PRESCRIBED ENERGY CONNECTING ORBITS FOR GRADIENT SYSTEMS

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Abstract. We are concerned with conservative systems $\ddot{q} = \nabla V(q)$, $q \in \mathbb{R}^N$ for a general class of potentials $V \in C^1(\mathbb{R}^N)$. Assuming that a given sublevel set $\{V \leq c\}$ splits in the disjoint union of two closed subsets $\mathcal{V}^c_-$ and $\mathcal{V}^c_+$, for some $c \in \mathbb{R}$, we establish the existence of bounded solutions q_c to the above system with energy equal to $-c$ whose trajectories connect $\mathcal{V}^c_-$ and $\mathcal{V}^c_+$. The solutions are obtained through an energy constrained variational method, whenever mild coerciveness properties are present in the problem. The connecting orbits are classified into brake, heteroclinic or homoclinic type, depending on the behavior of ∇V on $\partial\mathcal{V}^c_\pm$. Next, we illustrate applications of the existence result to double-well potentials V, and for potentials associated to systems of duffing type and of multiple-pendulum type. In each of the above cases we prove some convergence results of the family of solutions (q_c).

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1. Introduction

In the present paper we are concerned with second order conservative systems

$$\ddot{q} = \nabla V(q), \tag{1.1}$$

where potentials $V \in C^1(\mathbb{R}^N)$ are considered, for any dimension $N \geq 2$.

We study the existence of particular solutions to (1.1), for a class of potentials V for which there exists some value $c \in \mathbb{R}$ so that the sublevel set

$$\mathcal{V}^c := \{x \in \mathbb{R}^N : V(x) \leq c\},$$

is the union of two disjoint subsets. More precisely, we assume that for some $c \in \mathbb{R}$,

($\mathbf{V}^c$) There exist $\mathcal{V}^c_-$, $\mathcal{V}^c_+ \subset \mathbb{R}^N$ closed sets, such that $\mathcal{V}^c = \mathcal{V}^c_- \cup \mathcal{V}^c_+$ and $\operatorname{dist}(\mathcal{V}^c_-, \mathcal{V}^c_+) > 0$,

where $\operatorname{dist}(A, B) := \inf\{|x - y| : x \in A,\ y \in B\}$ refers to the Euclidean distance from a set $A \subset \mathbb{R}^N$ to a set $B \subset \mathbb{R}^N$.

Provided ($\mathbf{V}^c$) holds, we look for *bounded solutions* q of (1.1) on $\mathbb{R}$ with *prescribed mechanical energy* at level $-c$

$$E_q(t) := \tfrac{1}{2}|\dot{q}(t)|^2 - V(q(t)) = -c, \quad \text{for all } t \in \mathbb{R}, \tag{1.2}$$

which in addition *connect* the sets $\mathcal{V}^c_-$ and $\mathcal{V}^c_+$:

$$\inf_{t\in\mathbb{R}} \operatorname{dist}(q(t), \mathcal{V}^c_-) = \inf_{t\in\mathbb{R}} \operatorname{dist}(q(t), \mathcal{V}^c_+) = 0, \tag{1.3}$$

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where $\mathrm{dist}\,(x, A) := \inf\{|x-y| : y \in A\}$ denotes the distance from a point $x \in \mathbb{R}^N$ to a set $A \subset \mathbb{R}^N$.

To better describe which kind of solutions of (1.1) satisfying (1.2) and (1.3) one can get, it is better to make some simple qualitative reasoning. Note that condition (1.3) imposes $\inf_{t\in\mathbb{R}} \mathrm{dist}\,(q(t), \mathcal{V}^c_\pm) = 0$; this is true if either the solution q touches one (or both) of $\mathcal{V}^c_\pm$ in a point, or if it accumulates $\mathcal{V}^c_\pm$ at infinity, that is to say,

$$\liminf_{t\to-\infty} \mathrm{dist}\,(q(t), \mathcal{V}^c_\pm) = 0, \quad \text{or} \quad \liminf_{t\to+\infty} \mathrm{dist}\,(q(t), \mathcal{V}^c_\pm) = 0. \tag{1.4}$$

In the first case, there exists a time t_0 such that $q(t_0) \in \mathcal{V}^c_\pm$. In this situation we say that $q(t_0)$ is a *contact point* between the trajectory q and $\mathcal{V}^c_\pm$, and that t_0 is a *contact time*. Let us note right away that if t_0 is a *contact* time, since $V(q(t_0)) \le c$, then the energy condition (1.2) imposes that $V(q(t_0)) = c$ and $\dot{q}(t_0) = 0$. Hence t_0 is a *turning* time, i.e., q is symmetric with respect to t_0. From this we recover that q has at most two contact points.

The *connecting solutions* between $\mathcal{V}^c_\pm$ can therefore be classified into three types, corresponding to the different number of *contact points* they exhibit. Precisely, we have

(I) Two *contact points*: In this case the solution q has one contact point $q(\sigma)$ with $\mathcal{V}^c_-$ and one contact point $q(\tau)$ with $\mathcal{V}^c_+$. We can assume that $\sigma < \tau$ (by reflecting the time if necessary), and that the interval (σ, τ) does not contain other *contact times*. Since the solution is symmetric with respect to both σ and τ, it then follows that it has to be periodic, with period $2(\tau-\sigma)$. The solution oscillates back and forth in the configuration space along the arc $q([\sigma, \tau])$, and verifies $V(q(t)) > c$ for any $t \in (\sigma, \tau)$. This solution is said to be of *brake orbit* type (see [30], [32]). Let us remark that a *brake orbit* solution has *only one* contact point with each set $\mathcal{V}^c_\pm$.

(II) One *contact point*: In this case the solution q is symmetric with respect to the (unique) *contact* time σ, resulting that $V(q(t)) > c$ for any $t \in \mathbb{R} \setminus \{\sigma\}$ and $q(\sigma) \in \mathcal{V}^c_\pm$. Moreover $\liminf_{t\to\pm\infty} \mathrm{dist}\,(q(t), \mathcal{V}^c_\mp) = 0$. These solutions are said to be of *homoclinic type*.

(III) No *contact points*: In this last case the absence of contact times implies that $V(q(t)) > c$ for any $t \in \mathbb{R}$, being that $\liminf_{t\to-\infty} \mathrm{dist}\,(q(t), \mathcal{V}^c_\pm) = 0$ and $\liminf_{t\to+\infty} \mathrm{dist}\,(q(t), \mathcal{V}^c_\mp) = 0$. These solutions are said to be of *heteroclinic type*.

A great amount of work regards the existence and multiplicity of brake orbits when c is regular for V, and the set $\{V \ge c\}$ is non-empty and bounded; see [9, 12, 13, 15, 17, 20–22, 24, 25].

A unified approach for the study of general connecting solutions was first made via variational arguments in [1] for systems of Allen-Cahn type equations, where the author already builds solutions in the PDE setting analogous to the ones of heteroclinic type, homoclinic type and brake type solutions (cf. [1, Theorem 1.2] for details, and also [3–7] for related results and techniques).

Concerning the ODE case, the problem of existence of *connecting orbits* of (1.1) and their classification into heteroclinic, homoclinic and periodic type has been recently studied in [11], and subsequently in [18] (see also [19]) for potentials $V \in C^2(\mathbb{R}^N)$ that in addition to $(\mathbf{V}^c)$ satisfy $\partial\{x \in \mathbb{R}^N : V(x) > c\}$ is compact.

Our approach to the problem is variational and is an adaptation of the arguments developed in [1, 7] to the ODE setting. We work on the admissible class

$$\Gamma_c := \{q \in H^1_{loc}(\mathbb{R}, \mathbb{R}^N) : \quad \text{(i)} \quad V(q(t)) \ge c \text{ for all } t \in \mathbb{R}, \text{ and} \\ \text{(ii)} \quad \liminf_{t\to-\infty} \mathrm{dist}\,(q(t), \mathcal{V}^c_-) = \liminf_{t\to+\infty} \mathrm{dist}\,(q(t), \mathcal{V}^c_+) = 0\}, \tag{1.5}$$

and we look for minimizers in Γ_c of the Lagrangian functional

$$J_c(q) := \int_{-\infty}^{+\infty} \tfrac{1}{2}|\dot{q}(t)|^2 + (V(q(t)) - c)\, dt. \tag{1.6}$$

Note that the set Γ_c of admissible functions is defined via (i) and (ii). Condition (i) constitutes an energy constraint, in that, the function $V(q(t)) - c$ is non-negative over $\mathbb{R}$, so the functional J_c is well defined and bounded from below on Γ_c. If q is a minimizer of J_c on Γ_c, then q is a solution of (1.1) on any interval $I \subset \mathbb{R}$ for which condition (i) is strictly satisfied, i.e., $V(q(t)) > c$ for all $t \in I$ (see Lemma 2.10). Condition (ii) forces q to connect $\mathcal{V}^c_-$ to $\mathcal{V}^c_+$. Indeed, if q is a minimizer of J_c on Γ_c there exists an interval $I = (\alpha, \omega) \subset \mathbb{R}$ (possibly with $\alpha = -\infty$ or $\omega = +\infty$) for which $V(q(t)) > c$ for any $t \in I$ (see Lemma 2.11), and

$$\lim_{t\to\alpha^+} \operatorname{dist}(q(t), \mathcal{V}^c_-) = \lim_{t\to\omega^-} \operatorname{dist}(q(t), \mathcal{V}^c_+) = 0.$$

Thus, q is a solution of (1.1) on I and I is a *connecting time interval*, that is to say, an open interval $I \subset \mathbb{R}$ (not necessarily bounded) whose eventual extremes are *contact* times. The existence of a solution to our problem is then obtained by recognizing that the energy of such a minimizer q restricted to I equals $-c$ (see Lemma 2.14), from which we can proceed (by reflection and periodic continuation) to construct our entire connecting solution. Hence, we obtain brake orbit when the connecting interval I is bounded ($\alpha, \omega \in \mathbb{R}$), a homoclinic when I is an half-line (precisely one of α and ω is finite) and finally a heteroclinic if I is the entire real line.

In the present paper, we first establish a general existence result for solutions satisfying the aforementioned properties (see section §2). In fact, the existence of a minimizer of J_c on Γ_c is obtained whenever ($\mathbf{V}^c$) holds and J_c satisfies a mild coerciveness property on Γ_c, namely,

$$\exists R > 0 \;\text{ s.t. }\; \inf_{q\in\Gamma_c} J_c(q) = \inf\{J_c(q) : q \in \Gamma_c,\ \|q\|_{L^\infty(\mathbb{R},\mathbb{R}^N)} \le R\}. \tag{1.7}$$

From this minimizer we can reconstruct a solution $q_c \in C^2(\mathbb{R}, \mathbb{R}^N)$ to the problem (1.1),(1.3) satisfying the energy constraint $E_{q_c}(t) := \frac{1}{2}|\dot{q}(t)|^2 - V(q(t)) = -c$ for all $t \in \mathbb{R}$, see Theorem 2.1.

In order to have a better understanding of the scope of Theorem 2.1, which is presented in a very general form, it might be useful to illustrate some specific situations in which we can verify condition ($\mathbf{V}^c$) and (1.7). This is done in §3 where more explicit assumptions on the potential V are considered, including classical cases as double well, Duffing like and pendulum like potential systems. In all these situations the potential V has isolated minima at the level $c = 0$. The application of Theorem 2.1 to these cases allows us to obtain existence and multiplicity results of connecting orbits q_c at energy level $-c$ whenever c is sufficiently small (see propositions 3.1, 3.5 and 3.10). When $c = 0$ the corresponding connecting orbits are homoclinic or heteroclinic solutions connecting the different minima of the potential, while we get brake orbit solutions when c is a regular value for V. We then study convergence properties of the family of solutions q_c to homoclinic type solutions or heteroclinic type solutions as the energy level c goes to zero (see propositions 3.3, 3.7 and 3.12).

Our results extend recent studies made in [33] in the ODE framework, where for a certain class of two-well potentials, periodic orbits of (1.1) are shown to converge, in a suitable sense, to a heteroclinic solution joining the wells of such potential.

The issue of existence of heteroclinic solutions connecting the equilibria of multi-well potentials has been quite explored in the literature; the interested reader is referred to [8], [11], [19], [23], [28], and [26, 31] for different approaches on the subject.

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2. The general existence result

In this section we state and prove our general result concerning the existence of solutions to the conservative system (1.1) connecting the sublevels $\mathcal{V}^c_\pm$ and that satisfy a pointwise energy constraint, provided (1.7) and ($\mathbf{V}^c$) hold. The proof of Theorem 2.1 adapts, to the ODE case, arguments that were already developed in [1], [5] and [7] for (systems of) PDE.

Theorem 2.1. *Assume $V \in C^1(\mathbb{R}^N)$, and that there exists $c \in \mathbb{R}$ such that* ($\mathbf{V}^c$) *and the coercivity condition* (1.7) *of the energy functional J_c over Γ_c hold true. Then there exists a solution $q_c \in C^2(\mathbb{R}, \mathbb{R}^N)$ to* (1.1)-(1.3) *which in addition satisfies*

$$E_{q_c}(t) := \tfrac{1}{2}|\dot q_c(t)|^2 - V(q_c(t)) = -c, \quad \text{for all } t \in \mathbb{R}.$$

Furthermore, any such solution is classified in one of the following types

(a) *q_c is of* brake orbit type*: There exist $-\infty < \sigma < \tau < +\infty$ so that*
  (a.i) *$V(q_c(\sigma)) = V(q_c(\tau)) = c$, $V(q_c(t)) > c$ for every $t \in (\sigma, \tau)$ and $\dot q_c(\sigma) = \dot q_c(\tau) = 0$,*
  (a.ii) *$q_c(\sigma) \in \mathcal{V}^c_-$, $q_c(\tau) \in \mathcal{V}^c_+$, $\nabla V(q_c(\sigma)) \neq 0$ and $\nabla V(q_c(\tau)) \neq 0$,*
  (a.iii) *$q_c(\sigma + t) = q_c(\sigma - t)$ and $q_c(\tau + t) = q_c(\tau - t)$ for all $t \in \mathbb{R}$.*

(b) *q_c is of* homoclinic type*: There exist $\sigma \in \mathbb{R}$ and a component $\mathcal{V}^c_\pm$ of $\mathcal{V}^c$ so that*
  (b.i) *$V(q_c(\sigma)) = c$, $V(q_c(t)) > c$ for every $t \in \mathbb{R} \setminus \{\sigma\}$, $\dot q_c(\sigma) = 0$ and $\lim_{t\to\pm\infty} \dot q_c(t) = 0$,*
  (b.ii) *$q_c(\sigma) \in \mathcal{V}^c_\pm$, $\nabla V(q_c(\sigma)) \neq 0$ and there exists a closed connected set $\Omega \subset \mathcal{V}^c_\mp \cap \{x \in \mathbb{R}^N : V(x) = c,\ \nabla V(x) = 0\}$ so that $\lim_{t\to\pm\infty} \operatorname{dist}(q_c(t), \Omega) = 0$,*
  (b.iii) *$q_c(\sigma + t) = q_c(\sigma - t)$ for all $t \in \mathbb{R}$.*

(c) *q_c is of* heteroclinic type*: There holds*
  (c.i) *$V(q_c(t)) > c$ for all $t \in \mathbb{R}$ and $\lim_{t\to\pm\infty} \dot q_c(t) = 0$,*
  (c.ii) *There exist closed connected sets $\mathcal{A} \subset \mathcal{V}^c_- \cap \{x \in \mathbb{R}^N : V(x) = c,\ \nabla V(x) = 0\}$ and $\Omega \subset \mathcal{V}^c_+ \cap \{x \in \mathbb{R}^N : V(x) = c,\ \nabla V(x) = 0\}$ such that*

$$\lim_{t\to-\infty} \operatorname{dist}(q_c(t), \mathcal{A}) = 0 \quad \text{and} \quad \lim_{t\to+\infty} \operatorname{dist}(q_c(t), \Omega) = 0.$$

Remark 2.2. *Note that if c is a regular value for V then the corresponding solution q_c given by Theorem 2.1 is of brake type, while it may be of the heteroclinic or homoclinic type if c is a critical value of V.*

To prove Theorem 2.1, given $c \in \mathbb{R}$ and an interval $I \in \mathbb{R}$, we consider the *action functional*

$$J_{c,I}(q) := \int_I \tfrac{1}{2}|\dot q(t)|^2 + (V(q(t)) - c)\, dt,$$

defined on the space

$$\mathcal{X}_c := \{q \in H^1_{loc}(\mathbb{R}, \mathbb{R}^N) : \inf_{t\in\mathbb{R}} V(q(t)) \geq c\}.$$

We will write henceforth $J_c(q) := J_{c,\mathbb{R}}(q)$.

Remark 2.3. *Note that since the lower bound $V(q(t)) \geq c$ for all $t \in \mathbb{R}$ holds for any $q \in \mathcal{X}_c$, then we readily see that $J_{c,I}$ is non-negative on $\mathcal{X}_c$ for any given real interval I. Moreover, $\mathcal{X}_c$ is sequentially closed, and for any interval $I \subset \mathbb{R}$, $J_{c,I}$ is lower semicontinuous with respect to the weak topology of $H^1_{loc}(\mathbb{R}, \mathbb{R}^N)$.*

Remark 2.4. *If $q \in \mathcal{X}_c$ and $(\sigma, \tau) \subset \mathbb{R}$, then*

$$\begin{aligned} J_{c,(\sigma,\tau)}(q) &\geq \tfrac{1}{2(\tau-\sigma)}|q(\tau) - q(\sigma)|^2 + \int_\sigma^\tau (V(q(t)) - c)\, dt \\ &\geq \sqrt{\tfrac{2}{\tau-\sigma} \int_\sigma^\tau (V(q(t)) - c)\, dt}\, |q(\tau) - q(\sigma)|. \end{aligned}$$

In particular, if there exists some $\mu > 0$ for which $V(q(t)) - c \geq \mu \geq 0$ for all $t \in (\sigma, \tau)$, then we have

$$J_{c,(\sigma,\tau)}(q) \geq \sqrt{2\mu}\, |q(\tau) - q(\sigma)|. \tag{2.8}$$

Remark 2.5. *In view of* ($\mathbf{V}^c$)*, the sets $\mathcal{V}^c_-$ and $\mathcal{V}^c_+$ are disjoint and closed, and so they are locally well separated. Hence, if R denotes the constant introduced in the coerciveness assumption* (1.7)*, we have*

$$4\rho_0 := \mathrm{dist}\,(\mathcal{V}^c_- \cap B_R(0), \mathcal{V}^c_+ \cap B_R(0)) > 0. \tag{2.9}$$

The continuity of V ensures that for any $r > 0$ and $C > 0$, there exists $h_{r,C} > 0$ in such a way that

$$\inf\{V(x) : |x| \leq C \ \text{ and } \ \mathrm{dist}\,(x, \mathcal{V}^c) \geq r\} \geq c + h_{r,C}.$$

In what follows, we will simply denote $h_r := h_{r,R}$, where R is the constant given in (1.7).

The variational problem we are interested in studying involves the following admissible set

$$\Gamma_c := \Big\{u \in \mathcal{X}_c : \liminf_{t\to-\infty} \mathrm{dist}\,(q(t), \mathcal{V}^c_-) = \liminf_{t\to+\infty} \mathrm{dist}\,(q(t), \mathcal{V}^c_+) = 0\Big\},$$

and we will denote $m_c := \inf_{q\in\Gamma_c} J_c(q)$.

