Comparison principles for equations of Monge-Ampère type in Carnot groups: a direct proof *

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Dedicated to Ermanno Lanconelli for his 65th birthday

Abstract

We study fully nonlinear partial differential equations involving the determinant of the Hessian matrix of the unknown function with respect to a family of vector fields that generate a Carnot group. We prove a comparison theorem among viscosity sub- and supersolutions, for subsolutions uniformly convex with respect to the vector fields.

1 Introduction

We consider fully nonlinear partial differential equations of the form

$$-\det(D^2_{\mathcal{X}}u) + H(x, u, D_{\mathcal{X}}u) = 0, \text{ in } \Omega, \tag{1.1}$$

where $\Omega \subseteq I\!R^n$ is open and bounded, $D_{\mathcal{X}}u$ denotes the gradient of u with respect to a given family of $C^{1,1}$ vector fields $X_1, ..., X_m$, $D_{\mathcal{X}}u := (X_1u, ..., X_mu)$, $D^2_{\mathcal{X}}u$ denotes the symmetrized Hessian matrix of u with respect to the same vector fields

$$(D^2_{\mathcal{X}}u)_{ij} := (X_iX_ju + X_jX_iu)/2,$$

and H is a given Hamiltonian, at least continuous and nondecreasing in u. Our main examples are the vector fields that generate the homogeneous Carnot groups [18, 8, 11], and in that case $D_{\mathcal{X}}u$ and $D^2_{\mathcal{X}}u$ are called, respectively, the horizontal gradient and the horizontal Hessian.

A theory of fully nonlinear subelliptic equations was started recently by Bieske [9, 10] and Manfredi [28, 7], and Monge-Ampère equations of the form (1.1) with $H = f(x)$ are listed among the main examples. For such equations on the Heisenberg group Gutierrez and Montanari [21] proved, among other things, a comparison principle among smooth sub- and supersolutions (see also [19] for related results).

There are several motivations for studying H depending also on $D_{\mathcal{X}}u$. One is the prescribed horizontal Gauss curvature equation in Carnot groups, as defined by Danielli, Garofalo and Nhieu [16]. Another is the Monge-Ampère type equation derived by Stojanovic [30] for some stochastic

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control problems in financial mathematics. Finally, equations of this kind arise in optimal transportation problems between Carnot groups [2], Riemannian manifolds [34], and sub-Riemannian manifolds [17]. As for the dependence of H on u, it arises naturally in the Monge-Ampère equations of Riemannian geometry, see, e.g., [32, 22, 1] and the references therein.

In our paper [5] we began a study of the subelliptic Monge-Ampère-type equations (1.1) within the theory of viscosity solutions. We announced two comparison results that extend to the subelliptic setting a theorem of H. Ishii and P.-L. Lions for euclidean Monge-Ampère equations [23] (i.e., the case when the vector fields are the canonical basis of $\mathbf{R}^n$). For the large literature on the euclidean case we refer to the classical paper [25], the recent surveys [12, 31], the books [20, 33], and the references therein.

In the present paper we give the proof of one of these results, using a direct argument relying only on some basic tools of the theory of viscosity solutions [15, 14]. The proof outlined in [5] was different, since it transformed the PDE (1.1) into a Hamilton-Jacobi-Bellman equations and then used the comparison principle for such equations.

The new difficulties one encounters in the study of subelliptic Monge-Ampère equations are the following.

1. The PDE (1.1) is degenerate elliptic only on functions that are convex with respect to the vector fields $X_1, ..., X_m$, briefly $\mathcal{X}$*-convex*. Following Lu, Manfredi, and Stroffolini [26] such a function is an u.s.c. $u : \overline{\Omega} \to \mathbf{R}$ such that $-D^2_{\mathcal{X}} u \leq 0$ in Ω in viscosity sense, that is,

$$D^2_{\mathcal{X}} \varphi(x) \geq 0 \quad \forall\, \varphi \in C^2(\Omega), x \in \operatorname{argmax}(u - \varphi). \tag{1.2}$$

We refer to the nice survey in the book of Bonfiglioli, Lanconelli and Uguzzoni [11] for the recent literature on the notions of convexity in Carnot groups. Since $\mathcal{X}$-convex functions are not Lipschitz continuous, in general, we get better results in Carnot groups, where they are Lipschitz with respect to the intrinsic metric [26, 3, 16, 27, 29, 24].

