Lipschitz regularity for viscous Hamilton-Jacobi equations with L^p terms

Marco Cirant and Alessandro Goffi

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Abstract

We provide Lipschitz regularity for solutions to viscous time-dependent Hamilton-Jacobi equations with right-hand side belonging to Lebesgue spaces. Our approach is based on a duality method, and relies on the analysis of the regularity of the gradient of solutions to a dual (Fokker-Planck) equation. Here, the regularizing effect is due to the non-degenerate diffusion and coercivity of the Hamiltonian in the gradient variable.

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Contents

1 Introduction

We study the regularization effect of viscous Hamilton-Jacobi (briefly HJ) equations

$$\begin{cases} \partial_t u(x,t) - \displaystyle\sum_{i,j=1}^{d} a_{ij}(x,t)\partial_{ij}u(x,t) + H(x, Du(x,t)) = f(x,t) & \text{in } Q_T = \mathbb{T}^d \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbb{T}^d, \end{cases} \tag{1}$$

with rough right-hand side f. Our aim is to show that weak solutions (in a suitable sense) with bounded initial data u_0 become Lipschitz continuous at positive times. The regularization effect is based both on the non-degeneracy of the diffusion matrix a_{ij}, and on the strong coercivity assumption of the Hamiltonian H with respect to Du. We have indeed in mind Hamiltonians of the form

$$H(x,p) = h(x)|p|^\gamma + b(x) \cdot p, \tag{2}$$

for some $h, b \in C^1(\mathbb{T}^d)$, $\gamma > 1$ and $0 < h_0 \leq h(x)$. Depending on the growth of H with respect to the gradient variable, two main regimes are typically identified. If H is sub-quadratic, i.e. $1 < \gamma < 2$, then the second order diffusion is the dominating term at small scales. For $f \in L^\infty$, Lipschitz (and further) regularity of solutions for quasi-linear equations of the form (1) goes back to classical literature, see e.g. [16]. On the other hand, in the super-quadratic case $\gamma > 2$ the diffusion term is considered "weaker", and thus typically regarded as a perturbation of a first-order HJ equation. In this direction, Hölder regularity results with possibly unbounded f have been obtained in [5, 6] (where a_{ij} can indeed be degenerate).

Our goal is to combine the regularization effects of both the diffusion and the coercivity of the Hamiltonian. This is motivated by a remarkable result by P.-L. Lions [19], that states Lipschitz regularity of solutions to the *stationary* counterpart of (1) for $f \in L^q$, $q > d$ and *any* $\gamma > 1$. In [19], a Bernstein method that exploits both diffusion and coercivity is developed, but unfortunately it does not seem to generalize to time-dependent problems like (1).

Our analysis is based on a duality approach. The study of linear equations through their duals (adjoint) is a classical idea, which has been explored recently in the nonlinear framework of HJ equations by L.C. Evans [11]. Its application to viscous HJ equations, appearing in particular in so-called Mean-Field Games systems, has been then investigated in a series of papers by D. Gomes and collaborators, see [15] and references therein. Lipschitz bounds of solutions to equations of the form (1) with unbounded or rough data have been in particular considered in [13, 14]. In these works, limitations on the regularity of u itself (it is typically smooth), on the growth of H, i.e. γ, or on d are imposed. Here, we obtain results for all $\gamma > 1$ and $d \in \mathbb{N}$, and for weak solutions to (1).

A motivation of our analysis comes indeed from the theory of Mean-Field Games [17], where Hamilton-Jacobi equations of the form (1) appear naturally, and describe the value function of a typical player in a differential game involving a large population of agents. Here, f is a coupling term that may belong to a Lebesgue space. An important point in such systems is to prove boundedness of the gradient of u, that is crucial not only for PDE purposes, but also ensures boundedness of the optimal control-velocity of players that reached an equilibrium and regularity of their distribution. It is worth noting that Mean-Field games systems naturally exhibit the presence of an HJ equation and its dual Fokker-Planck: this feature somehow inspired the methods by duality presented here.

We now state our two main results. Assume that $d \geq 2$, and $A = (a_{ij}) : Q_T \to \mathrm{Sym}(\mathbb{R}^d)$, where $\mathrm{Sym}(\mathbb{R}^d)$ is the set of symmetric $d \times d$ real matrices, $a_{ij} \in C(0,T; W^{2,\infty}(\mathbb{T}^d))$ and

$$\text{for some } \lambda > 0, \quad \lambda|\xi|^2 \leq a_{ij}(x,t)\xi_i\xi_j \leq \lambda^{-1}|\xi|^2 \text{ for all } \xi \in \mathbb{R}^d \text{ and a. e. } (x,t) \in Q_T. \tag{A}$$

Here and in the sequel the summation over repeated indices is understood. We perform our analysis on the flat torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, to avoid boundary phenomena. A local analysis and a treatment in unbounded domains like the whole $\mathbb{R}^d$ will be matter of future work.
We suppose that $H(x,p)$ is $C^1(\mathbb{T}^d \times \mathbb{R}^N)$, convex in the second variable, and without loss of

generality $H \geq 0$ (if not, one may compensate by adding a positive constant to f). Moreover,

$$\begin{aligned}&\text{there exist constants } \gamma > 1 \text{ and } C_H > 0 \text{ such that}\\ &\quad C_H^{-1}|p|^\gamma - C_H \leq H(x,p) \leq C_H(|p|^\gamma + 1)\ ,\\ &\quad D_pH(x,p)\cdot p - H(x,p) \geq C_H^{-1}|p|^\gamma - C_H\ ,\\ &\quad |D_xH(x,p)| \leq C_H(|p|^\gamma + 1)\ ,\\ &\quad C_H^{-1}|p|^{\gamma-1} - C_H \leq |D_pH(x,p)| \leq C_H|p|^{\gamma-1} + C_H\ ,\end{aligned} \tag{H}$$

for every $x \in \mathbb{T}^d$, $p \in \mathbb{R}^d$. Note that our model Hamiltonian (2) satisfies (H); we mention that the assumptions on b in (2) could be relaxed, but this is beyond the scopes of this paper. Moreover, an explicit dependance with respect to the time variable t could be easily added to H provided that it respects the growth properties stated in (H).

The first result concerns the regularizing effect of the equation, namely Lipschitz regularity of *weak* solutions u for positive times. Below $\gamma' = \gamma/(\gamma-1)$ is the conjugate exponent of γ.

Theorem 1.1. *Suppose that*

- $a_{ij} \in C(0,T;W^{2,\infty}(\mathbb{T}^d))$ *and satisfies* (A),
- $H \in C^1(\mathbb{T}^d \times \mathbb{R}^d)$*, it is convex in the second variable, and satisfies* (H)*,*
- $f \in L^q(Q_T)$*, for some* $q > d+2$ *and* $q \geq \frac{d+2}{\gamma'-1}$*,*
- $u_0 \in L^\infty(\mathbb{T}^d)$*.*

Let u be a weak solution to (1) *(in the sense of Definition* (2.1)*) with $\mathcal{P} = \mathcal{Q}$ in* (12)*, i.e.*

$$D_pH(\cdot, Du) \in L^{\mathcal{P}}(\mathbb{T}^d \times (0,T)) \qquad \textit{for some} \quad \mathcal{P} \geq d+2.$$

Then, $u(\cdot,\tau) \in W^{1,\infty}(\mathbb{T}^d)$ for all $\tau \in (0,T]$. In particular, for all $t_1 \in (0,T)$ there exists a positive constant C_1 depending on t_1, λ, $\|a\|_{C(W^{2,\infty})}$, C_H, $\|u_0\|_{L^\infty(\mathbb{T}^d)}$, $\|f\|_{L^q(Q_T)}$, q, d, T such that

$$\|u(\cdot,\tau)\|_{W^{1,\infty}(\mathbb{T}^d)} \leq C_1 \qquad \textit{for all} \quad \tau \in [t_1, T]. \tag{3}$$

If, in addition, $u_0 \in W^{1,\infty}(\mathbb{T}^d)$, then there exists a positive constant C_2 depending on λ, $\|a\|_{C(W^{2,\infty})}$, C_H, $\|u_0\|_{W^{1,\infty}(\mathbb{T}^d)}$, $\|f\|_{L^q(Q_T)}$, q, d, T such that

$$\|u(\cdot,\tau)\|_{W^{1,\infty}(\mathbb{T}^d)} \leq C_2 \qquad \textit{for all} \quad \tau \in [0, T]. \tag{4}$$

Moreover, the same conclusions hold if u is a weak solution to (1) *with $\mathcal{P} \neq \mathcal{Q}$ in* (12) *whenever $a_{ij}(x,t) = A_{ij}$ on Q_T for some $A_{ij} \in \mathrm{Sym}(\mathbb{R}^d)$ satisfying* (A).

Note that if $\gamma \leq 2$ (i.e. the sub-quadratic/quadratic regime) f is required to be in $L^q(Q_T)$ for some $q > d+2$, while in the super-quadratic case $\gamma > 2$ conditions on f are more strict.

If we assume in addition that u is a *classical* solution to (1), we have the following a priori regularity results. Note that, with respect to the previous Theorem 1.1, Lipschitz bounds will depend on weaker properties of the data a, f.

Theorem 1.2. *Suppose that*

- $a_{ij} \in C([0,T];C^1(\mathbb{T}^d))$ *and satisfies* (A),
- $H \in C^2(\mathbb{T}^d \times \mathbb{R}^d)$ *and satisfies* (H),

- $f \in C([0,T]; C^1(\mathbb{T}^d))$,
- $u_0 \in C^1(\mathbb{T}^d)$.

Let

$$q > \min\left\{d+2, \frac{d+2}{2(\gamma'-1)}\right\}. \tag{5}$$

Then, there exists a positive constant C_3 depending on q, d, T, λ, C_H, $\|u_0\|_{W^{1,\infty}(\mathbb{T}^d)}$, $\|f\|_{L^q(Q_T)}$, $\|a\|_{C(0,T;W^{1,\infty}(\mathbb{T}^d))}$, such that every classical solution to (1) *satisfies*

$$\|u(\cdot,\tau)\|_{W^{1,\infty}(\mathbb{T}^d)} \le C_3 \qquad \text{for all} \quad \tau \in [0,T]. \tag{6}$$

Note that (5) reads

$$q > \begin{cases} d+2 & \text{if } 1 < \gamma \le 3, \\ \frac{d+2}{2(\gamma'-1)} & \text{if } \gamma > 3. \end{cases}$$

In particular, we obtain "maximal regularity" whenever $\gamma \le 3$, that is a control on $\partial_t u, \partial_{ij} u$ and $H(Du)$ in L^q with respect to the the L^q norm of f for any $q > d+2$. Also the results obtained for $\gamma > 3$ are new, since as far as we know, Lipschitz estimates in this regime are not available in the literature of parabolic viscous HJ equations. Anyhow, Lipschitz bounds in the regime $\gamma > 3$ and $d+2 < q < \frac{d+2}{2(\gamma'-1)}$ are at this stage an open problem.

In the next Section 1.1 we briefly describe our methods, and comment on crucial hypotheses that appear in Theorems 1.1, 1.2 and in the Definition 2.1 of weak solutions to (1). In Section 2 we present some preliminary facts and results on the adjoint equation. Sections 3 and 4 will be devoted mainly to the proofs of Theorems 1.1 and 1.2 respectively.

1.1 Heuristic derivation of Lipschitz estimates

The adjoint method implemented here can be heuristically described as follows. Let us assume that u is a smooth solution of the viscous HJ equation

$$\partial_t u(x,t) - \Delta u(x,t) + H(Du(x,t)) = f(x,t) \tag{7}$$

with $u(\cdot,0) \in C^1(\mathbb{T}^d)$ and f be C^1 in the space variable. We differentiate the equation to study the regularity of Du, namely, for any direction $\xi \in \mathbb{R}^d$ with $|\xi| = 1$, we consider $v = \partial_\xi u$. Then, v solves the linearized equation

$$\partial_t v - \Delta v + D_p H(Du) \cdot Dv = \partial_\xi f \ . \tag{8}$$

For any $\tau \in (0,T)$, $x_0 \in \mathbb{T}^d$, we then look at the adjoint equation with singular final datum

$$\begin{cases} -\partial_t \rho - \Delta\rho - \operatorname{div}(D_p H(Du)\rho) = 0 & \text{in } \mathbb{T}^d \times (0,\tau) \ , \\ \rho(\tau) = \delta_{x_0} & \text{on } \mathbb{T}^d \ . \end{cases} \tag{9}$$

By duality between (8) and (9) we immediately get

$$\partial_\xi u(x_0,\tau) = \langle v(\tau), \rho(\tau)\rangle = \iint_{\mathbb{T}^d\times(0,\tau)} \partial_\xi f\, \rho + \int_{\mathbb{T}^d} v\rho(0) = -\iint_{\mathbb{T}^d\times(0,\tau)} f\, \partial_\xi \rho + \int_{\mathbb{T}^d} \partial_\xi u\, \rho(0).$$

Thanks to integration by parts in the previous formula, we realize that our representation of $\partial_\xi u(x_0,\tau)$ roughly depends on $\|f\|_{L^q(Q_T)}$ and $\|D\rho\|_{L^{q'}(Q_T)}$, so, the more we know on the integrability of $D\rho$, the less we can assume on the integrability of the datum f. The difficulty here is that ρ depends on Du itself through the divergence term in (9), and has a final datum that is a Dirac measure. Therefore, even disregarding completely the divergence term in (9), and using as final datum an L^1 approximation of δ_{x_0}, the best we can expect is $\|D\rho\|_{L^{q'}(Q_T)}$ for $q' < (d+2)'$. This is actually an integrability limit on $D\rho$ imposed by the heat part of the equation. Therefore, we will always require f to be L^q with $q > d+2$ (which is optimal, see Remark 3.12).

