

On the relaxed area of the graph of discontinuous maps from the plane to the plane taking three values with no symmetry assumptions

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Abstract

In this paper we estimate from above the area of the graph of a singular map u taking a disk to three vectors, the vertices of a triangle, and jumping along three \mathcal{C}^2 -embedded curves that meet transversely at only one point of the disk. We show that the relaxed area can be estimated from above by the solution of a Plateau-type problem involving three entangled nonparametric area-minimizing surfaces. The idea is to “fill the hole” in the graph of the singular map with a sequence of approximating smooth two-codimensional surfaces of graph-type, by imagining three minimal surfaces, placed vertically over the jump of u , coupled together via a triple point in the target triangle. Such a construction depends on the choice of a target triple point, and on a connection passing through it, which dictate the boundary condition for the three minimal surfaces. We show that the singular part of the relaxed area of u cannot be larger than what we obtain by minimizing over all possible target triple points and all corresponding connections.

Key words: Relaxation, Cartesian currents, area functional, minimal surfaces, Plateau problem.

AMS (MOS) subject classification: 49Q15, 49Q20, 49J45.

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be an open set and $v = (v_1, v_2) : \Omega \rightarrow \mathbb{R}^2$ a Lipschitz map. It is well known that the area of the graph of v is given by

$$\mathbb{A}(v, \Omega) = \int_{\Omega} \sqrt{1 + |\nabla v_1|^2 + |\nabla v_2|^2 + \left(\frac{\partial v_1}{\partial x} \frac{\partial v_2}{\partial y} - \frac{\partial v_1}{\partial y} \frac{\partial v_2}{\partial x} \right)^2} dx dy. \quad (1.1)$$

Extending to nonsmooth maps via relaxation the definition of the area is a difficult question [12], and is motivated by rather natural problems in calculus of variations: we can mention for example

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the use of direct methods to face the two-codimensional Plateau problem in \mathbb{R}^4 in cartesian form, and the study of lower semicontinuous envelopes of polyconvex functionals with nonstandard growth [3], [10]. A crucial issue is to decide which topology one has to consider in order to compute the relaxed functional of $\mathbb{A}(\cdot, \Omega)$: of course, the weakest the topology, the most difficult should be the computation of the relaxed functional, but the easiest becomes the coerciveness. We recall that when v is scalar valued, the natural choice is the $L^1(\Omega)$ -convergence, and the relaxation problem is completely solved [8], [2]; the $L^1(\Omega)$ -relaxed functional in this case consists, besides the absolutely continuous part, of a singular part which is the total variation of the jump and Cantor parts of the distributional derivative of v in Ω ; in particular, the relaxed functional, when considered as a function of Ω , is a measure.

The case of interest here, namely when v takes values in \mathbb{R}^2 , is much more involved, due to the *nonconvexity* of the integrand in (1.1), and to the *unilateral linear growth*

$$\mathbb{A}(v, \Omega) \geq \int_{\Omega} \sqrt{|\nabla v_1|^2 + |\nabla v_2|^2} \, dx dy.$$

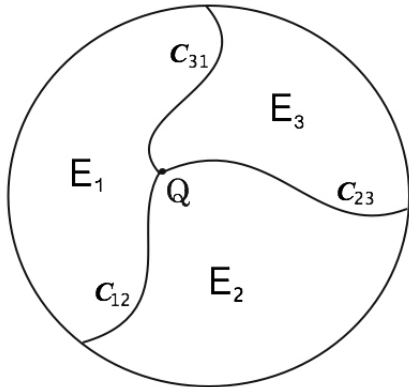
Choosing again the $L^1(\Omega; \mathbb{R}^2)$ -convergence (as we shall do in this paper), the relaxed functional $\mathcal{A}(\cdot, \Omega)$ of $\mathbb{A}(\cdot, \Omega)$, *i.e.*,

$$\mathcal{A}(v, \Omega) := \inf \left\{ \liminf_{\epsilon \rightarrow 0} \mathbb{A}(u^\epsilon, \Omega) : \{u^\epsilon\} \subset \text{Lip}(\Omega; \mathbb{R}^2), u^\epsilon \rightarrow u \text{ in } L^1(\Omega; \mathbb{R}^2) \right\}, \quad (1.2)$$

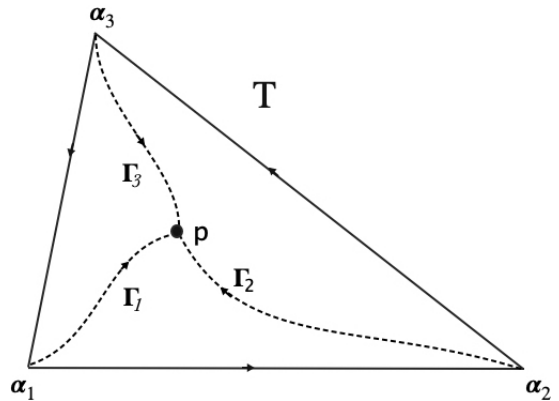
is, for $v \in L^1(\Omega; \mathbb{R}^2) \setminus W^{1,2}(\Omega; \mathbb{R}^2)$, far from being understood, and exhibits surprising features. One of the few known facts that must be pointed out is that, for a large class of nonsmooth maps v , the function $\Omega \rightarrow \mathcal{A}(v, \Omega)$ cannot be written as an integral [3], [5], [6]; this interesting phenomenon, related to nonlocality, has at least two sources. For simplicity, let us focus our attention on nonsmooth functions with jumps, thus neglecting the case of vortices. The first source of nonlocality has been enlightened answering to a conjecture in [9]. Specifically, consider the symmetric triple junction map u_{symm} , *i.e.*, the singular map from a disk D of $\mathbb{R}_S^2 = \mathbb{R}^2$ into $\mathbb{R}_T^2 = \mathbb{R}^2$, taking only three values – the vertices of an equilateral triangle $T_{\text{eq}} \subset \mathbb{R}_T^2$ – and jumping along three segments meeting at the origin in a triple junction at equal 120° angles: then $\mathcal{A}(u_{\text{symm}}, \cdot)$ is not subadditive. This result has been proven in [3]; subsequently in [4] it is shown that the value $\mathcal{A}(u_{\text{symm}}, D)$ is related to the solution of three one-codimensional Plateau-type problems in cartesian form suitably entangled together through the Steiner point in the triangle T_{eq} . Due to the special symmetry of the map u_{symm} , the three-problems collapse together to only one one-codimensional Plateau-type problem in cartesian form, on a fixed rectangle R whose sides are the radius of D and the side of T_{eq} . Positioning three copies of this minimal surface “vertically” (in the space of graphs, *i.e.*, in $D \times \mathbb{R}^2$) over the jump of u_{symm} allows, in turn, to construct a sequence $\{u_\epsilon\}$ of *Lipschitz* maps from D into \mathbb{R}^2 the limit area of which improves the upper estimate of [3]. Optimality of this construction has been shown in the recent paper [13], on the basis of a symmetrization procedure for currents.

It is one of the aims of the present paper to inspect solutions of the above mentioned three Plateau-type problems in more general situations, in order to provide upper estimates for $\mathcal{A}(u, D)$, for suitable piecewise constant maps u .

A second source of nonlocality for the functional $\mathcal{A}(u, \Omega)$ is given by the interaction of the jump set of a discontinuous map u with the boundary of the domain Ω . This phenomenon, already observed in [3] for the map with one-vortex at the center of a suitable disk, appears also for functions with jump discontinuities not piecewise constant [6]. More surprisingly, it appears also for piecewise constant



(a) The domain of u ; $u = \alpha_i$ on E_i .



(b) A Lipschitz graph-type connection in the target triangle T . $\Gamma_1 \cup \Gamma_2$ (resp. $\Gamma_2 \cup \Gamma_3$, $\Gamma_3 \cup \Gamma_1$) is graph over the segment $\overline{\alpha_1 \alpha_2}$ (resp. $\overline{\alpha_2 \alpha_3}$, $\overline{\alpha_3 \alpha_1}$) of a Lipschitz function φ_{12} (resp. φ_{23} , φ_{31}).

Figure 1

maps taking three values, provided the jump is sufficiently close to the boundary of Ω , as observed in [13], taking as Ω a sufficiently thin tubular neighbourhood of the jump itself. We shall not be concerned here with this second source of nonlocality.

As already mentioned above, in this paper we are interested in estimating from above the area of the graph of a singular map u taking three (non collinear) values and jumping along three embedded curves of class \mathcal{C}^2 that meet transversely at only one point, see Figure 1. Let us state this in a more precise way, referring to Sections 2 and 3 for all details. For simplicity, from now on we fix Ω to be an open disk D containing the origin 0_S in the source plane $\mathbb{R}^2 = \mathbb{R}_{x,y}^2 = \mathbb{R}_S^2$. Take three non-overlapping non-empty two-dimensional connected regions E_1 , E_2 , E_3 of D such that

$$E_1 \cup E_2 \cup E_3 = D. \quad (1.3)$$

The three regions are separated by three embedded curves of class \mathcal{C}^2 (up to the boundary) of length r_{12} , r_{23} , r_{31} respectively, that meet only at Q (source triple junction); moreover, each curve is supposed to meet the boundary of D transversely and we assume also that Q is a transversal intersection for the three curves, see Figure 1a. Let α_1 , α_2 , α_3 be the vertices of a closed triangle T with non empty interior in the target plane.

Set

$$\ell_{12} := |\alpha_1 - \alpha_2|, \quad \ell_{23} := |\alpha_2 - \alpha_3|, \quad \ell_{31} := |\alpha_1 - \alpha_3|. \quad (1.4)$$

We suppose that T contains the origin 0_T in its interior.

Let us introduce the space X of connections (Definition 3.1 and (3.3), (3.4)); a connection $\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3)$ consists of three rectifiable curves in T , that connect the vertices of T to some point inside the triangle (called target triple point). We shall suppose that *each curve can be written as a graph, possibly with vertical parts, over the corresponding two sides of T* . When Γ consists of three Lipschitz graphs, we write $\Gamma \in X_{\text{Lip}}$, and we say that Γ is a Lipschitz connection. We now show how to construct a new functional \mathcal{G} , consisting of the sum of the areas of three minimal surfaces

– graphs of three suitable area-minimizing functions m_{12} , m_{23} , m_{31} defined on certain rectangles – coupled together by the connection considered as a Dirichlet boundary condition, see Definition 3.4. Set

$$\mathbf{R}_{ij} := [0, \ell_{ij}] \times [0, r_{ij}], \quad ij \in \{12, 23, 31\}. \quad (1.5)$$

Assume $\Gamma \in X$. Then $\Gamma_{ij} := \Gamma_i \cup \Gamma_j$, $ij \in \{12, 23, 31\}$ are (generalized) graphs of functions φ_{ij} of bounded variation over $[0, \ell_{ij}]$. With a small abuse of notation, set

$$\varphi_{ij}(s, t) = \varphi_{ij}(s), \quad (s, t) \in \mathbf{R}_{ij}, \quad ij \in \{12, 23, 31\}. \quad (1.6)$$

The graph of φ_{12} on \mathbf{R}_{12} is depicted in Figure 2a.

Let $m_{ij} = m_{ij}(\Gamma)$ be the unique solution of the Dirichlet-Neumann minimum problem

$$\min \left\{ \int_{\mathbf{R}_{ij}} \sqrt{1 + |\nabla f|^2} \, dsdt : f \in W^{1,1}(\mathbf{R}_{ij}), f = \varphi_{ij} \mathcal{H}^1 - a.e. \text{ on } \partial_D \mathbf{R}_{ij} \right\}, \quad (1.7)$$

where

$$\partial_D \mathbf{R}_{ij} = \partial \mathbf{R}_{ij} \setminus ([0, \ell_{ij}] \times \{r_{ij}\}), \quad ij \in \{12, 23, 31\}.$$

Notice that the minimization is taken among all functions having a Dirichlet condition on three of the four sides of the rectangle \mathbf{R}_{ij} ; the missing side corresponds to the intersection points of the jump with the boundary of D .

From (1.6) it follows that the Dirichlet condition is zero on the sides $\{0\} \times [0, r_{ij}]$ and $\{\ell_{ij}\} \times [0, r_{ij}]$ of \mathbf{R}_{ij} ; see Figure 2b.

Set

$$\mathfrak{A}_{ij}(\Gamma) := \int_{\mathbf{R}_{ij}} \sqrt{1 + |\nabla m_{ij}|^2} \, dsdt, \quad ij \in \{12, 23, 31\}. \quad (1.8)$$

The main result of the present paper reads as follows (see Theorem 4.1 and Corollary 5.8).

Theorem 1.1. *Let $u : D \rightarrow \{\alpha_1, \alpha_2, \alpha_3\}$ be the discontinuous $BV(D; \mathbb{R}^2)$ function defined as*

$$u(x, y) := \begin{cases} \alpha_1 & \text{if } (x, y) \in E_1, \\ \alpha_2 & \text{if } (x, y) \in E_2, \\ \alpha_3 & \text{if } (x, y) \in E_3. \end{cases} \quad (1.9)$$

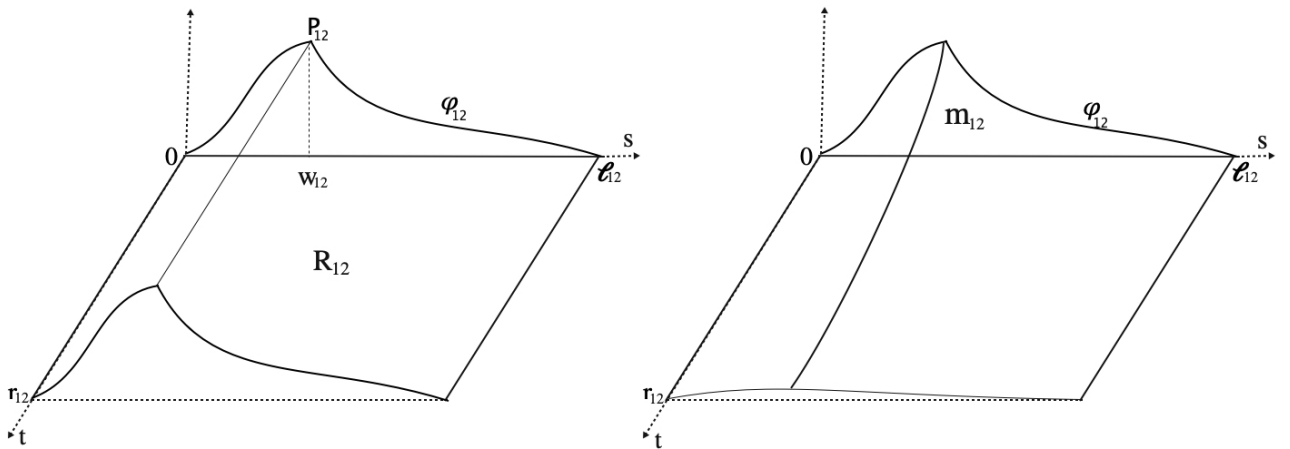
Then

$$\mathcal{A}(u, D) \leq |D| + \min \left\{ \mathfrak{A}_{12}(\Gamma) + \mathfrak{A}_{23}(\Gamma) + \mathfrak{A}_{31}(\Gamma) : \Gamma \in X \right\}. \quad (1.10)$$

This theorem says that the singular part of $\mathcal{A}(u, D)$ can be estimated from above by

$$\inf \left\{ \mathfrak{A}_{12}(\Gamma) + \mathfrak{A}_{23}(\Gamma) + \mathfrak{A}_{31}(\Gamma) : \Gamma \in X \right\} \quad (1.11)$$

and that such an infimum is a minimum. Intuitively, to “fill the hole” in the graph of u with smooth two-codimensional approximating surfaces of graph-type, we start to imagine three minimal surfaces, placed vertically over the jump of u , coupled together via a triple point in the target triangle T (notice that the union of these three minimal surfaces, viewed in $D \times \mathbb{R}^2$, is not smooth in correspondence of



(a) The graph of the function φ_{12} on R_{12} .

(b) The graph of m_{12} on R_{12} .

Figure 2

the source triple junction). Such a construction depends on the choice of a target triple point, and on a connection Γ passing through it, dictating the boundary condition for the three minimal surfaces, over the sides of the triangle T . Theorem 1.1 asserts that the interesting part of the relaxed area of u , namely its singular part, cannot be larger than what we obtain by minimizing over all possible target triple points and all corresponding connections. As a direct consequence of the results in [4], [13], when $u = u_{\text{symm}}$ (and 0_S is the center of D), the inequality in (1.10) is an equality, and the infimum in (1.11) is achieved by the Steiner graph connecting the three vertices of T (the optimal triple point being the Steiner point, i.e., the barycenter of T). This seems to be an interesting result that could be stated purely as a problem of three entangled area-minimizing surfaces (each of which lies in a half-space of \mathbb{R}^4 , the three half-spaces having only $\{0\} \times \mathbb{R}^2$ in common) without referring to the relaxation of the functional $\mathbb{A}(\cdot, D)$. We do not know whether, in general, the Steiner graph is still the solution of the minimization problem in (1.11), when no symmetry assumptions (the case we are considering here) are required. However it is reasonable to expect that, if in the source we have symmetry, i.e., the source triple junction is positioned at the center of D and u jumps along three segments meeting at equal 120° angles, and if the target triangle T is close to be equilateral, the inequality in (1.10) to be still an equality. In this respect, it is worthwhile to observe that showing a lower estimate, for instance showing that, in certain cases, the inequality in (1.10) is an equality, seems difficult. One of the main technical obstructions is due to the poor control on the tangential derivative of v^ε in proximity of the jump of a discontinuous L^1 -limit function v (see [6]), where $\{v^\varepsilon\}$ is a sequence of Lipschitz maps converging in $L^1(\Omega; \mathbb{R}^2)$ to v , and satisfying the uniform bound $\sup_\varepsilon \mathbb{A}(v^\varepsilon, \Omega) < +\infty$. We also notice that the symmetrization methods of [13] cannot be applied anymore, in view of the lackness of symmetry.

It is worth mentioning that the restriction that we assume on the connections Γ , namely that each Γ_i is a graph (possibly with vertical parts) on the corresponding two sides of T , cannot be avoided in our approach: indeed, only under this graphicality assumption we can solve the minimum problem in (1.10) in the class of surfaces which are graphs over the rectangles R_{ij} . In turn, the graphicality of such minimal surfaces allows to construct the sequence $\{u^\varepsilon\}$, see (4.11). Removing the graphicality

assumption on Γ requires some change of perspective, and needs further investigation.

