ } & \mathscr{N}_j \\
{\scriptstyle P_{ij}}\downarrow & & \downarrow{\scriptstyle Q_{ij}} \\
\mathscr{M}_i & \xrightarrow[\ \theta_i\ ]{} & \mathscr{N}_i
\end{array}
\tag{3.3}
$$

is a commutative diagram for every $i,j \in I$ with $i \leq j$.

With the above notion of morphism at our disposal, we can consider the category of inverse systems of normed $L^0(\mathfrak{m})$-modules. The correspondence associating to any inverse system of normed $L^0(\mathfrak{m})$-modules its inverse limit can be made into a functor, as we are going to see.

Theorem 3.4 (The inverse limit functor $\varprojlim$). *Let $\Theta = \{\theta_i\}_{i\in I}$ be a morphism between two inverse systems $\big(\{\mathscr{M}_i\}_{i\in I}, \{P_{ij}\}_{i\leq j}\big)$ and $\big(\{\mathscr{N}_i\}_{i\in I}, \{Q_{ij}\}_{i\leq j}\big)$ of normed $L^0(\mathfrak{m})$-modules, whose inverse limits are denoted by $\big(\varprojlim \mathscr{M}_\star, \{P_i\}_{i\in I}\big)$ and $\big(\varprojlim \mathscr{N}_\star, \{Q_i\}_{i\in I}\big)$, respectively. Then there exists a unique normed $L^0(\mathfrak{m})$-module morphism $\varprojlim \theta_\star \colon \varprojlim \mathscr{M}_\star \to \varprojlim \mathscr{N}_\star$ such that the diagram*

$$
\begin{array}{ccc}
\varprojlim \mathscr{M}_\star & \xRightarrow{\ \varprojlim \theta_\star\ } & \varprojlim \mathscr{N}_\star \\
{\scriptstyle P_i}\downarrow & & \downarrow{\scriptstyle Q_i} \\
\mathscr{M}_i & \xrightarrow[\ \theta_i\ ]{} & \mathscr{N}_i
\end{array}
\tag{3.4}
$$

commutes for every $i \in I$. In particular, the correspondence $\varprojlim$ is a covariant functor from the category of inverse systems of normed $L^0(\mathfrak{m})$-modules to the category of normed $L^0(\mathfrak{m})$-modules.

Proof. Pick any $v \in \varprojlim \mathscr{M}_\star$ and define $w_i := \theta_i\big(P_i(v)\big) \in \mathscr{N}_i$ for all $i \in I$. By (3.3) we see that

$$
Q_{ij}(w_j) = (Q_{ij} \circ \theta_j)\big(P_j(v)\big) = (\theta_i \circ P_{ij})\big(P_j(v)\big) = \theta_i\big(P_i(v)\big) = w_i \quad \text{for every } i,j \in I \text{ with } i \leq j.
$$

Then there is a unique element $\big(\varprojlim \theta_\star\big)(v) = w \in \varprojlim \mathscr{N}_\star$ such that $Q_i(w) = w_i$ for every $i \in I$, as observed in Remark 3.2. One can readily check that the resulting map $\varprojlim \theta_\star \colon \varprojlim \mathscr{M}_\star \to \varprojlim \mathscr{N}_\star$ is a morphism of normed $L^0(\mathfrak{m})$-modules. Finally, it clearly holds that $\varprojlim \theta_\star$ is the unique morphism for which the diagram (3.4) is commutative for all $i \in I$. Hence the statement is achieved. $\qquad\square$

3.2. **Main properties.** In this subsection we describe some important properties of inverse limits in the category of normed $L^0(\mathfrak{m})$-modules.