The transport (divergence) term in (9) is handled by exploiting a crucial information on the quantity

$$\iint |D_pH(Du)|^{\gamma'}\rho\,dxdt, \tag{10}$$

that is obtained using a sort of duality between (1) and (9), and has a very precise meaning in terms of optimality in stochastic control problems (see, e.g. [15] for further discussions). Such a quantity is actually a weighted $L^{\gamma'}(\rho)$ norm of the drift $-D_pH(Du)$ that appears in the divergence term, and turns out to be enough to derive bounds for $\|D\rho\|_{L^{q'}(Q_T)}$. This crucial result is stated in Proposition 2.6 and exploits a delicate combination of parabolic regularity, interpolation and embeddings of parabolic spaces. It is worth noting that such an $L^{\gamma'}(\rho)$ integrability deteriorates as γ grows. In particular, we observe that in the sub-quadratic regime $\gamma \le 2$, this information is strong enough to guarantee $\|D\rho\|_{L^{q'}(Q_T)}$ for $q' < (d+2)'$. We can then regard the div() term in (9) as perturbation of a heat equation. On the other hand, in the super-quadratic case $\gamma > 2$, we are just able to prove that $\|D\rho\|_{L^{q'}(Q_T)}$ for $q' \le q'_\gamma$, with $q'_\gamma < (d+2)'$, and actually $q'_\gamma \to 1$ as $\gamma \to \infty$. As expected, in the super-quadratic case the Hamiltonian term in (1) may overcome the regularizing effect of Laplacian. Still, under the additional hypothesis $f \in L^{q_\gamma}$, we obtain Lipschitz regularity results *for every* $\gamma > 1$. This is a major difference with respect to previous works [13, 14], where the techniques involved produce estimates on $D\rho$ only under the assumption that the drift entering into the dual equation is at least $L^2(\rho)$, thus limiting the range of γ.

In the next sections we make precise all the above formal computations, and for more general equations of the form (1). In the first part of the paper we aim at obtaining Lipschitz regularity of *weak solutions* to (1), in a sense specified below (see Definition 2.1). The main issues in this program are the following:

- To exploit duality between (1) and (9) in a weak framework, one has to understand the right weak setting for both equations. We realize here that a suitable weak notion guaranteeing Lipschitz regularity is basically the usual energy one for both equations (i.e. $u, \rho \in \mathcal{H}_2^1$). This relies strongly on the additional assumption $D_pH(Du) \in L^{\mathcal{Q}}((0,T); L^{\mathcal{P}})$, which can be considered a requirement for the adjoint equation (9) rather than for the given HJ equation (1), but one should always keep in mind the subtle interplay between the two equations. Of course this forces the final datum $\rho(\tau)$ to be in L^2, and therefore introduces an additional approximation step from L^2 to L^1 in our scheme.
  One may argue that, for γ very large, $|Du|^{\gamma-1} \approx D_pH(Du) \in L^{\mathcal{Q}}((0,T); L^{\mathcal{P}})$ is very close to $Du \in L^\infty$. We stress in Section 3.4 that to perform this (seemingly) small step, one cannot avoid in general this assumption on Du, and therefore our requirements on weak solutions are optimal to guarantee Lipschitz regularity.

- A weak solution u is not a priori a.e. differentiable, and $f \in L^q$, so no differentiation procedure of (1) is justified. This is circumvented by considering difference quotients of u in the x-variable, which are handled via a method that is again based on the optimality of $-D_pH(Du)$ in stochastic optimal control problems (though here PDE methods will be involved only).

- Though they are not our main focus, we have also to be careful with regularity of H and

a in the x-variable. Moreover, we are able to localize our estimates in time, thus assuming $u(0) \in L^\infty$ only.

The study of regularity, rather than the proof of a priori estimates of smooth solutions to (1), is a key difference with respect to works previously mentioned (e.g. [13, 14]). We take this different viewpoint in the final Section 4: assuming regularity of the solution, we can improve in some directions the previous procedure. First, it is possible to enhance (10) by absorbing part of the gradient term in the left hand side of the Lipschitz estimate. Second, rather than studying the equation for $\partial_\xi u$, we consider the equation for $|Du|^2$, following a classical idea of Bernstein. This yields a similar "linearized" equation, with additional information on D^2u coming from strict ellipticity of the operator. This allows us to prove *a priori* regularity of smooth solutions u to (1) that depend on weaker integrability properties of f and regularity of a_{ij} with respect to x.

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2 Functional spaces, weak solutions and basic properties

First, recall that the Lagrangian $L : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$, $L(x,\xi) := \sup_p\{p\cdot\xi - H(x,p)\}$, namely the Legendre transform of H in the p-variable, is well defined by the superlinear character of $H(x,\cdot)$. Moreover, by convexity of $H(x,\cdot)$,

$$H(x,p) = \sup_{\xi\in\mathbb{R}^d}\{\xi\cdot p - L(x,\xi)\},$$

and

$$H(x,p) = \xi\cdot p - L(x,\xi) \quad \text{if and only if} \quad \xi = D_pH(x,p). \tag{11}$$

The following properties of L are standard (see, e.g. [4]): for some $C_L > 0$,

$$C_L^{-1}|\xi|^{\gamma'} - C_L \le L(x,\xi) \le C_L|\xi|^{\gamma'} \tag{L1}$$

$$|D_xL(x,\xi)| \le C_L(|\xi|^{\gamma'} + 1). \tag{L2}$$

for all $\xi \in \mathbb{R}^d$.

Since we are working in the periodic setting, let us recall that $L^p(\mathbb{T}^d)$ is the space of all measurable and periodic functions on $\mathbb{R}^d$ belonging to $L^p_{\rm loc}(\mathbb{R}^d)$, with norm $\|\cdot\|_p = \|\cdot\|_{L^p((0,1)^d)}$. For positive integers k, $W^{k,p}(\mathbb{T}^d)$ is the space of those functions with (distributional) derivatives in $L^p(\mathbb{T}^d)$ up to order k.

For any time interval $I \subset \mathbb{R}$, let $Q = \mathbb{T}^d \times I$. For any integer k and $p \ge 1$, we denote by $W^{2,1}_p(Q)$ the space of functions u such that $\partial_t^r D_x^\beta u \in L^p(Q)$ for all multi-indices β and r such that $|\beta| + 2r \le 2$, endowed with the norm

$$\|u\|_{W^{2,1}_p(Q)} = \left(\iint_Q \sum_{|\beta|+2r\le 2} |\partial_t^r D_x^\beta u|^p dxdt\right)^{\frac{1}{p}}.$$

The space $W_p^{1,0}(Q)$ is defined similarly, and is endowed with the norm

$$\|u\|_{W_p^{1,0}(Q)} := \|u\|_{L^p(Q)} + \sum_{|\beta|=1} \left\|D_x^\beta u\right\|_{L^p(Q)} .$$

We define the space $\mathcal{H}_p^1(Q)$ as the space of functions $u \in W_p^{1,0}(Q)$ with $\partial_t u \in (W_{p'}^{1,0}(Q))'$, equipped with the norm

$$\|u\|_{\mathcal{H}_p^1(Q)} := \|u\|_{W_p^{1,0}(Q)} + \|\partial_t u\|_{(W_{p'}^{1,0}(Q))'} .$$

Denoting by $C(I;X)$ and $L^q(I;X)$ the usual spaces of continuous and Lebesgue functions respectively with values in a Banach space X, we have the following isomorphisms: $W_2^{1,0}(Q) \simeq L^2(I;W^{1,2}(\mathbb{T}^d))$, and

$$\begin{aligned}\mathcal{H}_2^1(Q) \simeq \{u \in L^2(I;W^{1,2}(\mathbb{T}^d)),\, \partial_t u \in (\, L^2(I;W^{1,2}(\mathbb{T}^d))\,)'\} \\ \simeq \left\{u \in L^2(I;W^{1,2}(\mathbb{T}^d)),\, \partial_t u \in L^2\big(I;(W^{1,2}(\mathbb{T}^d))'\big)\right\},\end{aligned}$$

and the latter is known to be continuously embedded into $C(I;L^2(\mathbb{T}^d))$ (see, e.g., [10, Theorem XVIII.2.1]). Sometimes, we will use the compact notation $C(X)$ and $L^q(X)$.

2.1 A notion of weak solution to viscous HJ equations

We will say that u is a *weak* solution to (1) in the following sense.

Definition 2.1. We say that a function $u \in \mathcal{H}_2^1(Q_T)$ satisfying

i) $H(x,Du) \in L^1(0,T;L^\sigma(\mathbb{T}^d))$ for some $\sigma > 1$,

ii) $D_pH(x,Du) \in L^{\mathcal{Q}}(0,T;L^{\mathcal{P}}(\mathbb{T}^d))$ for some $d \le \mathcal{P} \le \infty$, and $2 \le \mathcal{Q} \le \infty$ such that

$$\frac{d}{2\mathcal{P}} + \frac{1}{\mathcal{Q}} \le \frac{1}{2} \tag{12}$$

is a weak solution to (1) if

$$-\int_{\mathbb{T}^d} u_0\varphi(0)dx + \iint_{Q_T} -u\,\partial_t\varphi + \partial_i u\,\partial_j(a_{ij}\varphi) + H(x,Du)\varphi\,dxdt = \iint_{Q_T} f\varphi\,dxdt \tag{13}$$

for all $\varphi \in C^\infty(\mathbb{T}^d \times [0,T))$.

Note that $\mathcal{H}_2^1(Q_T)$ is continuously embedded in $C(0,T;L^2(\mathbb{T}^d))$, so this is equivalent to

$$\int_0^T \langle \partial_t u(t), \varphi(t)\rangle dt + \iint_{Q_T} \partial_i u\,\partial_j(a_{ij}\varphi) + H(x,Du)\varphi\,dxdt = \iint_{Q_T} f\varphi\,dxdt$$

for all $\varphi \in C^\infty(Q_T)$, and $u(0) = u_0$ in the L^2-sense (here, $\langle\cdot,\cdot\rangle$ is the duality pairing between $(W^{1,2}(\mathbb{T}^d))'$ and $W^{1,2}(\mathbb{T}^d)$). Note that for (13) to be meaningul, one could just require $H(x,Du) \in L^1(Q_T)$; we ask for slightly better integrability since we will use the adjoint variable ρ (see (14) below) as test function, that is not necessarily in $L^\infty(Q_T)$. In particular, (13) holds in general for $\varphi \in \mathcal{H}_2^1(Q_T) \cap L^\infty(0,T;L^{\sigma'}(\mathbb{T}^d))$. Anyway, as it will be pointed out in the following remark, *ii)* implies *i)* in many interesting cases. Though condition *ii)* appears to be unrelated to (1), it actually guarantees the existence of a weak (energy) solution of the adjoint equation (see Proposition 2.4 below), that will be crucial in our subsequent analysis.

Remark 2.2. Under the growth assumptions (H) on the Hamiltonian, one can easily verify the following implications: if $D_pH(x, Du)$ satisfies *ii)* for some $\mathcal{P} = \mathcal{Q} \geq d+2$, then *i)* holds for sure whenever $\gamma > \frac{d+2}{d+1}$. Or, if $D_pH(x, Du)$ satisfies *ii)* for $\mathcal{Q} = \infty$ and some $\mathcal{P} \geq d$, then *i)* always holds if $\gamma > \frac{d}{d-1}$.