The content of the paper is the following. In Section 2.1 we recall some properties of functions of bounded variation of one variable, the definition of generalized graph (formula (2.1)), and the chain rule. In Section 2.2 we recall some properties of Cartesian currents carried by a BV-function. The functional \mathcal{G} , appearing on the right hand side of (1.10), is introduced in Definition 3.4. In Section 4 we show that

$$\mathcal{A}(u, D) \leq |D| + \inf \{ \mathcal{G}(\Gamma) : \Gamma \in X_{\text{Lip}} \}, \quad (1.12)$$

see Theorem 4.1. The proof is rather involved, mainly due to technical difficulties: we first start by supposing that the jump of u is piecewise linear (Proposition 4.4). Some work is required to define u^ε on an ε -strip around the jump of u and avoiding a neighbourhood of the source triple junction (formula (4.11)) and to define u_ε in the missing neighbourhood of the source triple junction (step 3 of the proof of Proposition 4.4): the construction must be done in such a way that u^ε remains Lipschitz, and turns out to be rather involved in the three triangles $T_1^\varepsilon, T_2^\varepsilon, T_3^\varepsilon$, see Figure 4b. In Section 5 we prove that the infimum in (1.11) is a minimum. The proof is achieved by defining a topology in the space X which allows to prove the density of X_{Lip} in X (Lemma 5.2), the continuity of the functional \mathcal{G} (Proposition 5.4) and the sequential compactness of X (Theorem 5.6). This latter result is also based on a uniform bound on the length of the connections (Proposition 5.3), which is a consequence of the graphicality assumptions on the connections.

2 Some preliminaries

In this section we recall some results on functions of bounded variation of one variable [2], and on cartesian currents[12], needed in the sequel.

2.1 Functions of bounded variation in the interval

Let $(a, b) \subset \mathbb{R}$ be a bounded open interval and $\varphi \in \text{BV}((a, b))$; then

- φ is bounded, and it is continuous up to an at most countable set of points of (a, b) denoted by J_φ (jump set);
- the right and left limits $\varphi(s_\pm)$ of φ exist at any $s \in (a, b)$; the right limit $\varphi(a_+)$ and the left limit $\varphi(b_-)$ exist. Thus we may define

$$\varphi_+(s) := \max\{\varphi(s_+), \varphi(s_-)\}, \quad \varphi_-(s) := \min\{\varphi(s_+), \varphi(s_-)\}, \quad s \in (a, b);$$

- the distributional derivative φ' of φ splits as

$$\varphi' = \dot{\varphi}ds + \dot{\varphi}^{(j)} + \dot{\varphi}^{(c)},$$

where $\dot{\varphi}ds$ is the absolutely continuous part and $\dot{\varphi}$ is the differential of φ [2, p.138 and Cor.3.33], $\dot{\varphi}^{(j)}$ and $\dot{\varphi}^{(c)}$ are the jump and the Cantor part respectively.

We shall always assume that φ is a good representative in its L^1 class such that $\varphi(s) = \varphi_+(s)$ for all $s \in (a, b)$; the pointwise variation of φ is equal to the total variation $|\varphi'|((a, b))$.

The generalized graph of φ is defined as

$$\Gamma_\varphi := \{(s, \theta\varphi(s_-) + (1 - \theta)\varphi(s_+)) : s \in (a, b), \theta \in [0, 1]\}, \quad (2.1)$$

and the subgraph of φ as

$$S\mathcal{G}_{\varphi, (a, b)} := \{(s, t) \in (a, b) \times \mathbb{R} : t \leq \varphi(s)\}.$$

We recall that, if $\varphi \in L^1((a, b))$, then $\varphi \in \text{BV}((a, b))$ if and only if $S\mathcal{G}_{\varphi, (a, b)}$ has finite perimeter in $(a, b) \times \mathbb{R}$. We denote by $\partial^- S\mathcal{G}_{\varphi, (a, b)}$ the reduced boundary of $S\mathcal{G}_{\varphi, (a, b)}$.

We conventionally set $\varphi(a_-) = 0$, $\varphi(b_+) = 0$; in this case we can define Γ_φ as in (2.1) with (a, b) replaced by $[a, b]$, hence the generalized graph will always pass through the end points of the interval (with possibly vertical parts over a and b).

The following result can be found for instance in [12, p.486].

Theorem 2.1. *Let $g \in C^1(\mathbb{R})$ and $\varphi \in \text{BV}((a, b))$. Then $g \circ \varphi \in \text{BV}((a, b))$ and*

$$\begin{aligned} (g \circ \varphi)' &= g'(\varphi)\dot{\varphi}ds + g'(\varphi)\dot{\varphi}^{(c)} && \text{in } (a, b) \setminus J_\varphi \\ (g \circ \varphi)' &= \sum_{s \in J_\varphi} n(s, J_\varphi) \left[g(\varphi_+(s)) - g(\varphi_-(s)) \right] \delta_s && \text{in } J_\varphi, \end{aligned}$$

where $n(s, J_\varphi) := \frac{\varphi(s_+) - \varphi(s_-)}{|\varphi(s_+) - \varphi(s_-)|}$ and δ_s is the Dirac delta at s .

2.2 Cartesian currents

Let $I \subset \mathbb{R}$ be a bounded open interval and $\varphi \in \text{BV}(I)$. We denote by

$$\llbracket S\mathcal{G}_{\varphi, I} \rrbracket \in \mathcal{D}^2(\mathbb{R}^2)$$

the 2-current in $\mathbb{R} \times \mathbb{R}$ defined as the integration over the subgraph $S\mathcal{G}_{\varphi, I}$. The current $\llbracket S\mathcal{G}_{\varphi, I} \rrbracket \llcorner I \times \mathbb{R}$ can be also identified with an integer multiplicity current in $I \times \mathbb{R}$; moreover $S\mathcal{G}_{\varphi, I}$ has finite perimeter in $I \times \mathbb{R}$ so, if $\partial \llbracket S\mathcal{G}_{\varphi, I} \rrbracket \llcorner I \times \mathbb{R}$ denotes the 1-current in $I \times \mathbb{R}$ defined as its boundary, this results of finite mass.

For future purposes we recall the following result, see [12, Section 4.2.4].

Theorem 2.2. *Let $\varphi \in \text{BV}(I)$ and \mathcal{T} be the current defined by*

$$\mathcal{T} := -\partial \llbracket S\mathcal{G}_{\varphi, I} \rrbracket \llcorner I \times \mathbb{R}. \quad (2.2)$$

Then $\mathcal{T} \in \mathcal{D}^1(I \times \mathbb{R})$ is a Cartesian current, and

$$\mathcal{T}(\omega) = - \int \langle \omega(x), * \nu(x, S\mathcal{G}_{\varphi, I}) \rangle d\mathcal{H}^1 \llcorner \partial^- S\mathcal{G}_{\varphi, I}(x) \quad \forall \omega \in \mathcal{D}^1(I \times \mathbb{R}), \quad (2.3)$$

where $$ is the Hodge operator and $\nu(\cdot, S\mathcal{G}_{\varphi, I})$ is the inward generalized unit normal. Moreover \mathcal{T} can be decomposed into three mutually singular currents*

$$\mathcal{T} = \mathcal{T}^{(a)} + \mathcal{T}^{(j)} + \mathcal{T}^{(c)}, \quad (2.4)$$

such that

$$\mathcal{T}^{(a)}(\omega) = \int_I [\omega_1(s, \varphi(s)) + \omega_2(s, \varphi(s)) \dot{\varphi}(s)] ds, \quad (2.5)$$

$$\mathcal{T}^{(j)}(\omega) = \sum_{s \in J_\varphi} n(s, J_\varphi) \int_{\varphi_-(s)}^{\varphi_+(s)} \omega_2(s, \sigma) d\sigma, \quad (2.6)$$

$$\mathcal{T}^{(c)}(\omega) = \int_I \omega_2(s, \varphi(s)) \dot{\varphi}^{(c)}, \quad (2.7)$$

where $\omega = \omega_1 ds + \omega_2 d\sigma$.

The current \mathcal{T} is boundaryless in $I \times \mathbb{R}$, namely $\partial \mathcal{T} = 0$. Furthermore, if Γ_φ is the generalized graph of φ as defined in (2.1), it turns out that $\partial^- S\mathcal{G}_{\varphi, I} \cap (I \times \mathbb{R})$ is a subset of Γ_φ and they differ of a \mathcal{H}^1 -negligible set. Namely

$$\partial^- S\mathcal{G}_{\varphi, I} \cap (I \times \mathbb{R}) \subseteq \Gamma_\varphi, \quad \mathcal{H}^1(\Gamma_\varphi \setminus \partial^- S\mathcal{G}_{\varphi, I}) = 0.$$

It easily follows that the current \mathcal{T} coincides with the integration over the rectifiable set Γ_φ (with the correct orientation).

From now on, when the interval is clear from the context, we will simply denote $S\mathcal{G}_{\varphi, I}$ by $S\mathcal{G}_\varphi$.

3 The functional \mathcal{G}

In order to prove our main result (Theorem 1.1) we need some preparation. Take three open non-overlapping non-empty connected regions E_1, E_2, E_3 of an open disk D , each E_i with non empty interior and with $\overline{E_1} \cup \overline{E_2} \cup \overline{E_3} = \overline{D}$, and let C_{ij} be their boundaries in D as in the introduction.

Let $\alpha_1, \alpha_2, \alpha_3$ be the vertices of a closed triangle T as in Section 1; we suppose that T contains the origin 0_T in its interior, and let ℓ_{ij} be as in (1.4).

Definition 3.1 (Connections in T). *We say that $\Gamma := (\Gamma_1, \Gamma_2, \Gamma_3)$ is a BV graph-type (resp. Lip graph-type) connection in T if $\Gamma_i, i \in \{1, 2, 3\}$, are subsets of T such that $\Gamma_1 \cap \Gamma_2 = \Gamma_2 \cap \Gamma_3 = \Gamma_3 \cap \Gamma_1$ is one point p of T called target triple point of Γ , $\alpha_i \in \Gamma_i$ for any $i = 1, 2, 3$, and*

$$\Gamma_{ij} := \Gamma_i \cup \Gamma_j, \quad ij \in \{12, 23, 31\},$$

can be written as the generalized graph (resp. graph) of a function of bounded variation (resp. Lipschitz function) over the closed segment $\overline{\alpha_i \alpha_j}$ (see Figure 1b).

Note that the case $p \in \partial T$ is not excluded. However, by definition, if $\pi_{ij} : T \rightarrow \mathbb{R}_{\overline{\alpha_i \alpha_j}}$, $ij \in \{12, 23, 31\}$, is the orthogonal projection on the line $\mathbb{R}_{\overline{\alpha_i \alpha_j}}$ containing $\overline{\alpha_i \alpha_j}$, then $\pi_{ij}(p) \in \overline{\alpha_i \alpha_j}$. Set

$$w_{ij} := |\alpha_i - \pi_{ij}(p)|. \quad (3.1)$$

If necessary, in the sequel we will often identify Γ_{ij} with the (generalized) graph $\Gamma_{\varphi_{ij}}$ of a function

$$\varphi_{ij} : [0, \ell_{ij}] \rightarrow [0, \text{diam}T], \quad \varphi_{ij} = \varphi_{ij}(\Gamma_{ij}), \quad (3.2)$$

of bounded variation. If T is acute, choosing a suitable cartesian coordinate system where the s -axis is the line $\mathbb{R}_{\overline{\alpha_i \alpha_j}}$, we necessarily have $\varphi_{ij}(0) = \varphi_{ij}(\ell_{ij}) = 0$. In contrast, if the angle of T at α_i is greater than or equal to $\frac{\pi}{2}$ then φ_{ij} might have a vertical part over α_i and $\varphi_{ij}(0_+) > 0$. In this case the generalized graph of φ_{ij} does not pass through α_i .

In the sequel it will be often convenient to consider an extension of φ_{ij} on $(-\infty, 0) \cup (\ell_{ij}, +\infty)$. This extension is denoted by $\tilde{\varphi}_{ij}$. In the case of acute triangle $\tilde{\varphi}_{ij}$ is always set equal to 0 on $(-\infty, 0) \cup (\ell_{ij}, +\infty)$.

Remark 3.2. If for any $ij \in \{12, 23, 31\}$, w_{ij} in (3.1) is a point of continuity of φ_{ij} then the intersection of the generalized graph of φ_{ki} and the set $[w_{ki}, \ell_{ki}] \times \mathbb{R}$ coincides with Γ_i which is also the intersection of the generalized graph of φ_{ij} with the set $[0, w_{ij}] \times \mathbb{R}$, where $ij, ki \in \{12, 23, 31\}$, $ij \neq ki$. If w_{ij} is a discontinuity point of φ_{ij} this is in general not true, as in Figure 14b, when $i = 2$.

Remark 3.3. Assume that an angle of T is greater than $\frac{\pi}{2}$, say for instance the angle at α_1 ; as already said, the generalized graphs composing a connection Γ are allowed to have vertical parts over α_1 . The target triple point p of any connection Γ belongs to $T_{int} \subset T$, the part of the triangle T which is enclosed between the two lines passing through α_1 and orthogonal to $\overline{\alpha_1 \alpha_2}$ and $\overline{\alpha_1 \alpha_3}$ respectively.

Define the classes:

$$X_{\text{Lip}} := \{\Gamma : \Gamma \text{ Lip graph - type connection in } T\}, \quad (3.3)$$

$$X := \{\Gamma : \Gamma \text{ BV graph - type connection in } T\}. \quad (3.4)$$

Obviously $X_{\text{Lip}} \subset X$.

3.1 Useful results on one-codimensional area-minimizing cartesian surfaces

Let R_{ij} be as in (1.5), and $\Gamma \in X$. Then Γ_{ij} , $ij \in \{12, 23, 31\}$ are (generalized) graphs of functions φ_{ij} of bounded variation over $[0, \ell_{ij}]$. Let $B \subset \mathbb{R}^2$ be an open disk containing the doubled rectangle \widehat{R}_{ij} defined as

$$\widehat{R}_{ij} := [0, \ell_{ij}] \times [0, 2r_{ij}], \quad ij \in \{12, 23, 31\}. \quad (3.5)$$

We use for simplicity the same notation φ_{ij} for the extension of φ_{ij} to \widehat{R}_{ij} , defined as

$$\varphi_{ij}(s, t) = \varphi_{ij}(s), \quad (s, t) \in \widehat{R}_{ij} \quad ij \in \{12, 23, 31\}, \quad (3.6)$$

and for the extension of φ_{ij} to a $W^{1,1}$ function on $B \setminus \widehat{R}_{ij}$ as in [11, Theorem 2.16].

Let $\widehat{m}_{ij} = \widehat{m}_{ij}(\Gamma)$, $ij \in \{12, 23, 31\}$, be the solution of following Dirichlet minimum problem:

$$\min \left\{ \int_{\widehat{R}_{ij}} \sqrt{1 + |Df|^2} + \int_{\partial \widehat{R}_{ij}} |f - \varphi_{ij}| d\mathcal{H}^1 : f \in \text{BV}(B), f = \varphi_{ij} \text{ on } B \setminus \widehat{R}_{ij} \right\}, \quad (3.7)$$

where $\int_{\widehat{R}_{ij}} \sqrt{1 + |Df|^2}$ is the extension of the area functional to $\text{BV}(\widehat{R}_{ij})$ as defined in [11, Definition 14.1].

From [11, Theorem 15.9] and the fact that the restriction of φ_{ij} to $\partial \widehat{R}_{ij}$ is continuous up to a countable set of points, it follows that \widehat{m}_{ij} solves also

$$\min \left\{ \int_{\widehat{R}_{ij}} \sqrt{1 + |\nabla f|^2} dsdt : f \in W^{1,1}(\widehat{R}_{ij}), f = \varphi_{ij} \mathcal{H}^1 - a.e. \text{ on } \partial \widehat{R}_{ij} \right\}. \quad (3.8)$$

Let $m_{ij} = m_{ij}(\Gamma)$ be the restriction of \widehat{m}_{ij} to R_{ij} . Then, by the symmetry of φ_{ij} with respect to the line $\{t = r_{ij}\}$, m_{ij} is the unique solution of the Dirichlet-Neumann minimum problem (1.7). From (1.6) it follows that the Dirichlet condition is zero on the sides $\{0\} \times [0, r_{ij}]$ and $\{\ell_{ij}\} \times [0, r_{ij}]$ of the rectangle R_{ij} . Note that m_{ij} is analytic in the interior of R_{ij} but *not* necessarily Lipschitz in R_{ij} [11, Theorem 14.13], see Figure 2b.

Definition 3.4 (The functional \mathcal{G}). We define the functional $\mathcal{G} : X \rightarrow [0, +\infty)$ as

$$\mathcal{G}(\Gamma) := \mathfrak{A}_{12}(\Gamma) + \mathfrak{A}_{23}(\Gamma) + \mathfrak{A}_{31}(\Gamma), \quad (3.9)$$

where $\mathfrak{A}_{ij}(\Gamma)$ are as in (1.8).

The properties of the functional \mathcal{G} will be discussed in Section 5.

4 Infimum of \mathcal{G} as an upper bound of $\mathcal{A}(u, D)$

The aim of this section is to provide the following upper bound for $\mathcal{A}(u, D)$.

Theorem 4.1. Let $u \in \text{BV}(D; \{\alpha_1, \alpha_2, \alpha_3\})$ be the function defined in (1.9). Then

$$\mathcal{A}(u, D) \leq |D| + \inf \{ \mathcal{G}(\Gamma) : \Gamma \in X_{\text{Lip}} \}. \quad (4.1)$$

It is not difficult to see, by truncating the minimal surfaces in (1.7) with the lateral boundary of the prisms $[0, \ell_{ij}] \times \mathbb{T}$, that the infimum in (4.1) is the same as the infimum obtained without requiring in Definition 3.1 that $\Gamma_i \subset \mathbb{T}$, $i \in \{1, 2, 3\}$.

Lemma 4.2. Let $\ell \geq 0$, $p \geq 0$, $\varphi \in \text{Lip}([0, \ell]; [0, +\infty))$ be such that $\varphi(0) = \varphi(\ell) = 0$ and $w \in [0, \ell]$ so that $\varphi(w) = p$. Then there exists a sequence $\{\varphi^\sigma\}$ of \mathcal{C}^∞ equi-Lipschitz functions in $[0, \ell]$, converging to φ in $L^1([0, \ell])$ and uniformly on $[0, \ell]$ as $\sigma \rightarrow 0^+$, such that

$$\varphi^\sigma(0) = \varphi^\sigma(\ell) = 0, \quad \varphi^\sigma(w) = p, \quad \text{for any } \sigma > 0.$$

Proof. Let us extend φ in \mathbb{R} such that $\varphi(s) = 0$ in $\mathbb{R} \setminus [0, \ell]$, so that the extension (still denoted by φ) belongs to $\text{Lip}(\mathbb{R})$. Let $\widehat{\varphi}^\sigma(s) := \eta_\sigma * \varphi$ in \mathbb{R} , where $\{\eta_\sigma\}$ is a standard sequence of mollifiers. Hence $\widehat{\varphi}^\sigma \in \mathcal{C}^\infty(\mathbb{R})$, $\text{Lip}(\widehat{\varphi}^\sigma) \leq \text{Lip}(\varphi)$ and the sequence $\{\widehat{\varphi}^\sigma\}$ converges uniformly to φ on compact subsets of \mathbb{R} . Without loss of generality we may assume $\widehat{\varphi}^\sigma(s) = 0$ in $\mathbb{R} \setminus (-\sigma/2, \ell + \sigma/2)$ and $\widehat{\varphi}^\sigma(\frac{\ell+2\sigma}{\ell}w - \sigma) = p + c_\sigma$, $c_\sigma = o(1)$. Let us first suppose $p \neq 0$. We define

$$\varphi^\sigma : [0, \ell] \rightarrow [0, +\infty), \quad \varphi^\sigma(s) := \frac{p}{p + c_\sigma} \widehat{\varphi}^\sigma\left(\frac{\ell + 2\sigma}{\ell} s - \sigma\right). \quad (4.2)$$

It is easy to see that $\varphi^\sigma \in \mathcal{C}^\infty([0, \ell])$, $\varphi^\sigma(0) = \varphi^\sigma(\ell) = 0$, $\varphi^\sigma(w) = p$, φ^σ are equi-Lipschitz, and $\{\varphi^\sigma\}$ converges to φ in $L^1([0, \ell])$ as $\sigma \rightarrow 0$. Notice that the obtained approximation is constantly null in a neighborhood of 0 and ℓ .