The first observation in place is

Lemma 2.6. *There results $m_c \in (0, +\infty)$.*

Proof of Lemma 2.6. It is plain to observe that $m_c < +\infty$. Indeed, by ($\mathbf{V}^c$) we can choose two points $\xi_- \in \mathcal{V}^c_-$ and $\xi_+ \in \mathcal{V}^c_+$ such that $V(t\xi_+ + (1 - t)\xi_-) > c$ for any $t \in (0, 1)$. Then defining

$$q(t) = \begin{cases} \xi_- & \text{if } t \leq 0, \\ t\xi_+ + (1-t)\xi_- & \text{if } t \in (0,1), \\ \xi_+ & \text{if } 1 \leq t. \end{cases}$$

one plainly recognizes that $q \in \Gamma_c$ and $m_c \leq J_c(q) < +\infty$.
To show that $m_c > 0$, let us observe that in light of (1.7)

$$m_c = \inf\{J_c(q) : q \in \Gamma_c \ \text{ and } \ \|q\|_{L^\infty(\mathbb{R},\mathbb{R}^N)} \leq R\}.$$

Also, by Sobolev embedding theorems, any $q \in \Gamma_c$ is continuous over $\mathbb{R}$ and it verifies

$$\liminf_{t\to-\infty} \operatorname{dist}(q(t), \mathcal{V}^c_-) = \liminf_{t\to+\infty} \operatorname{dist}(q(t), \mathcal{V}^c_+) = 0.$$

Recalling that $\mathcal{V}^c = \mathcal{V}^c_- \cup \mathcal{V}^c_+$, we deduce from (2.9) and the fact that $\|q\|_{L^\infty(\mathbb{R},\mathbb{R}^N)} \le R$, that there exists a nonempty open interval $(\sigma, \tau) \subset \mathbb{R}$ depending on q, in such a way that

$$|q(\tau) - q(\sigma)| = 2\rho_0, \quad \text{and} \quad \operatorname{dist}(q(t), \mathcal{V}^c) \ge \rho_0 \ \text{ for any } \ t \in (\sigma, \tau).$$

But then Remark 2.5 yields a uniform lower bound $V(q(t)) - c \ge h_{\rho_0}$, for any $t \in (\sigma, \tau)$. This fact, combined with Remark 2.4 yields

$$J_{c,(\sigma,\tau)}(q) \ge \sqrt{2h_{\rho_0}}|q(\tau) - q(\sigma)| \ge \sqrt{2h_{\rho_0}}\, 2\rho_0.$$

Therefore, $m_c = \inf_{\Gamma_c} J_c(q) \ge \inf_{\Gamma_c} J_{c,(\sigma,\tau)}(q) \ge \sqrt{2h_{\rho_0}}\, 2\rho_0 > 0$. □

We now argue that finite energy elements in the admissible class Γ_c asymptotically approach the sublevel sets $\mathcal{V}^c_-$ and $\mathcal{V}^c_+$ in the following sense

Lemma 2.7. *Suppose $q \in \Gamma_c$ satisfies $\|q\|_{L^\infty(\mathbb{R},\mathbb{R}^N)} < +\infty$ and $J_c(q) < +\infty$. Then*

$$\lim_{t\to-\infty} \operatorname{dist}(q(t), \mathcal{V}^c_-) = \lim_{t\to+\infty} \operatorname{dist}(q(t), \mathcal{V}^c_+) = 0.$$

Proof of Lemma 2.7. Let us argue the case of the limit as $t \to -\infty$, the other limit can be argued similarly. By definition of Γ_c, q satisfies $\liminf_{t\to-\infty} \operatorname{dist}(q(t), \mathcal{V}^c_-) = 0$. Let us assume by contradiction that $\limsup_{t\to-\infty} \operatorname{dist}(q(t), \mathcal{V}^c_-) > 0$, so there must exist $\rho \in (0, \rho_0)$ and two sequences $\sigma_n \to -\infty$, $\tau_n \to -\infty$ such that $\tau_{n+1} < \sigma_n < \tau_n$ for which there results $|q(\tau_n) - q(\sigma_n)| = \rho$ and $\rho \le \operatorname{dist}(q(t), \mathcal{V}^c_-) \le 2\rho$ for any $t \in (\sigma_n, \tau_n)$. In particular, since $\rho < \rho_0$ it follows that $\operatorname{dist}(q(t), \mathcal{V}^c) > \rho$ for any $t \in (\sigma_n, \tau_n)$ and $n \in \mathbb{N}$. Since $M := \|q\|_{L^\infty(\mathbb{R},\mathbb{R}^N)} < +\infty$, Remark 2.5 yields $V(q(t)) > c + h_{\rho,M}$ for any $t \in (\sigma_n, \tau_n)$ and so, by Remark 2.4, we conclude

$$J_{c,(\sigma_n,\tau_n)}(q) \ge \sqrt{2h_{\rho,M}}\,|q(\tau_n) - q(\sigma_n)| = \sqrt{2h_{\rho,M}}\,\rho, \quad \text{for all } \ n \in \mathbb{N}.$$

But then $J_c(q) \ge \sum_{n=1}^{\infty} J_{c,(\sigma_n,\tau_n)}(q) = +\infty$, thus contradicting the assumption $J_c(q) < +\infty$. □

Moreover, by (2.9) we obtain the following concentration result

Lemma 2.8. *There exists $\bar{r} \in (0, \frac{\rho_0}{2})$ so that for any $r \in (0, \bar{r})$, there exist $L_r > 0$, $\nu_r > 0$, in such a way that for any $q \in \Gamma_c$ satisfying:* $\operatorname{dist}(q(0), \mathcal{V}^c) \ge \rho_0$, $\|q\|_{L^\infty(\mathbb{R},\mathbb{R}^N)} \le R$ *and* $J_c(q) \le m_c + \nu_r$, *one has*

(i) *There is $\tau \in (0, L_r)$ so that* $\operatorname{dist}(q(\tau), \mathcal{V}^c_+) \le r$, *and* $\operatorname{dist}(q(t), \mathcal{V}^c_+) < \rho_0$, *for all $t \ge \tau$.*

(ii) *There is $\sigma \in (-L_r, 0)$ so that* $\operatorname{dist}(q(\sigma), \mathcal{V}^c_-) \le r$, *and* $\operatorname{dist}(q(t), \mathcal{V}^c_-) < \rho_0$, *for all $t \le \sigma$.*

Proof of Lemma 2.8. Given any $r \in (0, \frac{\rho_0}{2})$ we define the following quantities

$$\mu_r := \max\{V(x) - c \,:\, |x| \le R \ \text{ and } \ \operatorname{dist}(x, \mathcal{V}^c) \le r\},$$
$$L_r := \tfrac{m_c+1}{h_r}, \quad \text{and} \quad \nu_r := \tfrac{1}{2}r^2 + \mu_r.$$

In view of the continuity of V we have $\lim_{r\to0^+} \nu_r = 0$. Hence, we can choose $\bar{r} \in (0, \frac{\rho_0}{2})$ so that

$$\nu_r < \min\left\{\sqrt{h_{\frac{\rho_0}{2}}}\,\frac{\rho_0}{4},\, 1\right\} \ \text{ for any } \ r \in (0, \bar{r}). \tag{2.10}$$

For $q \in \Gamma_c$ satisfying the assumptions of this lemma, let us define

$$\sigma := \sup\{t \in \mathbb{R} : \operatorname{dist}(q(t), \mathcal{V}_-^c) \le r\}, \quad \tau := \inf\{t > \sigma : \operatorname{dist}(q(t), \mathcal{V}_+^c) \le r\}.$$

We observe that $-\infty < \sigma < \tau < +\infty$, since $\lim_{t\to-\infty} \operatorname{dist}(q(t), \mathcal{V}_-^c) = \lim_{t\to+\infty} \operatorname{dist}(q(t), \mathcal{V}_+^c) = 0$, in light of the fact that the hypotheses of Lemma 2.7 are fulfilled for any q as above. Furthermore, the definition of σ and τ yield

$$\operatorname{dist}(q(t), \mathcal{V}^c) > r \quad \text{for any } t \in (\sigma, \tau). \tag{2.11}$$

Now we claim that

$$\operatorname{dist}(q(t), \mathcal{V}_-^c) < \rho_0 \quad \text{for any } t < \sigma. \tag{2.12}$$

Indeed, we can fix $\xi_\sigma \in \mathcal{V}_-^c$ so that $|q(\sigma) - \xi_\sigma| \le r$ and

$$V((1-s)q(\sigma) + s\xi_\sigma) > c \quad \text{for any } s \in (0,1). \tag{2.13}$$

Let us define

$$\underline{q}(t) := \begin{cases} q(t) & \text{if } t \le \sigma, \\ (\sigma + 1 - t)q(\sigma) + (t - \sigma)\xi_\sigma & \text{if } \sigma < t < \sigma + 1, \\ \xi_\sigma & \text{if } \sigma + 1 \le t. \end{cases}$$

and

$$\overline{q}(t) := \begin{cases} \xi_\sigma & \text{if } t \le \sigma - 1, \\ (t - \sigma + 1)q(\sigma) + (\sigma - t)\xi_\sigma & \text{if } \sigma - 1 < t < \sigma, \\ q(t) & \text{if } \sigma \le t. \end{cases}$$

First we note that by (2.13) and since $q \in \Gamma_c$, we have $\overline{q} \in \Gamma_c$ and so $J_c(\overline{q}) \ge m_c$. The latter, combined with the following inequality

$$J_{c,(-\infty,\sigma)}(\overline{q}) \le \int_{\sigma-1}^{\sigma} \tfrac{1}{2}|q(\sigma) - \xi_\sigma|^2 + \mu_r \, dt \le \tfrac{1}{2}r^2 + \mu_r = \nu_r,$$

shows that $J_{c,(\sigma,+\infty)}(\overline{q}) \ge m_c - \nu_r$, from which it follows

$$m_c + \nu_r \ge J_c(q) = J_{c,(-\infty,\sigma)}(\underline{q}) + J_{c,(\sigma,+\infty)}(\overline{q}) \ge J_{c,(-\infty,\sigma)}(\underline{q}) + m_c - \nu_r,$$

thus proving

$$J_{c,(-\infty,\sigma)}(\underline{q}) \le 2\nu_r. \tag{2.14}$$

To finish the proof of claim (2.12), let us assume by contradiction that there is $t_* < \sigma$ such that $\operatorname{dist}(q(t_*), \mathcal{V}_-^c) \ge \rho_0$. Since $\operatorname{dist}(q(\sigma), \mathcal{V}_-^c) \le r < \frac{\rho_0}{2}$, we deduce that there exists an interval $(\gamma, \delta) \subset (-\infty, \sigma)$ such that $|\underline{q}(\delta) - \underline{q}(\gamma)| = \frac{\rho_0}{2}$ and $\frac{\rho_0}{2} \le \operatorname{dist}(\underline{q}(t), \mathcal{V}_-^c) \le \rho_0$ for all $t \in (\gamma, \delta)$. Then, estimate (2.14) combined with Remark 2.4 and Remark 2.5 (since $\|q\|_{L^\infty(\mathbb{R},\mathbb{R}^N)} \le R$), yields

$$2\nu_r \ge J_{c,(-\infty,\sigma)}(\underline{q}) \ge J_{c,(\gamma,\delta)}(\underline{q}) \ge \sqrt{2h_{\frac{\rho_0}{2}}}\,\frac{\rho_0}{2}, \quad \text{for any } r \le \bar{r},$$

which contradicts (2.10) in view of the definition of $\bar{r}$. An analogous argument proves that

$$\operatorname{dist}(q(t), \mathcal{V}_+^c) < \rho_0 \quad \text{for any } t > \tau. \tag{2.15}$$

In this way, we have argued that the conditions (i)-(ii) are satisfied for the choice of τ and σ as above. We are left to prove the chain of inequalities$-L_r < \sigma < 0 < \tau < L_r$, for the choice of L_r as in the beginning of the proof. To see this, let us first note that $0 \in (\sigma, \tau)$. This follows from the way the time $t = 0$ was chosen: $\operatorname{dist}(q(0), \mathcal{V}^c) \geq \rho_0$ and from (2.12)-(2.15) combined. Also, from Remark 2.5 and (2.11) we have $V(q(t)) - c \geq h_r$ for $t \in (\sigma, \tau)$. This, and (2.10) yield the lower bound

$$m_c + 1 > m_c + \nu_r \geq J_{c,(0,\tau)}(q) \geq \int_0^\tau (V(q(t)) - c)dt \geq \tau h_r.$$

In other words, we have proved that $0 < \tau < \frac{m_c+1}{h_r} =: L_r$. Analogously, we derive that $m_c+1 > m_c+\nu_r \geq J_{c,(\sigma,0)}(q) \geq -\sigma h_r$ from which $0 < -\sigma < L_r$. The proof of Lemma 2.8 is now complete. □

We can now conclude that the minimal level m_c is achieved in Γ_c. Indeed, we have

Lemma 2.9. *There exists $q_0 \in \Gamma_c$ such that $J_c(q_0) = m_c$.*

Proof of Lemma 2.9. Let $(q_n) \subset \Gamma_c$ be a minimizing sequence, so $J_c(q_n) \to m_c$. The coerciveness assumption (1.7) allows us to assume that

$$\|q_n\|_{L^\infty(\mathbb{R},\mathbb{R}^N)} \leq R \quad \text{for any } n \in \mathbb{N}, \tag{2.16}$$

and hence Lemma 2.7 yields

$$\lim_{t\to-\infty} \operatorname{dist}(q_n(t), \mathcal{V}^c_-) = \lim_{t\to+\infty} \operatorname{dist}(q_n(t), \mathcal{V}^c_+) = 0 \quad \text{for any } n \in \mathbb{N}.$$

Thus, (2.9) combined with continuity arguments shows that there is $(t_n) \subset \mathbb{R}$ so that $\operatorname{dist}(q_n(t_n), \mathcal{V}^c) = \rho_0$. Since the variational problem is invariant under time translations, we can assume that

$$\operatorname{dist}(q_n(0), \mathcal{V}^c) = \rho_0 \quad \text{for any } n \in \mathbb{N}. \tag{2.17}$$

But then conditions (2.17),(2.16) together with $J_c(q_n) \to m_c$ allow us to use Lemma 2.8 to deduce the existence of $L > 0$, in a such way that

$$\sup_{t\in(-\infty,-L)} \operatorname{dist}(q_n(t), \mathcal{V}^c_-) \leq \rho_0 \quad \text{and} \quad \sup_{t\in(L,+\infty)} \operatorname{dist}(q_n(t), \mathcal{V}^c_+) \leq \rho_0, \tag{2.18}$$

for all but finitely many terms in the sequence (q_n). Observe now that $\inf_{t\in\mathbb{R}} V(q_n(t)) \geq c$ for all $n \in \mathbb{N}$, since $q_n \in \mathcal{X}_c$, whence

$$\|\dot{q}_n\|^2_{L^2(\mathbb{R},\mathbb{R}^N)} \leq 2m_c + o(1), \quad \text{as } n \to \infty. \tag{2.19}$$

By (2.16) and (2.19) we obtain the existence of $q_0 \in H^1_{loc}(\mathbb{R}, \mathbb{R}^N)$, such that along a subsequence (which we continue to denote q_n) $q_n \rightharpoonup q_0$ weakly in $H^1_{loc}(\mathbb{R}, \mathbb{R}^N)$. As $q_0 \in \mathcal{X}_c$, in view of Remark 2.3, we deduce $J_c(q_0) \leq m_c = \lim_{n\to\infty} J_c(q_n)$. On the other hand, the pointwise convergence, (2.16) and (2.18) yield that $V(q_0(t)) \geq c$ for any $t \in \mathbb{R}$, that $\|q_0\|_{L^\infty(\mathbb{R},\mathbb{R}^N)} \leq R$ and

$$\sup_{t\in(-\infty,-L)} \operatorname{dist}(q_0(t), \mathcal{V}^c_-) \leq \rho_0, \qquad \sup_{t\in(L,+\infty)} \operatorname{dist}(q_0(t), \mathcal{V}^c_+) \leq \rho_0. \tag{2.20}$$

Since $\int_L^{+\infty} (V(q_0(t)) - c)\, dt \leq J_{c,(L,+\infty)}(q_0) \leq m_c$, we obtain that

$$\liminf_{t\to+\infty} V(q_0(t)) - c = 0, \quad \text{which implies} \quad \liminf_{t\to+\infty} \operatorname{dist}(q_0(t), \mathcal{V}^c_+) = 0,$$

in view of (2.20). Analogously, we deduce that $\liminf_{t\to-\infty} \operatorname{dist}(q_0(t), \mathcal{V}^c_-) = 0$. Thus, we have argued that $q_0 \in \Gamma_c$, which in turn shows the reverse inequality $J_c(q_0) \geq m_c$. The proof of Lemma 2.9 is now complete. □

It will be convenient to introduce the following set of minimizers to the variational problem studied in Lemma 2.9,

$$\mathcal{M}_c := \{q \in \Gamma_c : \operatorname{dist}(q(0), \mathcal{V}^c) = \rho_0,\ J_c(q) = m_c,\ \|q\|_{L^\infty(\mathbb{R},\mathbb{R}^N)} \leq R\}.$$

The proof of the above lemma reveals that $\mathcal{M}_c \neq \emptyset$.

For any $q \in \mathcal{M}_c$, we introduce the contact times of the trajectory of q with the sublevel sets $\mathcal{V}_+^c$ and $\mathcal{V}_-^c$ of the potential by letting

$$\alpha_q := \begin{cases} -\infty & \text{if } q(\mathbb{R}) \cap \mathcal{V}_-^c = \emptyset, \\ \sup\{t \in \mathbb{R} : q(t) \in \mathcal{V}_-^c\} & \text{if } q(\mathbb{R}) \cap \mathcal{V}_-^c \neq \emptyset. \end{cases} \tag{2.21}$$

and

$$\omega_q := \begin{cases} \inf\{t > \alpha_q : q(t) \in \mathcal{V}_+^c\} & \text{if } q(\mathbb{R}) \cap \mathcal{V}_+^c \neq \emptyset, \\ +\infty & \text{if } q(\mathbb{R}) \cap \mathcal{V}_+^c = \emptyset. \end{cases} \tag{2.22}$$

Note that for all $q \in \mathcal{M}_c$, by Lemma 2.8 and by the definition of ρ_0 (2.9), it is simple to verify that

$$-\infty \leq \alpha_q < \omega_q \leq +\infty.$$

Moreover, by definition of α_q and ω_q, since $\operatorname{dist}(q(0), \mathcal{V}^c) = \rho_0$ for every $q \in \mathcal{M}_c$, we have $q(t) \in \mathbb{R}^N \setminus (\mathcal{V}_-^c \cup \mathcal{V}_+^c)$ for any $\alpha_q < t < \omega_q$, that is

$$V(q(t)) > c \quad \text{for any } t \in (\alpha_q, \omega_q).$$

Therefore we obtain

Lemma 2.10. *If $q \in \mathcal{M}_c$ then $q \in C^2((\alpha_q, \omega_q), \mathbb{R}^N)$. Furthermore, any such q is a solution to the system*

$$\ddot{q}(t) = \nabla V(q(t)), \quad \textit{for all } t \in (\alpha_q, \omega_q).$$

Proof of Lemma 2.10. Let $\psi \in C_0^\infty(\mathbb{R})$ be so that $\operatorname{supp}\psi \subset [a, b] \subset (\alpha_q, \omega_q)$. Since $V(q(t)) > c$ for any $t \in (\alpha_q, \omega_q)$ and $t \mapsto V(q(t))$ is continuous on $\mathbb{R}$, we derive that there exists $\lambda_0 > 0$ such that $\min_{t\in[a,b]} V(q(t)) = c + \lambda_0$. The continuity of V ensures that there exists $h_\psi > 0$ such that

$$\min_{t\in[a,b]} V(q(t) + h\psi(t)) > c \quad \text{for any } h \in (0, h_\psi).$$

In other words, for any $\psi \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}\psi \subset [a, b] \subset (\alpha_q, \omega_q)$ there exists $h_\psi > 0$ in such a way that $q + h\psi \in \Gamma_c$, provided $h \in (0, h_\psi)$. Since q is a minimizer of J_c over Γ_c, then

$$J_c(q + h\psi) - J_c(q) \geq 0 \quad \text{for any } h \in (0, h_\psi).$$

Writing the inequality explicitly, and using the Dominated Convergence Theorem as $h \to 0^+$, we readily see

$$\begin{aligned} &\lim_{h\to0^+} \frac{1}{h}\left(\int_{\alpha_q}^{\omega_q} (\tfrac{1}{2}|\dot{q} + h\dot{\psi}|^2 + V(q + h\psi) - c)\, dt - J_{c,(\alpha_q,\omega_q)}(q)\right) \\ &= \lim_{h\to0^+} \frac{1}{h} \int_{\operatorname{supp}\psi} \left(\tfrac{1}{2}(|\dot{q} + h\dot{\psi}|^2 - |\dot{q}|^2) + (V(q + h\psi) - V(q))\right) dt \\ &= \int_{\mathbb{R}} \dot{q} \cdot \dot{\psi} + \nabla V(q) \cdot \psi\, dt \geq 0. \end{aligned}$$

The same argument with $-\psi$ as test function shows $\int_{\mathbb{R}} \dot q \cdot \dot\psi + \nabla V(q)\cdot\psi\, dt = 0$, so q is a weak solution of $\ddot q = \nabla V(q)$ on (α_q, ω_q). Standard regularity arguments show that $q \in C^2((\alpha_q,\omega_q),\mathbb{R}^N)$, whence q is a strong solution to the above system. □

Moreover, we have

Lemma 2.11. *If $q \in \mathcal{M}_c$, then*

(i) $\lim_{t\to\alpha_q^+} \operatorname{dist}(q(t), \mathcal{V}^c_-) = 0$*, and if* $\alpha_q > -\infty$ *then* $V(q(\alpha_q)) = c$ *with* $q(\alpha_q) \in \mathcal{V}^c_-$*,*

(ii) $\lim_{t\to\omega_q^-} \operatorname{dist}(q(t), \mathcal{V}^c_+) = 0$*, and if* $\omega_q < +\infty$ *then* $V(q(\omega_q)) = c$ *with* $q(\omega_q) \in \mathcal{V}^c_+$*.*

Proof of Lemma 2.11. If $\alpha_q = -\infty$, by Lemma 2.7 we obtain $\lim_{t\to-\infty} \operatorname{dist}(q(t), \mathcal{V}^c_-) = 0$. If $\alpha_q > -\infty$, then the continuity of q and the definition of α_q imply that $q(\alpha_q) \in \mathcal{V}^c_-$, and $V(q(\alpha_q)) = c$. In particular, $\lim_{t\to\alpha_q^+} V(q(t)) = c$, and (i) follows. One argues (ii) in a similar fashion. □