2. The operator in (1.1) does not satisfy in general the standard structure conditions in viscosity theory [14]. Therefore we consider equations of the form

$$-\log\det(D^2_{\mathcal{X}} u) + K(x, u, Du) = 0, \text{ in } \Omega, \tag{1.3}$$

with a continuous K strictly increasing in u, and adapt the arguments of [15, 14] to this PDE. The main result of this paper states the comparison among uniformly $\mathcal{X}$-convex subsolutions and lower semicontinuous supersolutions of this equation. From this we get the same comparison result for the equation (1.1). We remark that the log of a Monge-Ampère operator is a natural object: it appears in problems arising in Riemannian geometry [1], [22], and in parabolic versions of the MA equation [32], [36], [13].

The variants needed for the case of H not strictly increasing in u are presented in the companion papers [5, 6]. We first observe that the comparison principle still holds in this case if the subsolution is strict, and then perturb a $\mathcal{X}$-convex subsolution to a uniformly $\mathcal{X}$-convex strict subsolution (cfr. [23, 4]). In [6] we also prove the existence of solutions to the Dirichlet problem via the Perron-Ishii method, as well as some extensions and variants, e.g., to vector fields that are not necessarily the generators of a Lie group.

In Section 2 we recall all necessary definitions, assumptions and preliminary results. In the final Section 3 we state and prove the Comparison Principle for the equations (1.3) and (1.1).

2 Definitions

Let us consider the following equation

$$-\det D^2_{\mathcal{X}} u + H(x, u, D_{\mathcal{X}} u) = 0, \quad \text{in } \Omega, \tag{2.1}$$

where the set $\Omega \subseteq I\!\!R^n$ is open and bounded,

$$D_{\mathcal{X}} u := (X_1 u, ..., X_m u)$$

is the intrinsic (or horizontal) gradient with respect of a family of $C^{1,1}$ vector fields $X_1, ..., X_m$, and

$$(D^2_{\mathcal{X}} u)_{ij} = \frac{X_i(X_j u) + X_j(X_i u)}{2}$$

is the symmetrized intrinsic Hessian. The Hamiltonian H is at least continuous and nondecreasing in the second entry.

We take the $n \times m$ $C^{1,1}$ matrix-valued function σ defined in $\overline{\Omega} \subseteq I\!R^n$ whose columns σ^j are the coefficients of X_j, $j = 1, \cdots, m$, that is, $X_j = \sigma^j \cdot \nabla$ and $\sigma_{ij} = \sigma^j_i$. Then, for any smooth u

$$D_{\mathcal{X}} u = \sigma^T(x) Du, \quad D^2_{\mathcal{X}} u = \sigma^T(x) D^2 u \, \sigma(x) + Q(x, Du), \tag{2.2}$$

where $Q(x, p)$ is the $m \times m$ matrix whose elements are

$$Q_{ij}(x, p) = \left(\frac{D\sigma^j(x)\, \sigma^i(x) + D\sigma^i(x)\, \sigma^j(x)}{2} \right) \cdot p. \tag{2.3}$$

We can now rewrite equation (2.1) by means of the matrix σ and the Euclidean gradient and Hessian:

$$-\det(\sigma^T(x) D^2 u \, \sigma(x) + Q(x, Du)) + H(x, u, \sigma^T(x) Du) = 0, \quad \text{in } \Omega. \tag{2.4}$$

Since the determinant of a matrix is increasing among positive definite matrices with the usual partial order, this equation is (degenerate) elliptic if it is restricted to a suitable set of candidate solutions u. Next we define such a set, within the viscosity solutions framework.

Let $USC(\overline{\Omega})$ and $LSC(\overline{\Omega})$ denote the sets of functions $\overline{\Omega} \to I\!R$ that are, respectively, upper semicontinuous and lower semicontinuous.

Definition 2.1 *If $\Psi : \overline{\Omega} \times I\!R^n \times S^n \to S^m$ and $M \in S^m$ we say that u is a (viscosity) subsolution of the matrix inequality*

$$\Psi(x, Du, D^2 u) \leq M, \text{ in } \Omega, \tag{2.5}$$

if u is $USC(\overline{\Omega})$ and

$$\Psi(x, D\phi(x), D^2\phi(x)) \leq M, \tag{2.6}$$

for all $\phi \in C^2(\Omega)$ and $x \in \operatorname{argmax}(u - \phi)$.