In the case $p = 0$, we argue differently. We consider the two intervals $[0, w]$ and $[w, \ell]$ and we repeat the same approximation above in the single intervals; more precisely we choose two points $w_1 \in (0, w)$ and $w_2 \in (w, \ell)$ with $\varphi(w_1) > 0$, $\varphi(w_2) > 0$ (if these points does not exist it means that the functions are constantly 0 and they are already smooth, so there is nothing to prove). Then we approximate the two functions $\varphi_{\lfloor}(0, w)$ and $\varphi_{\lfloor}[w, \ell]$ as before, and we glue them along w . Note that the glued function is smooth in w since both the two smooth approximations are constantly 0 in a neighborhood of w . □

To prove Theorem 4.1 we use the three area-minimizing functions $m_{ij}, ij \in \{12, 23, 31\}$, introduced in Section 3.1, to construct a sequence $\{u^\varepsilon\}$ of Lipschitz functions that converge to u in $L^1(D; \mathbb{R}^2)$. However $m_{ij}, ij \in \{12, 23, 31\}$, are only locally Lipschitz so we need the following smoothing lemma.

Lemma 4.3. *Let $\Gamma \in X_{\text{Lip}}$, $ij \in \{12, 23, 31\}$. Let $\varphi_{ij} = \varphi_{ij}(\Gamma_{ij}) \in \text{Lip}([0, \ell_{ij}])$, $m_{ij} = m_{ij}(\Gamma_{ij}) \in W^{1,1}(\mathbb{R}_{ij})$, be defined as in Section 3.1. Then there exists a sequence $\{m_{ij}^\sigma\}$ of Lipschitz functions such that $m_{ij}^\sigma : \mathbb{R}_{ij} \rightarrow \mathbb{R}$, $m_{ij}^\sigma = \varphi_{ij}$ on $\partial_D \mathbb{R}_{ij}$, and*

$$\left| \int_{\mathbb{R}_{ij}} \sqrt{1 + |\nabla m_{ij}|^2} dsdt - \int_{\mathbb{R}_{ij}} \sqrt{1 + |\nabla m_{ij}^\sigma|^2} dsdt \right| \leq O(\sigma). \quad (4.3)$$

Proof. This can be easily proved using an argument similar to the one in [4, p.378: p.381], and using also Lemma 4.2 with the choice $w = w_{ij}$ and $p = \varphi_{ij}(w_{ij})$. □

We start to prove Theorem 4.1 in the special case of a piecewise linear jump, as in Figure 3.

Proposition 4.4. *Let $u \in BV(D; \{\alpha_1, \alpha_2, \alpha_3\})$ be the map defined in (1.9) and assume that the jump set of u consists of three distinct segments that meet at the origin and reach the boundary of D . Then (4.1) holds.*

Proof. Let $\Gamma \in X_{\text{Lip}}$ be a connection passing through $p \in T$ and $\mathcal{G}(\Gamma) := \mathfrak{A}_{12}(\Gamma) + \mathfrak{A}_{23}(\Gamma) + \mathfrak{A}_{31}(\Gamma)$. To prove the proposition it is sufficient to construct a sequence $\{u^\varepsilon\} \subset \text{Lip}(D; \mathbb{R}^2)$ converging to u in $L^1(D; \mathbb{R}^2)$ such that

$$\lim_{\varepsilon \rightarrow 0} \mathcal{A}(u^\varepsilon, D) \leq |D| + \mathfrak{A}_{12}(\Gamma) + \mathfrak{A}_{23}(\Gamma) + \mathfrak{A}_{31}(\Gamma). \quad (4.4)$$

Case 1. Assume that the segments separating E_1, E_2, E_3 meet at the origin with angles less than π , as in Figure 3.

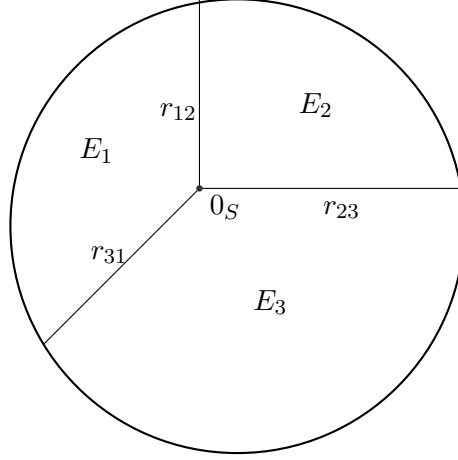


Figure 3: E_1 , E_2 , E_3 are separated by three segments of length r_{12} , r_{23} , r_{31} that meet at the origin.

To simplify the computation we may assume that $p = 0_T$, see Figure 1b. The idea of the proof is similar to the one used in [4], with however new difficulties, in particular in T^ε (step 3). We will specify various subsets of D and define the sequence $\{u^\varepsilon\}$ on each of these sets. Let $\varepsilon > 0$ be sufficient small and $\delta_\varepsilon > 0$ be such that $\delta_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$. Define T^ε to be the triangle with the origin 0_S in its interior, with vertices $\zeta^1 = \zeta_\varepsilon^1$, $\zeta^2 = \zeta_\varepsilon^2$, and $\zeta^3 = \zeta_\varepsilon^3$, and sides of lengths ε_{12} , ε_{23} , ε_{31} , $\varepsilon_{ij} := |\zeta^i - \zeta^j|$; the sides of T^ε are perpendicular to the lines containing r_{12} , r_{23} , r_{31} (respectively) and their distance from the origin 0_S equals δ_ε . Define three cygar-shaped sets S_{23}^ε , S_{31}^ε and S_{12}^ε as in Figure 4a: if for instance y is a coordinate on r_{12} and x is the perpendicular coordinate, then S_{12}^ε is defined as

$$S_{12}^\varepsilon := \{(x, y) \in D : x \in (\zeta_1^1, \zeta_1^2), y \geq \delta_\varepsilon\}, \quad (4.5)$$

where

$$\zeta^i = (\zeta_1^i, \zeta_2^i), \quad i = 1, 2, 3.$$

Let us set

$$E_1^\varepsilon := E_1 \setminus (S_{31}^\varepsilon \cup T^\varepsilon \cup S_{12}^\varepsilon), \quad E_2^\varepsilon := E_2 \setminus (S_{23}^\varepsilon \cup T^\varepsilon \cup S_{12}^\varepsilon), \quad E_3^\varepsilon := E_3 \setminus (S_{23}^\varepsilon \cup T^\varepsilon \cup S_{31}^\varepsilon). \quad (4.6)$$

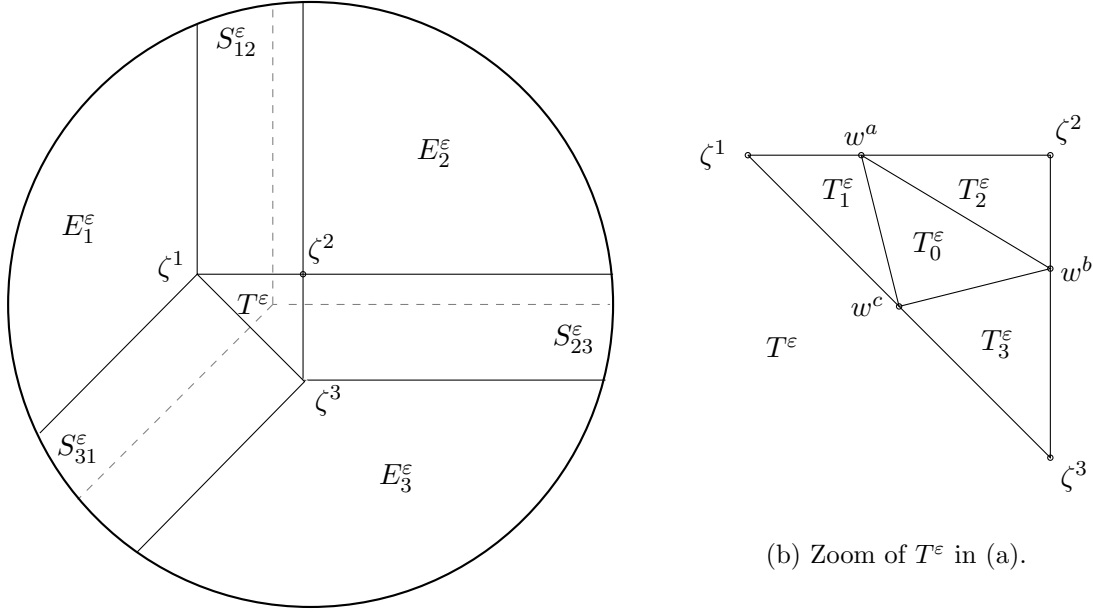
Step 1. Definition of u^ε on $E_1^\varepsilon \cup E_2^\varepsilon \cup E_3^\varepsilon$. We define

$$u^\varepsilon := \begin{cases} \alpha_1 & \text{in } E_1^\varepsilon, \\ \alpha_2 & \text{in } E_2^\varepsilon, \\ \alpha_3 & \text{in } E_3^\varepsilon. \end{cases} \quad (4.7)$$

Note that $\mathcal{A}(u^\varepsilon, E_1^\varepsilon \cup E_2^\varepsilon \cup E_3^\varepsilon) = |E_1^\varepsilon| + |E_2^\varepsilon| + |E_3^\varepsilon|$, hence

$$\lim_{\varepsilon \rightarrow 0^+} \mathcal{A}(u^\varepsilon, E_1^\varepsilon \cup E_2^\varepsilon \cup E_3^\varepsilon) = |D|. \quad (4.8)$$

Step 2. Definition of u^ε on $S_{23}^\varepsilon \cup S_{31}^\varepsilon \cup S_{12}^\varepsilon$.



(a) Case 1 of the proof of Proposition 4.4

Figure 4

We will start with the construction on S_{12}^ε . Set

$$\xi = (\xi_1, \xi_2) := \frac{\alpha_2 - \alpha_1}{\ell_{12}} \in \mathbb{S}^1, \quad \eta = (\eta_1, \eta_2) := \xi^\perp,$$

where $^\perp$ denotes the counterclockwise rotation of $\pi/2$.

Let $\psi_{12}^\varepsilon : [\delta_\varepsilon, r_{12} + c_\varepsilon] \rightarrow [0, r_{12}]$ be linear, increasing, surjective, where $c_\varepsilon > 0$ is the smallest number such that

$$S_{12}^\varepsilon \subset [\zeta_1^1, \zeta_1^2] \times [\delta_\varepsilon, r_{12} + c_\varepsilon], \quad \lim_{\varepsilon \rightarrow 0^+} c_\varepsilon = 0.$$

Note that for any $y \in [\delta_\varepsilon, r_{12} + c_\varepsilon]$ we have

$$(\psi_{12}^\varepsilon)'(y) = \frac{r_{12}}{r_{12} + c_\varepsilon - \delta} =: \kappa_\varepsilon, \quad \lim_{\varepsilon \rightarrow 0^+} \kappa_\varepsilon = 1. \quad (4.9)$$

Let m_{12}^σ be the map defined in Lemma 4.3, whose area on \mathbb{R}_{12} is by construction close to \mathfrak{A}_{12} , with $\{\sigma_\varepsilon\} \subset (0, +\infty)$ a sequence such that

$$\lim_{\varepsilon \rightarrow 0^+} \sigma_\varepsilon = 0. \quad (4.10)$$

We set, with $\sigma = \sigma_\varepsilon$ for simplicity,

$$u^\varepsilon(x, y) := \alpha_1 + \left(\frac{x - \zeta_1^1}{\varepsilon_{12}} \right) \ell_{12} \xi + m_{12}^\sigma \left(\frac{x - \zeta_1^1}{\varepsilon_{12}} \ell_{12}, \psi_{12}^\varepsilon(y) \right) \eta, \quad (x, y) \in S_{12}^\varepsilon. \quad (4.11)$$

Observe that $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon) \in \text{Lip}(S_{12}^\varepsilon; \mathbb{R}^2)$, $u^\varepsilon = \alpha_1$ on $\{(x, y) \in S_{12}^\varepsilon : x = \zeta_1^1\}$, and $u^\varepsilon = \alpha_2$ on $\{(x, y) \in S_{12}^\varepsilon : x = \zeta_1^2\}$. By the definition of m_{12}^σ , it is uniquely defined the point (depending on ε)

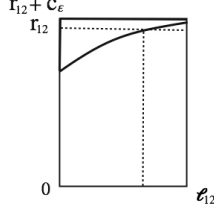


Figure 5: The set P_ϵ is bounded by the bold contour.

$w^a = (w_1^a, w_2^a) \in \overline{\zeta^1 \zeta^2}$ such that $u^\epsilon(w_1^a, w_2^a) = 0_T$ (see Figure 4b). Write for simplicity

$$\tilde{m} = m_{12}^\sigma.$$

Using that $|\xi| = |\eta| = 1$, $\xi_1 \eta_1 + \xi_2 \eta_2 = 0$, and $\xi_1 \eta_2 - \xi_2 \eta_1 = 1$, we compute

$$1 + |\nabla u_1^\epsilon|^2 + |\nabla u_2^\epsilon|^2 + \left(\frac{\partial u_1^\epsilon}{\partial x} \frac{\partial u_2^\epsilon}{\partial y} - \frac{\partial u_1^\epsilon}{\partial y} \frac{\partial u_2^\epsilon}{\partial x} \right)^2 = 1 + \frac{\ell_{12}^2}{\epsilon_{12}^2} \left(1 + (\tilde{m}_s)^2 + (\tilde{m}_t)^2 \kappa_\epsilon^2 \left(1 + \frac{\epsilon_{12}^2}{\ell_{12}^2} \right) \right),$$

where \tilde{m}_s, \tilde{m}_t denote, respectively, the partial derivatives of \tilde{m} with respect to $s := \frac{x - \zeta_1^1}{\epsilon_{12}} \ell_{12}$ and $t := \psi_{12}^\epsilon(y)$, and are evaluated at $\left(\frac{x - \zeta_1^1}{\epsilon_{12}} \ell_{12}, \psi_{12}^\epsilon(y) \right)$. As a consequence

$$\begin{aligned} & \mathcal{A}(u^\epsilon, S_{12}^\epsilon) \\ &= \frac{\ell_{12}}{\epsilon_{12}} \int_{S_{12}^\epsilon} \sqrt{1 + \left[\tilde{m}_s \left(\frac{x - \zeta_1^1}{\epsilon_{12}} \ell_{12}, \psi_{12}^\epsilon(y) \right) \right]^2 + \left[\tilde{m}_t \left(\frac{x - \zeta_1^1}{\epsilon_{12}} \ell_{12}, \psi_{12}^\epsilon(y) \right) \right]^2 \kappa_\epsilon^2 \left(1 + \frac{\epsilon_{12}^2}{\ell_{12}^2} \right) + O(\epsilon^2)} dx dy \\ &= \frac{1}{\kappa_\epsilon} \int_{\mathbf{R}_{12} \setminus P_\epsilon} \sqrt{1 + [\tilde{m}_s(s, t)]^2 + [\tilde{m}_t(s, t)]^2 \kappa_\epsilon^2 \left(1 + \frac{\epsilon_{12}^2}{\ell_{12}^2} \right) + O(\epsilon^2)} ds dt, \end{aligned} \quad (4.12)$$

where the last equality follows by the change of variables

$$\Phi : \mathbf{R}_{12} \ni (s, t) \mapsto \Phi(s, t) := \left(\frac{\epsilon_{12}}{\ell_{12}} s + \zeta_1^1, \psi_{12}^{\epsilon^{-1}}(t) \right) = (x, y) \in [\zeta_1^1, \zeta_1^2] \times [\delta_\epsilon, r_{12} + c_\epsilon] \supset S_{12}^\epsilon,$$

and $P_\epsilon := \mathbf{R}_{12} \setminus \Phi^{-1}(S_{12}^\epsilon)$ (see Figure 5). Hence, recalling also (4.9), we conclude

$$\lim_{\epsilon \rightarrow 0^+} \mathcal{A}(u^\epsilon, S_{12}^\epsilon) = \int_{\mathbf{R}_{12}} \sqrt{1 + (\tilde{m}_s)^2 + (\tilde{m}_t)^2} ds dt. \quad (4.13)$$

We recall that from (4.3) it follows that

$$\int_{\mathbf{R}_{12}} \sqrt{1 + (\tilde{m}_s)^2 + (\tilde{m}_t)^2} ds dt = \mathfrak{A}_{12}(\Gamma) + O(\epsilon). \quad (4.14)$$

Hence, employing the same construction in the strips S_{23}^ϵ and S_{31}^ϵ we obtain

$$\lim_{\epsilon \rightarrow 0^+} \mathcal{A}(u^\epsilon, S_{23}^\epsilon \cup S_{31}^\epsilon \cup S_{12}^\epsilon) = \mathfrak{A}_{12}(\Gamma) + \mathfrak{A}_{23}(\Gamma) + \mathfrak{A}_{31}(\Gamma). \quad (4.15)$$

Step 3. Definition of u^ε on T^ε . We divide T^ε into four closed triangles $T_1^\varepsilon, T_2^\varepsilon, T_3^\varepsilon$ and T_0^ε as in Figure 4b. We set

$$u^\varepsilon := 0_T \quad \text{in } T_0^\varepsilon. \quad (4.16)$$

We first define u^ε on ∂T_1^ε as follows:

- (i) the value of u^ε at ζ^1 is α_1 ;
- (ii) the value of u^ε on the side $\overline{w^c w^a}$ is 0_T .

Note that u^ε is already defined on the edges $\overline{\zeta^1 w^a}$ and $\overline{\zeta^1 w^c}$ and its graph over both edges is given by a rescaled version of the curve Γ_1 suitably parametrized.