By the previuos result we obtain

Lemma 2.12. *Any $q \in \mathcal{M}_c$ satisfies $J_c(q) = J_{c,(\alpha_q,\omega_q)}(q) = m_c$. Moreover, $q(t) \equiv q(\alpha_q)$ on $(-\infty, \alpha_q)$ if $\alpha_q \in \mathbb{R}$, and $q(t) \equiv q(\omega_q)$ on $(\omega_q, +\infty)$ if $\omega_q \in \mathbb{R}$.*

Proof of Lemma 2.12. Let us define $\tilde q$ to be equal to q on the interval (α_q, ω_q), and such that $\tilde q(t) = q(\alpha_q)$ on $(-\infty, \alpha_q)$ if $\alpha_q \in \mathbb{R}$, while $\tilde q(t) = q(\omega_q)$ on $(\omega_q, +\infty)$ if $\omega_q \in \mathbb{R}$ (so if neither of α_q or ω_q is finite, then $\tilde q = q$). In view of Lemma 2.11 we see that $\tilde q \in \Gamma_c$, whence $J_c(\tilde q) \geq m_c$. The latter implies that $\tilde q$ is also a minimizer of J_c, as $J_c(\tilde q) = J_{c,(\alpha_q,\omega_q)}(\tilde q) = J_{c,(\alpha_q,\omega_q)}(q) \leq m_c$, from which we deduce $m_c = J_c(\tilde q) = J_{c,(\alpha_q,\omega_q)}(q)$. In particular, since $\inf_{t\in\mathbb{R}} V(q(t)) \geq c$ we obtain $J_{c,(-\infty,\alpha_q)}(q) = J_{c,(\omega_q,+\infty)}(q) = 0$, which shows $\|\dot q\|_{L^2((-\infty,\alpha_q),\mathbb{R}^N)} = \|\dot q\|_{L^2((\omega_q,+\infty),\mathbb{R}^N)} = 0$. Therefore, q must be constant on $(-\infty, \alpha_q)$, and on $(\omega_q, +\infty)$, so the lemma is established. □

By the previous result, we obtain

Lemma 2.13. *Consider $q \in \mathcal{M}_c$, and let $(\tau, \sigma) \subseteq (\alpha_q, \omega_q)$ be arbitrary. Then,*

$$\tfrac{1}{2}\int_\tau^\sigma |\dot q(t)|^2\, dt = \int_\tau^\sigma (V(q(t)) - c)\, dt.$$

Proof of Lemma 2.13. Given any $\tau \in (\alpha_q, \omega_q)$, we will prove that

$$\tfrac{1}{2}\int_\tau^{\omega_q} |\dot q(t)|^2\, dt = \int_\tau^{\omega_q} (V(q(t)) - c)\, dt. \tag{2.23}$$

For any $s > 0$ and τ as above, we define

$$q_s(t) := \begin{cases} q(t+\tau) & \text{if } t \leq 0, \\ q(\frac{t}{s}+\tau) & \text{if } t > 0. \end{cases}$$

It is easy to check that $q_s \in \Gamma_c$, by Lemma 2.11 using that $q \in \mathcal{M}_c$. Furthermore, by Lemma 2.12, we deduce

$$J_c(q_s) = J_{c,(\alpha_q-\tau, s(\omega_q-\tau)))}(q_s) \geq m_c = J_{c,(\alpha_q,\omega_q)}(q) = J_{c,(\alpha_q-\tau,\omega_q-\tau)}(q(\cdot+\tau)).$$

In particular, for all $s>0$ we obtain the following

$$
\begin{aligned}
0 &\le J_{c,(\alpha_q-\tau,s(\omega_q-\tau))}(q_s) - J_{c,(\alpha_q-\tau,\omega_q-\tau)}(q(\cdot+\tau))\\
&= \int_0^{s(\omega_q-\tau)} (\tfrac{1}{2}|\dot q_s(t)|^2 + V(q_s(t)) - c)\,dt - \int_0^{\omega_q-\tau} (\tfrac{1}{2}|\dot q(t+\tau)|^2 + V(q(t+\tau)) - c)\,dt\\
&= \int_0^{s(\omega_q-\tau)} \Big(\tfrac{1}{2s^2}\big|\dot q(\tfrac{t}{s}+\tau)\big|^2 + V(q(\tfrac{t}{s}+\tau)) - c\Big)\,dt - J_{c,(\tau,\omega_q)}(q)\\
&= \frac{1}{s}\int_\tau^{\omega_q} \tfrac{1}{2}|\dot q(t)|^2\,dt + s\int_\tau^{\omega_q} (V(q(t)) - c)\,dt - J_{c,(\tau,\omega_q)}(q)\\
&= (\tfrac{1}{s}-1)\int_\tau^{\omega_q} \tfrac{1}{2}|\dot q(t)|^2\,dt + (s-1)\int_\tau^{\omega_q} (V(q(t)) - c)\,dt.
\end{aligned}
$$

Hence, setting $T := \int_\tau^{\omega_q} \frac{1}{2}|\dot q(t)|^2\,dt$ and $U := \int_\tau^{\omega_q} (V(q(t)) - c)\,dt$, we get that the real function $s \mapsto f(s) = (\frac{1}{s}-1)T + (s-1)U$ is non-negative over $(0,+\infty)$. Since it achieves a non-negative minimum at $s = \sqrt{T/U}$, where

$$0 \le f\Big(\sqrt{\tfrac{T}{U}}\Big) = \Big(\sqrt{\tfrac{U}{T}} - 1\Big)T + \Big(\sqrt{\tfrac{T}{U}} - 1\Big)U = -\Big(\sqrt{T} - \sqrt{U}\Big)^2,$$

we conclude $T = U$, i.e. (2.23).
A similar argument also shows that for any $\sigma \in (\alpha_q,\omega_q)$ one has the identity

$$\tfrac{1}{2}\int_{\alpha_q}^{\sigma} |\dot q(t)|^2\,dt = \int_{\alpha_q}^{\sigma} (V(q(t)) - c)\,dt. \tag{2.24}$$

Then, by (2.23) and (2.24), using the additivity property of the integral, we conclude the proof. □

We are now able to prove that every $q \in \mathcal{M}_c$ satisfies the following pointwise energy constraint:

Lemma 2.14. *Every $q \in \mathcal{M}_c$ verifies*

$$E_q(t) := \tfrac{1}{2}|\dot q(t)|^2 - V(q(t)) = -c, \quad \textit{for all } t \in (\alpha_q,\omega_q).$$

Proof of Lemma 2.14. By Lemma 2.10, q solves the system of differential equations (1.1) on (α_q,ω_q) and so the energy $E_q(t) = \frac{1}{2}|\dot q(t)|^2 - V(q(t))$ must be constant on (α_q,ω_q). We are left to show that the value of this constant is precisely $-c$. Let us first treat the case $\alpha_q = -\infty$. We observe that $J_c(q) = \int_{\mathbb{R}} \frac{1}{2}|\dot q(t)|^2 + (V(q(t)) - c) = m_c < +\infty$ direcly yields

$$\liminf_{t\to-\infty} \tfrac{1}{2}|\dot q(t)|^2 + V(q(t)) - c = 0.$$

Since $V(q(t)) \ge c$ for any $t \in \mathbb{R}$, we deduce that there exists a sequence (t_n) such that $t_n \to -\infty$ for which $\lim_{n\to+\infty} \frac{1}{2}|\dot q(t_n)|^2 = 0$ and $\lim_{n\to\infty} V(q(t_n)) = c$. So necessarily

$$\lim_{n\to+\infty} \tfrac{1}{2}|\dot q(t_n)|^2 - V(q(t_n)) = -c,$$

Hence $E_q(t) = -c$ for all $t \in (-\infty,\omega_q)$, proving the lemma in the case $\alpha_q = -\infty$. Clearly, the argument above can be easily applied when $\omega_q = +\infty$, to show $E_q(t) = -c$ for all $t \in (\alpha_q,+\infty)$.
Let us consider now the case $-\infty < \alpha_q < \omega_q < +\infty$. As $q \in \mathcal{M}_c$, Lemma 2.11 tells us that $V(q(\omega_q)) = c$,

and so by continuity of the potential it follows $\lim_{t\to\omega_q^-} V(q(t)) = c$. A similar continuity argument shows

$$\lim_{y\to\omega_q^-} \frac{1}{\omega_q - y}\int_y^{\omega_q} (V(q(t)) - c)\, dt = 0,$$

which, in light of the identity of Lemma 2.13, directly proves that

$$\lim_{y\to\omega_q^-} \frac{1}{\omega_q - y}\int_y^{\omega_q} \tfrac{1}{2}|\dot q(t)|^2\, dt = 0.$$

For then, $\liminf_{y\to\omega_q^-} \frac{1}{2}|\dot q(t)|^2 = 0$, from which

$$\liminf_{t\to\omega_q^-} \tfrac{1}{2}|\dot q(t)|^2 - V(q(t)) = \liminf_{y\to\omega_q^-} \tfrac{1}{2}|\dot q(t)|^2 - \lim_{t\to\omega_q^-} V(q(t)) = -c,$$

proving that $E_q(t) = -c$ for every $t \in (\alpha_q, \omega_q)$. □

We are now able to construct the connecting solutions, concluding the Proof of Theorem 2.1

For $q \in \mathcal{M}_c$ and provided $\omega_q < +\infty$, we denote q_+ the *extension of q by reflection* with respect to ω_q

$$q_+(t) := \begin{cases} q(t) & \text{if } t \in (\alpha_q, \omega_q], \\ q(2\omega_q - t) & \text{if } t \in (\omega_q, 2\omega_q - \alpha_q). \end{cases} \tag{2.25}$$

Similarly, for $q \in \mathcal{M}_c$ with $\alpha_q > -\infty$, let q_- be the *extension of q by reflection* with respect to α_q

$$q_-(t) := \begin{cases} q(2\alpha_q - t) & \text{if } t \in (2\alpha_q - \omega_q, \alpha_q), \\ q(t) & \text{if } t \in [\alpha_q, \omega_q). \end{cases}$$

We have

Lemma 2.15. *For $q \in \mathcal{M}_c$, the following properties hold:*

- *If $\omega_q < +\infty$, then $\lim_{t\to\omega_q^-} \dot q_+(t) = 0$, and q_+ is a solution of (1.1) on $(\alpha_q, 2\omega_q - \alpha_q)$. Furthermore, there results $\nabla V(q_+(\omega_q)) \neq 0$.*
- *If $\alpha_q > -\infty$, then $\lim_{t\to\alpha_q^+} \dot q_-(t) = 0$, and q_- is a solution of (1.1) on $(2\alpha_q - \omega_q, \omega_q)$. Furthermore, there results $\nabla V(q_-(\alpha_q)) \neq 0$.*

Proof of Lemma 2.15. Given $q \in \mathcal{M}_c$, let us assume $\omega_q < +\infty$; the other case where $\alpha_q > -\infty$ can be treated analogously. By Lemma 2.11 we already know that $V(q(\omega_q)) = c$, so by continuity, $\lim_{t\to\omega_q^-} V(q(t)) - c = 0$, which in turn shows $\lim_{t\to\omega_q^-} |\dot q(t)|^2 = \lim_{t\to\omega_q^-} 2(V(q(t) - c) = 0$, due to Lemma 2.14.

The system (1.1) is of second order and autonomous, so starting from the solution q of (1.1) over (α_q, ω_q) (in view of Lemma 2.10) we immediately get that q_+ is a solution of (1.1) on $(\alpha_q, \omega_q)\cup(\omega_q, 2\omega_q - \alpha_q)$. Since q_+ is continuous on the entire interval $(\alpha_q, 2\omega_q - \alpha_q)$ and $\lim_{t\to\omega_q} \dot q_+(t) = 0$ as argued in the preceding paragraph, we deduce that $q_+ \in C^1(\alpha_q, 2\omega_q - \alpha_q)$. Using now the fact that q_+ solves (1.1) on $(\alpha_q, 2\omega_q - \alpha_q) \setminus \{\omega_q\}$, we readily see that the second derivative exists $\ddot q_+(\omega_q) := \lim_{t\to\omega_q} \ddot q_+(t) = \nabla V(q_+(\omega_q))$. For then $q_+ \in C^2(\alpha_q, 2\omega_q - \alpha_q)$, and furthermore, it solves (1.1) on the entire interval $(\alpha_q, 2\omega_q - \alpha_q)$.

In order to conclude the proof, we need to argue that $\nabla V(q_+(\omega_q)) \neq 0$. Suppose on the contrary that $\nabla V(q_+(\omega_q)) = 0$, then $q(\omega_q)$ is an equilibrium of (1.1). Since $\dot q_+(\omega_q) = 0$ and $q_+(\omega_q) = q(\omega_q)$, the uniqueness of solutions to the Cauchy problem shows $q_+(t) = q(\omega_q)$ for any $t \in (\alpha_q, 2\omega_q - \alpha_q)$.

However, this contradicts Lemma 2.11, for which $q_+(\omega_q) = q(\omega_q) \in \mathcal{V}^c_+$ and

$$\lim_{t\to\alpha_q^+} \operatorname{dist}(q_+(t), \mathcal{V}^c_-) = \lim_{t\to\alpha_q^+} \operatorname{dist}(q(t), \mathcal{V}^c_-) = 0.$$

□

Thanks to Lemma 2.15, we know that in the event $q \in \mathcal{M}_c$ satisfies $-\infty < \alpha_q < \omega_q < +\infty$, then q_+ is a solution on the bounded interval $(\alpha_q, 2\omega_q - \alpha_q)$, and it verifies

$$\lim_{t\to\alpha_q^+} \dot{q}_+(t) = \lim_{t\to(2\omega_q-\alpha_q)^-} \dot{q}_+(t) = 0 \quad \text{and} \quad \lim_{t\to\alpha_q^+} q_+(t) = \lim_{t\to(2\omega_q-\alpha_q)^-} q_+(t) = q(\alpha_q).$$

This property implies, in particular, that the $2(\omega_q - \alpha_q)$-periodic extension of q_+ is well defined. In fact, by Lemma 2.10 this extension is a classical $2(\omega_q - \alpha_q)$-periodic solution of (1.1). Clearly, one also makes analogous statements for q_-.

Hence, in the case $q \in \mathcal{M}_c$ is so that $-\infty < \alpha_q < \omega_q < +\infty$, we denote $T := 2(\omega_q - \alpha_q)$, and we let q_c be the *T-periodic extension* of q_+ *(or q_-)* over $\mathbb{R}$, obtaining that q_c is a T-periodic classical solution of (1.1) over $\mathbb{R}$, that satisfies the pointwise energy constraint $E_{q_c}(t) = -c$ for all $t \in \mathbb{R}$. Furthermore, by Lemma 2.11 and Lemma 2.15, it connects $\mathcal{V}^c_-$ to $\mathcal{V}^c_+$, in the following sense

(i) $V(q_c(\alpha_q)) = V(q_c(\omega_q)) = c$, $V(q_c(t)) > c$ for any $t \in (\alpha_q, \omega_q)$ and $\dot{q}_c(\alpha_q) = \dot{q}_c(\omega_q) = 0$,
(ii) $q_c(\alpha_q) \in \mathcal{V}^c_-$, $q_c(\omega_q) \in \mathcal{V}^c_+$, $\nabla V(q_c(\alpha_q)) \neq 0$ and $\nabla V(q_c(\omega_q)) \neq 0$,
(iii) $q_c(\alpha_q - t) = q_c(\alpha_q + t)$ and $q_c(\omega_q - t) = q_c(\omega_q + t)$, for any $t \in \mathbb{R}$.

Therefore, when a minimizer $q \in \mathcal{M}_c$ satisfies $-\infty < \alpha_q < \omega_q < +\infty$, then it generates a solution which periodically oscillates back and forth between the boundary of the two sets $\mathcal{V}^c_-$ and $\mathcal{V}^c_+$, that is a *brake orbit type* solution of (1.1) connecting $\mathcal{V}^c_-$ and $\mathcal{V}^c_+$. Moreover, it is bounded, and in fact, verifies (1.3). Hence, denoting $\sigma = \alpha_q$ and $\tau = \omega_q$, the assertion (a) in Theorem 2.1 is proved.

The remaining cases, where the minimizer $q \in \mathcal{M}_c$ satisfies either $\omega_q = +\infty$, or $\alpha_q = -\infty$, are dealt in Lemma 2.16 below. Before stating this lemma, let us introduce some notation. For $q \in \mathcal{M}_c$ having $\omega_q = +\infty$, we will denote the ω-limit set of q by

$$\Omega_q := \bigcap_{t>\alpha_q} \overline{\{q(s) : s \geq t\}}.$$

The boundedness of any minimizer $q \in \mathcal{M}_c$, $\|q\|_{L^\infty(\mathbb{R},\mathbb{R}^N)} \leq R$, shows that $\Omega_q \subset \overline{B_R(0)}$. Also Ω_q is a closed, connected subset of $\mathbb{R}^n$, being the intersection of closed connected sets. Analogously, for $q \in \mathcal{M}_c$ having $\alpha_q = -\infty$, we will write

$$\mathcal{A}_q := \bigcap_{t<\omega_q} \overline{\{q(s) : s \leq t\}},$$

for the α-limit set of q, which is a closed, connected subset of $\overline{B_R(0)}$. Hence, we have

Lemma 2.16. *Suppose $q \in \mathcal{M}_c$ has either $\alpha_q = -\infty$, or $\omega_q = +\infty$. Then, $\dot{q}(t) \to 0$ as $t \to -\infty$, or as $t \to +\infty$, respectively. Moreover the α-limit set of q, or the ω-limit set of q, respectively, is constituted by critical points of V at level c, namely*

- $\mathcal{A}_q \subset \mathcal{V}^c_- \cap \{\xi \in \mathbb{R}^N : V(\xi) = c,\ \nabla V(\xi) = 0\}$*, or respectively,*
- $\Omega_q \subset \mathcal{V}^c_+ \cap \{\xi \in \mathbb{R}^N : V(\xi) = c,\ \nabla V(\xi) = 0\}$.

Proof of Lemma 2.16. Given $q \in \mathcal{M}_c$, let us assume $\omega_q = +\infty$; the case $\alpha_q = -\infty$ is treated similarly. First, note that $\lim_{t\to+\infty} \dot{q}(t) = 0$. Indeed, the fact that $\lim_{t\to+\infty} \operatorname{dist}(q(t), \mathcal{V}^c_+) = 0$ (cf. Lemma 2.7) combined with the uniform continuity of V on $\overline{B_R(0)}$, where $\|q\|_{L^\infty(\mathbb{R},\mathbb{R}^N)} \le R$, proves that $\lim_{t\to+\infty} V(q(t)) = c$. But this together with Lemma 2.14 show that $\lim_{t\to+\infty} |\dot{q}(t)|^2 = \lim_{t\to+\infty} 2(V(q(t)) - c) = 0$.
To prove the second statement, let $\xi \in \Omega_q$ so there is a sequence $t_n \to +\infty$ such that $q(t_n) \to \xi$ as $n \to +\infty$. Since $\lim_{t\to+\infty} \operatorname{dist}(q(t), \mathcal{V}^c_+) = 0$, it follows $\operatorname{dist}(\xi, \mathcal{V}^c_+) = \lim_{n\to\infty} \operatorname{dist}(q(t_n), \mathcal{V}^c_+) = 0$, whereby $\xi \in \mathcal{V}^c_+$ in light of the closeness of $\mathcal{V}^c_+$. On the other hand, we have already seen that $\lim_{t\to+\infty} V(q(t)) = c$, so the continuity of the potential yields $V(\xi) = \lim_{n\to\infty} V(q(t_n)) = c$. All of this shows that $\Omega_q \subset \mathcal{V}^c_+ \cap \{\xi \in \mathbb{R}^N : V(\xi) = c\}$. There just remains to be shown that $\nabla V(\xi) = 0$.
For $(t_n) \subset \mathbb{R}$ given as above, consider the following sequence of translates of q,

$$q_n(t) = q(t + t_n) \quad \text{for} \quad t \in (\alpha_q - t_n, +\infty) \quad \text{and} \quad n \in \mathbb{N},$$

and note that for any bounded interval $I \subset \mathbb{R}$ we have that $\sup_{t\in I} |\dot{q}_n(t)| \to 0$ as $n \to +\infty$, because $\dot{q}(t) \to 0$ as $t \to +\infty$. In particular, for any $t \in \mathbb{R}$ we deduce that

$$\lim_{n\to+\infty} q_n(t) = \lim_{n\to+\infty} q_n(0) + \lim_{n\to+\infty} \int_0^t \dot{q}_n(s)\, ds = \lim_{n\to+\infty} q(t_n) = \xi.$$

Put another way, $q(\cdot + t_n) \to \xi$ as $n \to +\infty$, with respect to the $C^1_{loc}(\mathbb{R}, \mathbb{R}^N)$-topology. Since $\ddot{q}(t + t_n) = \nabla V(q(t + t_n))$ for $t \in (\alpha_q - t_n, +\infty)$, we conclude that $\ddot{q}_n(t) \to \nabla V(\xi)$ as $n \to +\infty$, uniformly on bounded subsets of $\mathbb{R}$. On the other hand, given any $t \neq 0$, we can take the limit as $n \to +\infty$ in the identity

$$\dot{q}_n(t) - \dot{q}_n(0) = \int_0^t \ddot{q}_n(s)\, ds.$$

As argued before, the left side of the equation converges to 0, while the right side converges to $t\nabla V(\xi)$. Therefore, $\nabla V(\xi) = 0$, which concludes the proof. □

We remark that the previous result proves, in particular, that if c is a regular value for V then for every $q \in \mathcal{M}_c$ both α_q and ω_q must be finite.

