Measure rigidity of synthetic lower Ricci curvature bound on Riemannian manifolds

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Abstract

In this paper we investigate Lott-Sturm-Villani's synthetic lower Ricci curvature bound on Riemannian manifolds with boundary. We prove several measure rigidity results for some important functional and geometric inequalities, which completely characterize $\mathrm{CD}(K,\infty)$ condition and non-collapsed $\mathrm{CD}(K,N)$ condition on Riemannian manifolds with boundary. In particular, using L^1-optimal transportation theory, we prove that $\mathrm{CD}(K,\infty)$ condition implies geodesical convexity.

Keywords: measure rigidity, curvature-dimension condition, metric measure space, Riemannian manifold, boundary, Bakry-Émery theory, optimal transport.

Contents

1 Introduction

The synthetic theory of spaces with lower Ricci curvature bounds, initiated by Lott-Villani [21] and Sturm [26, 27], has remarkable developments in recent years. We refer the reader to the survey [2] for an overview of this topic and bibliography.

Many important results, previously known on Riemannian manifolds with lower Ricci curvature bound, now have their generalized versions in synthetic setting. However, we still do not fully understand synthetic lower Ricci curvature bound on

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Riemannian manifolds. In this paper we return to the starting point of this theory, and answer this question using new tools and results deveploed in recent years.

Firstly, we introduce some backgrouds and explain the motivations in more detail. Let (M, g) be a n-dimensional Riemannian manifold, and $\mathfrak{m} := e^{-V}\mathrm{Vol}_{\mathrm{g}}$ be a weighted measure for some smooth function V. The diffusion operator associated with the smooth metric measure space $(M, \mathrm{g}, \mathfrak{m})$ is $\mathrm{L} = \Delta - \nabla V$, and the well-known Bakry-Émery's Γ_2 operator is defined by

$$\Gamma_2(f) := \frac{1}{2}\mathrm{L}\Gamma(f,f) - \Gamma(f, \mathrm{L}f), \qquad \Gamma(f,f) := \frac{1}{2}\mathrm{L}(f^2) - f\mathrm{L}f.$$

It is known that $\Gamma(\cdot,\cdot) = \mathrm{g}(\nabla\cdot, \nabla\cdot)$, and we have the following generalized Bochner's formula

$$\Gamma_2(f) = \mathrm{Ricci}(\nabla f, \nabla f) + \mathrm{Hess}_V(\nabla f, \nabla f) + \|\mathrm{Hess}_f\|^2_{\mathrm{HS}} \tag{1.1}$$

for any $f \in C^\infty(M)$, where $\mathrm{Hess}_V = \mathrm{D}^2 V$ is the Hessian of V and $\|\mathrm{Hess}_f\|_{\mathrm{HS}}$ is the Hilbert-Schmidt norm of Hess_f. We say that $(M, \mathrm{g}, \mathfrak{m})$ satisfies (K, N)-Bakry-Émery's condition (or $\mathrm{BE}(K, N)$ condition for short), if the generalized Bochner inequality

$$\Gamma_2(f) \geq K\Gamma(f) + \frac{1}{N}(\mathrm{L}f)^2, \quad \forall f \in C^\infty(M) \tag{1.2}$$

holds. It is known that $\mathrm{BE}(K, N)$ condition yields many important geometric and analytic results on M. For example, when $n = \infty$, we have the following equivalent characterizations, which are also regarded as generalized lower Ricci curvature bound (c.f. [29]).

0) Modified Ricci tensor bound:

$$\mathrm{Ricci}_V(\nabla f, \nabla f) := \mathrm{Ricci}(\nabla f, \nabla f) + \mathrm{Hess}_V(\nabla f, \nabla f) \geq K|\nabla f|^2$$

for all $f \in C^\infty(M)$.

1) $\mathrm{BE}(K, \infty)$ condition: $\Gamma_2(f) \geq K\Gamma(f)$ for all $f \in C^\infty(M)$.

2) $\mathrm{CD}(K, \infty)$ condition: K-displacement convexity of the functional $\mathrm{Ent}(\,\cdot\,|\mathfrak{m})$ on L^2-Wasserstein space $\mathcal{W}_2(M) = (\mathcal{P}_2(M), W_2)$. That means, for any $\mu_0, \mu_1 \in \mathcal{P}_2(M)$ with $\mu_0, \mu_1 \ll \mathfrak{m}$, there is a L^2-Wasserstein geodesic $(\mu_t)_{t\in[0,1]}$ such that

$$\frac{K}{2}t(1-t)W_2^2(\mu_0, \mu_1) + \mathrm{Ent}(\mu_t|\mathfrak{m}) \leq t\mathrm{Ent}(\mu_1|\mathfrak{m}) + (1-t)\mathrm{Ent}(\mu_0|\mathfrak{m}) \tag{1.3}$$

where $\mathrm{Ent}(\mu_t|\mathfrak{m})$ is defined as $\int \ln \rho_t \, \mathrm{d}\mu_t$ for $\mu_t = \rho_t\, \mathfrak{m}$.

3) Gradient estimate of heat semi-group:

$$|\nabla \mathrm{H}_t(f)|^2 \leq e^{-2Kt}\mathrm{H}_t(|\nabla f|^2) \tag{1.4}$$

for any $f \in W^{1,2}(M, \mathfrak{m})$, where $(\mathrm{H}_t)_{t>0}$ is the semi-group generated by the diffusion operator L.

Let $(\overline{\Omega}, \mathrm{d}_\Omega, e^{-V}\mathrm{Vol}_\mathrm{g})$ be a smooth metric measure space with smooth boundary, where d_Ω is (the completion of) the intrinsic distance on a domain $\Omega \subset M$ induced by the Riemannian distance. One would ask the following questions.

Q-1 What is Bakry-Émery's Γ-calculus on $(\overline{\Omega}, \mathrm{d}_\Omega, e^{-V}\mathrm{Vol}_\mathrm{g})$, and what does $\Gamma_2 \geq K\Gamma$ mean in this case?

Q-2 What does $\mathrm{CD}(K,\infty)$ condition (1.3) imply? Can we say that Ω is geodesically convex?

Q-3 What does gradient estimate (1.4) of (Neumann) heat semi-group imply?

Firstly, in Section 2 we study the Bakry-Émery's Γ-calculus on smooth metric measure space with smooth boundary. Using the vocabularies and results devepoled in [25] and [16], we define a measure-valued Ricci tensor $\mathbf{Ricci}_\Omega$ by

$$\mathbf{Ricci}_\Omega(\cdot,\cdot) = \mathrm{Ricci}_V(\cdot,\cdot)\, e^{-V}\mathrm{dVol}_\mathrm{g} + II(\cdot,\cdot)\, e^{-V}\mathrm{d}\mathcal{H}^{n-1}|_{\partial\Omega} \tag{1.5}$$

where $\mathrm{Ricci}_V = \mathrm{Ricci} + \mathrm{Hess}_V$ is Bakry-Émery's modified Ricci tensor and II is the second fundamental form. Combining with the results in [6,7] and [16], we can see that the measure-valued Bochner inequality $\mathbf{Ricci}_\Omega \geq K$ is equivalent to non-smooth $\mathrm{BE}(K,\infty)$ condition and Lott-Sturm-Villani's $\mathrm{CD}(K,\infty)$ condition. More precisely, we have the following theorem.

Theorem 1.1 (Measure-valued Ricci tensor, Theorem 2.4 and Corollary 2.5)**.** *Let $(M, \mathrm{g}, e^{-V}\mathrm{Vol}_\mathrm{g})$ be a n-dimensional weighted Riemannian manifold and $\Omega \subset M$ be a domain with $(n-1)$-dimensional smooth orientable boundary. Then Gigli's measure-valued Ricci tensor (c.f. [16]) on $(\overline{\Omega}, \mathrm{d}_\Omega, e^{-V}\mathrm{Vol}_\mathrm{g})$ is given by*

$$\mathbf{Ricci}_\Omega(\nabla g, \nabla g) = \mathrm{Ricci}_V(\nabla g, \nabla g)\, e^{-V}\mathrm{dVol}_\mathrm{g} + II(\nabla g, \nabla g)\, e^{-V}\mathrm{d}\mathcal{H}^{n-1}|_{\partial\Omega} \tag{1.6}$$

for any $g \in C_c^\infty$ with $\mathrm{g}(\mathrm{N}, \nabla g) = 0$, where N is the outward normal vector field on $\partial\Omega$.

Furthermore, $(\overline{\Omega}, \mathrm{d}_\Omega, e^{-V}\mathrm{Vol}_\mathrm{g})$ is a $\mathrm{CD}(K,\infty)$ space if and only if $\mathrm{Ricci}_V \geq K$ and $II \geq 0$.

On the other side, from [5,6] we know that $(\overline{\Omega}, \mathrm{d}_\Omega, e^{-V}\mathrm{Vol}_\mathrm{g})$ is $\mathrm{CD}(K,\infty)$ if and only if the gradient estimate (1.4) holds. From a result of F.-Y. Wang (c.f. Chapter 3, [30]), we know the gradient estimate (1.4) holds if and only if $\mathrm{Ricci} \geq K$ and $II \geq 0$. Thus we completely answer the questions **Q-1**, **Q-2** and **Q-3**.

In the discussions above, we assume that $\partial\Omega$ is smooth and there is no mass on the boundary, none of them is necessary in metric measure setting. More precisely, given a domain Ω and a Borel measure $\mathfrak{m}$ with $\mathrm{supp}\,\mathfrak{m} = \overline{\Omega}$. Let d_Ω be the intrinsic distance induced by g on $\overline{\Omega}$. The K-displacement convexity of $\mathrm{Ent}(\ \cdot\ |\mathfrak{m})$ on L^2-Wasserstein space $\mathcal{W}_2(\overline{\Omega}, \mathrm{d}_\Omega)$ is always well-posed. It is exactly the definition of Lott-Sturm-Villani's synthetic lower Ricci curvature bound (i.e. $\mathrm{CD}(K,\infty)$ conditions).

More generally, we have the following notion of $\mathrm{CD}(K,N)$ condition.

Definition 1.2 (Definition 1.3 [27]). Let $K \in \mathbb{R}$ and $N \in [1, \infty]$. We say that $(\overline{\Omega}, \mathrm{d}_\Omega, \mathfrak{m})$ is a $\mathrm{CD}(K, N)$ space if for **any** pair $\mu_0, \mu_1 \in \mathcal{P}_2(\overline{\Omega})$ with $\mu_0, \mu_1 \ll \mathfrak{m}$, there **exists** a L^2-Wasserstein geodesic $(\mu_t)_{t\in[0,1]}$ connecting μ_0, μ_1 such that

$$\mathrm{S}_N(\mu_t|\mathfrak{m}) \le -\int \left[\tau_{K,N}^{(t)}\big(\mathrm{d}_\Omega(\gamma_0,\gamma_1)\big)\rho_1^{-\frac{1}{N}}(\gamma_1) + \tau_{K,N}^{(1-t)}\big(\mathrm{d}_\Omega(\gamma_0,\gamma_1)\big)\rho_0^{-\frac{1}{N}}(\gamma_0)\right] \mathrm{d}\Pi(\gamma) \tag{1.7}$$

for all $t \in [0, 1]$ and some distortion coefficients $\tau_{K,N}^{(t)}$, where $\Pi \in \mathcal{P}(\mathrm{Geo}(\Omega, \mathrm{d}_\Omega))$ is a lifting of (μ_t) satisfying $(e_t)_\sharp \Pi = \mu_t$, $\mathrm{S}_N(\mu_t|\mathfrak{m}) = -\int \rho_t^{-\frac{1}{N}} \,\mathrm{d}\mu_t$ and $\mu_t = \rho_t\, \mathfrak{m}$.

In addition, the heat semi-group on $(\overline{\Omega}, \mathrm{d}_\Omega, \mathfrak{m})$ is defined as the L^2-gradient flow of the energy form $\mathcal{E}(f) : W^{1,2}(\overline{\Omega}, \mathfrak{m}) \ni f \mapsto \int |\nabla f|^2 \,\mathrm{d}\mathfrak{m}$. So gradient estimate (1.4) is also well-posed for general Ω and $\mathfrak{m}$.

In [15], Gigli introduces *infinitesimally Hilbertian spaces*, whose Sobolev spaces (in metric measure sense) are Hilbert. In [6, 7], Ambrosio-Gigli-Savaré prove that a metric measure space is infinitesimally Hilbertian $\mathrm{CD}(K, \infty)$ space (i.e. $\mathrm{RCD}(K, \infty)$ space) if and only if the gradient estimate of gradient flows.

Therefore we have the following natural questions.

Q-4 If the boundary of Ω is not smooth or $\mathfrak{m}|_{\partial\Omega} \neq 0$, such that $(\overline{\Omega}, \mathrm{d}_\Omega, \mathfrak{m})$ is $\mathrm{CD}(K, \infty)$, can we say that Ω is geodesically convex as in Theorem 1.1?

Q-5 If there exists a measure $\mathfrak{m}$ with full support such that $(\overline{\Omega}, \mathrm{d}_\Omega, \mathfrak{m})$ is $\mathrm{CD}(K, \infty)$ (or $\mathrm{CD}(K, N)$). Does $\mathfrak{m} \ll \mathrm{Vol}_\mathrm{g}$? Is $(\overline{\Omega}, \mathrm{d}_\Omega, \mathfrak{m})$ a $\mathrm{RCD}(K, \infty)$ space?

In Section 3 we answer these questions completely. It should be noticed that Riemannian manifolds with boundary are actually non-smooth metric measure spaces, even if the boundary is smooth, since the regularity of the geodesics can not be better than C^1 in general (c.f. [1]). In addition, in many problems related to regularity, (assumption on) convexity plays essential roles. Therefore it is difficult to solve this problem with classical analysis and PDE method. However, using optimal transportation theory as an effective tool, it will not be more difficult to study non-smooth boundaries than smooth ones.