Remark 2.3. Notice that under the assumptions of Definition 2.1, weak solutions of (1) must be unique (except for a subtle endpoint case discussed below). This can be proven via a simple linearization argument: let $v(x,t) := u_1(x,t) - u_2(x,t)$ on Q_T, where u_i are two solutions of (1) in the sense of Definition 2.1. Then, $v \in \mathcal{H}_2^1(Q_T)$ is a weak (energy) solution to the linear equation

$$\partial_t v - \sum_{i,j=1}^{d} a_{ij}(x,t)\partial_{ij}v(x,t) + B(x,t) \cdot Dv(x,t) = 0,$$

satisfying $v(0) = 0$ in the L^2-sense, where $B(x,t)$ is some measurable vector field such that, in view of (H),

$$|B(x,t)| \leq C(|Du_1(x,t)|^{\gamma-1} + |Du_2(x,t)|^{\gamma-1} + 1).$$

Hence, again by (H) and (12), $B(x,t) \in L^{\mathcal{Q}}(0,T; L^{\mathcal{P}}(\mathbb{T}^d))$ for some $\mathcal{P}$, $\mathcal{Q}$ satisfying $\frac{d}{2\mathcal{P}} + \frac{1}{\mathcal{Q}} \leq \frac{1}{2}$. This is the typical assumption on coefficients of a linear equation that guarantees $0 \equiv v = u_1 - u_2$ on Q_T (see, e.g., [16, Theorem III.3.1], which can be readily adapted to the periodic setting, see [12]). Note that when $\mathcal{Q} = \infty$ and $\mathcal{P} = d$ one has to assume an additional hypothesis on B, namely that $|B(\cdot,t)|^d$ is uniformly integrable with respect to $t \in (0,T)$. We mention that, at least in the sub-quadratic case $\gamma < 2$, it is known that uniqueness holds under weaker conditions on u involving suitable powers of u itself in $L^2(0,T; W^{1,2})$, see [18] and references therein.

2.2 Well-posedness and regularity of the adjoint equation

This section is devoted to the analysis of the following Fokker-Planck equation

$$\begin{cases} -\partial_t \rho(x,t) - \sum_{i,j=1}^{d} \partial_{ij}(a_{ij}(x,t)\rho(x,t)) + \operatorname{div}(b(x,t)\,\rho(x,t)) = 0 & \text{in } Q_\tau\,, \\ \rho(x,\tau) = \rho_\tau(x) & \text{in } \mathbb{T}^d\,. \end{cases} \tag{14}$$

Note that when the vector field $b(x,t) = -D_pH(x, Du(x,t))$, then (14) becomes the adjoint equation of the linearization of (1).

Here, $\tau \in (0,T]$ and $Q_\tau := \mathbb{T}^d \times (0,\tau)$. For $b \in L^{\mathcal{Q}}(0,T; L^{\mathcal{P}}(\mathbb{T}^d))$ for some $\mathcal{P} \geq d$, and $\mathcal{Q} \geq 2$ satisfying (12), a (weak) solution $\rho \in \mathcal{H}_2^1(Q_\tau)$ is such that $\rho(\tau) = \rho_\tau$ in the L^2-sense, and

$$-\int_0^T \langle \partial_t \rho(t), \varphi(t)\rangle dt + \iint_{Q_\tau} \partial_j(a_{ij}\rho)\partial_i\varphi - b\rho \cdot D\varphi \, dxdt = 0 \tag{15}$$

for all $\varphi \in \mathcal{H}_2^1(Q_\tau)$.

Throughout this section we will assume that

$$\rho_\tau \in C^\infty(\mathbb{T}^d), \quad \rho_\tau \geq 0, \quad \text{and} \quad \int_{\mathbb{T}^d} \rho_\tau(x)\,dx = 1. \tag{16}$$

Note that $\rho \in C([0,\tau]; L^2(\mathbb{T}^d))$, so $\rho \in C([0,\tau]; L^1(\mathbb{T}^d))$, and

$$\int_{\mathbb{T}^d} \rho(x,t)\,dx = 1 \qquad \text{for all } t \in [0,\tau]. \tag{17}$$

This can be easily verified using $\varphi \equiv 1$ as a test function in (15).

Proposition 2.4. *Let* (A) *be in force,* $b \in L^{\mathcal{Q}}(0,\tau;L^{\mathcal{P}}(\mathbb{T}^d))$ *for some* $\mathcal{P} \geq d$, $\mathcal{Q} \geq 2$ *satisfying* (12)*, and* ρ_τ *be as in* (16)*. Then, there exists a weak solution* $\rho \in \mathcal{H}_2^1(Q_\tau)$ *to* (14)*. Moreover,* $\rho \in L^\infty(0,T;L^{\sigma'}(\mathbb{T}^d))$ *for all* $1 < \sigma' < \infty$ *and* ρ *is a. e. non-negative on* Q_τ*.*

Proof. Existence and regularity of weak solutions to linear equations in divergence form with $b \in L^{\mathcal{Q}}(0,\tau;L^{\mathcal{P}}(\mathbb{T}^d))$ is a classical matter that can be found in e.g. [1, 16]. Though well known references do not treat directly the periodic setting (but typically the Cauchy-Dirichlet problem), the adaptation of energy methods to $\mathbb{T}^d$ is straightforward, and can be checked for example following the lines of [3]. For additional details we refer to [12]. □

The previous proposition states the well-posedness of the Fokker-Planck equation for fixed ρ_τ. The main goal is now to derive estimates on ρ that are stable for any ρ_τ satisfying merely (16); one may have in mind that ρ_τ is an item of a sequence approaching a Dirac delta. These estimates will be achieved using some information on the integrability of the vector field b with respect to the solution ρ itself, that is a typical datum in the analysis of Hamilton-Jacobi equations.

The following proposition is a modification of [9, Proposition 2.4], and is a kind of parabolic regularity result.

Proposition 2.5. *Let* ρ *be a (non-negative) weak solution to* (14) *and*

$$1 < q' < \frac{d+2}{d+1}.$$

Then, there exists $C > 0$*, depending on* $\lambda, \|a\|_{C(W^{1,\infty})}, q', d, T$ *such that*

$$\|\rho\|_{\mathcal{H}^1_{q'}(Q_\tau)} \leq C(\|b\rho\|_{L^{q'}(Q_\tau)} + \|\rho\|_{L^{q'}(Q_\tau)} + \|\rho_\tau\|_{L^1(\mathbb{T}^d)}). \tag{18}$$

Note that C here does not depend on $\tau \in (0,T]$.

Proof. We assume that the coefficients a_{ij}, b_i are smooth, and therefore ρ is smooth as well on Q_τ. The general case $Da \in L^\infty(Q_\tau)$, $b \in L^{\mathcal{Q}}(0,T;L^{\mathcal{P}}(\mathbb{T}^d))$ follows by an approximation argument.

Fix $k = 1,...,d$. For $\delta > 0$, let $\psi = \psi_\delta$ be the classical solution to

$$\begin{cases} \partial_t\psi(x,t) - \sum_{i,j} a_{ij}(x,t)\partial_{ij}\psi(x,t) = (\delta + |\partial_k\rho(x,t)|^2)^{\frac{q'-2}{2}}\partial_k\rho(x,t) & \text{in } Q_\tau \ , \\ \psi(x,0) = 0 & \text{on } \mathbb{T}^d \ . \end{cases} \tag{19}$$

Since $q' < 2$, $\delta > 0$ serves as a regularizing perturbation. By standard parabolic regularity (see Lemma A.1), we have (for a positive constant not depending on $\tau \leq T$)

$$\|\psi\|_{W_q^{2,1}(Q_\tau)} \leq C\left\|(\delta + |\partial_k\rho|^2)^{\frac{q'-2}{2}}\partial_k\rho\right\|_{L^q(Q_\tau)} \leq C\left\||\partial_k\rho|^{q'-1}\right\|_{L^q(Q_\tau)} = C\,\|\partial_k\rho\|^{q'-1}_{L^{q'}(Q_\tau)}. \tag{20}$$

Set $\varphi(x,t) = \partial_{x_k}\psi(x,t)$. Then, φ is a classical solution to

$$\begin{cases} \partial_t\varphi - \sum_{i,j} a_{ij}\partial_{ij}\varphi = \partial_k\left[(\delta + |\partial_k\rho|^2)^{\frac{q'-2}{2}}\partial_k\rho\right] + \sum_{i,j}\partial_k(a_{ij})\partial_{ij}\psi & \text{in } Q_\tau \ , \\ \psi(x,0) = 0 & \text{on } \mathbb{T}^d \ . \end{cases} \tag{21}$$

Using φ as a test function for the equation satisfied by ρ,

$$\iint_{Q_\tau} \rho(\partial_t\varphi - a_{ij}\partial_{ij}\varphi - b\cdot D\varphi)dxdt = \int_{\mathbb{T}^d}\rho_\tau(x)\varphi(x,\tau)dx \ ,$$

and using the equation in (21) satisfied by φ we get, after integration by parts

$$\iint_{Q_\tau} (\delta + |\partial_k \rho|^2)^{\frac{q'-2}{2}} |\partial_k \rho|^2 - \partial_k(a_{ij})\partial_{ij}\psi\, \rho + b\rho \cdot D\varphi \, dxdt = -\int_{\mathbb{T}^d} \rho_\tau(x)\varphi(x,\tau)dx \ ,$$

Applying Hölder's inequality,

$$\iint_{Q_\tau} (\delta + |\partial_k \rho|^2)^{\frac{q'-2}{2}} |\partial_k \rho|^2 \, dxdt \le \|Da\|_{L^\infty(Q_\tau)} \|\psi\|_{W^{2,1}_q(Q_\tau)} \|\rho\|_{L^{q'}(Q_\tau)} \\ + \|b\rho\|_{L^{q'}(Q_\tau)} \|D\varphi\|_{L^q(Q_\tau)} + \|\rho_\tau\|_{L^1(\mathbb{T}^d)} \|\varphi(\cdot,\tau)\|_\infty.$$

Since $q > d+2$, by [16, Lemma II.3.3], the parabolic space $W^{2,1}_q(Q_\tau)$ is continuously embedded into $C([0,\tau]; C^1(\mathbb{T}^d))$, therefore $\|\varphi(\cdot,\tau)\|_\infty \le \|\psi(\cdot,\tau)\|_{C^1(\mathbb{T}^d)} \le C\|\psi\|_{W^{2,1}_q(Q_\tau)}$ (to be sure that C does not explode as $\tau \to 0$, one has to exploit that $\psi(0) = 0$, and argue as in the proof of Proposition A.2). Hence, since $\varphi = \partial_{x_k}\psi$,

$$\iint_{Q_\tau} (\delta + |\partial_k \rho|^2)^{\frac{q'-2}{2}} |\partial_k \rho|^2 \, dxdt \le C(\|\rho\|_{L^{q'}(Q_\tau)} + \|b\rho\|_{L^{q'}(Q_\tau)} + \|\rho_\tau\|_{L^1(\mathbb{T}^d)}) \|\psi\|_{W^{2,1}_q(Q_\tau)}.$$

by (20) and letting $\delta \to 0$,

$$\iint_{Q_\tau} |\partial_k \rho|^{q'} \, dxdt \le C(\|\rho\|_{L^{q'}(Q_\tau)} + \|b\rho\|_{L^{q'}(Q_\tau)} + \|\rho_\tau\|_{L^1(\mathbb{T}^d)}) \|\partial_k \rho\|^{q'-1}_{L^{q'}(Q_\tau)}.$$

Summarizing, we conclude

$$\|D\rho\|_{L^{q'}(Q_\tau)} \le C(\|\rho\|_{L^{q'}(Q_\tau)} + \|b\rho\|_{L^{q'}(Q_\tau)} + \|\rho_\tau\|_{L^1(\mathbb{T}^d)}) \ . \tag{22}$$

By Poincaré-Wirtinger inequality and (22), together with the fact that $\int_{\mathbb{T}^d} \rho(x,t)dx = 1$ for all $t \in [0,\tau]$, we obtain

$$\|\rho\|^{q'}_{L^{q'}(Q_\tau)} \le C(\|D\rho\|^{q'}_{L^{q'}(Q_\tau)} + \tau \|\rho_\tau\|^{q'}_{L^1(\mathbb{T}^d)}) \ ,$$

yielding, together with (22)

$$\|\rho\|_{W^{1,0}_{q'}(Q_\tau)} \le C(\|\rho\|_{L^{q'}(Q_\tau)} + \|b\rho\|_{L^{q'}(Q_\tau)} + \|\rho_\tau\|_{L^1(\mathbb{T}^d)}).$$

Finally, for any smooth test function φ (which may not vanish at the terminal time T), again by Hölder's inequality

$$\left| \int_0^\tau \langle \partial_t \rho(t), \varphi(t) \rangle dt \right| \le \iint_{Q_\tau} |\partial_j(a_{ij}\rho)\partial_i \varphi| + |b\rho| \, |D\varphi| \, dxdt \\ \le \left[(\|a\|_{L^\infty(Q_\tau)} + \|Da\|_{L^\infty(Q_\tau)}) \|\rho\|_{W^{1,0}_{q'}(Q_\tau)} + \|b\rho\|_{L^{q'}(Q_\tau)} \right] \|D\varphi\|_{L^q(Q_\tau)}.$$

Thus,

$$\|\partial_t \rho\|_{(W^{1,q}(Q_\tau))'} \le C(\|\rho\|_{L^{q'}(Q_\tau)} + \|b\rho\|_{L^{q'}(Q_\tau)} + \|\rho_\tau\|_{L^1(\mathbb{T}^d)}) \ .$$

