Rough traces of functions of bounded variation in metric measure spaces

Vito Buffa[*], Michele Miranda jr.[†]

April 18, 2019

Abstract

Following a Maz'ya-type approach, we re-adapt the theory of rough traces of functions of bounded variation (BV) in the context of doubling metric measure spaces supporting a Poincaré inequality. This eventually allows for an integration by parts formula involving the rough trace of such a function. We then compare our analysis with the discussion done in a recent work by P. Lahti and N. Shanmugalingam, where traces of BV functions are studied by means of the more classical Lebesgue-point characterization, and we determine the conditions under which the two notions coincide.

1 Introduction

This paper aims at investigating traces of BV functions and integration by parts formulæ in metric measure spaces. The setting is given by a complete and separable metric measure space $(\mathbb{X}, d, \mu)$ endowed with a doubling measure μ and supporting a weak $(1,1)$–Poincaré inequality. We prove an integration by parts formula on sets of finite perimeter with some regularity; the idea is to use the notion of essential boundary and to define the rough trace of a BV function on such boundary using its super–level sets.

Sets with finite perimeter in metric measure spaces were defined for instance in [24] and studied by L. Ambrosio in [1, 2]. The main fact we use is that for a set

[*]**Corresponding Author**; Aalto University, Espoo (Finland), e–mail: vito.buffa@aalto.fi
[†]University of Ferrara, Ferrara (Italy), e–mail: michele.miranda@unife.it

with finite perimeter, the perimeter measure is a Hausdorff measure concentrated on its essential boundary. The notion of essential boundary is good enough to perform the strategy given by V. Maz'ya in his book [23]; in the Euclidean case, the notion of reduced boundary was used and an integration by parts formula was proven. Also, the continuity of the trace operator was investigated and equivalent conditions for such continuity were given. In the metric space setting we have so far no good notion of reduced boundary, but for our aims the essential boundary suffices.

Properties of the trace operator have been recently investigated in [21] and sufficient conditions for the continuity of such operator were given in terms of a "measure-density condition" on the boundary of the selected domain. We compare this notion of trace with the rough trace proving almost-everywhere equality of the two functions on the boundary. In this way, two different characterizations of the trace values of a function with bounded variation are available, the two being equivalent.

The paper is organized as follows:

In Section 2 we review the basic tools of our analysis, namely the concept of a metric measure space $(\mathbb{X}, d, \mu)$ equipped with a doubling measure and supporting a weak Poincaré inequality, the notions of BV function and of Caccioppoli set, along with the fundamental results related to them, such as the Coarea Formula, the Isoperimetric Inequality and of course the remarkable Theorem by L. Ambrosio on the Hausdorff representation of the perimeter measure, [2, Theorem 5.3].

In Section 3 we rewrite, after [23], the notion and the properties of the rough trace of BV functions; in particular, we re-investigate the conditions under which a BV function admits a summable rough trace. The latter part of the Section is thus devoted to an integration by parts formula for functions of bounded variation in terms of essentially bounded divergence-measure vector fields, a formula which, as shown in Theorem 3.10, features implicitly the rough trace of $u \in BV(\Omega)$. In comparison with [8], an example of domains where the occurrence of the rough trace is explicit is also given.

Divergence-measure fields, namely the class of those vector fields whose distributional divergence is a finite Radon measure, have been an object of interest for quite a few decades now. After their introduction by G. Anzellotti [6] in 1983, they have been rediscovered in the early 2000's by several authors, whose studies have led to notable applications to sets of finite perimeter in the Euclidean setting, namely the validity of (generalized) Gauss-Green Formulæ in terms of normal traces of such vector fields ([25], [10], [11], and also the latest developments given in [13]). More recently, the issue has been attacked also in less regular settings, like metric measure spaces (for example for "regular balls" in metric spaces equipped with a doubling measure and supporting a Poincaré inequality, [22], but also in [8], where the differential machinery developed in [15] allowed for a Gauss-Green formula in the very abstract context of geodesic metric measure spaces satisfying no further structural assumption, an analysis which, in a work in progress by the authors of the present paper and G. E. Comi, is then specialized with particular emphasis to the case of $\mathsf{RCD}(K,\infty)$ spaces) and stratified groups, [12].

In Section 4, finally, we compare our approach with the results recently obtained in [21] about the trace operator for BV functions defined by means of Lebesgue points. As an intermediate byproduct of this discussion, we first establish an extension property for functions of bounded variation (Proposition 4.4) and then conclude by finding the optimal conditions on the domain Ω which ensure the coincidence in the $\mathcal{S}^h$-almost everywhere sense - Theorem 4.8 - between the rough and the "classical" traces, $u^*(x) = \mathrm{T}u(x)$.

Sections 3 and 4 extend and refine the results contained in [8, Section 7.2].

2 Preliminaries

Throughout this paper, $(\mathbb{X}, d, \mu)$ will be a complete and separable metric measure space equipped with a non-negative Borel measure μ such that $0 < \mu\left(B_\rho(x)\right) < \infty$ for any ball $B_\rho(x) \subset \mathbb{X}$ with radius $\rho > 0$ centered at $x \in \mathbb{X}$. We shall assume μ to be *doubling*: in other words, there exists a constant $c \geq 1$ such that

$$\mu(B_{2\rho}(x)) \leq c\mu\left(B_\rho(x)\right), \qquad \forall\, x \in \mathbb{X}, \forall\, \rho > 0. \tag{1}$$

The minimal constant appearing in (1) is called *doubling constant* and will be denoted by c_D; $s := \ln_2 c_D$ is the *homogeneous dimension* of the metric space $\mathbb{X}$ and it is known that the following property holds,

$$\frac{\mu(B_r(x))}{\mu(B_R(y))} \geq \frac{1}{c_D^2}\left(\frac{r}{R}\right)^s, \tag{2}$$

for every $y \in \mathbb{X}$, $x \in B_R(y)$, and for every $0 < r \leq R < \infty$ (see for example [7, Lemma 3.3]).

The Lebesgue spaces $L^p(\mathbb{X}, \mu)$, $1 \leq p \leq \infty$ are defined in the obvious way, [17]; since in a complete doubling metric measure space balls are totally bounded, we can equivalently set $L^p_{\mathrm{loc}}(\mathbb{X}, \mu)$ to denote the space of functions that belong to $L^p(K, \mu)$ for any compact set K or that belong to $L^p(B_\rho(x_0), \mu)$ for any $x_0 \in \mathbb{X}$ and any $\rho > 0$.

Given a Lipschitz function $f : \mathbb{X} \to \mathbb{R}$, we define its *pointwise Lipschitz contant* as

$$\mathrm{Lip}(f)(x) = \limsup_{r \to 0} \sup_{B_r(x)} \frac{|f(x) - f(y)|}{d(x, y)}.$$

We assume that the space supports a weak $(1, 1)$–Poincaré inequality, which means that there exist constants $c_P > 0, \lambda \geq 1$ such that for any Lipschitz function f

$$\int_{B_\rho(x)} |f(y) - f_{B_\rho(x)}| \mathrm{d}\mu(y) \leq c_P \rho \int_{B_{\lambda\rho}(x)} |\mathrm{Lip}(f)(y)| \mathrm{d}\mu(y), \tag{3}$$

where f_E is the mean value of f over the set E, i.e. if $\mu(E) \neq 0$

$$f_E = \frac{1}{\mu(E)} \int_E f(y) \mathrm{d}\mu(y).$$

We recall also the definition of upper gradient; we say that a Borel function $g : \mathbb{X} \to [0, +\infty]$ is an *upper gradient* for a measurable function f if for any rectifiable curve $\gamma : [0, 1] \to \mathbb{X}$ with endpoints $x, y \in \mathbb{X}$ we have that

$$|f(x) - f(y)| \leq \int_\gamma g = \int_0^1 g(\gamma(t)) \|\gamma'(t)\| \mathrm{d}t$$

where $\|\gamma'(t)\| = \mathrm{Lip}(\gamma)(t)$.

In what follows, we shall also need to quantify how "dense" is a set at a certain point of the space; then, the *upper and lower μ-densities* of $E \subset \mathbb{X}$ at $x \in \mathbb{X}$ are given by

$$\Theta^*_\mu(E, x) := \limsup_{\rho \to 0^+} \frac{\mu(E \cap B_\rho(x))}{\mu(B_\rho(x))},$$

and

$$\Theta_{*,\mu}(E, x) := \liminf_{\rho \to 0^+} \frac{\mu(E \cap B_\rho(x))}{\mu(B_\rho(x))}$$

respectively. The common value between the two limits will be called the *μ-density* of E at $x \in \mathbb{X}$, denoted by

$$\Theta_\mu(E, x) := \lim_{\rho \to 0} \frac{\mu\left(E \cap B_\rho(x)\right)}{\mu\left(B_\rho(x)\right)}.$$

When we work with the reference measure μ only and there is no ambiguity, we shall drop the suffix from the notation and the above will be simply referred to as the (upper, lower) *density* of E at x.

Observe that since the maps $x \mapsto \mu(E \cap B_\rho(x))$ are continuous for any Borel set E, the functions $\Theta^*(E,x)$ and $\Theta_*(E,x)$ are Borel.

Following the characterization given for instance in [4, Definition 3.60], for a Borel set $E \subset \mathbb{X}$ we shall denote by $E^{(t)}$, $t \in [0,1]$, the set of points where E has density t, namely

$$E^{(t)} := \left\{ x \in \mathbb{X};\ \Theta(E,x) = \lim_{\rho \to 0^+} \frac{\mu\left(B_\rho(x) \cap E\right)}{\mu\left(B_\rho(x)\right)} = t \right\}.$$

In particular, the sets $E^{(0)}$ and $E^{(1)}$ will be called the *measure-theoretic* (or, *essential*) *exterior* and *interior* of E, respectively.

The *measure-theoretic* (or, *essential*) *boundary* of E is then defined as

$$\partial^* E := \mathbb{X} \backslash (E^{(0)} \cup E^{(1)}) = \{x \in \mathbb{X};\ 0 < \Theta(E,x) < 1\}. \tag{4}$$

Note that we could equivalently characterize $\partial^* E$ as the set of points $x \in \mathbb{X}$ where both E and its complement E^c have positive upper density.

The *lower and upper approximate limits* of any measurable function $u : \mathbb{X} \to \mathbb{R}$ at $x \in \mathbb{X}$ are defined by

$$u^\wedge(x) := \sup \left\{ t \in \overline{\mathbb{R}};\ \lim_{\rho \to 0^+} \frac{\mu\left(B_\rho(x) \cap E_t\right)}{\mu\left(B_\rho(x)\right)} = 1 \right\} \tag{5}$$

and

$$u^\vee(x) := \inf \left\{ t \in \overline{\mathbb{R}};\ \lim_{\rho \to 0^+} \frac{\mu\left(B_\rho(x) \cap E_t\right)}{\mu\left(B_\rho(x)\right)} = 0 \right\} \tag{6}$$

respectively, where for $t \in \mathbb{R}$, E_t denotes the *super-level sets* of the function u, namely

$$E_t := \{x;\ u(x) \geq t\}.$$

We observe that the density condition in (5) is of course equivalent to ask that $\Theta(E_t^c, x) = 0$. The notion of approximate limits allows for the characterization of a *jump set* of the function u:

$$S_u := \{x \in \mathbb{X};\ u^\wedge(x) < u^\vee(x)\}. \tag{7}$$

So in particular, when $u = \mathbb{1}_E$, one gets $S_u = \partial^* E$.