More precisely, we recall that $\pi_{12} : \Gamma_1 \rightarrow \overline{\alpha_1 \alpha_2}$ and $\pi_{31} : \Gamma_1 \rightarrow \overline{\alpha_3 \alpha_1}$ are the orthogonal projections onto the edges $\overline{\alpha_1 \alpha_2}$ and $\overline{\alpha_3 \alpha_1}$. Since Γ_1 is, by hypothesis, a part of a Lipschitz graph, the maps $\pi_{12} \lfloor \Gamma_1$ and $\pi_{31} \lfloor \Gamma_1$ are bi-Lipschitz bijections between Γ_1 and the segments $\overline{\alpha_1 \pi_{12}(p)}$ and $\overline{\alpha_1 \pi_{31}(p)}$, respectively. We know that if (s, t) are coordinates on T with respect to the system with s -axis $\overline{\alpha_1 \alpha_2}$, then the inverse of $\pi_{12} \lfloor \Gamma_1$ is given by $\Phi_{12} : \overline{\alpha_1 \alpha_2} \rightarrow \Gamma_1$,

$$\Phi_{12}((s, 0)) = (s, \varphi_{12}(s)).$$

Let us denote by $L_{12} = L_{12}^\varepsilon : \overline{\zeta^1 w^a} \subset \mathbb{R}_S^2 \rightarrow \overline{\alpha_1 \pi_{12}(p)} \subset \mathbb{R}_T^2$ and $L_{31} = L_{31}^\varepsilon : \overline{\zeta^1 w^a} \subset \mathbb{R}_S^2 \rightarrow \overline{\alpha_1 \pi_{31}(p)} \subset \mathbb{R}_T^2$ the linear bijective maps

$$L_{12}(Q) = \alpha_1 + \frac{|Q - \zeta^1|}{\varepsilon_{12}}(\alpha_2 - \alpha_1), \quad L_{31}(Q') = \alpha_1 + \frac{|Q' - \zeta^1|}{\varepsilon_{31}}(\alpha_3 - \alpha_1).$$

Then we define

$$u^\varepsilon := \Phi_{12} \circ L_{12} \quad \text{on } \overline{\zeta^1 w^a}, \quad (4.17)$$

and

$$u^\varepsilon = \Phi_{31} \circ L_{31} \quad \text{on } \overline{\zeta^1 w^c}. \quad (4.18)$$

compare formula (4.11). Since Φ_{12} and Φ_{31} are Lipschitz with Lipschitz constant independent of ε , and the Lipschitz constants of L_{12} and L_{31} have order $\frac{1}{\varepsilon}$, it follows that the Lipschitz constants of u^ε over the segments $\overline{\zeta^1 w^a}$ and $\overline{\zeta^1 w^c}$ have order $\frac{1}{\varepsilon}$.

Now we want to define u^ε in the interior of T_1^ε . First we observe that the map $\pi_{31} \circ \Phi_{12} : \overline{\alpha_1 \pi_{12}(p)} \subset \mathbb{R}_T^2 \rightarrow \overline{\alpha_1 \pi_{31}(p)} \subset \mathbb{R}_T^2$ is a bi-Lipschitz bijection, with constant independent of ε . A direct computation then provides that the map $\Psi : \overline{\zeta^1 w^a} \subset \mathbb{R}_S^2 \rightarrow \overline{\zeta^1 w^c} \subset \mathbb{R}_S^2$ defined by

$$\Psi := (L_{31})^{-1} \circ \pi_{31} \circ \Phi_{12} \circ L_{12}, \quad (4.19)$$

is bi-Lipschitz between $\overline{\zeta^1 w^a}$ and $\overline{\zeta^1 w^c}$ with bi-Lipschitz constants of order 1 as $\varepsilon \rightarrow 0^+$. Given $Q \in \overline{\zeta^1 w^a}$ let $Q' := \Psi(Q) \in \overline{\zeta^1 w^c}$.

Now we show that T_1^ε is foliated by the segments $\overline{QQ'}$, *i.e.*, for any $R \in T_1^\varepsilon$ we can find a unique $Q \in \overline{\zeta^1 w^a}$ for which $R \in \overline{QQ'}$.

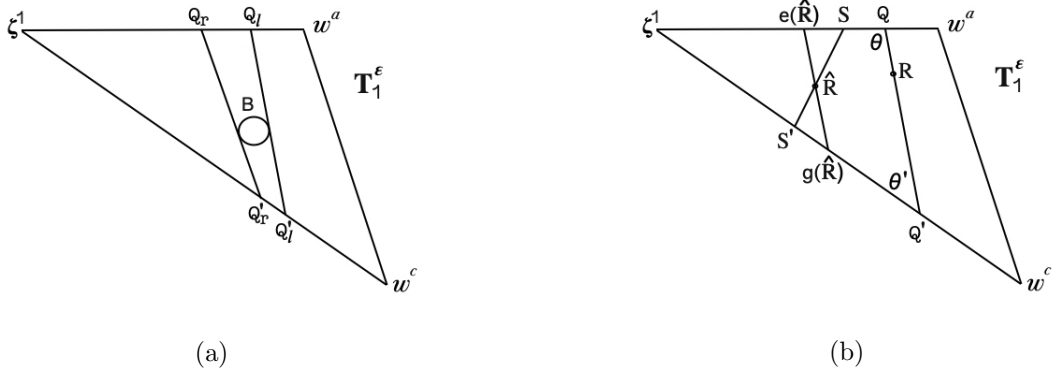


Figure 6: Surjectivity of the foliation in $T^\varepsilon \subset \mathbb{R}_S^2$.

First we notice that $\overline{QQ'} \cap \overline{SS'} = \emptyset$ for any $Q \neq S \in \overline{\zeta^1 w^a}$ with $Q' = \Psi(Q)$ and $S' = \Psi(S)$. Indeed, thanks to the fact that Ψ is a homeomorphism and that it keeps ζ^1 fixed, it is easy to see that if $S \in \overline{\zeta^1 Q}$ then $S' \in \overline{\zeta^1 Q'}$, or if $Q \in \overline{\zeta^1 S}$ then $Q' \in \overline{\zeta^1 S'}$. Consider the function

$$f(q, \sigma) = q\tau + \sigma\nu(q), \quad q \in [0, |w^a - \zeta^1|], \sigma \in [0, |\Psi(q\tau) - q\tau|],$$

where $\tau := \frac{w^a - \zeta^1}{|w^a - \zeta^1|}$ and $\nu(q) := \frac{\Psi(q\tau) - q\tau}{|\Psi(q\tau) - q\tau|}$. It is clear that the image of f is a closed set and $\text{Im}(f) = \{\overline{QQ'} : Q \in \overline{\zeta^1 w^a}, Q' = \Psi(Q)\}$. Now we show that $\text{Im}(f) = T_1^\varepsilon$. Assume by contradiction there is $R \in T_1^\varepsilon \setminus \text{Im}(f)$ and take a disk $B \subset T_1^\varepsilon \setminus \text{Im}(f)$ centered at R . Let $Q_r, Q_l \in \overline{\zeta^1 w^a}$ be such that $q_r := |Q_r - \zeta^1|$ (resp. $q_l := |Q_l - \zeta^1|$) be the supremum (resp. the infimum) parameter for which B lies on the right (resp. left) of $\overline{Q_r Q_r'}$ (resp. $\overline{Q_l Q_l'}$). Note that $Q_r \neq Q_l$ due to the injectivity of Ψ , thus for any $Q \in \overline{Q_r Q_l}$ the segment $\overline{QQ'}$ must intersect B , a contradiction, see Figure 6a. Hence we may define u^ε on T_1^ε as

$$u^\varepsilon(R) := u^\varepsilon(Q), \quad R \in \overline{QQ'}, \quad Q \in \overline{\zeta^1 w^a}. \quad (4.20)$$

We want now to show that on T_1^ε , u^ε is Lipschitz continuous with Lipschitz constant of order $\frac{1}{\varepsilon}$. To prove this let us fix $R \in T_1^\varepsilon$. By definition $u^\varepsilon(R) = u^\varepsilon(Q)$ for some $Q \in \overline{\zeta^1 w^a}$ and u^ε is constant on the segment $\overline{QQ'} \ni R$.

Let $e : T_1^\varepsilon \rightarrow \overline{\zeta^1 w^a}$ be the function taking $(x, y) \in T_1^\varepsilon$ to the intersection point of $\overline{\zeta^1 w^a}$ and the line passing through (x, y) parallel to $\overline{QQ'}$. Let $g : T_1^\varepsilon \rightarrow \overline{\zeta^1 w^c}$ be the function taking $(x, y) \in T_1^\varepsilon$ to the intersection point of $\overline{\zeta^1 w^c}$ and the line passing through (x, y) parallel to $\overline{QQ'}$. Let $\hat{R} \in T_1^\varepsilon$ a point in T_1^ε , we want to estimate the ratio

$$\frac{|u^\varepsilon(\hat{R}) - u^\varepsilon(R)|}{|\hat{R} - R|}.$$

Consider the two segments $\overline{Qe(\hat{R})}$ and $\overline{Qg(\hat{R})}$. By definition $\hat{R} \in \overline{SS'}$ and $u^\varepsilon(\hat{R}) = u^\varepsilon(S) = u^\varepsilon(S')$ for two points $S \in \overline{\zeta^1 w^a}$ and $S' \in \overline{\zeta^1 w^c}$. It is straightforward that either $S \in \overline{Qe(\hat{R})}$ or $S' \in \overline{Q'g(\hat{R})}$. Without loss of generality suppose the first case holds, see Figure 6b.

Finally, denote by θ the angle between $\overline{QQ'}$ and $\overline{\zeta^1 w^a}$ and by θ' the angle between $\overline{QQ'}$ and $\overline{\zeta^1 w^c}$. Using the fact that the homeomorphism in (4.19) is bi-Lipschitz with constant of order 1 it is not difficult to see that there is a constant $\theta_0 > 0$ independent of ε such that $\min\{\theta, \theta'\} \geq \theta_0$. This

is a consequence of the fact that the bi-Lipschitz constant of Ψ in (4.19) is of order 1. Indeed, if $L = \text{lip}(\Psi)$ and $1/L' = \text{lip}(\Psi^{-1})$, we see that

$$L' \leq \frac{|Q' - \zeta^1|}{|Q - \zeta^1|} \leq L,$$

hence

$$\frac{1/L + \cos \theta_{\zeta^1}}{\sin \theta_{\zeta^1}} \leq \frac{\cos \theta}{\sin \theta} \leq \frac{1/L' + \cos \theta_{\zeta^1}}{\sin \theta_{\zeta^1}},$$

where θ_{ζ^1} is the angle at ζ^1 (here we have used the law of sines and that $\theta' = \pi - \theta_{\zeta^1} - \theta$). A similar estimate holding for θ' , this readily provides the boundedness from below of $\min\{\theta, \theta'\}$.

As a consequence we have

$$|\hat{R} - R| \geq |Q - e(\hat{R})| |\sin \theta| \geq |Q - e(\hat{R})| |\sin \theta_0|.$$

Thus, we compute

$$\frac{|u^\varepsilon(\hat{R}) - u^\varepsilon(R)|}{|\hat{R} - R|} \leq \frac{|u^\varepsilon(Q) - u^\varepsilon(S)|}{|Q - e(\hat{R})| |\sin \theta|} \leq \frac{|u^\varepsilon(Q) - u^\varepsilon(S)|}{|Q - S| |\sin \theta|} \leq \frac{1}{|\sin \theta_0|} \frac{|u^\varepsilon(Q) - u^\varepsilon(S)|}{|Q - S|}, \quad (4.21)$$

that is bounded by the Lipschitz constant of $\Phi_{12} \circ L_{12}$ which is of order $\frac{1}{\varepsilon}$.

Eventually we compute the Jacobian of u^ε in (4.20). By construction the image of T_1^ε by u^ε is exactly the curve Γ_1 , which has zero Lebesgue measure in \mathbb{R}^2 . By a standard application of the area formula it follows that the Jacobian of u^ε vanishes a.e. in T_1^ε . We have concluded the definition of u^ε in T_1^ε . The constructions on T_2^ε and on T_3^ε are similar, and similar estimates of the derivatives and Jacobian hold.

Using that the area of the triangle T^ε is of order ε^2 , we have

$$\lim_{\varepsilon \rightarrow 0} \mathcal{A}(u^\varepsilon, T_\varepsilon) = \lim_{\varepsilon \rightarrow 0} O(\varepsilon) + O(\varepsilon^2) = 0. \quad (4.22)$$

From (4.7), (4.11), (4.16), (4.20), and the estimates above it follows that

$$\{u^\varepsilon\} \subset \text{Lip}(D; \mathbb{R}^2), \quad \lim_{\varepsilon \rightarrow 0} \int_D |u^\varepsilon - u| \, dx dy = 0. \quad (4.23)$$

Moreover

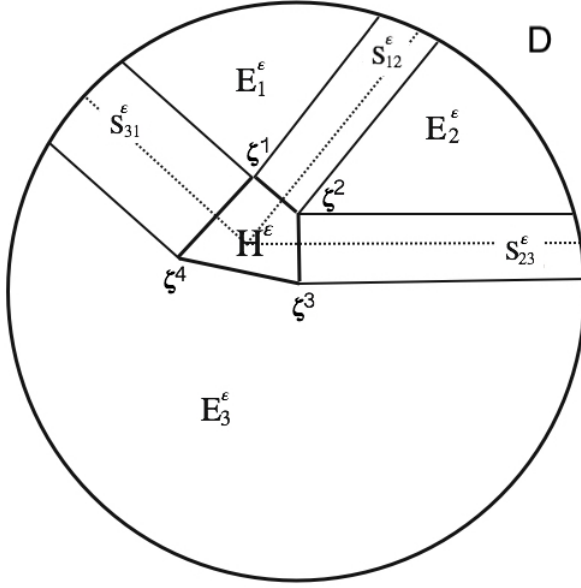
$$\mathcal{A}(u^\varepsilon, D) = \mathcal{A}(u^\varepsilon, E_1^\varepsilon \cup E_2^\varepsilon \cup E_3^\varepsilon) + \mathcal{A}(u^\varepsilon, S_{23}^\varepsilon) + \mathcal{A}(u^\varepsilon, S_{31}^\varepsilon) + \mathcal{A}(u^\varepsilon, S_{12}^\varepsilon) + \mathcal{A}(u^\varepsilon, T^\varepsilon). \quad (4.24)$$

Then (4.4) follows from (4.24), (4.8), (4.15), (4.10) and (4.22).

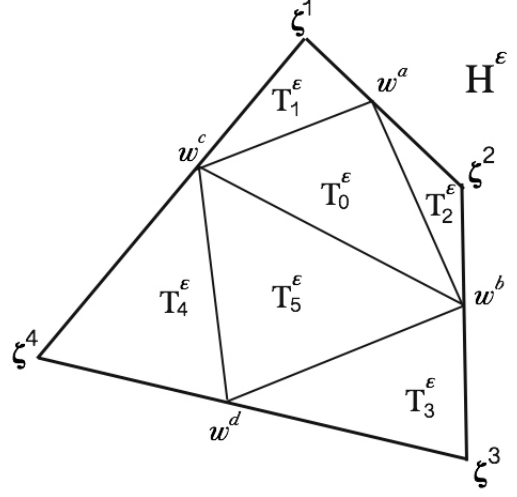
Case 2. Assume that two of the segments separating E_1 , E_2 , E_3 meet at the origin with an angle greater than or equal to π .

Similar to Case 1, we divide the domain D into a finite number of subsets and define the sequence $\{u^\varepsilon\}$ on each of these sets. Draw the normal to each segment at the point of distance δ from the origin. The normal lines meet at two points ζ^1 , ζ^2 . Divide D into three cygar-shape subsets S_{23}^ε , S_{31}^ε , S_{12}^ε (with widths of order $\delta = O(\varepsilon)$) and a quadrilateral H^ε as in Figure 7a. Let

$$E_1^\varepsilon := E_1 \setminus (S_{31}^\varepsilon \cup H^\varepsilon \cup S_{12}^\varepsilon), \quad E_2^\varepsilon := E_2 \setminus (S_{23}^\varepsilon \cup H^\varepsilon \cup S_{12}^\varepsilon), \quad E_3^\varepsilon := E_3 \setminus (S_{23}^\varepsilon \cup H^\varepsilon \cup S_{31}^\varepsilon).$$



(a) The non acute case.



(b) Zoom of H^ε in (a).

Figure 7

Set

$$u^\varepsilon := \begin{cases} \alpha_1 & \text{in } E_1^\varepsilon, \\ \alpha_2 & \text{in } E_2^\varepsilon, \\ \alpha_3 & \text{in } E_3^\varepsilon. \end{cases} \quad (4.25)$$

Define u^ε on $S_{23}^\varepsilon \cup S_{31}^\varepsilon \cup S_{12}^\varepsilon$ as in Step 2 in case 1. It remains to define u^ε on H^ε . Recall that by construction there exist uniquely determined three points $w^a \in \overline{\zeta^1 \zeta^2}$, $w^b \in \overline{\zeta^2 \zeta^3}$ and $w^c \in \overline{\zeta^1 \zeta^4}$ such that

$$u^\varepsilon(w^a) = u^\varepsilon(w^b) = u^\varepsilon(w^c) = 0_T.$$

Divide H^ε into six triangles $T_0^\varepsilon, T_1^\varepsilon, T_2^\varepsilon, T_3^\varepsilon, T_4^\varepsilon, T_5^\varepsilon$, as in Figure 7b, where w^d is any point in $\overline{\zeta^3 \zeta^4}$ and $w^d \neq \zeta^3, w^d \neq \zeta^4$.

