THE MULTIPHASE MUMFORD-SHAH PROBLEM

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Abstract. We perform a rigorous analysis of the multiphase version of the Mumford-Shah functional. A characteristic property of the formulation is the presence of a true partition of the image (so in two dimensions of *closed* contours), each cell of the partition possibly containing inner jumps. The nontrivial partitioning naturally occurs as a consequence of the presence in the energy functional of statistical terms or of phase depending weights. In particular, we prove a multiphase version of De Giorgi-Carriero-Leaci result.

1. Introduction and statement of the results

The Mumford-Shah functional is the prototype of a free discontinuity problem. It has been introduced by Mumford and Shah in [15] in order to reconstruct an image, essentially by enhancing the presence of contours corresponding to discontinuity sets of a grey-level image. Roughly speaking, given a rectangular box $D \subseteq \mathbb{R}^2$ and a function $f : D \to [0,1]$, one searches for a closed set $K \subseteq \overline{D}$ and a new image $u \in C^1(D \setminus K)$ which are obtained by minimizing

$$MS(K,u) = \int_{D\setminus K} |\nabla u|^2 dx + \int_{D\setminus K} (u-f)^2 dx + \mathcal{H}^1(K \cap D).$$

The first existence and qualitative results have been obtained by Ambrosio [1] and De Giorgi, Carriero, Leaci [10] passing through the relaxation of the functional in the space of special functions of bounded variation, and proving that the jump set of so-called "quasi-minimizers" is closed and Ahlfors regular. A second approach, of different nature, was proposed by Dal Maso, Morel and Solimini [9]. We refer the reader to the papers [12, 11] for an overview of the topic.

The purpose of this paper is to analyze the Mumford-Shah functional in a multiphase context. In the literature, a huge number of papers (mainly of numerical nature) refer to this problem, sometimes speaking about multi-region or multi-label Mumford-Shah functional. Even in the original paper by Mumford and Shah, the question was set as a partition problem of an original image, the contours where discontinuity occur being the boundaries of those sets. Nevertheless, a precise mathematical formulation of the multiphase problem in a good generality, as well as an existence result and the analysis of solutions in the same spirit of the one phase case, seems to be lacking. Our target is precisely to fill this gap. Before writing the model, and in order to put it into context, let us give a short account of the different formulations we have found in the literature.

The simplest, and maybe the most common, multi-phase model is the piecewise constant one. It was introduced by Mumford-Shah in [15], later on analyzed by Vese and Chan in

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[18] and numerically implemented and used by many authors. It reads

$$\min_{(\Omega_i)_{i=1,\dots,k}\in\mathcal{P}_k(D)} \sum_{i=1}^{k} \Big(\int_{\Omega_i} (f-c_i)^2 dx + \mathcal{H}^1(\partial^e\Omega_i \cap D)\Big),$$

where $\mathcal{P}_k(D)$ is the family of partitions of D into k measurable sets with finite perimeter, c_i are the averages of f on Ω_i and $\partial^e\Omega_i$ denotes the essential boundary. A more sophisticated version of this model was introduced in [17] and further analyzed in [14]. Roughly speaking, the sum of the length terms is replaced by

$$\left(\sum_{i=1}^{k} \mathcal{H}^1(\partial^e\Omega_i \cap D)\right)\left(\sum_{i=1}^{k} \frac{\mathcal{H}^1(\partial^e\Omega_i \cap D)}{|\Omega_i|}\right).$$

The main advantage of this formulation is that the number k of cells in the partition does not need to be a priori specified: in fact, very small sets are not competitive in the functional, having a large ratio perimeter/area, so that the number of phases turns out to be limited from above.

The situation in which the approximating function is not constant on each phase was discussed by Vese and Chan in [18] in the context of the approximation by a level-set method and separation in two regions. Instead of a true partition, there is a phase field function localizing the two regions: this formulation is called *soft*, in contrast to a true partition, which is called *hard*. This problem has been numerically studied in several papers, as for instance [13] and [8]. The multiphase Mumford-Shah problem for which numerical computations are performed is loosely written as

$$(1)\qquad \inf_{(\Omega_i)_{i=1,\dots,k}\in\mathcal{P}_k(D),\, u_i\in H^1(\Omega_i)} \sum_{i=1}^{k} \left(\int_{\Omega_i} |\nabla u_i|^2 dx + \int_{\Omega_i} (u_i-f)^2 dx + \mathcal{H}^1(\partial\Omega_i \cap D)\right).$$

The sets Ω_i are the different phases, which are supposed to be open. They are usually approached by phase functions, and those functions are also used in the computation of their perimeters, via a Modica-Mortola approximation. In particular, cracks are excluded to occur inside Ω_i, as the phase function has a constant sign on such set. Implicitly, the topological boundary of Ω_i is assumed to coincide with the measure theoretical one.

The existence of a solution to problem (1) is a topic which is not discussed in the literature. However we wish to emphasize that a global minimizer for problem (1) does not have any particular interest, as it is of one-phase type. In fact, denoting by (K, u) a minimizer of the Mumford-Shah functional, a solution to problem (1) is given by $(D \setminus K, \emptyset, \dots, \emptyset)$, associated with the functions $(u, 0, \dots, 0)$. In spite, local minimizers (which are the ones usually detected by numerical computations) do not correspond in general to restrictions of one-phase solutions, and as such they are definitely of interest.

In order that global minimizers become of interest, the cost functional must be suitably revised, through the addition of some extra-coefficient and/or extra-term.

In this perspective, the *multiphase Mumford-Shah energy* we are going to consider in this paper will be of the following form: for any $\mathbf{\Omega} \in \mathcal{P}_k(D)$ and $\mathbf{U} \in \mathcal{H}(\mathbf{\Omega})$ where

$$\mathcal{P}_k(D) := \Big\{ \mathbf{\Omega} := (\Omega_1, \dots, \Omega_k) \,:\, \Omega_i \subseteq D \text{ is open, } \partial\Omega_i \text{ is } \mathcal{H}^{d-1}\text{-countably rectifiable}$$
$$\text{with } \mathcal{H}^{d-1}(\partial\Omega_i) < +\infty,\ \Omega_i \cap \Omega_j = \emptyset \text{ for } i \neq j \text{ and } |D \setminus \bigcup_{i=1}^{k} \Omega_i| = 0 \Big\}$$

and

$$\mathcal{H}(\mathbf{\Omega}) := \Big\{ \mathbf{U} := (u_1, \dots, u_k) \in H^1(\Omega_1) \times \dots H^1(\Omega_k) \Big\},$$

we set

$$MMS(\mathbf{\Omega}, \mathbf{U}) := \sum_{i=1}^{k} \Big(\int_{\Omega_i} \alpha_i |\nabla u_i|^2 + \beta_i \mathcal{H}^{d-1}(\partial \Omega_i \cap D) \Big) + \sum_{i=1}^{k} \mathcal{E}_i(\Omega_i, u_i) \,. \tag{2}$$

(We recall that a set $E \subseteq \mathbb{R}^d$ is $\mathcal{H}^{d-1}$-countably rectifiable if it can be written as $E = E_0 \cup \bigcup_n E_n$, where $\mathcal{H}^{d-1}(E_0) = 0$ and E_n is a subset of a hypersurface of class C^1).
Notice firstly that, with the aim of separating the phases, we tune their contrast parameters through the extra-coefficients α_i and β_i (not necessarily different from each other) which appear respectively in front of the volume and the surface energy terms.
Moreover, we attach to each phase an extra-energy $\mathcal{E}_i$ (a priori nonzero). Part of these energies $\mathcal{E}_i$ is the classical fidelity term, while some other part will be typically issued from a statistical model, and serves to encode all the information that one would like to reconstruct in the segmentation procedure. As a prototype choice, $\mathcal{E}_i$ can be taken as the sum of a fidelity term plus a term of research for specific patterns: the former measures the distance from the u_i's to a given image f, the latter measures the distance between the normalized cumulative distributions of the hystograms of the u_i's and of a k-tuple of target objects g_i's (see the papers to [16, 7, 4] for a more detailed description in statistical terms). In practice, this amounts to take for every $u_i \in H^1(\Omega_i)$

$$\mathcal{E}_i(\Omega_i, u_i) := \gamma_i \int_{\Omega_i} |u_i - f|^2 \, dx + \delta_i(|\Omega_i|) \int_0^1 |G_{\Omega_i, u_i} - g_i| \, dt, \tag{3}$$

where $f : D \to [0, 1]$ and $g_1, \dots, g_k : [0, 1] \to [0, 1]$ are given measurable functions, and

$$G_{\Omega_i, u_i}(t) := \frac{|\Omega_i \cap \{u_i \leq t\}|}{|\Omega_i|}.$$

As well, $\gamma_i > 0$ are given, and $\delta_i : [0, |D|] \to [0, +\infty)$ are smooth functions satisfying $\delta_i(0) = 0$. The signification of this behaviour at 0 is that, if the measure of the phase i is small, then the histogram part of $\mathcal{E}_i$ becomes negligible, in accord with the eye vision.
We are now ready to state our existence result for the *multiphase Mumford-Shah problem*

$$\inf \Big\{ MMS(\mathbf{\Omega}, \mathbf{U}) \ : \ (\mathbf{\Omega}, \mathbf{U}) \in \mathcal{P}_k(D) \times \mathcal{H}(\mathbf{\Omega}) \Big\} \,. \tag{P}$$

