Optimal maps in non branching spaces with Ricci curvature bounded from below

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Abstract

We prove existence of optimal maps in non branching spaces with Ricci curvature bounded from below. The approach we adopt makes no use of Kantorovich potentials.

1 Introduction

The problem of existence of optimal maps is certainly central in the theory of optimal transportation. Brenier's theorem ([4]) concerning existence, uniqueness and characterization of optimal maps in the Euclidean case with cost=distance-squared is certainly one of the first and major achievements of the theory. Since then, such result has been generalized in many directions, the most important one being the one of McCann ([9]) proving the analogous result on Riemannian manifolds. We recall that on a Riemannian manifold M, given $\mu, \nu \in \mathscr{P}_2(M)$, with μ absolutely continuous w.r.t. the volume measure, not only there is existence and uniqueness of the optimal map T, but we also know that for μ-a.e. x there exists a unique geodesic connecting x to $T(x)$.

There is a natural way to express existence and uniqueness of optimal maps in conjunction with this uniqueness of geodesic: it consists in lifting the transport problem from M^2 to the space $\mathrm{Geo}(M)$ of constant speed geodesics in M, so that given $\mu, \nu \in \mathscr{P}_2(M)$ one looks for minimizers of

$$\int \mathsf{d}^2(\gamma_0, \gamma_1)\, \mathrm{d}\boldsymbol{\pi}(\gamma), \tag{1}$$

among all plans $\boldsymbol{\pi} \in \mathscr{P}(\mathrm{Geo}(M))$ such that $(\mathrm{e}_0)_\sharp\boldsymbol{\pi} = \mu$ and $(\mathrm{e}_1)_\sharp\boldsymbol{\pi} = \nu$, where $\mathrm{e}_t : \mathrm{Geo}(M) \to M$ is the evaluation map defined by $\mathrm{e}_t(\gamma) := \gamma_t$. Then McCann's theorem and its proof shows that if μ is absolutely continuous, then there exists a unique minimizer $\boldsymbol{\pi}$ of (1) and this minimizer is induced by a map from M to $\mathrm{Geo}(M)$.

In this paper we prove the same result on abstract spaces satisfying the so called *curvature dimension condition* $CD(K, N)$, $N < \infty$, introduced by Sturm in [11] (see also

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the work by Lott-Villani [8] for the case $K = 0$), under the non-branching assumption. The case $N = \infty$ is a bit more delicate and we only have a weaker version of the result for it.

As a side note, we remark that thanks to Eulerian calculus on the Wasserstein space developed in [10] and further analyzed in [5], it is possible to prove that the Euclidean space $\mathbb{R}^d$ is a $CD(0, d)$ space without relying on existence of optimal maps, so that the argument presented here allows for a new proof of Brenier's theorem (which clearly is overall much more complicated than the original one). The argument on which it is based, was somehow also present in an embryonal form in the paper by Lott-Villani [7], where the authors proved that the 'cut-locus' in non branching $CD(K, N)$ spaces is negligible.

Finally, at the level of speculation, we remark that our result, see in particular Corollary 2.8, suggests a way to define exponentiation on non branching $CD(K, N)$ spaces. More precisely, for every $\frac{\mathsf{d}^2}{2}$-concave function $\varphi : X \to \mathbb{R}$, the fact that $\partial^c\varphi(x)$ turns out to be single valued for $\mathfrak{m}$-a.e. x, suggests to define $\exp_x(-\nabla\varphi(x))$ as the only element in $\partial^c\varphi(x)$. Clearly with this notation neither 'exp' nor '$\nabla\varphi$' make sense by themselves, but only in the formal expression $\exp_x(-\nabla\varphi(x))$. In this direction, notice that if $(\mu_t) \subset \mathscr{P}_2(X)$ is a geodesic on a non branching $CD(K, N)$ space $(X, \mathsf{d}, \mathfrak{m})$, with $\mu_0 \ll \mathfrak{m}$ and φ is a Kantorovich potential inducing it, we know that $t\varphi$ is a Kantorovich potential for the couple (μ_0, μ_t), so that we can recover the basic formula

$$\mu_t = \big(\exp(-\nabla(t\varphi))\big)_\sharp \mu_0.$$

2 Preliminaries

2.1 Metric spaces and Wasserstein distance

In this paper, (X, d) will always denote a complete and separable metric space. The set $\mathscr{P}_2(X)$ is the set of probability measures on X with finite second moment, which we endow with the Wasserstein distance W_2, defined by