Following [2, 5, 24], we now briefly recall the basic notions and properties of functions of bounded variation on metric measure spaces. Given an open set $\Omega \subset \mathbb{X}$, we define the *total variation* of a measurable function $u : \Omega \to \mathbb{R}$ by setting

$$\|Du\|(\Omega) = \inf_{\mathcal{A}_u} \left\{ \liminf_{j\to+\infty} \int_\Omega \mathrm{Lip}(u_j)(y) \mathrm{d}\mu(y) \right\},$$

where

$$\mathcal{A}_u := \left\{ (u_j)_{j\in\mathbb{N}} \subset \mathrm{Lip}_{\mathrm{loc}}(\Omega),\ u_j \to u \text{ in } L^1_{\mathrm{loc}}(\Omega) \right\}.$$

Definition 2.1. *Given $u \in L^1(\Omega)$, we say that u has bounded variation in Ω, $u \in BV(\Omega)$, if $\|Du\|(\Omega) < +\infty$. A set $E \subset \mathbb{X}$ is said to have finite perimeter in $\mathbb{X}$ if $\mathbb{1}_E \in BV(\mathbb{X})$, and similarly to have finite perimeter in Ω if $\mathbb{1}_E \in BV(\Omega)$.*

Sets of finite perimeter will be also referred to as *Caccioppoli sets.*
A function $u \in BV(\Omega)$ defines a non-negative Radon measure $\|Du\|$, the *total variation measure*; when u is the characteristic function of some set E, $u = \mathbb{1}_E$, then $\|D\mathbb{1}_E\|$ is called *perimeter measure.*
A very important tool for our work will be the *Coarea Formula*, [24]; it asserts that for $u \in BV(\Omega)$, then for almost every $t \in \mathbb{R}$ the set E_t has finite perimeter in Ω and for any Borel set A

$$\|Du\|(A) = \int_\mathbb{R} \|D\mathbb{1}_{E_t}\|(A)\mathrm{d}t.$$

The Poincaré inequality together with the Sobolev embedding Theorem (see for instance [2], [16] or [24]) imply the following local isoperimetric inequality: for any set E with finite perimeter and for any ball $B_\rho(x)$, we have that

$$\min\{\mu(E\cap B_\rho), \mu(B_\rho \setminus E)\} \le c_I \left(\frac{\rho^s}{\mu(B_\rho(x))} \right)^{\frac{1}{s-1}} \|D\mathbb{1}_E\| (B_{\lambda\rho}(x))^{\frac{s}{s-1}}, \tag{8}$$

where $c_I > 0$ is known as the *isoperimetric constant.*
In our case, since in particular $(\mathbb{X}, d, \mu)$ supports a weak $(1,1)$-Poincaré inequality, (8) can be equivalently rewritten in the following form, [2, Remark 4.4]:

$$\min\{\mu(E\cap B_\rho), \mu(B_\rho \setminus E)\} \le c_I \left(\frac{\rho^s}{\mu(B_\rho(x))} \right)^{\frac{1}{s-1}} \|D\mathbb{1}_E\| (B_{\rho}(x))^{\frac{s}{s-1}}. \tag{9}$$

For the sake of completeness, we mention also that a weaker version of the Poincaré inequality holds for BV functions as well: given any ball $B_\rho(x) \subset \mathbb{X}$, for every $u \in BV(\mathbb{X})$ it holds

$$\int_{B_\rho(x)} |u - u_B| \,\mathrm{d}\mu \le c_P \rho \|Du\|(B_{\lambda\rho}(x)) \,. \tag{10}$$

Of course, both in (8) and (10) the notation is the same as in (3).

Two important properties of the perimeter measure of Caccioppoli sets, which we shall use extensively, are its absolute continuity with respect to the spherical Hausdorff measure and its localization inside the essential boundary, [2].
Let us denote by $\mathcal{S}^h$ the spherical Hausdorff measure defined in terms of the doubling function

$$h(B_\rho(x)) = \frac{\mu(B_\rho(x))}{\operatorname{diam}(B_\rho(x))}.$$

If $E \subset \mathbb{X}$ is a Caccioppoli set in $\mathbb{X}$, then we have the following

Theorem 2.1. [2, Theorem 5.3] *The measure $\|D\mathbb{1}_E\|$ is concentrated on the set*

$$\Sigma_\gamma = \left\{ x : \limsup_{\rho\to 0} \min \left\{ \frac{\mu\left(E \cap B_\rho(x)\right)}{\mu\left(B_\rho(x)\right)}, \frac{\mu\left(E^c \cap B_\rho(x)\right)}{\mu\left(B_\rho(x)\right)} \right\} \ge \gamma \right\} \subset \partial^* E,$$

where $\gamma = \gamma\left(c_D, c_I, \lambda\right)$. Moreover, $\mathcal{S}^h\left(\partial^* E \backslash \Sigma_\gamma\right) = 0$, $\mathcal{S}^h\left(\partial^* E\right) < \infty$ and

$$\|D\mathbb{1}_E\|(B) = \int_{B\cap\partial^* E} \theta_E \mathrm{d}\mathcal{S}^h$$

for any Borel set $B \subset \mathbb{X}$ and for some Borel map $\theta_E : \mathbb{X} \to [\alpha, \infty)$ with $\alpha = \alpha\left(c_P, c_I, \lambda\right) > 0$.

□

Remark 2.2. We explicitly observe that by [5, Theorem 4.6] one actually has $\theta_E \in [\alpha, c_D]$; thus, with the same notation as in Theorem 2.1, we are given the bounds

$$\alpha \mathcal{S}^h(B \cap \partial^* E) \le \|D\mathbb{1}_E\|(B) \le c_D \mathcal{S}^h(B \cap \partial^* E).$$

■

Definition 2.2. [5] *The space $(\mathbb{X}, d, \mu)$ will be called local if, given any two Caccioppoli sets $E, \Omega \subset \mathbb{X}$ with $E \subset \Omega$, one has that the constants arising from Theorem 2.1, θ_E and θ_Ω, coincide $\mathcal{S}^h$-almost everywhere on $\partial^*\Omega \cap \partial^* E$.*

Remark 2.3. In [5, Theorem 5.3], it was proven that for any function $u \in BV(\Omega)$, $\Omega \subset \mathbb{X}$ open set, the total variation measure $\|Du\|$ admits a decomposition into an "absolutely continuous" and a "singular" part, and that the latter is decomposable into a "Cantor" and a "jump" part. In other words, the following holds:

$$\begin{aligned}
\|Du\|(\Omega) &= \|Du\|^{\mathrm{a}}(\Omega) + \|Du\|^{\mathrm{s}}(\Omega) \\
&= \|Du\|^{\mathrm{a}}(\Omega) + \|Du\|^{\mathrm{c}}(\Omega) + \|Du\|^{\mathrm{j}}(\Omega) \\
&= \int_\Omega \mathfrak{a}\,\mathrm{d}\mu + \|Du\|^{\mathrm{c}}(\Omega) + \int_{\Omega\cap S_u} \int_{u^\wedge(x)}^{u^\vee(x)} \theta_{\{u\geq t\}}(x)\mathrm{d}t\mathrm{d}\mathcal{S}^h(x),
\end{aligned}$$

where $\mathfrak{a} \in L^1(\Omega)$ is the density of the absolutely continuous part, $\theta_{\{u\geq t\}}$ is given as in Theorem 2.1 and S_u is the jump set as in (7).

■

A further fact we shall use is the following localization property: if $E \subset \Omega$ has finite perimeter in an open set Ω, then the function

$$m_E(x,\rho) := \mu(E \cap B_\rho(x)) \tag{11}$$

is monotone non-decreasing as a function of ρ. If it is differentiable at $\rho > 0$, then

$$\|D\mathbb{1}_{E\cap B_\rho(x)}\|(\Omega) \leq m_E'(x,\rho) + \|D\mathbb{1}_E\| \left(\Omega \cap \overline{B}_\rho(x)\right). \tag{12}$$

The proof of (12) follows by considering a cut–off function

$$\eta_h(y) = \frac{1}{h}\min\{\max\{\rho + h - d(x,y), 0\}, 1\}$$

and defining $u_h = \eta_h \mathbb{1}_E$. Since

$$\|Du_h\|(\Omega) \leq \frac{1}{h}\int_{B_{\rho+h}(x)\setminus B_\rho(x)\cap\Omega} \mathbb{1}_E(y)\mathrm{d}\mu(y) + \|D\mathbb{1}_E\|\left(B_{\rho+h}(x)\cap\Omega\right),$$

passing to the limit as $h \to 0$ and using the lower semicontinuity of the total variation, we get

$$\|D\mathbb{1}_{E\cap B_\rho(x)}\|(\Omega) \leq m_E'(x,\rho) + \|D\mathbb{1}_E\| \left(\overline{B}_\rho(x) \cap \Omega\right).$$

If in particular $\|D\mathbb{1}_E\|\left(\Omega \cap \partial B_\rho(x)\right) = 0$, we then get

$$\|D\mathbb{1}_{E\cap B_\rho(x)}\|\left(\partial B_\rho(x) \cap \Omega\right) \leq m_E'(x,\rho).$$

3 Rough Trace

In this section we extend the notion of rough trace of a BV function to the metric measure space setting. The discussion will closely follow the monograph by V. Maz'ya [23, Section 9.5], whose results will be rephrased and re-proven accordingly; in particular, we shall focus on the issue of the integrability of rough trace with respect to the perimeter measure of the domain. We shall relate this issue with some geometric properties of the domain.

Below, $\Omega \subset \mathbb{X}$ shall always denote a bounded open set.

Definition 3.1. (Rough Trace) *Given $u \in BV(\Omega)$, we define its rough trace at $x \in \partial\Omega$ as the quantity*

$$u^*(x) := \sup\left\{t \in \mathbb{R} : \|D\mathbb{1}_{E_t}\|(\mathbb{X}) < \infty,\ x \in \partial^* E_t\right\}. \tag{13}$$

Of course, when u has a limit value inside Ω at $x \in \partial^*\Omega$, then

$$u^*(x) = \lim_{\Omega \ni y \to x} u(y).$$

We start with the following result:

Lemma 3.1. *If $\|D\mathbb{1}_\Omega\|(\mathbb{X}) < \infty$ and $u \in BV(\Omega)$, then u^* is $\mathcal{S}^h$- measurable on $\partial^*\Omega$ and*

$$\mathcal{S}^h\left(\{x \in \partial^*\Omega;\ u^*(x) \geq t\}\right) = \mathcal{S}^h\left(\partial^*\Omega \cap \partial^* E_t\right) \tag{14}$$

for almost every $t \in \mathbb{R}$.

Proof. We fix a measurable set $I \subset \mathbb{R}$ such that $|I| = 0$ and E_t has finite perimeter for any $t \in \mathbb{R}\backslash I$. Let us also fix $D \subset \mathbb{R}\backslash I$ countable and dense. Then the set

$$A_t = \{x \in \partial^*\Omega : u^*(x) \geq t\} = \bigcup_{t<s\in D} B_s$$

with $B_s = \partial^* E_s \cap \partial^*\Omega$. Since the sets B_s and $\partial^*\Omega$ are Borel sets, A_t is Borel and then u^* is a Borel function.

Now instead of (14), we shall prove that for every $t \in \mathbb{R}\backslash I$ - except at most a countable set - it holds

$$\mathcal{S}^h\Big(\{x \in \partial^*\Omega;\ u^*(x) \geq t\} \,\triangle (\partial^* E_t \cap \partial^*\Omega)\Big) = 0,$$

where $\triangle$ denotes the symmetric difference between two sets, $A\triangle B := (A\backslash B) \cup (B\backslash A)$.