Theorem 1. *Assume that the coefficients β_i appearing in the multiphase Mumford-Shah functional* (2) *satisfy the compatibility conditions*

$$\forall i \neq m \, : \, (k-1)\beta_i - \sum_{j \neq i, m} \beta_j > 0 \,, \tag{4}$$

and that the extra-energies $\mathcal{E}_i$ are given by (3)*, with the functions δ_i such that for every $i = 1, \dots, k$*

$$\sup_{t \in (0, |D|]} \Big| \frac{\delta_i(t)}{t} \Big| < +\infty \qquad \text{and} \qquad \sup_{t \in (0, |D|]} \Big| \Big(\frac{\delta_i(t)}{t} \Big)' \Big| < +\infty \,.$$

Then problem (P) admits a solution.

Our proof of Theorem 1 moves in the direction traced by De Giorgi, Carriero, Leaci in the pioneering paper [10]: similarly as in the one phase case, our strategy consists in minimizing a relaxed version of the energy MMS among a wider class of competitors, and then recovering the existence of a solution to the initial problem via a regularity result for minimizers to the relaxed one. More precisely, we consider the *relaxed multiphase Mumford-Shah problem*

$$(\overline{P}) \qquad \inf\left\{\overline{MMS}(\mathbf{\Omega},\mathbf{U})\ :\ (\mathbf{\Omega},\mathbf{U})\in\mathcal{A}_k(D)\times\mathcal{F}(\mathbf{\Omega})\right\},$$

where the enlarged class of competitors is given by

$$\mathcal{A}_k(D):=\left\{\mathbf{\Omega}:=(\Omega_1,\dots,\Omega_k)\ :\ (\Omega_1,\dots,\Omega_k)\text{ is a Caccioppoli partition of }D\right\}$$

$$\mathcal{F}(\mathbf{\Omega})=\left\{\mathbf{U}:=(u_1,\dots,u_k)\in(SBV(\mathbb{R}^d))^k\ :\ u_i=0\text{ a.e. on }\Omega_i^c\text{ for every }i=1,\dots,k\right\},$$

and the relaxed energy has the form

$$\overline{MMS}(\mathbf{\Omega},\mathbf{U}):=\sum_{i=1}^k\left(\int_{\Omega_i}\alpha_i|\nabla u_i|^2+\beta_i\mathcal{H}^{d-1}\left((\partial^e\Omega_i\cup J_{u_i})\cap D\right)\right)+\sum_{i=1}^k\mathcal{E}_i(\Omega_i,u_i)\,.$$

Here $\partial^e\Omega_i$ and J_{u_i} denote respectively the essential boundary of Ω_i and the jump set of u_i, while the energies $\mathcal{E}_i$ are still given by definition (3), now extended to functions $u_i\in SBV(\mathbb{R}^d)$ with $u_i=0$ a.e. on Ω_i^c. Moreover, by a Caccioppoli partition of D we mean that each Ω_i has finite perimeter, $|\Omega_i\cap\Omega_j|=0$ if $i\neq j$ and $|D\setminus\bigcup_i\Omega_i|=0$.
Some comments are in order about the choice of how to define the relaxed energy $\overline{MMS}$. In particular, we wish to draw attention on the fact that a point of $\partial^e\Omega_i$ may be, or not, a jump point of u_i. Actually, a natural question is whether in the length term one has to consider only the jump points of u_i or the jump points together with the full essential boundary of the phase Ω_i. Both from the mathematical point of view and from the modeling one, these two choices lead to definitely different problems. From the point of view of image segmentation the two issues are meaningful. Keeping only the jump part in the length penalization means that, if u_i is close to the value zero on some region, the separation of the phase from its complement relies only on the Dirichlet part of the energy, and not on the jump terms. This may sound reasonable for instance when the value of the background is zero and one has to extract objects which confound with the background itself. This situation in which only the jump points are considered in the length penalization was discussed by the authors in [5] in the context of a thermal insulation problem and applies as well to this kind of background enhanced image segmentation problems. In general, if the background does not play any particular role, there is no reason to focus on the zero level, the same treatment being applied to all levels. Consequently, a general formulation has to include the essential boundary of the phase in the penalty term. It is precisely this last issue that we shall discuss in the paper.
Our proof of Theorem 1 via the use of the relaxed problem $(\overline{P})$ can be outlined as follows:

- We prove the existence of a relaxed solution $(\mathbf{\Omega},\mathbf{U})$ to problem $(\overline{P})$ (Proposition 8). This is a straightforward consequence of the compactness theorem in the space SBV, and of the good convergence properties of the energies $\mathcal{E}_i$'s.
- We show that, if $(\mathbf{\Omega},\mathbf{U})$ is a solution to problem $(\overline{P})$,
  then each Ω_i is (equivalent to) an open set with $\partial^e\Omega_i\cap D=\partial\Omega_i\cap D$ and moreover
  $$\mathcal{H}^{d-1}\left((\overline{J_{u_i}}\setminus J_{u_i})\cap\Omega_i\right)=0.$$

This is proved by using as a crucial tool a uniform density estimate from below for the phases of a relaxed solution (Proposition 9). In view of the previous properties, by setting $\widetilde{\Omega}_i := \Omega_i \setminus \overline{J_{u_i}}$ and since $(u_i)_{|\widetilde{\Omega}_i} \in H^1(\widetilde{\Omega}_i)$, we obtain that $(\widetilde{\mathbf{\Omega}}, \mathbf{U})$ solves problem (P) (in particular, the infima of $(\overline{P})$ and (P) are the same).

Once established the existence of solutions, we turn our attention to their fine regularity properties, and we prove two different kinds of results.
The first one states that, under the same assumptions of Theorem 1, each of the sets $\partial^e\Omega_i \cup J_{u_i}$ enjoys Ahlfors regularity:

Theorem 2. *Under the same assumptions of Theorem 1, let* $(\mathbf{\Omega}, \mathbf{U})$ *be a solution to problem* $(\overline{P})$*. Then, for every* $i = 1, \dots, k$*, the set* $\partial^e\Omega_i \cup J_{u_i}$ *is Ahlfors regular in* D*, that is there exist a constant* $k_i > 0$ *and a radius* $r_i > 0$ *such that, for every* $x \in \partial^e\Omega_i \cup J_{u_i}$ *and every* $\rho \in (0, r_i]$ *such that* $\overline{B}_\rho(x) \subset D$*, there holds*

$$k_i\rho^{d-1} \le \mathcal{H}^{d-1}\big((\partial^e\Omega_i \cup J_{u_i}) \cap \overline{B}_\rho(x)\big) \le \frac{1}{k_i}\rho^{d-1}\,. \tag{5}$$

Let us notice that, as it can be checked by inspection of the proof, Theorem 2 (as well as Theorem 1) continues to hold for other multiphase Mumford-Shah models, in which the energies $\mathcal{E}_i$ are not given necessarily by (3), but may be replaced by any family of functionals satisfying the conditions (C1), (C2) and (C3) stated in Lemma 7 below.