For $q \in \mathcal{M}_c$ with $\alpha_q = -\infty$ or $\omega_q = +\infty$, Lemma 2.15 and Lemma 2.16 allow us to construct from it an entire solution of (1.1) with energy at level $-c$, connecting $\mathcal{V}^c_-$ and $\mathcal{V}^c_+$ in the sense of (1.3). This entire solution is either a *homoclinic type* solution or a *heteroclinic type* solution, depending on the finiteness of α_q and ω_q. Indeed, let us define

$$q_c(t) := \begin{cases} q_+ & \text{if } -\infty = \alpha_q \text{ and } \omega_q < +\infty, \\ q(t) & \text{if } -\infty = \alpha_q \text{ and } \omega_q = +\infty, \\ q_- & \text{if } -\infty < \alpha_q \text{ and } \omega_q = +\infty. \end{cases}$$

and observe that in light of Lemma 2.14 and Lemma 2.15, every subcase in the definition of q_c solves (1.1) and has energy $E_{q_c}(t) = -c$ for all $t \in \mathbb{R}$. The way the remaining condition (1.3) is fulfilled, depends on whether the trajectory of q has finite contact times with $\mathcal{V}^c_-$ and $\mathcal{V}^c_+$, or if it accumulates at infinity, as in (1.4).

In the case where $q \in \mathcal{M}_c$ has precisely one of α_q and ω_q finite, we say that q_c is a *homoclinic type* solution connecting $\mathcal{V}^c_-$ and $\mathcal{V}^c_+$. This solution satisfies the following properties

- If $\alpha_q = -\infty$ and $\omega_q < +\infty$, then we have $q_c := q_+$. In particular, adopting the notation of Theorem 2.1 we denote $\Omega := \mathcal{A}_q$ $(=\mathcal{A}_{q_c} = \Omega_{q_c})$ and $\sigma := \omega_q$, thus Ω is a closed connected set and $\sigma < +\infty$. From Lemma 2.11, Lemma 2.15 and Lemma 2.16, the definition of q_+ in (2.25) together with $\sigma = \omega_q$, it follows that

(i) $V(q_c(\sigma)) = V(q(\omega_q)) = c$, $V(q_c(t)) > c$ for any $t \neq \sigma$ and $\lim_{t\to\pm\infty} \dot{q}_c(t) = 0$,

(ii) $q_c(\sigma) \in \mathcal{V}^c_+$ and $\nabla V(q_c(\sigma)) = 0$. Moreover, $\Omega \subset \mathcal{V}^c_- \cap \{\xi \in \mathbb{R}^N : V(\xi) = c,\ \nabla V(\xi) = 0\}$, and $\lim_{t\to\pm\infty} \operatorname{dist}(q_c(t), \Omega) = 0$,

(iii) $q_c(\sigma + t) = q_c(\sigma - t)$ for any $t \in \mathbb{R}$.

Finally we remark that (1.3) is satisfied. Indeed, by Lemma 2.11, we get that the infimum $\inf_{t\in\mathbb{R}} \operatorname{dist}(q_c(t), \mathcal{V}^c_+) = 0$ is achieved at $t = \sigma$ and $\lim_{t\to\pm\infty} \operatorname{dist}(q_c(t), \mathcal{V}^c_-) = 0$, so in particular, $\inf_{t\in\mathbb{R}} \operatorname{dist}(q_c(t), \mathcal{V}^c_-) = 0$.

- A similar reasoning allows us to conclude that, when $\alpha_q > -\infty$ and $\omega_q = +\infty$, the function $q_c := q_-$ is a *homoclinic type* solution connecting $\mathcal{V}^c_-$ and $\mathcal{V}^c_+$, for which we have that $\Omega := \Omega_q$ $(= \Omega_{q_c} = \mathcal{A}_{q_c})$ is a connected closed set and setting $\sigma := \alpha_q$, we get

  (i) $V(q_c(\sigma)) = V(q(\alpha_q)) = c$, $V(q_c(t)) > c$ for any $t \neq \sigma$ and $\lim_{t\to\pm\infty} \dot{q}_c(t) = 0$,

  (ii) $q_c(\sigma) \in \mathcal{V}^c_-$ and $\nabla V(q_c(\sigma)) = 0$. Moreover, $\Omega \subset \mathcal{V}^c_+ \cap \{\xi \in \mathbb{R}^N : V(\xi) = c,\ \nabla V(\xi) = 0\}$, and $\lim_{t\to\pm\infty} \operatorname{dist}(q_c(t), \Omega) = 0$,

  (iii) $q_c(\sigma + t) = q_c(\sigma - t)$ for any $t \in \mathbb{R}$.

  Moreover, also in this case (1.3) is satisfied.

In the remaining case, where $q \in \mathcal{M}_c$ has $\alpha_q = -\infty$ and $\omega_q = +\infty$, we say that q_c is a *heteroclinic type* solution connecting $\mathcal{V}^c_-$ and $\mathcal{V}^c_+$.

- Clearly $q_c = q$. Adopting the notation of Theorem 2.1, we will write $\mathcal{A} := \mathcal{A}_q\,(= \mathcal{A}_{q_c})$ and $\Omega := \Omega_q\,(= \Omega_{q_c})$, whence $\mathcal{A}$ and Ω are closed connected sets. Then, since $\alpha_q = -\infty$ and $\omega_q = +\infty$, by definition and Lemma 2.16, we obtain

  (i) $V(q_c(t)) > c$ holds for any $t \in \mathbb{R}$ and $\lim_{t\to\pm\infty} \dot{q}_c(t) = 0$,

  (ii) $\mathcal{A} \subset \mathcal{V}^c_- \cap \{\xi \in \mathbb{R}^N : V(\xi) = c,\ \nabla V(\xi) = 0\}$, $\Omega \subset \mathcal{V}^c_+ \cap \{\xi \in \mathbb{R}^N : V(\xi) = c,\ \nabla V(\xi) = 0\}$ and $\lim_{t\to-\infty} \operatorname{dist}(q_c(t), \mathcal{A}) = \lim_{t\to+\infty} \operatorname{dist}(q_c(t), \Omega) = 0$.

  Finally, by Lemma 2.11 we get $\lim_{t\to\pm\infty} \operatorname{dist}(q_c(t), \mathcal{V}^c_\pm) = 0$, from which, condition (1.3) is satisfied.

The proof of Theorem 2.1 is now completed.

Remark 2.17. *Note that, by construction, the solution q_c given by Theorem 2.1 has a* connecting time interval *$(\alpha_{q_c}, \omega_{q_c}) \subseteq \mathbb{R}$, with $-\infty \leq \alpha_{q_c} < \omega_{q_c} \leq +\infty$ (coinciding in the statement with (σ, τ) in the case* (a)*, with $(-\infty, \sigma)$ in the case* (b) *and with $\mathbb{R}$ in the case* (c)*, respectively), such that*

(1) *$V(q_c(t)) > c$ for every $t \in (\alpha_{q_c}, \omega_{q_c})$,*

(2) *$\lim\limits_{t\to\alpha^+_{q_c}} \operatorname{dist}(q_c(t), \mathcal{V}^c_-) = 0$, and if $\alpha_{q_c} > -\infty$ then $\dot{q}_c(\alpha_{q_c}) = 0$, $V(q_c(\alpha_{q_c})) = c$ with $q_c(\alpha_{q_c}) \in \mathcal{V}^c_-$,*

(3) *$\lim\limits_{t\to\omega^-_{q_c}} \operatorname{dist}(q_c(t), \mathcal{V}^c_+) = 0$, and if $\omega_{q_c} < +\infty$ then $\dot{q}_c(\omega_{q_c}) = 0$, $V(q_c(\omega_{q_c})) = c$ with $q_c(\omega_{q_c}) \in \mathcal{V}^c_+$,*

(4) *$J_{c,(\alpha_{q_c},\omega_{q_c})}(q_c) = m_c$.*

Finally, the behavior of q_c on $\mathbb{R}$ is obtained by (eventual) reflection and periodic continuation of its restriction to the interval $(\alpha_{q_c}, \omega_{q_c}) \subseteq \mathbb{R}$. In particular, we have

(5) *$\|q_c\|_{L^\infty(\mathbb{R},\mathbb{R}^N)} = \|q_c\|_{L^\infty((\alpha_{q_c},\omega_{q_c}),\mathbb{R}^N)}$.*

3. Some applications

In this last section we illustrate some applications of Theorem 2.1 to certain classes of potentials V, extensively studied in the literature. More precisely, we establish existence of connecting orbits to (1.1), in the case of double-well potentials, as well as potentials associated to duffing-like systems and multiple-pendulum-like systems. Additionally, we show in each one of these cases that solutions of heteroclinic type and homoclinic type, at energy level 0, can be obtained as limits of sequences (q_c) of solutions to (1.1), as $c \to 0^+$.

3.1. **Double-well potential systems.** As a first example we consider *double-well potential systems* like the ones considered e.g. in [16] in the PDE (non-autonomous) setting and in [**?**, 8, 11, 31] in the ODE setting, among others. Precisely, we assume that $V \in C^1(\mathbb{R}^N)$ satisfies

(V1) There exist $a_- \neq a_+ \in \mathbb{R}^N$ such that $V(a_+) = V(a_-) = 0$, and $V(x) > 0$ for $x \in \mathbb{R}^N \setminus \{a_-, a_+\}$,

(V2) $\liminf_{|x|\to+\infty} V(x) =: \nu_0 > 0$.

As a consequence of Theorem 2.1 we have the following result.

Proposition 3.1. *Assume that $V \in C^1(\mathbb{R}^N)$ satisfies* (V1) *and* (V2)*. If $c \in [0, \nu_0)$ is such that* ($\mathbf{V}^c$) *holds, then the coercivity condition* (1.7) *of the energy functional J_c over Γ_c holds true. In particular, for such value of $c \in [0, \nu_0)$, Theorem 2.1 gives a solution $q_c \in C^2(\mathbb{R}, \mathbb{R}^N)$ to the problem* (1.1)-(1.3) *that satisfies the pointwise energy constraint $E_{q_c}(t) = -c$ for all $t \in \mathbb{R}$.*

Proof of Proposition 3.1. In order to prove that (1.7) holds true for $c \in [0, \nu_0)$ for which ($\mathbf{V}^c$) holds, we show that there exists $R > 0$ such that any minimizing sequence $(q_n) \subset \Gamma_c$, $J_c(q_n) \to m_c = \inf_{\Gamma_c} J_c(q)$, verifies the uniform bound $\|q_n\|_{L^\infty(\mathbb{R},\mathbb{R}^N)} \leq R$. Arguing by contradiction, let us assume that there is a value $c_* \in [0, \nu_0)$ for which ($\mathbf{V}^{c_*}$) holds true and that there exists a sequence $(q_n) \subset \Gamma_{c_*}$ for which $J_{c_*}(q_n) \to m_{c_*}$ but $\|q_n\|_{L^\infty(\mathbb{R},\mathbb{R}^N)} \to +\infty$.
Since $c_* < \nu_0$ and $\liminf_{|x|\to+\infty} V(x) = \nu_0$, we have that, denoting $\mu_0 := \frac{1}{2}(\nu_0 - c_*)$, there exists $R_0 > 0$ such that

$$V(x) > c_* + \mu_0 \;\text{ if }\; |x| \geq R_0. \tag{3.26}$$

Consequently, $\mathcal{V}^{c_*} = \mathcal{V}_-^{c_*} \cup \mathcal{V}_+^{c_*} \subset B_{R_0}(0)$. Since $(q_n) \subset \Gamma_{c_*}$ with $\|q_n\|_{L^\infty(\mathbb{R},\mathbb{R}^N)} \to +\infty$, we deduce that for any $R > R_0$ there exists $\bar{n} \in \mathbb{N}$ such that if $n \geq \bar{n}$ then q_n crosses (at least two times) the annulus $B_R(0) \setminus B_{R_0}(0)$. In particular, by (3.26), we obtain that for any $n \geq \bar{n}$ there is an interval $(s_n, t_n) \subset \mathbb{R}$ such that

$$|q_n(t_n) - q_n(s_n)| \geq R - R_0 \quad\text{and}\quad V(q_n(t)) \geq c_* + \mu_0 \;\text{ for }\; t \in (s_n, t_n).$$

By Remark 2.4 we then recover that for $n \geq \bar{n}$

$$m_{c_*} + o(1) = J_{c_*}(q_n) \geq \sqrt{2\mu_0}(R - R_0).$$

Since R is arbitrary, the latter contradicts the finiteness of m_{c_*}. Indeed, by Lemma 2.6 we know that $m_{c_*} < +\infty$ (in particular, the proof of Lemma 2.6 shows that this does not depend on (1.7)). □

Remark 3.2. *The continuity of V and the assumptions* (V1) *and* (V2) *imply that there always exists $c_{dw} \in (0, \nu_0)$ for which the condition* ($\mathbf{V}^c$) *is satisfied for every c in the interval* $[0, c_{dw})$*. Indeed, as seen in* (3.26)*, from* (V2) *we obtain $\exists R_0 > 0$ large so that $\mathcal{V}^c \subset B_{R_0}(0)$ for any $c \in (0, \nu_0)$. Hence, $\mathcal{V}^c$ is compact. Also, in view of* (V1) *and $V \in C(\mathbb{R}^N)$, we can choose c_{dw} sufficiently small so that $\mathcal{V}^c$ splits* *in the disjoint union of two compact sets $\mathcal{V}_-^c$ and $\mathcal{V}_+^c$, for any $c \in [0, c_{dw})$. In particular, we can assume that for $\varrho_0 := \frac{1}{4}|a_- - a_+|$ the condition below holds*

$$a_- \in \mathcal{V}_-^c \subset B_{\varrho_0}(a_-) \text{ and } a_+ \in \mathcal{V}_+^c \subset B_{\varrho_0}(a_+), \text{ for all } c \in [0, c_{dw}), \tag{3.27}$$

by reducing the value of $c_{dw} \in (0, \nu_0)$, if necessary. It follows that $\operatorname{dist}(\mathcal{V}_-^c, \mathcal{V}_+^c) \geq \frac{1}{2}|a_- - a_+| > 0$.

For any $c \in [0, \nu_0)$ for which ($\mathbf{V}^c$) is satisfied, Proposition 3.1 provides a solution q_c with energy $-c$ which connects $\mathcal{V}_-^c$ with $\mathcal{V}_+^c$. As noted in Remark 2.2, such a solution is of brake orbit type when c is a regular value for V, while it may be of the homoclinic or heteroclinic type if c is a critical value of V. Of particular interest is the case when $c = 0$, where we see that $\mathcal{V}_-^c = \{a_-\}$ with $\mathcal{V}_+^c = \{a_+\}$ (or viceversa) and since $a_\pm$ are critical points of V, we are in the case (c) of Theorem 2.1. Therefore the solution given by Proposition 3.1 for $c = 0$ is of heteroclinic type connecting the equilibria a_- and a_+.

We continue our analysis by studying the behavior of the solution q_c given by Proposition 3.1 as $c \to 0^+$ and we will prove that they converge, in a suitable sense, to a heteroclinic solution connecting the equilibria $a_\pm$. Precisely, we have

Proposition 3.3. *Assume that $V \in C^1(\mathbb{R}^N)$ satisfies* (V1)-(V2) *and that ∇V is locally Lipschitz continuous in $\mathbb{R}^N$. Let $c_n \to 0^+$ be any sequence and (q_{c_n}) be the sequence of solutions to the system* (1.1) *given by Proposition 3.1. Then, up to translations and a subsequence, $q_{c_n} \to q_0$ in $C^2_{loc}(\mathbb{R}, \mathbb{R}^N)$, where q_0 is a solution to* (1.1) *of heteroclinic type between a_- and a_+, i.e. it satisfies*

$$q_0(t) \to a_\pm \text{ and } \dot{q}_0(t) \to 0, \text{ as } t \to \pm\infty.$$

To prove Proposition 3.3, we begin by establishing in the next lemma a uniform estimate of the L^∞-norm of the solutions (q_c) for $0 \leq c < c_{dw}$, where c_{dw} is given in Remark 3.2.

Lemma 3.4. *Assume that $V \in C^1(\mathbb{R}^N)$ satisfies* (V1) *and* (V2) *and let $c_{dw} \in (0, \nu_0)$. Then there exists $R_{dw} > 0$ such that $\|q_c\|_{L^\infty(\mathbb{R},\mathbb{R}^N)} \leq R_{dw}$ for any $c \in [0, c_{dw})$.*

Proof of Lemma 3.4. First we claim that there exists $M_{dw} > 0$ such that

$$m_c = \inf_{q \in \Gamma_c} J_c(q) \leq M_{dw} \text{ for any } c \in [0, c_{dw}). \tag{3.28}$$

Indeed, consider the function $\xi(t) = (1 - t)a_- + ta_+$ for $t \in [0, 1]$. From (3.27) and the compactness of $\mathcal{V}_\pm^c$ we see that for each $c \in [0, c_{dw})$ there exist $0 \leq \sigma_c < \tau_c \leq 1$ that satisfy

$$\xi(\sigma_c) \in \mathcal{V}_-^c,\ \xi(\tau_c) \in \mathcal{V}_+^c \text{ and } V(\xi(t)) > c \text{ for any } t \in (\sigma_c, \tau_c).$$

Then, the function

$$q_{c,\xi}(t) = \begin{cases} \xi(\sigma_c) & \text{if } t \leq \sigma_c, \\ \xi(t) & \text{if } \sigma_c < t < \tau_c, \\ \xi(\tau_c) & \text{if } \tau_c \leq t. \end{cases}$$

belongs to Γ_c, and so our claim (3.28) follows by a plain estimate:

$$m_c \leq J_c(q_{c,\xi}) \leq \tfrac{1}{2}|a_+ - a_-|^2 + \max_{s \in [0,1]} V(\xi(s)) =: M_{dw}.$$

If q_c denotes the solution in Proposition 3.1, corresponding to a $c \in [0, c_{dw})$, by Remark 2.17 there is a *connecting time interval* $(\alpha_{q_c}, \omega_{q_c}) \subset \mathbb{R}$ for which properties (1)-(5) hold true. In particular, by (3.28)

$$J_{c,(\alpha_{q_c},\omega_{q_c})}(q_c) = m_c \leq M_{dw} \text{ for any } c \in [0, c_{dw}), \tag{3.29}$$

holds true. We now claim that there exists $R_{dw} > 0$ in such a way that

$$\|q_c\|_{L^\infty((\alpha_{q_c},\omega_{q_c}),\mathbb{R}^N)} \leq R_{dw} \quad \text{for any } c \in [0, c_{dw}). \tag{3.30}$$

The proof of Lemma 3.4 will be concluded upon establishing (3.30), since condition (5) in Remark 2.17 gives $\|q_c\|_{L^\infty((\alpha_{q_c},\omega_{q_c}),\mathbb{R}^N)} = \|q_c\|_{L^\infty(\mathbb{R},\mathbb{R}^N)}$. To prove (3.30), arguing by contradiction, we assume that for every $R > 0$ there exists $c_R \in [0, c_{dw})$ such that

$$\|q_{c_R}\|_{L^\infty((\alpha_{q_{c_R}},\omega_{q_{c_R}}),\mathbb{R}^N)} > R. \tag{3.31}$$

Let us observe that assumption (V2) ensures that for $h_0 := \frac{1}{2}(v_0 - c_{dw})$ there exists $R_0 > 0$ for which

$$V(x) > c_{dw} + h_0 \text{ if } |x| \geq R_0.$$

In particular, this shows that $\mathcal{V}^c = \mathcal{V}^c_- \cup \mathcal{V}^c_+ \subset B_{R_0}(0)$ for any $c \in [0, c_{dw})$. In the remaining of the proof we choose $R \in (R_0, +\infty)$ to satisfy $\sqrt{2h_0}(R - R_0) > M_{dw}$. For this choice of R, the contradiction assumption (3.31) implies that the trajectory of $q_{c_R} \in \mathcal{M}_{c_R}$ crosses the annulus $B_R(0) \setminus B_{R_0}(0)$; thus, there is an interval $[\sigma, \tau] \subset (\alpha_{q_{c_R}}, \omega_{q_{c_R}})$ in such a way that

$$|q_{c_R}(\tau) - q_{c_R}(\sigma)| \geq R - R_0 \quad \text{and} \quad V(q_{c_R}(t)) \geq c_{dw} + h_0 \text{ for any } t \in (\sigma, \tau).$$

Hence, Remark 2.4 along with the properties above, show the strict lower bound on the energy of q_{c_R}

$$m_{c_R} = J_{c_R,(\alpha_{q_{c_R}},\omega_{q_{c_R}})}(q_{c_R}) \geq \sqrt{2h_0}(R - R_0) > M_{dw}.$$

However, this contradicts the upper bound (3.29). In this way, we have argued that (3.30) follows, which in turn completes the proof of this lemma. □

We can now prove Proposition 3.3. Without loss of generality, let the sequence $c_n \to 0^+$ be so that $(c_n) \subset (0, c_{dw})$. Since $c_n < c_{dw}$, we know from Remark 3.2 that $(\mathbf{V}^{c_n})$ holds true for all $n \in \mathbb{N}$. By Remark 2.17, for each $n \in \mathbb{N}$, the solution q_{c_n} given by Proposition 3.1 has a *connecting time interval* $(\alpha_n, \omega_n) \subset \mathbb{R}$, with $-\infty \leq \alpha_n < \omega_n \leq +\infty$, in such a way that

- (1_n) $V(q_{c_n}(t)) > c_n$ for every $t \in (\alpha_n, \omega_n)$,
- (2_n) $\lim_{t\to\alpha_n^+} \operatorname{dist}(q_{c_n}(t), \mathcal{V}^{c_n}_-) = 0$, and if $\alpha_n > -\infty$ then $\dot q_{c_n}(\alpha_n) = 0$, $V(q_{c_n}(\alpha_n)) = c_n$ with $q_{c_n}(\alpha_n) \in \mathcal{V}^{c_n}_-$,
- (3_n) $\lim_{t\to\omega_n^-} \operatorname{dist}(q_{c_n}(t), \mathcal{V}^{c_n}_+) = 0$, and if $\omega_n < +\infty$ then $\dot q_{c_n}(\omega_n) = 0$, $V(q_{c_n}(\omega_n)) = c_n$ with $q_{c_n}(\omega_n) \in \mathcal{V}^{c_n}_+$.
- (4_n) $J_{c_n,(\alpha_n,\omega_n)}(q_{c_n}) = m_{c_n} = \inf_{q\in\Gamma_{c_n}} J_{c_n}(q)$.