Set

$$u^\varepsilon := 0_T \quad \text{in } T_0^\varepsilon.$$

We define u^ε in the triangles T_1^ε and T_2^ε as in Step 3; it remains to define u^ε on $T_3^\varepsilon, T_4^\varepsilon, T_5^\varepsilon$. Let us first define u^ε on the edges $\overline{w^c w^d}$ and $\overline{w^b w^d}$. The map u^ε is already defined on the other edges, and its graph over $\overline{\zeta^4 w^c}$ and $\overline{\zeta^3 w^b}$ is given by a suitable reparametrization of the curve Γ_3 , whereas u^ε on $\overline{\zeta^4 \zeta^3}$ is constantly α_3 . Therefore it suffices to define u^ε in such a way its graph over $\overline{w^c w^d}$ and $\overline{w^b w^d}$ coincides with Γ_3 as well, and then we can define u^ε inside T_3^ε and T_4^ε using the same construction for T_1^ε in step 3. Similarly, using that the graph of u^ε on $\overline{w^c w^d}$ and $\overline{w^b w^d}$ is again Γ_3 , we can repeat the construction in the triangle T_5^ε . Following the computation as in case 1 we get (4.4). This concludes the proof. \square

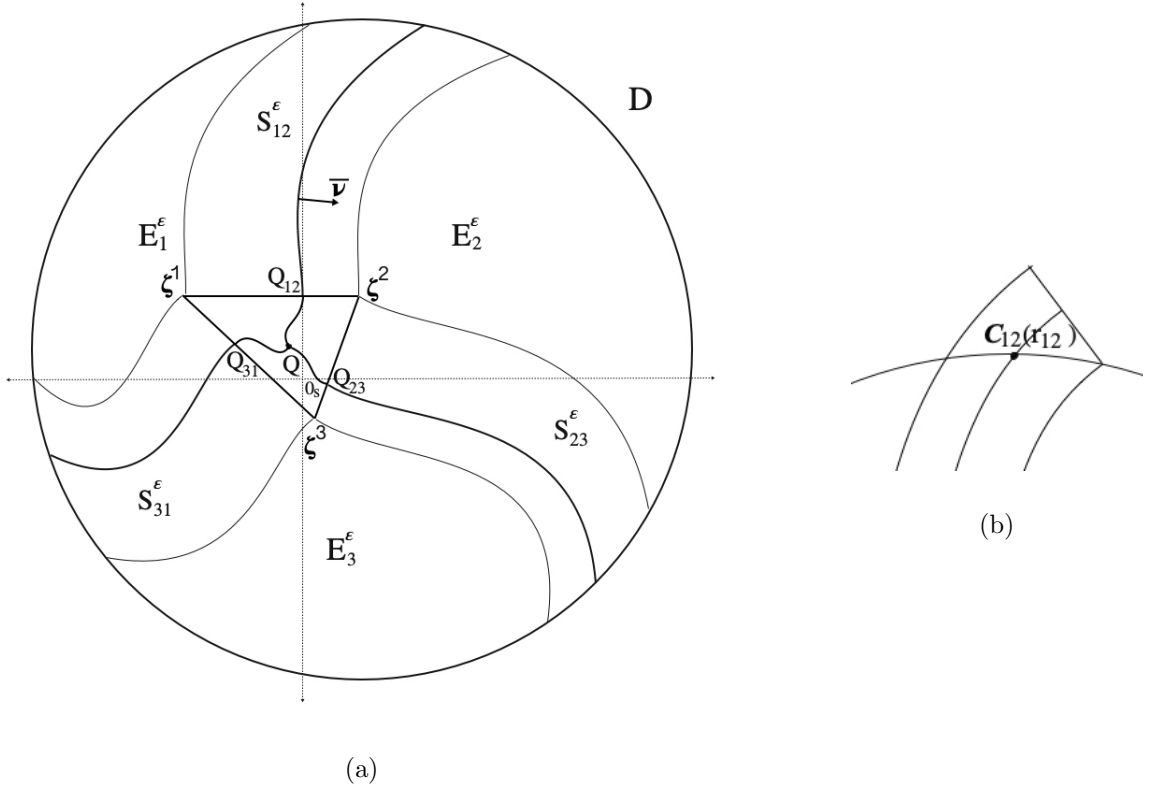


Figure 8: proof of Theorem 4.1.

Proof of Theorem 4.1. We will suitably adapt the construction made in the proof of Proposition 4.4. By hypothesis the regions E_1, E_2, E_3 are enclosed by C^2 -embedded curves C_{ij} , $ij \in \{12, 23, 31\}$, parametrized by arc length $c_{ij} : [0, r_{ij}] \rightarrow \mathbb{R}^2$, $ij \in \{23, 31, 12\}$. Moreover such curves meet ∂D transversely and intersect each other (transversely) only at one point Q . Suppose that the angles formed at Q by the three curves are all less than π (the other case is similarly adapted from the corresponding case in the proof of Proposition 4.4). We will divide the domain D into a finite number of subsets and define the sequence $\{u^\varepsilon\}$ on each of these sets.

Let $\delta_\varepsilon > 0$ be such that $\delta_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0^+$. Let $\tau \in [0, r_{ij}]$ be an arc length parameter on C_{ij} , with orthogonal coordinate d that coincides with the signed distance from C_{ij} negative in E_i and positive in E_j . Let $Q_{ij} \in C_{ij}$ be the point with arc distance $\tau = \delta_\varepsilon$ from the origin Q . Consider the three lines normal to C_{ij} at Q_{ij} . For δ_ε sufficiently small, since the angles at the origin are less than π and the curves are of class C^2 up to the closure, these lines mutually meet at points ζ^1 , ζ^2 , and ζ^3 . Let ε_{ij} be the length of $\overline{\zeta^i \zeta^j}$, which are of order ε . The tubular coordinates of the points ζ^1 and ζ^2 with respect to C_{12} are $(d_1, \delta_\varepsilon)$ and $(d_2, \delta_\varepsilon)$, with $d_2 - d_1 = \varepsilon_{12}$, $d_1 < 0, d_2 > 0$. For δ_ε small enough we can consider the cylindrical neighborhood of C_{12} defined as

$$S_{12}^\varepsilon := \{(x, y) \in D : \tau(x, y) \geq \delta_\varepsilon, d(x, y) \in (d_1, d_2)\}, \quad (4.26)$$

where we have prolonged C_{12} outside D for convenience. Similarly we define S_{23}^ε and S_{31}^ε . Let T^ε be the triangle with vertices ζ^1 , ζ^2 , and ζ^3 .

Finally, let E_1^ε , E_2^ε , E_3^ε be defined as in (4.6), and u^ε as in (4.7).

Step 1. Definition of u^ε on $S_{12}^\varepsilon \cup S_{23}^\varepsilon \cup S_{31}^\varepsilon$. We do the construction on S_{12}^ε , and u^ε will be defined similarly on S_{23}^ε and S_{31}^ε . We know that $c_{12}([\delta_\varepsilon, r_{12}]) = C_{12} \cap S_{12}^\varepsilon$. The system of coordinates (d, τ) defines a C^1 -diffeomorphism h between the rectangle $[d_1, d_2] \times [\delta_\varepsilon, \rho_{12}]$ and its image $\mathcal{N}_{12}^{\varepsilon, \delta}$ which contains S_{12}^ε , namely

$$h : [d_1, d_2] \times [\delta_\varepsilon, \rho_{12}] \rightarrow \mathcal{N}_{12}^{\varepsilon, \delta}; \quad h(d, \tau) := c_{12}(\tau) + d\bar{\nu}(\tau),$$

where $\bar{\nu}(\tau)$ is the unit normal vector pointing toward E_2 at $c_{12}(\tau)$ and $\rho_{12} = \rho_{12}^\varepsilon \geq r_{12}$ is the infimum of those ρ for which $S_{12}^\varepsilon \subset \mathcal{N}_{12}^{\varepsilon, \delta}$, see Figure 8b.

Since h is a C^1 -diffeomorphism we have that

$$h^{-1} : \mathcal{N}_{12}^{\varepsilon, \delta} \rightarrow [d_1, d_2] \times [\delta_\varepsilon, \rho_{12}]; \quad h^{-1}(x, y) := (d(x, y), \tau(x, y)),$$

is the inverse of h and is of class C^1 . We want to estimate the Jacobian of h^{-1} . To this aim, we first see that $\nabla d(c_{12}(\tau)) = \bar{\nu}(\tau)$ since $c_{12}([\delta_\varepsilon, \rho_{12}])$ is the zero level set of d and, from [1, Rem. 3(1)], we have

$$\nabla d(h(d, \tau)) = \bar{\nu}(\tau), \quad (d, \tau) \in [d_1, d_2] \times [\delta_\varepsilon, \rho_{12}]. \quad (4.27)$$

Fix $\tau \in [\delta_\varepsilon, \rho_{12}]$; by definition of tubular coordinates the segment $\{c_{12}(\tau) + d\bar{\nu}(\tau) : d \in [d_1, d_2]\}$ is a level set of the function $\tau(\cdot)$, hence,

$$\nabla \tau(x, y) \perp \bar{\nu}(\tau(x, y)), \quad (4.28)$$

therefore

$$\nabla \tau \cdot \nabla d = 0 \quad \text{in } S_{12}^\varepsilon. \quad (4.29)$$

Thus the Jacobian of h^{-1} will be

$$j(h^{-1}) = |\nabla \tau| |\nabla d| = |\nabla \tau|, \quad (4.30)$$

since $|\nabla d| = 1$ in $\mathcal{N}_{12}^{\varepsilon, \delta}$. Let us compute $\nabla \tau$; fix $d \in (d_1, d_2)$ and define $c_{12}^d(\tau) := c_{12}(\tau) + d\bar{\nu}(\tau)$. Now recall (4.28) and that $(c_{12}^d)'(\tau)$ is parallel to $\bar{\nu}^\perp(\tau)$, so that

$$|\nabla \tau| = \nabla \tau \cdot \bar{\nu}^\perp = \frac{\nabla \tau \cdot (c_{12}^d)'}{|(c_{12}^d)'|} \quad \text{in } S_{12}^\varepsilon. \quad (4.31)$$

Let us recall that C_{12} is parametrized by arc length, *i.e.*, $|c'_{12}(\tau)| = 1$, so that $\bar{\nu}'(\tau) = |c''_{12}(\tau)|c'_{12}(\tau)$. Thus $(c_{12}^d)'(\tau) = (1 + d|c''_{12}(\tau)|)c'_{12}(\tau)$. Since $\tau \circ c_{12}^d = \text{Id}$ it follows that $\nabla \tau(c_{12}^d(\tau))^T (c_{12}^d)'(\tau) = \nabla \tau(c_{12}^d(\tau)) \cdot (c_{12}^d)'(\tau) = 1$. Therefore, from (4.31), we deduce

$$|\nabla \tau| = \frac{1}{1 + d|c''_{12}(\tau)|} \quad \text{in } S_{12}^\varepsilon, \quad (4.32)$$

and in particular $\lim_{d \rightarrow 0} |\nabla \tau| = 1$ uniformly in S_{12}^ε .

We are ready to define u^ε in S_{12}^ε . We first set ψ_{12}^ε as in (4.9) with $r_{12} + c_\varepsilon = \rho_{12}$, *i.e.*, $\psi_{12}^\varepsilon(\tau) = \bar{\kappa}_\varepsilon(z - \delta_\varepsilon)$, setting $\bar{\kappa}_\varepsilon := \frac{\rho_{12}}{\rho_{12} - \delta_\varepsilon}$. Then we define \tilde{u}^ε on $[d_1, d_2] \times [\delta_\varepsilon, \rho_{12}]$ as in the right hand side of (4.11) and set

$$u^\varepsilon := \tilde{u}^\varepsilon \circ h^{-1} \quad \text{in } S_{12}^\varepsilon. \quad (4.33)$$

Explicitly, recalling that $\xi = \frac{\alpha_2 - \alpha_1}{\ell_{12}}$ and $\eta = \xi^\perp$, for $(x, y) \in S_{12}^\varepsilon$ we have

$$u^\varepsilon(x, y) := \alpha_1 + \left(\frac{d(x, y) - d_1}{\varepsilon_{12}} \right) \ell_{12} \xi + m_{12}^\sigma \left(\frac{d(x, y) - d_1}{\varepsilon_{12}} \ell_{12}, \bar{\kappa}_\varepsilon(\tau(x, y) - \delta_\varepsilon) \right) \eta. \quad (4.34)$$

Observe that $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon) \in \text{Lip}(S_{12}^\varepsilon; \mathbb{R}^2)$, $u^\varepsilon = \alpha_1$ on $\{(x, y) \in S_{12}^\varepsilon : d(x, y) = d_1\}$, $u^\varepsilon = \alpha_2$ on $\{(x, y) \in S_{12}^\varepsilon : d(x, y) = d_2\}$, and by construction there exists $w_a \in h([d_1, d_2] \times \{\delta_\varepsilon\})$ such that $u^\varepsilon(w_a) = 0_T$. Write for simplicity $\tilde{m} = m_{12}^\sigma$. We have

$$\begin{aligned} \nabla u_1^\varepsilon &= \left(\frac{\ell_{12} \xi^1}{\varepsilon_{12}} d_x + \frac{\ell_{12} \eta^1}{\varepsilon_{12}} \tilde{m}_s d_x + \bar{\kappa}_\varepsilon \tilde{m}_t \tau_x, \frac{\ell_{12} \xi^1}{\varepsilon_{12}} d_y + \frac{\ell_{12} \eta^1}{\varepsilon_{12}} \tilde{m}_s d_y + \bar{\kappa}_\varepsilon \tilde{m}_t \tau_y \right), \\ \nabla u_2^\varepsilon &= \left(\frac{\ell_{12} \xi^2}{\varepsilon_{12}} d_x + \frac{\ell_{12} \eta^2}{\varepsilon_{12}} \tilde{m}_s d_x + \bar{\kappa}_\varepsilon \tilde{m}_t \tau_x, \frac{\ell_{12} \xi^2}{\varepsilon_{12}} d_y + \frac{\ell_{12} \eta^2}{\varepsilon_{12}} \tilde{m}_s d_y + \bar{\kappa}_\varepsilon \tilde{m}_t \tau_y \right), \end{aligned}$$

where \tilde{m}_s, \tilde{m}_t denote the partial derivatives of \tilde{m} with respect to $s = \frac{d(x, y) - d_1}{\varepsilon_{12}} \ell_{12}$ and $t = \bar{\kappa}_\varepsilon(\tau(x, y) - \delta_\varepsilon)$ respectively, and are evaluated at $\left(\frac{d(x, y) - d_1}{\varepsilon_{12}} \ell_{12}, \bar{\kappa}_\varepsilon(\tau(x, y) - \delta_\varepsilon) \right)$. Hence

$$|\nabla u_1^\varepsilon|^2 + |\nabla u_2^\varepsilon|^2 = \frac{\ell_{12}^2}{\varepsilon_{12}^2} |\nabla d|^2 + \frac{\ell_{12}^2}{\varepsilon_{12}^2} |\nabla d|^2 (\tilde{m}_s)^2 + \bar{\kappa}_\varepsilon^2 |\nabla \tau|^2 (\tilde{m}_t)^2 + \frac{2\ell_{12}}{\varepsilon_{12}} \bar{\kappa}_\varepsilon (\nabla d \cdot \nabla \tau) \tilde{m}_s \tilde{m}_t$$

where we have used $|\xi| = |\eta| = 1$ and $\xi_1 \eta_1 + \xi_2 \eta_2 = 0$. From (4.29) we have

$$|\nabla u_1^\varepsilon|^2 + |\nabla u_2^\varepsilon|^2 = \frac{\ell_{12}^2}{\varepsilon_{12}^2} + \frac{\ell_{12}^2}{\varepsilon_{12}^2} (\tilde{m}_s)^2 + \bar{\kappa}_\varepsilon^2 |\nabla \tau|^2 (\tilde{m}_t)^2. \quad (4.35)$$

Moreover

$$\left(\frac{\partial u_1^\varepsilon}{\partial x} \frac{\partial u_2^\varepsilon}{\partial y} - \frac{\partial u_1^\varepsilon}{\partial y} \frac{\partial u_2^\varepsilon}{\partial x} \right)^2 = \frac{\ell_{12}^2}{\varepsilon_{12}^2} \bar{\kappa}_\varepsilon^2 (\tilde{m}_t)^2 (d_x \tau_y - d_y \tau_x)^2 (\xi^1 \eta^2 - \xi^2 \eta^1)^2 = \frac{\ell_{12}^2}{\varepsilon_{12}^2} \bar{\kappa}_\varepsilon^2 (\tilde{m}_t)^2 |\nabla \tau|^2, \quad (4.36)$$

where again \tilde{m}_s, \tilde{m}_t are evaluated at $\left(\frac{d(x, y) - d_1}{\varepsilon_{12}} \ell_{12}, \bar{\kappa}_\varepsilon(\tau(x, y) - \delta_\varepsilon) \right)$, and we have used (4.29), (4.30), and $\xi^1 \eta^2 - \xi^2 \eta^1 = 1$. Therefore from (4.35) and (4.36) we obtain

$$\begin{aligned} &1 + |\nabla u_1^\varepsilon|^2 + |\nabla u_2^\varepsilon|^2 + \left(\frac{\partial u_1^\varepsilon}{\partial x} \frac{\partial u_2^\varepsilon}{\partial y} - \frac{\partial u_1^\varepsilon}{\partial y} \frac{\partial u_2^\varepsilon}{\partial x} \right)^2 \\ &= 1 + \frac{\ell_{12}^2}{\varepsilon_{12}^2} \left(1 + (\tilde{m}_s)^2 + (\tilde{m}_t)^2 \bar{\kappa}_\varepsilon^2 \left(1 + \frac{\varepsilon_{12}^2}{\ell_{12}^2} \right) |\nabla \tau|^2 \right). \end{aligned}$$

As a consequence

$$\begin{aligned} \mathbb{A}(u^\varepsilon, S_{12}^\varepsilon) &= \frac{\ell_{12}}{\varepsilon_{12}} \int_{S_{12}^\varepsilon} \sqrt{1 + (\tilde{m}_s)^2 + (\tilde{m}_t)^2 \bar{\kappa}_\varepsilon^2 \left(1 + \frac{\varepsilon_{12}^2}{\ell_{12}^2} \right) |\nabla \tau|^2 + O(\varepsilon^2)} dx dy \\ &= \frac{1}{\bar{\kappa}_\varepsilon} \int_{\mathbb{R}_{12} \setminus P_\varepsilon} \frac{1}{|\nabla \tau|} \sqrt{1 + (\tilde{m}_s(s, t))^2 + (\tilde{m}_t(s, t))^2 \bar{\kappa}_\varepsilon^2 \left(1 + \frac{\varepsilon_{12}^2}{\ell_{12}^2} \right) |\nabla \tau|^2 + O(\varepsilon^2)} ds dt, \end{aligned} \quad (4.37)$$

where \tilde{m}_s, \tilde{m}_t in the first integral are evaluated at $\left(\frac{d(x,y)-d_1}{\varepsilon_{12}}\ell_{12}, \bar{\kappa}_\varepsilon(\tau(x,y) - \delta_\varepsilon)\right)$, $\nabla\tau$ in the second integral is evaluated at $(x,y) = \Phi^{-1}(s,t)$ and the last equality follows from the change of variables

$$\Phi : (x,y) \in \mathcal{N}_{12}^\varepsilon \rightarrow \left(\frac{d(x,y)-d_1}{\varepsilon_{12}}\ell_{12}, \bar{\kappa}_\varepsilon(\tau(x,y) - \delta_\varepsilon)\right) = (s,t) \in \mathbb{R}_{12}.$$

and $P_\varepsilon := \mathbb{R}_{12} \setminus \Phi(S_{12}^\varepsilon)$ (see Figure 5). Here one checks that $\Phi = H \circ h^{-1}$ with $H(d,\tau) = \left(\frac{d-d_1}{\varepsilon_{12}}\ell_{12}, \bar{\kappa}_\varepsilon(\tau - \delta_\varepsilon)\right)$ so that, using (4.30), the Jacobian of the change of variable is $\frac{1}{|\nabla\tau(\Phi^{-1}(s,t))|} \frac{\varepsilon_{12}}{\ell_{12}\bar{\kappa}_\varepsilon}$. Hence, recalling (4.32) and that $\bar{\kappa}_\varepsilon \rightarrow 1$ as $\varepsilon \rightarrow 0^+$,

$$\lim_{\varepsilon \rightarrow 0^+} \mathbb{A}(u^\varepsilon, S_{12}^\varepsilon) = \int_{\mathbb{R}_{12}} \sqrt{1 + (\tilde{m}_s)^2 + (\tilde{m}_t)^2} dsdt. \quad (4.38)$$

Now, let us recall that $\tilde{m} = m_{12}^\sigma$ is the approximating function as in (4.3); it follows that

$$\int_{\mathbb{R}_{12}} \sqrt{1 + (\tilde{m}_s)^2 + (\tilde{m}_t)^2} dsdt = \mathfrak{A}_{12}(\Gamma) + O(\sigma). \quad (4.39)$$

Hence, employing the same construction in the strips S_{23}^ε and S_{31}^ε , and using (4.39) we obtain from a diagonal argument with $\sigma = \sigma_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0^+$,

$$\lim_{\varepsilon \rightarrow 0^+} \mathbb{A}(u^\varepsilon, S_{23}^\varepsilon \cup S_{31}^\varepsilon \cup S_{12}^\varepsilon) = \mathfrak{A}_{12}(\Gamma) + \mathfrak{A}_{23}(\Gamma) + \mathfrak{A}_{31}(\Gamma). \quad (4.40)$$

Step 2. Definition of u^ε on T^ε . This is identical to Step 3 of the proof of Proposition 4.4 and therefore $\{u^\varepsilon\} \subset \text{Lip}(B_r; \mathbb{R}^2)$ and (4.23) holds. Following the same computations of Proposition 4.4 the conclusion follows.