$$W_2^2(\mu, \nu) := \inf_{\boldsymbol{\gamma} \in \mathrm{Adm}(\mu,\nu)} \int \mathsf{d}^2(x, y)\, \mathrm{d}\boldsymbol{\gamma}(x, y),$$

the set $\mathrm{Adm}(\mu, \nu)$ being the set of admissible transport plan, i.e. those measure $\boldsymbol{\gamma} \in \mathscr{P}_2(X^2)$ such that $\pi^1_\sharp\boldsymbol{\gamma} = \mu$, $\pi^2_\sharp\boldsymbol{\gamma} = \nu$.

$C([0, 1], X)$ is the space of continuous curves from $[0, 1]$ to X, endowed with the sup norm. It is complete and separable. For $t \in [0, 1]$, the evaluation map $\mathrm{e}_t : C([0, 1], X) \to X$ is defined by $\mathrm{e}_t(\gamma) := \gamma_t$.

A curve $\gamma : [0, 1] \to X$ is a minimizing constant speed geodesic (just geodesic in the following), provided

$$\mathsf{d}(\gamma_t, \gamma_s) = |t - s|\mathsf{d}(\gamma_0, \gamma_1), \qquad \forall t, s \in [0, 1].$$

We will denote by $\mathrm{Geo}(X)$ the space of geodesics on X, endowed with the sup norm. Notice that $\mathrm{Geo}(X)$ is complete and separable (regardless of any assumption on (X, d) beside completeness and separability).

If $\mu, \nu \in \mathscr{P}_2(X)$ are joined by a geodesic, then the distance $W_2(\mu, \nu)$ can be equivalently characterized as

$$W_2^2(\mu, \nu) = \min_{\boldsymbol{\pi}} \int_0^1 |\gamma_t|^2 \,\mathrm{d}t \,\mathrm{d}\boldsymbol{\pi}(\gamma),$$

the minimum being taken among all $\boldsymbol{\pi} \in \mathscr{P}(C([0,1], X))$ such that $(\mathrm{e}_0)_\sharp \boldsymbol{\pi} = \mu$, $(\mathrm{e}_1)_\sharp \boldsymbol{\pi} = \mu$. The set of minimizers will be denoted by $\mathrm{OptGeo}(\mu, \nu)$, and minimizers - which are always supported in $\mathrm{Geo}(X)$ - will be called optimal geodesic plan, or simply optimal plans. This point of view well adapts to the description of geodesics in $(\mathscr{P}_2(X), \mathsf{d})$, as it is known (see for instance Theorem 2.10 of [1]) that (μ_t) is a geodesic connecting μ to ν if and only if there exists $\boldsymbol{\pi} \in \mathrm{OptGeo}(\mu, \nu)$ such that

$$\mu_t = (\mathrm{e}_t)_\sharp \boldsymbol{\pi}.$$

We also recall the following basic and well known fact.

Lemma 2.1 *Let (X, d) be a metric space, $\mu, \nu \in \mathscr{P}_2(X)$, $\boldsymbol{\pi} \in \mathrm{OptGeo}(\mu, \nu)$ and $\gamma^1, \gamma^2 \in \mathrm{supp}(\boldsymbol{\pi})$. Assume that for some $t \in (0,1)$ it holds $\gamma^1_t = \gamma^2_t$. Then $\mathsf{d}(\gamma^1_0, \gamma^1_1) = \mathsf{d}(\gamma^2_0, \gamma^2_1) = \mathsf{d}(\gamma^2_0, \gamma^1_1) = \mathsf{d}(\gamma^1_0, \gamma^2_1)$.*

proof Let $D_i := \mathsf{d}(\gamma^i_0, \gamma^i_1)$, $i = 1, 2$, and notice that it holds (the first inequality coming from cyclical monotonicity):

$$\begin{aligned} D_1^2 + D_2^2 &\leq \mathsf{d}^2(\gamma^1_0, \gamma^2_1) + \mathsf{d}^2(\gamma^2_0, \gamma^1_1) \\ &\leq \big(tD_1 + (1-t)D_2\big)^2 + \big(tD_2 + (1-t)D_1\big)^2 \\ &= (2t^2 - 2t + 1)(D_1^2 + D_2^2) + (2t - 2t^2)D_1 D_2, \end{aligned}$$

and the thesis follows. $\square$

(X, d) is non-branching provided the the map $(\mathrm{e}_0, \mathrm{e}_t) : \mathrm{Geo}(X) \to X^2$ is injective for some, and thus any, $t \in (0,1)$. Notice that typically one imposes the non-branching assumption on geodesic spaces (i.e. spaces where any couple of points is connected by a geodesic), however here we don't make this assumption, because when dealing with $CD(K, \infty)$ spaces one only knows that geodesics exists for a dense set in X^2.