We first start by renormalizing the sequence (q_{c_n}) using the following phase shift procedure. In light of the properties (2_n) and (3_n), for any $n \in \mathbb{N}$ there exists $\zeta_n \in (\alpha_n, \omega_n)$ such that

$$\operatorname{dist}(q_{c_n}(\zeta_n), \{a_-, a_+\}) = \varrho_0,$$

for $\varrho_0 := \frac{1}{4}|a_- - a_+|$ as in (3.27). Hence, up to translations, eventually renaming q_{c_n} to be $q_{c_n}(\cdot - \zeta_n)$, we can assume

$$\alpha_n < 0 < \omega_n \text{ and } \operatorname{dist}(q_{c_n}(0), \{a_-, a_+\}) = \varrho_0, \quad \text{for any } n \in \mathbb{N}. \tag{3.32}$$

We now argue that (q_{c_n}) converges, in the C^2-topology on compact sets, to an entire solution q_0 of the system $\ddot q = \nabla V(q)$ over $\mathbb{R}$. To see this, let us observe that (q_{c_n}), $(\dot q_{c_n})$ and $(\ddot q_{c_n})$ are uniformly bounded in $L^\infty(\mathbb{R},\mathbb{R}^N)$, given the fact that $\|q_{c_n}\|_{L^\infty(\mathbb{R},\mathbb{R}^N)} \leq R_{dw}$ for any $n \in \mathbb{N}$ (by Lemma 3.4). More precisely, for every $n \in \mathbb{N}$ we have the bounds

$$\|\dot q_{c_n}\|_{L^\infty(\mathbb{R},\mathbb{R}^N)} \leq C_{dw} := (2\max\{V(x) : |x| \leq R_{dw}\})^{1/2} < +\infty, \quad \text{and}$$
$$\|\ddot q_{c_n}\|_{L^\infty(\mathbb{R},\mathbb{R}^N)} \leq C'_{dw} := \max\{|\nabla V(x)| : |x| \leq R_{dw}\} < +\infty,$$

where the former is a consequence of the pointwise energy constraint $E_{q_{c_n}}(t) = -c_n$ for all $t \in \mathbb{R}$, and the latter follows since $\ddot q_{c_n} = \nabla V(q_{c_n})$ on $\mathbb{R}$.

An application of the Ascoli-Arzelà Theorem shows that there exists $q_0 \in C^1(\mathbb{R},\mathbb{R}^N)$ and a subsequence of (q_{c_n}), still denoted (q_{c_n}), such that

$$q_{c_n} \to q_0 \ \text{ in } \ C^1_{loc}(\mathbb{R},\mathbb{R}^N), \ \text{ as } \ n \to +\infty.$$

In particular, (3.32) implies that

$$\operatorname{dist}(q_0(0), \{a_-, a_+\}) = \varrho_0. \tag{3.33}$$

Moreover, the above convergence can be improved to $q_{c_n} \to q_0$ in $C^2_{loc}(\mathbb{R},\mathbb{R}^N)$ as $n \to +\infty$, by using once more that $\ddot q_{c_n} = \nabla V(q_{c_n})$ on $\mathbb{R}$, for any $n \in \mathbb{N}$. In fact, the latter convergence shows, in turn, that

$$\ddot q_0 = \nabla V(q_0) \ \text{ on } \ \mathbb{R}. \tag{3.34}$$

To conclude the proof of Proposition 3.3, it will be enough to establish

$$q_0(t) \to a_\pm \ \text{ as } \ t \to \pm\infty. \tag{3.35}$$

Indeed, once the validity of (3.35) has been proved, we then get $\ddot q(t) \to 0$ as $t \to \pm\infty$ from (3.34), which in turn shows that $\dot q(t) \to 0$ as $t \to \pm\infty$ by interpolation inequalities.

To prove (3.35), let us first note that conditions (3.32), (3.33) and (3.34) imply altogether

$$\alpha_n \to -\infty \ \text{ and } \ \omega_n \to +\infty, \ \text{ as } n \to +\infty. \tag{3.36}$$

Indeed, arguing by contradiction, assume that along a subsequence α_n is bounded. As $\alpha_n < 0$ then, up to a subsequence, we deduce that $\alpha_n \to \alpha_0$ for some $\alpha_0 \leq 0$. By (2_n) and (3.27) we have $q_{c_n}(\alpha_n) \in \mathcal{V}_-^{c_n} \subset B_{\varrho_0}(a_-)$, which combined with $c_n \to 0$, (V1) and (V2) then yields $q_{c_n}(\alpha_n) \to a_-$. This, $\dot q_{c_n}(\alpha_n) = 0$ for any $n \in \mathbb{N}$, and the fact that $q_{c_n} \to q_0$ in $C^1_{loc}(\mathbb{R},\mathbb{R}^N)$ allow us to conclude $q_0(\alpha_0) = a_-$ and $\dot q_0(\alpha_0) = 0$. Nonetheless, the uniqueness of solutions to the Cauchy problem would then imply that $q_0(t) = a_-$ for any $t \in \mathbb{R}$. This is contrary to (3.33), thus showing $\alpha_n \to -\infty$, as claimed. Analogously, one can prove that $\omega_n \to +\infty$. Therefore, (3.36) follows.

In order to establish (3.35), let us observe that it is sufficient to prove: for any $r \in (0, \rho_0)$ there exists $L_r^\pm > 0$ and $n_r \geq \bar n$ such that for any $n \geq n_r$, there hold

$$|q_{c_n}(t) - a_-| < r \ \text{ for } \ t \in (\alpha_n, -L_r^-), \ \text{ and } \ |q_{c_n}(t) - a_+| < r \ \text{ for } \ t \in (L_r^+, \omega_n). \tag{3.37}$$

Indeed, by taking the limit as $n \to +\infty$ in (3.37), we then conclude in view of the pointwise convergence $q_{c_n} \to q_0$ and (3.36), that for any $r \in (0, \varrho_0)$ there exists $L_r := \max\{L_r^-, L_r^+\}$ such that

$$|q_0(t) - a_-| < r \ \text{ for } \ t \in (-\infty, -L_r), \ \text{ and } \ |q_0(t) - a_+| < r \ \text{ for } \ t \in (L_r, +\infty),$$

therefore, (3.35) follows.

To obtain the first estimate in (3.37) we argue by contradiction assuming that there exists $\bar r \in (0, \rho_0)$, a subsequence of (q_{c_n}), still denoted (q_{c_n}), and a sequence $s_n \to -\infty$ such that for any $n \in \mathbb{N}$

$$|q_{c_n}(s_n) - a_-| > \bar r \ \text{ with } \ s_n \in (\alpha_n, 0). \tag{3.38}$$

Then, by observing that (4_n) and (3.29) imply the inequality

$$M_{dw} \geq m_{c_n} = J_{c_n,(\alpha_n,\omega_n)}(q_{c_n}) \geq \int_{s_n}^{0} (V(q_{c_n}(t)) - c_n)\, dt \geq |s_n| \inf_{t\in(s_n,0)} (V(q_{c_n}(t)) - c_n),$$

we obtain that the contradiction hypothesis $s_n \to -\infty$ yields

$$\inf_{t\in(s_n,0)} V(q_{c_n}(t)) \leq c_n + \tfrac{M_{dw}}{|s_n|} \to 0 \quad \text{as} \quad n \to +\infty.$$

In particular, we deduce that for every $n \in \mathbb{N}$ there exists $t_n \in (s_n, 0)$ so that $V(q_{c_n}(t_n)) \to 0$ as $n \to +\infty$. This, in light of (V1) and (V2), shows

$$\text{dist}\,(q_{c_n}(t_n), \{a_-, a_+\}) \to 0 \quad \text{as} \quad n \to +\infty. \tag{3.39}$$

Let us show, if fact, that

$$\liminf_{n\to+\infty} \text{dist}\,(q_{c_n}(t_n), a_+) > 0. \tag{3.40}$$

Indeed, if (3.40) fails, along a subsequence (still denoted q_{c_n}) we have $q_{c_n}(t_n) \to a_+$. We connect the point $q_{c_n}(t_n)$ with a_+ with the segment $\{(1-\sigma)q_{c_n}(t_n) + \sigma a_+ \,:\, \sigma \in [0,1]\}$. By continuity, since $V(q_{c_n}(t_n)) > c_n > 0 = V(a_+)$, there is $\sigma_n \in (0,1)$ such that

$$V((1-\sigma_n)q_{c_n}(t_n) + \sigma_n a_+) = c_n \quad \text{and} \quad V((1-\sigma)q_{c_n}(t_n) + \sigma a_+) > c_n \;\text{ for any }\; \sigma \in [0, \sigma_n). \tag{3.41}$$

In particular $(1-\sigma_n)q_{c_n}(t_n) + \sigma_n a_+ \in \mathcal{V}_+^{c_n}$ and the function

$$q_{-,n}(t) = \begin{cases} q_{c_n}(\alpha_n) & \text{if } t < \alpha_n, \\ q_{c_n}(t) & \text{if } \alpha_n \leq t < t_n, \\ (1-(t-t_n))q_{c_n}(t_n) + (t-t_n)a_+ & \text{if } t_n \leq t < t_n + \sigma_n, \\ (1-\sigma_n)q_{c_n}(t_n) + \sigma_n a_+ & \text{if } t_n + \sigma_n \leq t. \end{cases}$$

(where we agree to omit the first item in the definition if $\alpha_n = -\infty$) is by construction an element of Γ_{c_n} for any $n \in \mathbb{N}$, whence $J_{c_n}(q_{-,n}) \geq m_{c_n}$. Since $q_{c_n}(t_n) \to a_+$ we have $V((1-\sigma)q_{c_n}(t_n) + \sigma a_+) \to 0$ as $n \to +\infty$ uniformly for $\sigma \in [0,1]$. Thus, we derive that,

$$J_{c_n,(t_n,+\infty)}(q_{-,n}) = J_{c_n,(t_n,t_n+\sigma_n)}(q_{-,n}) = \tfrac{\sigma_n}{2}|q_{c_n}(t_n) - a_+|^2 + \int_{t_n}^{t_n+\sigma_n} (V((1-\sigma)q_{c_n}(t_n) + \sigma a_+) - c_n) d\sigma \to 0,$$

as $n \to +\infty$. Since $q_{-,n} = q_{c_n}$ on (α_n, t_n) and since $q_{-,n}$ is constant outside $(\alpha_n, t_n + \sigma_n)$, the latter shows

$$J_{c_n,(\alpha_n,t_n)}(q_{c_n}) = J_{c_n,(\alpha_n,t_n)}(q_{-,n}) = J_{c_n}(q_{-,n}) - J_{c_n,(t_n,+\infty)}(q_{-,n}) \geq m_{c_n} - o(1).$$

This bound with (4_n) implies that

$$m_{c_n} = J_{c_n,(\alpha_n,\omega_n)}(q_{c_n}) = J_{c_n,(\alpha_n,t_n)}(q_{c_n}) + J_{c_n,(t_n,\omega_n)}(q_{c_n}) \geq m_{c_n} - o(1) + J_{c_n,(t_n,\omega_n)}(q_{c_n}),$$

and so

$$J_{c_n,(t_n,\omega_n)}(q_{c_n}) \to 0 \quad \text{as} \quad n \to +\infty. \tag{3.42}$$

On the other hand, since $q_{c_n}(t_n) \to a_+$ and, by (3.32), $\text{dist}\,(q_{c_n}(0), \{a_-, a_+\}) = \varrho_0$, when n is large the trajectory of q_{c_n} crosses the annulus $B_{\varrho_0/2}(a_+) \setminus B_{\varrho_0/4}(a_+)$ at least one in the interval $(t_n, 0)$, i.e., there exists $(t_{1,n}, t_{2,n}) \subset (t_n, 0)$ such that $|q_{c_n}(t_{2,n}) - q_{c_n}(t_{1,n})| = \frac{\varrho_0}{4}$ and $\text{dist}\,(q_{c_n}(t), \{a_-, a_+\}) \geq \frac{\varrho_0}{4}$ for any $t \in (t_{1,n}, t_{2,n})$. If we set $\mu(\frac{\varrho_0}{4}) := \inf_{\mathbb{R}^N \setminus (B_{\varrho_0/4}(a_-) \cup B_{\varrho_0/4}(a_+))} V$ it follows that for every $t \in (t_{1,n}, t_{2,n})$, $V(q_{c_n}(t)) \geq \mu(\frac{\varrho_0}{4})$, and hence Remark 2.4 yields the bound

$$J_{c_n,(t_n,\omega_n)}(q_{c_n}) \geq J_{c_n,(t_{1,n},t_{2,n})}(q_{c_n}) \geq \sqrt{2(\mu(\tfrac{\varrho_0}{4}) - c_n)}|q_{c_n}(t_{2,n}) - q_{c_n}(t_{1,n})| > \sqrt{\mu(\tfrac{\varrho_0}{4})}\tfrac{\varrho_0}{4}$$

for n large enough. This last inequality contradicts (3.42) proving (3.40).

By (3.39) and (3.40) we obtain $q_{c_n}(t_n) \to a_-$. We now show that this case is not possible either, thus obtaining a contradiction with (3.39). This will establish the first estimate of (3.37).

To prove that $q_{c_n}(t_n) \to a_-$ cannot occur, we use an argument similar to the one used above. If $q_{c_n}(t_n) \to a_-$, we fix $\sigma_n \in (0,1)$ such that

$$V((1-\sigma_n)a_- + \sigma_n q_{c_n}(t_n)) = c_n \quad \text{and} \quad V((1-\sigma)a_- + \sigma q_{c_n}(t_n)) > c_n \text{ for any } \sigma \in (\sigma_n, 1].$$

Then the function

$$q_{+,n}(t) := \begin{cases} (1-\sigma_n)a_- + \sigma_n q_{c_n}(t_n) & \text{if } t \leq t_n - (1-\sigma_n), \\ (t_n - t)a_- + (t - t_n + 1)q_{c_n}(t_n) & \text{if } t_n - (1-\sigma_n) < t < t_n, \\ q_{c_n}(t) & \text{if } t_n < t \leq \omega_n, \\ q_{c_n}(\omega_n) & \text{if } \omega_n < t. \end{cases}$$

(where we omit the last item if $\omega_n = +\infty$) belongs to Γ_{c_n} and $J_{c_n}(q_{+,n}) \geq m_{c_n}$. Since $q_{c_n}(t_n) \to a_-$, with an argument analogous to the one used for $q_{-,n}$, we obtain

$$J_{c_n,(-\infty,t_n)}(q_{+,n}) = J_{c_n,(t_n-(1-\sigma_n),t_n)}(q_{+,n}) \to 0 \text{ as } n \to +\infty.$$

A reasoning similar to the one that lead to (3.42), then shows

$$J_{c_n,(\alpha_n,t_n)}(q_{c_n}) \to 0 \text{ as } n \to +\infty. \tag{3.43}$$

By (3.38) and (3.40), we are now in the situation where

$$\alpha_n < s_n < t_n < 0, \ |q_{c_n}(s_n) - a_-| > \bar{r}, \text{ and } q_{c_n}(t_n) \to a_- \text{ as } n \to +\infty.$$

Then, when n is large, the trajectory of q_{c_n} crosses the annulus $B_{\bar{r}/2}(a_-) \setminus B_{\bar{r}/4}(a_-)$ in the interval (s_n, t_n) and so there exists $(t_{3,n}, t_{4,n}) \subset (s_n, t_n)$ such that $|q_{c_n}(t_{3,n}) - q_{c_n}(t_{4,n})| = \frac{\bar{r}}{4}$ and $\operatorname{dist}(q_{c_n}(t), \{a_-, a_+\}) \geq \frac{\bar{r}}{4}$ for $t \in (t_{3,n}, t_{4,n})$. As above, since $V(q(t)) \geq \mu(\frac{\bar{r}}{4}) := \inf_{\mathbb{R}^N \setminus (B_{\bar{r}/4}(a_-) \cup B_{\bar{r}/4}(a_+))} V$ for any $t \in (t_{3,n}, t_{4,n})$, we can use Remark 2.4 to obtain

$$J_{c_n,(\alpha_n,t_n)}(q_{c_n}) \geq J_{c_n,(t_{3,n},t_{4,n})}(q_{c_n}) > \sqrt{\mu(\tfrac{\bar{r}}{4})}\tfrac{\bar{r}}{4} \quad \text{for } n \text{ large.}$$

This contradicts (3.43) and so the case $q(t_n) \to a_-$ cannot occur either. Then the first estimate of (3.37) follows.

One can readily verify that a strategy similar as the one above can be used to establish the second estimate of (3.37). In conclusion, (3.37) follows, and as a consequence (3.35) has been established. The proof of Proposition 3.3 is now complete.

3.2. **Duffing like systems.** A second application of Theorem 2.1 include *Duffing like* systems. More precisely, we follow the assumptions made in [10] and [27]: let V be a $C^1(\mathbb{R}^N)$ potential satisfying

(V3) V has a strict local minimum at $x_0 := 0$, with value $V(0) = 0$:

$$\exists r_0 > 0 \text{ such that } V(x) > 0 \text{ for any } x \in \overline{B_{4r_0}(0)} \setminus \{0\};$$

(V4) The set $C_0 := \{x \in \mathbb{R}^N : V(x) > 0\} \cup \{0\}$ is bounded, and such that $\nabla V(x) \neq 0$ for any $x \in \partial C_0$.

We observe that for any $c \geq 0$, J_c satisfies the coercivity property (1.7) on Γ_c. To see this, in view of (V3) and (V4), note that $\{x \in \mathbb{R}^N : V(x) \geq c\} \subset \overline{C_0}$ thus, by definition of Γ_c, $q(\mathbb{R}) \subset \overline{C_0}$ for any $q \in \Gamma_c$. Hence, since C_0 is a bounded set, we conclude that (1.7) holds.