Definition 2.2 *$u \in USC(\overline{\Omega})$ is* **$\mathcal{X}$-convex** *in Ω with respect to the fields $X_1, \cdots, X_m$ if it is a viscosity subsolution of*

$$-\sigma^T(x) D^2 u \, \sigma(x) - Q(x, Du) \leq 0, \text{ in } \Omega, \tag{2.7}$$

u is **uniformly $\mathcal{X}$-convex** *in Ω if it is a viscosity subsolution of*

$$-\sigma^T(x) D^2 u \, \sigma(x) - Q(x, Du) \leq -\gamma I, \quad \text{for some } \gamma > 0, \text{ in } \Omega. \tag{2.8}$$

Note that, for smooth u, the inequalities (2.7), (2.8) can be written as $D^2_{\mathcal{X}} u \geq 0$ and $D^2_{\mathcal{X}} u \geq \gamma I$, in Ω.

The definition of (viscosity) subsolution u of 2.4 is given in a standard way, as in [23]:

Definition 2.3 *A function $u \in USC(\overline{\Omega})$ is a (viscosity) subsolution of (2.1) if for all $\phi \in C^2(\overline{\Omega})$ such that $u - \phi$ has a maximum point at x_0 we have*

$$-\det(\sigma^T(x_0) D^2\phi(x_0)\, \sigma(x_0) + Q(x_0, D\phi(x_0)) + H(x_0, v(x_0), D\phi(x_0)) \leq 0.$$

The definition of (viscosity) supersolution v is modified, as was done in [23] for the Euclidean case, by restricting the test functions to the C^2 functions ϕ with $D^2_{\mathcal{X}}\phi(x) > 0$ at points $x \in \arg\min(v - \phi)$.

Definition 2.4 *A function $v \in LSC(\overline{\Omega})$ is a (viscosity) supersolution of (2.1) if for all $\phi \in C^2(\overline{\Omega})$ such that $v - \phi$ has a minimum point at x_0 and*

$$\sigma^T(x_0)D^2\phi(x_0)\,\sigma(x_0) + Q(x_0, D\phi(x_0)) > 0, \tag{2.9}$$

we have

$$-\det(\sigma^T(x_0)D^2\phi(x_0)\,\sigma(x_0) + Q(x_0, D\phi(x_0)) + H(x_0, v(x_0), D\phi(x_0)) \geq 0. \tag{2.10}$$

Note that if u is a uniformly $\mathcal{X}$-convex subsolution, condition (2.9) is automatically satisfied because any $\phi \in C^2$ touching u from above satisfies (2.9).

The equation (2.4) is proper in the sense of the theory of the viscosity solutions [15] if it is restricted to $\mathcal{X}$-convex subsolutions. In fact the function

$$F(x, r, p, X) := -\det(\sigma^T(x)X\,\sigma(x) + Q(x, p)) + H(x, r, \sigma^T(x)p)$$

satisfies the following properties

$$\begin{aligned}
&F : \overline{\Omega} \times I\!R \times I\!R^n \times S^n \to I\!R \ \text{ is continuous;}\\
&F(x, r, p, X) \leq F(x, r, p, Y),\ \forall x, r, p, X, Y \in S^n \text{ such that } \sigma^T X\sigma + Q \geq \sigma^T Y\sigma + Q \geq 0,\\
&F(x, r, p, X) \geq F(x, s, p, X),\ \forall x \in \overline{\Omega}, r, s \in I\!R, p \in I\!R^n, X \in S^n,\ \text{ if } r \geq s,
\end{aligned}$$

where we denote with S^n the set of the symmetric $n \times n$ matrices, and with $\geq$ the usual partial order of matrices.

We recall now some definitions concerning Carnot groups. We adopt the terminology and notations of the recent book [11]. Consider a group operation $\circ$ on $I\!R^n = I\!R^{n_1} \times ... \times I\!R^{n_r}$ with identity 0, such that

$$(x, y) \mapsto y^{-1} \circ x \quad \text{ is smooth,}$$

and the dilation $\delta_\lambda : I\!R^n \to I\!R^n$

$$\delta_\lambda(x) = \delta_\lambda(x^{(1)}, ..., x^{(r)}) := (\lambda x^{(1)}, \lambda^2 x^{(2)}, ..., \lambda^r x^{(r)}), \quad x^{(i)} \in I\!R^{n_i}.$$

If δ_λ is an automorphism of the group $(I\!R^n, \circ)$ for all $\lambda > 0$, then $(I\!R^n, \circ, \delta_\lambda)$ is a homogeneous Lie group on $I\!R^n$. We say that $m = n_1$ smooth vector fields $X_1, ..., X_m$ on $I\!R^n$ generate $(I\!R^n, \circ, \delta_\lambda)$, and that this is a (homogeneous) Carnot group, if

- $X_1, ..., X_m$ are invariant with respect to the left translations on $I\!R^n$ $\tau_\alpha(x) := \alpha \circ x$ for all $\alpha \in I\!R^n$,
- $X_i(0) = \partial/\partial x_i$, $i = 1, ..., m$,
- the rank of the Lie algebra generated by $X_1, ..., X_m$ is n at every point $x \in I\!R^n$.