Lemma 3.5. *Let $\mathscr{M} \neq \{0\}$ be a given normed $L^0(\mathfrak{m})$-module. Call $\mathscr{M}_n := \mathscr{M}$ for every $n \in \mathbb{N}$. For any $n, m \in \mathbb{N}$ with $n \leq m$, we define the morphism $P_{nm} \colon \mathscr{M}_m \to \mathscr{M}_n$ as*

$$
P_{nm}(v) := \frac{n}{m}\, v \quad \text{for every } v \in \mathscr{M}_m.
$$

Then $\big(\{\mathscr{M}_n\}_{n\in\mathbb{N}}, \{\mathrm{P}_{nm}\}_{n\le m}\big)$ *is an inverse system of normed* $L^0(\mathfrak{m})$*-modules, with inverse limit*

$$\varprojlim \mathscr{M}_\star = \{0\}.$$

Proof. It immediately follows from its very definition that $\big(\{\mathscr{M}_n\}_{n\in\mathbb{N}}, \{\mathrm{P}_{nm}\}_{n\le m}\big)$ is an inverse system of normed $L^0(\mathfrak{m})$-modules. Moreover, its inverse limit $\big(\mathscr{M}_{\mathrm{Alg}}, \{\mathrm{P}'_n\}_{n\in\mathbb{N}}\big)$ in the category of algebraic $L^0(\mathfrak{m})$-modules is given by

$$\mathscr{M}_{\mathrm{Alg}} = \Big\{(nv)_{n\in\mathbb{N}} \in \prod\nolimits_{n\in\mathbb{N}} \mathscr{M}_n \;\Big|\; v \in \mathscr{M}\Big\},$$
$$\mathrm{P}'_m\big((nv)_{n\in\mathbb{N}}\big) = mv \quad \text{for every } m \in \mathbb{N} \text{ and } (nv)_{n\in\mathbb{N}} \in \mathscr{M}_{\mathrm{Alg}}.$$

Therefore for any element $v \in \mathscr{M}_{\mathrm{Alg}}$ it $\mathfrak{m}$-a.e. holds that

$$|v| = \operatorname*{ess\,sup}_{n\in\mathbb{N}} |nv| = +\infty \cdot \chi_{\{|v|>0\}},$$

whence $\varprojlim \mathscr{M}_\star = \{0\}$. This proves the statement. $\qquad\square$

The category of inverse systems of normed $L^0(\mathfrak{m})$-modules is a pointed category, whose zero object is the inverse system $\big(\{\mathscr{M}_i\}_{i\in I}, \{\mathrm{P}_{ij}\}_{i\le j}\big)$ given by $\mathscr{M}_i := \{0\}$ for all $i \in I$ and $\mathrm{P}_{ij} := 0$ for all $i, j \in I$ with $i \le j$. Given a morphism $\Theta = \{\theta_i\}_{i\in I}$ of two inverse systems $\big(\{\mathscr{M}_i\}_{i\in I}, \{\mathrm{P}_{ij}\}_{i\le j}\big)$ and $\big(\{\mathscr{N}_i\}_{i\in I}, \{\mathrm{Q}_{ij}\}_{i\le j}\big)$ of normed $L^0(\mathfrak{m})$-modules, it holds that:

 a) The kernel $\ker(\Theta)$ of Θ is given by $\big(\{\ker(\theta_i)\}_{i\in I}, \{\mathrm{P}_{ij}|_{\ker(\theta_j)}\}_{i\le j}\big)$.
 b) The image $\mathrm{im}(\Theta)$ of Θ is given by $\big(\{\mathrm{im}(\theta_i)\}_{i\in I}, \{\mathrm{Q}_{ij}|_{\mathrm{im}(\theta_j)}\}_{i\le j}\big)$.

Items a) and b) make sense, as $\mathrm{P}_{ij}\big(\ker(\theta_j)\big) \subseteq \ker(\theta_i)$ and $\mathrm{Q}_{ij}\big(\mathrm{im}(\theta_j)\big) \subseteq \mathrm{im}(\theta_i)$ whenever $i \le j$.