This discussion shows that Theorem 2.1 applies, and so we have

Proposition 3.5. *Assume that $V \in C^1(\mathbb{R}^N)$ satisfies* (V3) *and* (V4). *If $c \geq 0$ is such that* ($\mathbf{V}^c$) *holds true then Theorem 2.1 gives a solution $q_c \in C^2(\mathbb{R}, \mathbb{R}^N)$ to the problem* (1.1)-(1.3) *with energy $E_{q_c}(t) = -c$ for all $t \in \mathbb{R}$.*

We now remark that condition ($\mathbf{V}^c$) is verified if $c > 0$ is chosen sufficiently small. Indeed, by (V3) there exists $\nu_1 > 0$ such that $V(x) \geq \nu_1$ for $|x| = r_0$. Then $\mathcal{V}^c \cap \partial B_{r_0}(0) = \emptyset$ for any $c \in [0, \nu_1)$ and ($\mathbf{V}^c$) is satisfied by the sets

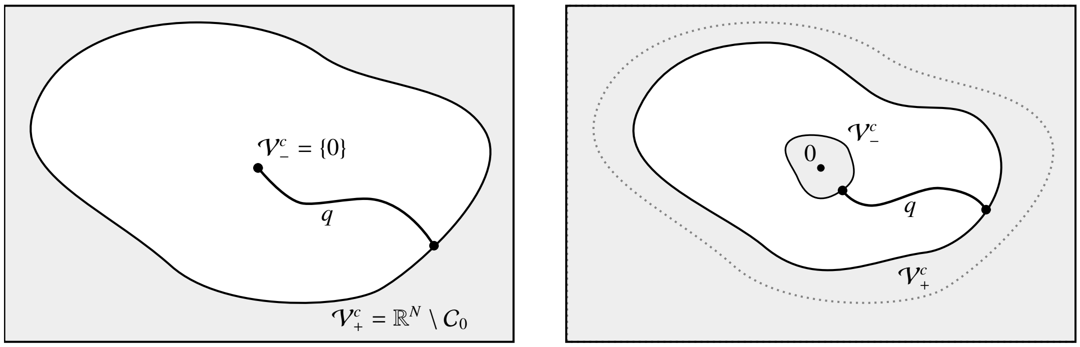
$$\mathcal{V}^c_- := \mathcal{V}^c \cap B_{r_0}(0) \quad \text{and} \quad \mathcal{V}^c_+ := \mathcal{V}^c \setminus B_{r_0}(0). \tag{3.44}$$

In particular, the fact that $\operatorname{dist}(\mathcal{V}^c_-, \mathcal{V}^c_+) > 0$ is guaranteed by the continuity of V.

It is worth distinguishing among the different type of solutions q_c that we can obtain from Proposition 3.5, for suitable choices of $c \in [0, +\infty)$. In the case $c = 0$, we have from (3.44) that $\mathcal{V}^c_- = \{0\}$ and $\mathcal{V}^c_+ = \mathbb{R}^N \setminus C_0$. Then Proposition 3.5 above states the existence of a solution q_0 to (1.1) connecting 0 with ∂C_0. This solution *cannot* be of *brake orbit* type since $\nabla V(0) = 0$, and so by Theorem 2.1-(a) it cannot have a *contact* point with $\mathcal{V}^c_-$. Analogously, q_0 *cannot* be of *heteroclinic* type, since in this case, by Theorem 2.1-(c), there must exist a set $\Omega \subset \partial C_0$ consisting of critical points of V; thus contradicting the hypothesis made on ∂C_0 in (V4). We conclude that q_0 must be a *homoclinic type* solution with energy $E_{q_0}(t) = 0$ for all $t \in \mathbb{R}$ satisfying $q_0(t) \to 0$, $\dot{q}_0(t) \to 0$ as $t \to \pm\infty$, and that reaches ∂C_0 at a *contact* time $\sigma \in \mathbb{R}$, with respect to which it is symmetric (see Figure 1(a)). This gives back the result already proved in [10], [11], [18] and [27]. In contrast, let us now consider the cases $0 < c < \nu_1$, where $\nu_1 := \min\{V(x) : |x| = r_0\}$ as above. Relative to the separation property (3.44), Proposition 3.5 provides for any such value c the existence of a connecting orbit q_c between $\mathcal{V}^c_-$ and $\mathcal{V}^c_+$. Arguing as above, we recognize that *if c is a regular value of V* then q_c is a *brake orbit type* solution of (1.1) with energy $E_{q_c}(t) = -c$ for all $t \in \mathbb{R}$, connecting $\mathcal{V}^c_-$ and $\mathcal{V}^c_+$ (see Figure 1(b)).

To analyze the case where c is not a regular value of V, we use an approximation argument. Let us first note that $\operatorname{dist}(\mathcal{V}^c_+, \partial C_0) \to 0$ as $c \to 0^+$ in view of compactness of these sets, and the continuity of V. Since $\nabla V \neq 0$ on ∂C_0, the continuity of ∇V and compactness of $\mathcal{V}^c_+$ show $\nabla V \neq 0$ on $\partial \mathcal{V}^c_+$ if c is small, say $c < c_D$ for some number $c_D \in (0, \nu_1)$. So by arguing as above, the solution q_c can be either of brake orbit type or of homoclinic type (depending on whether $\nabla V \neq 0$ on $\partial \mathcal{V}^c_-$), reaching $\partial \mathcal{V}^c_+$ at a *contact* time with respect to which it is symmetric. Using Remark 2.17 we conclude that for any $c \in (0, c_D)$ the solution q_c given by Proposition 3.5 relative to the decomposition (3.44) has a *connecting time interval* $(\alpha_{q_c}, \omega_{q_c}) \subset \mathbb{R}$, with $-\infty \leq \alpha_{q_c} < \omega_{q_c} < +\infty$, such that

(1_D) $V(q_c(t)) > c$ for every $t \in (\alpha_{q_c}, \omega_{q_c})$,



(a) $c = 0$*: The homoclinic case*

(b) $c \in (0, \nu_1)$ *regular value: The brake orbit case*

FIGURE 1. Possible configurations in Duffing like systems.

(2_D) $\lim_{t\to\alpha^+_{q_c}} \operatorname{dist}(q_c(t), \mathcal{V}^c_-) = 0$, and if $\alpha_{q_c} > -\infty$ then $\dot q_c(\alpha_{q_c}) = 0$, $V(q_c(\alpha_{q_c})) = c$ with $q_c(\alpha_{q_c}) \in \mathcal{V}^c_-$,

(3_D) $\dot q_c(\omega_{q_c}) = 0$, $V(q_c(\omega_{q_c})) = c$ with $q_c(\omega_{q_c}) \in \mathcal{V}^c_+$,

(4_D) $J_{c,(\alpha_{q_c},\omega_{q_c})}(q_c) = m_c$.

We continue by observing that

Remark 3.6. *There exists $M_D > 0$ such that*

$$m_c = \inf_{\Gamma_c} J_c \le M_D \quad \text{for any } c \in [0, c_D).$$

This is obtained along the lines of the proof of (3.28)*. Let us fix $\zeta \in \mathbb{R}^N$ with $|\zeta| = 1$, then we may consider the ray $\{t\zeta : t \ge 0\} \subset \mathbb{R}^N$. From* (V3)*, there must exist $\bar t > 0$ such that $\{t\zeta : 0 \le t < \bar t\} \subset C_0$, while $\bar t\zeta \in \partial C_0$. Moreover for any $c \in [0, c_D)$ there exist $0 \le \sigma_c < \tau_c \le \bar t$ such that*

$$\sigma_c\zeta \in \mathcal{V}^c_-,\ \tau_c\zeta \in \mathcal{V}^c_+ \ \text{ and } \ V(t\zeta) > c \ \text{ for any } t \in (\sigma_c, \tau_c).$$

As in the proof of (3.28)*, we readily see the function*

$$q_{c,\zeta}(t) := \begin{cases} \sigma_c\zeta & \text{if } t < \sigma_c, \\ t\zeta & \text{if } \sigma_c \le t < \tau_c, \\ \tau_c\zeta & \text{if } \tau_c \le t. \end{cases}$$

belongs to Γ_c. Hence, setting $\operatorname{diam}(C_0) := \sup\{|x - y| : x, y \in C_0\}$*, we obtain that*

$$\sup_{c\in[0,c_D)} m_c \le \sup_{c\in[0,c_D)} J_c(q_{c,\zeta}) \le \operatorname{diam}(C_0)\left(\tfrac{1}{2} + \max_{x\in C_0} V(x)\right) =: M_D.$$

Arguments similar to the ones used in the case of double well potentials show that the solutions (q_c) accumulate a near solution of homoclinic type, as c approaches zero.

Proposition 3.7. *Assume that $V \in C^1(\mathbb{R}^N)$ satisfies* (V3)-(V4)*, and that ∇V is locally Lipschitz continuous in $\mathbb{R}^N$. For any sequence $c_n \to 0^+$, consider the sequence of solutions (q_{c_n}) to the system* (1.1) *given by Proposition 3.5. Then, up to translations and a subsequence, $q_{c_n} \to q_0$ in $C^2_{loc}(\mathbb{R}, \mathbb{R}^N)$ where q_0 is a solution to* (1.1) *of homoclinic type at $x_0 = 0$.*

Proof of Proposition 3.7. Without loss of generality assume the sequence $c_n \to 0^+$ is so $(c_n) \subset (0, c_D)$. Since $c_n < c_D$ we have $(\mathbf{V}^c)$ is satisfied by (3.44), and let us denote $q_n := q_{c_n}$ the solution given by Proposition 3.5. Recalling (1_D) through (4_D), we denote $(\alpha_n, \omega_n) := (\alpha_{c_n}, \omega_{c_n})$ the *connecting time interval* of q_n. Since $\omega_n \in \mathbb{R}$ for $n \in \mathbb{N}$, we can assume, up to translations, that $\omega_n = 0$. That is to say,

$$\alpha_n < 0 = \omega_n \quad \text{and} \quad \dot{q}_n(0) = 0,\ q_n(0) \in \partial\mathcal{V}_+^{c_n} \ \text{ for all } n \in \mathbb{N}. \tag{3.45}$$

Since C_0 is bounded and by construction $q_n(\mathbb{R}) \subset C_0$ for every $n \in \mathbb{N}$, there exists $R_D > 0$ such that $\sup_{n\in\mathbb{N}} \|q_n\|_{L^\infty(\mathbb{R},\mathbb{R}^N)} \le R_D$. Since $\ddot{q}_n = \nabla V(q_n)$ on $\mathbb{R}$, the same arguments in the proof of Proposition 3.3 yield that there exists $q_0 \in C^2(\mathbb{R}, \mathbb{R}^N)$ such that $\ddot{q}_0 = \nabla V(q_0)$ on $\mathbb{R}$, and $q_n \to q_0$ in $C^2_{loc}(\mathbb{R}, \mathbb{R}^N)$, along a subsequence, as $n \to +\infty$. The pointwise convergence and (3.45), together with the fact that $\operatorname{dist}(\partial\mathcal{V}_+^{c_n}, \partial C_0) \to 0$ as $n \to +\infty$, imply furthermore that

$$q_0(0) \in \partial C_0 \ \text{ and } \ \dot{q}_0(0) = 0. \tag{3.46}$$

Our next goal is to show, similarly to (3.36), that

$$\alpha_n \to -\infty \ \text{ as } \ n \to +\infty. \tag{3.47}$$

If (3.47) were false then we could assume, up to a subsequence, that $\alpha_n \to \alpha_0 \in \mathbb{R}$. By (2_D) and (3.44), since $c_D \in (0, \nu_1)$, then yields $q_n(\alpha_n) \in \mathcal{V}_-^{c_n} \subset B_{r_0}(0)$. Since $c_n \to 0$, it would follow that $q_n(\alpha_n) \to 0$ and, recalling $\dot{q}_n(\alpha_n) = 0$ for any $n \in \mathbb{N}$, we would obtain $q_0(\alpha_0) = 0$ and $\dot{q}_0(\alpha_0) = 0$. By uniqueness of solutions to the Cauchy problem, necessarily $q_0(t) \equiv 0$ for any $t \in \mathbb{R}$, a contradiction with (3.46). This shows (3.47).

To prove the proposition, we are left to show that

$$q_0(t) \to 0 \ \text{ as } \ t \to \pm\infty. \tag{3.48}$$

Note that by (3.46), q_0 is symmetric with respect to the contact time $\omega_0 = 0$. Hence, (3.48) follows once we show that

$$q_0(t) \to 0 \ \text{ as } \ t \to -\infty. \tag{3.49}$$

The latter reduces to prove that for any $r \in (0, r_0)$, with r_0 as in (V3), there exist $L_r > 0$ and $n_r \in \mathbb{N}$ s.t.

$$|q_n(t)| < r \quad \text{for } \ t \in (\alpha_n, -L_r), \quad \text{for any } \ n \ge n_r. \tag{3.50}$$

In order to establish (3.50) we assume by contradiction that there exist $\bar{r} \in (0, r_0)$, a subsequence of (q_n), still denoted (q_n) and a sequence $(s_n) \subset \mathbb{R}$, in such a way that

$$|q_n(s_n)| > \bar{r} \ \text{ with } \ s_n \in (\alpha_n, 0), \quad \text{and } \ s_n \to -\infty. \tag{3.51}$$

The rest of the proof is devoted to obtain a contradiction with (3.51), following very similar steps as the proof of (3.37) in Proposition 3.3. We will only sketch the main ideas and spare some details. The contradiction will be reached by arguing that if (3.51) holds true, then there exists (t_n) so that $t_n \in (s_n, 0)$ for $n \in \mathbb{N}$ with

$$\lim_{n\to+\infty} \operatorname{dist}(q_n(t_n), \partial C_0 \cup \{0\}) = 0. \tag{3.52}$$

But for such sequence there hold furthermore

$$\liminf_{n\to+\infty} \operatorname{dist}(q_n(t_n), \partial C_0) > 0 \quad \text{and} \quad \liminf_{n\to+\infty} \operatorname{dist}(q_n(t_n), 0) > 0, \tag{3.53}$$

which is in contradiction with (3.52). This will prove (3.49), completing the proof of Proposition 3.7. To establish (3.52), we remark that the uniform bound on the energies: $J_{c_n,(\alpha_n,0)}(q_n) = m_{c_n} \le M_D$ for all $n \in \mathbb{N}$ (see Remark 3.6 since $c_n < c_D$) together with $s_n \to -\infty$ yield that for any $n \in \mathbb{N}$ there exists $t_n \in (s_n, 0)$ so that $V(q_n(t_n)) \to 0$ as $n \to +\infty$. This, in light of (V3)-(V4), shows (3.52).

We next argue (3.53). If $\liminf_{n\to+\infty} \operatorname{dist}(q_n(t_n), \partial C_0) > 0$ fails to hold, then there exists $\xi_0 \in \partial C_0$ so that $q_n(t_n) \to \xi_0$ as $n \to +\infty$, along a subsequence that we continue to denote (q_n). We first note that this behavior of q_n is energetically inexpensive, in that

$$J_{c_n,(t_n,0)}(q_n) \to 0 \;\text{ as }\; n \to +\infty. \tag{3.54}$$

This is a consequence of an energy analysis with a suitable competitor $(q_{-,n})$ as follows. By continuity, since $V(q_n(t_n)) > c_n > 0 = V(\xi_0)$, there is $\sigma_n \in (0, 1)$ such that

$$V((1-\sigma_n)q_n(t_n) + \sigma_n\xi_0) = c_n \quad\text{and}\quad V((1-\sigma)q_n(t_n) + \sigma\xi_0) > c_n \;\text{ for any } \sigma \in [0, \sigma_n).$$

Hence, the curve

$$q_{-,n}(t) := \begin{cases} q_n(\alpha_n) & \text{if } t < \alpha_n, \\ q_n(t) & \text{if } \alpha_n \le t < t_n, \\ (1-(t-t_n))q_n(t_n) + (t-t_n)\xi_0 & \text{if } t_n \le t < t_n + \sigma_n, \\ (1-\sigma_n)q_n(t_n) + \sigma_n\xi_0 & \text{if } t_n + \sigma_n \le t. \end{cases}$$

is an element of Γ_{c_n}, thus $J_{c_n}(q_{-,n}) \ge m_{c_n}$. Since $V((1-\sigma)q_n(t_n) + \sigma\xi_0) \to 0$ as $n \to +\infty$ uniformly for $\sigma \in [0, 1]$, we derive that

$$J_{c_n,(t_n,+\infty)}(q_{-,n}) = J_{c_n,(t_n,t_n+\sigma_n)}(q_{-,n}) = \tfrac{\sigma_n}{2}|q_n(t_n) - \xi_0|^2 + \int_{t_n}^{t_n+\sigma_n} (V((1-\sigma)q_n(t_n) + \sigma\xi_0) - c_n)d\sigma \to 0.$$

as $n \to +\infty$. Hence, $J_{c_n,(\alpha_n,t_n)}(q_n) = J_{c_n,(\alpha_n,t_n)}(q_{-,n}) = J_{c_n}(q_{-,n}) - J_{c_n,(t_n,+\infty)}(q_{-,n}) \ge m_{c_n} - o(1)$, and since $m_{c_n} = J_{c_n,(\alpha_n,t_n)}(q_n) + J_{c_n,(t_n,0)}(q_n)$ we conclude (3.54).

As an intermediate step, we now claim that

$$\sup_{t\in(t_n,0)} \operatorname{dist}(q_n(t), \partial C_0) \to 0 \;\text{ as }\; n \to +\infty. \tag{3.55}$$

Indeed, if not, there is a $\rho_* \in (0, r_0/3)$ and a sequence (τ_n) with $\tau_n \in (t_n, 0)$ for $n \in \mathbb{N}$, such that $\operatorname{dist}(q_n(\tau_n), \partial C_0) = 3\rho_*$, up to subsequence. In light of (3.45), this implies that there exists $(t_{1,n}, t_{2,n}) \subset (\tau_n, 0)$ such that $|q_n(t_{2,n}) - q_n(t_{1,n})| = \rho_*$ and $2\rho_* \ge \operatorname{dist}(q_n(t), \partial C_0) \ge \rho_*$ for every $t \in (t_{1,n}, t_{2,n})$. Hence

$$\inf_{t\in(t_{1,n},t_{2,n})} V(q(t)) \ge \mu(\rho_*) := \min\{V(x) : x \in C_0 \;\text{ and }\; 2\rho_* \ge \operatorname{dist}(x, \partial C_0) \ge \rho_*\} > 0,$$

and by Remark 2.4 we obtain

$$J_{c_n,(t_n,0)}(q_n) \ge J_{c_n,(t_{1,n},t_{2,n})}(q_n)\, \sqrt{2(\mu(\rho_*) - c_n)}|q_n(t_{2,n}) - q_n(t_{1,n})| > \sqrt{\mu(\rho_*)}\tfrac{\rho_*}{4},$$

for n large enough. This last inequality contradicts (3.54), proving (3.55).
Since $t_n \to -\infty$ and $q_n \to q_0$ in $C^2_{loc}(\mathbb{R}, \mathbb{R}^N)$, by (3.55) we conclude

$$q_0(t) \in \partial C_0 \;\text{ for any }\; t \le 0. \tag{3.56}$$

Also, the energy constraint $E_{q_n}(t) = -c_n$ for every $t \in \mathbb{R}$, together with the pointwise convergence show that $E_{q_0}(t) = 0$ for any $t \in \mathbb{R}$. That is to say, $\frac{1}{2}|\dot{q}_0(t)|^2 = V(q_0(t))$ for $t \in \mathbb{R}$ and since $V(x) = 0$ for $x \in \partial C_0$, by (3.56) we obtain $\dot{q}_0(t) = 0$ for any $t < 0$. Thus, q_0 is constant with $\ddot{q}(t) = 0$ for $t < 0$. Nonetheless, using that $\nabla V \neq 0$ on $\partial\mathcal{C}_0$, see (V4), we would simultaneously have that $\ddot{q}_0(t) = \nabla V(q_0(t)) \neq 0$ for $t < 0$ (by (3.56)). This contradiction proves that

$$\liminf_{n\to+\infty} \operatorname{dist}(q_n(t_n), \partial\mathcal{C}_0) > 0.$$

To conclude the proof, let us finally show that $\liminf_{n\to+\infty} \operatorname{dist}(q_n(t_n), 0) > 0$. Assume by contradiction that $q_n(t_n) \to 0$ and, as in the two-well case, for any $n \in \mathbb{N}$ we fix $\sigma_n \in (0,1)$ such that

$$V((1-\sigma_n)0 + \sigma_n q_n(t_n)) = c_n \quad \text{and} \quad V((1-\sigma)0 + \sigma q_n(t_n)) > c_n \text{ for any } \sigma \in (\sigma_n, 1].$$

Then

$$q_{+,n}(t) := \begin{cases} \sigma_n q_n(t_n) & \text{if } t < t_n - (1-\sigma_n), \\ (t - t_n + 1) q_n(t_n) & \text{if } t_n - (1-\sigma_n) \leq t < t_n, \\ q_n(t) & \text{if } t_n \leq t < 0, \\ q_n(0) & \text{if } 0 \leq t. \end{cases}$$

is in Γ_{c_n}, whence $J_{c_n}(q_{+,n}) \geq m_{c_n}$, and just as in the two-well case, we obtain as $n \to +\infty$,

$$J_{c_n,(\alpha_n,t_n)}(q_n) \to 0 \quad \text{as} \quad n \to +\infty. \tag{3.57}$$

In summary, we are in a situation where

$$\alpha_n < s_n < t_n < 0, \ |q_n(s_n)| > \bar{r} \text{ and } q_n(t_n) \to 0 \text{ as } n \to +\infty.$$

Then, when n is large, there exists $(t_{3,n}, t_{4,n}) \subset (s_n, t_n)$ such that $|q_n(t_{4,n}) - q_n(t_{3,n})| = \frac{\bar{r}}{4}$ and $|q_n(t)| \geq \frac{\bar{r}}{4}$ for any $t \in (t_{3,n}, t_{4,n})$. Letting $\bar{\mu}(\bar{r}) := \min\{V(x) : \frac{\bar{r}}{4} \leq |x| \leq \bar{r}\}$ we can apply Remark 2.4 similarly as above, to obtain for n sufficiently large:

$$J_{c_n,(\alpha_n,t_n)}(q_n) \geq J_{c_n,(t_{3,n},t_{4,n})}(q_n) > \sqrt{\bar{\mu}(\bar{r})}\tfrac{\bar{r}}{4},$$

This contradicts (3.57), which proves that $q_n(t_n) \to 0$ is not possible either; $\liminf_{n\to+\infty} \operatorname{dist}(q_n(t_n), 0) > 0$ is now established. □

3.3. **The multiple pendulum type systems.** As a last classical example, we consider the case of *multiple pendulum type* systems. That is, we assume (see e.g. [2], [14], [29], for analogous assumptions)

(V5) $V \in C^1(\mathbb{R}^N)$ is $\mathbb{Z}^N$-periodic,

(V6) $V(x) \geq 0$, and $V(x) = 0$ if and only if $x \in \mathbb{Z}^N$.