We refer to [11] for the connections of this definition with the classical one in the context of abstract Lie groups and for the properties of the generators. We will use only the following property, and refer to Remark 1.4.6, p. 59 of [11] for more precise informations.

Proposition 2.1 *If $X_1, ..., X_m$ are generators of a Carnot group, then*

$$X_j(x) = \frac{\partial}{\partial x_j} + \sum_{i=m+1}^{n} \sigma_{ij}(x)\frac{\partial}{\partial x_i}$$

with $\sigma_{ij}(x) = \sigma_{ij}(x_1, ..., x_{i-1})$ homogeneous polynomials of a degree $\leq n - m$.

The previous Proposition implies that $\sigma(x) = \begin{pmatrix} I \\ A(x) \end{pmatrix}$ where I is the $m \times m$ identity matrix and $A(x)$ is a suitable $(n-m) \times m$ matrix.

If $X_1, \cdots, X_m$ are the generators of a Carnot group $\mathcal{G}$, the definition (2.2) of $\mathcal{X}$-convexity coincides with the definition of convexity in $\mathcal{G}$ in viscosity sense (v-convexity) of Lu, Manfredi, Stroffolini [26]. A more geometric notion of convexity in $\mathcal{G}$, called *horizontal convexity* (or weak H-convexity), was introduced and studied in the same seminal paper [26] and, independently, by Danielli, Garofalo, and Nhieu [16]. The equivalence of the two notions was studied by several authors, first in the Heisenberg groups [26], [3], and then in general Carnot groups [35], [27], [24], see also the survey in [11].

In the special case of Carnot groups, various authors proved, under different assumptions, the Lipschitz continuity of $\mathcal{X}$-convex functions with respect to the intrinsic metric of the group and bounds on their horizontal gradient in the sense of distributions [26], [16], [27], [29], [24]. From those results we obtain the following gradient bound in viscosity sense.

Proposition 2.2 *Let u be $\mathcal{X}$-convex in Ω with respect to the generators of a Carnot group. Then, for every open Ω_1 with $\overline{\Omega}_1 \subseteq \Omega$, there exists a constant C such that*

$$|\sigma^T(x)\, Du| \le C, \;\; in\ \Omega_1$$

in viscosity sense.

Proof. Since u is USC, it is locally bounded from above. Then $\mathcal{X}$-convexity implies local Lipschitz continuity with respect to Carnot-Caratheodory distance, by a result of Magnani [27] and Rickly [29], see also [24]. In particular, u is continuous in Ω. We mollify u by convolution with kernels adapted to the group structure, as in [16, 11]. The approximating u_ϵ converge to u uniformly on compact subsets of Ω, and they are smooth and $\mathcal{X}$-convex. Moreover, from the proof of Theorem 9.1 of [16] we get, for R small enough,

$$\sup_{B_C(x_0,R)} \left(\sum_{j=1}^{m} (X_j u_\epsilon)^2 \right)^{1/2} \le \frac{2}{R} \sup_{B_C(x_0,3R)} |u|,$$

where the balls B_C are taken with respect to the gauge pseudo-distance and $X_j u$ denotes the derivative of u along the trajectory of the vector field X_j. Since u_ϵ is C^∞, $X_j u_\epsilon(x) = \sigma^j(x) D u_\epsilon(x)$. Therefore there is a constant C depending only on $\sup_{\overline{\Omega}} |u|$ and the pseudo-distance of Ω_1 from $\partial\Omega$ such that

$$|\sigma^T(x)\, D u_\epsilon| \le C \text{ in } \Omega_1.$$

By letting $\epsilon \to 0$, we obtain that u is a viscosity subsolution of the same inequality. □

3 A comparison principle

In this section we prove a Comparison Principle for the subelliptic Monge-Ampère equation in Carnot groups (2.1). We assume H positive, continuous, and strictly increasing in r. More precisely, for a suitable $M > 0$,

$$\begin{aligned} &H : \overline{\Omega} \times I\!R \times I\!R^m \to (0, +\infty) \text{ is continuous;} \\ &H(x,r,q) - H(x,s,q) \ge C(r-s), \\ &\text{for some } C > 0 \text{ and all } r,s \in [-M,M], x \in \overline{\Omega},\ q \in I\!R^m. \end{aligned} \tag{3.1}$$