Proposition 3.6. *Let* $\Theta = \{\theta_i\}_{i\in I}$ *be a morphism between inverse systems* $\big(\{\mathscr{M}_i\}_{i\in I}, \{\mathrm{P}_{ij}\}_{i\le j}\big)$ *and* $\big(\{\mathscr{N}_i\}_{i\in I}, \{\mathrm{Q}_{ij}\}_{i\le j}\big)$ *of normed* $L^0(\mathfrak{m})$*-modules such that* $\ker(\Theta) = 0$. *Then* $\ker\big(\varprojlim \theta_\star\big) = 0$.

Proof. Pick $v \in \varprojlim \mathscr{M}_\star$ with $\big(\varprojlim \theta_\star\big)(v) = 0$. This implies $\theta_i\big(\mathrm{P}_i(v)\big) = \mathrm{Q}_i(0) = 0$ for every $i \in I$ by (3.4), whence $\mathrm{P}_i(v) = 0$ as $\ker(\theta_i) = 0$ by assumption. Then $v = 0$ by Remark 3.2. $\qquad\square$

Remark 3.7. The dual statement of that of Proposition 3.6 is false, as one can build a morphism of inverse systems $\Theta = \{\theta_i\}_{i\in I}$ with $\mathrm{im}(\Theta) = \big(\{\mathscr{N}_i\}_{i\in I}, \{\mathrm{Q}_{ij}\}_{i\le j}\big)$ such that $\mathrm{im}\big(\varprojlim \theta_\star\big) \ne \varprojlim \mathscr{N}_\star$.

For instance, fix any normed $L^0(\mathfrak{m})$-module $\mathscr{M} \ne \{0\}$. Let us define the inverse systems of normed $L^0(\mathfrak{m})$-modules $\big(\{\mathscr{M}_n\}_{n\in\mathbb{N}}, \{\mathrm{P}_{nm}\}_{n\le m}\big)$ and $\big(\{\mathscr{N}_n\}_{n\in\mathbb{N}}, \{\mathrm{Q}_{nm}\}_{n\le m}\big)$ as follows:

$$\mathscr{M}_n = \mathscr{N}_n := \mathscr{M} \quad \text{for every } n \in \mathbb{N},$$
$$\mathrm{P}_{nm}(v) := \frac{n}{m}\, v \quad \text{for every } n \le m \text{ and } v \in \mathscr{M}_m,$$
$$\mathrm{Q}_{nm}(w) := w \quad \text{for every } n \le m \text{ and } w \in \mathscr{N}_m.$$

The morphism $\Theta = \{\theta_n\}_{n\in\mathbb{N}}$ between $\big(\{\mathscr{M}_n\}_{n\in\mathbb{N}}, \{\mathrm{P}_{nm}\}_{n\le m}\big)$ and $\big(\{\mathscr{N}_n\}_{n\in\mathbb{N}}, \{\mathrm{Q}_{nm}\}_{n\le m}\big)$ we consider is given by

$$\theta_n(v) := \frac{1}{n}\, v \quad \text{for every } n \in \mathbb{N} \text{ and } v \in \mathscr{M}_n.$$

Therefore $\varprojlim \mathscr{M}_\star = \{0\}$ by Lemma 3.5 and $\varprojlim \mathscr{N}_\star = \mathscr{M}$. This yields the desired counterexample, as all the maps θ_n are surjective but $\mathrm{im}\big(\varprojlim \theta_\star\big) = \{0\} \ne \mathscr{M}$. $\qquad\blacksquare$

Lemma 3.8. *Suppose that the directed set* $(I, \le)$ *admits a greatest element* $m \in I$. *Then for any inverse system* $\big(\{\mathscr{M}_i\}_{i\in I}, \{\mathrm{P}_{ij}\}_{i\le j}\big)$ *of normed* $L^0(\mathfrak{m})$*-modules it holds that*

$$\big(\mathscr{M}_m, \{\mathrm{P}_{im}\}_{i\in I}\big) \text{ is the inverse limit of } \big(\{\mathscr{M}_i\}_{i\in I}, \{\mathrm{P}_{ij}\}_{i\le j}\big). \tag{3.5}$$