Firstly we show that the absolute continuity of the reference measure and the regularity of its density. The following theorem improves the results proved by Cavalletti-Mondino [9] and Kell [19] in non-smooth framework. To the knowledge of the author, this is the first measure rigidity result without dimension bound.

Theorem 1.3 (Measure rigidity: absolute continuity and regularity, Proposition 3.1 3.3 3.4). *Let $(M, \mathrm{g}, \mathrm{Vol}_\mathrm{g})$ be a complete n-dimensional Riemannian manifold, $|\cdot|_\mathrm{g}$ be the Riemannian distance induced by* g *and $\mathfrak{m}$ be a Borel measure with full support on M. Then we have the following results.*

1) *Assume $(M, |\cdot|_\mathrm{g}, \mathfrak{m})$ satisfies $\mathrm{CD}(K, \infty)$ condition. Then there is a locally Lipschitz semi-convex functions V on M such that $\mathfrak{m} = e^{-V}\mathrm{Vol}_\mathrm{g}$.*

2) Assume $(M, |\cdot|_{\mathrm{g}}, \mathfrak{m})$ *satisfies* $\mathrm{MCP}(K, N)$ *for some* $K \in \mathbb{R}$ *and* $N < \infty$*. Then* $\mathfrak{m} = e^{-V}\mathrm{Vol}_{\mathrm{g}}$ *for some locally bounded function* V*.*

In the next theorem, we prove that there is no non-trivial measure other than volume measure on a n-dimensional Riemannian manifold, such that the corresponding metric measure space satisfies $\mathrm{CD}(k, n)$. On a weighted Riemannian manifold with C^2 weight, it is known that Bakry-Émery condition $\mathrm{BE}(k, n)$ holds if and only if the weight is a constant. However in general case this is still an open problem. Recently De Philippis-Gigli [12] propose two definitions of non-collapsed metric measure space. We say that a $\mathrm{CD}(K, N)$ space $(X, \mathrm{d}, \mathfrak{m})$ is weakly non-collapsed if $\mathfrak{m} \ll \mathcal{H}^N$, and we call it (strongly) non-collapsed if $\mathfrak{m} = c\mathcal{H}^N$ for some constant c. In case X is a Riemannian manifold, the following Theorem tells us that non-collapsing and weakly non-collapsing are equivalent.

Theorem 1.4 (Measure Rigidity: non-collapsed spaces, Theorem 3.5)**.** *Let* (M, g) *be a* n*-dimensional Riemannian manifold. Assume there exists a measure* $\mathfrak{m}$ *with full support such that* $(M, |\cdot|_{\mathrm{g}}, \mathfrak{m})$ *is* $\mathrm{CD}(k, n)$*. Then* $\mathfrak{m} = c\mathrm{Vol}_{\mathrm{g}}$ *for some constant* $c > 0$*.*

In the last theorem, we study dimension-free $\mathrm{CD}(K, \infty)$ condition on Lipschitz domain. We prove that the reference measure must support on a convex set, and answer the question why there is no mass on the boundary. In particular, we have fully understood the curvature-dimension condition on smooth metric measure space with boundary (c.f. Theorem 2.4).

In this problem, we assume neither infinitesimally Hilbertian nor non-branching property, which are often used in the study of related problems. So we do not know whether the L^2-Wasserstein geodeisc is unique or not (i.e. Brenier's theorem). In the proof, we make full use of L^1-optimal transport theory and its connection with L^2-optimal transport, which is developed by Klartag [20] and Cavalletti-Mondino (c.f. [10]).

Theorem 1.5 (Measure rigidity: $\mathrm{CD}(K, \infty)$ condition, Theorem 3.6)**.** *Let* (M, g) *be a complete Riemannian manifold,* $\Omega \subset M$ *be a Lipschitz domain. Let* d_Ω *be the intrinsic distance induced by Riemannian distance* $|\cdot|_{\mathrm{g}}$ *on* $\overline{\Omega}$*, and* $\mathfrak{m}$ *be a reference measure with* $\mathrm{supp}\,\mathfrak{m} = \overline{\Omega}$*. Assume that* $(\overline{\Omega}, \mathrm{d}_\Omega, \mathfrak{m})$ *satisfies* $\mathrm{CD}(K, \infty)$ *condition, then we have the following rigidity results.*

1) $\overline{\Omega}$ *is* g*-geodesically convex, that is, any shortest path in* $(\overline{\Omega}, \mathrm{d}_\Omega)$ *is a geodesic segment in* (M, g)*;*

2) $\mathfrak{m}|_{\partial\Omega} = 0$ *and* $\mathfrak{m} = e^{-V}\mathrm{Vol}_{\mathrm{g}}$ *for some semi-convex, locally Lipschitz function* V*;*

3) $(\overline{\Omega}, \mathrm{d}_\Omega, \mathfrak{m})$ *is a* $\mathrm{RCD}(K, \infty)$ *space.*

In particular, $(\overline{\Omega}, \mathrm{d}_\Omega, \mathrm{Vol}_{\mathrm{g}})$ *is* $\mathrm{CD}(K, \infty)$ *if and only if* $\overline{\Omega}$ *is* g*-geodesically convex and* $\mathrm{Ricci} \geq K$ *on* Ω*.*

At last, we remark that most of the measure rigidity results obtained in this paper are still true on Alexandrov spaces with bounded curvature, which can be proved in similar ways. Compared with previous works studying curvature-dimension conditions with finite dimension, we have used some new methods to solve the infinite dimensional problems. We believe that these methods have potential applications on more general metric measure spaces.

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2 Smooth metric measure spaces with boundary

Let $M := (X, \mathrm{d}, \mathfrak{m})$ be an abstract metric measure space. We say that $f \in L^2(X, \mathfrak{m})$ is a Sobolev function in $W^{1,2}(M)$ if there exists a sequence of Lipschitz functions $(f_n)_n \subset L^2$, such that $f_n \to f$ and $\mathrm{lip}(f_n) \to G$ in L^2 for some G, where $\mathrm{lip}(f)$ is the local Lipschitz constant of f. It is known that there exists a minimal function in $\mathfrak{m}$-a.e. sense, denoted it by $|\mathrm{D}f|$, called minimal weak upper gradient. If (X, d) is a Riemannian manifold and $\mathfrak{m}$ is volume measure, we know that $|\mathrm{D}f| = |\nabla f| = \mathrm{lip}(f)$ for any $f \in C^\infty$ (c.f. Theorem 6.1 [11]). Furthermore, let $\Omega \subset X$ be a domain such that $\mathfrak{m}(\partial\Omega) = 0$, by locality we have $|\mathrm{D}f|_{\overline{\Omega}} = |\nabla f|$ $\mathfrak{m}$-a.e..

We equip $W^{1,2}(M)$ with the norm

$$\|f\|_{W^{1,2}(X,\mathrm{d},\mathfrak{m})} := \sqrt{\|f\|^2_{L^2(X,\mathfrak{m})} + \||\mathrm{D}f|\|^2_{L^2(X,\mathfrak{m})}}.$$

It is known that $W^{1,2}(M)$ is a Banach space, but not necessarily a Hilbert space. We say that $(X, \mathrm{d}, \mathfrak{m})$ is an **infinitesimally Hilbertian space** if $W^{1,2}$ is a Hilbert space. Obviously, Riemannian manifolds equipped with volume measure are infinitesimally Hilbertian spaces. In general, infinitesimal Hilbertianity is not trivial even if the base space is a Riemannian manifold.

On an infinitesimally Hilbertian space, we have a pointwise bilinear map defined by

$$[W^{1,2}]^2 \ni (f, g) \mapsto \langle \nabla f, \nabla g \rangle := \frac{1}{4}\Big(|\mathrm{D}(f+g)|^2 - |\mathrm{D}(f-g)|^2\Big).$$

It can be seen that $\langle \cdot, \cdot \rangle = \mathrm{g}(\cdot, \cdot)$ on a Riemannian manifold (M, g).

Then we can define the measure-valued Laplacian by integration by part.

Definition 2.1 (Measure-valued Laplacian, [15, 16])**.** The space $\mathrm{D}(\mathbf{\Delta}) \subset W^{1,2}(M)$ is the space of $f \in W^{1,2}(M)$ such that there is a measure μ satisfying

$$\int h \,\mathrm{d}\mu = -\int \langle \nabla h, \nabla f \rangle \,\mathrm{d}\mathfrak{m}, \quad \forall h \text{ Lipschitz with bounded support.}$$

In this case the measure μ is unique and we shall denote it by $\mathbf{\Delta} f$. If $\mathbf{\Delta} f \ll m$, we denote its density by Δf.

The following proposition links the curvature-dimension condition $\mathrm{RCD}(K,\infty)$ and non-smooth Bakry-Émery theory. We say that a space is $\mathrm{RCD}(K,\infty)$ if it is a $\mathrm{CD}(K,\infty)$ space defined by Lott-Sturm-Villani in [21, 26, 27], equipped with an infinitesimally Hilbertian Sobolev space. For more details, see [6] (also [4]).

We define $\mathrm{TestF}(M) \subset W^{1,2}(M)$, the set of test functions by

$$\mathrm{TestF}(M) := \Big\{ f \in \mathrm{D}(\mathbf{\Delta}) \cap L^\infty : |\mathrm{D}f| \in L^\infty \quad \text{and} \quad \Delta f \in W^{1,2}(M) \cap L^\infty(M) \Big\}.$$

It is known (c.f. [25]) that $\mathrm{TestF}(M)$ is dense in $W^{1,2}(M)$ when M is $\mathrm{RCD}(K,\infty)$.

Let $f, g \in \mathrm{TestF}(M)$. It is known from [25] that measure $\mathbf{\Gamma}_2(f,g)$ is well-defined by

$$\mathbf{\Gamma}_2(f,g) = \frac{1}{2}\mathbf{\Delta}\langle \nabla f, \nabla g\rangle - \frac{1}{2}\big(\langle \nabla f, \nabla \Delta g\rangle + \langle \nabla g, \nabla \Delta f\rangle\big)\,\mathfrak{m},$$

and we put $\mathbf{\Gamma}_2(f) := \mathbf{\Gamma}_2(f,f)$. Then we have the following Bochner inequality on metric measure spaces.

Proposition 2.2 (Bakry-Émery condition, [6, 7], [13]). *Let $M = (X, \mathrm{d}, \mathfrak{m})$ be a $\mathrm{RCD}(K,\infty)$ space with $K \in \mathbb{R}$. Then*

$$\mathbf{\Gamma}_2(f) \geq K|\mathrm{D}f|^2\,\mathfrak{m}$$

for any $f \in \mathrm{TestF}(M)$.

Let $f \in \mathrm{TestF}(M)$. We define the Hessian $\mathrm{Hess}_f : \big\{\nabla g : g \in \mathrm{TestF}(M)\big\}^2 \mapsto L^0(M)$ by

$$2\mathrm{Hess}_f(\nabla g, \nabla h) = \langle \nabla g, \nabla\langle \nabla f, \nabla h\rangle\rangle + \langle \nabla h, \nabla\langle \nabla f, \nabla g\rangle\rangle - \langle \nabla f, \nabla\langle \nabla g, \nabla h\rangle\rangle$$

for any $g, h \in \mathrm{TestF}(M)$. It can be proved (see Theorem 3.4 [25] and Theorem 3.3.8 [16]) that Hess_f can be extended to a symmetric $L^\infty(M)$-bilinear map on $L^2(TM)$ and continuous with values in $L^0(M)$. On Riemannian manifolds, Hess_f coincides with the usual Hessian $\mathrm{D}^2 f$.

Furthermore, we have the following proposition, dues to Gigli [16].

Proposition 2.3 (Theorem 3.6.7 [16]). *Let M be a $\mathrm{RCD}(K,\infty)$ space. Then*

$$\mathbf{Ricci}(\nabla f, \nabla f) \geq K|\mathrm{D}f|^2\,\mathfrak{m}$$

for any $f \in \mathrm{TestF}(M)$, where the measure-valued Ricci tensor **Ricci** *is defined by*

$$\mathbf{Ricci}(\nabla f, \nabla f) := \mathbf{\Gamma}_2(f) - \|\mathrm{Hess}_f\|^2_{\mathrm{HS}}\,\mathfrak{m}.$$

Now we introduce our first theorem.

Theorem 2.4 (Measure-valued Ricci tensor). *Let $(M, \mathrm{g}, e^{-V}\mathrm{Vol}_\mathrm{g})$ be a n-dimensional weighted Riemannian manifold and $\Omega \subset M$ be a domain with $(n-1)$-dimensional smooth orientable boundary. Then the measure valued Ricci tensor on $(\overline{\Omega}, \mathrm{d}_\Omega, e^{-V}\mathrm{Vol}_\mathrm{g})$ is given by*

$$\mathbf{Ricci}_\Omega(\nabla g, \nabla g) = \mathrm{Ricci}_V(\nabla g, \nabla g)\,e^{-V}\mathrm{dVol_g}|_\Omega + II(\nabla g, \nabla g)\,e^{-V}\mathrm{d}\mathcal{H}^{n-1}|_{\partial\Omega} \tag{2.1}$$

for any $g \in C_c^\infty$ with $\mathrm{g}(\mathrm{N}, \nabla g) = 0$, where N is the outward normal vector field on $\partial\Omega$, and Ricci_V is Bakry-Émery's generalized Ricci tensor.