Theorem 3.1 *Suppose the vector fields $X_1, ..., X_m$ are the generators of a Carnot group on $I\!R^n$. Let $u : \overline{\Omega} \to I\!R$ be a bounded, uniformly $\mathcal{X}$-convex, USC subsolution of* (2.1) *and $v : \overline{\Omega} \to I\!R$ be a bounded LSC supersolution of* (2.1). *Assume H satisfies (3.1) with $M = \max\{\|u\|_\infty, \|v\|_\infty\}$. Then*

$$\sup_\Omega (u - v) \le \max_{\partial\Omega} (u - v)^+.$$

Remark 3.1 The same result holds without the assumption that $X_1, ..., X_m$ generate a Carnot group if H is uniformly continuous and bounded, e.g., it does not depend on $D_{\mathcal{X}}u$, as in the equation

$$\lambda u - \det(D^2_{\mathcal{X}}u) + f(x) = 0 \qquad \text{in } \Omega,$$

with $\lambda > 0$ and $f \in C(\overline{\Omega})$, $f > 0$. In fact that assumption is used only to get a gradient bound for u that allows to treat H first as a uniformly continuous and then as a bounded function. This is easily checked in the proof given below.

To prove Theorem 3.1 we use a general comparison result in Carnot groups for the equation

$$-\log\det(D^2_{\mathcal{X}}u) + K(x, u, D_{\mathcal{X}}u) = 0 \qquad \text{in } \Omega. \tag{3.2}$$

For equation (3.2) the definition of viscosity sub- and supersolution is analogous to the case without log.

Theorem 3.2 *Assume that $X_1, \cdots, X_m$ are the generators of a Carnot group. Suppose $u \in USC(\overline{\Omega})$ is a bounded, uniformly $\mathcal{X}$-convex, i.e.,*

$$-\sigma^T(x)D^2u\sigma(x) - Q(x, Du) + \gamma I \leq 0, \text{ in } \Omega, \text{ for some } \gamma > 0, \tag{3.3}$$

and a subsolution of (3.2), whereas $v \in LSC(\overline{\Omega})$ is a bounded supersolution of (3.2). Assume $K : \overline{\Omega} \times I\!R \times I\!R^m \to I\!R$ is continuous and strictly increasing in r, i.e. $K(x, r, q) - K(x, s, q) \geq C(r - s)$, for some $C > 0$ and all $r, s \in [-M, M]$, $M = \max\{\|u\|_\infty, \|v\|_\infty\}$. Then

$$\sup_{\Omega}(u - v) \leq \max_{\partial\Omega}(u - v)^+.$$

Remark 3.2 The following proof gives the same result also for the equation

$$-\log\det(D^2_{\mathcal{X}}u) + K(x, u, D_{\mathcal{X}}u, D^2_{\mathcal{X}}u) = 0$$

if K is proper and satisfies the structure conditions for fully nonlinear second order operators of the viscosity theory, see [15]. Under these conditions there is no need of the gradient bound, so the result holds for general vector fields and not only for generators of Carnot groups. On the other hand the structure conditions of [15] are more restrictive on the regularity of H in x, uniformly in the other entries, than the mere continuity assumed in Theorem 3.2. See [6] for more details.

Proof. (of Theorem 3.2). For $\epsilon > 0$ the function $\Phi_\epsilon(x, y) = u(x) - v(y) - \frac{1}{2\epsilon}|x - y|^2$ has a maximum point (x_ϵ, y_ϵ). A standard argument gives

$$\frac{|x_\epsilon - y_\epsilon|^2}{\epsilon} \to 0, \text{ as } \epsilon \to 0^+. \tag{3.4}$$

If there is a sequence $\epsilon_k \to 0$ such that $x_{\epsilon_k} \to \hat{x} \in \partial\Omega$, then $y_{\epsilon_k} \to \hat{x}$, and by the upper semicontinuity of $u(x) - v(y)$

$$\max_{\Omega}(u - v) \leq \Phi_\epsilon(x_\epsilon, y_\epsilon) \to \max_{\partial\Omega}(u - v), \quad \text{as } \epsilon \to 0.$$

The case of $y_{\epsilon_j} \to \hat{y} \in \partial\Omega$ for some $\epsilon_j \to 0$ is analogous. Therefore we are left with the case that $\text{dist}\big((x_\epsilon, y_\epsilon), \partial(\Omega \times \Omega)\big) \geq \delta > 0$. Then there exists Ω_1 open, $\overline{\Omega}_1 \subset \Omega$ such that $(x_\epsilon, y_\epsilon) \in \Omega_1 \times \Omega_1$, for all small ϵ.