In particular, given any morphism $\Theta = \{\theta_i\}_{i\in I}$ *between two inverse systems* $\big(\{\mathscr{M}_i\}_{i\in I}, \{\mathrm{P}_{ij}\}_{i\le j}\big)$ *and* $\big(\{\mathscr{N}_i\}_{i\in I}, \{\mathrm{Q}_{ij}\}_{i\le j}\big)$ *of normed* $L^0(\mathfrak{m})$*-modules, it holds that* $\varprojlim \theta_\star = \theta_m$.

Proof. Fix any couple $\big(\mathscr{N}, \{Q_i\}_{i \in I}\big)$ – where $\mathscr{N}$ is a normed $L^0(\mathfrak{m})$-module and $Q_i \colon \mathscr{N} \to \mathscr{M}_i$ are morphisms – satisfying $Q_i = P_{ij} \circ Q_j$ for all $i, j \in I$ with $i \leq j$. Hence $\Phi \coloneqq Q_m$ is the unique morphism from $\mathscr{N}$ to $\mathscr{M}_m$ such that $Q_i = P_{im} \circ \Phi$ holds for every $i \in I$, which proves (3.5). $\qquad\square$

Remark 3.9. By slightly modifying the examples provided in Remark 2.11, it can be readily checked that the inverse limit functor $\varprojlim$ is neither faithul nor full. $\qquad\blacksquare$

Proposition 3.10. *Let* $\big(\{\mathscr{M}_i\}_{i \in I}, \{\varphi_{ij}\}_{i \leq j}\big)$ *be a direct system of normed $L^0(\mathfrak{m})$-modules, whose direct limit is denoted by* $\big(\varinjlim \mathscr{M}_\star, \{\varphi_i\}_{i \in I}\big)$. *Given any normed $L^0(\mathfrak{m})$-module $\mathscr{N}$ and $i, j \in I$ with $i \leq j$, we define the morphism* $P_{ij} \colon \mathrm{HOM}(\mathscr{M}_j, \mathscr{N}) \to \mathrm{HOM}(\mathscr{M}_i, \mathscr{N})$ *as*

$$P_{ij}(T) \coloneqq T \circ \varphi_{ij} \quad \text{for every } T \in \mathrm{HOM}(\mathscr{M}_j, \mathscr{N}).$$

Then $\big(\{\mathrm{HOM}(\mathscr{M}_i, \mathscr{N})\}_{i \in I}, \{P_{ij}\}_{i \leq j}\big)$ *is an inverse system of normed $L^0(\mathfrak{m})$-modules. Moreover, it holds that*

$$\varprojlim \mathrm{HOM}(\mathscr{M}_\star, \mathscr{N}) \cong \mathrm{HOM}\big(\varinjlim \mathscr{M}_\star, \mathscr{N}\big),$$

the natural projections $P_i \colon \mathrm{HOM}\big(\varinjlim \mathscr{M}_\star, \mathscr{N}\big) \to \mathrm{HOM}(\mathscr{M}_i, \mathscr{N})$ *being defined as* $P_i(T) \coloneqq T \circ \varphi_i$ *for every $i \in I$ and $T \in \mathrm{HOM}\big(\varinjlim \mathscr{M}_\star, \mathscr{N}\big)$.*