Our second regularity result is of different fashion and applies to a general form of local almost quasi-minimizers for the multiphase Mumford-Shah functional (including solutions to problem (P)), intended according to the following definition. For simplicity of notation, for every $\mathbf{\Omega} = (\Omega_1, \dots, \Omega_k), \mathbf{\Lambda} = (\Lambda_1, \dots, \Lambda_k) \in \mathcal{A}_k(D)$, we denote

$$\partial^e\mathbf{\Omega} := \bigcup_{i=1}^k \partial^e\Omega_i, \qquad \mathbf{\Omega}\Delta\mathbf{\Lambda} := \bigcup_{i=1}^k (\Omega_i\Delta\Lambda_i)$$

where $A\Delta B$ stands for the symmetric difference of the sets A, B. Moreover for every $\mathbf{U} = (u_1, \dots, u_k) \in \mathcal{F}(\mathbf{\Omega}), \mathbf{V} = (v_1, \dots, v_k) \in \mathcal{F}(\mathbf{\Lambda})$, we set

$$J_{\mathbf{U}} := \bigcup_{i=1}^k J_{u_i}, \qquad \{\mathbf{U} \neq \mathbf{V}\} := \bigcup_{i=1}^k \{u_i \neq v_i\}.$$

Definition 3 (Multiphase local almost quasi-minimizers). Let $c_1, c_2, c_3, \alpha > 0$ with $c_1 \le c_2$. We say that $(\mathbf{\Omega}, \mathbf{U}) \in \mathcal{A}_k(D) \times \mathcal{F}(\mathbf{\Omega})$ is a *local almost quasi-minimizer of the multiphase Mumford-Shah functional at point* $x_0 \in D$ if there exists $\rho_0 > 0$, such that for every $0 < \rho < \rho_0$, for every ball $\overline{B}_\rho(x_0) \subseteq D$ and for every $(\mathbf{\Lambda}, \mathbf{V}) \in \mathcal{A}_k(D) \times \mathcal{F}(\mathbf{\Lambda})$ such that

$$\mathbf{\Omega}\Delta\mathbf{\Lambda} \subseteq B_\rho(x_0), \quad \{\mathbf{U} \neq \mathbf{V}\} \subseteq B_\rho(x_0)$$

there holds

$$\sum_{i=1}^k \int_{B_\rho(x_0)} |\nabla u_i|^2 dx + c_1\mathcal{H}^{d-1}\Big((\partial^e\mathbf{\Omega} \cup J_{\mathbf{U}}) \cap \overline{B}_\rho(x_0)\Big)$$
$$\le \sum_{i=1}^k \int_{B_\rho(x_0)} |\nabla v_i|^2 dx + c_2\mathcal{H}^{d-1}\Big((\partial^e\mathbf{\Lambda} \cup J_{\mathbf{V}}) \cap \overline{B}_\rho(x_0)\Big) + c_3\rho^{d-1+\alpha}\,.$$

In the single phase setting when $k = 1$, the above definition reduces precisely to that of local almost quasi-minimizer of the Mumford-Shah functional given in [6, Definition 2.1], and, if additionally $c_1 = c_2$, it gives back the classical notion of quasi minimizer according to [3, Definition 7.17].
In the multiphase setting, thanks in particular to the presence of possibly different constants $c_1 \leq c_2$, the class of local almost quasi-minimizers of the multiphase Mumford-Shah functional contains as typical examples solutions to problem (P) (suitably rescaled to get rid of the weights in front of the gradient terms). At the same time, it is clearly much larger, as it includes also minimizers of more general energies, possibly containing terms depending on the normal field at the jump sets, the values of traces on the jump sets, and so on. For instance, one can consider minimizers $(\mathbf{\Omega}, \mathbf{U})$ in $\mathcal{A}_k(D) \times \mathcal{F}(\mathbf{\Omega})$ to the multiphase energy

$$\sum_{i=1}^{k} \int_{\Omega_i} |\nabla u_i|^2 + \mathcal{H}^{d-1}\left((\partial^e \mathbf{\Omega} \cup J_{\mathbf{U}}) \cap D\right) + \sum_{i=1}^{k} \mathcal{E}_i(\Omega_i, u_i)\,. \tag{6}$$

Notice that, in (6), no distinction is made between the perimeter of each phase and the jumps of u_i "inside" the phase. This is in contrast to what happens in the formulation of problem $(\overline{P})$, where the common boundary between Ω_i and Ω_j is somehow counted twice, while the inner jumps only once.
Though Definition 3 is very weak, from any pair $(\mathbf{\Omega}, \mathbf{U}) \in \mathcal{A}_k(D) \times \mathcal{F}(\mathbf{\Omega})$ which fits it, there is a way to build a local minimizer u of the classical one phase Mumford-Shah functional such that $J_u = J_{\mathbf{U}} \cup \partial^e \mathbf{\Omega}$. This construction is possible under the additional condition that all the phase functions are in L^∞. As a consequence, under such (natural) assumption, we obtain the Ahlfors regularity for the union of the jumps and the essential boundaries of the phases for any local almost quasi-minimizer of the multiphase Mumford-Shah functional. More precisely, we prove:

Theorem 4. *Let $(\mathbf{\Omega}, \mathbf{U}) \in \mathcal{A}_k(D) \times \mathcal{F}(\mathbf{\Omega})$ be a local almost quasi-minimizer of the multiphase Mumford-Shah functional with constants (c_1, c_2, c_3, α) at point x_0, such that, $u_i \in L^\infty(D)$ for every $i = 1, \dots, k$. Then, there exist constants $a_1, \dots, a_k$ such that the function*

$$u(x) = \sum_{i=1}^{k} (u_i(x) + a_i) 1_{\Omega_i}$$

is a local almost quasi-minimizer of the (classical) Mumford-Shah functional at x_0, with constants $(c_1, c_2, \tilde{c}_3, \alpha)$, where $\tilde{c}_3$ depends only on (c_2, c_3) and the dimension of the space.

Corollary 5. *Let $(\mathbf{\Omega}, \mathbf{U}) \in \mathcal{A}_k(D) \times \mathcal{F}(\mathbf{\Omega})$ be a local almost quasi-minimizer of the multiphase Mumford-Shah functional at every point $x \in D$. Assume that $u_i \in L^\infty(D)$ for every $i = 1, \dots, k$. Then the set $J_{\mathbf{U}} \cup \partial^e \mathbf{\Omega}$ is closed and Ahlfors regular.*

We stress that Theorem 4 is obtained under much weaker assumptions with respect to Theorem 2, but, contrarily to this latter, the resulting Corollary 5 is only *global*, namely it does not provide any kind of information about the regularity of each phase individually. Finally let us point out that, in the particular case $c_1 = c_2$, Theorem 4 allows to recover the regularity result proved by Ambrosio, Fusco and Pallara in [2].

Corollary 6. *Under the same hypotheses as Corollary 5, assume moreover that $c_1 = c_2$. Then there exists a closed set $\Gamma \subseteq J_{\mathbf{U}} \cup \partial^e \mathbf{\Omega}$ such that $\mathcal{H}^{d-1}(\Gamma) = 0$, and $(J_{\mathbf{U}} \cup \partial^e \mathbf{\Omega}) \setminus \Gamma$ is locally a $C^{1,\delta}$ manifold for every $\delta \in (0, 1)$ (and $C^{1,1}$ in two dimensions).*

The proofs of Theorem 1, Theorem 2, and Theorem 4 (with related corollaries) are given respectively in Sections 2, 3 and 4 below.

Notation. If $E \subseteq \mathbb{R}^d$, E^c will denote its complement, 1_E will stand for its characteristic function, while $|E|$ and $\mathcal{H}^{d-1}(E)$ will denote its Lebesgue and its $(d-1)$-dimensional Hausdorff measure respectively. $B_\rho(x)$ will denote the open ball of center $x \in \mathbb{R}^d$ and radius $\rho > 0$: we set $\omega_d := |B_1(x)|$.
Concerning functional spaces, if $A \subseteq \mathbb{R}^d$ is open, $H^1(A)$ will stand for the usual space Sobolev functions which are square integrable together with their weak partial derivatives, while $SBV(A)$ will denote the space of special functions of bounded variation on A. In our analysis we will employ also sets with finite perimeter: if $E \subseteq \mathbb{R}^d$ has finite perimeter, $\partial^e E$ will denote its essential boundary. We refer the reader to the monograph [3] for an exhaustive treatment of the space SBV and of the family of sets with finite perimeter.

2. Proof of Theorem 1

Our route to the proof of Theorem 1 has been outlined in the Introduction. The intermediate steps are precisely the following:

- we establish some useful properties of the functionals $\mathcal{E}_i$ (Lemma 7);
- we show the existence of a solution to problem $(\overline{P})$ (Proposition 8);
- we prove a uniform lower bound for the density of the phases of a relaxed solution (Proposition 9);
- we obtain some regularity properties for the phases and the jump sets (Proposition 10).