Upon assuming (V5) and (V6) we observe that $\mathcal{V}^c = \mathbb{Z}^N$ for $c = 0$. Thus, by continuity and periodicity there exists $c_p > 0$ so that

$$\mathcal{V}^c \subset \bigcup_{\xi\in\mathbb{Z}^n} B_{1/3}(\xi) \text{ for any } c \in [0, c_p).$$

Hence, by denoting for any such $c \in [0, c_p)$

$$\mathcal{V}^c_\xi := \mathcal{V}^c \cap B_{1/3}(\xi) \text{ for } \xi \in \mathbb{Z}^N,$$

we observe the following properties hold:

($\mathcal{V}$i) $\mathcal{V}^c = \bigcup_{\xi\in\mathbb{Z}^N} \mathcal{V}^c_\xi$,

($\mathcal{V}$ii) $\mathcal{V}^c_\xi$ is compact for any $\xi \in \mathbb{Z}^N$,

($\mathcal{V}$iii) $\exists r_c > 0$ such that $\operatorname{dist}(\mathcal{V}^c_\xi, \mathcal{V}^c_{\xi'}) \geq 2r_c$ for any pair $\xi \neq \xi' \in \mathbb{Z}^N$,

($\mathcal{V}$iv) $\forall r \in (0, r_c)$, $\exists \mu_r > 0$ such that $V(x) > c + \mu_r$ for any $x \in \mathbb{R}^N \setminus \bigcup_{\xi\in\mathbb{Z}^N} B_r(\mathcal{V}^c_\xi)$.

Remark 3.8. *Let us denote the elements of the canonical basis of* $\mathbb{R}^N$ *by* $\boldsymbol{e}_\ell$, *for* $\ell = 1, \ldots, N$, *and define associated functions* $\zeta_\ell : \mathbb{R} \to \mathbb{R}^N$ *by* $\zeta_\ell(t) = t\boldsymbol{e}_\ell$ *for* $t \in [0,1]$, $\zeta_\ell(t) = 0$ *for* $t \le 0$ *and* $\zeta_\ell(t) = \boldsymbol{e}_\ell$ *for* $t \ge 1$. *Since by definition* $\mathcal{V}^c_\xi \subset B_{1/3}(\xi)$ *for any* $\xi \in \mathbb{Z}^N$ *and* $c \in [0, c_p)$, *elementary geometric considerations give that for any* $\ell \in \{1, \ldots, N\}$ *and* $c \in [0, c_p)$ *there exists* $(s_{c,\ell}, t_{c,\ell}) \subset [0,1]$ *in such a way that*

$$V(\zeta_\ell(t)) > c \ \text{ for any } \ t \in (s_{c,\ell}, t_{c,\ell}),\ \zeta_\ell(s_{c,\ell}) \in \partial\mathcal{V}^c_0 \ \text{ and } \ \zeta_\ell(t_{c,\ell}) \in \partial\mathcal{V}^c_{\boldsymbol{e}_\ell}.$$

Next, it will be convenient to introduce the following test function $\eta_{c,\ell} : \mathbb{R} \to \mathbb{R}^N$ *given by* $\eta_{c,\ell}(t) = \zeta_\ell(t)$ *for* $t \in (s_{c,\ell}, t_{c,\ell})$, $\eta_{c,\ell}(t) = \zeta_\ell(s_{c,\ell})$ *for* $t \le s_{c,\ell}$ *and* $\eta_{c,\ell}(t) = \zeta_\ell(t_{c,\ell})$ *for* $t \ge t_{c,\ell}$. *We readily get the bound*

$$J_c(\eta_{c,\ell}) \le J_0(\zeta_\ell) = \int_0^1 \tfrac{1}{2}|\boldsymbol{e}_\ell|^2 + V(t\boldsymbol{e}_\ell)\,dt \le M_p := \tfrac{1}{2} + \max_{|x|\le 1} V(x), \tag{3.58}$$

for any $\ell \in \{1, \ldots, N\}$ *and* $c \in [0, c_p)$ *as above.*

We first observe

Lemma 3.9. *There exists* $R_p > 0$ *so that for any* $c \in [0, c_p)$, *if* $q \in W^{1,2}_{loc}(\mathbb{R}, \mathbb{R}^N)$ *and* $(\sigma, \tau) \subset \mathbb{R}$ *are such that* $V(q(t)) \ge c$ *for* $t \in (\sigma, \tau)$ *and* $J_{c,(\sigma,\tau)}(q) \le M_p + 1$, *then*

$$|q(\tau) - q(\sigma)| \le R_p.$$

Proof of Lemma 3.9. Let us write $r_p := \frac{1}{2} r_{c_p}$, and $\mu_p := \mu_{r_p}$ as in ($\mathcal{V}$iii)-($\mathcal{V}$iv). Then it follows that $V(x) > c_p + \mu_p$ for any $x \in \mathbb{R}^N \setminus \bigcup_{\xi\in\mathbb{Z}^N} B_{r_p}(\mathcal{V}^{c_p}_\xi)$. Now, to establish Lemma 3.9 let us assume by contradiction that there are sequences $(c_n) \subset [0, c_p)$ and $(q_n) \subset W^{1,2}_{loc}(\mathbb{R}, \mathbb{R}^N)$ with corresponding intervals $(\sigma_n, \tau_n) \subset \mathbb{R}$ such that $V(q_n(t)) \ge c_n$ for any $t \in (\sigma_n, \tau_n)$, $J_{c_n,(\sigma_n,\tau_n)}(q_n) \le M_p + 1$, and

$$|q_n(\tau_n) - q_n(\sigma_n)| \ge 2n\sqrt{N}.$$

In particular, since $c_n \le c_p$ we see from ($\mathcal{V}$iii) that dist$(\mathcal{V}^{c_n}_\xi, \mathcal{V}^{c_n}_{\xi'}) \ge 2r_{c_p}$ for all $n \in \mathbb{N}$, if $\xi \ne \xi'$. These inequalities, along with basic geometric considerations, cf. (V5), imply the existence of n disjoint intervals $(s_i, t_i) \subset (\sigma_n, \tau_n)$ for $1 \le i \le n$, such that $q_n(t) \notin \bigcup_{\xi\in\mathbb{Z}^N} B_{r_p}(\mathcal{V}^{c_p}_\xi)$ if $t \in \bigcup_i (s_i, t_i)$, while $|q_n(t_i) - q_n(s_i)| \ge 2(r_{c_p} - r_p) = r_{c_p}$. Then, by Remark 2.4, $J_{c_n,(\sigma_n,\tau_n)}(q_n) \ge n\sqrt{2\mu_p}\, r_{c_p}$ for any $n \in \mathbb{N}$. But this goes in contradiction with $J_{c_n,(\sigma_n,\tau_n)}(q_n) \le M_p + 1$, and the Lemma follows. □

The above properties allow us to apply Theorem 2.1, giving the next

Proposition 3.10. *Assume that* V *satisfies* (V5) *and* (V6). *Then for every* $c \in [0, c_p)$ *there exists* $k_c \in \mathbb{N}$ *and a finite set* $\{\xi^1, \ldots, \xi^{k_c}\} \subset \mathbb{Z}^N \setminus \{0\}$, *satisfying*

$$\xi^i \ne \xi^j \ \text{ for } \ i \ne j, \ \text{ and } \ \{n_1\xi^1 + \ldots + n_{k_c}\xi^{k_c} : n_1, \ldots, n_{k_c} \in \mathbb{Z}\} = \mathbb{Z}^N, \tag{3.59}$$

for which, given any $j \in \{1, \ldots, k_c\}$, *there is a solution* $q_{c,j} \in C^2(\mathbb{R}, \mathbb{R}^N)$ *to* (1.1) *with energy* $E_{q_{c,j}}(t) = -c$ *for any* $t \in \mathbb{R}$, *verifying*

$$\inf_{t\in\mathbb{R}} \operatorname{dist}(q_{c,j}(t), \mathcal{V}^c_0) = \inf_{t\in\mathbb{R}} \operatorname{dist}(q_{c,j}(t), \mathcal{V}^c_{\xi^j}) = 0 \ \text{ and } \ \|q_{c,j}\|_{L^\infty(\mathbb{R},\mathbb{R}^N)} \le R_p + 1, \tag{3.60}$$

where R_p *is given by Lemma 3.9.*

Proof of Proposition 3.10. For any $c \in [0, c_p)$, let us set

$$\mathcal{V}^c_{-,1} := \mathcal{V}^c_0 \quad \text{and} \quad \mathcal{V}^c_{+,1} := \bigcup_{\xi \in \mathbb{Z}^N \setminus \{0\}} \mathcal{V}^c_\xi = \mathcal{V}^c \setminus \mathcal{V}^c_{-,1}. \tag{3.61}$$

In particular, we observe in light of ($\mathcal{V}$iii) that

$$\operatorname{dist}(\mathcal{V}^c_{-,1}, \mathcal{V}^c_{+,1}) \geq 2r_c,$$

so ($\mathbf{V}^c$) holds. Then defining $\Gamma_{c,1}$ as in (1.5), relative to the partition (3.61), we let

$$m_{c,1} := \inf_{q \in \Gamma_{c,1}} J_c(q).$$

Recall the definition of $\eta_{c,\ell}$ in Remark 3.8 and observe that $\eta_{c,1} \in \Gamma_{c,1}$. This, together with (3.58), yields

$$m_{c,1} \leq J_c(\eta_{c,1}) \leq M_p.$$

Consider now any minimizing sequence $(q_n) \subset \Gamma_{c,1}$, so that $J_c(q_n) \to m_{c,1}$. Eventually passing to a subsequence, we can assume that $J_c(q_n) \leq M_p + 1$ for any $n \in \mathbb{N}$, and so by Lemma 3.9

$$|q_n(t) - q_n(s)| \leq R_p \ \text{ for any } (s, t) \subset \mathbb{R}, \text{ and } n \in \mathbb{N}.$$

Therefore the coercivity condition (1.7) of J_c is satisfied for the division (3.61), hence Theorem 2.1 gives the existence of a solution $q_{c,1} \in C^2(\mathbb{R}, \mathbb{R}^N)$ to (1.1) with energy $E_{q_{c,1}}(t) = -c$ for all $t \in \mathbb{R}$, satisfying

$$\liminf_{t \to -\infty} \operatorname{dist}(q_{c,1}(t), \mathcal{V}^c_{-,1}) = 0 \quad \text{and} \quad \liminf_{t \to +\infty} \operatorname{dist}(q_{c,1}(t), \mathcal{V}^c_{+,1}) = 0.$$

This solution is either of brake orbit type, case (a) of the theorem, of homoclinic type, case (b), or of heteroclinic type in the case (c). We now continue with the proof of Proposition 3.10 by checking that (3.60) is satisfied, regardless of the case in consideration.

If we are in the case (a) there exists $-\infty < \sigma < \tau < +\infty$ such that

(a_0) $q_{c,1}(\sigma) \in \mathcal{V}^c_{-,1}$, $q_{c,1}(\tau) \in \mathcal{V}^c_{+,1}$, $V(q_{c,1}(t)) > c$ for $t \in (\sigma, \tau)$, $q_{c,1}(\sigma + t) = q_{c,1}(\sigma - t)$ and $q_{c,1}(\tau + t) = q_{c,1}(\tau - t)$ for all $t \in \mathbb{R}$.

Let us now point out that (a_0), (3.61) and ($\mathcal{V}$i) yield that $q_{c,1}(\sigma) \in \mathcal{V}^c_0$ and that there exists $\xi^1 \in \mathbb{Z}^N \setminus \{0\}$ with $q_{c,1}(\tau) \in \mathcal{V}^c_{\xi^1}$. Moreover, from Remark 2.17 we have that for any $(s, t) \subset (\sigma, \tau)$, $J_{c,(s,t)}(q_{c,1}) \leq J_{c,(\sigma,\tau)}(q_{c,1}) = m_{c,1} \leq M_p$ and so Lemma 3.9 gives in particular $|q_{c,1}(\sigma) - q_{c,1}(t)| \leq R_p$ for any $t \in (\sigma, \tau)$. By periodicity we then obtain $\operatorname{dist}(q_{c,1}(t), \mathcal{V}^c_0) \leq R_p$ for any $t \in \mathbb{R}$. Since $\mathcal{V}^c_0 \subset B_{1/3}(0)$ we get $\|q_{c,1}\|_{L^\infty(\mathbb{R},\mathbb{R}^N)} \leq R_p + 1$, therefore property (3.60) is satisfied by $q_{c,1}$.

If we are in the case (b), then there exists $\sigma \in \mathbb{R}$ such that

(b_0) $q_{c,1}(\sigma) \in \mathcal{V}^c_{\pm,1}$, $\lim_{t\to\pm\infty} \operatorname{dist}(q_{c,1}(t), \mathcal{V}^c_{\mp,1}) = 0$, $V(q_{c,1}(t)) > c$ for $t \in \mathbb{R} \setminus \{\sigma\}$, and $q_{c,1}(\sigma + t) = q_{c,1}(\sigma - t)$ for all $t \in \mathbb{R}$.

Once again, we point out that (b_0), (3.61) and ($\mathcal{V}$i) imply that there is $\xi^1 \in \mathbb{Z}^N \setminus \{0\}$ such that either $q_{c,1}(\sigma) \in \mathcal{V}^c_0$ and $\lim_{t\to\pm\infty} \operatorname{dist}(q_{c,1}(t), \mathcal{V}^c_{\xi^1}) = 0$, or $q_{c,1}(\sigma) \in \mathcal{V}^c_{\xi^1}$ and $\lim_{t\to\pm\infty} \operatorname{dist}(q_{c,1}(t), \mathcal{V}^c_0) = 0$. By Remark 2.17 we have $J_{c,(s,t)}(q_{c,1}) \leq J_{c,(-\infty,\sigma)}(q_{c,1}) = m_{c,1} \leq M_p$ for any $(s, t) \subset (-\infty, \sigma)$. This, combined with Lemma 3.9 and the reflection $q_{c,1}(\sigma + t) = q_{c,1}(\sigma - t)$ for all $t \in \mathbb{R}$, shows that $|q_{c,1}(t) - q_{c,1}(s)| \leq R_p$ for any $s < t \in \mathbb{R}$. But since $q_{c,1}(\sigma) \in \mathcal{V}^c_{\pm,1}$ and $\lim_{t\to\pm\infty} \operatorname{dist}(q_{c,1}(t), \mathcal{V}^c_{\mp,1}) = 0$ we obtain $\operatorname{dist}(q_{c,1}(t), \mathcal{V}^c_0) \leq R_p$ for any $t \in \mathbb{R}$ and so $\|q_{c,1}\|_{L^\infty(\mathbb{R},\mathbb{R}^N)} \leq R_p + 1$. This shows again (3.60) for $q_{c,1}$.

Finally if we are in the case (c) we have

(c_0) $V(q_{c,1}(t)) > c$ for all $t \in \mathbb{R}$, $\lim\limits_{t\to-\infty} \operatorname{dist}(q_{c,1}(t), \mathcal{V}^c_{-,1}) = 0$ and $\lim\limits_{t\to+\infty} \operatorname{dist}(q_{c,1}(t), \mathcal{V}^c_{+,1}) = 0$.

As before (c_0), (3.61) and $(\mathcal{V}i)$ show that there is $\xi^1 \in \mathbb{Z}^N \setminus \{0\}$ such that $\lim_{t\to+\infty} \text{dist}\,(q_{c,1}(t), \mathcal{V}^c_{\xi^1}) = 0$. From Remark 2.17 we have $J_{c,(s,t)}(q_{c,1}) \le J_c(q_{c,1}) = m_{c,1} \le M_p$ for any $(s,t) \subset \mathbb{R}$, which combined with Lemma 3.9 gives $|q_{c,1}(t) - q_{c,1}(s)| \le R_p$ for any $s < t \in \mathbb{R}$. Since $\lim_{t\to-\infty} \text{dist}\,(q_{c,1}(t), \mathcal{V}^c_0) = 0$, we deduce that $\text{dist}\,(q_{c,1}(s), \mathcal{V}^c_0) \le R_p$ for any $s \in \mathbb{R}$ and hence $\|q_{c,1}\|_{L^\infty(\mathbb{R},\mathbb{R}^N)} \le R_p + 1$. Whence, even in case (c) the condition (3.60) holds for $q_{c,1}$.
The above argument shows the existence of a solution $q_{c,1}$ with energy $E_{q_{c,1}} = -c$ with

$$\inf_{t\in\mathbb{R}} \text{dist}\,(q_{c,1}(t), \mathcal{V}^c_0) = \inf_{t\in\mathbb{R}} \text{dist}\,(q_{c,1}(t), \mathcal{V}^c_{\xi^1}) = 0 \quad \text{and} \quad \|q_{c,1}\|_{L^\infty(\mathbb{R},\mathbb{R}^N)} \le R_p + 1.$$

In order to establish the multiplicity of solutions in Proposition 3.10, we will proceed by induction. Let us assume that for $j \ge 1$ we have

(I) There are $\xi^1, \xi^2, ..., \xi^j \in \mathbb{Z}^N \setminus \{0\}$ such that $\xi^h \neq \xi^k$ for $h \neq k$ and if we set $\mathcal{L}_j := \{\sum_{i=1}^j n_i\xi^i : n_1, \dots, n_j \in \mathbb{Z}\}$, then $\mathcal{L}_j \neq \mathbb{Z}^N$.

(II) For any $i \in \{1, \dots, j\}$ there exists $q_{c,i}$ with energy $E_{q_{c,i}} = -c$ such that

$$\inf_{t\in\mathbb{R}} \text{dist}\,(q_{c,i}(t), \mathcal{V}^c_0) = \inf_{t\in\mathbb{R}} \text{dist}\,(q_{c,i}(t), \mathcal{V}^c_{\xi^i}) = 0 \quad \text{and} \quad \|q_{c,i}\|_{L^\infty(\mathbb{R},\mathbb{R}^N)} \le R_p + 1.$$

Proposition 3.10 will follow once we show that (I) and (II) together imply the existence of $\xi^{j+1} \in \mathbb{Z}^N \setminus \mathcal{L}_j$ and a solution $q_{c,j+1}$ with energy $E_{q_{c,j+1}} = -c$, in such a way that

$$\inf_{t\in\mathbb{R}} \text{dist}\,(q_{c,j+1}(t), \mathcal{V}^c_0) = \inf_{t\in\mathbb{R}} \text{dist}\,(q_{c,j+1}(t), \mathcal{V}^c_{\xi^{j+1}}) = 0 \quad \text{and} \quad \|q_{c,j+1}\|_{L^\infty(\mathbb{R},\mathbb{R}^N)} \le R_p + 1. \tag{3.62}$$

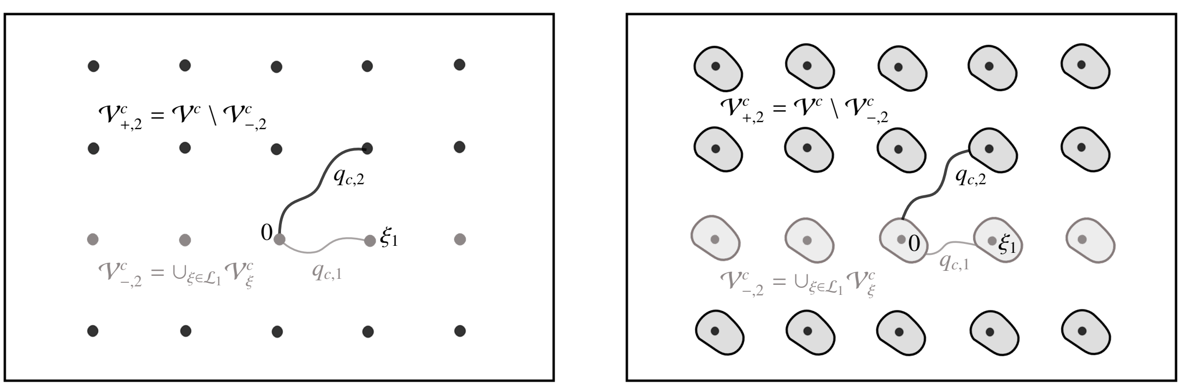
To see this, we consider in view of (I), the following decomposition of $\mathcal{V}^c$:

$$\mathcal{V}^c_{-,j+1} := \textstyle\bigcup_{\xi\in\mathcal{L}_j} \mathcal{V}^c_\xi \quad \text{and} \quad \mathcal{V}^c_{+,j+1} := \mathcal{V}^c \setminus \mathcal{V}^c_{-,j+1} = \textstyle\bigcup_{\xi\in\mathbb{Z}^N\setminus\mathcal{L}_j} \mathcal{V}^c_\xi. \tag{3.63}$$

In light of (I) both sets $\mathcal{V}^c_{-,j+1}, \mathcal{V}^c_{+,j+1}$ are non-empty, they clearly verify $\text{dist}\,(\mathcal{V}^c_{-,j+1}, \mathcal{V}^c_{+,j+1}) \ge r_c$, and

$$\mathcal{V}^c_{\pm,j+1} = \xi + \mathcal{V}^c_{\pm,j+1} \quad \text{for any } \xi \in \mathcal{L}_j, \tag{3.64}$$

see Figure 2 below.