Proof. From integration by part formula (Green's formula) on Riemannian manifold, we know

$$\int \mathrm{g}(\nabla f, \nabla g)\, e^{-V} \mathrm{dVol}_{\mathrm{g}} = -\int f \Delta_V g\, e^{-V} \mathrm{dVol}_{\mathrm{g}} + \int_{\partial\Omega} f \mathrm{g}(\mathrm{N}, \nabla g)\, e^{-V} \mathrm{d}\mathcal{H}^{n-1}|_{\partial\Omega}$$

for any $f, g \in C_c^\infty$, where $\Delta_V := (\Delta - \nabla V)$ and $\mathcal{H}^{n-1}|_{\partial\Omega}$ is the $(n-1)$-dimensional Hausdorff measure on $\partial\Omega$.

Hence $g \in \mathrm{D}(\mathbf{\Delta}_\Omega)$ and the measure-valued Laplacian is given by the following formula

$$\mathbf{\Delta}_\Omega g = \Delta_V g\, e^{-V} \mathrm{Vol}_{\mathrm{g}}|_\Omega - \mathrm{g}(\mathrm{N}, \nabla g)\, e^{-V} \mathcal{H}^{n-1}|_{\partial\Omega}.$$

Therefore for any $g \in C_c^\infty$ with $\mathrm{g}(\mathrm{N}, \nabla g) = 0$ on $\partial\Omega$, we have $g \in \mathrm{TestF}(\Omega)$.

Now we can compute the measure-valued Ricci tensor. Let $g \in C_c^\infty$ be a test function with $\mathrm{g}(\mathrm{N}, \nabla g) = 0$ on $\partial\Omega$. We have

$$\begin{aligned}
\mathbf{Ricci}_\Omega(\nabla g, \nabla g) &= \frac{1}{2}\mathbf{\Delta}_\Omega |\mathrm{D}g|_\Omega^2 - \langle \nabla g, \nabla \Delta_\Omega g\rangle_\Omega\, e^{-V}\mathrm{Vol}_{\mathrm{g}} - \|\mathrm{Hess}_g\|_{\mathrm{HS}}^2\, e^{-V}\mathrm{Vol}_{\mathrm{g}} \\
&= \frac{1}{2}\Delta_V |\nabla g|^2\, e^{-V}\mathrm{Vol}_{\mathrm{g}} - \mathrm{g}(\nabla g, \nabla \Delta_V g)\, e^{-V}\mathrm{Vol}_{\mathrm{g}} - \|\mathrm{Hess}_g\|_{\mathrm{HS}}^2\, e^{-V}\mathrm{Vol}_{\mathrm{g}} \\
&\quad -\frac{1}{2}\mathrm{g}(\mathrm{N}, \nabla |\nabla g|^2)\, e^{-V}\mathcal{H}^{n-1}|_{\partial\Omega} \\
&= \mathrm{Ricci}(\nabla g, \nabla g)\, e^{-V}\mathrm{Vol}_{\mathrm{g}} + \mathrm{Hess}_V(\nabla g, \nabla g)\, e^{-V}\mathrm{Vol}_{\mathrm{g}} \\
&\qquad\qquad -\frac{1}{2}\mathrm{g}(\mathrm{N}, \nabla |\nabla g|^2)\, e^{-V}\mathcal{H}^{n-1}|_{\partial\Omega} \\
&= \mathrm{Ricci}_V(\nabla g, \nabla g)\, e^{-V}\mathrm{Vol}_{\mathrm{g}} - \frac{1}{2}\mathrm{g}(\mathrm{N}, \nabla |\nabla g|^2)\, e^{-V}\mathcal{H}^{n-1}|_{\partial\Omega},
\end{aligned}$$

where we use Bochner's formula at the third equality, and $\mathrm{Ricci}_V = \mathrm{Ricci} + \mathrm{Hess}_V$ is Bakry-Émery's generalized Ricci tensor on weighted Riemannian manifold with weight e^{-V}.

By definition of second fundamental form, we have

$$II(\nabla g, \nabla g) = \mathrm{g}(\nabla_{\nabla g}\mathrm{N}, \nabla g) = \mathrm{g}\big(\nabla \mathrm{g}(\mathrm{N}, \nabla g), \nabla g\big) - \frac{1}{2}\mathrm{g}(\mathrm{N}, \nabla |\nabla g|^2).$$

Recall that $\mathrm{g}(N, \nabla g) = 0$ on $\partial\Omega$, we have $\mathrm{g}(\nabla_{\nabla g}\mathrm{N}, \nabla g) = -\frac{1}{2}\mathrm{g}(\mathrm{N}, \nabla |\nabla g|^2)$.

Finally, we obtain

$$\mathbf{Ricci}_\Omega(\nabla g, \nabla g) = \mathrm{Ricci}_V(\nabla g, \nabla g)\, \mathrm{Vol}_{\mathrm{g}} + II(\nabla g, \nabla g) e^{-V}\, \mathcal{H}_{n-1}|_{\partial\Omega} \tag{2.2}$$

for any $g \in C_c^\infty$ with $\mathrm{g}(N, \nabla g) = 0$. □

At the last, we list some simple applications of Theorem 2.4 without proof. It can be checked that $\{g : g \in C_c^\infty, \mathrm{g}(\mathrm{N}, \nabla g) = 0\} \subset \mathrm{TestF}(\Omega)$. So we have the following corollaries concerning Ricci curvature and the mean curvature.

Corollary 2.5 (Rigidity: convexity of the boundary). *Let* $(\overline{\Omega}, \mathrm{d}_\Omega, e^{-V}\mathrm{Vol}_{\mathrm{g}})$ *be a space as stated in Theorem 2.4. Then it is* $\mathrm{RCD}(K, \infty)$ *if and only if* $\partial\Omega$ *is convex and* $\mathrm{Ricci}_V \geq K$ *on* Ω.

The next result tells us that the boundary does not influence the dimension bound of the smooth metric measure space.

Corollary 2.6. *A n-dimensional Riemannian manifold with boundary is* $\mathrm{RCD}(K,\infty)$ *if and only if it is* $\mathrm{RCD}(K,n)$.

The last corollary characterizes metric measure spaces with upper Ricci curvature bound, see [14] for another kind of rigidity result concerning upper Ricci bound.

Corollary 2.7. *If* $\mathbf{Ricci}_\Omega \ll \mathrm{Vol}_\mathrm{g}$*, then* $\partial\Omega$ *is a minimal (hyper)surface.*

3 Main results: measure rigidity theorems

At first, we recall some important properties of $\mathrm{CD}(K,\infty)$, $\mathrm{CD}(K,N)$ and $\mathrm{RCD}(K,\infty)$ spaces. To readers who are not familiar with Lott-Sturm-Villani's synthetic lower Ricci curvature bound, these properties can be regarded as alternative definitions.

1) [Density bound of intermediate measures on $\mathrm{CD}(K,\infty)$ spaces, Lemma 3.1 [23].] Let $\mu_0,\mu_1 \in \mathcal{P}(X)$ be a pair of measures with bounded densities and so that $W_2(\mu_0,\mu_1)<\infty$. Suppose also that $\mathrm{diam}(\operatorname{supp}\mu_0 \cup \operatorname{supp}\mu_1)<\infty$. Then there exists a L^2-Wasserstein geodesic (μ_t) connecting μ_0 and μ_1 such that the densities $\frac{\mathrm{d}\mu_t}{\mathrm{d}\mathfrak{m}}$ are uniformly bounded.

2) [Generalized Brunn–Minkowski inequality on $\mathrm{CD}(K,N)$ spaces, Proposition 2.1 [27].] Given $K,N\in\mathbb{R}$, with $N\geq 1$, we set for $(t,\theta)\in[0,1]\times\mathbb{R}^+$,

$$\tau_{K,N}^{(t)}(\theta) := \begin{cases} \infty, & \text{if } K\theta^2 \geq (N-1)\pi^2, \\ t^{\frac{1}{N}}\left(\frac{\sin(t\theta\sqrt{K/(N-1)})}{\sin(\theta\sqrt{K/(N-1)})}\right)^{1-\frac{1}{N}}, & \text{if } 0<K\theta^2<(N-1)\pi^2, \\ t, & \text{if } K\theta^2=0 \text{ or if } K\theta^2<0 \text{ and } N=0, \\ t^{\frac{1}{N}}\left(\frac{\sinh(t\theta\sqrt{-K/(N-1)})}{\sinh(\theta\sqrt{-K/(N-1)})}\right)^{1-\frac{1}{N}}, & \text{if } K\theta^2<0 \text{ and } N>1. \end{cases}$$

Then for any measurable sets $A_0,A_1\subset X$ with $\mathfrak{m}(A_0)+\mathfrak{m}(A_1)>0$, $t\in[0,1]$ and $N'\geq N$, we have

$$\mathfrak{m}\big(A_t\big)^{\frac{1}{N'}} \geq \tau_{K,N'}^{(1-t)}\big(\Theta\big)\mathfrak{m}\big(A_0\big)^{\frac{1}{N'}} + \tau_{K,N'}^{(t)}\big(\Theta\big)\mathfrak{m}\big(A_1\big)^{\frac{1}{N'}}, \tag{3.1}$$

where A_t denotes the set of points which divide geodesics starting in A_0 and ending in A_1 with ratio $\frac{t}{1-t}$ and where Θ denotes the minimal ($K\geq 0$) or maximal ($K<0$) length of such geodesics, In particular, when A_0 is a single point, we have the following (K,N)-measure contraction property (or $\mathrm{MCP}(K,N)$ condition for short):

$$\mathfrak{m}\big(A_t\big) \geq \left[\tau_{K,N}^{(t)}\big(\Theta\big)\right]^N \mathfrak{m}\big(A_1\big). \tag{3.2}$$

3) [Riemannian-Curvature-Dimension condtion (RCD condition), [6, 15]] We say that a space is $\mathrm{RCD}(K,\infty)$ (or $\mathrm{RCD}(K,N)$) if it is a infinitesimally Hilbertian $\mathrm{CD}(K,\infty)$ (or $\mathrm{CD}(K,N)$ respectively) space. It is known that Riemannian manifolds with lower Ricci curvature bound, Riemannian limit spaces and Alexandrov spaces are RCD spaces.

   It is proved in [24] that $\mathrm{RCD}(K,\infty)$ condition implies essentially non-branching property. For any $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ which are absolutely continuous with respect to $\mathfrak{m}$, there exists $\Pi \in \mathcal{P}(\mathrm{Geo}(\mathrm{X}, \mathrm{d}))$ which is the lifting of a geodesic connecting μ_0, μ_1 (c.f. Theorem 2.10 [3]). Then Π is concentrated on a set of non-branching geodesics. We say that a set $\Gamma \subset \mathrm{Geo}(\mathrm{X}, \mathrm{d})$ is non-branching if for any $\gamma^1, \gamma^2 \in \Gamma$, it holds:

$$\exists t \in (0,1) \ \text{ s.t. } \ \forall s \in [0,t] \ \gamma^1_t = \gamma^2_t \Rightarrow \gamma^1_s = \gamma^2_s, \ \forall s \in [0,1].$$

   Furthermore, there exits a unique L^2-Wasserstein geodesic connecting μ_0, μ_1 (c.f. [17]) which is induced by an optimal transport map.

Let $(X, \mathrm{d}, \mathfrak{m}_1)$ and $(X, \mathrm{d}, \mathfrak{m}_2)$ be two metric measure spaces satisfying essentially non-branching MCP condition. In [9], Cavalletti and Mondino (c.f. Proposition 8.1 and Corollary 8.2) prove the mutual absolute continuity of the reference measures $\mathfrak{m}_1, \mathfrak{m}_2$ (see [19] for another proof by Kell). In case X is a Riemannian manifold, we extend such result to $\mathrm{CD}(K,\infty)$ condition and improve their result by proving a quantitative density estimate under $\mathrm{CD}(K,N)$ condition.

Proposition 3.1 (Measure rigidity: absolute continuity). *Let $(M, \mathrm{g}, \mathrm{Vol}_\mathrm{g})$ be a complete n-dimensional Riemannian manifold, $|\cdot|_\mathrm{g}$ be the Riemannian distance induced by g and $\mathfrak{m}$ be a Borel measure with full support on M. Then we have the following results.*

1) *Assume that $(M, |\cdot|_\mathrm{g}, \mathfrak{m})$ satisfies $\mathrm{CD}(K,\infty)$ condition for some $K \in \mathbb{R}$. Then $\mathfrak{m} \ll \mathrm{Vol}_\mathrm{g}$.*

2) *Assume that $(M, |\cdot|_\mathrm{g}, \mathfrak{m})$ satisfies $\mathrm{MCP}(K,N)$ for some $K \in \mathbb{R}$, $N < \infty$. Then we have*

$$\frac{\mathfrak{m}\big(B_r(x)\big)}{\mathrm{Vol}_\mathrm{g}\big(B_r(x)\big)} \in L^\infty_{\mathrm{loc}} \quad \textit{uniformly in } r > 0.$$

   In particular, $\mathfrak{m} = e^{-V}\,\mathrm{Vol}_\mathrm{g}$ for some $e^{-V} \in L^\infty_{\mathrm{loc}}$.

3) *Assume that $(M, |\cdot|_\mathrm{g}, \mathfrak{m})$ satisfies $\mathrm{MCP}(K,N)$ for some $K \in \mathbb{R}$, $N < \infty$. Then we have*

$$\frac{r^{N-n}\mathrm{Vol}_\mathrm{g}\big(B_r(x)\big)}{\mathfrak{m}\big(B_r(x)\big)} \in L^\infty_{\mathrm{loc}} \quad \textit{uniformly in } r > 0.$$

   In particular, when $N = n$, we know $\mathfrak{m} = e^{-V}\,\mathrm{Vol}_\mathrm{g}$ for some $V \in L^\infty_{\mathrm{loc}}$.