We define the Borel set $F_t \coloneqq A_t \backslash B_t$. If $x \in B_t$, the definition of u^* implies that $u^*(x) \geq t$ and then the inclusion $B_t \subset A_t$ holds. We then reduce ourselves to prove that $\mathcal{S}^h(F_t) = 0$. Since for $s < t$ we have that $E_t \subset E_s \subset \Omega$, we also have that

$$\frac{\mu(E_t \cap B_\rho(x))}{\mu(B_\rho(x))} \leq \frac{\mu(E_s \cap B_\rho(x))}{\mu(B_\rho(x))} \leq \frac{\mu(\Omega \cap B_\rho(x))}{\mu(B_\rho(x))},$$

whence

$$\Theta_*(E_t, x) \leq \Theta_*(E_s, x) \leq \Theta^*(E_s, x) \leq \Theta^*(\Omega, x)$$

and so the inclusion $B_t \subset B_s$ holds true. From this we deduce that the sets F_t are disjoint; indeed if $s < t$,

$$F_t \cap F_s = (A_t \backslash B_t) \cap (A_s \backslash B_s) = A_t \cap A_s \cap B_t^c \cap B_s^c = A_t \backslash B_s = \emptyset$$

since if $x \in A_t$ then there exists $\tau \in (t, u^*(x)]$ such that $x \in \partial^* E_\tau \cap \partial^* \Omega = B_\tau \subset B_s$. The inclusion $F_t \subset \partial^* \Omega$ then implies that the set

$$\left\{t \in \mathbb{R} \backslash I;\, \mathcal{S}^h(F_t) > 0\right\}$$

is at most countable, and this concludes the proof. □

The result below is simply a combination of Lemma 4 and Corollary 2 in [23, Section 9.5], so we just state it with no proof:

Proposition 3.2. *For any $u \in BV(\Omega)$ and for $\mathcal{S}^h$-almost every $x \in \partial^* \Omega$, one has*

$$-u^*(x) = (-u)^*(x).$$

Consequently if we decompose $u = u^+ - u^-$ in its positive and negative part, then

$$(u^*)^+ = \left(u^+\right)^*, \qquad (u^*)^- = \left(u^-\right)^*$$

and then

$$u^* = \left(u^+\right)^* - \left(u^-\right)^*.$$

□

Remark 3.3. Throughout the remainder of this section, we shall always work in the hypothesis that $\Omega \subset \mathbb{X}$ is a bounded open set with finite perimeter in $\mathbb{X}$, $\|D\mathbb{1}_\Omega\|(\mathbb{X}) < \infty$.

Moreover, $E \subset \Omega$ will always be a Caccioppoli set in Ω, $\|D\mathbb{1}_E\|(\Omega) < \infty$; this, by the locality of the perimeter measure, implies that $\|D\mathbb{1}_E\|(\mathbb{X}) < \infty$ as well.

Indeed, given any three sets $A, B, C \subset \mathbb{X}$ such that $\mu((A \triangle B) \cap C) = 0$, one has $\|D\mathbb{1}_A\|(C) = \|D\mathbb{1}_B\|(C)$ by the remarks in [2, Section 4] and [24]; therefore, if we take $A = E$, $B = \Omega$ and $C = \Omega^c$ we obviously have $\mu((E \triangle \Omega) \cap \Omega^c) = 0$ and then $\|D\mathbb{1}_E\|(\Omega^c) = \|D\mathbb{1}_\Omega\|(\Omega^c)$ which is finite since Ω is a Caccioppoli set in $\mathbb{X}$.

All in all, this forces

$$\|D\mathbb{1}_E\|(\mathbb{X}) = \|D\mathbb{1}_E\|(\Omega) + \|D\mathbb{1}_E\|(\Omega^c) < \infty,$$

which is our claim.

For completeness of information, we also recall that by [18, Proposition 6.3], whenever $\Omega \subset \mathbb{X}$ is an open set such that $\mathcal{S}^h(\partial\Omega) < \infty$, then one has $\|D\mathbb{1}_E\|(\mathbb{X}) < \infty$ for any Borel set $E \subset \Omega$ with $\|D\mathbb{1}_E\|(\Omega) < \infty$; therefore, by [20, Theorem 1.1] - and then also by [20, Corollary 5.1], in particular - we could equivalently ask that $\|D\mathbb{1}_\Omega\|(\mathbb{X}) < \infty$ or that $\mathcal{S}^h(\partial\Omega) < \infty$ to infer that the set $E \subset \Omega$ in our configuration, namely $\|D\mathbb{1}_E\|(\Omega) < \infty$, is a Caccioppoli set in $\mathbb{X}$ as well.

For the sake of clarity, we observe that we will always operate in the situation where $\partial^* E$ intersects $\partial^*\Omega$ and $\partial^*\Omega \backslash \partial^* E$ is non-empty.

■

In the next results, we will often make use of the following simple property of the rough trace:

Remark 3.4. Let $\Omega \subset \mathbb{X}$ be such that $\|D\mathbb{1}_\Omega\|(\mathbb{X}) < \infty$. If $E \subset \Omega$ is a Caccioppoli set in Ω, then $\mathbb{1}_E^*(x) = 0$ for all $x \in \partial^*\Omega \backslash \partial^* E$ and $\mathbb{1}_E^*(x) = 1$ for all $x \in \partial^*\Omega \cap \partial^* E$.

Indeed, when considering the characteristic function of E one of course has

$$E_t = \{x \in \Omega,\ \mathbb{1}_E \geq t\} = \begin{cases} \emptyset, & t > 1 \\ \Omega, & t \leq 0 \\ \Omega \cap E = E, & t \in (0, 1]. \end{cases}$$

This means, obviously,

$$\partial^* E_t = \begin{cases} \emptyset, & t > 1 \\ \partial^*\Omega, & t \leq 0 \\ \partial^*(\Omega \cap E) = \partial^* E, & t \in (0, 1]. \end{cases}$$

So, when $t \in (0,1]$, the definition of rough trace (13) forces $\mathbb{1}_E^*(x) = 1$ for every $x \in \partial^* E$.

Let us then assume $t \leq 0$; again by (13), in order to have $x \in \partial^* E_t = \partial^* \Omega$, it must be $\mathbb{1}_E^*(x) = 0$ for every x therein. Thus, combining with the conclusion right above, as $\partial^* E$ intersects $\partial^* \Omega$, we infer that $\mathbb{1}_E^*(x) = 0$ for all $x \in \partial^* \Omega \setminus \partial^* E$ and $\mathbb{1}_E^*(x) = 1$ for all $x \in \partial^* \Omega \cap \partial^* E$, proving the claim.

■

With these preliminary facts at our disposal, we can start discussing the summability of the rough trace.

Theorem 3.5. *Let $\|D\mathbb{1}_\Omega\|(\mathbb{X}) < \infty$ and assume $\mathcal{S}^h(\partial\Omega \setminus \partial^*\Omega) = 0$. In order for any $u \in BV(\Omega)$ to satisfy*

$$\inf_{c \in \mathbb{R}} \int_{\partial\Omega} |u^*(x) - c| \, \mathrm{d}\mathcal{S}^h(x) \leq k \, \|Du\|(\Omega)$$

with $k > 0$ independent of u, it is necessary and sufficient that the inequality

$$\min \left\{ \|D\mathbb{1}_E\|(\Omega^c), \|D\mathbb{1}_{\Omega \setminus E}\|(\Omega^c) \right\} \leq k \|D\mathbb{1}_E\|(\Omega)$$

holds for any $E \subset \Omega$ with finite perimeter in Ω.

Proof. We start with necessity. Let $E \subset \Omega$ be such that $\|D\mathbb{1}_E\|(\Omega) < \infty$, and apply Remark 3.3 to infer that $\|D\mathbb{1}_E\|(\mathbb{X}) < \infty$. Then, since $\mathcal{S}^h(\partial\Omega \setminus \partial^*\Omega) = 0$, by Remark 3.4 we get

$$\inf_{c \in \mathbb{R}} \int_{\partial^*\Omega} |\mathbb{1}_E^*(x) - c| \, \mathrm{d}\mathcal{S}^h(x) = \min_{c \in \mathbb{R}} \left\{ |1 - c| \mathcal{S}^h(\partial^* E \cap \partial^* \Omega) + |c| \mathcal{S}^h(\partial^* \Omega \setminus \partial^* E) \right\}.$$

Now observe that the function

$$\Phi(c) \coloneqq |1 - c| \mathcal{S}^h(\partial^* E \cap \partial^* \Omega) + |c| \mathcal{S}^h(\partial^* \Omega \setminus \partial^* E),$$

$c \in \mathbb{R}$, clearly attains its minima when $c = 0$ and $c = 1$, so we actually have

$$\begin{aligned} \inf_{c \in \mathbb{R}} \int_{\partial^*\Omega} |\mathbb{1}_E^*(x) - c| \, \mathrm{d}\mathcal{S}^h(x) &= \min \left\{ \mathcal{S}^h(\partial^*\Omega \cap \partial^* E), \mathcal{S}^h(\partial^*\Omega \setminus \partial^* E) \right\} \\ &\leq \frac{1}{\alpha} \min \left\{ \|D\mathbb{1}_E\|(\Omega^c), \|D\mathbb{1}_{\Omega \setminus E}\|(\Omega^c) \right\} \end{aligned}$$

by Remark 2.2.

Since by hypothesis

$$\inf_{c\in\mathbb{R}} \int_{\partial\Omega} |\mathbb{1}^*_E(x) - c| \,\mathrm{d}\mathcal{S}^h(x) \le k \|D\mathbb{1}_E\|(\Omega),$$

we then obtain our claim.

We now pass to sufficiency. Let $u \in BV(\Omega)$; then for every t, $\mathcal{S}^h(\partial\Omega \cap \partial^* E_t)$ is a non-increasing function of t. In fact, if $x \in \partial^*\Omega \cap \partial^* E_t$ and $\tau < t$, then $\Omega \supset E_\tau \supset E_t$ and the same holds as well for the essential boundaries; moreover,

$$\Theta(E_t, x) \le \Theta(E_\tau, x) \le \Theta(\Omega, x).$$

This means, by hypothesis and by the definition of essential boundary (4), that $0 < \Theta(E_\tau, x) < 1$ and then $x \in \partial^*\Omega \cap \partial^* E_\tau$. In a similar manner we can show that $\mathcal{S}^h(\partial\Omega \setminus \partial^* E_t)$ is a non-decreasing function of t. By the Coarea Formula and by Remark 2.2,

$$k\|Du\|(\Omega) = k\int_{\mathbb{R}} \|D\mathbb{1}_{E_t}\|(\Omega)\mathrm{d}t \ge k\alpha \int_{\mathbb{R}} \min\left\{\mathcal{S}^h(\partial\Omega \cap \partial^* E_t), \mathcal{S}^h(\partial\Omega\setminus\partial^* E_t)\right\} \mathrm{d}t.$$

If we now set

$$t_0 \coloneqq \sup\left\{t;\ \|D\mathbb{1}_{E_t}\|(\mathcal{X}) < \infty,\ \mathcal{S}^h(\partial\Omega \cap \partial^* E_t) \ge \mathcal{S}^h(\partial\Omega\setminus\partial^* E_t)\right\},$$

then we get, by recalling Lemma 3.1,

$$\begin{aligned}
k\|Du\|(\Omega) &\ge k\alpha\left(\int_{t_0}^{+\infty} \mathcal{S}^h(\partial\Omega \cap \partial^* E_t)\,\mathrm{d}t + \int_{-\infty}^{t_0} \mathcal{S}^h(\partial\Omega\setminus\partial^* E_t)\,\mathrm{d}t\right) \\
&= k\alpha\left(\int_{t_0}^{+\infty} \mathcal{S}^h(\{x;\ u^*(x) \ge t\})\,\mathrm{d}t + \int_{-\infty}^{t_0} \mathcal{S}^h(\{x;\ u^*(x) \le t\})\,\mathrm{d}t\right) \\
&= k\alpha\left(\int_{\partial\Omega} [u^*(x) - t_0]^+ \,\mathrm{d}\mathcal{S}^h(x) + \int_{\partial\Omega} [u^*(x) - t_0]^- \,\mathrm{d}\mathcal{S}^h(x)\right) \\
&= k\alpha \int_{\partial\Omega} |u^*(x) - t_0| \,\mathrm{d}\mathcal{S}^h(x).
\end{aligned}$$

In other words,

$$\frac{k}{\alpha}\|Du\|(\Omega) = k'\|Du\|(\Omega) \ge \inf_{c\in\mathbb{R}} \int_{\partial\Omega} |u^*(x) - c| \,\mathrm{d}\mathcal{S}^h(x).$$

□

Definition 3.2. *Let $A \subset \overline{\Omega}$. We shall denote by $\zeta_A^{(\alpha)}$ the infimum of those $k > 0$ such that $[\|D\mathbb{1}_E\|(\Omega^c)]^\alpha \leq k \|D\mathbb{1}_E\|(\Omega)$ for all sets $E \subset \Omega$ which satisfy $\mu(E \cap A) + \mathcal{S}^h(A \cap \partial^* E) = 0$.*

Theorem 3.6. *Let $\|D\mathbb{1}_\Omega\|(\mathbb{X}) < \infty$ and assume $\mathcal{S}^h(\partial\Omega \backslash \partial^*\Omega) = 0$. Then, if A is as in Definition 3.2, for every $u \in BV(\Omega)$ such that $u|_{A\cap\Omega} = 0$ and $u^*|_{A\cap\partial^*\Omega} = 0$, then there is a constant $c > 0$, depending on $\zeta_A^{(1)}$ and on c_D, such that*

$$\int_{\partial\Omega} |u^*(x)| \,\mathrm{d}\mathcal{S}^h(x) \leq c \|Du\|(\Omega).$$

Moreover, the constant c is sharp.