(a) $c = 0$: *The homoclinic case* (b) $c \in (0, c_p)$ *regular value: The brake orbit case*

FIGURE 2. Possible configurations in pendulum like systems.

Then, according to the partition (3.63) of $\mathcal{V}^c$, define $\Gamma_{c,j+1}$ as in (1.5) and let $m_{c,j+1} := \inf_{q\in\Gamma_{c,j+1}} J_c(q)$. Since $\mathcal{L}_j \neq \mathbb{Z}^N$ there must exist $\ell_{j+1} \in \{1,\dots,N\}$ so that $e_{\ell_{j+1}} \notin \mathcal{L}_j$. Then, by Remark 3.8, $\eta_{c,\ell_{j+1}} \in \Gamma_{c,j+1}$, and so by (3.58)

$$m_{c,j+1} \le J_c(\eta_{c,\ell_{j+1}}) \le M_p.$$

Let $(q_n) \subset \Gamma_{c,j+1}$ be such that $J_c(q_n) \to m_{c,j+1}$. With no loss of generality we can assume that $J_c(q_n) < M_p + 1$ for any $n \in \mathbb{N}$, and so by Lemma 3.9 we obtain

$$|q_n(t) - q_n(s)| \le R_p \ \text{ for any } (s,t) \subset \mathbb{R}, \text{ and } n \in \mathbb{N}. \tag{3.65}$$

Since $q_n \in \Gamma_{c,j+1}$ we have $\liminf_{t\to-\infty} \operatorname{dist}(q_n(t), \mathcal{V}^c_{-,j+1}) = 0$ for any $n \in \mathbb{N}$. Hence, there exist sequences $(s_n) \subset \mathbb{R}$ and $(\zeta_n) \subset \mathcal{L}_j$ for which

$$|q_n(s_n) - \zeta_n| < 1 \ \text{ for all } n \in \mathbb{N}. \tag{3.66}$$

By periodicity of V, (3.64) and by (3.65)-(3.66), we have

$$\tilde q_n := q_n - \zeta_n \in \Gamma_{c,j+1},\ J_c(\tilde q_n) \to m_{c,j+1} \ \text{ and } \ \|\tilde q_n\|_{L^\infty(\mathbb{R},\mathbb{R}^N)} \le R_p + 1 \ \text{ for any } n \in \mathbb{N},$$

from which we conclude that

$$m_{c,j+1} = \inf\{J_c(q) : q \in \Gamma_{c,j+1},\ \|q\|_{L^\infty(\mathbb{R},\mathbb{R}^N)} \le R_p + 1\}, \tag{3.67}$$

namely, the coercivity condition (1.7) of J_c over $\Gamma_{c,j+1}$ follows. Thus, Theorem 2.1 yields the existence of a solution $\bar q_{c,j+1} \in C^2(\mathbb{R},\mathbb{R}^N)$ to (1.1) with energy $E_{\bar q_{c,j+1}} = -c$, satisfying

$$\liminf_{t\to-\infty} \operatorname{dist}(\bar q_{c,j+1}(t), \mathcal{V}^c_{-,j+1}) = 0 \quad \text{and} \quad \liminf_{t\to+\infty} \operatorname{dist}(\bar q_{c,j+1}(t), \mathcal{V}^c_{+,j+1}) = 0.$$

This solution is either of brake orbit type, case (a), of homoclinic type, case (b), or of heteroclinic type in the case (c). If we are in the case (a), then there exists $-\infty < \sigma < \tau < +\infty$ such that

(a_j) $\bar q_{c,j+1}(\sigma) \in \mathcal{V}^c_{-,j+1}$, $\bar q_{c,j+1}(\tau) \in \mathcal{V}^c_{+,j+1}$, $V(\bar q_{c,j+1}(t)) > c$ for $t \in (\sigma,\tau)$, $\bar q_{c,j+1}(\sigma+t) = \bar q_{c,j+1}(\sigma-t)$ and $\bar q_{c,j+1}(\tau+t) = \bar q_{c,j+1}(\tau-t)$ for all $t \in \mathbb{R}$.

By (a_j) there exists $\xi_- \in \mathcal{L}_j$, $\xi_+ \in \mathbb{Z}^N \setminus \mathcal{L}_j$ such that $\bar q_{c,j+1}(\sigma) \in \mathcal{V}^c_{\xi_-}$ and $\bar q_{c,j+1}(\tau) \in \mathcal{V}^c_{\xi_+}$. Then $\xi^{j+1} = \xi_+ - \xi_- \in \mathbb{Z}^N \setminus \mathcal{L}_j$, and by periodicity of V, the function $q_{c,j+1} := \bar q_{c,j+1} - \xi_-$ is a solution to (1.1) with energy $E_{q_{c,j+1}} = -c$. By Remark 2.17 we have moreover $J_{c,(s,t)}(q_{c,j+1}) \le J_{c,(\sigma,\tau)}(q_{c,j+1}) = m_{c,j+1} \le M_p$ for any $(s,t) \subset (\sigma,\tau)$, and so, arguing as in the case (a_0) above we deduce $\|q_{c,j+1}\|_{L^\infty(\mathbb{R},\mathbb{R}^N)} \le R_p + 1$. Then property (3.62) with respect to ξ^{j+1} is satisfied by $q_{c,j+1}$.
In case (b) there exists $\sigma \in \mathbb{R}$ such that

(b_j) $\bar q_{c,j+1}(\sigma) \in \mathcal{V}^c_{\pm,j+1}$, $\lim_{t\to\pm\infty} \operatorname{dist}(\bar q_{c,j+1}(t), \mathcal{V}^c_{\mp,j+1}) = 0$, $V(\bar q_{c,j+1}(t)) > c$ for $t \in \mathbb{R} \setminus \{\sigma\}$, and $\bar q_{c,j+1}(\sigma+t) = \bar q_{c,j+1}(\sigma-t)$ for all $t \in \mathbb{R}$.

In particular, from (b_j) we deduce the existence of $\bar\xi \in \mathbb{Z}^N$ so that $\bar q_{c,j+1}(\sigma) \in \mathcal{V}^c_{\bar\xi}$. Let us say that $\bar\xi \in \mathcal{L}_j$, then $\lim_{t\to\pm\infty} \operatorname{dist}(\bar q_{c,j+1}(t), \mathcal{V}^c_{+,j+1}) = 0$. The symmetry of $\bar q_{c,j+1}$ with respect to σ together with the discreteness of $\mathcal{V}^c_{+,j+1}$ implies that there is $\xi_\infty \in \mathbb{Z}^N \setminus \mathcal{L}_j$ so that $\lim_{t\to\pm\infty} \operatorname{dist}(\bar q_{c,j+1}(t), \mathcal{V}^c_{\xi_\infty}) = 0$. In this case we set $\xi^{j+1} := \xi_\infty - \bar\xi \in \mathbb{Z}^N \setminus \mathcal{L}_j$ and the function $q_{c,j+1} := \bar q_{c,j+1} - \bar\xi$ is a solution to (1.1) with energy $E_{\bar q_{c,j+1}} = -c$, which verifies the first part of (3.62) with respect to ξ^{j+1}. Otherwise, if $\bar\xi \in \mathbb{Z}^N \setminus \mathcal{L}_j$, then $\lim_{t\to\pm\infty} \operatorname{dist}(\bar q_{c,j+1}(t), \mathcal{V}^c_{-,j+1}) = 0$. The symmetry of $\bar q_{c,j+1}$ with respect to σ and the discreteness of $\mathcal{V}^c_{-,j+1}$ imply the existence of $\xi_\infty \in \mathcal{L}_j$ so that $\lim_{t\to\pm\infty} \operatorname{dist}(\bar q_{c,j+1}(t), \mathcal{V}^c_{\xi_\infty}) = 0$. In this case we let $\xi^{j+1} := \bar\xi - \xi_\infty \in \mathbb{Z}^N \setminus \mathcal{L}_j$, and again the function $q_{c,j+1} := \bar q_{c,j+1} - \xi_\infty$ is a solution to (1.1) with energy $E_{q_{c,j+1}} = -c$ which verifies the first part of (3.62) with respect to ξ^{j+1}.

To get the second part of (3.62) observe that by Remark 2.17 we have $J_{c,(s,t)}(q_{c,j+1}) \le J_{c,(-\infty,\sigma)}(q_{c,j+1}) = m_{c,j+1} \le M_p$ for any $(s,t) \subset (-\infty,\sigma)$. Then Lemma 3.9 and the symmetry property of $q_{c,j+1}$ with respect to σ allow us to argue as in the case (b_0) above to deduce $\|q_{c,j+1}\|_{L^\infty(\mathbb{R},\mathbb{R}^N)} \le R_p + 1$, thus showing (3.62).

Finally, let us assume that we are in case (c). Then

(c_j) $V(\bar{q}_{c,j+1}(t)) > c$ for all $t \in \mathbb{R}$, and $\lim_{t\to\pm\infty} \operatorname{dist}(\bar{q}_{c,j+1}(t), \mathcal{V}^c_{\pm,j+1}) = 0$.

By invoking once again the discreteness of the sets $\mathcal{V}^c_\pm$, (c_j) shows the existence of $\xi_- \in \mathcal{L}_j$ and $\xi_+ \in \mathbb{Z}^N \setminus \mathcal{L}_j$ in such a way that $\lim_{t\to\pm\infty} \operatorname{dist}(\bar{q}_{c,j+1}(t), \mathcal{V}^c_{\xi_\pm}) = 0$. Setting $\xi^{j+1} := \xi_+ - \xi_- \in \mathbb{Z}^N \setminus \mathcal{L}_j$ we see from the periodicity of V that the function $q_{c,j+1} := \bar{q}_{c,j+1} - \xi_-$ is a solution to (1.1) with energy $E_{q_{c,j+1}} = -c$, verifying the first part of (3.62) with respect to ξ^{j+1}. By Remark 2.17 we have $J_{c,(s,t)}(q_{c,j+1}) \le J_c(q_{c,j+1}) = m_{c,j+1} \le M_p$ for any $(s,t) \subset \mathbb{R}$. Using Lemma 3.9 and arguing as in the case (c_0) above we obtain again $\|q_{c,j+1}\|_{L^\infty(\mathbb{R},\mathbb{R}^N)} \le R_p + 1$. Then (3.62) follows for $q_{c,j+1}$.

This concludes the proof of the inductive step, and hence the proof of Proposition 3.10. □

Remark 3.11. *Proposition 3.10 constitutes a multiplicity result. It asserts the existence of k_c elements $\xi^1, \dots, \xi^{k_c}$ in the lattice $\mathbb{Z}^N \setminus \{0\}$, for each of which there exists a connecting orbit between $\mathcal{V}^c_0$ and $\mathcal{V}^c_{\xi^j}$. In particular, we necessarily have $k_c \ge N$, due to* (3.59).

As for the preceding cases, we finalize by analyzing the convergence properties of the family of solutions $q_{c,j}$ given by Proposition 3.10 as $c \to 0^+$.

Proposition 3.12. *Assume that $V \in C^1(\mathbb{R}^N)$ satisfies* (V5)-(V6) *and that ∇V is locally Lipschitz continuous in $\mathbb{R}^N$. For any sequence $c_n \to 0^+$, let $k_{c_n} \in \mathbb{N}$, $\{\xi^1_n, \dots, \xi^{k_{c_n}}_n\} \subset \mathbb{Z}^N \setminus \{0\}$ and $q_{c_n,j}$ be the solution given by Proposition 3.10 associated to ξ^j_n for $j = 1, \dots, k_{c_n}$. Then, along a subsequence, we have that*

(i) There exists $\kappa \in \mathbb{N}$ such that $k_{c_n} = \kappa$ for all $n \in \mathbb{N}$;

(ii) There exist distinct elements $\hat{\xi}^1, \dots, \hat{\xi}^\kappa$ in $\mathbb{Z}^N \setminus \{0\}$ so that $\{n_1\hat{\xi}^1 + \dots + n_\kappa\hat{\xi}^\kappa : n_1, \dots, n_\kappa \in \mathbb{Z}\} = \mathbb{Z}^N$, and

$$\xi^1_n = \hat{\xi}^1, \dots, \xi^{k_{c_n}}_n = \hat{\xi}^\kappa \quad \text{for all } n \in \mathbb{N};$$

(iii) For any $j \in \{1, \dots, \kappa\}$ there is a solution $q^j \in C^2(\mathbb{R}, \mathbb{R}^N)$ to (1.1) *of heteroclinic type between* 0 *and $\hat{\xi}^j$, and there exists $(\tau_{n,j}) \subset \mathbb{R}$ such that*

$$q_{c_n,j}(\cdot - \tau_{n,j}) \to q^j \quad \text{in} \quad C^2_{loc}(\mathbb{R}, \mathbb{R}^N), \quad \text{as} \quad n \to +\infty.$$

Proof of Proposition 3.12. With no loss of generality we can assume $c_n < c_p$ for all $n \in \mathbb{N}$. Let $k_{c_n} \in \mathbb{N}$, $\{\xi^1_n, \dots, \xi^{k_{c_n}}_n\} \subset \mathbb{Z}^N \setminus \{0\}$ and $q_{c_n,j}$, for $j = 1, \dots, k_{c_n}$ be given by Proposition 3.10 associated to ξ^j_n. We know $\|q_{c_n,j}\|_{L^\infty(\mathbb{R},\mathbb{R}^N)} \le R_p + 1$ for any $n \in \mathbb{N}$. Moreover $\inf_{t\in\mathbb{R}} \operatorname{dist}(q_{c_n,j}(t), \mathcal{V}^{c_n}_0) = \inf_{t\in\mathbb{R}} \operatorname{dist}(q_{c_n,j}(t), \mathcal{V}^{c_n}_{\xi^j_n}) = 0$ for any $j \in \{1, \dots, k_{c_n}\}$ and all $n \in \mathbb{N}$.

In particular, the above shows that $\{\xi^1_n, \dots, \xi^{k_{c_n}}_n\} \subset B_{R_p+1}(0) \cap \mathbb{Z}^N$ for any $n \in \mathbb{N}$. Since $B_{R_p+1}(0) \cap \mathbb{Z}^N$ is a finite set, all the sequences (k_{c_n}), (ξ^j_n) for $1 \le j \le k_{c_n}$ take their values in a finite set, hence they are all constant along a common subsequence: there exists $\kappa \in \mathbb{N}$, $\{\hat{\xi}^1, \dots, \hat{\xi}^\kappa\} \subset \mathbb{Z}^N \setminus \{0\}$ and an increasing sequence $(n_i) \subset \mathbb{N}$, such that $k_{c_{n_i}} = \kappa$, $\xi^j_{n_i} = \hat{\xi}^j$ for any $i \in \mathbb{N}$ and $1 \le j \le \kappa$. Thus (i) and (ii) follow.

For $j \in \{1, \dots, \kappa\}$ fixed, and let us simplify the notation by allowing

$$q_i := q_{c_{n_i},j}, \ c_i := c_{n_i} \ \text{and} \ \xi := \hat{\xi}^j = \xi^j_{n_i} \ \text{for all } i \in \mathbb{N}.$$

Since any q_i is given by Theorem 2.1, we can invoke Remark 2.17 to see that, for each $i \in \mathbb{N}$, q_i has a connecting time interval $(\alpha_i, \omega_i) \subset \mathbb{R}$ with $-\infty \le \alpha_i \le \omega_i \le +\infty$ in such a way that

(1_i) $V(q_i(t)) > c_i$ for every $t \in (\alpha_i, \omega_i)$,

(2_i) $\lim_{t\to\alpha_i^+} \operatorname{dist}(q_i(t), \mathcal{V}_0^{c_i}) = 0$, and if $\alpha_i > -\infty$ then $\dot q_i(\alpha_i) = 0$, $V(q_i(\alpha_i)) = c_i$ with $q_i(\alpha_i) \in \mathcal{V}_0^{c_i}$,

(3_i) $\lim_{t\to\omega_i^-} \operatorname{dist}(q_i(t), \mathcal{V}_\xi^{c_i}) = 0$, and if $\omega_i < +\infty$ then $\dot q_i(\omega_i) = 0$, $V(q_i(\omega_i)) = c_l$ with $q_i(\omega_i) \in \mathcal{V}_\xi^{c_i}$,

(4_i) $J_{c_i,(\alpha_i,\omega_i)}(q_i) = m_{c_i} = \inf_{q\in\Gamma_{c_i}} J_{c_i}(q)$.

The rest of our argument goes along the same lines as the proof convergence to a heteroclinic type solution in Proposition 3.3, so we will briefly review it. We first renormalize the sequence (q_i) by a phase shift procedure. From (1_i)-(2_i)-(3_i) it follows that for all $i \in \mathbb{N}$ there is $t_i \in (\alpha_i, \omega_i)$ so that $|q_i(t_i)| = \frac{1}{2}$. Renaming, if necessary, q_i to be $q_i(\cdot - t_i)$, we can assume

$$\alpha_i < 0 < \omega_i \text{ and } |q_i(0)| = \tfrac{1}{2}, \text{ for any } i \in \mathbb{N}.$$

Just like before, the bound $\sup_{i\in\mathbb{N}} \|q_i\|_{L^\infty(\mathbb{R},\mathbb{R}^N)} \le R_p + 1$ combined with energy constraint $E_{q_i} = -c_i$ and the fact that q_i solves the system $\ddot q = \nabla V(q)$ on $\mathbb{R}$, allows us to conclude that $(\dot q_i)$ and $(\ddot q_i)$ are uniformly bounded in $L^\infty(\mathbb{R}, \mathbb{R}^N)$. Whence, by the Ascoli-Arselà Theorem there is $q_0 \in C^1(\mathbb{R}, \mathbb{R}^N)$ so that (q_i) has a subsequence, yet denoted by (q_i), for which $q_i \to q_0$ in $C^1_{loc}(\mathbb{R}, \mathbb{R}^N)$ as $i \to +\infty$. This convergence is then bootstrapped into the equation $\ddot q_i = \nabla V(q_i)$ in order to enhance it to $C^2_{loc}(\mathbb{R}, \mathbb{R}^N)$. This shows, in turn, that

$$\ddot q_0 = \nabla V(q_0) \text{ on } \mathbb{R}.$$

In addition, by taking the limit $i \to +\infty$ in $|q_i(0)| = \frac{1}{2}$ and $\|q_i\|_{L^\infty(\mathbb{R},\mathbb{R}^N)} \le R_p + 1$, we learn that this solution satisfies

$$|q_0(0)| = \tfrac{1}{2} \text{ and } \|q_0\|_{L^\infty(\mathbb{R},\mathbb{R}^N)} \le R_p + 1. \tag{3.68}$$

Furthermore, the same argument used to establish (3.36) can be applied to our case, to yield

$$\alpha_i \to -\infty \text{ and } \omega_i \to +\infty, \text{ as } i \to +\infty,$$

for potentials satisfying (V5)-(V6): arguing by contradiction, we show that q_0 solves the Cauchy problem $\ddot q = \nabla q$ and $q(0) = 0, \dot q(0) = 0$, whence $q_0 \equiv 0$, which is contrary to (3.68).

To conclude the proof of Proposition 3.7 we are left to show that q_0 is of heteroclinic type. More precisely, our goal is to show that

$$q_0(t) \to 0 \text{ as } t \to -\infty, \text{ and } q_0(t) \to \xi \text{ as } t \to +\infty. \tag{3.69}$$

As before, interpolation inequalities would then prove that q_0 is a solution to (1.1) of heteroclinic type between 0 and ξ. By analogy with our previous analysis, (3.69) reduces to proving that for any $r \in (0, \frac{1}{3})$ there exist $L_r^-, L_r^+ > 0$ and $i_r \in \mathbb{N}$, in such a way that

$$|q_i(t)| < r \text{ for } t \in (\alpha_i, -L_r^-) \text{ and } |q_i(t) - \xi| < r \text{ for } t \in (L_r^+, \omega_i),$$

for all $i \ge i_r$. The proof of this assertion can be obtained by rephrasing the argument used to prove (3.37) in Proposition 3.3, and we omit it. □

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