□

Proposition 2.6. *Let ρ be the (non-negative) weak solution to* (14) *and*

$$1 < q' < \frac{d+2}{d+1}.$$

Then, there exists $C > 0$, depending on $\lambda, \|a\|_{C(W^{1,\infty})}, T, q', d$ such that

$$\|\rho\|_{\mathcal{H}^1_{q'}(Q_\tau)} \leq C \left(\iint_{Q_\tau} |b(x,t)|^{r'} \rho(x,t)\, dxdt + 1 \right), \tag{23}$$

where

$$r' = 1 + \frac{d+2}{q}. \tag{24}$$

Proof. Inequality (18), (16) and the generalized Hölder's inequality yield

$$\begin{aligned} \|\rho\|_{\mathcal{H}^1_{q'}(Q_\tau)} &\leq C(\|b\rho^{1/r'}\rho^{1/r}\|_{L^{q'}(Q_\tau)} + \|\rho\|_{L^{q'}(Q_\tau)} + 1) \\ &\leq C \left(\left(\iint_{Q_\tau} |b|^{r'} \rho\, dxdt \right)^{1/r'} \|\rho\|^{1/r}_{L^p(Q_\tau)} + \|\rho\|_{L^{q'}(Q_\tau)} + 1 \right), \end{aligned} \tag{25}$$

for $p > q'$ satisfying

$$\frac{1}{q'} = \frac{1}{r'} + \frac{1}{rp}. \tag{26}$$

Then, by Young's inequality, for all $\varepsilon > 0$

$$\|\rho\|_{\mathcal{H}^1_{q'}(Q_\tau)} \leq C \left(\frac{1}{\varepsilon} \iint_{Q_\tau} |b|^{r'} \rho\, dxdt + \varepsilon \|\rho\|_{L^p(Q_\tau)} + \|\rho\|_{L^{q'}(Q_\tau)} + 1 \right), \tag{27}$$

Since $\|\rho\|_{L^1(Q_\tau)} = \tau$, by interpolation between $L^1(Q_\tau)$ and $L^p(Q_\tau)$ we have $\|\rho\|_{L^{q'}(Q_\tau)} \leq \tau^{1/r'} \|\rho\|^{1/r}_{L^p(Q_\tau)}$, and again by Young's inequality

$$\|\rho\|_{\mathcal{H}^1_{q'}(Q_\tau)} \leq \widetilde{C} \left(\frac{1}{\varepsilon} \iint_{Q_\tau} |b|^{r'} \rho\, dxdt + \varepsilon \|\rho\|_{L^p(Q_\tau)} + 1 \right), \tag{28}$$

One can verify that (24) and (26) yield

$$\frac{1}{p} = \frac{1}{q'} - \frac{1}{d+2}.$$

The continuous embedding of $\mathcal{H}^1_{q'}(Q_\tau)$ in $L^p(Q_\tau)$ stated in Proposition A.2 then implies

$$\|\rho\|_{L^p(Q_\tau)} \leq C_1 \big(\|\rho\|_{\mathcal{H}^1_{q'}(Q_\tau)} + \tau \big)\,.$$

Hence, the term $\varepsilon \|\rho\|_{L^p(Q_\tau)}$ can be absorbed by the left hand side of (28) by choosing $\varepsilon = (2\widetilde{C} C_1)^{-1}$, thus providing the assertion. □

3 Lipschitz regularity

This section is devoted to the proof of Lipschitz regularity of u, stated in Theorem 1.1. We will assume that the assumptions of Theorem 1.1 are in force: $a_{ij} \in C(0,T; W^{2,\infty}(\mathbb{T}^d))$ and satisfies (A), $H \in C^1(\mathbb{T}^d \times \mathbb{R}^d)$, it is convex in the second variable, and satisfies (H) and $u_0 \in L^\infty(\mathbb{T}^d)$. Moreover, $f \in L^q(Q_T)$ for some $q > d+2$. At a certain stage we will require $q \geq \frac{d+2}{\gamma'-1}$ also.

The result will be obtained using regularity properties of the adjoint variable ρ, i.e. the solution to

$$\begin{cases} -\partial_t \rho(x,t) - \sum_{i,j=1}^{d} \partial_{ij}(a_{ij}(x,t)\rho(x,t)) - \operatorname{div}\left(D_pH(x,Du(x,t))\,\rho(x,t)\right) = 0 & \text{in } Q_\tau \ , \\ \rho(x,\tau) = \rho_\tau(x) & \text{on } \mathbb{T}^d \ , \end{cases} \tag{29}$$

for $\tau \in (0,T)$, $\rho_\tau \in C^\infty(\mathbb{T}^d)$, $\rho_\tau \geq 0$ with $\|\rho_\tau\|_{L^1(\mathbb{T}^d)} = 1$. Recall that u is a weak solution to the viscous Hamilton-Jacobi equation (1). By the integrability assumptions on D_pH, the adjoint state $\rho \in \mathcal{H}_2^1(Q_\tau)$ is, for any ρ_τ, well-defined, non-negative and bounded in $L^\infty(0,\tau;L^{\sigma'}(\mathbb{T}^d))$ for all $\sigma' > 1$, by a straightforward application of Proposition 2.4.

In what follows, we establish bounds on ρ that are independent on the choice of τ and ρ_τ.

3.1 Estimates on the adjoint variable

Let us point out first that from now on we will denote by $C, C_1, \ldots$ positive constants that may depend on λ, C_H, $\|u_0\|_{L^\infty(\mathbb{T}^d)}$, $\|f\|_{L^q(Q_T)}$, $\|a\|_{C(W^{1,\infty})}$, $\|D^2a\|_{L^\infty(Q_\tau)}, T, q, d$, but do not depend on τ, ρ_τ.

We first bound from above the solution of the Hamilton-Jacobi equation (1), using a duality argument that involves solutions of a backward heat equation.

Proposition 3.1. *There exists $C > 0$ (depending on $\lambda, \|a\|_{C(W^{1,\infty})}, T, q', d$) such that any weak solution u to* (1) *satisfies*

$$u(x,\tau) \leq \|u_0\|_{L^\infty(\mathbb{T}^d)} + C\|f\|_{L^q(Q_T)} \tag{30}$$

for all $\tau \in (0,T)$ and a.e. $x \in \mathbb{T}^d$.

Proof. Let $\tau \in (0,T)$. Consider the (strong) non-negative solution of the following backward problem

$$\begin{cases} -\partial_t \mu(x,t) - \sum_{i,j} \partial_{ij}(a_{ij}(x,t)\mu(x,t)) = 0 & \text{on } Q_\tau \ , \\ \mu(x,\tau) = \mu_\tau(x) & \text{on } \mathbb{T}^d \ . \end{cases}$$

with $\mu_\tau \in C^\infty(\mathbb{T}^d)$, $\mu_\tau \geq 0$ and $\|\mu_\tau\|_{L^1(\mathbb{T}^d)} = 1$. Note that μ is a solution of a Fokker-Planck equation of the form (14) with drift $b \equiv 0$. Then, since $q' < (d+2)/(d+1)$, by Proposition 2.6 there exists a positive constant C (not depending on μ_τ) such that $\|\mu\|_{\mathcal{H}_{q'}^1(Q_\tau)} \leq C$.

Use μ as a test function in the weak formulation of the Hamilton-Jacobi equation (1) to get

$$\int_{\mathbb{T}^d} u(x,\tau)\mu_\tau(x)dx = \int_{\mathbb{T}^d} u_0(x)\mu(x,0)dx + \iint_{Q_\tau} f\mu dxdt - \iint_{Q_\tau} H(x,Du)\mu dxdt \ .$$

Applying Hölder's inequality to the second term of the right-hand side of the above inequality and the fact that $\|\mu(t)\|_{L^1(\mathbb{T}^d)} = 1$ for all $t \in (0,\tau)$, we get

$$\int_{\mathbb{T}^d} u(x,0)\mu(x,0)dx + \int_0^\tau \int_{\mathbb{T}^d} f\mu dxdt \leq \|u_0\|_{L^\infty(\mathbb{T}^d)} + C\|f\|_{L^q(Q_\tau)} \ ,$$

By the assumption $H(x,Du) \geq 0$, we then conclude

$$\int_{\mathbb{T}^d} u(x,\tau)\mu_\tau(x)dx \leq \|u_0\|_{L^\infty(\mathbb{T}^d)} + C\|f\|_{L^q(Q_\tau)} \ .$$

Finally, by passing to the supremum over $\mu_\tau \geq 0$, $\|\mu_\tau\|_{L^1(\mathbb{T}^d)} = 1$, one deduces the estimate (30) by duality. $\square$

Lemma 3.2. *Let u be a weak solution to* (1). *Assume that ρ is a weak solution to* (29). *Then,*

$$\int_{\mathbb{T}^d} u(x,\tau)\rho_\tau(x)dx = \int_{\mathbb{T}^d} u(x,0)\rho(x,0) + \iint_{Q_\tau} L(x, D_pH(x,Du))\rho dxdt + \iint_{Q_\tau} f\rho\, dxdt. \quad (31)$$

Proof. Using $-\rho \in \mathscr{H}_2^1(Q_\tau) \cap L^\infty(0,\tau; L^{\sigma'}(\mathbb{T}^d))$ as a test function in the weak formulation of problem (1), $u \in \mathscr{H}_2^1(Q_\tau)$ as a test function for the corresponding adjoint equation (29) and summing both expressions, one obtains

$$-\int_0^\tau \langle \partial_t u(t), \varphi(t)\rangle dt - \int_0^\tau \langle \partial_t \rho(t), u(t)\rangle dt \\ + \iint_{Q_\tau} (D_pH(x,Du)\cdot Du - H(x,Du))\rho dxdt + \iint_{Q_\tau} f\rho\, dxdt = 0\ .$$

The desired equality follows after integrating by parts in time and using property (11) of L. Note that since $H(x,Du) \in L^1(0,T; L^\sigma(\mathbb{T}^d))$, then $L(x, D_pH(Du)) \in L^1(0,T; L^\sigma(\mathbb{T}^d))$ by (L1) and (H), so all the terms in (31) make sense. □

We are now ready to prove a crucial estimate on the the integrability of D_pH with respect to ρ.

Proposition 3.3. *Let u be a weak solution to* (1) *and ρ be a weak solution to* (29). *Then, there exist positive constants C and C_1 (depending on $\lambda, \|a\|_{C(W^{1,\infty})}, C_H$, $\|u_0\|_{L^\infty(\mathbb{T}^d)}$, $\|f\|_{L^q(Q_T)}$, q, d, T) such that*

$$\iint_{Q_\tau} |D_pH(x, Du(x,t))|^{\gamma'}\rho(x,t)\, dxdt \le C \quad (32)$$

and

$$\|u(\cdot,\tau)\|_{L^\infty(\mathbb{T}^d)} \le C \qquad \textit{for all } \tau \in [0,T]. \quad (33)$$

Remark 3.4. Note that as a straightforward consequence of (32), one has

$$\iint_{Q_\tau} |Du(x,t)|^\beta \rho(x,t)\, dxdt \le C_\beta \qquad \text{for all } 1 \le \beta \le \gamma. \quad (34)$$

Indeed, by (H) and (17), $\iint_{Q_\tau} |Du(x,t)|^\gamma \rho(x,t)\, dxdt \le C$, which yields (34) for $\beta = \gamma$. For $\beta < \gamma$ it is sufficient to use Young's inequality and (17).

Proof. Rearrange the representation formula (31) to get, for $\tau \in [0,T]$,

$$\iint_{Q_\tau} L(x, D_pH(x,Du))\rho\, dxdt = \int_{\mathbb{T}^d} u(x,\tau)\rho_\tau(x)dx - \int_{\mathbb{T}^d} u(x,0)\rho(x,0) - \iint_{Q_\tau} f\rho\, dxdt. \quad (35)$$

Fix some s such that $(d+2)/\gamma' < s < d+2$ $(< q)$. Use now bounds on the Lagrangian (L1), (30) and Hölder's inequality to obtain

$$C_L^{-1} \iint_{Q_\tau} |D_pH(x,Du)|^{\gamma'}\rho\, dxdt \le \iint_{Q_\tau} L(x, D_pH(x,Du))\rho\, dxdt \\ \le 2\|u_0\|_{L^\infty(\mathbb{T}^d)} + C\|f\|_{L^q(Q_\tau)} + \|f\|_{L^s(Q_\tau)}\|\rho\|_{L^{s'}(Q_\tau)} \\ \le 2\|u_0\|_{L^\infty(\mathbb{T}^d)} + \|f\|_{L^q(Q_\tau)}(C + \|\rho\|_{L^{s'}(Q_\tau)}), \quad (36)$$

Let $\bar{q}$ be such that

$$\frac{1}{s'} = \frac{1}{\bar{q}'} - \frac{1}{d+2}$$

By Proposition A.2, $\mathcal{H}^1_{\bar{q}'}(Q_\tau)$ is continuously embedded in $L^{s'}(Q_\tau)$. Moreover, choosing $s > (d+2)/2$ guarantees $\bar{q}' < (d+2)/(d+1)$, so by inequality (23) (with q replaced by $\bar{q}$),

$$\|\rho\|_{L^{s'}(Q_\tau)} \le C(\|\rho\|_{\mathcal{H}^1_{\bar{q}'}(Q_\tau)} + 1) \le C_1 \left(\iint_{Q_\tau} |D_pH(x, Du)|^{r'} \rho(x,t)\, dxdt + 1 \right), \qquad (37)$$

where $r' = 1 + \frac{d+2}{\bar{q}}$. Finally, the right hand side of (37) can be absorbed in the left hand side of (36) whenever $r' < \gamma'$ by Young's inequality: this is assured by

$$r' = 1 + \frac{d+2}{\bar{q}} = \frac{d+2}{s} < \gamma'.$$

One then obtains (32).