The following is the basic result which justifies the introduction of the non-branching assumption in the setting of optimal transport.

Theorem 2.2 *Let $(X, \mathsf{d}, \mathfrak{m})$ be a non branching metric space, $\mu, \nu \in \mathscr{P}_2(X)$, $\boldsymbol{\pi} \in \mathrm{OptGeo}(\mu, \nu)$ and $\gamma^1, \gamma^2 \in \mathrm{supp}(\boldsymbol{\pi})$. Assume that for some $t \in (0,1)$ it holds $\gamma^1_t = \gamma^2_t$. Then $\gamma^1 = \gamma^2$.*

proof By Lemma 2.1 we know that $\mathsf{d}(\gamma^1_0, \gamma^1_1) = \mathsf{d}(\gamma^2_0, \gamma^2_1)$. Now let $\gamma : [0,1] \to X$ be defined by $\gamma_s := \gamma^1_s$ for $s \in [0, t]$ and $\gamma_s := \gamma^2_t$ for $s \in [t, 1]$. By assumption we know that γ is continuous and by Lemma 2.1 that $\gamma \in \mathrm{Geo}(X)$. Since γ coincides with γ_1 in the non trivial interval $[0, t]$. $\square$

An important consequence of this theorem is the next corollary: a proof of it can be found, for instance, in [1] - see Proposition 2.16).

Corollary 2.3 *Let (X, d) be a non branching metric space and $(\mu_t) \subset \mathscr{P}_2(X)$ a geodesic. Then for any $t \in (0,1)$, $s \in [0,1]$ there exists a unique $\boldsymbol{\pi}_t^s \in \mathrm{OptGeo}(\mu_t, \mu_s)$ and it is induced by a map.*

2.2 Weak Ricci curvature bound

Here we recall the definition of metric measure spaces with Ricci curvature bounded below, given in the seminal papers of Lott-Villani [8] (for the case $K = 0$ or $N = \infty$) and of Sturm [11], [12] (for the general case).

Let (X, d) be a complete separable metric space and $\mathfrak{m} \in \mathscr{P}(X)$. For $N \in (1, \infty)$ we define $E_N : \mathscr{P}_2(X) \to \mathbb{R}$ by

$$E_N(\mu) := -\int \rho^{1-\frac{1}{N}}\, \mathrm{d}\mathfrak{m}, \qquad \mu = \rho\mathfrak{m} + \mu^s.$$

For $N = \infty$ the functional $E_\infty : \mathscr{P}_2(X) \to \mathbb{R} \cup \{+\infty\}$ is defined as

$$E_\infty(\mu) := \int \rho \log \rho \, \mathrm{d}\mathfrak{m}, \qquad \text{if } \mu = \rho\mathfrak{m},$$

and $+\infty$ otherwise. We denote by $D(E_\infty)$ the domain of E_∞. i.e. $D(E_\infty) := \{\mu \in \mathscr{P}(X) : E_\infty(\mu) < \infty\}$.

For $N \in (1, \infty)$, $K \in \mathbb{R}$ and $(t, \theta) \in [0,1] \times [0, \infty)$ we put

$$\tau_{K,N}^{(t)}(\theta) := \begin{cases} +\infty, & \text{if } K\theta^2 \geq (N-1)\pi^2, \\ t^{\frac{1}{N}} \left(\dfrac{\sin(t\theta\sqrt{K(N-1)})}{\sin(\theta\sqrt{K(N-1)})} \right)^{1-\frac{1}{N}}, & \text{if } 0 < K\theta^2 < (N-1)\pi^2, \\ t, & \text{if } K\theta^2 = 0, \\ t^{\frac{1}{N}} \left(\dfrac{\sinh(t\theta\sqrt{-K(N-1)})}{\sinh(\theta\sqrt{-K(N-1)})} \right)^{1-\frac{1}{N}}, & \text{if } K\theta^2 < 0. \end{cases}$$