We use the Theorem on Sums in [14] and get $X, Y \in S^n$ such that, for $p_\epsilon := \frac{|x_\epsilon - y_\epsilon|}{\epsilon}$,

$$u(x) \leq u(x_\epsilon) + p_\epsilon \cdot (x - x_\epsilon) + \frac{1}{2}(x - x_\epsilon)^T X(x - x_\epsilon) + o(|x - x_\epsilon|^2), \qquad x \to x_\epsilon,$$

$$v(y) \geq v(y_\epsilon) + p_\epsilon \cdot (y - y_\epsilon) + \frac{1}{2}(y - y_\epsilon)^T Y(y - y_\epsilon) + o(|y - y_\epsilon|^2), \qquad y \to y_\epsilon,$$

and

$$-\frac{3}{\epsilon}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le \frac{3}{\epsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{3.5}$$

Assumption (3.3) implies

$$G(x_\epsilon, p_\epsilon, X) := \sigma^T(x_\epsilon)X\,\sigma(x_\epsilon) + Q(x_\epsilon, p_\epsilon) \ge \gamma I.$$

We seek a similar inequality for $G(y_\epsilon, p_\epsilon, Y) := \sigma^T(y_\epsilon)Y\,\sigma(y_\epsilon) + Q(y_\epsilon, p_\epsilon)$. To this end we multiply on the left the second inequality in (3.5) by the $m \times 2n$ matrix whose first n columns are $\sigma^T(x_\epsilon)$ and the last n are $\sigma^T(y_\epsilon)$, and then on the right by the transpose of such matrix. Since the operation preserves the inequality, we get

$$\begin{aligned} &\sigma^T(x_\epsilon)X\sigma(x_\epsilon) - \sigma^T(y_\epsilon)Y\sigma(y_\epsilon) \le \frac{3}{\epsilon}(\sigma(x_\epsilon) - \sigma(y_\epsilon))^T(\sigma(x_\epsilon) - \sigma(y_\epsilon)) \\ &\le \frac{3}{\epsilon}C_\sigma|x_\epsilon - y_\epsilon|^2\, I, \end{aligned} \tag{3.6}$$

where C_σ is a suitable constant related to the Lipschitz constant of σ. From the definition of $Q(x,p)$ and σ (see (2.3) and Proposition (2.1)), taking $p_\epsilon := \frac{x_\epsilon - y_\epsilon}{\epsilon}$

$$-C_1\frac{|x_\epsilon - y_\epsilon|^2}{\epsilon}I \le Q(y_\epsilon, p_\epsilon) - Q(x_\epsilon, p_\epsilon) \le C_1\frac{|x_\epsilon - y_\epsilon|^2}{\epsilon}I. \tag{3.7}$$

Then, by (3.6) and (3.7),

$$\begin{aligned} &G(y_\epsilon, p_\epsilon, Y) = \sigma^T(y_\epsilon)Y\,\sigma(y_\epsilon) + Q(y_\epsilon, p_\epsilon) \ge \\ &\ge G(x_\epsilon, p_\epsilon, X) - \frac{3}{\epsilon}C_\sigma|x_\epsilon - y_\epsilon|^2\, I + Q(y_\epsilon, p_\epsilon) - Q(x_\epsilon, p_\epsilon) \ge \\ &\ge \big(\gamma - \frac{3}{\epsilon}C_\sigma|x_\epsilon - y_\epsilon|^2 - C_1\frac{|x_\epsilon - y_\epsilon|^2}{\epsilon}\big)\, I \ge \frac{\gamma}{2}\, I, \end{aligned} \tag{3.8}$$

for ϵ small enough, by (3.4). Now we use the fact that u and v are sub- and supersolutions of (3.2):

$$\begin{aligned} &-\log\det(\sigma^T(x_\epsilon)X\,\sigma(x_\epsilon) + Q(x_\epsilon, p_\epsilon)) + K(x_\epsilon, u(x_\epsilon), \sigma^T(x_\epsilon)p_\epsilon) \le 0, \\ &-\log\det(\sigma^T(y_\epsilon)Y\,\sigma(y_\epsilon) + Q(y_\epsilon, p_\epsilon)) + K(y_\epsilon, v(y_\epsilon), \sigma^T(y_\epsilon)p_\epsilon) \ge 0. \end{aligned} \tag{3.9}$$