Lemma 7. *Let $f : D \to [0,1]$ and $g : [0,1] \to [0,1]$ be given measurable functions, and let $\delta : [0,|D|] \to [0,+\infty)$ be a smooth function such that*

$$\sup_{t\in(0,|D|]} \Big|\frac{\delta(t)}{t}\Big| < +\infty \quad \textit{and} \quad \sup_{t\in(0,|D|]} \Big|\Big(\frac{\delta(t)}{t}\Big)'\Big| < +\infty\,.$$

The functional defined, for any set $A \subset D$ of finite perimeter and any $v \in SBV(\mathbb{R}^d)$ with $v = 0$ a.e. on A^c, by

$$\mathcal{E}(A,v) := \int_A |v-f|^2\,dx + \delta(|A|)\int_0^1 |G_{A,v} - g|\,dt\,, \quad \textit{with } G_{A,v}(t) := \frac{|A\cap\{v\le t\}|}{|A|}$$

satisfies the following properties:

(C1) *Truncation: setting $v^T = \min\{\max\{v,0\},1\}$, it holds*
$$\mathcal{E}(A,v^T) \le \mathcal{E}(A,v)\,;$$

(C2) *Continuity: if $(A^n,v^n) \to (A,v)$ strongly in $L^1(D)\times L^1(D)$, with $\sup_n \|v^n\|_{L^\infty(D)} < +\infty$, then*
$$\mathcal{E}(A^n,v^n) \to \mathcal{E}(A,v);$$

(C3) *Controllability: there exist positive constants C, C' such that the following items hold true.*

(i) *If $v = 0$ a.e. on A^c and $A \cap B = \emptyset$*
$$|\mathcal{E}(A\cup B,v) - \mathcal{E}(A,v)| \le C|B|.$$

(ii) *If* $0 \le v, w \le 1$ *with* $v, w = 0$ *a.e. on* A^c

$$|\mathcal{E}(A, v) - \mathcal{E}(A, w)| \le C'|A \cap \{v \ne w\}|.$$

Proof. Concerning property (C1), the inequality $\int_A |v^T - f|^2\,dx \le \int_A |v - f|^2\,dx$ follows immediately from the fact that f takes values into $[0,1]$, while it holds $\int_0^1 |G_{A,v^T} - g|\,dt = \int_0^1 |G_{A,v} - g|\,dt$ because $G_{A,v}(t) = G_{A,v^T}(t)$ for a.e. $t \in [0,1]$.
Let us pass to property (C2). Let $(A^n, v^n) \to (A, v)$ strongly in $L^1(D) \times L^1(D)$, with $\sup_n \|v^n\|_{L^\infty(D)} < +\infty$. Then up to a subsequence we have

$$|v^n - f|^2 1_{A^n} \to |v - f|^2 1_A \qquad \text{a.e. in } D,$$

and by dominated convergence we obtain

$$\int_{A^n} |v^n - f|^2\,dx \to \int_A |v - f|^2\,dx.$$

Also the continuity of the second term in $\mathcal{E}$ follows by dominated convergence, provided we show that for $A \ne \emptyset$

$$G_{A^n,v^n} \to G_{A,v} \qquad \text{for a.e. } t \in [0,1]\,. \tag{7}$$

Since

$$\int_D |v^n - v|\,dx = \int_{-\infty}^{+\infty} \left[\int_D |1_{\{v^n \le t\}} - 1_{\{v \le t\}}|\,dx\right] dt,$$

we infer that for a.e. $t \in \mathbb{R}$ we have

$$1_{\{v^n \le t\}} \to 1_{\{v \le t\}} \qquad \text{strongly in } L^1(D)$$

from which (7) easily follows.
Let us come finally to property (C3), item (i). If $v = 0$ on A^c and $A \cap B = \emptyset$, we have

$$\begin{aligned}
&\big|\mathcal{E}(A \cup B, v) - \mathcal{E}(A, v)\big| \\
&\le \left|\frac{\delta(|A \cup B|)}{|A \cup B|} \int_0^1 \big||(A \cup B) \cap \{v \le t\}| - g(t)|A \cup B|\big|\,dt \right. \\
&\quad \left. - \frac{\delta(|A|)}{|A|} \int_0^1 \big||A \cap \{v \le t\}| - g(t)|A|\big|\,dt \right| + \int_B |f|^2\,dx \\
&= \left|\frac{\delta(|A| + |B|)}{|A| + |B|} \int_0^1 \big||A \cap \{v \le t\}| + |B \cap \{v \le t\}| - g(t)|A| - g(t)|B|\big|\,dt \right. \\
&\quad \left. - \frac{\delta(|A|)}{|A|} \int_0^1 \big||A \cap \{v \le t\}| - g(t)|A|\big|\,dt \right| + \int_B |f|^2\,dx \\
&\le \left|\frac{\delta(|A| + |B|)}{|A| + |B|} - \frac{\delta(|A|)}{|A|}\right| \int_0^1 \big||A \cap \{v \le t\}| - g(t)|A|\big|\,dt \\
&\quad + \frac{\delta(|A| + |B|)}{|A| + |B|} \int_0^1 \big||B \cap \{v \le t\}| - g(t)|B|\big|\,dt + |B| \\
&\le \left[2|D| \sup_{t \in (0,|D|]} \left|\left(\frac{\delta(t)}{t}\right)'\right| + 2 \sup_{t \in (0,|D|]} \left|\frac{\delta(t)}{t}\right| + 1\right] |B| =: C|B|\,.
\end{aligned}$$

Let us come to item (ii). If $v, w = 0$ on A^c and $0 \le v, w \le 1$, we have

$$\begin{aligned}
&\big|\mathcal{E}(A,v) - \mathcal{E}(A,w)\big| \\
&\le \frac{\delta(|A|)}{|A|}\left|\int_0^1 \Big(\big||A\cap\{v\le t\}| - g(t)|A|\big| - \big||A\cap\{w\le t\}| - g(t)|A|\big|\Big)\,dt\right| \\
&\quad + \int_A |(-2f+v+w)(v-w)|\,dx \\
&\le \frac{\delta(|A|)}{|A|}\int_0^1 \Big(\big||A\cap\{v\le t\}| - |A\cap\{w\le t\}|\big|\Big)\,dt + 4|A\cap\{v\ne w\}| \\
&= \frac{\delta(|A|)}{|A|}\int_0^1 \Big(\big||A\cap\{v\ne w\}\cap\{v\le t\}| - |A\cap\{v\ne w\}\cap\{w\le t\}|\big|\Big)\,dt + 4|A\cap\{v\ne w\}| \\
&\le 2\sup_{t\in(0,|D|]}\left|\frac{\delta(t)}{t}\right|\big|A\cap\{v\ne w\}\big| + 4|A\cap\{v\ne w\}| =: C'\big|A\cap\{v\ne w\}\big|\,.
\end{aligned}$$

□

Proposition 8. *The relaxed problem* $(\overline{P})$ *admits a solution.*

Proof. Let $(\mathbf{\Omega}^n, \mathbf{U}^n)$ be a minimizing sequence, with $\mathbf{\Omega}^n$ in $\mathcal{A}_k(D)$ and $\mathbf{U}^n \in \mathcal{F}(\mathbf{\Omega}^n)$. Since the data f and g_i are in $L^\infty(D;[0,1])$, by truncation (in particular since the functionals $\mathcal{E}_i$ satisfy condition (C1) in Lemma 7), we can assume that the functions u_i^n belong as well to $L^\infty(D;[0,1])$ for every $i = 1,\dots,k$ and every $n \in \mathbb{N}$.
Then, by Ambrosio's compactness and semicontinuity theorem [3, Thms. 4.7 and 4.8], up to (not relabeled) subsequences, we have $(\mathbf{\Omega}^n, \mathbf{U}^n) \to (\mathbf{\Omega}, \mathbf{U})$ strongly in $(L^1(D))^k \times (L^1(D))^k$, with $(\mathbf{\Omega}, \mathbf{U}) \in \mathcal{A}_k(D) \times \mathcal{F}(\mathbf{\Omega})$, and

$$\int_{\Omega_i} |\nabla u_i|^2 \le \liminf_n \int_{\Omega_i^n} |\nabla u_i^n|^2\,.$$

To see that also the second term in the energy is lower semicontinuous we take a positive parameter ε, and for every $i = 1,\dots,k$, we consider the sequence $v_i^n := u_i^n + \varepsilon 1_{\Omega_i^n}$. Since u_i^n are non-negative, with $u_i^n = 0$ a.e. on $(\Omega_i^n)^c$, we have $\partial^e\Omega_i^n \subseteq J_{v_i^n}$, so that

$$J_{v_i^n} = \partial^e\Omega_i^n \cup J_{u_i^n}\,. \tag{8}$$

Moreover, we have

$$v_i^n \to v_i := u_i + \varepsilon 1_{\Omega_i} \qquad \text{strongly in } L^1(D)\,. \tag{9}$$

Since $(\Omega_i^n, u_i^n) \to (\Omega_i, u_i)$ strongly in $L^1(D) \times L^1(D)$, we have that u_i is non-negative with $u_i = 0$ a.e. on Ω_i^c, so that condition (8) continues to hold for the limit functions v_i's, namely

$$J_{v_i} = \partial^e\Omega_i \cup J_{u_i}\,. \tag{10}$$

Thanks to (8), (9) and (10), we can apply again Ambrosio's semicontinuity theorem to obtain

$$\begin{aligned}
\mathcal{H}^{d-1}((\partial^e\Omega_i \cup J_{u_i}) \cap D) = \mathcal{H}^{d-1}(J_{v_i} \cap D) &\le \liminf_n \mathcal{H}^{d-1}(J_{v_i^n} \cap D) \\
&= \liminf_n \mathcal{H}^{d-1}((\partial^e\Omega_i^n \cup J_{u_i^n}) \cap D)\,.
\end{aligned}$$