Proof. Given any $i, j, k \in I$ with $i \leq j \leq k$ and $T \in \mathrm{HOM}(\mathscr{M}_k, \mathscr{N})$, it holds that

$$P_{ik}(T) = T \circ \varphi_{ik} = T \circ \varphi_{jk} \circ \varphi_{ij} = P_{ij}(T \circ \varphi_{jk}) = (P_{ij} \circ P_{jk})(T),$$

whence $\big(\{\mathrm{HOM}(\mathscr{M}_i, \mathscr{N})\}_{i \in I}, \{P_{ij}\}_{i \leq j}\big)$ is an inverse system. Analogously, $P_i = P_{ij} \circ P_j$ holds for all $i, j \in I$ with $i \leq j$, thus to conclude it only remains to show that $\big(\mathrm{HOM}\big(\varinjlim \mathscr{M}_\star, \mathscr{N}\big), \{P_i\}_{i \in I}\big)$ satisfies the universal property defining the inverse limit. Fix any $\big(\mathscr{O}, \{Q_i\}_{i \in I}\big)$, where $\mathscr{O}$ is a normed $L^0(\mathfrak{m})$-module, while the morphisms $Q_i \colon \mathscr{O} \to \mathrm{HOM}(\mathscr{M}_i, \mathscr{N})$ satisfy $Q_i = P_{ij} \circ Q_j$ for every $i, j \in I$ with $i \leq j$. Given any element $w \in \mathscr{O}$, we consider the family $\big\{Q_i(w)\big\}_{i \in I}$. Call f_w the function $\chi_{\{|w|>0\}} \frac{1}{|w|} \in L^0(\mathfrak{m})$ and observe that the morphisms $f_w \cdot Q_i(w) \colon \mathscr{M}_i \to \mathscr{N}$ satisfy

$$\big(f_w \cdot Q_i(w)\big)(v) = f_w \cdot Q_i(w)(v) = f_w \cdot P_{ij}\big(Q_j(w)\big)(v) = f_w \cdot \big(Q_j(w) \circ \varphi_{ij}\big)(v)$$
$$= f_w \cdot Q_j(w)\big(\varphi_{ij}(v)\big) = \big(f_w \cdot Q_j(w)\big)\big(\varphi_{ij}(v)\big)$$

for every $i, j \in I$ with $i \leq j$ and $v \in \mathscr{M}_i$. This shows that $\big(\mathscr{N}, \{f_w \cdot Q_i(w)\}_{i \in I}\big)$ is a target for the direct system $\big(\{\mathscr{M}_i\}_{i \in I}, \{\varphi_{ij}\}_{i \leq j}\big)$, whence there exists a unique morphism $\Phi_0(w) \colon \varinjlim \mathscr{M}_\star \to \mathscr{N}$ such that $f_w \cdot Q_i(w)(v) = \Phi_0(w)\big(\varphi_i(v)\big)$ holds for every $i \in I$ and $v \in \mathscr{M}_i$, thus accordingly the element $\Phi(w) \coloneqq |w| \cdot \Phi_0(w) \in \mathrm{HOM}\big(\varinjlim \mathscr{M}_\star, \mathscr{N}\big)$ satisfies $Q_i(w)(v) = \Phi(w)\big(\varphi_i(v)\big) = P_i\big(\Phi(w)\big)(v)$ for all $i \in I$ and $v \in \mathscr{M}_i$. Hence $\Phi \colon \mathscr{O} \to \mathrm{HOM}\big(\varinjlim \mathscr{M}_\star, \mathscr{N}\big)$ is the unique morphism such that

$$\mathscr{O} \xrightarrow{\ \Phi\ } \mathrm{HOM}\big(\varinjlim \mathscr{M}_\star, \mathscr{N}\big)$$
$$Q_i \searrow \qquad \downarrow P_i$$
$$\mathrm{HOM}(\mathscr{M}_i, \mathscr{N})$$

is a commutative diagram for any $i \in I$. Then the statement is achieved. $\qquad\square$