Proof. **Part 1):**

Fix a bounded convex domain $U \subset M$. By definition, $(\overline{U}, |\cdot|_g, \mathfrak{m}|_U)$ satisfies $\mathrm{CD}(K,\infty)$ condition, where $|\cdot|_g$ is the Riemannian distance on M. Given a point $x \in U$ and a parameter $r > 0$, we define $\epsilon_r : U \mapsto \mathbb{R}^+$ by

$$\epsilon_r(x) := \frac{\mathfrak{m}\big(B_r(x)\big)}{\mathrm{Vol}_{\mathrm{g}}\big(B_r(x)\big)}. \tag{3.3}$$

Let $\mathfrak{m} := \mathfrak{m}_{\mathrm{ac}} + \mathfrak{m}_{\mathrm{s}}$ be the Lebesgue decomposition of $\mathfrak{m}$. We firstly show that $\mathfrak{m}_{\mathrm{ac}} \neq 0$. Assume by contradiction that $\mathfrak{m} = \mathfrak{m}_{\mathrm{s}}$, then for any constant $c > 0$, we have

$$\mathfrak{m}\big(\{x : \varliminf_{r\to 0} \epsilon_r(x) \leq c\}\big) = 0,$$

otherwise $\mathfrak{m}_{\mathrm{ac}} \neq 0$. In particular, we know

$$\varliminf_{r\to 0} \epsilon_r(y) = \infty \qquad \mathfrak{m} - \text{a.e. } y \in U.$$

Furthermore, by Fatou's lemma we have

$$\varlimsup_{r\to 0} \mathfrak{m}\big(\{x : \epsilon_r(x) < c\}\big) \leq \mathfrak{m}\big(\{x : \varliminf_{r\to 0} \epsilon_r(x) \leq c\}\big) = 0. \tag{3.4}$$

Fix a point $y_0 \in U$ with $\varliminf_{r\to 0} \epsilon_r(y_0) = \infty$. By Rauch's (and Toponogov's) comparison theorem, there exists a small $R > 0$ and constants $c_1, c_2 > 0$ such that

$$c_1 t|yz|_g < |\gamma_t^{xz}\gamma_t^{xy}|_{\mathrm{g}} < c_2 t|yz|_g \quad \forall t \in (0,1] \tag{3.5}$$

and

$$|\gamma_s^{xz}\gamma_t^{xy}|_{\mathrm{g}} > c_1 s \wedge t\Big(|yz|_{\mathrm{g}} - \big||xy|_{\mathrm{g}} - |xz|_{\mathrm{g}}\big|\Big) \quad \forall s,t \in (0,1)$$

for any $x, y, z \in B_{3R}(x_0)$, where γ^{xz} is the geodeisic from x to z and γ^{xy} is the geodeisic from x to y. Thus for any geodesic γ^1, γ^2 with endpoints in $B_{3R}(y_0) \subset U$, the following comparison principle holds:

$$|\gamma_t^1\gamma_t^2|_{\mathrm{g}} < c_2 \max\Big\{|\gamma_0^1\gamma_0^2|_{\mathrm{g}}, |\gamma_1^1\gamma_1^2|_{\mathrm{g}}\Big\} \quad \forall t \in (0,1). \tag{3.6}$$

$$|\gamma_s^1\gamma_t^2|_{\mathrm{g}} > c_2(s-1)|\gamma_0^1\gamma_0^2|_{\mathrm{g}} + c_1 s \wedge t\Big(|\gamma_1^1\gamma_1^2|_{\mathrm{g}} - \big||\gamma_0^2\gamma_1^1| - |\gamma_0^2\gamma_1^2|\big|\Big) \tag{3.7}$$

for all $s, t \in (0,1)$.

Let $y \in B_{3R}(y_0) \setminus \overline{B_{2R}(y_0)}$. Now we consider the L^2-Wasserstein geodesic $(\mu_t^{r,y})_t$ from $\mu_0^{r,y} := \frac{1}{\mathfrak{m}(B_r(y_0))}\mathfrak{m}|_{B_r(y_0)}$ to $\mu_1^{r,y} := \frac{1}{\mathfrak{m}(B_r(y))}\mathfrak{m}|_{B_r(y)}$. By density bound of intermediate measures on $\mathrm{CD}(K,\infty)$ space (c.f. Lemma 3.1 [23]), we get the following (uniform) estimate

$$\mathfrak{m}\big(\operatorname{supp}\mu_t^{r,y}\big) \gtrsim \min\Big\{\mathfrak{m}\big(B_r(y_0)\big), \mathfrak{m}\big(B_r(y)\big)\Big\} \qquad \forall t \in [0,1], \tag{3.8}$$

where we adopt the notation $A \lesssim B$ if there is a constant $C > 0$ such that $A < CB$. Combining (3.8) with the fact $\mathrm{Vol}_{\mathrm{g}}(B_r) \gtrsim r^n$, we get

$$\mathfrak{m}\big(\operatorname{supp}\mu_t^{r,y}\big) \gtrsim r^n \min\Big\{\epsilon_r(y_0), \epsilon_r(y)\Big\} \qquad \forall t \in [0,1]. \tag{3.9}$$

Let T_t be the optimal transport map which induces $(\mu_t^{r,y})_t$. By (3.6) we know

$$|T_t(x)\gamma_t^{y_0y}|_{\mathrm{g}} \le c_2 \max\Big\{|xy_0|_{\mathrm{g}}, |T_1(x)y|_{\mathrm{g}}\Big\} \le c_2 r$$

for any $x \in B_r(y_0)$, where γ^{y_0y} is the geodesic from y_0 to y. Therefore

$$\mu_t^{r,y}\big(B_{c_2r}(\gamma_t^{y_0y})\big) = 1 \qquad \forall t \in [0,1]. \tag{3.10}$$

In other words, $\cup_t \operatorname{supp}\mu_t^{r,y} \subset \overline{(\gamma^{y_0y})_{c_2r}}$, where $(\gamma_t^{y_0y})_{c_2r}$ is the c_2r-neighbourhood of γ^{y_0y} w.r.t. Hausdorff distance.

For any $c > 0$, by (3.4) we know there is $0 < r \ll R$ such that

$$\frac{\mathfrak{m}\big(\{x : \epsilon_r(x) \ge c\} \cap B_{3R}(y_0) \setminus B_{2R}(y_0)\big)}{\mathfrak{m}\big(B_{3R}(y_0) \setminus B_{2R}(y_0)\big)} > \frac{1}{2}. \tag{3.11}$$

Consider the projection map $\mathrm{Prj} : B_{3R}(y_0) \mapsto \partial B_{2R}(y_0)$ along the radius. By (3.11), Fubini's theorem and (3.4) we know

$$\mathcal{H}^{n-1}\Big(\mathrm{Prj}\big(\{x : \epsilon_r(x) \ge c\} \cap B_{3R}(y_0) \setminus B_{2R}(y_0)\big)\Big) \ge \frac{1}{2}\frac{c_1}{c_2}\mathcal{H}^{n-1}(\partial B_{2R}(y_0)).$$

Therefore there exist N points $y_1, y_2, ..., y_N \subset \{x : \epsilon_r(x) \ge c\} \cap B_{3R}(y_0) \setminus B_{2R}(y_0)$, where $N \gtrsim \frac{1}{r^{n-1}}$ is an integer independent of c, such that

$$|\mathrm{Prj}(y_i)\mathrm{Prj}(y_j)|_{\mathrm{g}} > \frac{4c_2}{c_1} r \qquad \forall 1 \le i < j \le N.$$

where $\mathrm{Prj}(y_i), i = 1, ..., N$ are the projections of $y_i, i = 1, ..., N$ on $\partial B_{2R}(y_0)$. So by (3.7) we know $|\gamma_s^{y_0\mathrm{Prj}(y_i)}\gamma_t^{y_0\mathrm{Prj}(y_j)}|_{\mathrm{g}} > 2c_2 r$ for any $s, t \in [\frac{1}{2}, 1]$. From (3.10), we also have

$$\bigcup_{t\in[\frac{1}{2},\frac{2}{3}]} \operatorname{supp}\mu_t^{r,y_i} \subset \overline{\Big(\gamma_t^{y_0y_i}\big|_{t\in[\frac{1}{2},\frac{2}{3}]}\Big)_{c_2r}} \subset \overline{\Big(\gamma_t^{y_0\mathrm{Prj}(y_i)}\big|_{t\in[\frac{1}{2},1]}\Big)_{c_2r}}.$$

So $\bigcup_{t\in[\frac{1}{2},\frac{2}{3}]} \operatorname{supp}\mu_t^{r,y_i}$, $i = 1, ..., N$ are essentially disjoint.

Furthermore, consider the following partition

$$N(y_i, r) := \Big\{\mathbf{t} = (t_1, t_2, ...) : t_i \in [\frac{1}{2}, \frac{2}{3}], \operatorname{supp}\mu_{t_1}^{r,y_i}, \operatorname{supp}\mu_{t_2}^{r,y_i}, ... \subset B_{2R}(y_0) \text{ are disjoint}\Big\},$$

it can be seen that $\max_{\mathbf{t}\in N(y_i,r)} |\mathbf{t}| \gtrsim \frac{1}{r}$, $i = 1, ..., N$.

In conclusion, we can find approximate $\frac{1}{r^n}$ measures whose supports are disjoint in $B_{2R}(y_0)$. Combining with (3.9) and local finiteness of $\mathfrak{m}$ (c.f. Theorem 4.24 [26]) we obtain the following estimate

$$r^n \min\Big\{\epsilon_r(y_0), \epsilon_r(y_1), ..., \epsilon_r(y_N)\Big\}\frac{1}{r^n} < C\mathfrak{m}\big(B_{2R}(y_0)\big) < \infty \tag{3.12}$$

where C is independent of c. By the choice of $\{y_1, ..., y_N\}$, we know

$$\min\Big\{\epsilon_r(y_1), ..., \epsilon_r(y_N)\Big\} \ge c.$$

Letting $c \to \infty$, by (3.12) we get $\epsilon_r(y_0) < C\mathfrak{m}(B_R(y_0))$, which is the contradiction. Therefore $\mathfrak{m}_{\mathrm{ac}} \neq 0$.

Finally, we will prove $\mathfrak{m}_{\mathrm{s}} = 0$ by contradiction. Assume that $\mathfrak{m}|_U$ is not absolutely continuous w.r.t. $\mathrm{Vol_g}$, then there exists a compact singular set $\mathtt{N} \subset U$ such that $\mathfrak{m}(\mathtt{N}) = \mathfrak{m}_{\mathrm{s}}(\mathtt{N}) > 0$ and $\mathrm{Vol_g}(\mathtt{N}) = 0$.

Since $\mathfrak{m}_{\mathrm{ac}} \neq 0$, there exists a bounded set E_L with positive $\mathfrak{m}$-measure such that $\frac{\mathrm{d}\mathfrak{m}}{\mathrm{dVol_g}} < L$ on E_L. Considering the L^2-Wasserstein geodesic (μ_t) from $\mu_0 := \frac{1}{\mathfrak{m}(\mathtt{N})}\mathfrak{m}|_{\mathtt{N}}$ to $\mu_1 := \frac{1}{\mathfrak{m}(E_L)}\mathfrak{m}|_{E_L}$. By the choice of E_L, we know $\mu_1 \ll \mathrm{Vol_g}$ with bounded density. By measure contraction property of $(\overline{U}, |\cdot|, \mathrm{Vol_g})$, we know $\mu_t \ll \mathrm{Vol_g}$ for any $t > 0$. In particular $\mu_t(\mathtt{N}) = 0$, so there is a Borel set $A_t \subset \operatorname{supp}\mu_t$ such that $A_t \cap \mathtt{N} = \emptyset$ and $\mu_t(A_t) = 1$. However, by Lemma 3.1 [23] again, we also have $\mu_t \le C_1\mathfrak{m}$ for some constant $C_1 > 0$. Next we will show the contradiction using the argument in [19] (c.f. Lemma 6.4 therein). Given $\epsilon > 0$, we know

$$A_t \subset \operatorname{supp}\mu_t \subset (\operatorname{supp}\mu_0)_\epsilon = (\mathtt{N})_\epsilon$$

for t small enough. Then

$$\begin{aligned}
\mathfrak{m}(\mathtt{N}) &= \lim_{\epsilon\to 0}\mathfrak{m}\big((\mathtt{N})_\epsilon\big)\\
&\ge \overline{\lim_{t\to 0}}\,\mathfrak{m}(\operatorname{supp}\mu_t)\\
&\ge \overline{\lim_{t\to 0}}\,\mathfrak{m}(A_t \setminus \mathtt{N}) + \mathfrak{m}(\mathtt{N})\\
&= \overline{\lim_{t\to 0}}\,\mathfrak{m}(A_t) + \mathfrak{m}(\mathtt{N})\\
&\ge \overline{\lim_{t\to 0}}\,\frac{1}{C_1}\mu_t(A_t) + \mathfrak{m}(\mathtt{N})\\
&\ge \frac{1}{C_1} + \mathfrak{m}(\mathtt{N})
\end{aligned}$$

which is the contradiction. As U is arbitrary, we know $\mathfrak{m} \ll \mathrm{Vol_g}$ on whole M.