Definition 2.4 (Weak Ricci curvature bound) *We say that $(X, \mathsf{d}, \mathfrak{m})$ is a $CD(K,N)$ space, $K \in \mathbb{R}$ $N \in (1, \infty)$ provided for any μ, ν there exists $\boldsymbol{\pi} \in \mathrm{OptGeo}(\mu, \nu)$ such that*

$$E_{N'}((\mathrm{e}_t)_\sharp \boldsymbol{\pi}) \leq -\int \tau_{K,N'}^{(1-t)}(\mathsf{d}(\gamma_0, \gamma_1))\rho^{-\frac{1}{N'}}(\gamma_0) + \tau_{K,N'}^{(t)}(\mathsf{d}(\gamma_0, \gamma_1))\eta^{-\frac{1}{N'}}(\gamma_1)\, \mathrm{d}\boldsymbol{\pi}(\gamma), \quad \forall t \in [0,1], \tag{2}$$

for any $N' \geq N$, where $\mu = \rho\mathfrak{m} + \mu^s$ and $\nu = \eta\mathfrak{m} + \nu^s$.

We say that $(X, \mathsf{d}, \mathfrak{m})$ is a $CD(K, \infty)$ space provided for any couple of measures $\mu, \nu \in D(E_\infty)$ there exists $\boldsymbol{\pi} \in \mathrm{OptGeo}(\mu, \nu)$ such that

$$E_\infty((\mathrm{e}_t)_\sharp \boldsymbol{\pi}) \leq (1-t)E_\infty(\mu) + tE_\infty(\nu) - \frac{K}{2}t(1-t)W_2^2(\mu, \nu), \qquad \forall t \in [0,1]. \tag{3}$$

Lemma 2.5 *Let $(X, \mathsf{d}, \mathfrak{m})$ be a non branching $CD(K,N)$ space. If $N < \infty$, then any optimal geodesic plan $\boldsymbol{\pi}$ satisfies* (2).

If $N = \infty$, $\boldsymbol{\pi}$ is an optimal geodesic plan with $(\mathrm{e}_0)_\sharp\boldsymbol{\pi}, (\mathrm{e}_1)_\sharp\boldsymbol{\pi} \in D(E_\infty)$ satisfying (3) *and $\tilde{\boldsymbol{\pi}} \ll \boldsymbol{\pi}$ with bounded density, then $\tilde{\boldsymbol{\pi}}$ satisfies* (3) *as well and $(\mathrm{e}_0)_\sharp\tilde{\boldsymbol{\pi}}, (\mathrm{e}_1)_\sharp\tilde{\boldsymbol{\pi}} \in D(E_\infty)$.*

proof Let $N = \infty$. By our assumptions on $\boldsymbol{\pi}$ we know that $(\mathrm{e}_t)_\sharp\boldsymbol{\pi} \in D(E_\infty)$ for any $t \in [0,1]$. Thus, since $\frac{\mathrm{d}\tilde{\boldsymbol{\pi}}}{\mathrm{d}\boldsymbol{\pi}}$ is bounded, we also have $\mu_t := (\mathrm{e}_t)_\sharp\tilde{\boldsymbol{\pi}} \in D(E_\infty)$ for any $t \in [0,1]$. By Corollary 2.3 we know that for $s \in (0,1)$ there exists a unique geodesic from μ_0 to μ_s, which therefore must be the restriction (and rescaling) of (μ_t) to $[0,s]$. Since $(X, \mathsf{d}, \mathfrak{m})$ is $CD(K,\infty)$, and $\mu_0, \mu_s \in D(E_\infty)$ we get

$$E_\infty(\mu_{\mu_{ts}}) \le (1-t)E_\infty(\mu_0) + tE_\infty(\mu_s) - \frac{K}{2}t(1-t)W_2^2(\mu_0,\mu_s), \qquad \forall t \in [0,1]. \tag{4}$$

Letting $t \downarrow 0$ and using the lower semicontinuity of E_∞ we get that $E_\infty(\mu_t) \to E(\mu_0)$. Arguing symmetrically we also get $E_\infty(\mu_t) \to E_\infty(\mu_1)$ as $t \uparrow 1$. Hence, letting $s \uparrow 1$ in (4) we conclude.