Regarding (33), we already know from Proposition 3.1 that $u(\cdot, \tau)$ is essentially bounded from above. To prove the bound from below, use formula (31) and bounds from below for the Lagrangian (L1) to get

$$\int_{\mathbb{T}^d} u(x,\tau)\rho_\tau(x)dx \ge \int_{\mathbb{T}^d} u(x,0)\rho(x,0) - C_L \iint_{Q_\tau} \rho(x,t)dxdt + \iint_{Q_\tau} f\rho\, dxdt.$$

Since $\iint f\rho$ can be bounded from below using as before Hölder's inequality and (37),

$$\int_{\mathbb{T}^d} u(x,\tau)\rho_\tau(x)dx \ge \|u(\cdot,0)\|_{L^\infty(\mathbb{T}^d)} - C_L\tau - C,$$

that holds for any smooth ρ_τ with $\|\rho_\tau\|_{L^1(\mathbb{T}^d)} = 1$, implying the desired result.

□

Integrability of D_pH with respect to ρ provides finally $L^{q'}$ regularity of $D\rho$. From now on, we will suppose that $q > d+2$ and $q \ge \frac{d+2}{\gamma'-1}$.

Corollary 3.5. *Let u be a weak solution to* (1) *and ρ be a weak solution to* (29). *Let $\bar{q}$ be such that*

$$\bar{q} > d+2 \quad \text{and} \quad \bar{q} \ge \frac{d+2}{\gamma'-1}.$$

Then, there exists a positive constant C such that

$$\|\rho\|_{\mathcal{H}^1_{\bar{q}'}(Q_\tau)} \le C\ ,$$

where C depends in particular on λ, $\|a\|_{C(W^{1,\infty})}$, C_H, $\|u_0\|_{L^\infty(\mathbb{T}^d)}$, $\|f\|_{L^q(Q_\tau)}$, $\bar{q}, d, T$ (but not on τ, ρ_τ).

Proof. Since $\bar{q}' < \frac{d+2}{d+1}$, (23) applies (with $q = \bar{q}$), yielding

$$\|\rho\|_{\mathcal{H}^1_{\bar{q}'}(Q_\tau)} \le C \left(\iint_{Q_\tau} |D_pH(x, Du(x,t))|^{r'} \rho(x,t)\, dxdt + 1 \right),$$

with

$$r' = 1 + \frac{d+2}{\bar{q}} \le \gamma'.$$

If $r' = \gamma'$, use Proposition 3.3 to conclude, otherwise. If $r' < \gamma'$ use Young's inequality first to control $\iint |D_pH(x, Du(x,t))|^{r'} \rho\, dxdt$ with $\iint |D_pH(x, Du)|^{\gamma'}\, dxdt + \tau$. □

3.2 Proof of Theorem 1.1

Theorem 3.6. *Let u be a weak solution to* (1)*. Suppose also that $\mathcal{P} = \mathcal{Q}$ holds in* (12)*.*

Let $\eta = \eta(t)$ be a positive smooth function on $(0,T)$ satisfying $\eta(t) \le 1$ for all t. Then, $(\eta u)(\cdot,\tau) \in W^{1,\infty}(\mathbb{T}^d)$ for all $\tau \in (0,T)$, and there exists $C > 0$ depending on $\lambda, \|a\|_{C(W^{1,\infty})}$, $\|D^2 a\|_{L^\infty(Q_\tau)}$, C_H, $\|u_0\|_{L^\infty(\mathbb{T}^d)}$, $\|f\|_{L^q(Q_T)}$, q, d, T such that

$$\eta(\tau)\|u(\cdot,\tau)\|_{W^{1,\infty}(\mathbb{T}^d)} \le C\Big(\eta(0)\|Du_0\|_{L^\infty(Q_T)} + \|Da\|_{L^\infty(Q_\tau)}\|\eta Du\|_{L^{\mathcal{P}}(Q_\tau)} + \sup_{(0,T)}|\eta'(t)| + 1\Big)$$

for all $\tau \in (0,T]$.

Note finally that if $Da \equiv 0$ on Q_T, then the conclusion of the theorem holds for any weak solution u, i. e. without the requirement $\mathcal{P} = \mathcal{Q}$ in (12)*.*

Proof. **Step 1.** Since H is convex and superlinear we can write for a.e. $(x,t) \in Q_T$

$$H(x, Du(x,t)) = \sup_{\xi\in\mathbb{R}^d}\{\xi \cdot Du(x,t) - L(x,\xi)\}.$$

Hence we get

$$\int_0^T \langle \partial_t u(t), \varphi(t)\rangle dt + \iint_{Q_T} \partial_i u(x,t)\, \partial_j(a_{ij}(x,t)\varphi(x,t)) + [\Xi(x,t)\cdot Du(x,t) - L(x,\Xi(x,t))]\varphi\, dxdt \\ \le \iint_{Q_T} f(x,t)\varphi(x,t)\, dxdt \quad (38)$$

for all test functions $\varphi \in \mathcal{H}_2^1(Q_T) \cap L^\infty(0,\tau; L^{\sigma'}(\mathbb{T}^d))$ and measurable $\Xi : Q_T \to \mathbb{R}^d$ such that $L(\cdot,\Xi(\cdot,\cdot)) \in L^1(0,\tau; L^\sigma(\mathbb{T}^d))$ and $\Xi \cdot Du \in L^1(0,\tau; L^\sigma(\mathbb{T}^d))$. Note that the previous inequality becomes an equality if $\Xi(x,t) = D_p H(x, Du(x,t))$ in Q_T.

We first fix $\tau \in (0,T)$, ρ_τ as in (16) and $0 \ne h \in \mathbb{R}^d$. Set

$$w(x,t) = \eta(t)u(x,t).$$

Use now (38) with $\Xi(x,t) = D_p H(x, Du(x,t))$ and $\varphi = \eta\rho \in \mathcal{H}_2^1(Q_\tau) \cap L^\infty(0,\tau; L^{\sigma'}(\mathbb{T}^d))$ for all $1 < \sigma' < \infty$, where ρ is the adjoint variable (i.e. the weak solution to (29)) to find

$$\int_0^\tau \langle \partial_t w(t), \rho(t)\rangle dt + \iint_{Q_\tau} \partial_i w\, \partial_j(a_{ij}\rho) + D_p H(x,Du)\cdot Dw\rho - L(x, D_p H(x,Du))\eta\rho\, dxdt \\ = \iint_{Q_\tau} f\eta\rho\, dxdt + \iint_{Q_\tau} u\eta'\rho\, dxdt. \quad (39)$$

Then, use $w \in \mathcal{H}_2^1(Q_T)$ as a test function in the weak formulation of the equation satisfied by ρ to ge

$$-\int_0^\tau \langle \partial_t \rho(t), w(t)\rangle dt + \iint_{Q_\tau} \partial_j(a_{ij}\rho)\partial_i w + D_p H(x,Du)\rho \cdot Dw\, dxdt = 0 \quad (40)$$

We obtain, subtracting the previous equality to (39), and integrating by parts in time

$$\int_{\mathbb{T}^d} w(x,\tau)\rho_\tau(x)dx = \int_{\mathbb{T}^d} w(x,0)\rho(x,0)dx + \iint_{Q_\tau} \eta(t)f(x,t)\rho(x,t)dxdt \\ + \iint_{Q_\tau} \eta(t)L\big(x, D_p H(x,Du(x,t))\big)\rho(x,t)dxdt + \iint_{Q_\tau} \eta'(t)u(x,t)\rho(x,t)dxdt. \quad (41)$$

For $h > 0$ and $\xi \in \mathbb{R}^N$, $|\xi| = 1$ define $\hat{\rho}(x,t) := \rho(x-h,t)$. After a change of variables in (29), it can be seen that $\hat{\rho}$ satisfies, using w as a test function,

$$
\begin{aligned}
&-\int_0^T \langle \partial_t \hat{\rho}(t), w(t) \rangle dt \\
&+ \iint_{Q_T} \partial_j \big(a_{ij}(x-h,t)\hat{\rho}(x,t) \big) \partial_i w + D_p H(x-h, Du(x-h,t))\hat{\rho}(x,t) \cdot Dw(x,t)\, dxdt = 0. \qquad (42)
\end{aligned}
$$

As before, plugging $\Xi(x,t) = D_p H(x-h, Du(x-h,t))$ and $\varphi = \eta\hat{\rho}$ in (38) yields

$$
\begin{aligned}
&\int_0^\tau \langle \partial_t w(t), \hat{\rho}(t) \rangle dt + \\
&\iint_{Q_\tau} \partial_i w\, \partial_j (a_{ij}\hat{\rho}) + D_p H(x-h, Du(x-h,t)) \cdot Dw \hat{\rho} - L(x, D_p H(x-h, Du(x-h,t)))\eta\hat{\rho}\, dxdt \\
&\qquad \leq \iint_{Q_\tau} f\eta\hat{\rho}\, dxdt + \iint_{Q_\tau} u\eta'\hat{\rho}\, dxdt.
\end{aligned}
$$

Hence, subtracting (42) to the previous inequality,

$$
\begin{aligned}
\int_{\mathbb{T}^d} w(x,\tau)\hat{\rho}_\tau(x) dx - \int_{\mathbb{T}^d} w(x,0)\hat{\rho}(x,0) dx \leq \iint_{Q_T} \partial_j \Big(\big(a_{ij}(x-h,t) - a_{ij}(x,t) \big) \hat{\rho}(x,t) \Big) \partial_i w\, dxdt \\
+ \iint_{Q_\tau} L(x, D_p H(x-h, Du(x-h,t)))\eta\hat{\rho}\, dxdt + \iint_{Q_\tau} f\eta\hat{\rho}\, dxdt + \iint_{Q_\tau} u\eta'\hat{\rho}\, dxdt,
\end{aligned}
$$

which, after the change of variables $x \mapsto x + h$, becomes

$$
\begin{aligned}
&\int_{\mathbb{T}^d} w(x+h,\tau)\rho_\tau(x) dx - \int_{\mathbb{T}^d} w(x+h,0)\rho(x,0) dx \\
&\qquad \leq \iint_{Q_T} \partial_j \Big(\big(a_{ij}(x-h,t) - a_{ij}(x,t) \big) \rho(x,t) \Big) \partial_i w\, dxdt \\
&\qquad + \iint_{Q_\tau} \eta(t) L(x+h, D_p H(x, Du(x,t)))\rho(x,t)\, dxdt \\
&\qquad\qquad + \iint_{Q_\tau} f\eta\hat{\rho}\, dxdt + \iint_{Q_\tau} u\eta'\hat{\rho}\, dxdt, \qquad (43)
\end{aligned}
$$

Taking the difference between (43) and (41) we obtain

$$
\begin{aligned}
\int_{\mathbb{T}^d} (w(x+h,\tau) - w(x,\tau))\rho_\tau(x) dx &\leq \int_{\mathbb{T}^d} (w(x+h,0) - w(x,0))\rho(x,0) dx \\
&+ \iint_{Q_\tau} \partial_j \Big(\big(a_{ij}(x-h,t) - a_{ij}(x,t) \big) \rho(x,t) \Big) \partial_i w\, dxdt \\
&+ \iint_{Q_\tau} \eta(t) \Big(L(x+h, D_p H(x, Du(x,t))) - L(x, D_p H(x, Du(x,t))) \Big) \rho(x,t)\, dxdt \qquad (44) \\
&+ \iint_{Q_\tau} \eta(t) f(x,t) \big(\rho(x-h,t) - \rho(x,t) \big)\, dxdt \\
&+ \iint_{Q_\tau} \eta'(t) u(x,t) \big(\rho(x-h,t) - \rho(x,t) \big)\, dxdt.
\end{aligned}
$$

Step 2. We now estimate al the right hand side terms of (44). We stress that constants $C, C_1, \ldots$ are not going to depend on τ, ρ_τ, h. First, since $\|\rho(x,0)\|_{L^1(\mathbb{T}^d)} = 1$,

$$\left| \int_{\mathbb{T}^d} (w(x+h,0) - w(x,0))\rho(x,0)dx \right| \leq \eta(0)\|Du_0\|_{L^\infty(Q_T)}|h|.$$

Regarding the following term, by Young's and Holder's inequality

$$\left| \iint_{Q_\tau} \partial_j \Big(\big(a_{ij}(x-h,t) - a_{ij}(x,t)\big)\rho(x,t) \Big) \partial_i w \, dxdt \right| =$$

$$\left| \iint_{Q_\tau} \big(\partial_j a_{ij}(x-h,t) - \partial_j a_{ij}(x,t)\big)\rho \, \partial_i w \, dxdt + \iint_{Q_\tau} (a_{ij}(x-h,t) - a_{ij}(x,t))\partial_j \rho \, \partial_i w \, dxdt \right|$$

$$\leq \|D^2 a\|_{L^\infty(Q_\tau)}|h| \iint_{Q_\tau} |Du|\rho \, dxdt + \|Da\|_{L^\infty(Q_\tau)}|h| \iint_{Q_\tau} |\eta Du| \, |D\rho| \, dxdt$$

$$\leq C|h| \left(\iint_{Q_\tau} |Du|^\gamma \rho \, dxdt + \tau + \|Da\|_{L^\infty(Q_\tau)}\|\eta Du\|_{L^{\mathcal{P}}(Q_\tau)}\|D\rho\|_{L^{\mathcal{P}'}(Q_\tau)} \right)$$

$$\leq C_1|h| \left(\|Da\|_{L^\infty(Q_\tau)}\|\eta Du\|_{L^{\mathcal{P}}(Q_\tau)} + 1 \right), \quad (45)$$

where in the last inequality we used (34) and Corollary 3.5 (it applies since $\bar{q} = \mathcal{P} \geq (d+2)(\gamma - 1) = (d+2)/(\gamma' - 1)$).