Part 2): Given $x \in M$. For any $y \in B_{3R}(x) \setminus \overline{B_{2R}(x)}$, let us consider the L^2-Wasserstein geodesic $(\mu_t^r)_t$ from $\mu_0^r := \frac{1}{\mathfrak{m}(B_r(x))}\mathfrak{m}|_{B_r(x)}$ to $\mu_1^r := \delta_y$. By measure contraction property, we have the following (uniform) estimate

$$\mathfrak{m}\big(\operatorname{supp}\mu_t^r\big) \gtrsim \mathfrak{m}\big(B_r(x)\big) \qquad \forall t \in [0, \frac{2}{3}]. \tag{3.13}$$

Combining (3.3) and (3.13), we know

$$\mathfrak{m}\big(\operatorname{supp}\mu_t^r\big) \gtrsim r^n\epsilon_r(x) \qquad \forall t \in [0, \frac{2}{3}]. \tag{3.14}$$

As previously shown in Part 1), there exist (approximate) $\frac{1}{r^n}$ measures whose supports are disjoint. Combining with (3.14) we get

$$\epsilon_r(x) = \big(r^n\epsilon_r(x)\big)\frac{1}{r^n} \lesssim \mathfrak{m}(B_{2R}(x)). \tag{3.15}$$

Since $\mathrm{MCP}(K,N)$ condition implies measure doubling property, we know $\mathfrak{m}(B_{2R}(x))$ is locally bounded. Hence $\epsilon_r \in L^\infty_{\mathrm{loc}}$ uniformly in r.

Letting $r \to 0$ in (3.15). By Lebesgue differentiation theorem, there is $e^{-V} \in L^\infty_{\mathrm{loc}}(\mathrm{Vol}_{\mathrm{g}})$ such that

$$e^{-V(x)} = \lim_{r\to 0} \frac{\mathfrak{m}\big(B_r(x)\big)}{\mathrm{Vol}_{\mathrm{g}}\big(B_r(x)\big)}, \quad \mathrm{Vol}_{\mathrm{g}} - \text{a.e. } x.$$

Part 3):

Let $x \in M$, we define

$$\delta_r(x) := \frac{r^{N-n}\mathrm{Vol}_{\mathrm{g}}\big(B_r(x)\big)}{\mathfrak{m}\big(B_r(x)\big)}.$$

Denote by $(\mu_t^r)_t$ the Wasserstein geodesic from $\nu_0^r := \frac{1}{\mathrm{Vol}_{\mathrm{g}}(B_r(x))}\mathrm{Vol}_{\mathrm{g}}|_{B_r(x)}$ to $\nu_1^r := \delta_y$, where $0 < r \ll R$. By measure contraction property, we obtain

$$\mathrm{Vol}_{\mathrm{g}}\big(\operatorname{supp}\nu_t^r\big) \gtrsim r^{n-N}\delta_r\mathfrak{m}\big(B_r(x)\big) \qquad \forall t \in [0, \frac{2}{3}]. \tag{3.16}$$

By Bishop-Gromov inequality (c.f. Corollary 2.4 [27]), we know $\mathfrak{m}(B_r(x)) \gtrsim \big(\frac{r}{R}\big)^N \mathfrak{m}(B_{2R}(x))$. Therefore (3.16) implies

$$\mathrm{Vol}_{\mathrm{g}}\big(\operatorname{supp}\nu_t^r\big) \gtrsim \delta_r r^n \qquad \forall t \in [0, \frac{2}{3}]. \tag{3.17}$$

Similarly, we can find approximate $\frac{1}{r^n}$ measures whose supports are disjoint inside $B_{2R}(x)$, then we obtain

$$\delta_r \lesssim \mathrm{Vol}_{\mathrm{g}}\big(B_{2R}(x)\big). \tag{3.18}$$

Therefore we have the following uniformly L^∞_{loc} estimate

$$\frac{r^{N-n}\mathrm{Vol}_{\mathrm{g}}\big(B_r(x)\big)}{\mathfrak{m}\big(B_r(x)\big)} \in L^\infty_{\mathrm{loc}}.$$

If $N = n$, we have $\frac{\mathrm{Vol}_{\mathrm{g}}(B_r(x))}{\mathfrak{m}(B_r(x))} \in L^\infty_{\mathrm{loc}}$. Letting $r \to 0$, by Lebesgue differentiation theorem we know

$$e^V = \lim_{r\to 0} \frac{\mathrm{Vol}_{\mathrm{g}}\big(B_r(x)\big)}{\mathfrak{m}\big(B_r(x)\big)} \in L^\infty_{\mathrm{loc}}.$$

Combining with $e^{-V} \in L^\infty_{\mathrm{loc}}$, we obtain $V \in L^\infty_{\mathrm{loc}}$ (see also Proposition 3.4).

□

The following result has been proved in Part 1) of the proof above (see also Lemma 6.4 [19]). For convenience of later applications, we extract it as a separate lemma.

Lemma 3.2. *Let μ_0, μ_1 be two probability measures with compact support. Assume that $\mu_1 \ll \mathrm{Vol}_{\mathrm{g}}$ and $(\mu_t) \subset \mathcal{W}_2(M, \mathrm{g})$ is the unique L^2-Wasserstein geodesic connecting μ_0 and μ_1. If there exists a locally finite measure $\mathfrak{m}$ such that the density functions $\frac{\mathrm{d}\mu_t}{\mathrm{d}\mathfrak{m}}, t \in [0,1]$ are uniformly bounded. Then $\mu_0 \ll \mathrm{Vol}_{\mathrm{g}}$.*

In the following two propositions, we will improve the regularity of density functions.

Proposition 3.3. *Let $V : M \mapsto \mathbb{R} \cup \{+\infty\}$ be an extended-valued function on a complete Riemannian manifold (M, g), such that $(M, |\cdot|_{\mathrm{g}}, e^{-V}\mathrm{Vol}_{\mathrm{g}})$ is a $\mathrm{CD}(K, \infty)$ space. Then V has a semi-convex, locally Lipschitz representative.*

Proof. Denote $\mathfrak{m} := e^{-V}\mathrm{Vol}_{\mathrm{g}}$. Let $\Omega \subset M$ be a convex compact set such that points in Ω do not have cut-locus inside Ω. Given $L > 0$ such that the sub-level set $\{V \le L\} \cap \Omega$ has positive $\mathfrak{m}$-measure. Denote $E_L := \{V \le L\} \cap \operatorname{supp} \mathfrak{m}|_{\{V \le L\}} \cap \Omega$. Let $E_L^0, E_L^1 \subset E_L$ be two closed sets with $\mathrm{diam} E_L^i < \delta \ll 1$ and $\mathfrak{m}(E_L^i) > 0$, $i = 0, 1$. Denote by $(\mu_t)_{t\in[0,1]}$ the L^2-Wasserstein geodesic from $\mu_0 := \frac{1}{\mathrm{Vol}_{\mathrm{g}}(E_L^0)}\mathrm{Vol}_{\mathrm{g}}|_{E_L^0}$ to $\mu_1 := \frac{1}{\mathrm{Vol}_{\mathrm{g}}(E_L^1)}\mathrm{Vol}_{\mathrm{g}}|_{E_L^1}$.

When δ is small, from the proof of Theorem 1.1 in [29], we know that the entropy functional $\mathrm{Ent}(\,\cdot\,|\mathrm{Vol}_{\mathrm{g}})$ is k-concave along (μ_t) for some constant $k \in \mathbb{R}$. Combining with the following formula

$$\mathrm{Ent}(\,\cdot\,|\mathfrak{m}) = \mathrm{Ent}(\,\cdot\,|\mathrm{Vol}) + \int V \,\mathrm{d}(\cdot),$$

we know that the potential functional $\mu \mapsto \int V \,\mathrm{d}\mu$ is $(K-k)$-convex along (μ_t). By replacing V with $V + H$ for some locally $|K-k|$-convex function H, we may assume $k = 0$ without loss of generality.

Therefore,

$$\int V \,\mathrm{d}\mu_t \le t \int V \,\mathrm{d}\mu_1 + (1-t) \int V \,\mathrm{d}\mu_0 \le L. \tag{3.19}$$

So $E_L \cap \operatorname{supp} \mu_t \neq \emptyset$. By non-branching property of (μ_t) (or existence and uniqueness of optimal transport map, c.f. [22]), we know $\mu_t(E_L) = 1$. Denote the lifting of (μ_t) in $\mathcal{P}(\mathrm{Geod}(M, g))$ by Π. By Fubini's theorem, for Π-a.e. γ, we have $\gamma_t \in E_L$ for L^1-a.e. $t \in [0,1]$. Let $x \in E_L$ be arbitrary. Letting $E_L^1 \to x$ in Hausdorff topology, by Rauch's theorem we know the set of t-intermediate points between E_L^1 and E_L^2 converges to set of t-intermediate points between x and E_L^2. Hence we know $\gamma_t^{xy} \in E_L$ a.e. $t \in [0,1]$ for $\mathfrak{m}$-a.e. $y \in E_L^2$, where γ^{xy} is the geodesic from x to y. Therefore for $\mathfrak{m} \times \mathfrak{m}$-a.e. $(x, y) \in E_L \times E_L$, we know $\gamma^{xy} \in E_L$ for L^1-a.e. $t \in [0,1]$. In conclusion, we have proved the following assertion:

Assertion 1): Let $\mathrm{Conv}(E_L)$ be the convex hull of E_L in Ω. Then

$$\mathfrak{m}\big(\mathrm{Conv}(E_L) \setminus E_L\big) = 0. \tag{3.20}$$

We define a family $\mathcal{V}$ of measurable sets in the following way. We say that $U \in \mathcal{V}$ if there exists a ball $B_r(y)$ with radius $r > 0$, and a point $x_0 \in M \setminus B_r(y)$ such that $U = \operatorname{supp}\mu_t$ for some $t \in [\frac{1}{2}, 1]$, where $(\mu_t)_{t\in[0,1]}$ is the unique L^2- Wasserstein geodesic from $\mu_0 := \delta_{x_0}$ to $\mu_1 := \frac{1}{\mathrm{Vol_g}(B_r(y))}\mathrm{Vol_g}|_{B_r(y)}$. Let

$$T_t(z) := \exp_z\big(-t\nabla\varphi(z)\big)$$

be the optimal transport map (c.f. [22]), where $\varphi(z) = \frac{1}{2}|x_0 z|_{\mathrm{g}}^2$, such that $\mu_t = (T_{1-t})_\sharp\mu_1$. Denote the geodesic from x to y by γ. On one hand, by Rauch's comparison theorem, there exists a constant $C_1 > 0$ such that

$$|\gamma_t T_{1-t}(z)|_{\mathrm{g}} \le C_1|\gamma_1 z|_{\mathrm{g}} \le C_1 r$$

for any $z \in \operatorname{supp}\mu_1 = B_r(y)$. So we have

$$\operatorname{supp}\mu_t \subset B_{C_1 r}(\gamma_t).$$

On the other hand, given $\eta > 0$, by Toponogov's theorem there exists $C_2 > 0$ such that

$$C_2\big|\big(T_{1-t}\big)^{-1}(z)\gamma_1\big|_{\mathrm{g}} \le |\gamma_t z|_{\mathrm{g}} \le \eta$$

for any $z \in B_\eta(\gamma_t)$. Let $\eta = C_2 r$, we get

$$\big|\big(T_{1-t}\big)^{-1}(z)\gamma_1\big|_{\mathrm{g}} \le r$$

for any $z \in B_{C_2 r}(\gamma_t)$. Therefore

$$B_{C_2 r}(\gamma_t) \subset \operatorname{supp}\mu_t.$$

In conclusion, we have

$$B_{C_2 r}(\gamma_t) \subset \operatorname{supp}\mu_t \subset B_{C_1 r}(\gamma_t). \tag{3.21}$$

Therefore the sets in $\mathcal{V}$ have bounded eccentricity. This means that there exists some constant $c > 0$ such that each set $U \in \mathcal{V}$ is contained in a ball B_r and $\mathrm{Vol_g}(U) \ge c\mathrm{Vol_g}(B_r)$. It can be seen from (3.21) that $\mathcal{V}$ is a fine cover of M, every point $x \in M$ is covered by sets in $\mathcal{V}$ with arbitrarily small diameter.

We define an extended real-valued function $\bar{V} : M \mapsto \mathbb{R} \cup \{\pm\infty\}$ by

$$\bar{V}(x) := \varliminf_{\mathcal{V}\ni U\to x} \frac{1}{\mathrm{Vol_g}(U)}\int_U V\,\mathrm{dVol_g}.$$

Denote $V^+ := V \vee 0$ and $V^- := V \wedge 0$. By the inequality $t \le e^t$ on $[0, +\infty)$, we know $|V^-| \le e^{-V}$. Therefore $V^- \in L^1_{\mathrm{loc}}(\mathrm{Vol_g})$ and $V^+ \le L$ on E_L, so $V \in L^1(\overline{\mathrm{Conv}(E_L)}^\circ, \mathrm{Vol_g})$. By Lebesgue differentiation theorem, we know $\bar{V} = V$ $\mathfrak{m}$-a.e. on $\overline{\mathrm{Conv}(E_L)}^\circ$ and there is $M^* \subset \overline{\mathrm{Conv}(E_L)}^\circ$ with full measure such that

$$\lim_{\mathcal{V}\ni U\to x} \frac{1}{\mathrm{Vol_g}(U)}\int_U V\,\mathrm{dVol_g} = \bar{V}(x) \in \mathbb{R} \qquad \forall x \in M^*.$$

We just need to show that $\bar{V}$ is geodesically convex on $\overline{\mathrm{Conv}(E_L)}^\circ$, then from [18] (see also Corollary 3.10 [28]) we know $\bar{V}$ is locally Lipschitz on M.