The case $N < \infty$ follows along similar lines, taking into account that $E_N(\mu) \in \mathbb{R}$ for any $N \in (1,\infty)$, $\mu \in \mathscr{P}(X)$. $\square$

Lemma 2.6 *Let $(X, \mathsf{d}, \mathfrak{m})$ be a non branching $CD(K,N)$ (resp. $CD(K,\infty)$) space and $\boldsymbol{\pi} \in \mathscr{P}(\mathrm{Geo}(X))$ be a plan satisfying* (2) *(resp.* (3) *with $(\mathrm{e}_i)_\sharp\boldsymbol{\pi} \in D(E_\infty)$, $i = 0,1$). Let $(\mathrm{e}_t)_\sharp\boldsymbol{\pi} = \rho_t\mathfrak{m} + \mu^s$. Then*

$$\mathfrak{m}(\{\rho_0 > 0\}) \le \varliminf_{t\downarrow 0}\ \mathfrak{m}(\{\rho_t > 0\})$$

proof Assume $N < \infty$. If $(\mathrm{e}_0)_\sharp\boldsymbol{\pi} \perp \mathfrak{m}$ there is nothing to prove, so we can assume $\mathfrak{m}(\{\rho_0 > 0\}) > 0$. Let $A := \{\rho_0 > 0\}$, $\mathcal{A} := \{\gamma : \rho_0(\gamma_0) > 0\}$ and define the plans $\boldsymbol{\pi}', \boldsymbol{\pi}'' \in \mathscr{P}(\mathrm{Geo}(X))$ by

$$\boldsymbol{\pi}' := \boldsymbol{\pi}(\mathcal{A})^{-1}\boldsymbol{\pi}|_{\mathcal{A}},$$
$$\mathrm{d}\boldsymbol{\pi}''(\gamma) := \frac{\mathfrak{m}(A)}{\rho_0(\gamma_0)}\mathrm{d}\boldsymbol{\pi}'(\gamma).$$

Since $\mathrm{supp}(\boldsymbol{\pi}'') \subset \mathrm{supp}(\boldsymbol{\pi})$, $\boldsymbol{\pi}''$ is optimal as well and thus satisfies (2). Also, by construction, we have $(\mathrm{e}_0)_\sharp\boldsymbol{\pi}'' = \mathfrak{m}(A)^{-1}\mathfrak{m}|_A$. Furthermore, putting $(\mathrm{e}_t)_\sharp\boldsymbol{\pi}'' = \tilde{\mu}_t = \tilde{\rho}_t\mathfrak{m} + \tilde{\mu}_t^s$ we have $\mathfrak{m}(\{\tilde{\rho}_t > 0\}) \le \mathfrak{m}(\{\rho_t > 0\})$ for any $t \in [0,1]$. Hence the conclusion follows from

$$E_N(\tilde{\mu}_t) \ge -\mathfrak{m}(\{\tilde{\rho}_t > 0\})^{\frac{1}{N}},$$
$$\lim_{t\downarrow 0} E_n(\tilde{\mu}_t) = E_N(\tilde{\mu}_0) = -\mathfrak{m}(\{\tilde{\rho}_0 > 0\})^{\frac{1}{N}},$$

which are easy consequences of the definition of E_N and of (2).

For the case $N = \infty$ we argue as follows. First of all we know from the assumptions that $(\mathrm{e}_t)_\sharp\boldsymbol{\pi} \ll \mathfrak{m}$ for any $t \in [0,1]$. Let ρ_t be its density and fix $\varepsilon > 0$. Then we can find a Borel set $A_\varepsilon \subset \{\rho_0 > 0\}$ such that $\mathfrak{m}(\{\rho_0 > 0\}) - \mathfrak{m}(A) < \varepsilon$ and $0 < c \le \rho_0(x) \le C < \infty$ for some c, C. We now proceed as before defining $\mathcal{A}_\varepsilon := \{\gamma : \gamma_0 \in A_\varepsilon\}$, and $\boldsymbol{\pi}', \boldsymbol{\pi}'' \in \mathscr{P}(\mathrm{Geo}(X))$ by

$$\begin{aligned} \boldsymbol{\pi}' &:= \boldsymbol{\pi}(\mathcal{A}_\varepsilon)^{-1} \boldsymbol{\pi}|_{\mathcal{A}_\varepsilon}, \\ \mathrm{d}\boldsymbol{\pi}''(\gamma) &:= \frac{\mathfrak{m}(A_\varepsilon)}{\rho_0(\gamma_0)} \mathrm{d}\boldsymbol{\pi}'(\gamma). \end{aligned}$$