Finally, since the functionals $\mathcal{E}_i$ satisfy the continuity condition (C2) in Lemma 7, we have $\mathcal{E}_i(\Omega_i^n, u_i^n) \to \mathcal{E}_i(\Omega_i, u_i)$. We conclude that $(\mathbf{\Omega}, \mathbf{U})$ is a solution to $(\overline{P})$. □

Proposition 9 (**Density lower bound for the phases).** *Let $(\mathbf{\Omega}, \mathbf{U}) \in \mathcal{A}_k(D) \times \mathcal{F}(\mathbf{\Omega})$ be a solution to problem $(\overline{P})$. For every $i = 1, \dots, k$ there exist a constant $c_i > 0$ and a radius $\rho_i > 0$ such that the following property holds true: for every $x \in D$ such that $|\Omega_i \cap B_\rho(x)| > 0$ for every $\rho > 0$, we have*

$$\frac{|\Omega_i \cap B_\rho(x)|}{\rho^d} \geq c_i \qquad \text{for every } \rho \in (0, \rho_i). \tag{11}$$

In particular the set

$$\Omega_i^{(0)} := \Big\{ x \in D \ : \ \lim_{\rho \to 0^+} \frac{|\Omega_i \cap B_\rho(x_n)|}{\rho^d} = 0 \Big\}$$

is open.

Proof. Let $(\mathbf{\Omega}, \mathbf{U}) \in \mathcal{A}_k(D) \times \mathcal{F}(\mathbf{\Omega})$ be a solution to problem $(\overline{P})$. Thanks to property (C1) in Lemma 7, it is not restrictive to assume $0 \leq u_i \leq 1$ a.e. for every $i = 1, \dots, k$. Let $i \in \{1, \dots, k\}$ be fixed, and let $x \in D$ be such that $|\Omega_i \cap B_\rho(x)| > 0$ for every $\rho > 0$. We fix an index $j \neq i$, and we consider the competitor $(\mathbf{\Lambda}, \mathbf{V}) \in \mathcal{A}_k(D) \times \mathcal{F}(\mathbf{\Lambda})$ defined by

$$\Lambda_i := \Omega_i \setminus \overline{B}_\rho(x)\,, \quad \Lambda_j := \Omega_j \cup (\Omega_i \cap B_\rho(x))\,, \quad \Lambda_h = \Omega_h \text{ for every } h \neq i, j$$

and

$$v_i = u_i 1_{\mathbb{R}^d \setminus \overline{B}_\rho(x)}\,, \qquad v_h = u_h \text{ for every } h \neq i\,.$$

In order to compare the energies of $(\mathbf{\Omega}, \mathbf{U})$ and $(\mathbf{\Lambda}, \mathbf{V})$, let us set for every $l, m \in \{1, \dots, k\}$

$$\begin{aligned}
&\zeta_l = (J_{u_l} \setminus \partial^e \Omega_l) \cap \overline{B}_\rho(x) \cap D \\
&\gamma_l = \Omega_l \cap \partial B_\rho(x) \\
&\gamma_{lm} = (\partial^e \Omega_l \cap \partial^e \Omega_m) \cap \overline{B}_\rho(x) \cap D, \quad l \neq m.
\end{aligned}$$

The optimality of $(\mathbf{\Omega}, \mathbf{U})$ gives

$$\begin{aligned}
&\sum_{l \in \{1 \dots, k\}} \left[\int_{\Omega_l \cap B_\rho(x)} \alpha_l |\nabla u_l|^2 \, dx + \beta_l \mathcal{H}^{d-1}(\zeta_l) + \sum_{m \neq l} \beta_l \mathcal{H}^{d-1}(\gamma_{lm}) + \mathcal{E}_l(\Omega_l, u_l) \right] \\
&\leq \beta_i \mathcal{H}^{d-1}(\gamma_i) + \mathcal{E}_i(\Omega_i \setminus \overline{B}_\rho(x), v_i) \\
&+ \int_{\Omega_j \cap B_\rho(x)} \alpha_j |\nabla u_j|^2 \, dx + \beta_j \mathcal{H}^{d-1}(J_{u_j} \cap \overline{B}_\rho(x) \cap D) \\
&+ \sum_{m \neq i,j} \beta_j \mathcal{H}^{d-1}(\gamma_{jm}) + \beta_j \mathcal{H}^{d-1}(\gamma_i) + \sum_{m \neq i,j} \beta_j \mathcal{H}^{d-1}(\gamma_{im}) \\
&+ \mathcal{E}_j(\Omega_j \cup (\Omega_i \cap B_\rho(x))), u_j) \\
&+ \sum_{l \neq i,j} \left[\int_{\Omega_l \cap B_\rho(x)} \alpha_l |\nabla u_l|^2 \, dx + \beta_l \mathcal{H}^{d-1}(\zeta_l) + \sum_{m \neq l} \beta_l \mathcal{H}^{d-1}(\gamma_{lm}) + \mathcal{E}_l(\Omega_l, u_l) \right].
\end{aligned}$$

Simplifying we obtain

$$
\begin{aligned}
&\int_{\Omega_i\cap B_\rho(x)} \alpha_i|\nabla u_i|^2\,dx + \beta_i\mathcal{H}^{d-1}(\zeta_i) + \sum_{m\neq i}\beta_i\mathcal{H}^{d-1}(\gamma_{im}) + \mathcal{E}_i(\Omega_i,u_i)\\
&+\beta_j\mathcal{H}^{d-1}(\zeta_j) + \beta_j\mathcal{H}^{d-1}(\gamma_{ij}) + \mathcal{E}_j(\Omega_j,u_j)\\
&\le \beta_i\mathcal{H}^{d-1}(\gamma_i) + \mathcal{E}_i(\Omega_i\setminus\overline{B}_\rho(x),v_i)\\
&+\beta_j\mathcal{H}^{d-1}(J_{u_j}\cap\overline{B}_\rho(x)\cap D) + \beta_j\mathcal{H}^{d-1}(\gamma_i) + \sum_{m\neq i,j}\beta_j\mathcal{H}^{d-1}(\gamma_{im})\\
&+\mathcal{E}_j(\Omega_j\cup(\Omega_i\cap B_\rho(x))),u_j).
\end{aligned}
\tag{12}
$$

Notice that

$$\beta_j\mathcal{H}^{d-1}(\zeta_j) + \beta_j\mathcal{H}^{d-1}(\gamma_{ij}) \ge \beta_j\mathcal{H}^{d-1}(J_{u_j}\cap\overline{B}_\rho(x)\cap D).$$

Moreover thanks to property (C3) in Lemma 7 we have

$$
\begin{aligned}
\mathcal{E}_i(\Omega_i\setminus\overline{B}_\rho(x),v_i) &\le \mathcal{E}_i(\Omega_i,v_i) + C_i|\Omega_i\cap B_\rho(x)|\\
&\le \mathcal{E}_i(\Omega_i,u_i)| + C_i'|\Omega_i\cap\{v_i\neq u_i\}| + C_i|\Omega_i\cap B_\rho(x)|\\
&\le \mathcal{E}_i(\Omega_i,u_i)| + (C_i'+C_i)|\Omega_i\cap B_\rho(x)|
\end{aligned}
$$

and

$$\mathcal{E}_j(\Omega_j\cup(\Omega_i\cap B_\rho(x))),u_j) \le \mathcal{E}_j(\Omega_j,u_j) + C_j|\Omega_i\cap B_\rho(x)|$$

so that we infer from (12)

$$\sum_{m\neq i}\beta_i\mathcal{H}^{d-1}(\gamma_{im}) - \sum_{m\neq i,j}\beta_j\mathcal{H}^{d-1}(\gamma_{im}) \le (\beta_i+\beta_j)\mathcal{H}^{d-1}(\gamma_i) + \hat{C}_j|\Omega_i\cap B_\rho(x)| \tag{13}$$

for some $\hat{C}_j > 0$. The above inequality holds for an arbitrary index $j\neq i$. Therefore, we can sum over $j\neq i$ and obtain, for some constant $C>0$,

$$
\begin{aligned}
(k-1)\sum_{m\neq i}\beta_i\mathcal{H}^{d-1}(\gamma_{im}) - \sum_{j\neq i}\sum_{m\neq i,j}\beta_j\mathcal{H}^{d-1}(\gamma_{im})&\\
\le [(k-1)\beta_i + \sum_{j\neq i}\beta_j]\mathcal{H}^{d-1}(\gamma_i) + C|\Omega_i\cap B_\rho(x)|&
\end{aligned}
$$

or equivalently

$$\sum_{m\neq i}\big[(k-1)\beta_i - \sum_{j\neq i,m}\beta_j\big]\mathcal{H}^{d-1}(\gamma_{im}) \le [(k-1)\beta_i + \sum_{j\neq i}\beta_j]\mathcal{H}^{d-1}(\gamma_i) + C|\Omega_i\cap B_\rho(x)|\,.$$