Next, using (32) and property (L2) of $D_x L$

$$\left| \iint_{Q_\tau} \eta(t)\Big(L(x+h, D_pH(x, Du(x,t))) - L(x, D_pH(x, Du(x,t))) \Big)\rho(x,t) \, dxdt \right|$$

$$\leq |h| \iint_{Q_\tau} \|D_x L(\cdot, D_pH(x, Du(x,t)))\|_{L^\infty(\mathbb{T}^d)}\rho(x,t) \, dxdt$$

$$\leq |h| \iint_{Q_\tau} \big(|D_pH(x, Du(x,t))|^{\gamma'} + 1 \big)\rho(x,t) \, dxdt \leq C|h|.$$

Denote by $D^h\rho(x,t) := |h|^{-1}(\rho(x+h,t) - \rho(x,t))$. Then, for the term involving f we use again Corollary 3.5, with $\bar{q} = q$, and control the $L^{q'}$ norm of difference quotient $D^h\rho$ via $D\rho$ (as in, e.g. [21, Theorem 2.1.6]), to get

$$\left| \iint_{Q_\tau} \eta(t) f(x,t)\big(\rho(x-h,t) - \rho(x,t)\big) \, dxdt \right|$$

$$\leq |h| \iint_{Q_\tau} |f(x,t)| \, |D^h\rho(x,t)| \, dxdt \leq |h|\|f\|_{L^q(Q_\tau)}\|D\rho\|_{L^{q'}(Q_\tau)} \leq C|h|.$$

Finally, by boundedness of u stated in (33) and again Corollary 3.5

$$\left| \iint_{Q_\tau} \eta'(t) u(x,t)\big(\rho(x-h,t) - \rho(x,t)\big) \, dxdt \right| \leq |h|\big(\sup_{(0,T)} |\eta'(t)| \big)\|u\|_{L^\infty(Q_\tau)}\|D\rho\|_{L^1(Q_\tau)}$$

$$\leq C|h| \sup_{(0,T)} |\eta'(t)|.$$

Plugging all the estimates in (44) we obtain

$$\int_{\mathbb{T}^d} (w(x+h,\tau) - w(x,\tau))\rho_\tau(x)dx$$

$$\leq C|h| \Big(\eta(0)\|Du_0\|_{L^\infty(Q_T)} + \|Da\|_{L^\infty(Q_\tau)}\|\eta Du\|_{L^{\mathcal{P}}(Q_\tau)} + \sup_{(0,T)} |\eta'(t)| + 1 \Big) \quad (46)$$

Step 3. Since (46) holds for all smooth $\rho_\tau \geq 0$ with $\|\rho_\tau\|_{L^1(\mathbb{T}^d)} = 1$, we get

$$\eta(\tau)[u(x+h,\tau)-u(x,\tau)] \leq C|h|\Big(\eta(0)\|Du_0\|_{L^\infty(Q_T)} + \|Da\|_{L^\infty(Q_\tau)}\|\eta Du\|_{L^{\mathcal{P}}(Q_\tau)} + \sup_{(0,T)}|\eta'(t)|+1\Big) \tag{47}$$

for a.e. $x \in \mathbb{T}^d$. Note that the previous inequality holds for any $0 \neq h \in \mathbb{R}^d$. Therefore, one may select a continuous representative of $u(\cdot,\tau)$ such that (48) holds for all $x \in \mathbb{T}^d$ and $h \in \mathbb{R}^d$ (e.g. the uniform limit as $\delta \to 0$ of $u \star \delta^{-d}\chi(\cdot/\delta)$, where χ is a smooth mollifier). Thus, $u(\cdot,\tau)$ has a Lipschitz continuous representative, and

$$\eta(\tau)\|u(\cdot,\tau)\|_{W^{1,\infty}(\mathbb{T}^d)} \leq C\Big(\eta(0)\|Du_0\|_{L^\infty(Q_T)} + \|Da\|_{L^\infty(Q_\tau)}\|\eta Du\|_{L^{\mathcal{P}}(Q_\tau)} + \sup_{(0,T)}|\eta'(t)|+1\Big). \tag{48}$$

Since C does not depend on $\tau \in (0,T)$, we have proved the theorem.

Finally, for the special case $Da \equiv 0$ on Q_T, one may follow the very same lines, with the difference that there is no need to control the term appearing in (45) (which is identically zero). Therefore, there is no need to keep track of $\|\eta Du\|_{L^{\mathcal{P}}(Q_\tau)}$, and therefore the theorem is proven without assuming the constraint $\mathcal{P} = \mathcal{Q}$ in (12).

□

The following lemma shows that $\|Du\|_{L^\gamma(Q_T)}$ can be bounded by a constant depending on the data only.

Lemma 3.7. *Let u be a weak solution. Then, there exists a constant C depending on C_H, $\|u_0\|_{L^\infty(\mathbb{T}^d)}$, $\|f\|_{L^q(Q_T)}$, $\|Da\|_{L^\infty(Q_\tau)}$, q, d, T such that*

$$\|Du\|_{L^\gamma(Q_T)} \leq C.$$

Proof. Plugging $\varphi \equiv 1$ as a test function in the weak formulation of (1) we obtain

$$\int_{\mathbb{T}^d} u(x,T)\,dx - \int_{\mathbb{T}^d} u(x,0)\,dx + \iint_{Q_T} \partial_i u\,\partial_j(a_{ij}) + H(x,Du)\,dxdt = \iint_{Q_T} f dxdt$$

Hence, using (H), and Young's inequality we get

$$C_H \iint_{Q_T} |Du|^\gamma\,dxdt \leq \|u(\cdot,T)\|_{L^\infty(\mathbb{T}^d)} + \|u(\cdot,0)\|_{L^\infty(\mathbb{T}^d)} + \frac{C_H}{2}\iint_{Q_T} |Du|^\gamma\,dxdt$$
$$+ CT\|Da_{ij}\|^{\gamma'}_{L^\infty(Q_T)} + \iint_{Q_T} |f|^q dxdt + T + C_H^{-1}.$$

Therefore, we conclude using (33) and boundedness of $\|f\|_{L^q(Q_T)}$.

□

We are now ready to prove the main theorem on Lipschitz regularity of u.

Proof of Theorem 1.1. For $t_1 \in (0,T)$, let $\eta = \eta(t)$ be a positive smooth function on $(0,T)$ satisfying $\eta(t) \leq 1$ for all t, $\eta(t) \equiv 1$ on $[t_1,T]$ and $\eta(0) = 0$. Then, Theorem 3.6 yields $u(\cdot,\tau) \in W^{1,\infty}(\mathbb{T}^d)$ for all $\tau \in (0,T)$, and the existence of $C > 0$ (depending on the data and η, so t_1 itself) such that

$$\eta(\tau)\|u(\cdot,\tau)\|_{W^{1,\infty}(\mathbb{T}^d)} \leq C\big(\|Da\|_{L^\infty(Q_\tau)}\|\eta Du\|_{L^{\mathcal{P}}(Q_\tau)} + 1\big)$$

for all $\tau \in [0,T]$. If $\mathcal{P} \leq \gamma$, we immediately conclude (6) using Lemma 3.7. Otherwise, by interpolation of $L^{\mathcal{P}}(Q_\tau)$ between $L^\gamma(Q_\tau)$ and $L^\infty(Q_\tau)$ we get

$$\eta(\tau)\|Du(\cdot,\tau)\|_{L^\infty(\mathbb{T}^d)} \leq C\left(\|Da\|_{L^\infty(Q_\tau)}\|\eta Du\|^{1-\frac{\gamma}{\mathcal{P}}}_{L^\infty(Q_\tau)}\|\eta Du\|^{\frac{\gamma}{\mathcal{P}}}_{L^\gamma(Q_\tau)}+1\right),$$

that implies (6) after passing to the supremum with respect to $\tau \in (0,T)$ and again using Lemma 3.7 to control $\|\eta Du\|_{L^\gamma(Q_\tau)}$.

To prove the global in time bound (4) one may follow the same lines, using $\eta \equiv 1$ on $[0,T]$ instead.

Finally, if $a_{ij}(x,t) = A_{ij}$ on Q_T for some A_{ij} satisfying (A), then $Da \equiv 0$ on Q_T, and we obtain the same conclusion, exploiting the fact that Theorem 3.6 does not require anymore $\mathcal{P} = \mathcal{Q}$.

□

3.3 Some consequences of Lipschitz regularity

Once Lipschitz regularity is established, one can deduce further properties of weak solutions. Indeed, the viscous HJ equation (1) can be treated in terms of regularity as a linear equation, being the $H(x,Du)$ term (locally in time) bounded in L^∞. Thus, the classical Calderón-Zygmund parabolic theory applies, and the so-called maximal regularity for u follows, i.e.: $\partial_t u, \partial_{ij}u, H(x,Du) \in L^q$.

Corollary 3.8. *Under the assumptions of Theorem 1.1, any weak solution u of* (1) *is a strong solution belonging to $W^{2,1}_q(\mathbb{T}^d \times (\tau,T))$ for all $\tau \in (0,T)$, namely it solves* (1) *almost everywhere in Q_T.*

Proof. For any $\tau > 0$, Theorem 1.1 yields $H(x,Du(x,t)) \in L^\infty(\mathbb{T}^d \times (\tau/2,T))$. Therefore, since $f \in L^q(\mathbb{T}^d \times (\tau(2,T))$ and $q > d+2$, there exists a weak (energy) solution v to the linear equation

$$\partial_t v(x,t) - \sum_{i,j=1}^d a_{ij}(x,t)\partial_{ij}v(x,t) = -H(x,Du(x,t)) + f(x,t) \qquad \in L^q(\mathbb{T}^d \times (\tau/2,T)), \quad (49)$$

that satisfies $v(\tau/2) = u(\tau/2)$ in the L^2-sense, and enjoys the additional strong regularity property $W^{2,1}_q(\mathbb{T}^d \times (\tau,T))$. This can be proven using, e.g., local estimates in [16, Theorem IV.10.1]. Since weak solutions to (49) are unique, u coincides a.e. with v on $\mathbb{T}^d \times (\tau,T)$, and we obtain the assertion. □

Remark 3.9. Another consequence of the Lipschitz estimates that we obtained is the possibility to derive existence results for (1). In particular, under the assumptions of Theorem 1.1, one may obtain the existence of a unique solution $u \in \mathcal{H}^1_2(Q_T)$ that is also strong (i.e. $u \in W^{2,1}_q((\tau,T) \times \mathbb{T}^d)$ for all $\tau \in (0,T)$), satisfying the initial condition in the L^2-sense. To this aim, one has first to regularize the initial datum (e.g. via convolution) and use the global Lipschitz estimate (4) to set up a fixed point argument to get existence for small T first and arbitrary T by continuation (as in e.g. [8]). The local in time estimate (6) can be used then to remove the regularization of the initial datum, that is achieved in the limiting procedure only in the L^2-sense.

The same procedure can be applied using Theorem 1.2, to obtain a strong solution under the assumption that $f \in L^q(Q_T)$ for some $q > \min\{d+2, (d+2)/[2(\gamma'-1)]\}$. Note that, since Theorem 1.2 states a global in time a priori estimate, one has to assume that $u_0 \in W^{1,\infty}(\mathbb{T}^d)$.