Step 3. For the case where two of the curves $C_{ij}, ij \in \{12, 23, 31\}$ meet at Q with an angle larger than or equal to π we replace T^ε with H^ε defined in case 2 of Proposition 4.4, in the above construction. \square

5 Existence of minimizers for the functional \mathcal{G}

Let D be an open disk centered at the origin such that E_1, E_2, E_3 are circular sectors with 120° angles and let T be an equilateral triangle. Let p be the barycenter of T and $\tilde{\Gamma}_i$ be the segment connecting α_i and p , $i \in \{1, 2, 3\}$. Hence $\tilde{\Gamma} = (\tilde{\Gamma}_1, \tilde{\Gamma}_2, \tilde{\Gamma}_3) \in X_{\text{Lip}}$ so that

$$\inf \{ \mathcal{G}(\Gamma) : \Gamma \in X_{\text{Lip}} \} \leq \mathcal{G}(\tilde{\Gamma}).$$

Moreover we have

$$|D| + \mathcal{G}(\tilde{\Gamma}) = \mathcal{A}(u, D) \leq |D| + \inf \{ \mathcal{G}(\Gamma) : \Gamma \in X_{\text{Lip}} \},$$

where $u = u_{\text{symm}}$ (see Section 1), and the equality follows from [13, Section 3] and the inequality follows from Proposition 4.4. Thus

$$\mathcal{G}(\tilde{\Gamma}) = \min \{ \mathcal{G}(\Gamma) : \Gamma \in X_{\text{Lip}} \}.$$

Hence in this symmetric situation the optimal connection is obtained through the Steiner graph connecting α_1 , α_2 and α_3 . This motivates the analysis of this section, which is carried on without symmetry assumptions.

We recall that given a connection $\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3) \in X$ we denote by $\varphi_{ij} = \varphi_{ij}(\Gamma_{ij}) : [0, \ell_{ij}] \rightarrow \mathbb{R}$ the function whose graph is $\Gamma_{ij} = \Gamma_i \cup \Gamma_j$ (see (3.2)).

Definition 5.1 (Convergence in X). *We say that a sequence $\{\Gamma^n\} \subset X$ converges to $\Gamma \in X$ in X , and we write $\Gamma^n \rightarrow \Gamma$ in X , if*

$$\varphi_{ij}(\Gamma_{ij}^n) \rightarrow \varphi_{ij}(\Gamma_{ij}) \quad \text{in } L^1([0, \ell_{ij}]), \quad ij \in \{12, 23, 31\}. \quad (5.1)$$

5.1 Density and approximation

We start to show that a BV connection $\Gamma \in X$ can be approximated by Lipschitz connections; the difficulty is to keep graphicality of each branch of the approximating connections with respect to the two corresponding edges of T at the same time.

Recall that Γ_i is the branch of the connection Γ connecting α_i to p and that by Definition 3.1 we have

$$\Gamma_i \llcorner T \setminus \overline{p\pi_{ij}(p)} \cup \overline{p\pi_{ki}(p)} = \Gamma_{\varphi_{ij} \llcorner [0, w_{ij}]} = \Gamma_{\varphi_{ki} \llcorner [w_{ki}, \ell_{ki}]}$$

Note that we excluded the vertical parts over the points $\pi_{ij}(p)$, $ij \in \{12, 23, 31\}$, due to Remark 3.2; however we still have

$$\Gamma_i \cup \Gamma_j = \Gamma_{\varphi_{ij}}.$$

Lemma 5.2 (Piecewise linear approximation). *For any $\Gamma \in X$ with target triple point $p \in T$ there exists a sequence $\{\Gamma^n\} \subset X_{\text{Lip}}$ of connections with target triple point p such that $\varphi_{ij}(\Gamma_{ij}^n)$, $ij \in \{12, 23, 31\}$, is a piecewise linear¹ function,*

$$\mathcal{H}^1(\Gamma_{ij}^n) \leq \mathcal{H}^1(\Gamma_{ij})$$

and

$$\Gamma^n \rightarrow \Gamma \quad \text{in } X. \quad (5.2)$$

Proof. Let $ij = 12$ and let w_{12} be defined as in (3.1). Let $n_{12} := (0, 1) \in \mathbb{R}^2$ be the inward unit normal to $\overline{\alpha_1\alpha_2}$, $n_{31} := (\alpha, \beta)$ be the inward unit normal to $\overline{\alpha_3\alpha_1}$, and $\nu(\bar{s}) := (\nu^1(\bar{s}), \nu^2(\bar{s}))$ be the generalized outward unit normal at the point $(\bar{s}, \varphi_{12}(\bar{s}))$ to the generalized graph $\Gamma_{\varphi_{12} \llcorner [0, w_{12}]}$ of $\varphi_{12} \llcorner [0, w_{12}]$ (for all \bar{s} where it exists), see Figure 9. Without loss of generality we may assume $\Gamma_1 = \Gamma_{\varphi_{12} \llcorner [0, w_{12}]}$. We start to show that φ_{12} cannot have too negative slope, otherwise Γ_1 loses graphicality with respect to $\overline{\alpha_3\alpha_1}$.

Step 1. We claim that

$$\varphi'_{12} \llcorner [0, w_{12}] \geq \frac{\beta}{\alpha}$$

in the sense of measures, *i.e.*,

$$\varphi'_{12}(B) \geq \frac{\beta}{\alpha} \mathcal{L}^1(B), \quad \forall B \subseteq [0, w_{12}] \text{ Borel set.} \quad (5.3)$$

¹This means that it is Lipschitz piecewise linear with at most finitely many points of nondifferentiability.

From the graphicality with respect to $\overline{\alpha_1 w_{31}}$ we have, for all \bar{s} where $\nu(\bar{s})$ exists,

$$\nu(\bar{s}) \cdot n_{31} \leq 0. \quad (5.4)$$

Set

$$\begin{aligned} I^r &:= \{\bar{s} \in [0, w_{12}] : \nu(\bar{s}) \text{ is defined, and } \nu^2(\bar{s}) > 0\}, \\ I^s &:= \{\bar{s} \in [0, w_{12}] : \nu(\bar{s}) \text{ is defined, and } \nu^2(\bar{s}) = 0\}, \end{aligned}$$

note that $\nu(\bar{s}) = \frac{1}{\sqrt{1+(\dot{\varphi}_{12}(\bar{s}))^2}}(-\dot{\varphi}_{12}(\bar{s}), 1)$ for any $\bar{s} \in I^r$. From [12, Thm. 7 p. 301 and Thm. 5 p. 379], we have

$$\dot{\varphi}_{12} d\bar{s} = \varphi'_{12} \llcorner I^r, \quad \dot{\varphi}_{12}^{(j)} + \dot{\varphi}_{12}^{(c)} = \varphi'_{12} \llcorner I^s. \quad (5.5)$$

From (5.4) it follows that

$$\nu(\bar{s}) = (-1, 0) \quad \forall \bar{s} \in I^s \quad \text{and} \quad \dot{\varphi}_{12}(\bar{s}) \geq \frac{\beta}{\alpha} \quad \forall \bar{s} \in I^r. \quad (5.6)$$

From [12, Thm. 4, p. 378] we have

$$\dot{\varphi}_{12}^{(j)} + \dot{\varphi}_{12}^{(c)} = -\nu^1 |\mu| \llcorner I^s = |\mu| \llcorner I^s,$$

where $\mu := (\varphi'_{12}, -\mathcal{L}^1) = (-\nu^1, -\nu^2) |\mu|$ and the second equality follows from the first formula in (5.6).

For any Borel set $B \subseteq [0, w_{12}]$ we deduce

$$\varphi'_{12}(B) = \int_B \dot{\varphi}_{12} d\bar{s} + \dot{\varphi}_{12}^{(j)}(B) + \dot{\varphi}_{12}^{(c)}(B) \geq \frac{\beta}{\alpha} \mathcal{L}^1(B) + |\mu| \llcorner I^s(B) \geq \frac{\beta}{\alpha} \mathcal{L}^1(B).$$

Step 2. Given $\epsilon \in (0, 1)$, we choose $n = n(\epsilon) \in \mathbb{N}$ and points

$$\xi_0 = 0 < \xi_1 < \cdots < \xi_{n-1} < \xi_n = w_{12},$$

such that each ξ_i , $i \in \{1, \dots, n-1\}$, is a point of continuity of φ_{12} , and if we define $\varphi^n \in \text{Lip}([0, w_{12}])$ as the piecewise linear interpolation with

$$\varphi^n(\xi_i) = \varphi_{12}(\xi_i), \quad i = 0, \dots, n,$$

then

$$\|\varphi^n - \varphi_{12}\|_{L^1((0, w_{12}))} < \epsilon.$$

The graph of φ^n may still have vertical parts over $\overline{w_{31} \alpha_1}$. Indeed from [2, Theorem 3.30], and the fact that ξ_i are continuity points of φ_{12} we have

$$(\varphi^n)'(\xi) = \frac{\varphi_{12}(\xi_i) - \varphi_{12}(\xi_{i-1})}{\xi_i - \xi_{i-1}} = \frac{\varphi'_{12}(\xi_{i-1}, \xi_i)}{\xi_i - \xi_{i-1}} \geq \frac{\beta}{\alpha}, \quad \xi \in (\xi_{i-1}, \xi_i), \quad (5.7)$$

and equality may hold, hence the graph of φ^n over $\overline{\pi_{31}(p) \alpha_1}$ may have finitely many vertical parts. It is now sufficient to repeat the argument with φ^n in place of φ_{12} , choosing a suitable partition of $[w_{31}, \ell_{31}]$, so to ensure that (out of finitely many points)

$$(\varphi^n)' > \frac{\beta}{\alpha} \quad \text{on } [0, w_{12}].$$

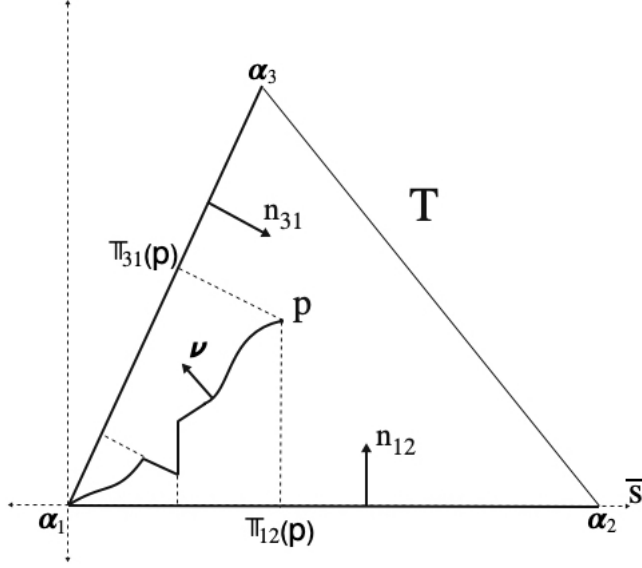


Figure 9: For convenience we choose $\alpha_1 = 0$.

In this way φ^n is a Lipschitz graph also with respect to $\overline{\pi_{31}(p)\alpha_1}$.

Step 3. We have

$$\begin{aligned}
\mathcal{H}^1(\Gamma_{\varphi^n}) &= \sum_{i=1}^n \int_{\xi_{i-1}}^{\xi_i} \sqrt{1 + |\varphi^{n'}(s)|^2} ds = \sum_{i=1}^n |(\xi_i, \varphi^n(\xi_i)) - (\xi_{i-1}, \varphi^n(\xi_{i-1}))| \\
&\leq \sup \left\{ \sum_{i=1}^m |(\eta_i, \varphi_{12}(\eta_i)) - (\eta_{i-1}, \varphi_{12}(\eta_{i-1}))| : m \in \mathbb{N}, \eta_0 = 0 < \eta_1 < \dots < \eta_{m-1} < \eta_m = w_{12} \right\} \quad (5.8) \\
&= \int_{[0, w_{12}]} |\Phi'_{12}| = \mathcal{H}^1(\Gamma_{\varphi_{12} \llcorner [0, w_{12}]}) ,
\end{aligned}$$

where $\Phi_{12} \in \text{BV}([0, w_{12}]; \mathbb{R}^2)$ is defined as $\Phi_{12}(\xi) := (\xi, \varphi_{12}(\xi))$, and the last equality follows from [2, (3.24), p.136].

Step 4. Define

$$\Gamma_1^n := \Gamma_{\varphi^n}.$$

Similarly we define Γ_2^n and Γ_3^n , and we set $\Gamma_{ij}^n := \Gamma_i^n \cup \Gamma_j^n$. Then $\Gamma^n := (\Gamma_1^n, \Gamma_2^n, \Gamma_3^n)$, satisfies the required properties. \square

Proposition 5.3 (Uniform estimate of the length). *There exists $c > 0$ such that for all $\Gamma \in X$ we have*

$$\mathcal{H}^1(\Gamma_{ij}) \leq c, \quad ij \in \{12, 23, 31\}. \quad (5.9)$$

Proof. Let $\Gamma \in X$ be a connection through $p \in T$. Without loss of generality we may assume that

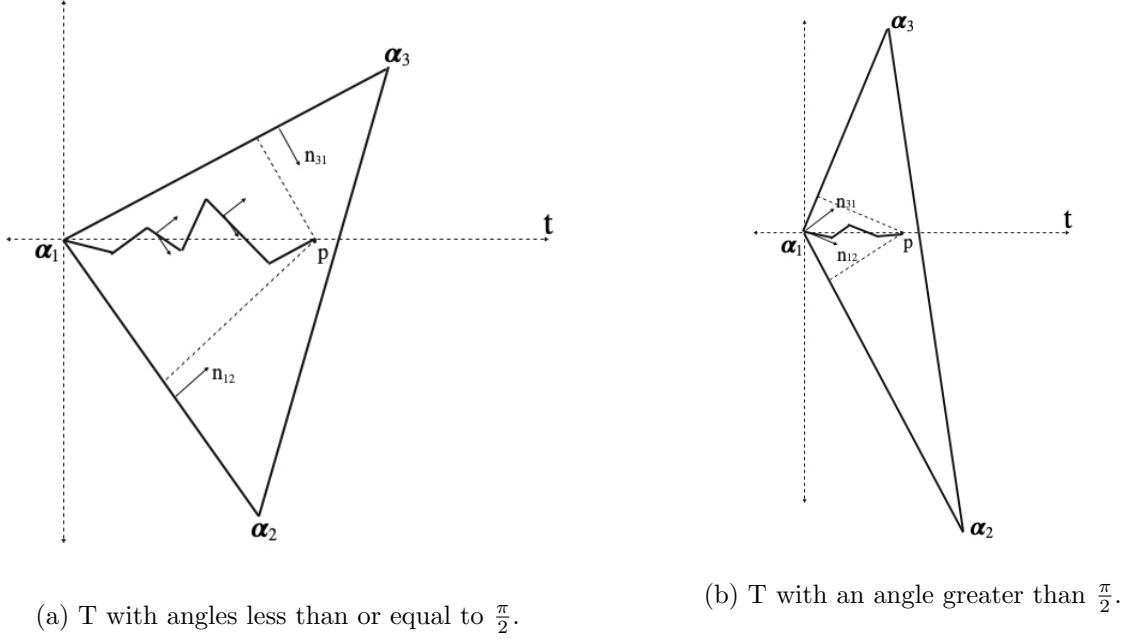


Figure 10

$p \neq \alpha_1$. From (5.8) we have

$$\mathcal{H}^1(\Gamma_1) = \sup \left\{ \sum_{i=1}^m |(\eta_i, \varphi_{12}(\eta_i)) - (\eta_{i-1}, \varphi_{12}(\eta_{i-1}))| : m \in \mathbb{N}, \eta_0 = 0 < \eta_1 < \dots < \eta_{m-1} < \eta_m = w_{12} \right\}. \quad (5.10)$$

Choose a partition

$$\xi_0 = 0 < \xi_1 < \dots < \xi_{h-1} < \xi_h = w_{12}.$$

Let Γ_1^h be the piecewise linear interpolation connecting $(\xi_{i-1}, \varphi_{12}(\xi_{i-1}))$ and $(\xi_i, \varphi_{12}(\xi_i))$, $i \in \{1, \dots, h\}$. The unit tangent to Γ_1^h is enclosed in the angle formed by n_{12} and n_{31} , the unit normals to $\overline{\alpha_1 \alpha_2}$ and $\overline{\alpha_3 \alpha_1}$ (due to the graphicality condition with respect to $\overline{\alpha_1 \alpha_2}$ and $\overline{\alpha_3 \alpha_1}$), see Figure 10. It follows that Γ_1^h is the graph of a function ϕ_{12}^h over the segment $\overline{\alpha_1 p}$. Fix a Cartesian coordinate system in which the t -axis is the line $\overline{\alpha_1 p}$ and the origin is α_1 . For any $t \in [0, |\alpha_1 - p|]$ (up to a finite set) let $\tau(t)$ be the unit tangent to Γ_1^h at $(t, \phi_{12}^h(t))$ and let $n = (1, 0)$ and $n^\perp = (0, 1)$. Hence $\phi_{12}^h{}' = \frac{\tau \cdot n^\perp}{\tau \cdot n}$ satisfies

$$c_1^- := \frac{n_{31} \cdot n^\perp}{n_{31} \cdot n} \leq \phi_{12}^h{}' \leq \frac{n_{12} \cdot n^\perp}{n_{12} \cdot n} =: c_1^+.$$

Note that one between $|c_1^-|$ and $|c_1^+|$ might be $+\infty$, since one of the sides $\overline{\alpha_1 \alpha_2}$ or $\overline{\alpha_3 \alpha_1}$ can be horizontal (this happens only if the point p is on one side of the triangle). However we always have that $c_1^- \leq 0$, $c_1^+ \geq 0$. Furthermore, when the angle $\hat{\alpha}_1$ in α_1 is less or equal to $\frac{\pi}{2}$, it follows that $\tilde{c}_1 := \min\{|c_1^-|, |c_1^+|\} \leq |\tan(\frac{\pi}{2} - \hat{\alpha}_1)|$. In the case that $\hat{\alpha}_1 > \frac{\pi}{2}$, thanks to the fact that $p \in T_{int}$, we have $\max\{|c_1^-|, |c_1^+|\} \leq |\tan(\pi - \hat{\alpha}_1)|$. Thus the only difficulty to prove that the length of Γ_1^h is controlled when $\hat{\alpha}_1 \leq \frac{\pi}{2}$. So let us assume this and in addition that $|c_1^-| = \tilde{c}_1$ (the other case is

similar). Since $\phi_{12}^h(|\alpha_1 - p|) = \phi_{12}^h(0) = 0$ we have

$$0 = \phi_{12}^h{}'([0, |\alpha_1 - p|]) = (\phi_{12}^h{}')^+([0, |\alpha_1 - p|]) - (\phi_{12}^h{}')^-([0, |\alpha_1 - p|]),$$

where $(\phi_{12}^h{}')^+$ and $(\phi_{12}^h{}')^-$ are the positive and negative parts of the measure $\phi_{12}^h{}' = \dot{\phi}_{12}^h dt$, thus we estimate

$$\begin{aligned} |\mathcal{H}^1(\Gamma_1^h)| &= \int_0^{|\alpha_1 - p|} \sqrt{1 + \dot{\phi}_{12}^h(t)^2} dt \\ &\leq |\alpha_1 - p| + |\phi_{12}^h{}'|([0, |\alpha_1 - p|]) = |\alpha_1 - p| + 2(\phi_{12}^h{}')^-([0, |\alpha_1 - p|]) \\ &\leq |\alpha_1 - p| + 2|\alpha_1 - p|\tilde{c}_1. \end{aligned} \quad (5.11)$$

Defining c_1 as the right-hand side of the last inequality we see that c_1 is a positive constant depending only on the geometry of T .