Corollary 3.11. *Let* $\big(\{\mathscr{M}_i\}_{i \in I}, \{\varphi_{ij}\}_{i \leq j}\big)$ *be a direct system of normed $L^0(\mathfrak{m})$-modules, whose direct limit is denoted by* $\big(\varinjlim \mathscr{M}_\star, \{\varphi_i\}_{i \in I}\big)$. *Then* $\big(\{\mathscr{M}_i^*\}_{i \in I}, \{\varphi_{ij}^{\mathrm{adj}}\}_{i \leq j}\big)$ *is an inverse system of normed $L^0(\mathfrak{m})$-modules (cf. (1.3) for the definition of the adjoint $\varphi_{ij}^{\mathrm{adj}}$). Moreover, it holds that*

$$\varprojlim \mathscr{M}_\star^* \cong \big(\varinjlim \mathscr{M}_\star\big)^*, \tag{3.6}$$

the natural projections being given by $\varphi_i^{\mathrm{adj}} \colon \big(\varinjlim \mathscr{M}_\star\big)^* \to \mathscr{M}_i^*$ *for every $i \in I$.*

Proof. Just apply Proposition 3.10 with $\mathscr{N} \coloneqq L^0(\mathfrak{m})$. $\qquad\square$

Remark 3.12 (Pullback and inverse limit do not commute)**.** Let $(X, d_X, \mathfrak{m}_X)$, $(Y, d_Y, \mathfrak{m}_Y)$ be metric measure spaces. Let $f\colon X \to Y$ be a Borel map with $f_*\mathfrak{m}_X \ll \mathfrak{m}_Y$. Let $\big(\{\mathscr{M}_i\}_{i\in I}, \{P_{ij}\}_{i\leq j}\big)$ be an inverse system of normed $L^0(\mathfrak{m}_Y)$-modules, whose inverse limit is denoted by $\big(\varprojlim \mathscr{M}_\star, \{P_i\}_{i\in I}\big)$. Then $\big(\{f^*\mathscr{M}_i\}_{i\in I}, \{f^*P_{ij}\}_{i\leq j}\big)$ is an inverse system of normed $L^0(\mathfrak{m}_X)$-modules, as the diagram

$$
\begin{array}{ccccccc}
\varprojlim \mathscr{M}_\star & \xrightarrow{\ P_k\ } & \mathscr{M}_k & \xrightarrow{\ P_{jk}\ } & \mathscr{M}_j & \xrightarrow{\ P_{ij}\ } & \mathscr{M}_i \\
{\scriptstyle f^*}\big\downarrow & & {\scriptstyle f^*}\big\downarrow & & {\scriptstyle f^*}\big\downarrow & & {\scriptstyle f^*}\big\downarrow \\
f^*\varprojlim \mathscr{M}_\star & \xrightarrow[f^*P_k]{} & f^*\mathscr{M}_k & \xrightarrow[f^*P_{jk}]{} & f^*\mathscr{M}_j & \xrightarrow[f^*P_{ij}]{} & f^*\mathscr{M}_i
\end{array}
$$

commutes for every $i, j, k \in I$ with $i \leq j \leq k$ by Theorem 1.9. Nevertheless, it might happen that

$$\varprojlim f^*\mathscr{M}_\star \not\cong f^*\varprojlim \mathscr{M}_\star.$$

For instance, consider the constant map $\pi\colon [0,1] \to \{0\}$, where the domain is endowed with the Euclidean distance and the restricted Lebesgue measure $\mathcal{L}_1 = \mathcal{L}^1\big|_{[0,1]}$, while the target is endowed with the trivial distance and the Dirac delta measure δ_0. Notice that trivially $\pi_*\mathcal{L}_1 \ll \delta_0$. Moreover, by combining Example 1.11 with Example 1.3 we deduce that

$$\pi^*\mathbb{B} \cong L^0\big([0,1], \mathbb{B}\big) \quad \text{for every Banach space } \mathbb{B}. \tag{3.7}$$