Proof. We know that

$$\int_{\partial\Omega} |u^*(x)| \,\mathrm{d}\mathcal{S}^h(x) = \int_0^{+\infty} \left[\mathcal{S}^h(\{x;\ u^*(x) \geq t\}) + \mathcal{S}^h(\{x;\ -u^*(x) \geq t\})\right] \mathrm{d}t.$$

Notice that, by Lemma 3.1 and Remark 2.2,

$$\begin{aligned} \int_0^{+\infty} \mathcal{S}^h(\{x;\ u^*(x) \geq t\}) \,\mathrm{d}t &= \int_0^{+\infty} \mathcal{S}^h(\partial^*\Omega \cap \partial^* E_t) \mathrm{d}t \\ &\leq c_D \int_0^{+\infty} \|D\mathbb{1}_{E_t}\|(\Omega^c) \,\mathrm{d}t \\ &\leq c_D \zeta_A^{(1)} \int_0^{+\infty} \|D\mathbb{1}_{E_t}\|(\Omega) \mathrm{d}t, \end{aligned}$$

where we used the definition of $\zeta_A^{(1)}$ and the fact that, by our hypotheses, we get $\mu(A \cap E_t) + \mathcal{S}^h(A \cap \partial^* E_t) = 0$ for almost every t.
Similarly, we find

$$\begin{aligned} \int_0^{+\infty} \mathcal{S}^h(\{x;\ -u^*(x) \geq t\}) \,\mathrm{d}t \leq& c_D \int_{-\infty}^0 \left\|D\mathbb{1}_{\Omega\backslash E_t}\right\|(\Omega^c) \,\mathrm{d}t \\ \leq& c_D \zeta_A^{(1)} \int_{-\infty}^0 \|D\mathbb{1}_{E_t}\|(\Omega) \mathrm{d}t. \end{aligned}$$

Therefore, letting $c \coloneqq c_D \zeta_A^{(1)}$ gives the assertion.

To deduce that c is sharp, it suffices to substitute u with $\mathbb{1}_E$, taking E as in Definition 3.2. Indeed, in this case by Remark 3.4 we would simply have

$$\begin{aligned}\|\mathbb{1}_E^*(x)\|_{L^1(\partial\Omega,\mathcal{S}^h)} &= \int_{\partial\Omega} |\mathbb{1}_E^*(x)| \mathrm{d}\mathcal{S}^h(x) \\ &= \mathcal{S}^h(\partial^*\Omega \cap \partial^* E) \leq c_D \|D\mathbb{1}_E\|(\Omega^c) \\ &\leq c_D \zeta_A^{(1)} \|D\mathbb{1}_E\|(\Omega) = c\|D\mathbb{1}_E\|(\Omega),\end{aligned}$$

where we explicitly used the assumption $\mathcal{S}^h(\partial\Omega \cap \partial^*\Omega) = 0$.

□

The most important result of the present section is the following re-adaptation of [23, Theorem 9.5.4]:

Theorem 3.7. *Let* $\|D\mathbb{1}_\Omega\|(\mathbb{X}) < \infty$ *and assume* $\mathcal{S}^h(\partial\Omega\backslash\partial^*\Omega) = 0$*. Then, every* $u \in BV(\Omega)$ *satisfies*

$$\|u^*\|_{L^1(\partial\Omega,\mathcal{S}^h)} \leq c\,\|u\|_{BV(\Omega)}$$

with a constant $c > 0$ *independent of* u*, if and only if there exists* $\delta > 0$ *such that for every* $E \subset \Omega$ *with* $\mathrm{diam}(E) \leq \delta$ *and with* $\|D\mathbb{1}_E\|(\Omega) < \infty$ *there holds*

$$\|D\mathbb{1}_E\|(\Omega^c) \leq c' \|D\mathbb{1}_E\|(\Omega) \tag{15}$$

for some constant $c' > 0$ *independent of* E*.*

Proof. We start by recalling that by Remark 3.4, one has $\mathbb{1}_E^*(x) = 1$ on $\partial^*\Omega \cap \partial^* E$ and $\mathbb{1}_E^*(x) = 0$ on $\partial^*\Omega\backslash\partial^* E$. Therefore,

$$\|\mathbb{1}_E^*\|_{L^1(\partial\Omega)} = \int_{\partial\Omega} \mathbb{1}_E^*(x)\mathrm{d}\mathcal{S}^h(x) = \mathcal{S}^h(\partial^*\Omega \cap \partial^* E),$$

since by hypothesis $\mathcal{S}^h(\partial\Omega\backslash\partial^*\Omega) = 0$.

Now, assume $\|u^*\|_{L^1(\partial\Omega)} \leq c\,\|u\|_{BV(\Omega)} = c\Big(\|u\|_{L^1(\Omega)} + \|Du\|(\Omega)\Big)$; for $u = \mathbb{1}_E$,

$$\begin{aligned}\|D\mathbb{1}_E\|(\Omega^c) &= \|D\mathbb{1}_E\|(\partial\Omega) \\ &\leq c_D \mathcal{S}^h(\partial\Omega \cap \partial^* E) \\ &= c_D \mathcal{S}^h(\partial^*\Omega \cap \partial^* E) \\ &= c_D \|\mathbb{1}_E^*\|_{L^1(\partial\Omega)} \leq c\left(\mu(\Omega \cap E) + \|D\mathbb{1}_E\|(\Omega)\right),\end{aligned}$$

where we labeled by c the product $c \cdot c_D$ and $\mu(\Omega \cap E) = \mu(E)$ since $E \subset \Omega$. By the boundedness of Ω, there exists a $\delta > 0$ such that, for a fixed $x_0 \in \mathbb{X}$ one has $E \Subset B_\delta(x_0)$, meaning that $E \subset B_\delta(x_0)$ and $\mathrm{dist}(E, B_\delta(x_0)^c) > 0$. We choose δ in such a way that can write

$$\mu(E) = \mu\left(E \cap B_\delta\left(x_0\right)\right) = \min\left\{\mu\left(E \cap B_\delta\left(x_0\right)\right), \mu\left(B_\delta\left(x_0\right) \setminus E\right)\right\}.$$

Note that if we take $\rho > \delta$ such that $\mu\left(B_\rho\left(x_0\right) \setminus \mu\left(B_\delta\left(x_0\right)\right)\right) \geq \mu\left(B_\delta\left(x_0\right)\right)$, then the above still holds.

Applying the Poincaré inequality for BV functions,

$$\begin{aligned}\int_{B_\rho(x_0)} \left|\mathbb{1}_E - \left(\mathbb{1}_E\right)_{B_\rho(x_0)}\right| \mathrm{d}\mu &= \int_{B_\rho(x_0)} \left|\mathbb{1}_E - \frac{\mu\left(E \cap B_\rho\left(x_0\right)\right)}{\mu\left(B_\rho\left(x_0\right)\right)}\right| \mathrm{d}\mu \\ &\leq c\rho \left\|D\mathbb{1}_E\right\|\left(B_{\lambda\rho}\left(x_0\right)\right).\end{aligned}$$

Since $B_\rho\left(x_0\right) = E \cup \left(B_\rho\left(x_0\right) \setminus E\right)$, computing the integral gives

$$\begin{aligned}&\mu\left(E \cap B_\rho\left(x_0\right)\right)\left(1 - \frac{\mu\left(E \cap B_\rho\left(x_0\right)\right)}{\mu\left(B_\rho\left(x_0\right)\right)}\right) + \frac{\mu\left(E \cap B_\rho\left(x_0\right)\right)}{\mu\left(B_\rho\left(x_0\right)\right)}\mu\left(B_\rho\left(x_0\right) \setminus E\right) = \\ &\quad= \frac{\mu\left(E \cap B_\rho\left(x_0\right)\right)}{\mu\left(B_\rho\left(x_0\right)\right)}\mu\left(B_\rho\left(x_0\right) \setminus E\right) + \frac{\mu\left(E \cap B_\rho\left(x_0\right)\right)}{\mu\left(B_\rho\left(x_0\right)\right)}\mu\left(B_\rho\left(x_0\right) \setminus E\right) \\ &\quad= 2\frac{\mu\left(E \cap B_\rho\left(x_0\right)\right)}{\mu\left(B_\rho\left(x_0\right)\right)}\mu\left(B_\rho\left(x_0\right) \setminus E\right) \\ &\quad= 2\mu\left(E \cap B_\rho\left(x_0\right)\right)\left(1 - \frac{\mu\left(B_\rho\left(x_0\right) \cap E\right)}{\mu\left(B_\rho\left(x_0\right)\right)}\right).\end{aligned}$$

As $\mu\left(B_\delta\left(x_0\right)\right) \leq \mu\left(B_\rho\left(x_0\right) \setminus B_\delta\left(x_0\right)\right)$, again by the Poincaré inequality we get

$$\begin{aligned}\mu\left(E \cap B_\rho\left(x_0\right)\right) &\leq 2\mu\left(E \cap B_\rho\left(x_0\right)\right)\left(1 - \frac{\mu\left(B_\rho\left(x_0\right) \cap E\right)}{\mu\left(B_\rho\left(x_0\right)\right)}\right) \\ &\leq c\rho \left\|D\mathbb{1}_E\right\|\left(B_{\lambda\rho}\left(x_0\right)\right) \\ &= c\rho \left\|D\mathbb{1}_E\right\|\left(\mathbb{X}\right) \\ &= c\rho\left(\left\|D\mathbb{1}_E\right\|\left(\Omega\right) + \left\|D\mathbb{1}_E\right\|\left(\partial\Omega\right)\right) \\ &= c\rho\left(\left\|D\mathbb{1}_E\right\|\left(\Omega\right) + \left\|D\mathbb{1}_E\right\|\left(\Omega^c\right)\right),\end{aligned}$$

which, by the estimate $\left\|D\mathbb{1}_E\right\|\left(\Omega^c\right) \leq c_D\left(\left\|D\mathbb{1}_E\right\|\left(\Omega\right) + \mu(E)\right)$ previously found, entails

$$\begin{aligned}\|D\mathbb{1}_E\|(\Omega^c) &\le c_D\left(\|D\mathbb{1}_E\|(\Omega)+\mu\left(E\cap B_\rho\left(x_0\right)\right)\right)\\ &\le c_D\left(\|D\mathbb{1}_E\|(\Omega)+c\rho\,\|D\mathbb{1}_E\|(\Omega)+c\rho\,\|D\mathbb{1}_E\|(\Omega^c)\right)\\ &= c_D\left(1+c\rho\right)\|D\mathbb{1}_E\|(\Omega)+c_D\cdot c\rho\,\|D\mathbb{1}_E\|(\Omega^c)\,,\end{aligned}$$

whence

$$\|D\mathbb{1}_E\|(\Omega^c)\le c_D\frac{1+\rho}{1-c_D\cdot c\rho}\|D\mathbb{1}_E\|(\Omega)=c'\,\|D\mathbb{1}_E\|(\Omega),$$

where we of course require $\rho<\frac{1}{c_D\cdot c}$.