From the (5.11) and (5.10) it then follows

$$\mathcal{H}^1(\Gamma_1) \leq c_1. \quad (5.12)$$

Similarly we may show that $\mathcal{H}^1(\Gamma_2) \leq c_2$ and $\mathcal{H}^1(\Gamma_3) \leq c_3$ for $c_2, c_3 > 0$ depending only on T . This proves (5.9) with $c = c_1 + c_2 + c_3$. \square

The next lemma shows continuity of the sum of the three areas of area minimizing surfaces defining \mathcal{G} in (3.9), with respect to the L^1 convergence of the traces in T .

Proposition 5.4 (Continuity of \mathcal{G}). *Let $\Gamma \in X$, and let $\{\Gamma^n\} \subset X$ be a sequence converging to Γ in X . Then*

$$\lim_{n \rightarrow +\infty} \mathcal{G}(\Gamma^n) = \mathcal{G}(\Gamma). \quad (5.13)$$

Proof. Since $\Gamma \in X$ and $\{\Gamma^n\} \subset X$ we have $\varphi_{ij} \in \text{BV}([0, \ell_{ij}])$ and $\{\varphi_{ij}^n\} \subset \text{BV}([0, \ell_{ij}])$ where $\varphi_{ij} := \varphi_{ij}(\Gamma_{ij})$, $\varphi_{ij}^n := \varphi_{ij}(\Gamma_{ij}^n)$.

Hence from (1.6) and Section 3.1 it follows that there exist $\hat{m}_{ij}, \hat{m}_{ij}^n \in W^{1,1}(\hat{R}_{ij})$ such that

$$\begin{aligned} 2\mathfrak{A}_{ij}^n &= \int_{\hat{R}_{ij}} \sqrt{1 + |\nabla \hat{m}_{ij}^n|^2} ds dt \\ &= \min \left\{ \int_{\hat{R}_{ij}} \sqrt{1 + |Df|^2} + \int_{\partial \hat{R}_{ij}} |f - \varphi_{ij}^n| d\mathcal{H}^1 : f \in \text{BV}(B), f = \varphi_{ij}^n \text{ on } B \setminus \hat{R}_{ij} \right\}, \end{aligned} \quad (5.14)$$

$$\begin{aligned} 2\mathfrak{A}_{ij} &= \int_{\hat{R}_{ij}} \sqrt{1 + |\nabla \hat{m}_{ij}|^2} ds dt \\ &= \min \left\{ \int_{\hat{R}_{ij}} \sqrt{1 + |Df|^2} + \int_{\partial \hat{R}_{ij}} |f - \varphi_{ij}| d\mathcal{H}^1 : f \in \text{BV}(B), f = \varphi_{ij} \text{ on } B \setminus \hat{R}_{ij} \right\}, \end{aligned} \quad (5.15)$$

where we recall that \hat{R}_{ij} is the double rectangle defined in (3.5) and $\varphi_{ij}, \varphi_{ij}^n$ are extended on a disk B containing \hat{R}_{ij} as in Section 3.1.

Define \tilde{m}_{ij}^n and \tilde{m}_{ij} as

$$\tilde{m}_{ij}^n := \begin{cases} \hat{m}_{ij}^n & \text{in } \hat{R}_{ij}, \\ \varphi_{ij} & \text{in } B \setminus \hat{R}_{ij}, \end{cases} \quad \tilde{m}_{ij} := \begin{cases} \hat{m}_{ij} & \text{in } \hat{R}_{ij}, \\ \varphi_{ij}^n & \text{in } B \setminus \hat{R}_{ij}, \end{cases}$$

so that $\tilde{m}_{ij}^n, \tilde{m}_{ij} \in \text{BV}(B)$. Since \tilde{m}_{ij}^n is competitor in (5.15) and \tilde{m}_{ij} is competitor in (5.14) we have, recalling also the discussion leading to (3.8),

$$\begin{aligned} 2\mathfrak{A}_{ij} &\leq \int_{\widehat{\mathbb{R}}_{ij}} \sqrt{1 + |\nabla \widehat{m}_{ij}^n|^2} \, dsdt + \int_{\partial \widehat{\mathbb{R}}_{ij}} |\varphi_{ij}^n - \varphi_{ij}| d\mathcal{H}^1 = 2\mathfrak{A}_{ij}^n + \int_{\partial \widehat{\mathbb{R}}_{ij}} |\varphi_{ij}^n - \varphi_{ij}| d\mathcal{H}^1, \\ 2\mathfrak{A}_{ij}^n &\leq \int_{\widehat{\mathbb{R}}_{ij}} \sqrt{1 + |\nabla \widehat{m}_{ij}|^2} \, dsdt + \int_{\partial \widehat{\mathbb{R}}_{ij}} |\varphi_{ij} - \varphi_{ij}^n| d\mathcal{H}^1 = 2\mathfrak{A}_{ij} + \int_{\partial \widehat{\mathbb{R}}_{ij}} |\varphi_{ij} - \varphi_{ij}^n| d\mathcal{H}^1. \end{aligned}$$

Thus

$$|2\mathfrak{A}_{ij}^n - 2\mathfrak{A}_{ij}| \leq \int_{\partial \widehat{\mathbb{R}}_{ij}} |\varphi_{ij}^n - \varphi_{ij}| d\mathcal{H}^1. \quad (5.16)$$

Recall that m_{ij}^n (resp. m_{ij}) is the restriction of \widehat{m}_{ij}^n (resp. \widehat{m}_{ij}) to \mathbb{R}_{ij} . Hence, from (3.9), (5.1) and (5.16), (5.13) follows. \square

Corollary 5.5. *We have*

$$\inf \{ \mathcal{G}(\Gamma) : \Gamma \in X \} = \inf \{ \mathcal{G}(\Gamma) : \Gamma \in X_{\text{Lip}} \}. \quad (5.17)$$

5.2 Compactness of the class X

The aim of this section is to show that the infimum in (5.17) is attained. To do this we need the following result.

Theorem 5.6 (Compactness). *Any sequence $\{\Gamma^n\} \subset X$ admits a subsequence converging in X to some $\Gamma \in X$.*

Remark 5.7. In Definition 5.1 it is required convergence of $\{\Gamma^n\}$ to Γ in L^1 . For this reason, if Γ^n has target triple point p_n , it is not guaranteed that the point $b := \lim_{n \rightarrow +\infty} p_n$ (it exists up to subsequences) still belongs to Γ_{ij} for all ij , see Figures 12 and 15a. As a consequence, if $\{\Gamma^n\}$ converges to Γ it is not true, in general, that $p_n \rightarrow p$, where p is the target triple point of Γ .

Proof. Let $\{\Gamma^n\} \subset X$ and $\varphi_{ij}^n = \varphi_{ij}(\Gamma_{ij}^n)$, $ij \in \{12, 23, 31\}$. From Proposition 5.3 $\{\varphi_{ij}^n\}$ is uniformly bounded in $\text{BV}([0, \ell_{ij}])$ for any $ij \in \{12, 23, 31\}$. Thus, up to a not relabelled subsequence, there exists $\varphi_{ij} \in \text{BV}([0, \ell_{ij}])$ such that

$$\varphi_{ij}^n \rightarrow \varphi_{ij} \quad \text{in } L^1((0, \ell_{ij})) \text{ and pointwise a.e.,} \quad (5.18)$$

$$(\varphi_{ij}^n)' \rightharpoonup \varphi_{ij}' \quad \text{weakly* as measures.} \quad (5.19)$$

We shall adopt our usual convention

$$\varphi_{ij}^n(0_-) = \varphi_{ij}(0_-) = \varphi_{ij}^n(\ell_{ij+}) = \varphi_{ij}(\ell_{ij+}) = 0, \quad \varphi_{ij}^n = \varphi_{ij+}^n, \quad \varphi_{ij} = \varphi_{ij+}, \quad \varphi = \varphi_+. \quad (5.20)$$

Denote by $\Gamma_{ij} \subset \mathbb{R}^2$ the limit graph over (the closed segment) $\overline{\alpha_i \alpha_j}$ that we identify with the generalized graph of φ_{ij} over $[0, \ell_{ij}]$. Since T is closed and convex we have $\Gamma_{ij} \subset T$; moreover, by construction, α_i and α_j are the endpoints of Γ_{ij} . Notice that if we assume that T is acute, this excludes the presence of vertical parts over its vertices.

It remains to prove that the three obtained curves Γ_{ij} , $ij \in \{12, 23, 31\}$, form a BV connection; in particular that they intersect mutually in a unique well-defined point.

We claim that

there exists a unique $p \in \bigcap_{ij} \Gamma_{ij}$ that divides each Γ_{ij} into two curves Γ_{ij}^l and Γ_{ij}^r such that

$$\Gamma_{ij}^l = \Gamma_{ki}^r, \quad ij, ki \in \{12, 23, 31\}, \quad ij \neq ki.$$

Let us denote by $\tilde{\varphi}_{ij}^n$ the extension to \mathbb{R} of the function φ_{ij}^n vanishing in $(-\infty, 0) \cup (\ell_{ij}, +\infty)$. Similarly $\tilde{\varphi}_{ij}$ is the extension of φ_{ij} vanishing in $(-\infty, 0) \cup (\ell_{ij}, +\infty)$. Consider the sequence $\{\llbracket SG_{\tilde{\varphi}_{ij}^n} \rrbracket\}_n \subset \mathcal{D}_2(\mathbb{R}^2)$ of 2-currents regarded in \mathbb{R}^2 and the 2-current $\llbracket SG_{\tilde{\varphi}_{ij}} \rrbracket$. Their boundaries are the currents carried by the graphs of $\tilde{\varphi}_{ij}^n$ and $\tilde{\varphi}_{ij}$, as defined in Theorem 2.2. The 1-currents carried by the graph of φ_{ij}^n and φ_{ij} , by convention (5.20), coincide with the restrictions of $\partial \llbracket SG_{\tilde{\varphi}_{ij}^n} \rrbracket$ and $\partial \llbracket SG_{\tilde{\varphi}_{ij}} \rrbracket$ to the closed set $[0, \ell_{ij}] \times \mathbb{R}$. Namely, if we denote by

$$\llbracket \Gamma_{ij}^n \rrbracket := \partial \llbracket SG_{\tilde{\varphi}_{ij}^n} \rrbracket \llcorner [0, \ell_{ij}] \times \mathbb{R}, \quad \llbracket \Gamma_{ij} \rrbracket := \partial \llbracket SG_{\tilde{\varphi}_{ij}} \rrbracket \llcorner [0, \ell_{ij}] \times \mathbb{R},$$

then

$$\llbracket \Gamma_{ij}^n \rrbracket = \partial \llbracket SG_{\tilde{\varphi}_{ij}^n} \rrbracket - \mathcal{L}_{ij} \quad \text{and} \quad \llbracket \Gamma_{ij} \rrbracket = \partial \llbracket SG_{\tilde{\varphi}_{ij}} \rrbracket - \mathcal{L}_{ij}, \quad (5.21)$$

where \mathcal{L}_{ij} is the 1-current given by integration over the two halflines $(-\infty, 0) \times \{0\} \cup (\ell_{ij}, +\infty) \times \{0\}$. The curves Γ_{ij}^n and Γ_{ij} coincide with the support of $\llbracket \Gamma_{ij}^n \rrbracket$ and $\llbracket \Gamma_{ij} \rrbracket$, respectively.

We now prove our claim in three steps.

Step 1. The currents $\llbracket \Gamma_{ij}^n \rrbracket$ converge (up to a not relabelled subsequence) weakly in the sense of currents to $\llbracket \Gamma_{ij} \rrbracket$, *i.e.*,

$$\llbracket \Gamma_{ij}^n \rrbracket(\omega) \rightarrow \llbracket \Gamma_{ij} \rrbracket(\omega) \quad \forall \omega \in \mathcal{D}^1(\mathbb{R}^2). \quad (5.22)$$

Moreover

$$\mathcal{H}^1(\Gamma_{ij}) \leq c, \quad (5.23)$$

where $c > 0$ is the constant in (5.9).

Indeed, thanks to (5.18), the characteristic functions $\chi_{SG_{\varphi_{ij}^n}}$ converge to $\chi_{SG_{\varphi_{ij}}}$ in $L^1_{\text{loc}}(\mathbb{R}^2)$, hence

$$\llbracket SG_{\tilde{\varphi}_{ij}^n} \rrbracket \rightharpoonup \llbracket SG_{\tilde{\varphi}_{ij}} \rrbracket \quad \text{weakly as currents,}$$

since

$$\int_{SG_{\tilde{\varphi}_{ij}^n}} \hat{\omega}(s, t) ds dt \rightarrow \int_{SG_{\tilde{\varphi}_{ij}}} \hat{\omega}(s, t) ds dt \quad \forall \hat{\omega} \in \mathcal{D}^2(\mathbb{R}^2).$$

This implies

$$\partial \llbracket SG_{\tilde{\varphi}_{ij}^n} \rrbracket \rightharpoonup \partial \llbracket SG_{\tilde{\varphi}_{ij}} \rrbracket \quad \text{weakly in the sense of currents,}$$

and (5.22) follows from (5.21).

Finally (5.23) follows from Lemma 5.3 and the weak lower semicontinuity of the mass of currents, and the proof of step 1 is concluded.

It is not restrictive to assume that $w_{ij}^n = |\alpha_i - \pi_{ij}(p_n)|$ is a point of continuity of φ_{ij}^n for all $n \in \mathbb{N}$ and all $ij \in \{12, 23, 31\}$. Indeed given a sequence $\{\Gamma^n\} \subset X$ converging to Γ , from Lemma 5.2 for all n we can assign a sequence $\{\Gamma^{m,n}\} \subset X_{\text{Lip}}$ such that $\Gamma^{m,n} \rightarrow \Gamma^n$ as $m \rightarrow +\infty$. Thus by a diagonal argument, we find a sequence $\{\Gamma^{m(n),n}\} \subset X_{\text{Lip}}$ which tends to Γ and satisfies the above requirement (we can also assume that Γ^n is Lipschitz, but this will not be needed in the proof).

Without loss of generality (up to a not relabeled subsequence) we may further assume

$$p_n \rightarrow b \in \mathbb{T},$$

$\{w_{ij}^n\}$ is a monotone sequence, and

$$w_{ij}^n \rightarrow w_{ij} := |\alpha_i - \pi_{ij}(b)|, \quad ij \in \{12, 23, 31\}.$$

Before passing to the second step, it is convenient to divide the target triangle \mathbb{T} into various regions. Assume first that \mathbb{T} is acute. The point b , together with the heights

$$h_{ij} := \overline{\pi_{ij}(b)b}, \quad ij \in \{12, 23, 31\}, \quad (5.24)$$

divides \mathbb{T} into three regions \mathcal{P}_i , $i \in \{1, 2, 3\}$, as shown in Figure 11a; precisely, if $\overline{\mathcal{P}_i}$ denotes the closed region enclosed by h_{ij} , h_{ki} , $\alpha_i\pi_{ij}(b)$ and $\alpha_i\pi_{ki}(b)$, then \mathcal{P}_i is defined by

$$\mathcal{P}_i := \overline{\mathcal{P}_i} \setminus (h_{ij} \cup h_{ki}), \quad i = 1, 2, 3. \quad (5.25)$$

Similarly we define h_{ij}^n and \mathcal{P}_i^n by replacing b with p_n in (5.24) and (5.25).

Assume now that \mathbb{T} is not acute. Without loss of generality we may assume that the angle at α_1 is greater than $\frac{\pi}{2}$. The only difference here is with the definition of \mathcal{P}_1^n and \mathcal{P}_1 since each Γ_{ij} has to satisfy the graphicality condition with respect to $\overline{\alpha_i\alpha_j}$; hence we define $\overline{\mathcal{P}_1}$ as the closed quadrilateral bounded by h_{12} , h_{31} , m_{12} and m_{31} , where m_{12} and m_{31} are the normals to $\overline{\alpha_1\alpha_2}$ and $\overline{\alpha_3\alpha_1}$, respectively, passing through α_1 (see Figure 11b). Similarly we define $\overline{\mathcal{P}_1}^n$. Finally we set $\mathcal{P}_1 := \overline{\mathcal{P}_1} \setminus (h_{12} \cup h_{31})$ and $\mathcal{P}_1^n := \overline{\mathcal{P}_1}^n \setminus (h_{12} \cup h_{31})$.