Thanks to the assumed compatibility conditions (4), this implies, for some constant $\delta>0$:

$$\delta\sum_{m\neq i}\mathcal{H}^{d-1}(\gamma_{im}) \le [(k-1)\beta_i + \sum_{j\neq i}\beta_j]\mathcal{H}^{d-1}(\gamma_i) + C|\Omega_i\cap B_\rho(x)|\,.$$

We now add to both sides $\delta\mathcal{H}^{d-1}(\gamma_i)$ and we divide by δ. We obtain, for some constants $C', C''>0$:

$$\mathcal{H}^{d-1}(\gamma_i) + \sum_{m\neq i}\mathcal{H}^{d-1}(\gamma_{im}) \le C'\mathcal{H}^{d-1}(\gamma_i) + C''|\Omega_i\cap B_\rho(x)|\,.$$

We observe that the left hand side of the above inequality is precisely the perimeter of the set $\Omega_i\cap B_\rho(x)$ in D. Hence, invoking the isoperimetric inequality, we obtain that, for ρ

sufficiently small (independent of x), the function $\theta_i(\rho) := |\Omega_i \cap B_\rho(x)|$ satisfies for some constant $C''' > 0$ the differential inequality

$$\theta_i(\rho)^{\frac{d-1}{d}} \leq C'''\mathcal{H}^{d-1}(\gamma_i) = C'''\frac{d}{d\rho}\theta_i(\rho)\,.$$

Dividing by $\theta_i(\rho)^{\frac{d-1}{d}}$ (which is strictly positive by assumption) and integrating we obtain (11).

The openness of $\Omega_i^{(0)}$ follows immediately arguing by contradiction: indeed, if $x \in \Omega_i^{(0)}$ can be approximated by a sequence of points $x_n \in D \setminus \Omega_i^{(0)}$, by applying (11), we get for ρ sufficiently small

$$\frac{|\Omega_i \cap B_\rho(x_n)|}{\rho^d} \geq c_i$$

and passing to the limit as $n \to +\infty$

$$\frac{|\Omega_i \cap B_\rho(x)|}{\rho^d} \geq c_i,$$

against the assumption $x \in \Omega_i^{(0)}$. □

Proposition 10 (Regularity). *Let* $(\mathbf{\Omega}, \mathbf{U}) \in \mathcal{A}_k(D) \times \mathcal{F}(\mathbf{\Omega})$ *be a solution to problem* $(\overline{P})$. *For every* $i = 1, \ldots, k$ *the following items hold true.*

(a) *The phase* Ω_i *is equivalent to an open set with*

$$\partial\Omega_i \cap D = \partial^e\Omega_i \cap D.$$

(b) *The restriction of* u_i *to* Ω_i *is a quasi-minimizer of the Mumford-Shah energy in* Ω_i. *In particular the jump set* J_{u_i} *is essentiallly closed in* Ω_i, *that is*

$$\mathcal{H}^{d-1}\left((\overline{J_{u_i}} \setminus J_{u_i}) \cap \Omega_i\right) = 0. \tag{14}$$

Proof. By Proposition 9 the sets $C_i := D \setminus \Omega_i^{(0)}$ are relatively closed in D. Since we can write

$$\partial^e\Omega_i \cap D = \bigcup_{j\neq i}(\partial^e\Omega_i \cap \partial^e\Omega_j \cap D) = \bigcup_{j\neq i}(C_i \cap C_j),$$

we conclude that $\partial^e\Omega_i \cap D$ is closed in D.

To prove item (a), notice that Ω_i is equivalent to the union of a family of connected components of the open set $D \setminus (\partial^e\Omega_i \cap D)$, so that it is itself open with

$$\partial\Omega_i \cap D \subseteq \partial^e\Omega_i \cap D.$$

Since the opposite inclusion is always true, the conclusion follows.

Let us come to item (b). Thanks to property (C1) in Lemma 7, it is not restrictive to assume $0 \leq u_i \leq 1$ a.e. for every $i = 1, \ldots, k$. We claim that u_i is a local quasi-minimizer for the (weighted) Mumford-Shah energy in the open set Ω_i, namely there exists a constant $C_i > 0$ such that, for every $y \in \Omega_i$ and every $v_i \in SBV(\Omega_i)$ with $\{v_i \neq u_i\} \subset \overline{B}_\rho(y) \subset \Omega_i$,

$$\begin{aligned}\int_{B_\rho(y)} \alpha_i|\nabla u_i|^2 + \beta_i\mathcal{H}^{d-1}(J_{u_i} \cap \overline{B}_\rho(y)) \\ \leq \int_{B_\rho(y)} \alpha_i|\nabla v_i|^2 + \beta_i\mathcal{H}^{d-1}(J_{v_i} \cap \overline{B}_\rho(y)) + C_i\rho^d.\end{aligned} \tag{15}$$

To check this fact, it is enough to consider the competitor $(\mathbf{\Lambda}, \tilde{\mathbf{V}})$ defined by $\mathbf{\Lambda} = \mathbf{\Omega}$ and

$$\tilde{v}_j := \begin{cases} u_j & \text{if } j \neq i \\ u_i 1_{\mathbb{R}^d \setminus B_\rho(y)} + [|v_i| \wedge \|u_i\|_\infty] 1_{B_\rho(y)} & \text{if } j = i\,. \end{cases}$$

By the optimality of $(\mathbf{\Omega}, \mathbf{U})$, we have

$$\int_{B_\rho(y)} \alpha_i |\nabla u_i|^2 + \beta_i \mathcal{H}^{d-1}(J_{u_i} \cap \overline{B}_\rho(y)) + \mathcal{E}_i(u_i, \Omega_i)$$
$$\leq \int_{B_\rho(y)} \alpha_i |\nabla v_i|^2 + \beta_i \mathcal{H}^{d-1}(J_{v_i} \cap \overline{B}_\rho(y)) + \mathcal{E}_i(\tilde{v}_i, \Omega_i)\,,$$

which implies (15) thanks to the controllability condition (C3) (ii) in Lemma 7, since $\{u_i \neq \tilde{v}_i\} \subseteq B_\rho(y)$. Relation (14) is now a consequence of the local quasi-minimality property (15) in view of the De Giorgi-Carriero-Leaci result [10] (see also [6, Theorem 3.1]). □

We are now in a position to give:

Proof of Theorem 1. Since $(\mathbf{\Omega}, \mathbf{U}) \in \mathcal{P}_k(D) \times \mathcal{H}(\mathbf{\Omega})$ is also an element of $\mathcal{A}_k(D) \times \mathcal{F}(\mathbf{\Omega})$ with

$$\overline{MMS}(\mathbf{\Omega}, \mathbf{U}) \leq MMS(\mathbf{\Omega}, \mathbf{U})$$

being $(\partial^e \Omega_i \cup J_{u_i}) \cap D \subseteq \partial \Omega_i \cap D$ for every $i = 1, \ldots, k$, it is clear that

$$\inf(\overline{P}) \leq \inf(P)\,.$$

Let now $(\mathbf{\Omega}, \mathbf{U})$ be a solution to problem $(\overline{P})$ whose existence is guaranteed by Proposition 8. In view of Proposition 10 we can consider the open sets

$$\widetilde{\Omega}_i := \Omega_i \setminus \overline{J_{u_i}}.$$

Notice that since

$$\partial \widetilde{\Omega}_i \cap D = (\partial^e \Omega_i \cup \overline{J_{u_i}}) \cap D,$$

we deduce that the topological boundary of $\widetilde{\Omega}_i$ is $\mathcal{H}^{d-1}$-countably rectifiable (see [3, Theorem 3.59 and Theorem 3.78]), so that $\widetilde{\mathbf{\Omega}} := (\widetilde{\Omega}_1, \ldots, \widetilde{\Omega}_k) \in \mathcal{P}_k(D)$. Since the restriction of u_i to $\widetilde{\Omega}_i$ belongs to $H^1(\widetilde{\Omega}_i)$, we get that $(\widetilde{\mathbf{\Omega}}, \mathbf{U})$ is an admissible competitor for problem (P). Since clearly

$$MMS(\widetilde{\mathbf{\Omega}}, \mathbf{U}) = \overline{MMS}(\mathbf{\Omega}, \mathbf{U})$$

we get

$$\inf(P) \leq MMS(\widetilde{\mathbf{\Omega}}, \mathbf{U}) = \overline{MMS}(\mathbf{\Omega}, \mathbf{U}) = \inf(\overline{P})\,.$$

We conclude that the above inequalities hold as equalities, which shows in particular that $(\widetilde{\mathbf{\Omega}}, \mathbf{U})$ solves problem (P) and achieves the proof. □

3. Proof of Theorem 2

Let $(\mathbf{\Omega}, \mathbf{U})$ be a solution to problem $(\overline{P})$, let $i \in \{1, \dots, k\}$ be fixed, and let $x \in (\partial^e \Omega_i \cup J_{u_i}) \cap D$. Let us divide the proof in two steps.