Let us now show the reverse implication; assume then that $\|D\mathbb{1}_E\|(\Omega^c)\le c\,\|D\mathbb{1}_E\|(\Omega)$ for any finite perimeter set $E\subset\Omega$ with diameter less then δ. This in particular implies that, by Remark 2.2,

$$\begin{aligned}\mathcal{S}^h(\partial^*E\cap\partial^*\Omega)\le&\frac{1}{\alpha}\|D\mathbb{1}_E\|(\partial^*\Omega)=\frac{1}{\alpha}\|D\mathbb{1}_E\|(\partial\Omega)\\ =&\frac{1}{\alpha}\|D\mathbb{1}_E\|(\Omega^c)\le\frac{c'}{\alpha}\|D\mathbb{1}_E\|(\Omega).\end{aligned}$$

Let us fix then $u\in BV(\mathbb{X})$ and assume $u\ge 0$; by Lemma 3.1 and Cavalieri's Principle, we obtain that

$$\|u^*\|_{L^1(\partial\Omega,\mathcal{S}^h)}=\int_0^\infty\mathcal{S}^h(\{x\in\partial^*\Omega: u^*(x)\ge t\})\mathrm{d}t=\int_0^\infty\mathcal{S}^h(\partial^*E_t\cap\partial^*\Omega)\mathrm{d}t. \quad (16)$$

Take $t\in[0,\infty)$ such that E_t has finite perimeter in Ω and set $E=E_t$. We fix $r>0$ such that $2r<\delta$ and consider a covering of $\mathbb{X}$ made of balls of the type $B_r(x_i)$, $i\in I\subset\mathbb{N}$ such that $B_{2r}(x_i)$ have overlapping bounded by $c_o>0$. We also select $r_i\in(r,2r)$ such that $m_E(x,\cdot)$ is differentiable at r_i and

$$m_E'(x,r_i)\le 2m_E(x,2r);$$

This is possible since $\rho\mapsto m_E(x,\rho)$ is monotone non decreasing and

$$\int_r^{2r}m_E'(x,\rho)d\rho\le m_E(x,2r)-m_E(x,r)\le m_E(x,2r),$$

so that

$$|\{t\in(r,2r): m_E'(x,t)>2m_E(x,2r)\}|<\frac{r}{2}.$$

We shall denote $B_i := B_{r_i}(x_i)$. Notice that for any set E, $\partial^* E \cap B_i \subset \partial^*(E \cap B_i)$; indeed for any $x \in E \cap B_i$, there exists $\rho_0 > 0$ such that $B_\rho(x) \subset B_i$ for any $\rho < \rho_0$, hence

$$\frac{\mu(E \cap B_i \cap B_\rho(x))}{\mu(B_\rho(x))} = \frac{\mu(E \cap B_\rho(x))}{\mu(B_\rho(x))}.$$

So we have that

$$\mathcal{S}^h(\partial^* E \cap \partial^* \Omega) \leq \sum_{i \in I} \mathcal{S}^h(\partial^* E \cap \partial^* \Omega \cap B_i) \leq \sum_{i \in I} \mathcal{S}^h(\partial^*(E \cap B_i) \cap \partial^* \Omega).$$

From this, using (12) and the fact that $r_i < \delta$, by assumption,

$$\begin{aligned}
\mathcal{S}^h(\partial^* E \cap \partial^* \Omega) &\leq \sum_{i \in I} \mathcal{S}^h(\partial^*(E \cap B_i) \cap \partial^* \Omega) \leq \frac{1}{\alpha} \sum_{i \in I} \|D\mathbb{1}_{E \cap B_i}\| (\partial^* \Omega) \\
&= \frac{1}{\alpha} \sum_{i \in I} \|D\mathbb{1}_{E \cap B_i}\| (\Omega^c) \leq \frac{c'}{\alpha} \sum_{i \in I} \|D\mathbb{1}_{E \cap B_i}\|(\Omega) \\
&\leq \frac{c'}{\alpha} \sum_{i \in I} \left(m'_E(x, r_i) + \|D\mathbb{1}_E\|(\Omega \cap \overline{B}_{r_i}) \right) \\
&\leq \frac{c'}{\alpha} \sum_{i \in I} \left(m_E(x, 2r) + \|D\mathbb{1}_E\|(\Omega \cap B_{2r}(x_i)) \right) \\
&\leq \frac{c' c_o}{\alpha} \left(\mu(E \cap \Omega) + \|D\mathbb{1}_E\|(\Omega) \right).
\end{aligned}$$

So, recalling that $E = E_t$, we have obtained the estimate

$$\mathcal{S}^h(\partial^* E_t \cap \partial^* \Omega) \leq \frac{c' c_o}{\alpha} \left(\mu(E_t \cap \Omega) + \|D\mathbb{1}_{E_t}\|(\Omega) \right).$$

Integrating this inequality and using Coarea formula we then conclude that

$$\|u^*\|_{L^1(\partial\Omega, \mathcal{S}^h)} \leq \frac{c' c_o}{\alpha} \Big(\|u\|_{L^1(\Omega)} + \|Du\|(\Omega) \Big).$$

The general case $u \in BV(\Omega)$ can be done by splitting $u = u^+ - u^-$ into its positive and negative part.

□

We end this discussion by considering *en passant* the issue of the extendability of a BV function in terms of its rough trace.

Definition 3.3. *Let $u \in BV(\Omega)$. We define its β-extension to $\mathbb{X}$, $\beta \in \mathbb{R}$, by setting*

$$u_\beta(x) := \begin{cases} u(x), & x \in \Omega \\ \\ \beta, & x \in \Omega^c. \end{cases}$$

We then have the following:

Lemma 3.8. *Assume $\Omega \subset \mathbb{X}$ is such that $\|D\mathbb{1}_\Omega\|(\mathbb{X}) < \infty$ and $\mathcal{S}^h(\partial\Omega \setminus \partial^*\Omega) = 0$. Let $\beta \in \mathbb{R}$ and $u \in BV(\Omega)$. Then, one has*

$$\|Du_\beta\|(\mathbb{X}) \leq \|Du\|(\Omega) + c_D\|u^* - \beta\|_{L^1(\partial\Omega, \mathcal{S}^h)}.$$

Proof. By Coarea, one obviously has

$$\|Du\|(\Omega) = \int_{\mathbb{R}} \|D\mathbb{1}_{E_t}\|(\Omega)\mathrm{d}t.$$

We observe that one may also write

$$\|Du\|(\Omega) = \int_0^{+\infty} \|D\mathbb{1}_{\{|u|\geq t\}}\|(\Omega)\mathrm{d}t = \|D(|u|)\|(\Omega).$$

Moreover, since any two functions differing by an additive constant have the same total variation, the following holds as well:

$$\|Du\|(\Omega) = \|D(|u-\beta|)\|(\Omega) = \int_0^{+\infty} \|D\mathbb{1}_{\{|u-\beta|\geq t\}}\|(\Omega)\mathrm{d}t.$$

Therefore, if we consider the β-extension of u from Ω to the whole of $\mathbb{X}$,

$$\begin{aligned}\|Du_\beta\|(\mathbb{X}) &= \int_0^{+\infty} \|D\mathbb{1}_{\{|u_\beta-\beta|\geq t\}}\|(\mathbb{X})\mathrm{d}t \\ &= \int_0^{+\infty} \|D\mathbb{1}_{\{|u_\beta-\beta|\geq t\}}\|(\Omega)\mathrm{d}t + \int_0^{+\infty} \|D\mathbb{1}_{\{|u_\beta-\beta|\geq t\}}\|(\Omega^c)\mathrm{d}t.\end{aligned}$$

Now, by the hypothesis $\mathcal{S}^h(\partial\Omega \setminus \partial^*\Omega) = 0$ and by Remark 2.2,

$$\int_0^{+\infty} \|D\mathbb{1}_{\{|u_\beta-\beta|\geq t\}}\|(\Omega^c)\mathrm{d}t$$

$$\leq c_D \left(\int_{-\infty}^{0} \mathcal{S}^h \left(\{x;\ (u-\beta)^* \leq t\} \right) \mathrm{d}t + \int_{0}^{+\infty} \mathcal{S}^h \left(\{x;\ (u-\beta)^* \geq t\} \right) \mathrm{d}t \right),$$

which in turn equals

$$c_D \int_{\partial\Omega} |(u-\beta)^*(x)| \mathrm{d}\mathcal{S}^h(x)$$
$$= c_D \int_{\partial\Omega} |u^*(x) - \beta| \mathrm{d}\mathcal{S}^h(x) = c_D \|u^* - \beta\|_{L^1(\partial\Omega, \mathcal{S}^h)}$$

since of course $(u-\beta)^*(x) = u^*(x) - \beta$.

□

Remark 3.9. Lemma 3.8 reformulated in the hypotheses of Theorem 3.7 tells us that the zero-extension of $u \in BV(\Omega)$ to the whole of $\mathbb{X}$, u_0, has BV norm $\|u_0\|_{BV(\mathbb{X})}$ bounded by the BV norm of u in Ω. In other words, $u_0 \in BV(\mathbb{X})$.

■

3.1 An Integration by Parts Formula for BV functions

Summarizing the previous results, we can state that

$$\|D\mathbb{1}_\Omega\|(\mathbb{X}) < \infty \qquad \text{and} \qquad \mathcal{S}^h \left(\partial\Omega \backslash \partial^*\Omega \right) = 0$$

are the underlying conditions for the domain Ω which ensure that the rough trace $u^*(x)$ of any function $u \in BV(\Omega)$ is in $L^1\left(\partial\Omega, \mathcal{S}^h\right)$.
This conclusion motivates us, as already done in [8], to proceed towards an integration by parts formula for functions of bounded variation by means of essentially bounded divergence-measure vector fields.

For the purpose, we shall refer to the differential structure introduced in [15], where the author constructs an L^2-theory of differential forms and vector fields on metric measure spaces through the notions of L^2-cotangent and tangent modules, tools which in turn find their roots in the formalism of L^p-normed modules also discussed in [15]. Also, the notion of BV functions we implicitly consider below, given in a work in progress by the authors of this work and G. E. Comi, is analogous to the one introduced by [14], which makes use of bounded Lipschitz derivations. The characterization we are using relies on vector fields in $L^\infty(T\mathbb{X})$ - the tangent module of essentially bounded vector fields - whose divergence is an L^∞ function, a class of objects which, by the results of [9], in the framework of a doubling metric measure space supporting a Poincaré inequality is equivalent - in the μ-almost everywhere sense - to the space of bounded Lipschitz derivations. Then, thanks to the equivalence results shown in [3] and then in [14], our notion of BV is equivalent to the relaxation procedure of [24] we have been using until now; of course, given that we are going to deal with vector fields, it is more convenient to us to use now the formalism of the aforementioned work in progress, whence the motivation of our choice.

We will not discuss the tools and ideas behind this characterization of BV functions, addressing the reader to the papers already mentioned and the references therein, along with [8], which gives a survey on the topics summarized above and also a straightforward generalization of the differential structure of [15] to any exponent $p \in [1, \infty]$.