3.4 Some remarks on the exponents $\mathcal{P}$, $\mathcal{Q}$, q

In the following remarks, we stress the importance of the condition $D_pH \in L^{\mathcal{Q}}(0,T;L^{\mathcal{P}}(\mathbb{T}^d))$ with $\mathcal{P}$, $\mathcal{Q}$ satisfying

$$\frac{d}{2\mathcal{P}} + \frac{1}{\mathcal{Q}} \leq \frac{1}{2}. \tag{50}$$

Not only it guarantees Lipschitz regularity of u, but is also related to uniqueness of solutions in the distributional sense. In the following examples it is indeed possible to observe multiple solutions; among them, there is one that satisfies (50) and is Lipschitz continuous, while the other(s) are not, showing therefore that Lipschitz regularity for positive times stated in Theorem 1.1 fails in general without extra integrability properties of $D_pH(x, Du)$.
We will also comment on the condition $f \in L^q(Q_T)$ for some $q > d + 2$.

Remark 3.10. We consider the super-quadratic regime $\gamma > 2$. For $\mathcal{Q} = \infty$, (50) reads

$$D_pH(x, Du) \in L^\infty(0,T;L^{\mathcal{P}}(\mathbb{T}^d)) \qquad \text{for some} \quad \mathcal{P} \geq d.$$

Let $a_{ij} = \delta_{ij}$ and $H(x,p) = |p|^\gamma$, $\gamma > 2$. For $c, \alpha > 0$, we consider the function

$$u_1(x,t) = c\psi(x)|x|^\alpha \qquad \text{on } Q_T,$$

where ψ is a smooth function having support on $B_{1/2}(0)$ and is identically one in $B_{1/4}(0)$. Note that ψ has the role of a localizing term only, so that $u_1(x,t)$ is a representative on $[-1/2, 1/2]^d$ of a periodic function on $\mathbb{R}^d$. If we let

$$\alpha = \frac{\gamma-2}{\gamma-1}, \qquad c = \frac{(d+\alpha-2)^{\frac{1}{\gamma-1}}}{\alpha}$$

then u_1 is a solution in the distributional sense to

$$\begin{cases} \partial_t u - \Delta u(x,t) + |Du(x,t)|^\gamma = f_1(x,t) & \in L^\infty(Q_T) \\ u(x,0) = c\psi(x)|x|^\alpha & \in L^\infty(\mathbb{T}^d), \end{cases} \tag{51}$$

namely it satisfies *i)* (if $d > 2$) and (13) in Definition 2.1. On the other hand, *ii)* fails, and in particular

$$(\gamma-1)|Du|^{\gamma-1} = |D_pH(x,Du)| \in L^\infty(0,T;L^{\mathcal{P}}(\mathbb{T}^d)) \qquad \text{if and only if} \quad \mathcal{P} < d.$$

Moreover, $u_1(\cdot, \tau)$ is not Lipschitz continuous for all $\tau \in [0,T]$.

Note that $u(x,0) \in L^\infty(\mathbb{T}^d)$ and $f_1 \in L^\infty(Q_T)$, so arguing as in Remark 3.9, one could obtain the existence of a strong and Lipschitz solution to (51). Thus, (51) admits two distinct distributional solutions, but only the one satisfying fully the Definition 2.1, in particular *ii)*, enjoys Lipschitz regularity.

Remark 3.11. We consider now the sub-quadratic regime $1 + (d+1)^{-1} < \gamma < 2$. For $a_{ij} = \delta_{ij}$ and $H(x,p) = |p|^\gamma$, $1 < \gamma < 2$, there exists a weak solution to (1) satisfying all requirements of Definition 2.1 except *ii)*, namely $D_pH(x,Du) \in L^{\mathcal{Q}}(0,T;L^{\mathcal{P}}(\mathbb{T}^d))$ if and only if

$$\frac{d}{2\mathcal{P}} + \frac{1}{\mathcal{Q}} > \frac{1}{2},$$

that is not Lipschitz continuous (and not even bounded in $L^\infty(\mathbb{T}^d)$) with respect to x on any time interval $(0,t)$ for all $t > 0$. The construction of such a u is based on the existence, for $k > 0$

small, of $U \in C^2(0,\infty)\cap C^1[0,\infty)$ to the Cauchy problem

$$\begin{cases} U''(y) + \left(\frac{d-1}{y} + \frac{y}{2}\right) U'(y) + U(y) + |U'(y)|^\gamma = 0 \quad \text{for } 0 < y < \infty \\ U'(0) = 0 \\ U(0) = \alpha_0, \end{cases}$$

for some $\alpha_0 > 0$, that satisfies for some positive c

$$|U(y)| + |U'(y)| + |U''(y)| \le ce^{-y} \qquad \text{as } y \to \infty.$$

The existence of such a U is proven in [2, Section 3], see in particular Theorem 3.5, Proposition 3.11, Proposition 3.14 and Remark 3.8. As in our Remark 3.10, we need a smooth localization term ψ having support on $(-1/2, 1/2)$ and identically one in $[-1/4, 1/4]$. Let then

$$u_2(x,t) = -t^{-\sigma} U(|x|\, t^{-1/2})\psi(|x|), \qquad \sigma = \frac{2-\gamma}{2(\gamma-1)}.$$

We have that u_2 is a classical solution to

$$\partial_t u(x,t) - \Delta u(x,t) + |Du(x,t)|^\gamma = f_2(x,t),$$

where $u_2(0) = 0$ in the L^2-sense whenever $\gamma > 1 + 2/(d+2)$. Moreover,

$$\begin{aligned} f_2(x,t) = -t^{-\sigma-1}\Bigg\{ &\left[U''(|x|\, t^{-1/2}) + \left(\frac{d-1}{|x|\, t^{-1/2}} + \frac{|x|\, t^{-1/2}}{2} \right) U'(|x|\, t^{-1/2}) + kU(|x|\, t^{-1/2}) \right] \psi(|x|) \\ &+ \left| U'(|x|\, t^{-1/2})\psi(|x|) + t^{1/2} U(|x|\, t^{-1/2})\psi'(|x|) \right|^\gamma \\ &+ 2t^{1/2} U'(|x|\, t^{-1/2})\psi'(|x|) + tU(|x|\, t^{-1/2})\psi''(|x|) + \frac{d-1}{|x|} tU(|x|\, t^{-1/2})\psi'(|x|) \Bigg\}. \end{aligned}$$

Note that $f_2(x,t)$ is identically zero on $|x| \le 1/4$ and $|x| \ge 1/2$; otherwise, it is bounded in L^∞, since $|U(|x|\, t^{-1/2})| + |U'(|x|\, t^{-1/2})| + |U''(|x|\, t^{-1/2})| \le ce^{-t^{-1/2}/4}$. Therefore, one should expect the existence of a weak solution to the HJ equation with zero initial datum that is Lipschitz continuous (as in Remark 3.9), but such a solution cannot be u_2, since $u_2(t)$ becomes unbounded as $t \to 0$.

Remark 3.12. To have Lipschitz bounds for solutions to (1), one cannot avoid in general the condition

$$f \in L^q(Q_T) \quad \text{for some} \quad q > d+2. \tag{52}$$

This constraint is actually imposed by the linear (heat) part of (1). Consider indeed $a_{ij} = \delta_{ij}$ and $H(x,p) = |p|^\gamma$, $\gamma > 1$. For $T > 0$, let $\chi \in C_0^\infty(\mathbb{R}^d)$, $\Gamma(x,t)$ be fundamental solution of the heat equation in $\mathbb{R}^d$, $f_3(x,t) := \chi(x/\sqrt{T-t})[\sqrt{T-t}\,\log(T-t)]^{-1}$ and u_3 be the function

$$u_3(x,t) := \iint_{\mathbb{R}^d\times(0,t)} f_3(y,s)\Gamma(x-y, t-s)\, dyds \qquad \text{on } Q_T$$

Clearly, u_3 is a classical solution to

$$\begin{cases} \partial_t u(x,t) - \Delta u(x,t) + |Du(x,t)|^\gamma = f_3(x,t) + |Du_3(x,t)|^\gamma \\ u(x,0) = 0, \end{cases}$$

$f_3 \in L^q(Q_T)$ for all $q \le d+2$ and $|Du_3|^\gamma \in L^\infty(0,T;L^\beta(\mathbb{T}^d))$ for all $\beta < \infty$. In turn, we have that $\|Du_3(\cdot,t)\|_\infty \to \infty$ as $t \to T$. Note that this example can be recast into the periodic setting by multiplying u_3 by a cut-off function ψ, as in the previous remarks.

Therefore, with respect to integrability requirements on f, Theorem 1.2 is optimal, at least when $\gamma < 3$, namely when $d+2 \ge \frac{d+2}{2(\gamma'-1)}$. We do not know whether (52) is enough also when $\gamma \ge 3$.

4 A priori estimates: Bernstein's and the adjoint methods

This section is devoted to the proof of Theorem 1.2, and complements regularity results of the previous section. Here, u is a classical solution to (1). This will allow to perform the Bernstein's method, namely to analyse the equation satisfied by $|Du|^2$. The adjoint of such an equation is basically (29). As before we will exploit the interplay between the equation itself and its adjoint.

We will assume that $a_{ij} \in C([0,T];C^1(\mathbb{T}^d))$ and satisfies (A), $H \in C^2(\mathbb{T}^d \times \mathbb{R}^d)$ and satisfies (H), $f \in C([0,T];C^1(\mathbb{T}^d))$, $u_0 \in C^1(\mathbb{T}^d)$ and

$$q > \min\left\{d+2, \frac{d+2}{2(\gamma'-1)}\right\}.$$

As before, for any fixed $\tau \in (0,T)$, $\rho_\tau \in C^\infty(\mathbb{T}^d)$, $\rho_\tau \ge 0$ with $\|\rho_\tau\|_{L^1(\mathbb{T}^d)} = 1$, let ρ be the (classical) solution to (29). Note that Proposition 3.1, Lemma 3.2 and Proposition 3.3 apply. We start with a revised version of Corollary 3.5.

Corollary 4.1. *Let u and ρ be solutions to* (1) *and* (29) *respectively. Let $\bar q$ be such that*

$$\bar q > \frac{d+2}{2(\gamma'-1)}.$$

Then, there exist constants $C > 0$ and $0 < \delta < 1$ such that

$$\|\rho\|_{\mathcal{H}^1_{\bar q'}(Q_\tau)} \le C\big(\|Du\|^{1-\delta}_{L^\infty(Q_\tau)} + 1\big)\ ,$$

where C depends in particular on $\lambda, \|a\|_{C(W^{1,\infty})}$, C_H, $\|u_0\|_{L^\infty(\mathbb{T}^d)}$, $\|f\|_{L^q(Q_T)}$, $\bar q, d, T$ (but not on τ, ρ_τ).

A straightforward consequence of the corollary is that

$$\|\rho\|_{L^{\bar p'}(Q_\tau)} \le C\big(\|Du\|^{1-\delta}_{L^\infty(Q_\tau)} + 1\big), \qquad \text{for all} \quad \bar p > \frac{d+2}{2(\gamma'-1)+1}. \tag{53}$$

Indeed, since $\bar q' < \frac{d+2}{d+1}$, Proposition A.2 gives the result.

Proof. Since $\bar q' < \frac{d+2}{d+1}$, (23) applies (with $q = \bar q$), yielding by (H)

$$\begin{aligned}\|\rho\|_{\mathcal{H}^1_{\bar q'}(Q_\tau)} &\le C\left(\iint_{Q_\tau} |D_pH(x,Du)|^{r'}\rho\, dxdt + 1\right)\\ &\le C_1\left(\iint_{Q_\tau} |Du|^{(\gamma-1)r'}\rho\, dxdt + 1\right)\\ &\le C_1\left(\|Du\|^{1-\delta}_{L^\infty(Q_\tau)}\iint_{Q_\tau} |Du|^{(\gamma-1)r'-1+\delta}\rho\, dxdt + 1\right),\end{aligned}$$

with $r' = 1 + (d+2)\bar q^{-1}$. Note that $\delta > 0$ can be chosen small so that $(\gamma-1)r' - 1 + \delta \le \gamma$. One then uses the estimate (34) on $\iint |Du|^\gamma \rho$ to conclude. □

We are now ready to prove our main a priori Lipschitz regularity result.