Let $x, y \in M^*$, and γ be the geodesic from x to y. Given two parameters $0 < \epsilon, \delta \ll 1$, we define $\mu_0^{\epsilon,\delta} := \frac{1}{\mathrm{Vol_g}(B_\epsilon(x))}\mathrm{Vol_g}|_{B_\epsilon(x)}$, $\mu_1^{\epsilon,\delta} := \frac{1}{\mathrm{Vol_g}(B_\delta(y))}\mathrm{Vol_g}|_{B_\delta(y)}$, and $\mu_0^{0,\delta} = \delta_x$, $\mu_1^{\epsilon,0} = \delta_y$. Let $(\mu_t^{\epsilon,\delta})_{t\in[0,1]}$ be the L^2-Wasserstein geodesic from $\mu_0^{\epsilon,\delta}$ to $\mu_1^{\epsilon,\delta}$. By (3.19) we know

$$\int V\,\mathrm{d}\mu_t^{\epsilon,\delta} \le t\int V\,\mathrm{d}\mu_1^{\epsilon,\delta} + (1-t)\int V\,\mathrm{d}\mu_0^{\epsilon,\delta}. \tag{3.22}$$

Fix $\delta > 0$ and $t > 0$, by measure contraction property (3.2), we know that $\frac{\mathrm{d}\mu_t^{\epsilon,\delta}}{\mathrm{dVol_g}}$ is uniformly bounded for $\epsilon \ge 0$. By the existence and uniqueness of optimal transpot map (c.f. [22]), combining with Rauch's comparison theorem, we know $\mu_t^{\epsilon,\delta} \to \mu_t^{0,\delta}$ in L^1-Wasserstein distance. By an approximation argument, we can prove

$$\lim_{\epsilon\to 0}\int V\,\mathrm{d}\mu_t^{\epsilon,\delta} = \int V\,\mathrm{d}\mu_t^{0,\delta} \qquad \forall t\in(0,1).$$

Letting $\epsilon \to 0$ in (3.22), we obtain

$$\int V\,\mathrm{d}\mu_t^{0,\delta} \le t\int V\,\mathrm{d}\mu_1^{0,\delta} + (1-t)\bar{V}(x). \tag{3.23}$$

Assume $\mu_t^{0,\delta} = \rho_t^\delta\,\mathrm{Vol_g}$, by change of variable formula we know

$$\rho_t^\delta(T_{1-t})\det\big(\mathrm{D}T_{1-t}\big) = \rho_1^\delta \tag{3.24}$$

where

$$T_t(z) := \exp_z\big(-t\nabla\varphi(z)\big)$$

is the optimal transport map (c.f. [22]) and $\varphi(z) = \frac{1}{2}|xz|_{\mathrm{g}}^2$, such that $\mu_t^{0,\delta} = (T_{1-t})_\sharp\mu_1^{0,\delta}$.

Since the sets in $\mathcal{V}$ have bounded eccentricity (3.21), we have

$$\mathrm{supp}\,\mu_t^{0,\delta} \to \gamma_t$$

in Hausdorff topology. By (3.24) we know $\rho_t^\delta(T_{1-t}) = \frac{\rho_1^\delta}{\det(\mathrm{d}T_{1-t})}$. It can be seen that $\det(\mathrm{d}T_{1-t})$ is smooth and strictly positive on $B_\delta(y)$. Therefore

$$\begin{aligned}\sup_{z_1,z_2\in B_\delta(y)}\frac{\rho_t^\delta(T_{1-t}(z_1))}{\rho_t^\delta(T_{1-t}(z_2))} &\le 1 + \sup_{z_1,z_2\in B_\delta(y)}\frac{|\rho_t^\delta(T_{1-t}(z_1)) - \rho_t^\delta(T_{1-t}(z_2))|}{\rho_t^\delta(T_{1-t}(z_2))}\\ &= 1 + O(\delta).\end{aligned}$$

So we know ρ_t^δ is almost a constant up to $O(\delta)$, that is

$$\rho_t^\delta = \frac{1+O(\delta)}{\mathrm{Vol_g}(\mathrm{supp}\,\mu_t^{0,\delta})}.$$

Combining the results above, we obtain

$$\begin{aligned}\varliminf_{\delta\to 0}\int V\,\mathrm{d}\mu_t^{0,\delta} &= \varliminf_{\delta\to 0}\int V\,\mathrm{d}\Big(\frac{1}{\mathrm{Vol_g}(\mathrm{supp}\,\mu_t^{0,\delta})}\mathrm{Vol_g}|_{\mathrm{supp}\,\mu_t^{0,\delta}}\Big)\\ &\ge \bar{V}(\gamma_t).\end{aligned}$$

Letting $\delta \to 0$ in (3.23), we obtain

$$\bar{V}(\gamma_t) \le t\bar{V}(y) + (1-t)\bar{V}(x) \le L. \tag{3.25}$$

By Fubini's theorem, for any $z \in \overline{\mathrm{Conv}(E_L)}^{\circ}$, there are $x, y \in M^*$ such that z is an intermediate point on the geodesic connecting x and y. So $\bar{V}(z) \le L$ for all $z \in \overline{\mathrm{Conv}(E_L)}^{\circ}$. Furthermore, let $y \in \overline{\mathrm{Conv}(E_L)}^{\circ}$ be arbitrary and $\gamma_t, x \in M^*$. Repeat the argument above, combining with bounded eccentricity (3.21), we can see that $\bar{V}(y) > -\infty$ for all $y \in \overline{\mathrm{Conv}(E_L)}^{\circ}$.

For any geodesic γ, by (3.25) we know $\bar{V}$ is Lipschitz continuous on $\gamma \cap M^*$. Given $x \in \overline{\mathrm{Conv}(E_L)}^{\circ}$, by Fubini's theorem we know there exists $S_x \subset \{v \in S^1(T_xM)\}$ with full $\mathcal{H}^{n-1}$-measure and a continuous function $\tau : \overline{\mathrm{Conv}(E_L)}^{\circ} \mapsto (0,1]$, such that $\exp_x(tv) \in M^*$ for all $v \in S_x$ and a.e. $t \in [0, \tau(x))$. We define a set of geodesic segments Γ_x by

$$\Gamma_x := \left\{ \big(\exp_x(tv)\big)_{t\in[0,\tau(x))} : v \in S_x \right\}.$$

Then for any $\gamma \in \Gamma_x$, (3.25) yields that $\bar{V}$ is convex on $\gamma \cap M^*$.

To prove the convexity of $\bar{V}$ on whole $\overline{\mathrm{Conv}(E_L)}^{\circ}$, we just need to prove the continuity of $\bar{V}$. Then by an approximate argument, we can see that $\bar{V}$ satisfies (3.25) on all geodesics. With this aim, we will prove the following assertions.

Assertion 2): Let $x \in \overline{\mathrm{Conv}(E_L)}^{\circ}$. Then $\mathrm{Lip}(\bar{V}|_{\gamma\cap M^*})$ is uniformly bounded for $\gamma \in \Gamma_x$.

By (3.25) we know

$$\bar{V}(\gamma_{0^+}) := \lim_{y \in \gamma\cap M^* \to x} \bar{V}(y)$$

is well-defined for any $\gamma \in \Gamma_x$, and $\bar{V}(x) \le \bar{V}(\gamma_{0^+}) \le L$. Therefore, $\bar{V}$ is $\frac{2(L-\bar{V}(x)+1)}{\tau}(x)$-Lipschitz on $\{(\gamma_t)_{t\in[0,\frac{\tau(x)}{2}]} \cap M^* : \gamma \in \Gamma_x\}$, which is the thesis.

We define a (possibly multi-valued) function $\bar{V}' : \overline{\mathrm{Conv}(E_L)}^{\circ} \mapsto [\bar{V}(x), L]$ by

$$\bar{V}'(x) := \bar{V}(\gamma_{0^+}), \quad \text{if } \gamma \in \Gamma_x.$$

For any $x \in M^*$, by (3.25) we know the valued of $\bar{V}'(x)$ is independent of the choice of the geodesic $\gamma \in \Gamma_x$ and $\bar{V}'(x) = \bar{V}(x)$, so $\bar{V}' = V$ almost everywhere. Furthermore, assume $\bar{V}'$ is continuous, by definition we know $\bar{V} = \bar{V}'$ on whole $\overline{\mathrm{Conv}(E_L)}^{\circ}$. Therefore it is sufficient to prove the following assertion.

Assertion 3): $\bar{V}'$ is single-valued and continuous on $\overline{\mathrm{Conv}(E_L)}^{\circ}$.

Let $x \in \overline{\mathrm{Conv}(E_L)}^{\circ}$ be an arbitrary point. Assume by contradiction that

$$-\infty < \bar{V}(x) \le \bar{V}(\gamma^1_{0^+}) < \bar{V}(\gamma^2_{0^+}) \le L \tag{3.26}$$

for some $\gamma^1, \gamma^2 \in \Gamma_x$. By Fubini's theorem, we can find sequences $(x_n) \subset \gamma^1 \cap M^*$, $(y_n) \subset \gamma^2 \cap M^*$ such that $x_n, y_n \to x$, and $y_n \in \gamma^{x_n} \in \Gamma_{x_n}$. By (3.25), we know $\bar{V}'(x_n) \to \bar{V}(\gamma^1_{0^+})$, $\bar{V}'(y_n) \to \bar{V}(\gamma^2_{0^+})$, and $\bar{V}'$ is convex and Lipschitz on $\gamma^{x_n} \cap M^*$. From (3.26) we also know $\mathrm{Lip}\, \bar{V}'|_{\gamma_{x_n}} \to +\infty$, which contradicts to the local finiteness of $\bar{V}'$.

Finally, let $(z_n) \subset \overline{\mathrm{Conv}(E_L)}^\circ$ be an arbitrary sequence with $z_n \to x$. We can find $z'_n \in M^*$ such that $|z'_n z_n|_{\mathrm{g}} < \frac{1}{n}$ and $|\bar{V}(z'_n) - \bar{V}'(z_n)| < \frac{1}{n}$. By uniqueness of $\bar{V}'(x)$ and **Assertion 2)**, we know $\bar{V}(z'_n) \to \bar{V}'(x)$. So $\bar{V}'$ is continuous at x.

□

Proposition 3.4. *Let $V : M \mapsto \mathbb{R} \cup \{+\infty\}$ be an extended-valued function on a compact Riemannian manifold (M, g), and $\mathfrak{m} = e^{-V}\mathrm{Vol}_{\mathrm{g}}$. If $(M, |\cdot|_{\mathrm{g}}, \mathfrak{m})$ satisfies $\mathrm{MCP}(K, N)$ for some $K \in \mathbb{R}$ and $N < \infty$. Then V is locally bounded. In particular, $(M, |\cdot|_{\mathrm{g}}, \mathfrak{m})$ is infinitesimally Hilbertian.*

Proof. We define the following family of functions with parameter $r > 0$, as in the proof of Proposition 3.1,

$$\epsilon_r(x) := \frac{\mathfrak{m}(B_r(x))}{\mathrm{Vol}_{\mathrm{g}}(B_r(x))}.$$

Let $x \in M$ be a Lebesgue point of e^{-V}, i.e. $\lim_{r\to 0} \epsilon_r(x) = e^{-V(x)}$. We define a family $\mathcal{V}$ of Borel sets as follows. We say that $U \in \mathcal{V}$ if there exist $0 < r \ll \frac{R}{2}$, $x_0 \in B_{2R}(x) \setminus B_R(x)$, and a L^2- Wasserstein geodesic $(\mu_t)_{t\in[0,1]}$ with $\mu_0 := \delta_{x_0}$ and $\mu_s := \frac{1}{\mathfrak{m}(B_r(x))}\mathfrak{m}|_{B_r(x)}$ for some $s \in [\frac{1}{3}, 1]$, such that $U = \mathrm{supp}\,\mu_1$. For the same reason as in the proof of Proposition 3.3, we know the sets in $\mathcal{V}$ is a covering of $H_R(x) := B_{2R}(x) \setminus B_R(x)$ with bounded eccentricity.

By Lebesgue differentiation theorem, there exists $H^*_R(x) \subset H_R(x)$ with full measure such that

$$\lim_{\mathcal{V}\ni U\to y} \frac{\mathfrak{m}(U)}{\mathrm{Vol}_{\mathrm{g}}(U)} = \lim_{\mathcal{V}\ni U\to y} \frac{1}{\mathrm{Vol}_{\mathrm{g}}(U)} \int_U e^{-V}\,\mathrm{dVol}_{\mathrm{g}} = e^{-V(y)} > 0 \qquad \forall y \in H^*_R(x).$$

Let $y \in H^*_R(x)$ and $0 < \delta \ll 1$. There is a L^2- Wasserstein geodesic $(\mu_t)_{t\in[0,1]}$ with $\mu_0 := \delta_{x_0}$ and $\mu_s := \frac{1}{\mathfrak{m}(B_r(x))}\mathfrak{m}|_{B_r(x)}$ for some $s \in [\frac{1}{3}, 1]$, such that $U = \mathrm{supp}\,\mu_1$ and

$$1 - \delta < \left| \frac{\frac{\mathfrak{m}(U)}{\mathrm{Vol}_{\mathrm{g}}(U)}}{e^{-V(y)}} \right| < 1 + \delta.$$

By measure contraction property, there is a universal constant $C > 0$ such that

$$\epsilon_r(x)\mathrm{Vol}_{\mathrm{g}}\big(B_r(x)\big) = \mathfrak{m}\big(B_r(x)\big) > C\mathfrak{m}(U) > C(1-\delta)\mathrm{Vol}_{\mathrm{g}}(U)e^{-V(y)}.$$

Dividing r^n on both sides and letting $r \to 0$, we get $e^{-V(x)} \gtrsim e^{-V(y)}$. Recall that $H^*_R(x)$ has full measure in $H_R(x)$, we have the following weak mean-value property

$$e^{-V(x)} \gtrsim \mathfrak{m}\big(B_R(x)\big) > 0.$$

Combining with Proposition 3.1, we know $V \in L^\infty_{\mathrm{loc}}$.