Definition 3.4. *We say that $F \in L^\infty(T\mathbb{X})$ is an essentially bounded divergence-measure vector field, and we write $F \in \mathcal{DM}^\infty(\mathbb{X})$, if its distributional divergence, which we denote as $\mathrm{div}(F)$, is a finite Radon measure; that is, $\mathrm{div}(F)$ is a measure satisfying*

$$-\int_{\mathbb{X}} g \mathrm{d}\mathrm{div}(F) = \int_{\mathbb{X}} \mathrm{d}g(F)\mathrm{d}\mu$$

for every $g \in \mathrm{Lip}_b(\mathbb{X})$.

The following Theorem was originally given in [8]; below, the notation $\mathrm{d}u(F)$, $u \in BV(\Omega)$, is just the pairing measure which appears in the underlying definition of BV functions we are currently using.

Theorem 3.10. [8, Theorem 7.2.12] *Let $\Omega \subset \mathbb{X}$ be a bounded open set such that $\|D\mathbb{1}_\Omega\|(\mathbb{X}) < \infty$ and $\mathcal{S}^h(\partial\Omega \backslash \partial^*\Omega) = 0$. Then, for every $u \in BV(\Omega)$ and every $F \in \mathcal{DM}^\infty(\mathbb{X})$ one has*

$$\int_\Omega \mathrm{d}u(F) + \int_\Omega u \mathrm{div}(F) \mathrm{d}\mu = -\int_{\mathbb{X}} \Theta\left(u^*(x)\right) \mathrm{d}\mathcal{S}^h(x),$$

where $\Theta(u^*(x))$ is clarified in (18) below.

Proof. To get things started, we remark that if $E \subset \Omega$ is a Caccioppoli set in Ω, then the pairing measure in the definition of BV as given in the work in progress, $\mathrm{d}\mathbb{1}_E(X)$, satisfies $\mathrm{d}\mathbb{1}_E(X)(\mathbb{X}) = 0$ (where X is a vector field as in the definition of BV, and the claim holds as X is compactly supported).
Now, an integration by parts formula holds for the whole of $\mathbb{X}$, namely

$$\int_{\mathbb{X}} \mathrm{d}u(F) = -\int_{\mathbb{X}} u div(F) \mathrm{d}\mu.$$

Moreover, clearly,

$$\int_{\Omega} \mathrm{d}u(F) = \int_{\Omega} \mathrm{d}u(F) - \int_{\mathbb{X}\setminus\Omega} \mathrm{d}u(F) = -\int_{\mathbb{X}} u div(F) \mathrm{d}\mu - \int_{\mathbb{X}\setminus\Omega} \mathrm{d}u(F).$$

Now, suppose $u \equiv \mathbb{1}_E$ with E as above. The previous equalities become

$$\begin{aligned}\int_{\Omega} \mathrm{d}\mathbb{1}_E(F) &= -\int_{\mathbb{X}} \mathbb{1}_E div(f) \mathrm{d}\mu - \int_{\mathbb{X}\setminus\Omega} \mathrm{d}\mathbb{1}_E(F) \\ &= -\int_{\mathbb{X}} \mathbb{1}_E div(F) \mathrm{d}\mu - \int_{\partial\Omega\cap\partial^* E} \mathrm{d}\mathbb{1}_E(F).\end{aligned}$$

We used the locality property of the perimeter measure, which is concentrated on the essential boundary of E by Theorem 2.1; then,

$$\mathrm{d}\mathbb{1}_E(F)(\Omega) = -\mathrm{d}\mathbb{1}_E(F)(\partial\Omega) = -\mathrm{d}\mathbb{1}_E(F)(\partial\Omega \cap \partial^* E).$$

Let now $u \geq 0$ for simplicity; the proof for a general $u \in BV(\Omega)$ will follow by considering separately its positive and negative parts. Using Coarea Formula we obtain

$$\begin{aligned}\int_{\Omega} \mathrm{d}u(F) &= \int_0^{+\infty} \mathrm{d}t \int_{\Omega} \mathrm{d}\mathbb{1}_{E_t}(F) \\ &= -\int_0^{+\infty} \mathrm{d}t \left(\int_{\mathbb{X}} \mathbb{1}_{E_t} div(F) \mathrm{d}\mu + \int_{\partial\Omega\cap\partial^* E_t} \mathrm{d}\mathbb{1}_{E_t}(F) \right).\end{aligned}$$

Above, we have used the fact that the pairing $\mathrm{d}\mathbb{1}_E(F)$ defines a measure which is absolutely continuous with respect to the perimeter measure: indeed, setting

$$\nu_E^F : A \mapsto \int_A \mathrm{d}\mathbb{1}_E(F) = \nu_E^F(A),$$

one has $\left|\nu_E^F\right|(A) \leq \|F\|_{L^\infty(T\mathbb{X})} \|D\mathbb{1}_E\|(A)$ and then again by Theorem 2.1

$$\nu_E^F(A) = \int_A \sigma_E^F(x)\mathrm{d}\|D\mathbb{1}_E\|(x) = \int_{A\cap\partial^* E} \sigma_E^F \theta_E \mathrm{d}\mathcal{S}^h(x).$$

So, $\mathrm{d}\mathbb{1}_E(F) = \sigma_E^F\theta_E\mathcal{S}^h \llcorner \partial^* E$ and similarly $\mathrm{d}\mathbb{1}_\Omega(F) = \sigma_\Omega^F\theta_\Omega\mathcal{S}^h \llcorner \partial^*\Omega$. Let us set

$$f_{E,\Omega} := \frac{\sigma_E^F\theta_E}{\sigma_\Omega^F\theta_\Omega} = \frac{\lambda_E^F}{\lambda_\Omega^F}. \tag{17}$$

Summing up, we find

$$\int_{\partial^*\Omega} \mathrm{d}\mathbb{1}_E(F)$$
$$= \int_{\partial^*\Omega\cap\partial^* E} \lambda_E^F \mathrm{d}\mathcal{S}^h(x) = \int_{\partial^*\Omega\cap\partial^* E} \lambda_\Omega^F \mathrm{d}\mathcal{S}^h(x) = \int_{\partial^* E} f_{E,\Omega}\mathrm{d}\mathcal{S}^h(x).$$

Applying the same argument to our case,

$$\begin{aligned}
\int_\Omega \mathrm{d}u(F) &= -\int_0^{+\infty} \mathrm{d}t \left(\int_{\mathbb{X}} \mathbb{1}_{E_t} div(F)\mathrm{d}\mu + \int_{\partial^* E_t} f_{E_t,\Omega}\mathrm{d}\mathcal{S}^h(x)\right) \\
&= -\int_0^{+\infty} \mathrm{d}t \left(\int_\Omega \mathbb{1}_{E_t} div(F)\mathrm{d}\mu + \int_{\{u^*\geq t\}} f_{E_t,\Omega}\mathrm{d}\mathcal{S}^h(x)\right) \\
&= -\int_\Omega u div(F)\mathrm{d}\mu - \int_{\partial\Omega} \Theta\left(u^*(x)\right)\mathrm{d}\mathcal{S}^h(x),
\end{aligned}$$

where

$$\Theta\left(u^*(x)\right) = \int_0^{u^*(x)} \mathbb{1}_{E_t} f_{E_t,\Omega}\mathrm{d}t. \tag{18}$$

□

Remark 3.11. *i)* In the same spirit of Definition 2.2, we shall say that $(\mathbb{X}, d, \mu)$ is *strongly local* if, besides the condition $\theta_E = \theta_\Omega$ $\mathcal{S}^h$-almost everywhere on $\partial^*\Omega\cap\partial^* E$, one also has $\sigma_E^F = \sigma_\Omega^F$ $\mathcal{S}^h$-almost everywhere on $\partial^*\Omega\cap\partial^* E$. Observe that if $(\mathbb{X}, d, \mu)$ is strongly local, then the function $f_{E,\Omega}$ in (17) is identically equal to 1.
ii) If we change the statement of Theorem 3.10 assuming that Ω is a regular domain in the sense of [8], namely an open set of finite perimeter coinciding with the upper inner Minkowski content of its boundary,

$$\|D\mathbb{1}_\Omega\|(\mathbb{X}) = \mathfrak{M}^*_{\mathrm{in}}(\partial\Omega) := \limsup_{t\to 0} \frac{\mu\left(\Omega\backslash\Omega_t\right)}{t},$$

where for $t > 0$ we set $\Omega_t \coloneqq \{x \in \Omega : \operatorname{dist}(x, \Omega^c) \geq t\}$, and we also require that $\mathcal{S}^h(\partial\Omega \setminus \partial^*\Omega) = 0$, then we can show that for every $u \in BV(\Omega)$ there exists a *trace operator* $\mathrm{T} : BV(\Omega) \to L^1(\partial\Omega, \|D\mathbb{1}_\Omega\|)$ such that for every $F \in \mathcal{DM}^\infty(\mathbb{X})$ one has

$$\int_\Omega \mathrm{d}u(F) + \int_\Omega u div(F)\mathrm{d}\mu = -\int_{\partial\Omega} u^*(x)\,(F\cdot\nu)^-_{\partial\Omega}\,\mathrm{d}\,\|D\mathbb{1}_\Omega\|(x) \coloneqq \left\langle \mathrm{T}u, (F\cdot\nu)^-_{\partial\Omega}\right\rangle.$$

Indeed, in this case we can use the defining sequence $(\varphi_\varepsilon)_{\varepsilon>0} \subset \mathrm{Lip}_c(\Omega)$ of the regular domain Ω, [8, Remark 7.1.5], and we are entitled to repeat the proof of [8, Theorem 7.1.7].

As a concrete example, we observe that for all $x \in \mathbb{X}$ and for almost-every $\rho > 0$, any ball $B_\rho(x)$ is a regular domain, [8].

■

4 Trace comparison

In this last section we compare the foregoing discussion on the rough trace with [21], where the authors investigate the properties of the trace operator for BV functions by means of the more classical Lebesgue-points characterization.

We start by summarizing the salient definitions and results of [21] which will be of relevance to us.

Definition 4.1. *Let $\Omega \subset \mathbb{X}$ be an open set and let u be a μ-measurable function on Ω. Then, we shall say that a function $\mathrm{T}u : \partial\Omega \to \mathbb{R}$ is a trace of u if for $\mathcal{S}^h$-almost every $x \in \partial\Omega$ one has*

$$\lim_{\rho\to 0^+} \fint_{\Omega\cap B_\rho(x)} |u - \mathrm{T}u(x)|\,\mathrm{d}\mu = 0.$$

The *zero extension* of μ from Ω to $\overline{\Omega}$, $\bar{\mu}$, is given by $\bar{\mu}(A) \coloneqq \mu(A \cap \Omega)$ whenever $A \subset \overline{\Omega}$; the same notation will of course apply to the spherical Hausdorff measure, so we shall write $\overline{\mathcal{S}}^h$ to intend the zero-extension of $\mathcal{S}^h$ from Ω to $\overline{\Omega}$.

Accordingly, for any measurable function u in Ω, its zero-extension to $\overline{\Omega}$ will be written as $\bar{u}$; $\bar{u}^\vee$ and $\bar{u}^\wedge$ will therefore denote the approximate limits of $\bar{u}$ computed in terms of the extended measure $\bar{\mu}$.