Now consider the Banach space ℓ^1, that is the direct limit of some direct system $\big(\{\mathbb{B}_i\}_{i\in I}, \{\varphi_{ij}\}_{i\leq j}\big)$ of finite-dimensional Banach spaces, for instance by Lemma 2.5. Since the spaces $\mathbb{B}_i'$ have the Radon-Nikodým property while $\ell^\infty = (\ell^1)'$ does not (cf. [2]), we know from (1.4) that

$$
\begin{aligned}
L^0\big([0,1], \mathbb{B}_i\big)^* &\cong L^0\big([0,1], \mathbb{B}_i'\big) \quad \text{for every } i \in I, \\
L^0\big([0,1], \ell^1\big)^* &\not\cong L^0\big([0,1], \ell^\infty\big).
\end{aligned}
\tag{3.8}
$$

Therefore it holds that

$$
\begin{aligned}
\pi^*\varprojlim \mathbb{B}_\star' &\overset{(3.6)}{\cong} \pi^*\big(\varinjlim \mathbb{B}_\star\big)' \cong \pi^*\ell^\infty \overset{(3.7)}{\cong} L^0\big([0,1], \ell^\infty\big) \overset{(3.8)}{\not\cong} L^0\big([0,1], \ell^1\big)^* \overset{(3.7)}{\cong} (\pi^*\ell^1)^* \\
&\cong \big(\pi^*\varinjlim \mathbb{B}_\star\big)^* \overset{(2.6)}{\cong} \big(\varinjlim \pi^*\mathbb{B}_\star\big)^* \overset{(3.6)}{\cong} \varprojlim(\pi^*\mathbb{B}_\star)^* \overset{(3.7)}{\cong} \varprojlim L^0\big([0,1], \mathbb{B}_\star\big)^* \\
&\overset{(3.8)}{\cong} \varprojlim L^0\big([0,1], \mathbb{B}_\star'\big) \overset{(3.7)}{\cong} \varprojlim \pi^*\mathbb{B}_\star'.
\end{aligned}
$$

Summing up, we have found an inverse system $\big(\{\mathbb{B}_i'\}_{i\in I}, \{\varphi_{ij}^{\mathrm{adj}}\}_{i\leq j}\big)$ of Banach spaces for which it holds that $\pi^*\varprojlim \mathbb{B}_\star' \not\cong \varprojlim \pi^*\mathbb{B}_\star'$. ∎

References

[1] S. Di Marino, D. Lučić, and E. Pasqualetto, *Representation of separable normed modules*. Work in progress, (2019).

[2] J. Diestel and J. J. Uhl, Jr., *Vector measures*, American Mathematical Society, Providence, R.I., 1977. With a foreword by B. J. Pettis, Mathematical Surveys, No. 15.

[3] N. Gigli, *Nonsmooth differential geometry - an approach tailored for spaces with Ricci curvature bounded from below*. Mem. Amer. Math. Soc., 236 (2015), pp. vi+91.

[4] ———, *Lecture notes on differential calculus on* RCD *spaces.* Publ. RIMS Kyoto Univ. 54, (2018).

[5] N. Gigli, E. Pasqualetto, and E. Soultanis, *Differential of metric valued Sobolev maps*, (2018). Submitted, arXiv:1807.10063.

[6] J. Heinonen, P. Koskela, N. Shanmugalingam, and J. Tyson, *Sobolev spaces on metric measure spaces: An approach based on upper gradients*, Cambridge University Press, United States, 2015.

[7] S. Lang, *Algebra*, Addison-Wesley Publishing Company, Advanced Book Program, Reading, MA, second ed., 1984.

[8] D. Lučić and E. Pasqualetto, *The Serre–Swan theorem for normed modules*, Rendiconti del Circolo Matematico di Palermo Series 2, (2018).

[9] S. Mac Lane, *Categories for the working mathematician*, vol. 5 of Graduate Texts in Mathematics, Springer-Verlag, New York, second ed., 1998.

Department of Mathematics and Statistics, P.O. Box 35 (MaD), FI-40014 University of Jyväskylä
E-mail address: enpasqua@jyu.fi