If there is a sequence $\epsilon_k \to 0$ such that $\lim_k \big(u(x_{\epsilon_k}) - v(y_{\epsilon_k})\big) \le 0$, then we conclude that

$$\max_{\overline{\Omega}}(u - v) \le \lim_k \big(u(x_{\epsilon_k}) - v(y_{\epsilon_k})\big) \le 0.$$

Otherwise, if $u(x_\epsilon) - v(y_\epsilon) \ge \gamma_1 > 0$, by the strict monotonicity of K with respect to the 2nd entry r, we get

$$-\log\det(\sigma^T(y_\epsilon)Y\,\sigma(y_\epsilon) + Q(y_\epsilon, p_\epsilon)) + K(y_\epsilon, u(x_\epsilon), \sigma^T(y_\epsilon)p_\epsilon) - C\gamma_1 \ge 0. \tag{3.10}$$

Now we subtract this inequality from the first of (3.9):

$$\begin{aligned} &-\log\det G(x_\epsilon, p_\epsilon, X) + \log\det G(y_\epsilon, p_\epsilon, Y) + K(x_\epsilon, u(x_\epsilon), \sigma^T(x_\epsilon)p_\epsilon) - \\ &-K(y_\epsilon, u(x_\epsilon), \sigma^T(y_\epsilon)p_\epsilon) + C\gamma_1 \le 0. \end{aligned} \tag{3.11}$$

By inequalities (3.6) and (3.7) we know that

$$G(x_\epsilon, p_\epsilon, X) \le G(y_\epsilon, p_\epsilon, Y) + (3C_\sigma + C_1)\frac{|x_\epsilon - y_\epsilon|^2}{\epsilon}I.$$

Hence, by the monotonicity of *det* over positive definite matrices,

$$
\begin{aligned}
&-\log\det G(x_\epsilon,p_\epsilon,X)+\log\det G(y_\epsilon,p_\epsilon,Y)\geq\\
&-\log\det\left(G(y_\epsilon,p_\epsilon,Y)+(3C_\sigma+C_1)\frac{|x_\epsilon-y_\epsilon|^2}{\epsilon}I\right)+\log\det G(y_\epsilon,p_\epsilon,Y)=\\
&=-\log\det\left(I+(3C_\sigma+C_1)\frac{|x_\epsilon-y_\epsilon|^2}{\epsilon}G^{-1}(y_\epsilon,p_\epsilon,Y)\right).
\end{aligned}
$$

Since, from (3.8), we have that

$$G^{-1}(y_\epsilon,p_\epsilon,Y))\leq\frac{2}{\gamma}I,$$

by the monotonicity of *det* we obtain that

$$-\log\det G(x_\epsilon,p_\epsilon,X)+\log\det G(y_\epsilon,p_\epsilon,Y)\geq-\log\left(1+\frac{2}{\gamma}(3C_\sigma+C_1)\frac{|x_\epsilon-y_\epsilon|^2}{\epsilon}\right) \tag{3.12}$$

and the right hand side tends to 0 as $\epsilon\to0^+$, by (3.4).

Next we consider the term

$$K(x_\epsilon,u(x_\epsilon),\sigma^T(x_\epsilon)p_\epsilon)-K(y_\epsilon,u(x_\epsilon),\sigma^T(y_\epsilon)p_\epsilon)$$

in (3.11). Since $p_\epsilon=\frac{x_\epsilon-y_\epsilon}{\epsilon}$ is in the superdifferential of u at $x_\epsilon\in\Omega_1$, with $\overline{\Omega}_1\subset\Omega$, Proposition 2.2 gives

$$|\sigma^T(x_\epsilon)p_\epsilon|\leq C. \tag{3.13}$$

Moreover

$$|\sigma^T(x_\epsilon)p_\epsilon-\sigma^T(y_\epsilon)p_\epsilon|\leq L_\sigma\frac{|x_\epsilon-y_\epsilon|^2}{\epsilon}\to0\ \text{ as }\epsilon\to0,$$

where L_σ is a Lipschitz constant of σ, and therefore, for ϵ small,

$$|\sigma^T(y_\epsilon)p_\epsilon|\leq C+1. \tag{3.14}$$

Let ω_1 be the modulus of continuity of K on $\overline{\Omega}\times[-M,M]\times\overline{B}(0,C+1)$. Then