Step 2. We will prove that we can decompose $\Gamma_{12} \cup \Gamma_{23} \cup \Gamma_{31}$ as three currents meeting at a point b . It is easy to see that the sets $\overline{\mathcal{P}_i}^n$ are converging to $\overline{\mathcal{P}_i}$ with respect to the Hausdorff distance. It is not true in general that Γ_{ij}^n is converging to Γ_{ij} with respect to the Hausdorff distance (see Figure 12); however, since

$$\Gamma_i^n = \Gamma_{ij}^n \cap \overline{\mathcal{P}_i}^n = \Gamma_{ki}^n \cap \overline{\mathcal{P}_i}^n, \quad \Gamma_i^n \subset \overline{\mathcal{P}_i}^n, \quad (5.26)$$

for all $ij, ki \in \{12, 23, 31\}$, $ij \neq ki$, it is readily seen that

$$\begin{aligned} \Gamma_{ij} &\subset \mathbb{T} \setminus \mathcal{P}_k, \\ \llbracket \Gamma_{ij} \rrbracket &= \llbracket \Gamma_{ij} \rrbracket \llcorner \mathcal{P}_i + \llbracket \Gamma_{ij} \rrbracket \llcorner \mathcal{P}_j + \llbracket \Gamma_{ij} \rrbracket \llcorner h_{ki} + \llbracket \Gamma_{ij} \rrbracket \llcorner h_{ij} + \llbracket \Gamma_{ij} \rrbracket \llcorner h_{jk}. \end{aligned} \quad (5.27)$$

For $i \in \{1, 2, 3\}$, the integral 1-current $\llbracket \Gamma_i^n \rrbracket = \llbracket \Gamma_{ij}^n \rrbracket \llcorner [0, w_{ij}] \times \mathbb{R}$ has boundary $\delta_{p_n} - \delta_{\alpha_i}$ in \mathbb{R}^2 . By the compactness theorem for integral currents [12, Theorem 2, p.141] there exists an integral current $\mathcal{T}_i \in \mathcal{D}_1(\mathbb{R}^2)$, $i = 1, 2, 3$, such that, up to a not relabeled subsequence,

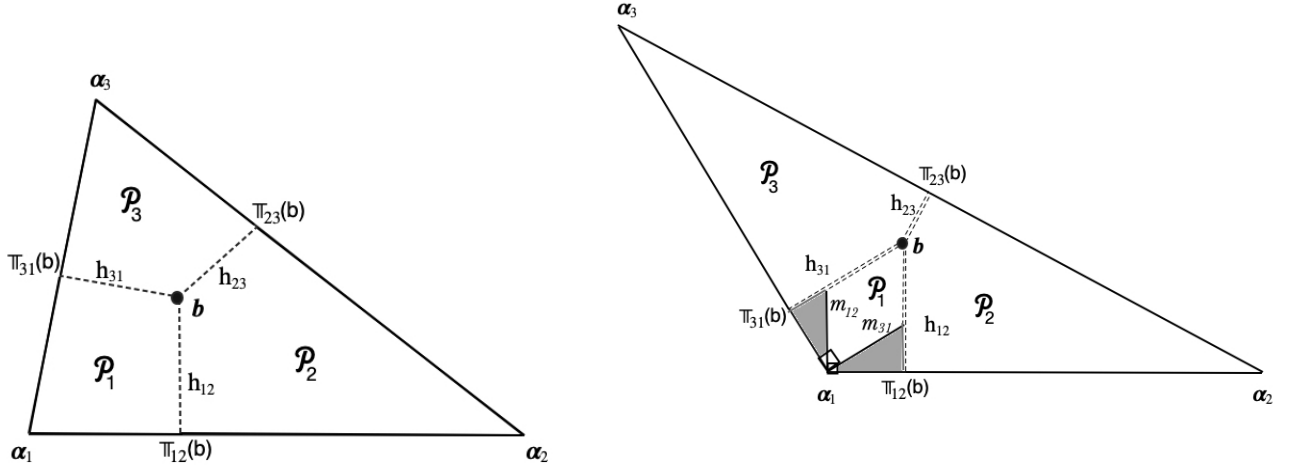
$$\llbracket \Gamma_i^n \rrbracket(\omega) \rightarrow \mathcal{T}_i(\omega) \quad \forall \omega \in \mathcal{D}^1(\mathbb{R}^2). \quad (5.28)$$

Clearly

$$\partial \mathcal{T}_i = \delta_b - \delta_{\alpha_i}. \quad (5.29)$$

From (5.26) and thanks to the convergence of $\overline{\mathcal{P}_i}^n$ to $\overline{\mathcal{P}_i}$ with respect to the Hausdorff distance, we infer

$$\text{spt } \mathcal{T}_i \subset \overline{\mathcal{P}_i}, \quad \text{hence } \mathcal{T}_i = \mathcal{T}_i \llcorner \mathcal{P}_i + \mathcal{T}_i \llcorner h_{ij} + \mathcal{T}_i \llcorner h_{ki}, \quad (5.30)$$



(a) Partitions of T into $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ and three segments. (b) Partitions of T minus the two grey triangles into $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ and three segments for a non acute triangle.

Figure 11

where $ij, ki \in \{12, 23, 31\}$. Note that \mathcal{T}_i is not necessarily equal to $[\Gamma_{ij}] \llcorner \overline{\mathcal{P}_i}$, due to a possible cancellation of a vertical part over $\pi_{ij}(b)$, $ij \in \{12, 23, 31\}$ (that is, on h_{ij}), see Figure 12. However from $[\Gamma_{ij}^n] = [\Gamma_i^n] + [\Gamma_j^n]$ and (5.28) we have

$$[\Gamma_{ij}] = \mathcal{T}_i - \mathcal{T}_j, \quad ij \in \{12, 23, 31\}, \quad (5.31)$$

as currents in \mathbb{R}^2 . Notice that \mathcal{T}_i and \mathcal{T}_j have multiplicity one, and in (5.31) they contribute with opposite orientation. This allows, if necessary, to identify \mathcal{T}_i , $i = 1, 2, 3$, with its support. Note also

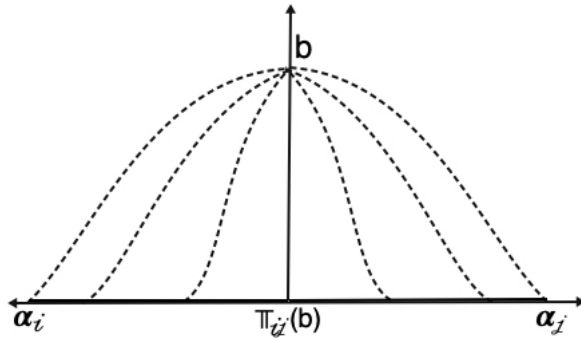


Figure 12: In dots a sequence of graphs $[\Gamma_{ij}^n]$ of functions that pass through a fixed point $b \in T$. In bold the graph of the limit function (the horizontal segment) $[\Gamma_{ij}]$. The limit in the sense of currents of the left branches of the sequence $\{\Gamma_i^n\}$ is $\overline{\alpha_i \pi_{ij}(b)} \cup \overline{\pi_{ij}(b) b}$ while the limit of the right branches $\{\Gamma_j^n\}$ is $\overline{\pi_{ij}(b) b} \cup \overline{\pi_{ij}(b) \alpha_j}$.

that \mathcal{T}_i may have vertical part over α_i , see Figure 13. Now, since $[\Gamma_i^n]$ is Cartesian with respect to

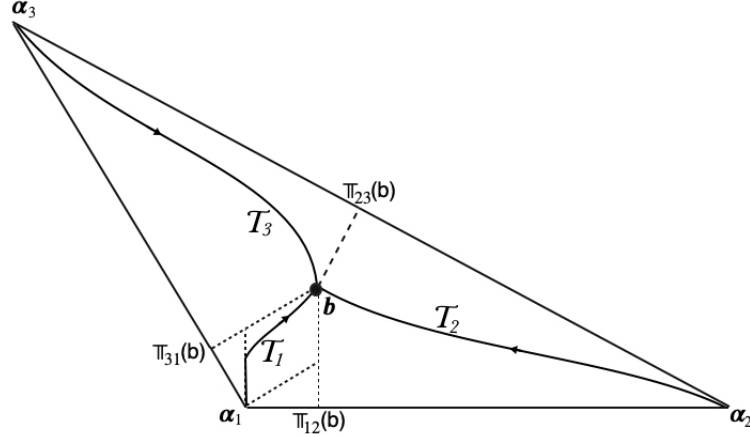


Figure 13

both the edges $\overline{\alpha_i \alpha_j}$ and $\overline{\alpha_k \alpha_i}$, from (5.28) it follows that $\mathcal{T}_i \llcorner \mathcal{P}_i$ is part of two generalized graphs over the same edges, *i.e.*,

$$\mathcal{T}_i \llcorner \mathcal{P}_i = \llbracket \Gamma_{ij} \rrbracket \llcorner \mathcal{P}_i = -\llbracket \Gamma_{ki} \rrbracket \llcorner \mathcal{P}_i. \quad (5.32)$$

Moreover, we infer that \mathcal{T}_i cannot have vertical part over h_{ij} and h_{ki} at the same time; in other words once the current \mathcal{T}_i touches one of the heights h_{ij} or h_{ki} it stays there until it reaches b , and \mathcal{T}_i cannot have a nonempty support in more than one height, see Figures 14b-15b. We conclude the following statement:

- (A) The supports of the three currents \mathcal{T}_i , $i = 1, 2, 3$, have as common point b . Moreover, if there are $i \neq j$ such that the supports of \mathcal{T}_i and \mathcal{T}_j intersect in a point different from b , then this intersection occurs on the mutual height h_{ij} . Finally, if the supports of \mathcal{T}_i and \mathcal{T}_j intersect on h_{ij} outside b , then they intersect on a closed segment and the intersection of the supports of \mathcal{T}_i and \mathcal{T}_j with \mathcal{T}_k is only the point b .

Step 3. To conclude the proof of our claim we now analyse the possible cases arising from (A).

Case (i). Assume that the three supports of the currents \mathcal{T}_i , $i = 1, 2, 3$, intersect only at the point b . This includes the case

$$\mathcal{T}_i \llcorner h_{ij} = \mathcal{T}_j \llcorner h_{ij} = 0, \quad \text{for all } ij \in \{12, 23, 31\},$$

as in Figure 14a. But it may also happen that \mathcal{T}_i has vertical part over h_{ij} , provided that \mathcal{T}_j does not have vertical part over the same height (see for instance Figure 14b). In any case we may set

$$p := b, \quad \Gamma_{ij}^l = \mathcal{T}_i, \quad \Gamma_{ij}^r := -\mathcal{T}_j, \quad ij \in \{12, 23, 31\},$$

where we have identified the currents \mathcal{T}_i with their supports. By (5.29) and (5.32), the claim is achieved.

Case (ii). The second case which must be discussed is the one considering possible overlapping of the support of the currents \mathcal{T}_i . By condition (A) such overlapping, giving rise to cancellations, can

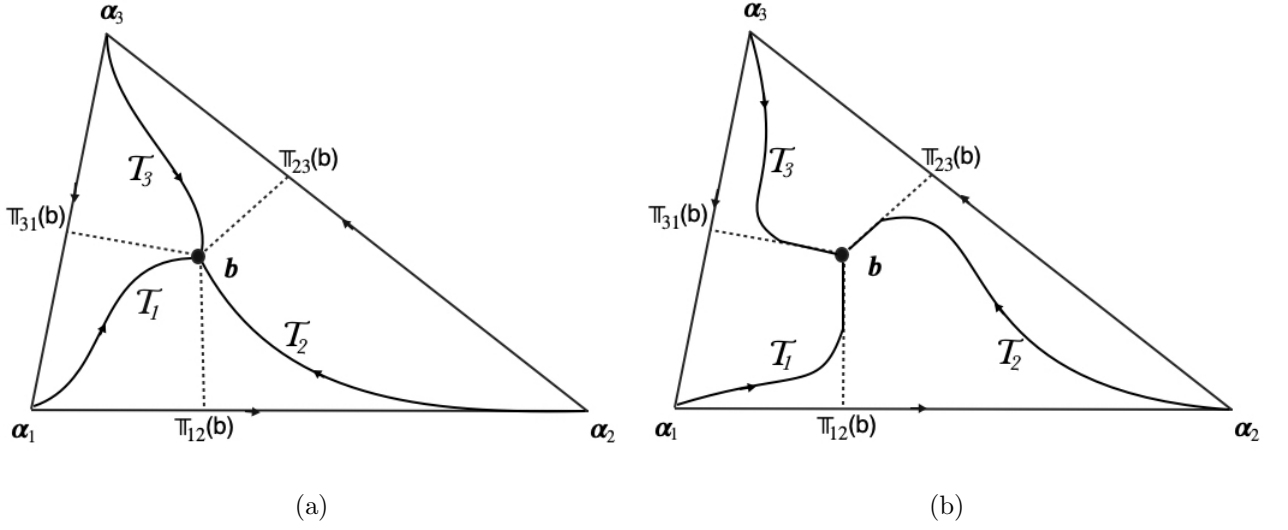


Figure 14: Case (i) of step 3 in the proof of Theorem 5.6.

occur only on one height h_{ij} . Hence, assume there exists one (and only one) $ij \in \{12, 23, 31\}$ such that

$$\mathcal{T}_i \llcorner h_{ij} \neq 0 \quad \text{and} \quad \mathcal{T}_j \llcorner h_{ij} \neq 0.$$

Thus we have $\mathcal{T}_i \llcorner h_{ki} = 0$ and $\mathcal{T}_j \llcorner h_{jk} = 0$.

First assume that $[\Gamma_{ij}] \llcorner h_{ij} = 0$, *i.e.*, φ_{ij} is continuous at w_{ij} . Then $\mathcal{T}_i \llcorner h_{ij} = \mathcal{T}_j \llcorner h_{ij}$, see Figure 15a. We set, identifying \mathcal{T}_i with its support,

$$\begin{aligned} p &:= \varphi_{ij}(w_{ij}), \\ \Gamma_{ij}^l &:= \mathcal{T}_i \llcorner \mathcal{P}_i, & \Gamma_{ij}^r &:= \mathcal{T}_j \llcorner \mathcal{P}_j, \\ \Gamma_{jk}^l &:= \mathcal{T}_j \llcorner \mathcal{P}_i, & \Gamma_{jk}^r &:= \mathcal{T}_k \cup \mathcal{T}_j \llcorner h_{ij}, \\ \Gamma_{ki}^l &:= \mathcal{T}_k \cup \mathcal{T}_i \llcorner h_{ij}, & \Gamma_{ki}^r &:= \mathcal{T}_i \llcorner \mathcal{P}_i. \end{aligned}$$

One checks that the connection built above is a BV graph type connection, addressing the claim. Now assume that

$$[\Gamma_{ij}] \llcorner h_{ij} \neq 0,$$

i.e., φ_{ij} jumps at w_{ij} . Thus either $\text{spt } [\Gamma_{ij}] \llcorner h_{ij} \subseteq \text{spt } \mathcal{T}_i \llcorner h_{ij}$ or $\text{spt } [\Gamma_{ij}] \llcorner h_{ij} \subseteq \text{spt } \mathcal{T}_j \llcorner h_{ij}$. Without loss of generality we may assume that $\text{spt } [\Gamma_{ij}] \llcorner h_{ij} \subseteq \text{spt } \mathcal{T}_i \llcorner h_{ij}$ hence $\mathcal{T}_i \llcorner (h_{ij} \setminus \text{spt } [\Gamma_{ij}]) = -\mathcal{T}_j \llcorner h_{ij} \neq 0$ (note that $\text{spt } [\Gamma_{ij}] \llcorner h_{ij} = \{t\varphi_{ij}(w_{ij+}) + (1-t)\varphi_{ij}(w_{ij-}) : t \in [0, 1]\}$). We set

$$\begin{aligned} p &:= \varphi_{ij}(w_{ij}) = \varphi_{ij+}(w_{ij}), \\ \Gamma_{ij}^l &:= \mathcal{T}_i \llcorner \mathcal{P}_i \cup (h_{ij} \cap \text{spt } [\Gamma_{ij}]), & \Gamma_{ij}^r &:= \mathcal{T}_j \llcorner \mathcal{P}_j, \\ \Gamma_{jk}^l &:= \mathcal{T}_j \llcorner \mathcal{P}_i, & \Gamma_{jk}^r &:= \mathcal{T}_k \cup \mathcal{T}_j \llcorner h_{ij}, \\ \Gamma_{ki}^l &:= \mathcal{T}_k \cup \mathcal{T}_j \llcorner h_{ij}, & \Gamma_{ki}^r &:= \mathcal{T}_i \llcorner \mathcal{P}_i \cup (h_{ij} \cap \text{spt } [\Gamma_{ij}^r]). \end{aligned}$$

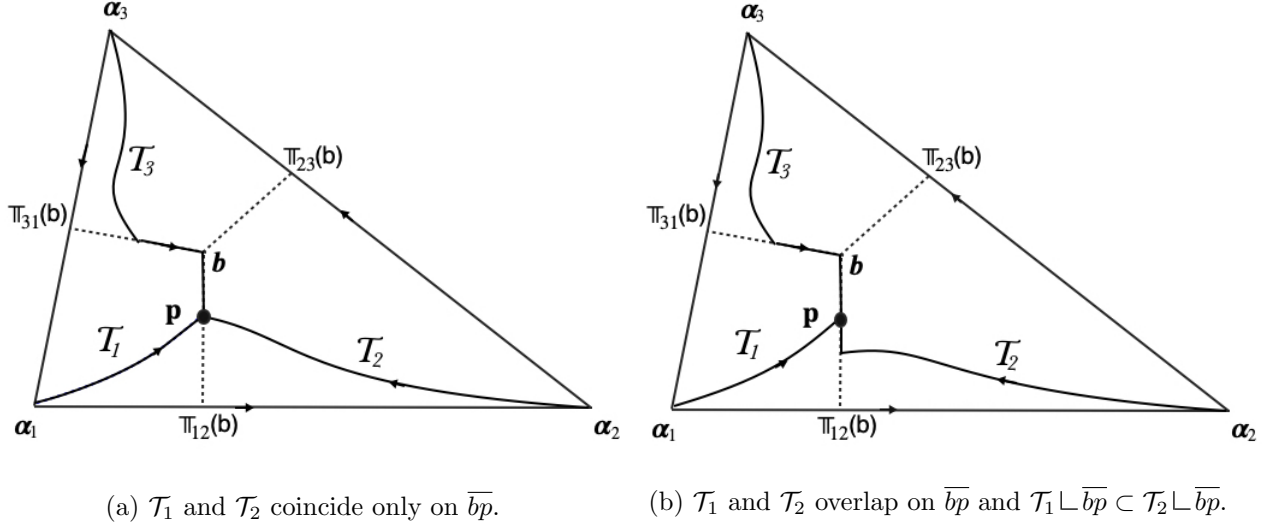


Figure 15: Case (ii) of step 3 in the proof of Theorem 5.6.

see Figure 15b. Also in this case the conclusion follows.
In the end we define

$$\Gamma := (\Gamma_1, \Gamma_2, \Gamma_3), \quad \Gamma_i := \Gamma_{ij}^l = \Gamma_{ki}^r, \quad i = 1, 2, 3. \quad (5.33)$$

The proof is achieved. □

From compactness of the space of BV connection, combining with Proposition 5.4, we see that the infimum in (4.1) is attained. As a consequence, we can conclude the proof of Theorem 1.1.

Corollary 5.8. *We have*

$$\mathcal{A}(u, D) \leq |D| + \inf\{\mathcal{G}(\Gamma) : \Gamma \in X_{\text{Lip}}\} = |D| + \min\{\mathcal{G}(\Gamma) : \Gamma \in X\}. \quad (5.34)$$

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