Proposition 4.1. [21, Proposition 3.3] *Let $\Omega \subset \mathbb{X}$ be a bounded open set supporting a $(1,1)$-Poincaré inequality and assume that μ is doubling on Ω. Let $\overline{\Omega}$ be equipped with the extended measure $\bar{\mu}$. If $u \in BV(\Omega)$, then its zero-extension $\bar{u}$ to $\overline{\Omega}$ is such that $\|\bar{u}\|_{BV(\overline{\Omega})} = \|u\|_{BV(\Omega)}$, whence $\|D\bar{u}\|(\partial\Omega) = 0$.*

□

Definition 4.2. *We say that an open set Ω satisfies a measure-density condition if there exists a constant $C > 0$ such that*

$$\mu\left(B_\rho(x) \cap \Omega\right) \geq C\mu\left(B_\rho(x)\right) \tag{19}$$

for $\mathcal{S}^h$-almost every $x \in \partial\Omega$ and for every $\rho \in (0, \operatorname{diam}(\Omega))$.

Theorem 4.2. [21, Theorem 3.4] *Let $\Omega \subset \mathbb{X}$ be a bounded open set that supports a $(1,1)$-Poincaré inequality, and assume that μ is doubling on Ω. Then, there exist $q > 1$ depending only on the doubling constant in Ω and a linear trace operator* T *on $BV(\Omega)$ such that, given $u \in BV(\Omega)$, for $\overline{\mathcal{S}}^h$-almost every $x \in \partial\Omega$ we have*

$$\lim_{\rho \to 0^+} \fint_{\Omega \cap B_\rho(x)} |u - \mathrm{T}u(x)|^{\frac{q}{q-1}} \,\mathrm{d}\mu = 0. \tag{20}$$

If Ω also satisfies the measure-density condition (19)*, the above holds for $\mathcal{S}^h$-almost every $x \in \partial\Omega$.*

□

Remark 4.3. In the proof of [21, Theorem 3.4], the authors used the condition $\|D\bar{u}\|(\partial\Omega) = 0$ found in Proposition [21, Proposition 3.3] to infer that $\overline{\mathcal{S}}^h(S_{\bar{u}} \cap \partial\Omega)$ $= 0$; this of course arises from the decomposition of the total variation measure given in Remark 2.3 and entails that the equality

$$\bar{u}^\wedge(x) = \bar{u}^\vee(x)$$

holds for $\overline{\mathcal{S}}^h$-almost every $x \in \partial\Omega$.

Always in the proof of [21, Theorem 3.4], (20) was actually found to hold in the form

$$\lim_{\rho \to 0^+} \fint_{\Omega \cap B_\rho(x)} |u - \bar{u}^\wedge(x)|^{\frac{q}{q-1}} \mathrm{d}\mu = 0;$$

then one sets $\mathrm{T}u(x) = \bar{u}^{\wedge}(x)$, which in turn equals $\bar{u}^{\vee}(x)$ for $\overline{\mathcal{S}}^h$-almost every $x \in \partial\Omega$ by the considerations right above. So in particular, when Ω satisfies the measure-density condition (19), such equalities are fulfilled for $\mathcal{S}^h$-almost every $x \in \partial\Omega$ as well.

■

We now give an extension result for the zero-extension of $u \in BV(\Omega)$ from Ω to $\overline{\Omega}$ and then proceed towards the proof of the $\mathcal{S}^h$-almost everywhere equivalence $u^*(x) = \mathrm{T}u(x)$ on $\partial\Omega$.

For the sake of clarity we explicitly observe that in Proposition 4.4 and in Proposition 4.6 below we are **not** going to assume that the domain Ω is such that (Ω, d, μ) is a doubling metric measure space supporting a $(1,1)$-Poincaré inequality.

Proposition 4.4. *Suppose $\Omega \subset \mathbb{X}$ is an open set such that $\mathcal{S}^h(\partial\Omega) < \infty$ and $\mathcal{S}^h(\partial\Omega \backslash \partial^*\Omega) = 0$; let $E \subset \Omega$ be a set of finite perimeter in Ω. Then, $\bar{\mathbb{1}}_E \in BV(\mathbb{X})$. Under the same hypotheses, for any $u \in BV(\Omega)$ one has $\bar{u} \in BV(\mathbb{X})$.*

Proof. Let us start with $u = \mathbb{1}_E \in BV(\Omega)$. We can write

$$\begin{aligned}
\|D\bar{\mathbb{1}}_E\|(\mathbb{X}) &= \|D\mathbb{1}_E\|(\Omega) + \|D\mathbb{1}_E\|(\mathbb{X}\backslash\Omega) \\
&= \|D\mathbb{1}_E\|(\Omega) + \|D\mathbb{1}_E\|(\partial\Omega) \\
&\leq \|D\mathbb{1}_E\|(\Omega) + c_D\mathcal{S}^h(\partial\Omega \cap \partial^* E) = \|D\mathbb{1}_E\|(\Omega) + c_D\,\mathcal{S}^h\left(\partial^*\Omega \cap \partial^* E\right),
\end{aligned}$$

which is finite by our assumptions and by the previous remarks. Then, the assertion follows for $\mathbb{1}_E \in BV(\Omega)$.

Let us now take $u \in BV(\Omega)$; we shall make use of the above argument and of the Coarea Formula.

For simplicity, assume $u \geq 0$. Then, denoting by $\bar{E}_t$ the super-level sets for $\bar{u}$,

$$\|D\bar{u}\|(\mathbb{X}) = \int_0^{+\infty} \|D\bar{\mathbb{1}}_{\bar{E}_t}\|(\mathbb{X})\mathrm{d}t \leq \int_0^{+\infty} \left[\|D\mathbb{1}_{E_t}\|(\Omega) + c_D\,\mathcal{S}^h\left(\partial^*\Omega \cap \partial^*\bar{E}_t\right)\right]\mathrm{d}t,$$

by the previous inequality. We already know that

$$\int_0^{+\infty} \|D\mathbb{1}_{E_t}\|(\Omega)\mathrm{d}t < \infty$$

by Coarea, since $u \in BV(\Omega)$; let us then estimate

$$\int_0^{+\infty} \mathcal{S}^h\left(\partial^*\Omega \cap \partial^*\bar{E}_t\right)\mathrm{d}t$$

and recall that the integrand is finite for almost every t. Using Fubini's Theorem,

$$\begin{aligned}\int_0^{+\infty} \mathcal{S}^h\left(\partial^*\Omega \cap \partial^* \bar{E}_t\right) \mathrm{d}t &= \int_0^{+\infty} \mathrm{d}t \int_{\partial^*\Omega} \mathbb{1}_{\partial^* \bar{E}_t}(x) \mathrm{d}\mathcal{S}^h(x) \\ &= \int_{\partial^*\Omega} \mathrm{d}\mathcal{S}^h(x) \int_0^{+\infty} \mathbb{1}_{\partial^* \bar{E}_t}(x) \mathrm{d}t.\end{aligned}$$

Now observe that since we are integrating over $\partial^*\Omega \cap \partial^* \bar{E}_t$ and $t \geq 0$, then $t \in [0, \bar{u}^\vee(x)]$ by [5, Proposition 5.2].

We claim that actually it must be $t \in [0, \bar{u}^\vee(x))$. Indeed, in the limit case $t = \bar{u}^\vee(x)$, then by the definition of upper approximate limit (6) applied to $\bar{u}$,

$$t = \bar{u}^\vee(x) = \inf\left\{ s \in \overline{\mathbb{R}};\ \Theta_{\bar{\mu}}(\bar{E}_s, x) = \lim_{\rho \to 0^+} \frac{\mu\left(B_\rho(x) \cap \bar{E}_s\right)}{\mu\left(B_\rho(x) \cap \Omega\right)} = 0 \right\},$$

we would have

$$\begin{aligned}0 &= \lim_{\rho \to 0^+} \frac{\mu\left(B_\rho(x) \cap \bar{E}_t\right)}{\mu\left(B_\rho(x) \cap \Omega\right)} \\ &= \lim_{\rho \to 0^+} \frac{\mu\left(B_\rho(x) \cap \bar{E}_t\right)}{\mu((B_\rho(x))} \lim_{\rho \to 0^+} \frac{\mu\left(B_\rho(x)\right)}{\mu\left(B_\rho(x) \cap \Omega\right)},\end{aligned}$$

the decomposition being possible as the two limits exist because by hypothesis $x \in \partial^*\Omega \cap \partial^* \bar{E}_t$. So, by the definition of essential boundary (4), the above equalities would lead to a contradiction, forcing $t < \bar{u}^\vee(x)$ as claimed. Therefore, $t \in [0, \lambda]$ for some $0 < \lambda < \bar{u}^\vee(x)$.

All in all, by applying again Fubini,

$$\begin{aligned}\int_{\partial^*\Omega} \mathrm{d}\mathcal{S}^h(x) \int_0^{+\infty} \mathbb{1}_{\partial^* \bar{E}_t}(x) \mathrm{d}t &= \int_{\partial^*\Omega} \mathrm{d}\mathcal{S}^h(x) \int_0^{\lambda} \mathbb{1}_{\partial^* \bar{E}_t}(x) \mathrm{d}t \\ &= \int_0^{\lambda} \mathcal{S}^h(\partial^*\Omega \cap \partial^* \bar{E}_t) \mathrm{d}t \\ &< \infty.\end{aligned}$$

In conclusion, we have found

$$\|D\bar{u}\|(\mathbb{X}) \leq \|Du\|(\Omega) + c_D \int_0^{\lambda} \mathcal{S}^h(\partial^*\Omega \cap \partial^* \bar{E}_t) \mathrm{d}t < \infty.$$

The proof for a general $u \in BV(\Omega)$ follows by applying the above arguments to the positive and negative parts of u.

□

Remark 4.5. Proposition 4.4 for $u = \mathbb{1}_E \in BV(\Omega)$ can be actually seen as a particular case of [18, Proposition 6.3].
In [19, Lemma 3.2] it was proven that for any function $u \in BV(\mathbb{X})$ its approximate limits satisfy

$$-\infty < u^\wedge(x) \leq u^\vee(x) < \infty$$

for $\mathcal{S}^h$-almost every $x \in \mathbb{X}$. Consequently, if we assume the hypotheses of Proposition 4.4 to be satisfied, we can conclude that

$$-\infty < \bar{u}^\wedge(x) \leq \bar{u}^\vee(x) < \infty$$

for $\mathcal{S}^h$-almost every $x \in \mathbb{X}$.

■

Next, we prove that the rough trace of a BV function is bounded by the approximate limits of its zero-extension to $\overline{\Omega}$:

Proposition 4.6. *Let $\Omega \subset \mathbb{X}$ be an open set. Then, for every $x \in \partial^*\Omega$ and for every $u \in BV(\Omega)$, assuming $\bar{u}^\wedge(x), \bar{u}^\vee(x) \in \mathbb{R}$ one has $u^*(x) \in \mathbb{R}$ with $\bar{u}^\wedge(x) \leq u^*(x) \leq \bar{u}^\vee(x)$.*

Proof. Recall that by definition, (13), $u^*(x)$ is the supremum of those t for which $\|D\mathbb{1}_{E_t}\|(\Omega) < \infty$ and $x \in \partial^* E_t$.
We first assume $t \geq \bar{u}^\wedge(x)$. This gives, by the definition of lower approximate limit (5) and of extended measure,

$$\begin{aligned} 1 &= \lim_{\rho\to 0^+} \frac{\mu\left(E_t \cap B_\rho(x) \cap \Omega\right)}{\mu\left(B_\rho(x) \cap \Omega\right)} = \lim_{\rho\to 0^+} \frac{\mu\left(E_t \cap B_\rho(x)\right)}{\mu\left(\Omega \cap B_\rho(x)\right)} \\ &= \lim_{\rho\to 0^+} \frac{\mu\left(E_t \cap B_\rho(x)\right)}{\mu\left(B_\rho(x)\right)} \lim_{\rho\to 0^+} \frac{\mu\left(B_\rho(x)\right)}{\mu\left(\Omega \cap B_\rho(x)\right)} \\ &> \lim_{\rho\to 0^+} \frac{\mu\left(E_t \cap B_\rho(x)\right)}{\mu\left(B_\rho(x)\right)} = \Theta_\mu(E_t, x), \end{aligned}$$

whence $x \notin E_t^{(1)}$. Let us then show that $x \notin E_t^{(0)}$. Since by hypothesis $x \in \partial^*\Omega$, by (4) there exists $\sigma \in (0,1)$ such that $\sigma \leq \Theta_\mu(\Omega, x) \leq 1 - \sigma$. Then, we find

$$\begin{aligned} \sigma \leq \Theta_\mu(\Omega, x) &= \lim_{\rho\to 0^+} \frac{\mu\left(\Omega \cap B_\rho(x)\right)}{\mu(B_\rho(x))} \\ &= \underbrace{\lim_{\rho\to 0^+} \frac{\mu(\Omega \cap B_\rho(x))}{\mu(E_t \cap B_\rho(x))}}_{\leq 1,\, E_t \subset \Omega} \lim_{\rho\to 0^+} \frac{\mu(E_t \cap B_\rho(x))}{\mu(B_\rho(x))} \\ &\leq \lim_{\rho\to 0^+} \frac{\mu(E_t \cap B_\rho(x))}{\mu(B_\rho(x))} = \Theta_\mu(E_t, x), \end{aligned}$$

forcing $x \notin E_t^{(0)}$. In other words, $x \in \mathbb{X} \backslash \big(E_t^{(0)} \cup E_t^{(1)}\big) = \partial^* E_t$ and $t \leq u^*(x)$. Since we are assuming $t \leq \bar{u}^\wedge(x)$, this entails $\bar{u}^\wedge(x) \leq u^*(x)$.