□

Theorem 3.5 (Measure Rigidity: non-collapsed spaces)**.** *Let* (M, g) *be a* n*-dimensional Riemannian manifold. Assume there exists a measure* $\mathfrak{m}^*$ *with full support such that* $(M, |\cdot|_{\mathrm{g}}, \mathfrak{m}^*)$ *is* $\mathrm{CD}(K, n)$ *for some* $K \in \mathbb{R}$*. Then there exists a constant* $c > 0$ *such that* $\mathfrak{m}^* = c\mathrm{Vol}_{\mathrm{g}}$*.*

Proof. By Proposition 3.1 and Proposition 3.3 we know there exists a positive continuous function φ such that $\mathfrak{m}^* = \varphi^n \mathrm{Vol}_{\mathrm{g}}$. So we just need to prove that φ is a constant.

Given two points $x, y \in M$. Let γ be a geodesic from x to y. Without loss of generality, we assume that x has no cut-locus on γ and $(M, |\cdot|_{\mathrm{g}}, \mathrm{Vol}_{\mathrm{g}})$ is $\mathrm{CD}(k, n)$ for some k. Let $\mathfrak{m} = \mathfrak{m}^* = \varphi^n \mathrm{Vol}_{\mathrm{g}}$ and $\mathfrak{m} = \mathrm{Vol}_{\mathrm{g}}$ respectively. By Brunn–Minkowski inequality on $\mathrm{CD}(K, n)$ spaces (c.f. [27] Proposition 2.1) and Rauch's comparison theorem, there is $C > 0$ such that

$$\mathfrak{m}\big(B_{\frac{\sigma}{2}(1+C|xy|^2_{\mathrm{g}})}(\gamma_{\frac{1}{2}})\big)^{\frac{1}{n}} \geq \tau^{(\frac{1}{2})}_{K,n}(\Theta)\mathfrak{m}\big(B_\sigma(x)\big)^{\frac{1}{n}} \tag{3.27}$$

where $0 < \sigma \ll 1$ and $\big|\Theta - |xy|_{\mathrm{g}}\big| \leq \sigma$.

We define $\mathfrak{J}_{\mathfrak{m}}(x)$ by

$$\mathfrak{J}_{\mathfrak{m}}(x) := \lim_{r \to 0} \left(\frac{\mathfrak{m}(B_r(x))}{r^n} \right)^{\frac{1}{n}}.$$

Dividing $\frac{\sigma}{2}(1 + C|xy|^2_{\mathrm{g}})$ on both sides of (3.27) and letting $\sigma \to 0$, we obtain

$$\mathfrak{J}_{\mathfrak{m}}(\gamma_{\frac{1}{2}}) \geq \frac{2}{1 + C|xy|^2_{\mathrm{g}}} \tau^{(\frac{1}{2})}_{K,n}(|xy|_{\mathrm{g}}) \mathfrak{J}_{\mathfrak{m}}(x).$$

When $|xy|_{\mathrm{g}}$ is small, by Taylor expansion of $\tau^{(\frac{1}{2})}_{K,n}(\theta)$ we obtain

$$\mathfrak{J}_{\mathfrak{m}}(\gamma_{\frac{1}{2}}) \geq \frac{1 + O(|xy|^2_{\mathrm{g}})}{1 + C|xy|^2_{\mathrm{g}}} \mathfrak{J}_{\mathfrak{m}}(x).$$

For any $N > 0$, we divide γ equally into N parts. Repeating the argument above on each interval with length $\frac{1}{N}|xy|_{\mathrm{g}}$ we get

$$\mathfrak{J}_{\mathfrak{m}}(\gamma_{\frac{i+1}{N}}) \geq \Big(1 + o(\frac{1}{N})\Big) \mathfrak{J}_{\mathfrak{m}}(\gamma_{\frac{i}{N}}) \qquad i = 0, ..., N-1.$$

Therefore

$$\mathfrak{J}_{\mathfrak{m}}(y) \geq \Big(1 + o(\frac{1}{N})\Big)^N \mathfrak{J}_{\mathfrak{m}}(x).$$

Letting $N \to \infty$, we obtain $\mathfrak{J}_{\mathfrak{m}}(y) \geq \mathfrak{J}_{\mathfrak{m}}(x)$. By symmetry, we can also prove $\mathfrak{J}_{\mathfrak{m}}(y) \leq \mathfrak{J}_{\mathfrak{m}}(x)$, hence $\mathfrak{J}_{\mathfrak{m}}(y) = \mathfrak{J}_{\mathfrak{m}}(x)$. So $\mathfrak{J}_{\mathfrak{m}}$ is a constant for both $\mathfrak{m} = \mathrm{Vol}_{\mathrm{g}}$ and $\mathfrak{m} = \mathfrak{m}^*$. By Proposition 3.4 we know φ is continuous, so we also have

$$\mathfrak{J}_{\mathfrak{m}^*} = \varphi \mathfrak{J}_{\mathrm{Vol}_{\mathrm{g}}}.$$

Therefore φ is a constant.

□

In the last theorem, we study the $\mathrm{CD}(K,\infty)$ condition on Lipschitz domain. A domain $\Omega \subset M$ is called Lipschitz domain (or domain with Lipschitz boundary) if its boundary $\partial\Omega$ can be written locally as the graph of a Lipschitz continuous function on $\mathbb{R}^{n-1}$.

Theorem 3.6 (Measure rigidity: $\mathrm{CD}(K,\infty)$ condition)**.** *Let (M,g) be a complete Riemannian manifold, $\Omega \subset M$ be a Lipschitz domain. Let d_Ω be the intrinsic distance induced by Riemannian distance $|\cdot|_{\mathrm{g}}$ on $\overline{\Omega}$, and $\mathfrak{m}$ be a reference measure with* $\operatorname{supp}\mathfrak{m} = \overline{\Omega}$. *Assume that $(\overline{\Omega},\mathrm{d}_\Omega,\mathfrak{m})$ satisfies $\mathrm{CD}(K,\infty)$ condition, then we have the following rigidity results.*

1) $\overline{\Omega}$ is g-geodesically convex, i.e. any shortest path in $(\overline{\Omega},\mathrm{d}_\Omega)$ is a geodesic segment in (M,g);

2) $\mathfrak{m}|_{\partial\Omega} = 0$ and $\mathfrak{m} = e^{-V}\mathrm{Vol}_{\mathrm{g}}$ for some semi-convex, locally Lipschitz function V;

3) $(\overline{\Omega},\mathrm{d}_\Omega,\mathfrak{m})$ is a $\mathrm{RCD}(K,\infty)$ space.

In particular, $(\overline{\Omega},\mathrm{d}_\Omega,\mathrm{Vol}_{\mathrm{g}})$ is $\mathrm{CD}(K,\infty)$ if and only if $\overline{\Omega}$ is g-geodesically convex and $\mathrm{Ricci} \geq K$ on Ω.

Proof. Since all the assertions are local, without loss of generality, we may assume that $\overline{\Omega}$ is compact and points in $\overline{\Omega}$ do not have cut-locus inside $\overline{\Omega}$.

Given $x, y \in \Omega$ and a parameter $\epsilon > 0$ such that $B_\epsilon(x), B_\epsilon(y) \subset \Omega$. By Proposition 3.1 we know $\mathfrak{m}|_{\overline{\Omega}^\circ} \ll \mathrm{Vol}_{\mathrm{g}}$. We firstly consider the L^1-optimal transportation on $(\overline{\Omega},\mathrm{d}_\Omega)$ between $\mu_0^\epsilon := \frac{1}{\mathfrak{m}(B_\epsilon(x))}\mathfrak{m}|_{B_\epsilon(x)}$ and $\mu_1^\epsilon := \frac{1}{\mathfrak{m}(B_\epsilon(y))}\mathfrak{m}|_{B_\epsilon(y)}$. Let $(\mu_t^\epsilon)_t$ be a geodesic from μ_0^ϵ to μ_1^ϵ in L^1-Wasserstein space $\mathcal{W}_1(\overline{\Omega},\mathrm{d}_\Omega)$. Denote by Π^ϵ its lifting in $\mathcal{P}(\mathrm{Geod}(\overline{\Omega},\mathrm{d}_\Omega))$ satisfying $(e_t)_\sharp\Pi^\epsilon = \mu_t^\epsilon$. By L^1-optimal transport theory, there exists a Kantorovich potential φ associated with such optimal transportation, which is a 1-Lipschitz function. Let Γ^φ be the subset of $\mathrm{C}([0,1];(\overline{\Omega},\mathrm{d}_\Omega))$ containing all the trajectories of the gradient flow of φ. It is known that $\Pi^\epsilon(\Gamma^\varphi) = 1$.

For $\delta > 0$ small enough, $(\mu_t^\epsilon)_{t\in[0,\delta]}$ and $(\mu_t^\epsilon)_{t\in[1-\delta,1]}$ are also L^1-Wasserstein geodesics (segments) in $\mathcal{W}_1(M,\mathrm{g})$. By needle decomposition via L^1-optimal transport (c.f. Theorem 3.8, Theorem 5.1 [10]), there is $\Gamma \subset \Gamma^\varphi$ such that $\Pi^\epsilon(\Gamma^\varphi \setminus \Gamma) = 0$ and $(\gamma_t)_{t\in[0,\delta]\cup[1-\delta,1]}, \gamma \in \Gamma$ are pairwisely disjoint. In addition, the measure $\mathrm{Vol}_{\mathrm{g}}|_{B_\epsilon(x)}$ has a decomposition

$$\mathrm{Vol}_{\mathrm{g}}|_{B_\epsilon(x)} = \int_{\mathfrak{Q}} \mathfrak{m}_q \,\mathrm{d}\mathfrak{q} \tag{3.28}$$

where $\mathfrak{Q}$ can be written locally as a level set of φ, and $(\mathfrak{m}_q)_q$ support on disjoint geodesic segments $(X_q)_q$ such that $\mathfrak{m}_q \ll \mathcal{H}^1$ and $h_q = \frac{\mathrm{d}\mathfrak{m}_q}{\mathrm{d}\mathcal{H}^1}$ is a $\mathrm{CD}(k,n)$ density for $\mathfrak{q}$-a.e. q. Similarly, $\mathrm{Vol}_{\mathrm{g}}|_{B_\epsilon(y)}$ has a similar decomposition.

Next we will construct a L^2-optimal transportation based on Π^ϵ and Γ. Denote the level set $\{\varphi = T\}$ by φ_T. For any $z \in B_\epsilon(x)$, there exist $\gamma^z \in \Gamma$ and $T_z \in \mathbb{R}$ such that $z = \varphi_{T_z} \cap \gamma^z$. In addition, by Fubini's theorem, there exists T_0 such that $B^* := \{z : \gamma^z \cap \varphi_{T_z - T_0} \in B_\epsilon(y)\} \cap B_\epsilon(x)$ has positive $\mathrm{Vol}_{\mathrm{g}}$-volume. It can be seen that

$\mathrm{Cpl} := \{(z_1, z_2) : z_1 \in B^*, z_2 \in \gamma^{z_1} \cap \varphi_{T_{z_1} - T_0}\} \subset B_\epsilon(x) \times B_\epsilon(y)$ is still a L^1-optimal transport coupling. Furthermore, we have

$$\begin{aligned} & \Big(\varphi(y_1) - \varphi(y_0)\Big)\Big(\varphi(x_1) - \varphi(x_0)\Big) \\ = & \Big(\big(\varphi(x_1) - T_0\big) - \big(\varphi(x_0) - T_0\big)\Big)\Big(\varphi(x_1) - \varphi(x_0)\Big) \\ = & \Big(\varphi(x_1) - \varphi(x_0)\Big)^2 \geq 0 \end{aligned}$$

for any $(x_0, y_0), (x_1, y_1) \in \mathrm{Cpl}$. By Lemma 4.6 in [8], we know Cpl is d^2_Ω-cyclically monotone, so that it is also a L^2-optimal transport coupling. From the construction above we know $(\pi^1)_\sharp \mathrm{Cpl} = B^*$ has positive $\mathfrak{m}$-measure, and by measure decomposition (3.28), we also know $\mathfrak{m}((\pi^2)_\sharp \mathrm{Cpl}) > 0$. Then by renormalization we obtain a curve, still denote it by (μ^ϵ_t), which is a L^1-Wasserstein geodesic, as well as a L^2-Wasserstein geodesic. From the construction above, we can see that both $\mu^\epsilon_0, \mu^\epsilon_1$ have bounded $\mathfrak{m}$-densities.

To prove the geodesical convexity of $\overline{\Omega}$, we just need to show that $\Pi^\epsilon\big(\mathrm{Geod}(\overline{\Omega}, \mathrm{d}_\Omega) \setminus \mathrm{Geod}(M, \mathrm{g})\big) = 0$, then letting $\epsilon \to 0$ we know that x and y are connected by a geodesic in (M, g).

Let R be the set of regular points where $\partial\Omega$ is differentiable. It can be seen that $\mathrm{Geod}(\overline{\Omega}, \mathrm{d}_\Omega)$ is covered by $\Gamma^1 \cup \Gamma^2 \cup \Gamma^3$, where

a) $\Gamma^1 = \big\{\gamma : \mathcal{H}^1(\gamma \cap \mathrm{R}) > 0\big\}$;

b) $\Gamma^2 = \big\{\gamma : \mathcal{H}^1(\gamma \cap \mathrm{R}) = 0,\ \gamma \cap \partial\Omega \setminus \mathrm{R} \neq \varnothing\big\}$;

c) $\Gamma^3 = \big\{\gamma : \mathcal{H}^1(\gamma \cap \mathrm{R}) = 0,\ \gamma \cap \partial\Omega \subset \mathrm{R}\big\}$.