Step 1: Upper bound inequality. We follow the proof of Proposition 9, considering the competitor $(\mathbf{\Lambda}, \mathbf{V})$ defined by

$$\Lambda_i := \Omega_i \setminus \overline{B}_\rho(x)\,, \quad \Lambda_j := \Omega_j \cup (\Omega_i \cap B_\rho(x))\,, \quad \Lambda_h = \Omega_h \text{ for every } h \neq i, j$$

with

$$v_i = u_i 1_{\mathbb{R}^d \setminus \overline{B}_\rho(x)}\,, \qquad v_h = u_h \text{ for every } h \neq i\,.$$

Setting again for every $l \in \{1, \dots, k\}$

$$\begin{aligned} \zeta_l &= (J_{u_l} \setminus \partial^e \Omega_l) \cap \overline{B}_\rho(x) \cap D \\ \gamma_l &= \Omega_l \cap \partial B_\rho(x) \\ \gamma_{lm} &= (\partial^e \Omega_l \cap \partial^e \Omega_m) \cap \overline{B}_\rho(x) \cap D, \quad l \neq m, \end{aligned}$$

we proceed as in the proof of Proposition 9 until we arrive at inequality (12). Such inequality implies a fortiori that (using the same arguments leading to (13), but keeping the term $\beta_i \mathcal{H}^{d-1}(\zeta_i)$)

$$\beta_i \mathcal{H}^{d-1}(\zeta_i) + \sum_{m \neq i} \beta_i \mathcal{H}^{d-1}(\gamma_{im}) - \sum_{m \neq i,j} \beta_j \mathcal{H}^{d-1}(\gamma_{im}) \leq (\beta_i + \beta_j) \mathcal{H}^{d-1}(\gamma_i) + \hat{C}_j |\Omega_i \cap B_\rho(x)|$$

for some $\hat{C}_j > 0$. Summing over $j \neq i$, we arrive at

$$(k-1)\beta_i \mathcal{H}^{d-1}(\zeta_i) + \sum_{m \neq i} \big[(k-1)\beta_i - \sum_{j \neq i,m} \beta_j\big] \mathcal{H}^{d-1}(\gamma_{im}) \leq \big[(k-1)\beta_i + \sum_{j \neq i} \beta_j\big] \mathcal{H}^{d-1}(\gamma_i) + C|\Omega_i \cap B_\rho(x)|.$$

.

Thanks to the assumed compatibility conditions (4) this implies, for some constants $\delta, C, C' > 0$ (independent of x):

$$\begin{aligned} \delta\Big(\mathcal{H}^{d-1}(\zeta_i) + \sum_{m \neq i} \mathcal{H}^{d-1}(\gamma_{im})\Big) &\leq \big[(k-1)\beta_i + \sum_{j \neq i} \beta_j\big] \mathcal{H}^{d-1}(\gamma_i) + C|\Omega_i \cap B_\rho(x)| \\ &\leq C' \rho^{d-1} + C\rho^d\,. \end{aligned}$$

Since

$$(\partial^e \Omega_i \cup J_{u_i}) \cap \overline{B}_\rho(x) = \zeta_i \cup \bigcup_{m \neq i} \gamma_{im}\,,$$

this implies, for ρ sufficiently small (independent of x), the validity of the upper bound inequality in (5).

Step 2: Lower bound inequality. Let us fix $x \in D$, and let us distinguish the two cases

$$x \in \partial^e \Omega_i \qquad \text{and} \qquad x \in J_{u_i} \setminus \partial^e \Omega_i$$

• Let $x \in \partial^e\Omega_i$. By the relative isoperimetric inequality we have, for a dimensional constant $c_d > 0$,

$$\Big(\min\Big\{\big|B_\rho(x)\cap\Omega_i\big|\,,\,\big|B_\rho(x)\setminus\Omega_i\big|\Big\}\Big)^{\frac{d-1}{d}} \le c_d\mathcal{H}^{d-1}\big(\partial^e\Omega_i\cap B_\rho(x)\big)\,. \tag{16}$$

We choose an index j so that $x \in \partial^e\Omega_i \cap \partial^e\Omega_j$. In view of the inclusion

$$B_\rho(x)\cap\Omega_j \subset B_\rho(x)\setminus\Omega_i$$

and of the density lower bound (11) for Ω_i and Ω_j at the point x, the inequality (16) implies that that there exist $\rho_i' > 0$ and $k_i' > 0$ (independent of x) such that for every $\rho < \rho_i'$

$$\mathcal{H}^{d-1}\big((\partial^e\Omega_i\cup J_{u_i})\cap\overline{B}_\rho(x)\big) \ge \mathcal{H}^{d-1}(\partial^e\Omega_i\cap\overline{B}_\rho(x)) \ge k_i'\rho^{d-1}.$$

• Let $x \in J_{u_i}\setminus\partial^e\Omega_i$. Then thanks to Proposition 10 we have $x \in J_{u_i}\cap\Omega_i$. Since u_i is a local quasi-minimizer for the (weighted) Mumford-Shah functional in the open set Ω_i, this entails the Ahlfors regularity of its jump set (see [6, Section 3.2]). Thus, setting $\delta_i(x) := \operatorname{dist}(x,\partial\Omega_i)$, there exist a radius $\rho_i'' > 0$ and a constant $k_i'' > 0$ (independent of x) such that for every $\rho < \rho_i''\wedge\delta_i(x)$

$$\mathcal{H}^{d-1}\big((\partial^e\Omega_i\cup J_{u_i})\cap\overline{B}_\rho(x)\big) \ge \mathcal{H}^{d-1}(J_{u_i}\cap\overline{B}_\rho(x)) \ge k_i''\rho^{d-1}. \tag{17}$$

We have to prove that the same kind of lower bound continues to hold for balls contained in D (and not necessarily in $D\cap\Omega_i$).
To that aim we first remark that, up to changing k_i'' into k_i''/m^{d-1}, the validity of (17) can be extended to radii $\rho \in (0, m(\rho_i''\wedge\delta_i(x)))$ for any $m \in \mathbb{N}$. Indeed by applying (17) with $\frac{\rho}{m}$ in place of ρ we get:

$$\mathcal{H}^{d-1}\big((\partial^e\Omega_i\cup J_{u_i})\cap\overline{B}_\rho(x)\big) \ge \mathcal{H}^{d-1}(J_{u_i}\cap\overline{B}_\rho(x)) \ge \mathcal{H}^{d-1}(J_{u_i}\cap\overline{B}_{\frac{\rho}{m}}(x)) \ge \frac{k_i'}{m^{d-1}}\rho^{d-1}$$

for every $\rho \in (0, m(\rho_i''\wedge\delta_i(x)))$. Since we can choose m large enough (independent of x) so that

$$m(\rho_i''\wedge\delta_i(x)) \ge 2\delta_i(x)\,,$$

we are reduced to show the lower bound inequality in (5) for radii $\rho \ge 2\delta_i(x)$ such that $\overline{B}_\rho(x) \subset D$. For such a radius we observe that, denoting by $x_j \in \partial^e\Omega_j$ a point such that $|x_j - x| = \delta_i(x)$, thanks to the density lower bound (11) for Ω_j at x_j, we have

$$|B_\rho(x)\setminus\Omega_i| \ge |\Omega_j\cap B_\rho(x)| \ge |\Omega_j\cap B_{\rho-\delta_i(x)}(x_j)| \ge c_j(\rho-\delta_i(x))^d \ge \frac{c_j}{2^d}\rho^d. \tag{18}$$

Then, by using the isoperimetric inequality (16), inequality (18), and the density lower bound (11) for Ω_i at x, we obtain that, for any $\rho \ge 2\delta_i(x)$, the quantity $\mathcal{H}^{d-1}\big(\partial^e\Omega_i\cap\overline{B}_\rho(x)\big)$, and hence $\mathcal{H}^{d-1}\big((\partial^e\Omega_i\cup J_{u_i})\cap\overline{B}_\rho(x)\big)$, is bounded from below by a constant (independent of x) times ρ^{d-1}. □

4. Proofs of Theorem 4, Corollary 5, and Corollary 6

Proof of Theorem 4. Since all functions u_i are assumed to belong to $L^\infty(D)$, and since possible vertical shifts of $u_i|_{\Omega_i}$ on Ω_i do not affect the set $J_{u_i}\cup\partial^e\Omega_i$, we may assume without loss of generality that

$$(a+1)i \le u_i|_{\Omega_i} \le (a+1)i + a \qquad \text{and} \qquad u_i = 0 \quad \text{on } D\setminus\Omega_i$$

for some $a > 0$. We build the function

$$u = \sum_{i=1}^{k} u_i 1_{\Omega_i} \in SBV(D),$$

and claim that u is a local almost quasi-minimizer of the (scalar) Mumford-Shah functional, with constants to be determined.