By construction, $(\mathrm{e}_0)_\sharp \boldsymbol{\pi}'' = \mathfrak{m}(A_\varepsilon)^{-1} \mathfrak{m}|_{A_\varepsilon}$, $\boldsymbol{\pi}'' \ll \boldsymbol{\pi}$ (so in particular $\boldsymbol{\pi}''$ is optimal) and $\frac{\mathrm{d}\boldsymbol{\pi}''}{\mathrm{d}\boldsymbol{\pi}}$ is bounded. Thus from Lemma 2.5 we know that $\boldsymbol{\pi}''$ satisfies (3) with marginals in the domain of E_∞ so that arguing as before we get

$$\mathfrak{m}(\{\rho_0 > 0\}) - \varepsilon \le \varliminf_{t \downarrow 0} \ \mathfrak{m}(\{\rho_t > 0\}).$$

Since $\varepsilon > 0$ was arbitrary, the proof is concluded. $\square$

Theorem 2.7 (Optimal maps) *Let $(X, \mathsf{d}, \mathfrak{m})$ be a complete, separable, geodesic and non branching space. Then the following is true.*

i) Assume it is a $CD(K, N)$ space, $\mu, \nu \in \mathscr{P}_2(X)$ and $\mu \ll \mathfrak{m}$. Then there is a unique plan $\boldsymbol{\pi} \in \mathrm{OptGeo}(\mu, \nu)$ and this $\boldsymbol{\pi}$ is induced by a map.

ii) Assume it is a $CD(K, \infty)$ space, $\mu, \nu \in D(E_\infty) \cap \mathscr{P}_2(X)$. Then there is a unique plan $\boldsymbol{\pi} \in \mathrm{OptGeo}(\mu, \nu)$ satisfying (3) and this $\boldsymbol{\pi}$ is induced by a map.

proof In case (i) we know by Lemma 2.5 that any $\boldsymbol{\pi} \in \mathrm{OptGeo}(\mu, \nu)$ satisfies (2). Thus in either of the cases we have to prove that there exists a unique $\boldsymbol{\pi} \in \mathrm{OptGeo}(\mu, \nu)$ for which the corresponding convexity inequality is satisfied. We claim that in order to do so, it is sufficient to prove that any $\boldsymbol{\pi} \in \mathrm{OptGeo}(\mu, \nu)$ for which the convexity inequality is satisfied, is actually induced by map. Indeed, if we have this and, by absurdum, there are two different plans $\boldsymbol{\pi}^1, \boldsymbol{\pi}^2 \in \mathrm{OptGeo}(\mu, \nu)$ for which (2) (resp. (3)) is fulfilled, then by convexity w.r.t. linear interpolation of E_N (resp. E_∞), the same would be true for $\frac{1}{2}(\boldsymbol{\pi}_1 + \boldsymbol{\pi}_2)$, which is not induced by a map.

Thus let $\boldsymbol{\pi} \in \mathrm{OptGeo}(\mu, \nu)$ be satisfying (2) (resp. (3) with extrema in the domain of E_∞). Assume by contradiction that $\boldsymbol{\pi}$ is not induced by a map. This is the same as to say that $\boldsymbol{\pi}_x$ is not a Dirac mass for a set of x of positive μ-measure, where $\{\boldsymbol{\pi}_x\}$ is the disintegration of $\boldsymbol{\pi}$ w.r.t. e_0.

By a measurable selection argument we can find a μ-measurable map $T : X \to \mathrm{Geo}(X)$ such that $T(x) \in \mathrm{supp}(\boldsymbol{\pi}_x)$ for μ-a.e. x. Our assumption guarantees that $\boldsymbol{\pi} \neq T_\sharp \mu$. Therefore we can find $r > 0$ small enough so that the set

$$A := \Big\{ x \in X \ : \ \boldsymbol{\pi}_x(B_r(T(x))) > 0, \ \boldsymbol{\pi}_x(B_{2r}(T(x))) < 1 \Big\},$$

has positive μ-measure. Similarly, we can find $R > 2r$ large enough so that the set

$$B := \Big\{ x \in A \ : \ \boldsymbol{\pi}_x(B_R(T(x)) \setminus B_{2r}(T(x))) > 0 \Big\},$$