Proof of Theorem 1.2. **Step 1.** Set $z(x,t) := \frac{|Du(x,t)|^2}{2}$ on Q_T. Straightforward computations yield

$$z_{x_i} = Du \cdot Du_{x_i} \ , \quad z_{x_i x_j} = Du_{x_j} \cdot Du_{x_i} + Du \cdot Du_{x_i x_j} \ , \quad \partial_t z = Du \cdot D(\partial_t u) \ ,$$

which give

$$\mathrm{Tr}(AD^2 z) = \sum_{k=1}^{d} ADu_{x_k} \cdot Du_{x_k} + Du \cdot D\{\mathrm{Tr}(AD^2 u)\} - \sum_{k=1}^{d} u_{x_k} \mathrm{Tr}(A_{x_k} D^2 u) \ . \tag{54}$$

Then, differentiating the HJ equation (1) with respect to x_k, multiplying the resulting equation by u_{x_k}, and summing for $k = 1, \dots, d$, one finds

$$Du \cdot D(\partial_t u) - Du \cdot D\{\mathrm{Tr}(AD^2 u)\} + D_p H \cdot Dz + D_x H \cdot Du = Df \cdot Du \ .$$

Therefore, by plugging (54) into the previous equality we obtain the following equation satisfied by z

$$\partial_t z - \mathrm{Tr}(AD^2 z) + \sum_{k=1}^{d} ADu_{x_k} \cdot Du_{x_k} + D_p H \cdot Dz = \sum_{k=1}^{d} u_{x_k} \mathrm{Tr}(A_{x_k} D^2 u) - D_x H \cdot Du + Df \cdot Du \ . \tag{55}$$

Using the uniform ellipticity condition (A) we estimate the third term on the left-hand side as

$$\sum_{k=1}^{d} ADu_{x_k} \cdot Du_{x_k} \geq \lambda \mathrm{Tr}((D^2 u)^2).$$

Multiply (55) by the adjoint variable ρ and integrate by parts in space-time to get

$$\begin{aligned} \int_{\mathbb{T}^d} z(x,\tau)\rho_\tau(x)\,dx + \lambda \iint_{Q_\tau} \mathrm{Tr}((D^2u)^2)\rho\,dxdt &\leq \int_{\mathbb{T}^d} z(x,0)\rho(x,0)\,dxdt + \\ \iint_{Q_\tau} |D_x H||Du|\rho\,dxdt + \iint_{Q_\tau} Df \cdot Du\rho\,dxdt &+ \iint_{Q_\tau} u_{x_k} \mathrm{Tr}(A_{x_k} D^2 u)\rho\,dxdt. \end{aligned} \tag{56}$$

Step 2. We proceed by estimating the four terms on the right hand side of (56). First,

$$\int_{\mathbb{T}^d} z(x,0)\rho(x,0)\,dxdt \leq \frac{1}{2}\|Du(\cdot,0)\|^2_{L^\infty(\mathbb{T}^d)}. \tag{57}$$

Second, thanks to (H), Proposition 3.3 and Young's inequality,

$$\iint_{Q_\tau} |D_x H||Du|\rho \leq \|Du\|_{L^\infty(Q_\tau)} \left[C_H \iint_{Q_\tau} |Du|^\gamma \rho\,dxdt + C_H \tau \right] \leq C_2 + \frac{1}{8}\|Du\|^2_{L^\infty(Q_\tau)}. \tag{58}$$

We then consider $\iint Df \cdot Du\rho$. Integrating by parts,

$$\begin{aligned} \left| \iint_{Q_\tau} Df \cdot Du\rho\,dxdt \right| &= \left| \iint_{Q_\tau} f \mathrm{div}(Du\rho)\,dxdt \right| \\ &\leq \left| \iint_{Q_\tau} f Du \cdot D\rho\,dxdt \right| + \left| \iint_{Q_\tau} f \mathrm{Tr}(D^2 u)\rho\,dxdt \right| =: I_1 + I_2 \end{aligned}$$

The term I_1 can be controlled by means of Hölder's and Young's inequalities, and the control on $\|\rho\|_{\mathcal{H}^1_{\bar{q}'}}$ stated in Corollary 4.1:

$$I_1 \le \|Du\|_{L^\infty(Q_\tau)}\|f\|_{L^{\bar{q}}(Q_\tau)}\|D\rho\|_{L^{\bar{q}'}(Q_\tau)} \le C\|Du\|_{L^\infty(Q_\tau)}\|f\|_{L^{\bar{q}}(Q_\tau)}\big(\|Du\|^{1-\delta}_{L^\infty(Q_\tau)}+1\big)$$
$$\le C_3 + \frac{1}{16}\|Du\|^2_{L^\infty(Q_\tau)}. \quad (59)$$

We apply to I_2 also Hölder's and Young's inequalities to get, for a $\bar{p}>1$ to be chosen,

$$I_2 \le \frac{1}{2\lambda}\iint_{Q_\tau} f^2\rho\,dxdt + \frac{\lambda}{2}\iint_{Q_\tau}\mathrm{Tr}((D^2u)^2)\rho\,dxdt$$
$$\le \frac{1}{2\lambda}\|f\|^2_{L^{2\bar{p}}(Q_\tau)}\|\rho\|_{L^{\bar{p}'}(Q_\tau)} + \frac{\lambda}{2}\iint_{Q_\tau}\mathrm{Tr}((D^2u)^2)\rho\,dxdt.$$

Let us focus on the first term of the right-hand side of the above inequality: it can be bounded by (53) and $\|f\|_{L^q(Q_T)}$ whenever there exists $\bar{p}$ such that

$$\frac{2(d+2)}{2(\gamma'-1)+1} < 2\bar{p} \le q.$$

Such a $\bar{p}$ indeed exists, since $q > \min\left\{d+2, \frac{d+2}{2(\gamma'-1)}\right\}$. Therefore,

$$I_2 \le C_3 + \frac{1}{16}\|Du\|^2_{L^\infty(Q_\tau)} + \frac{\lambda}{2}\iint_{Q_\tau}\mathrm{Tr}((D^2u)^2)\rho\,dxdt. \quad (60)$$

For the last term $\iint u_{x_k}\mathrm{Tr}(A_{x_k}D^2u)\rho$, Cauchy-Schwartz and Young's inequalities yield

$$\iint_{Q_\tau} u_{x_k}\mathrm{Tr}(A_{x_k}D^2u)\rho\,dxdt \le C\|Da\|^2_\infty\iint_{Q_\tau}|Du|^2\rho\,dxdt + \frac{\lambda}{2}\iint_{Q_\tau}\mathrm{Tr}((D^2u)^2)\rho\,dxdt$$

We distinguish two cases: if $\gamma \ge 2$, we have by (34) (with $\beta = 2$) that $\iint_{Q_\tau}|Du|^2\rho \le C$. Otherwise, if $1<\gamma<2$,

$$\iint_{Q_\tau}|Du|^2\rho \le \|Du\|^{2-\gamma}_{L^\infty(Q_\tau)}\iint_{Q_\tau}|Du|^\gamma\rho\,dxdt \le C\|Du\|^{2-\gamma}_{L^\infty(Q_\tau)}.$$

In both cases we end up with

$$\iint_{Q_\tau} u_{x_k}\mathrm{Tr}(A_{x_k}D^2u)\rho\,dxdt \le C_4 + \frac{1}{8}\|Du\|^2_{L^\infty(Q_\tau)} + \frac{\lambda}{2}\iint_{Q_\tau}\mathrm{Tr}((D^2u)^2)\rho\,dxdt. \quad (61)$$

Step 3. Plugging (57), (58), (59), (60) and (61) into (56) we get

$$\frac{1}{2}\int_{\mathbb{T}^d}|Du(x,\tau)|^2\rho_\tau(x)\,dx = \int_{\mathbb{T}^d} z(x,\tau)\rho_\tau(x)\,dx \le \frac{1}{2}\|Du(\cdot,0)\|^2_{L^\infty(\mathbb{T}^d)} + C + \frac{3}{8}\|Du\|^2_{L^\infty(Q_\tau)}.$$

Since this inequality holds for all smooth $\rho_\tau \ge 0$ with $\|\rho_\tau\|_{L^1(\mathbb{T}^d)} = 1$, we obtain

$$\frac{1}{2}\|Du(\cdot,\tau)\|^2_{L^\infty(\mathbb{T}^d)} \le \frac{1}{2}\|Du(\cdot,0)\|^2_{L^\infty(\mathbb{T}^d)} + C + \frac{3}{8}\|Du\|^2_{L^\infty(Q_\tau)},$$

and we conclude by passing to the supremum with respect to $\tau \in (0,T)$. □

A Some auxiliary results

Lemma A.1. *Let $p > 1$, and suppose that $a_{ij} \in C(Q_T)$ saitsfies* (A). *Then, there exists a unique solution in $W^{2,1}_p(Q_T)$ to*

$$\begin{cases} \partial_t u(x,t) - \sum_{i,j=1}^d a_{ij}(x,t)\partial_{ij} u(x,t) = f(x,t) & \text{in } Q_T\ , \\ u(x,0) = 0 & \text{in } \mathbb{T}^d\ . \end{cases}$$

Moreover, there exists a constant C (depending on λ, p, and the modulus of continuity of a on Q_T) such that

$$\|u\|_{W^{2,1}_p(Q_T)} \le C\|f\|_{L^p(Q_T)}\ . \tag{62}$$

Proof. This is a classical maximal L^p regularity statement for uniformly elliptic equations with continuous coefficients, that can be deduced from results contained in [16]; see [7] for additional details. One can also rely on abstract results on maximal regularity for parabolic equations in [20]. □

The following continuous embedding result of $\mathcal{H}^1_\sigma(Q_T)$ into $L^p(Q_T)$ is rather known and can be found for example in [9]. However, we need its stability as $T \to 0$: this requires an additional control on the trace at some time (e.g. $t = 0$). We provide a proof here for the reader's convenience, that does not make use of fractional Sobolev spaces.

Proposition A.2. *If $1 < \sigma < (d+2)/(d+1)$, then $\mathcal{H}^1_\sigma(Q_T)$ is continuously embedded into $L^p(Q_T)$ for*

$$\frac{1}{p} = \frac{1}{\sigma} - \frac{1}{d+2}.$$

Moreover, if $u \in \mathcal{H}^1_\sigma(Q_T)$ and $u(\cdot,0) \in L^1(\mathbb{T}^d)$, we have

$$\|u\|_{L^p(Q_T)} \le C\big(\, \|u\|_{\mathcal{H}^1_\sigma(Q_T)} + \|u(0)\|_{L^1(\mathbb{T}^d)}\,\big)\ , \tag{63}$$

where the constant C depends on d, p, σ, T, but remains bounded for bounded values of T.

Proof. Let $f \in L^{p'}(Q_T)$ and φ be the solution to

$$\begin{cases} -\partial_t \varphi(x,t) - \Delta\varphi(x,t) = f(x,t) & \text{in } Q_T\ , \\ \varphi(x,T) = 0 & \text{in } \mathbb{T}^d\ . \end{cases}$$

By Theorem A.1, φ satisfies

$$\|\varphi\|_{W^{2,1}_{p'}(Q_T)} \le C\|f\|_{L^{p'}(Q_T)}\ . \tag{64}$$

Note that C here may depend on T, but it is the same for all $T \le 1$ (if $T < 1$, it is sufficient to extend trivially f on $\mathbb{T}^d \times (T,1)$ and use (62) on $\mathbb{T}^d \times (0,1)$). Note that $(d+2)/2 < p' < d+2$. Therefore, by the embedding results in [16, Lemma II.3.3],

$$\|\varphi\|_{C(Q_T)} \le C\|\varphi\|_{W^{2,1}_{p'}(Q_T)}, \qquad \|\varphi\|_{W^{1,0}_{\sigma'}(Q_T)} \le C\|\varphi\|_{W^{2,1}_{p'}(Q_T)} \tag{65}$$

Note that a straightforward application of [16, Lemma II.3.3] yields bounded constants in (65) as $T \to 0$, plus an additional term on the right-hand sides of the form $C_1 T^{-1}\|\varphi\|_{L^{p'}(Q_T)}$; this term can be removed using the fact that $\varphi(T) = 0$, that guarantees $\|\varphi\|_{L^{p'}(Q_T)} \le T\|\partial_t\varphi\|_{L^{p'}(Q_T)} \le \|\varphi\|_{W^{2,1}_{p'}(Q_T)}$. Note also that here we can identify norms on $\mathbb{T}^d$ with norms on $\Omega = (0,1)^d$.

Therefore, integrating by parts in time and using (64) and (65),

$$\left|\iint_{Q_T} uf\,dxdt\right| = \left|\iint_{Q_T} u(-\partial_t\varphi - \Delta\varphi)\,dxdt\right|$$
$$\leq \int_{\mathbb{T}^d} |\varphi(x,0)u(x,0)|dx + \left|\iint_{Q_T} \partial_t u\,\varphi\,dxdt\right| + \iint_{Q_T} |D\varphi|\,|Du|\,dxdt$$
$$\leq C\Big(\|\varphi(0)\|_{L^\infty(\mathbb{T}^d)}\|u(0)\|_{L^1(\mathbb{T}^d)} + \|\partial_t u\|_{\left(W^{1,0}_{\sigma'}(Q_T)\right)'}\|\varphi\|_{W^{1,0}_{\sigma'}(Q_T)} + \|Du\|_{L^\sigma(Q_T)}\|D\varphi\|_{L^{\sigma'}(Q_T)}\Big)$$
$$\leq C\Big(\|u(0)\|_{L^1(\mathbb{T}^d)} + \|\partial_t u\|_{\left(W^{1,0}_{\sigma'}(Q_T)\right)'} + \|Du\|_{L^\sigma(Q_T)}\Big)\|f\|_{L^{p'}(Q_T)},$$

yielding the desired result.

□

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cirant@math.unipd.it
Dipartimento di Matematica "Tullio Levi-Civita"
Università di Padova
via Trieste 63, 35121 Padova (Italy)
alessandro.goffi@gssi.it
Gran Sasso Science Institute
viale Francesco Crispi 7, 67100 L'Aquila (Italy)