$$|K(x_\epsilon,u(x_\epsilon),\sigma^T(x_\epsilon)p_\epsilon)-K(y_\epsilon,u(x_\epsilon),\sigma^T(y_\epsilon)p_\epsilon)|\leq\omega_1(|x_\epsilon-y_\epsilon|+L_\sigma\frac{|x_\epsilon-y_\epsilon|^2}{\epsilon})\to0 \tag{3.15}$$

as $\epsilon\to0$. Then from (3.12) and (3.15), (3.11) becomes

$$0<C\gamma_1\leq\log\det(1+\frac{2}{\gamma}(3C_\sigma+C_1)\frac{|x_\epsilon-y_\epsilon|^2}{\epsilon})I+\omega_1(|x_\epsilon-y_\epsilon|+L_\sigma\frac{|x_\epsilon-y_\epsilon|^2}{\epsilon}),$$

and we get a contradiction by letting $\epsilon\to0^+$.

□

Proof. (of Theorem 3.1) We rewrite (2.1) in the form

$$-\log\det(D^2_{\mathcal{X}}u)+\log H(x,u,D_{\mathcal{X}}u)=0,\ \text{ in }\Omega. \tag{3.16}$$

We follow the steps of the proof of Theorem 3.2 with $K(x,r,q):=\log H(x,r,q)$.

The only assumption that we have to check is the strict monotonicity with respect to r of $\log H(x,r,q)$ from the strict monotonicity of $H(x,r,q)$. In this case, to obtain estimate (3.10) for $(x_\epsilon,y_\epsilon)\in\Omega_1\times\Omega_1$, we proceed as follows:

$$
\begin{aligned}
&\log H(y_\epsilon,u(x_\epsilon),\sigma^T(y_\epsilon)p_\epsilon)-\log H(y_\epsilon,v(y_\epsilon),\sigma^T(y_\epsilon)p_\epsilon)\geq\\
&\log\left(H(y_\epsilon,v(y_\epsilon),\sigma^T(y_\epsilon)p_\epsilon)+C\left(u(x_\epsilon)-v(y_\epsilon)\right)\right)-\log H(y_\epsilon,v(y_\epsilon),\sigma^T(y_\epsilon)p_\epsilon)=\\
&\log\left(1+\frac{C\left(u(x_\epsilon)-v(y_\epsilon)\right)}{H(y_\epsilon,v(y_\epsilon),\sigma^T(y_\epsilon)p_\epsilon)}\right).
\end{aligned}
$$

From the continuity of H and (3.14) we get $H(y_\epsilon, v(y_\epsilon), \sigma^T(y_\epsilon)p_\epsilon) \le \overline{C}$, for $y_\epsilon \in \Omega_1$, and therefore

$$K(y_\epsilon, u(x_\epsilon), \sigma^T(y_\epsilon)p_\epsilon) - K\left(y_\epsilon, v(y_\epsilon), \sigma^T(y_\epsilon)p_\epsilon\right) \ge \log\left(1 + \frac{C}{\overline{C}}(u(x_\epsilon) - v(y_\epsilon))\right) \ge \tilde{C}\big(u(x_\epsilon) - v(y_\epsilon)\big),$$

for a suitable $\tilde{C} > 0$, for all $(x_\epsilon, y_\epsilon) \in \Omega_1 \times \Omega_1$. The rest of the proof is the same at that of Theorem 3.2.

□

Example 3.1 Consider the equation

$$\lambda u - \det(D^2_{\mathcal{X}} u) + k(x)(1 + |D_{\mathcal{X}} u|^2)^{(m+2)/2} = 0, \quad \text{in } \Omega, \tag{3.17}$$

with $\lambda \ge 0$, $k(x) > 0$ continuous. For $\lambda = 0$ this equation prescribes the horizontal Gaussian curvature $k(x)$ of the graph of u. It is classical in the euclidean case ($n = m$) and it was introduced in [16] for Carnot groups. By taking log we rewrite it as

$$-\log\det(D^2_{\mathcal{X}} u) + K(x, u, D_{\mathcal{X}} u) = 0$$

with

$$K(x, r, p) = \log\left(k(x)(1 + |p|^2)^{(m+2)/2} + \lambda\, r\right).$$

For $\lambda > 0$, $\frac{\partial K}{\partial r}$ is positive and bounded away from 0 for bounded $r \ge 0$ and bounded p. Therefore the preceding Comparison Principle applies. In particular, for $\lambda > 0$, there is at most one uniformly horizontally convex viscosity solution of (3.17) with prescribed boundary conditions. The case $\lambda = 0$ is studied in the forthcoming paper [6].

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