has positive μ-measure. Also, we can find $c > 0$ small enough so that the set

$$C := \Big\{ x \in B \ : \ c \le \frac{\mathrm{d}\mu}{\mathrm{d}\mathfrak{m}}(x) \le c^{-1} \Big\},$$

has positive $\mathfrak{m}$-measure. Finally, we find a compact set $D \subset C$ such that

$$L := \sup_{x \in D} \mathsf{d}(x, \mathrm{e}_1(T(x))) < \infty.$$

By construction, the measures

$$\begin{aligned}
\boldsymbol{\pi}^1_x &:= \Big(\boldsymbol{\pi}_x\big(B_r(T(x))\big)\Big)^{-1} \boldsymbol{\pi}_x|_{B_r(T(x))},\\
\boldsymbol{\pi}^2_x &:= \Big(\boldsymbol{\pi}_x\big(B_R(T(x)) \setminus B_{2r}(T(x))\big)\Big)^{-1} \boldsymbol{\pi}_x|_{B_R(T(x)))\setminus B_{2r}(T(x))},
\end{aligned}$$

are well defined for $\mathfrak{m}$-a.e. $x \in D$. Define the plans

$$\begin{aligned}
\boldsymbol{\pi}^1 &:= \int_D \boldsymbol{\pi}^1_x \,\mathrm{d}\mathfrak{m}(x),\\
\boldsymbol{\pi}^2 &:= \int_D \boldsymbol{\pi}^2_x \,\mathrm{d}\mathfrak{m}(x),
\end{aligned}$$

Clearly $\boldsymbol{\pi}^1, \boldsymbol{\pi}^2 \ll \boldsymbol{\pi}$, $\boldsymbol{\pi}^1 \perp \boldsymbol{\pi}^2$ and $(\mathrm{e}_0)_\sharp \boldsymbol{\pi}^1 = (\mathrm{e}_0)_\sharp \boldsymbol{\pi}^2 = \mathfrak{m}|_D$. From the definitions of A, B, C we also get that both $\frac{\mathrm{d}\boldsymbol{\pi}^1}{\mathrm{d}\boldsymbol{\pi}}$ and $\frac{\mathrm{d}\boldsymbol{\pi}^2}{\mathrm{d}\boldsymbol{\pi}}$ are bounded. Hence from Lemma 2.5 we have that $\boldsymbol{\pi}^1, \boldsymbol{\pi}^2$ satisfy (2) (resp. (3) with marginals in the domain of E_∞). Therefore from Lemma 2.6 we deduce

$$\varliminf_{t \downarrow 0} \mathfrak{m}\big(\{\rho^i_t > 0\}\big) \ge \mathfrak{m}(D), \qquad i = 1, 2, \tag{5}$$

where ρ^i_t is the density of $(\mathrm{e}_t)_\sharp \boldsymbol{\pi}^i$.

Here it comes the main point of the proof. For $\varepsilon > 0$ let D_ε be the ε-neighborhood of D, i.e. $D_\varepsilon := \{x \in X : \mathsf{d}(x, D) < \varepsilon\}$. Since D is compact, for ε sufficiently small, we certainly have that $\mathfrak{m}(D_\varepsilon) \le \frac{3}{2}\mathfrak{m}(D)$. Fix such ε and notice that by definition of D, we know that $\mathrm{supp}((\mathrm{e}_t)_\sharp \boldsymbol{\pi}^i)$ is contained in the $t(L+R)$-neighborhood of D, $i = 1, 2$. Hence if $t < \frac{\varepsilon}{L+R}$ is small enough, by (5) we get that the two sets $\{\rho^1_t > 0\}$ and $\{\rho^2_t > 0\}$ both have $\mathfrak{m}$-measure at least comparable with that of D, and contained in a set of measure bounded by $\frac{3}{2}\mathfrak{m}(D)$. Thus they must intersect in a set of positive $\mathfrak{m}$-measure E. This means that for any $x \in E$ there are $\gamma^{1,x} \in \mathrm{supp}(\boldsymbol{\pi}^1)$ and $\gamma^{2,x} \in \mathrm{supp}(\boldsymbol{\pi}^2)$ such that $\gamma^1_t = \gamma^2_t = x$. By Theorem 2.2 we deduce that $\gamma^{1,x} = \gamma^{2,x}$ for any $x \in E$, which is absurdum because we know that $\boldsymbol{\pi}^1 \perp \boldsymbol{\pi}^2$. $\square$