We will prove $\Pi^\epsilon(\Gamma^i \setminus \mathrm{Geod}(M, \mathrm{g}) = 0, i = 1, 2, 3$ in the following three steps.

Step 1): $\Pi^\epsilon(\Gamma^1) = 0$.

By Rademacher's theorem we know Lipschitz functions on $\mathbb{R}^{n-1}$ are differentiable $\mathcal{H}^{n-1}$-almost everywhere, so $\mathcal{H}^{n-1}(\partial\Omega \setminus \mathrm{R}) = 0$. By Proposition 3.1 and Proposition 3.3, there exists a locally Lipschitz and semi-convex function V such that

$$\mathfrak{m} = e^{-V}\mathrm{Vol}_{\mathrm{g}}|_\Omega + \mathfrak{m}|_{\partial\Omega \setminus \mathrm{R}} + \mathfrak{m}|_{\mathrm{R}}.$$

In particular, $\mathfrak{m}|_\Omega < C_0 \mathrm{Vol}_{\mathrm{g}}|_\Omega$ for some $C_0 > 0$.

Claim: $\mathfrak{m}(\mathrm{R}) = 0$, therefore $\mathfrak{m} = e^{-V}\mathrm{Vol}_{\mathrm{g}}|_\Omega + \mathfrak{m}|_{\partial\Omega\setminus\mathrm{R}}$.

Assume that $\partial\Omega$ is locally represented as the graph of a bi-Lipschitz function φ on $U \subset \mathbb{R}^{n-1}$, and

$$\frac{1}{L}|ab| < |xy|_{\mathrm{g}} < L|ab| \quad \forall a, b \in U,\ x = (a, \varphi(a)), y = (b, \varphi(b))$$

for some $L > 0$. Let $a \in U$ with $(a, \varphi(a)) \in \mathrm{R}$. We know there is a unique tangent plan of $\partial\Omega$ at this point and we have

$$\lim_{r \to 0} \sup_{b \in U, |b - a| < r} \frac{\big|\varphi(b) - \varphi(a) - \mathrm{d}\varphi(a)(b - a)\big|}{|b - a|} = 0.$$

So (unit) inward normal vector fields $\mathrm{R} \ni x \mapsto \mathrm{N}_x$ is uniquely defined on regular points. Furthermore, there is $\delta > 0$ such that if

$$\sup_{b\in U, |b-a|<r_0} \frac{\big|\varphi(b) - \varphi(a) - \mathrm{d}\varphi(a)(b-a)\big|}{|b-a|} < \delta$$

for some $r_0 > 0$ and $a \in U$. Then for any $r \leq r_0$, all d_Ω-geodesics from $B_{\frac{Lr}{5}}(\exp_x(\frac{3Lr}{5}\mathrm{N_x}))$ to $x = (a, \varphi(a))$ are g-geodesics. By measurable selection theorem, there is a measurable function $r : \mathrm{R} :\mapsto (0,1)$ such that d_Ω-geodesics from $B_{\frac{Lr(x)}{5}}(\exp_x(\frac{3Lr(x)}{5}\mathrm{N_x}))$ to x are g-geodesics. Assume by contradiction that $\mathfrak{m}(\mathrm{R}) \neq 0$. By Lusin's theorem there exists $\mathrm{R}^* \subset \mathrm{R}$ with $\mathfrak{m}(\mathrm{R}^*) > 0$, such that the map $x \to (\mathrm{N}_x, r(x))$ is continuous on R^*.

Let $x \in \mathrm{R}^* \cap \operatorname{supp} \mathfrak{m}|_{\mathrm{R}^*}$. There is a neighbourhood $U_x \subset \partial\Omega$ of x, such that all the d_Ω-geodesics connecting U_x and $B_{\frac{Lr(x)}{10}}(\exp_x(\frac{3Lr(x)}{5}\mathrm{N_x}))$ are g-geodesics. By Lemma 3.2, we get the contradiction. Therefore $\mathfrak{m}(\mathrm{R}) = 0$.

Assume by contradiction that $\Pi^\epsilon(\Gamma^1) > 0$. By Fubini's theorem we know

$$(\Pi^\epsilon \times L^1)(\{(\gamma, t) : \gamma \in \Gamma_1, t \in [0,1], \gamma_t \in \mathrm{R}\}) > 0,$$

and there is $t_0 \in [0,1]$ such that

$$\Pi^\epsilon(\{\gamma : \gamma \in \Gamma^1, \gamma_{t_0} \in \mathrm{R}\}) > 0,$$

so $\mu^\epsilon_{t_0}(\mathrm{R}) > 0$, which contradicts to the facts that $\mu^\epsilon_{t_0} \ll \mathfrak{m}$ and $\mathfrak{m}|_{\mathrm{R}} = 0$. Therefore $\Pi^\epsilon(\Gamma^1) = 0$.

Step 2): $\Pi^\epsilon(\Gamma^2) = 0$.

For any $t \in (0,1]$, we define

$$\Gamma^{2,t} := \Big\{\gamma : \gamma \in \Gamma^2, (\gamma_s)_{s\in[0,t)} \subset \Omega \cup \mathrm{R}, \gamma_t \in \partial\Omega \setminus \mathrm{R}\Big\}.$$

Then we have the following decomposition of Γ^2:

$$\Gamma^2 = \bigcup_{t\in(0,1]} \Gamma^{2,t}.$$

Assume $\Pi^\epsilon(\Gamma^2) > 0$, by Fubini's theorem and the decomposition (3.28) we know there exists $t_0 \in (0,1)$ such that

$$\mathcal{H}^{n-1}(\{\gamma_0 : \gamma \in \Gamma^{2,t_0}\}) \neq 0.$$

Given $\sigma \in (0, \epsilon)$, we define couplings

$$\mathrm{Cpl}^{1,2}_\sigma := \Big\{(z_1, z_2) : z_1 \in \gamma \cap \varphi_{T_{\gamma_0} - s\sigma}, z_2 \in \gamma \cap \varphi_{T_{\gamma_{t_0}} + (1-s)\sigma}, \gamma \in \Gamma^{2,t_0}, s \in (0,1)\Big\}$$

and

$$\mathrm{Cpl}^{1,3}_\sigma := \Big\{(z_1, z_3) : z_1 \in \gamma \cap \varphi_{T_{\gamma_0} - s\sigma}, z_3 \in \gamma \cap \varphi_{T_{\gamma_0} - s\sigma - T_0}, \gamma \in \Gamma^{2,t_0}, s \in (0,1)\Big\}.$$

From the construction, we know these couplings are both L^1-optimal and L^2-optimal. By renormalization and reparameterization, we can find a Wasserstein geodesic (ν^σ_t) in $\mathcal{W}_2(\overline{\Omega}, \mathrm{d}_\Omega) \cap \mathcal{W}_1(\overline{\Omega}, \mathrm{d}_\Omega)$, such that $\mathrm{Cpl}^{1,2}_\sigma$ is the optimal coupling for $(\nu^\sigma_0, \nu^\sigma_{\frac{1}{2}})$ and $\mathrm{Cpl}^{1,3}_\sigma$ is the optimal coupling for $(\nu^\sigma_0, \nu^\sigma_1)$ and

1) $(\nu_t^\sigma)_{t\in[0,\frac12]}$ is a geodesic segment in Wasserstein space $\mathcal{W}_2(M,\mathrm{g})$;

2) Given $\delta\in(0,\frac12)$, $(\nu_t^\sigma)_{t\in[0,\frac12-\delta]\cup\{1\}}$ have uniformly bounded $\mathfrak{m}$-densities;

3) $\mathfrak{m}(\operatorname{supp}\nu_0^\sigma)=O(\sigma)$ and $\mathfrak{m}(\operatorname{supp}\nu_1^\sigma)=O(\sigma)$.

Moreover, since $\mathcal{H}^{n-1}(\partial\Omega\setminus\mathrm{R})=0$, by Rauch's comparison theorem we know $\mathrm{Vol}_{\mathrm{g}}(\operatorname{supp}\nu_{\frac12}^\sigma)\lesssim\sigma^n$, so that $\mathfrak{m}(\operatorname{supp}\nu_{\frac12}^\sigma)\lesssim\sigma^n$.

Since $(\overline{\Omega},\mathrm{d}_\Omega,\mathfrak{m})$ is $\mathrm{CD}(K,\infty)$, by Lemma 3.1 [23] there exists a L^2-Wasserstein geodesic $(\bar\nu_t^\sigma)_{t\in[0,1]}\subset\mathcal{W}_2(\overline{\Omega},\mathrm{d}_\Omega)$ with uniformly bounded densities, connecting $\nu_{\frac14}^\sigma$ and ν_1^σ such that

$$\mathfrak{m}(\operatorname{supp}\bar\nu_t^\sigma)\gtrsim\min\Big\{\mathfrak{m}(\operatorname{supp}\nu_{\frac14}^\sigma),\mathfrak{m}(\operatorname{supp}\nu_1^\sigma)\Big\},\quad t\in[0,1].$$

It is known that Riemannian manifolds are essentially non-branching, hence $\nu_s^\sigma\in(\bar\nu_t^\sigma)$ for all $s\in[\frac14,\frac12]$. In particular, there exists $t_1\in(0,1)$ such that $\nu_{\frac12}^\sigma=\bar\nu_{t_1}^\sigma$. Therefore we have

$$\sigma^n\gtrsim\mathfrak{m}(\operatorname{supp}\nu_{\frac12}^\sigma)=\mathfrak{m}(\operatorname{supp}\bar\nu_{t_1}^\sigma)\gtrsim O(\sigma)$$

which is the contradiction. Therefore $\Pi^\epsilon(\Gamma^2)=0$.

Step 3): $\Pi^\epsilon\big(\Gamma^3\setminus\mathrm{Geod}(M,\mathrm{g})\big)=0$.

Let $\gamma\in\Gamma^3$. For any $t\in[0,1]$ with $\gamma_t\in\Omega$, there is $\delta_t>0$ such that $(\gamma_s)_{s\in(0\vee(t-\delta_t),1\wedge(t+\delta_t))}$ is a g-geodesic segment. Therefore, for any $\delta_1,\delta_2\in(0,\delta_t]$ we have

$$|\gamma_{0\vee(t-\delta_1)}\gamma_{1\wedge(t+\delta_2)}|_{\mathrm{g}}=\mathrm{d}_\Omega\big(\gamma_{0\vee(t-\delta_1)},\gamma_{1\wedge(t+\delta_2)}\big).\tag{3.29}$$

For any $t\in[0,1]$ with $\gamma_t\in\partial\Omega$. By definition of Γ^3 we know $\gamma_t\in\mathrm{R}$, so

$$\lim_{\delta\to0}\frac{|\gamma_{0\vee(t-\delta)}\gamma_{1\wedge(t+\delta)}|_{\mathrm{g}}}{\mathrm{d}_\Omega\big(\gamma_{0\vee(t-\delta)},\gamma_{1\wedge(t+\delta)}\big)}=1.\tag{3.30}$$

Given $\epsilon>0$, from (3.30) we know there is $\delta_t>0$ such that

$$1-\epsilon<\frac{|\gamma_{0\vee(t-\delta_1)}\gamma_{1\wedge(t+\delta_2)}|_{\mathrm{g}}}{\mathrm{d}_\Omega\big(\gamma_{0\vee(t-\delta_1)},\gamma_{1\wedge(t+\delta_2)}\big)}\le1\tag{3.31}$$

for any $\delta_1,\delta_2\in(0,\delta_t]$.

By compactness, there is a finite covering of $[0,1]$, denoted by $\{(t_i-\delta_{t_i},t_i+\delta_{t_i})\}_{i=1,\dots,N}$ for some $N\in\mathbb{N}$. Combining (3.29) and (3.31), we get

$$\begin{aligned}(1-\epsilon)\mathrm{d}_\Omega(\gamma_0,\gamma_1)&=(1-\epsilon)\Big(\mathrm{d}_\Omega(\gamma_0,\gamma_{t_1+\delta_{t_1}})+\sum_{i=1}^{N-1}\mathrm{d}_\Omega\big(\gamma_{t_i+\delta_{t_i}},\gamma_{1\wedge(t_{i+1}+\delta_{t_{i+1}})}\big)\Big)\\&<|\gamma_0\gamma_{t_1+\delta_{t_1}}|_{\mathrm{g}}+\sum_{i=1}^{N-1}|\gamma_{t_i+\delta_{t_i}}\gamma_{1\wedge(t_{i+1}+\delta_{t_{i+1}})}|_{\mathrm{g}}\\&=|\gamma_0\gamma_1|_{\mathrm{g}}\\&\le\mathrm{d}_\Omega(\gamma_0,\gamma_1).\end{aligned}$$

Letting $\epsilon \to 0$ we have $|\gamma_0\gamma_1|_{\mathrm{g}} = \mathrm{d}_\Omega(\gamma_0, \gamma_1)$. So $\gamma \in \mathrm{Geod}(M, \mathrm{g})$ and $\Gamma^3 \subset \mathrm{Geod}(M, \mathrm{g})$

Combining the results proved above, we know $(\overline{\Omega}, \mathrm{d}_\Omega, \mathfrak{m})$ is (M, g)-geodesically convex. By Lemma 3.2 we have $\mathfrak{m}|_{\partial\Omega} = 0$.

Therefore $\mathfrak{m} = e^{-V}\mathrm{Vol}_{\mathrm{g}}|_\Omega$ for some Lipschitz function V, so that $(\overline{\Omega}, \mathrm{d}_\Omega, \mathfrak{m})$ is infinitesimally Hilbertian and it satisfies $\mathrm{RCD}(K, \infty)$ condition.

□

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