We note first that $J_u = J_{\mathbf{U}} \cup \partial^e \mathbf{\Omega}$. In order to prove the local almost quasi-minimality of u, let $v \in SBV(D)$ be a perturbation of u in the ball $B_\rho(x_0) \subset D$ (we simply write B_ρ below, as x_0 is fixed). Without restricting generality, we can assume that $a+1 \leq v \leq k(a+1)+a$, a.e. in D. The function v is a perturbation of the function u only, and does not take into account the presence of the partition. In order to gather information for u from the quasi-minimality of $(\mathbf{\Omega}, \mathbf{U})$, we have to produce a perturbation $(\mathbf{\Lambda}, \mathbf{V})$ for $(\mathbf{\Omega}, \mathbf{U})$, in relationship with v. In other words, we have to identify k functions v_i and k corresponding phases, associated to v, which build a local perturbation of $(\mathbf{\Omega}, \mathbf{U})$. The construction of the k phases from the unique function v is the main difficulty, since forcing the presence of different phases is based on cutting the support of the function v which may naturally introduce new boundaries outside the set J_v. This procedure leads to possible new terms of dimension $(d-1)$ at the right-hand side of the inequality in Definition 3.
For any $t \in (0,1)$, we consider the perturbation $(\mathbf{\Lambda}, \mathbf{V})$ defined by

$$\text{(19)} \quad v_i(x) := \begin{cases} \min\{\max\{(a+1)i, v\}, (a+1)i+a\} \cdot 1_{\{(a+1)i-1+t<v<(a+1)i+a+t\}} \text{ in } B_\rho \\ u_i \text{ in } B_\rho^c \end{cases}$$

and

$$\text{(20)} \qquad \Lambda_i := (\{(a+1)i - 1 + t < v < (a+1)i + a + t\} \cap B_\rho) \cup (\Omega_i \setminus B_\rho)$$

Note that outside B_ρ the function v coincides with u, so that v_i coincides with u_i on $\Omega_i \setminus B_\rho$. In particular we have

$$(a+1)i < v < (a+1)i + a \qquad \text{a.e. on } \Lambda_i \setminus B_\rho$$

which yields

$$\bigcup_{i=1}^{k} \partial^e \Lambda_i \cap \overline{B}_\rho \subseteq \bigcup_{i=1}^{k} \partial^e \{v < (a+1)i + a + t\} \cap \overline{B}_\rho.$$

Moreover by construction the function v_i is piecewise constant on the set

$$(\Lambda_i \cap B_\rho) \setminus \{(a+1)i < v < (a+1)i + a\},$$

where its gradient vanishes almost everywhere.
By testing the local almost quasi minimality of $(\mathbf{\Omega}, \mathbf{U})$ with $(\mathbf{\Lambda}, \mathbf{V})$, we get

$$\sum_{i=1}^{k} \int_{B_\rho} |\nabla u_i|^2 dx + c_1 \mathcal{H}^{d-1}\Big((\partial^e \mathbf{\Omega} \cup J_{\mathbf{U}}) \cap \overline{B}_\rho\Big)$$
$$\leq \sum_{i=1}^{k} \int_{B_\rho} |\nabla v_i|^2 dx + c_2 \mathcal{H}^{d-1}\Big((\partial^e \mathbf{\Lambda} \cup J_{\mathbf{V}}) \cap \overline{B}_\rho\Big) + c_3 \rho^{d-1+\alpha},$$

or, in terms of u, v,

$$\text{(21)} \quad \int_{B_\rho} |\nabla u|^2 dx + c_1 \mathcal{H}^{d-1}\Big(J_u \cap \overline{B}_\rho\Big)$$
$$\leq \int_{B_\rho} |\nabla v|^2 dx + c_2 \mathcal{H}^{d-1}\Big(J_v \cap \overline{B}_\rho\Big) - \int_{B_\rho \setminus \cup_{i=1}^k \{(a+1)i<v<(a+1)i+a\}} |\nabla v|^2 dx$$
$$+ c_2 \mathcal{H}^{d-1}\left(\bigcup_{i=1}^{k-1} \partial^e \{v < (a+1)i + a + t\} \cap \overline{B}_\rho\right) + c_3 \rho^{d-1+\alpha}.$$

There are two possibilities.

Case 1. There exists some $t' \in (0,1)$ such that

$$c_2 \mathcal{H}^{d-1}\left(\bigcup_{i=1}^{k-1} \partial^e \{v < (a+1)i + a + t'\} \cap \overline{B}_\rho\right) \leq \int_{B_\rho \setminus \cup_{i=1}^k \{(a+1)i<v<(a+1)i+a\}} |\nabla v|^2 dx.$$

In this case, we consider the perturbation $(\mathbf{\Lambda}, \mathbf{V})$ defined in (19)-(20) precisely for $t = t'$. From the inequality (21), written for $t = t'$, we see the local almost quasi-minimality of $(\mathbf{\Omega}, \mathbf{U})$ with $(\mathbf{\Lambda}, \mathbf{V})$ as test implies the local almost-quasi minimality of u with v as test (with the same constants).

Case 2. For almost every $t \in (0,1)$ we have

$$\int_{B_\rho \setminus \cup_{i=1}^k \{(a+1)i<v<(a+1)i+a\}} |\nabla v|^2 dx \leq c_2 \mathcal{H}^{d-1}\left(\bigcup_{i=1}^{k-1} \partial^e \{v < (a+1)i + a + t\} \cap \overline{B}_\rho\right).$$

We integrate over $t \in (0,1)$ and use the co-area formula to get

$$\int_{B_\rho \setminus \cup_{i=1}^k \{(a+1)i<v<(a+1)i+a\}} |\nabla v|^2 dx \leq c_2 \int_{B_\rho \setminus \cup_{i=1}^k \{(a+1)i<v<(a+1)i+a\}} |\nabla v| dx.$$

Using Cauchy-Schwarz inequality at the right hand side, we get

$$\int_{B_\rho \setminus \cup_{i=1}^k \{(a+1)i<v<(a+1)i+a\}} |\nabla v|^2 dx \leq c_2 \Big(\int_{B_\rho(x) \setminus \cup_{i=1}^k \{(a+1)i<v<(a+1)i+a\}} |\nabla v|^2 dx\Big)^{\frac{1}{2}} |B_\rho|^{\frac{1}{2}},$$

or equivalently, since $|B_\rho| = \omega_d \rho^d$,

$$\int_{B_\rho \setminus \cup_{i=1}^k \{(a+1)i<v<(a+1)i+a\}} |\nabla v|^2 dx \leq c_2^2 \omega_d \rho^d.$$

Using again Cauchy-Schwarz inequality, this time at the left-hand side, we get

$$\int_{B_\rho \setminus \cup_{i=1}^k \{(a+1)i<v<(a+1)i+a\}} |\nabla v| dx \leq c_2 \omega_d \rho^d.$$

As a consequence of the co-area formula and of the mean value theorem, we deduce that there exists some value $t'' \in (0,1)$ such that

$$\mathcal{H}^{d-1}\left(\bigcup_{i=1}^{k-1} \partial^e \{v < (a+1)i + a + t''\} \cap \overline{B}_\rho\right) \leq c_2 \omega_d \rho^d.$$

Then, we consider the perturbation $(\mathbf{\Lambda}, \mathbf{V})$ defined in (19)-(20) precisely for $t = t''$. From the inequality (21), written for $t = t''$, we see that the local almost quasi-minimality of $(\mathbf{\Omega}, \mathbf{U})$ with $(\mathbf{\Lambda}, \mathbf{V})$ as test implies the local almost quasi minimality of u with v as test, provided the quantity $c_2 \omega_d \rho^d$ is added at the right hand side.

Taking into account cases 1 and 2, by choosing $\tilde{\rho}_0 = \min\{\rho_0, 1\}$ we conclude that u is a local almost quasi-minimizer for the Mumford-Shah functional at the point x_0 with constants $(c_1, c_2, c_3 + c_2\omega_d, \alpha)$. □

Proof of Corollaries 5 and 6. The conclusions follow immediately by applying Theorem 4 combined respectively with the regularity results proved in [6] and [2]. □

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