Now, assume $t \geq \bar{u}^\vee(x)$. Using the previous arguments with the definition of upper approximate limit (6), we get

$$\begin{aligned} 0 &= \lim_{\rho\to 0^+} \frac{\mu(E_t \cap B_\rho(x))}{\mu(\Omega \cap B_\rho(x))} \\ &= \lim_{\rho\to 0^+} \frac{\mu(E_t \cap B_\rho(x))}{\mu(B_\rho(x))} \underbrace{\lim_{\rho\to 0^+} \frac{\mu(B_\rho(x))}{\mu(\Omega \cap B_\rho(x))}}_{>1,\, x\in\partial^*\Omega}. \end{aligned}$$

But this would force

$$\lim_{\rho\to 0^+} \frac{\mu(E_t \cap B_\rho(x))}{\mu(B_\rho(x))} = \Theta_\mu(E_t, x) = 0,$$

meaning that $x \in E_t^{(0)}$ and therefore $x \notin \partial^* E_t$.

Thus, $t \geq u^*(x)$, and since by hypothesis $t \geq \bar{u}^\vee(x)$, we conclude that $u^*(x) \leq \bar{u}^\vee(x)$.

□

Remark 4.7. The condition $\bar{u}^\wedge(x) \leq u^*(x) \leq \bar{u}^\wedge(x)$ is actually always fullfilled, regardless of the approximate limits of $\bar{u}$ being finite. However, if we rephrase Proposition 4.6 by imposing on Ω the same conditions as in Proposition 4.4, namely $\mathcal{S}^h(\partial\Omega) < \infty$ and $\mathcal{S}^h(\partial\Omega \backslash \partial^*\Omega) = 0$, then by Remark 4.5 we can drop the assumption that $\bar{u}^\wedge(x), \bar{u}^\vee(x) \in \mathbb{R}$.

■

In conclusion, if we now put together the above results and remarks with [21, Proposition 3.3] and [21, Theorem 3.4], we are entitled to state the following

Theorem 4.8. *Let $\Omega \subset \mathbb{X}$ be a bounded open set such that $\mathcal{S}^h(\partial\Omega) < \infty$ and $\mathcal{S}^h(\partial\Omega \backslash \partial^*\Omega) = 0$. Further, assume that Ω supports a $(1,1)$-Poincaré inequality, μ is doubling on Ω, and that the measure-density condition* (19) *holds.*

Then, the equality

$$u^*(x) = \mathrm{T}u(x)$$

is fulfilled for $\mathcal{S}^h$-almost every $x \in \partial\Omega$.

□

Indeed, since by Proposition 4.6 we get $\bar{u}^\wedge(x) \le u^*(x) \le \bar{u}^\vee(x)$ on $\partial^*\Omega$, then by [21, Theorem 3.4] and Remark 4.3 we are given

$$\bar{u}^\wedge(x) = \mathrm{T}u(x) \le u^*(x) \le \bar{u}^\vee(x) = \mathrm{T}u(x)$$

for $\mathcal{S}^h$-almost every $x \in \partial\Omega$; therefore, the inequalities are actually equalities and then the rough trace $u^*(x)$ defines *a fortiori* a linear operator.

Remark 4.9. (Comments and Open Problems) In conclusion, our discussion allowed us to find the conditions to impose on a domain $\Omega \subset \mathbb{X}$ in order to ensure that the "classical" trace $\mathrm{T}u$ and the rough trace u^* of a BV function coincide $\mathcal{S}^h$-almost everywhere on the boundary of such domain.

Actually, our results also address the L^1-summability of the trace $\mathrm{T}u$; indeed, if in Theorem 4.8 we introduce the additional assumption that for some $\delta > 0$ and for any set $E \subset \Omega$ with finite perimeter in Ω and $\mathrm{diam}(E) \le \delta$ it holds

$$\|D\mathbb{1}_E\|(\Omega^c) \le c\|D\mathbb{1}_E\|(\Omega)$$

for some constant $c > 0$ independent of E, which is namely the fundamental condition (15) of Theorem 3.7, then we get that $\mathrm{T}u \in L^1(\partial\Omega, \mathcal{S}^h)$ as well.

In [21, Section 5], the authors attack the issue of the summability of the trace $\mathrm{T}u$ by working again in terms of the measure-density condition (19) and assuming an additional "surface-density" condition for $\partial\Omega$, namely that there is a constant $c = c_{\partial\Omega} > 0$ such that

$$\mathcal{S}^h(B_\rho(x) \cap \partial\Omega) \le c\frac{\mu(B_\rho(x))}{\rho}$$

for any $x \in \partial\Omega$ and any $\rho \in (0, 2\mathrm{diam}(\Omega))$.

Thus said, one question arises naturally: *how does the requirement* (15) *in Theorem 3.7 relate with the measure-density condition* (19) *and with the surface-density condition above?*

Answering to such a question would be of general interest as it would provide us with a better understanding of the domains where the "nice" properties of traces of BV functions are satisfied, and therefore we would have a more consistent and more comprehensive theory of traces of BV functions.

■

References

[1] L. Ambrosio, *Some fine properties of sets of finite perimeter in Ahlfors regular metric measure spaces*, Adv. Math. 159 (2001), no. 1, 51–67.

[2] L. Ambrosio, *Fine properties of sets of finite perimeter in doubling metric measure spaces*, Calculus of variations, nonsmooth analysis and related topics. Set-Valued Anal. 10 (2002), no. 2-3, 111–128.

[3] L. Ambrosio, S. Di Marino, *Equivalent definitions of BV space and of total variation on metric measure spaces*, J. Funct. Anal. 266 (2014), no. 7, 4150–4188.

[4] L. Ambrosio, N. Fusco, D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000. xviii+434 pp.

[5] L. Ambrosio, M. Miranda Jr., D. Pallara, *Special functions of bounded variation in doubling metric measure spaces*, Calculus of variations: topics from the mathematical heritage of E. De Giorgi, 1–45, Quad. Mat., 14, Dept. Math., Seconda Univ. Napoli, Caserta, 2004.

[6] G. Anzellotti, *Pairings between measures and bounded functions and compensated compactness*, Ann. Mat. Pura Appl. (4) 135 (1983), 293–318 (1984).

[7] A. Björn, J. Björn, *Nonlinear potential theory on metric spaces*, EMS Tracts in Mathematics, 17. European Mathematical Society (EMS), Zürich, 2011.

[8] V. Buffa, *BV Functions in Metric Measure Spaces: Traces and Integration by Parts Formulæ*, Ph.D. Thesis, Università degli Studi di Ferrara (Ferrara, Italy, 2017).

[9] J. Cheeger, *Differentiability of Lipschitz functions on metric measure spaces*, Geom. Funct. Anal. 9 (1999), no. 3, 428–517.

[10] G. Q. Chen, M. Torres, *Divergence-measure fields, sets of finite perimeter, and conservation laws*, Arch. Ration. Mech. Anal. 175 (2005), no. 2, 245–267.

[11] G. Q. Chen, M. Torres, W. P. Ziemer, *Gauss-Green theorem for weakly differentiable vector fields, sets of finite perimeter, and balance laws*, Comm. Pure Appl. Math. 62 (2009), no. 2, 242–304.

[12] G. E. Comi, V. Magnani, *The Gauss-Green theorem in stratified groups*, Submitted Paper, arXiv:1806.04011.

[13] G. E. Comi, K. R. Payne, *On locally essentially bounded divergence measure fields and sets of locally finite perimeter*, Adv. Calc. Var, 2017, `doi: 10.1515/acv-2017-0001`.

[14] S. Di Marino, *Recent advances on BV and Sobolev Spaces in metric measure spaces*, Ph.D. Thesis, Scuola Normale Superiore (Pisa, Italy, 2014).

[15] N. Gigli, *Nonsmooth differential geometry—an approach tailored for spaces with Ricci curvature bounded from below*, Mem. Amer. Math. Soc. 251 (2018), no. 1196, v+161 pp.

[16] P. Hajłasz, P. Koskela, *Sobolev met Poincaré*, Mem. Amer. Math. Soc. 145 (2000), no. 688, x+101 pp.

[17] J. Heinonen, P. Koskela, N. Shanmugalingam, J. T. Tyson, *Sobolev spaces on metric measure spaces. An approach based on upper gradients*, New Mathematical Monographs, 27. Cambridge University Press, Cambridge, 2015. xii+434 pp.

[18] J. K. Kinnunen, R. Korte, N. Shanmugalingam, H. Tuominen, *A characterization of Newtonian functions with zero boundary values*, Calc. Var. Partial Differential Equations 43 (2012), no. 3-4, 507–528.

[19] J. K. Kinnunen, R. Korte, N. Shanmugalingam, H. Tuominen, *Pointwise properties of functions of bounded variation in metric spaces*, Rev. Mat. Complut. 27 (2014), no. 1, 41–67.

[20] P. Lahti, *Federer's characterization of sets of finite perimeter in metric spaces*, Preprint, arXiv:1804.11216.

[21] P. Lahti, N. Shanmugalingam, *Trace theorems for functions of bounded variation in metric spaces*, J. Funct. Anal. 274 (2018), no. 10, 2754–2791.

[22] N. Marola, M. Miranda Jr., N. Shanmugalingam, *Boundary measures, generalized Gauss-Green formulas, and mean value property in metric measure spaces*, Rev. Mat. Iberoam. 31 (2015), no. 2, 497–530.

[23] V. G. Maz'ya, *Sobolev spaces with applications to elliptic partial differential equations. Second, revised and augmented edition*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 342. Springer, Heidelberg, 2011. xxviii+866 pp.

[24] M. Miranda Jr., *Functions of bounded variation on "good" metric spaces*, J. Math. Pures Appl. (9) 82 (2003), no. 8, 975–1004.

[25] M. Šilhavý, *Divergence measure fields and Cauchy's stress theorem*, Rend. Sem. Mat. Univ. Padova 113 (2005), 15–45.

Vito Buffa:

Aalto University, Department of Mathematics and Systems Analysis - P.O. Box 11100, FI-00076 Aalto, Finland.

Michele Miranda jr:

Università degli Studi di Ferrara, Dipartimento di Matematica e Informatica - Via Machiavelli 30, 44121 Ferrara, Italy.