Corollary 2.8 *Let $(X, \mathsf{d}, \mathfrak{m})$ be a non branching $CD(K,N)$ space, $N < \infty$, and $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ a $\frac{\mathsf{d}^2}{2}$-concave function. Let $D := \{x : \varphi(x) \in \mathbb{R}\}$ and Ω the interior of D. Then for $\mathfrak{m}$-a.e. $x \in \Omega$ the set $\partial^c\varphi(x)$ contains exactly one point and there is only one geodesic connecting this point to x.*

Note that the last part of the statement is not a consequence of the negligibility of the cut locus proved by Lott-Villani in [7].

proof
Non emptiness. To prove that $\partial^c\varphi(x) \neq \emptyset$ for $\mathfrak{m}$-a.e $x \in \Omega$ we argue exactly as in Step 1 and Step 3 of the proof of Theorem 1 of [6]. The proof works verbatim as it is, thanks to the fact that closed balls in $CD(K,N)$ spaces are compact (see [12]). Notice that we actually get that $\partial^c\varphi(x) \neq \emptyset$ for any (and not just $\mathfrak{m}$-a.e.) $x \in \Omega$.
Uniqueness. Argue by contradiction and assume there is a Borel set $A \subset \Omega$ of positive measure such that

$$G_x := \Big\{\gamma \in \mathrm{Geo}(X) \ : \ \gamma_0 = x,\ \gamma_1 \in \partial^c\varphi(x)\Big\},$$

contains more than one point for any $x \in A$. Up to a restriction we can assume that both A and $\partial^c\varphi(A)$ are bounded.

Since the map $G : X \to \mathrm{Geo}(X)$ has closed graph, by a measurable selection argument we can find 2 $\mathfrak{m}$-measurable maps $T^1, T^2 : A \to \mathrm{Geo}(X)$ such that $T^1(x), T^2(x) \in G_x$ and $T^1(x) \neq T^2(x)$ for any $x \in A$.

Now define $\mu := \mathfrak{m}(A)^{-1}\mathfrak{m}|_A$ and $\boldsymbol{\pi} \in \mathscr{P}(\mathrm{Geo}(X))$ by

$$\boldsymbol{\pi} := \frac{T^1_\sharp\mu + T^2_\sharp\mu}{2}.$$

By construction, $\mathrm{supp}((\mathrm{e}_0, \mathrm{e}_1)_\sharp\boldsymbol{\pi}) \subset \partial^c\varphi$, and hence $\boldsymbol{\pi}$ is an optimal plan. Also, by definition $\boldsymbol{\pi}$ is not induced by a map. Since $(\mathrm{e}_0)_\sharp\boldsymbol{\pi} = \mu \ll \mathfrak{m}$, these facts contradict Theorem 2.7 above. □

We conclude with some comments.

The same argument presented here can be applied to spaces satisfying the reduced curvature dimension condition $CD^*(K,N)$ introduced by Bacher-Sturm in [3]. The proof is exactly the same.

The assumption $\mathfrak{m} \in \mathscr{P}(X)$ is not really necessary, but has been made to simplify the exposition. The problem with reference measures with infinite mass is the fact that the entropy functionals E_N, E_∞ might be not lower semicontinuous in $(\mathscr{P}_2(X), W_2)$, a property which has been used in the proof of Lemma 2.5.

In the case $N = \infty$ it has been showed in [2] that a sufficient condition on $\mathfrak{m}$ which grants the desired lower semicontinuity is the existence of $C > 0$ such that

$$\int e^{-C\mathsf{d}^2(x,x_0)}\,\mathrm{d}\mathfrak{m}(x) < \infty \tag{6}$$

for some $x_0 \in X$. Thus if $\mathfrak{m}$ is a σ-finite measure satisfying (6) and $(X, \mathsf{d}, \mathfrak{m})$ is a $CD(K,\infty)$ space, Theorem 2.7 remains true.

The case $N < \infty$ is somehow more delicate to discuss. It should be noticed, for instance, that on the hyperbolic plane $\mathbb{H}^2$, due to the exponential volume growth, for any $N \in (1,\infty)$

it is possible to construct a measure $\mu_N \in \mathscr{P}_2(\mathbb{H}^2)$ such that $E_N(\mu_N) = -\infty$, which easily yields that E_N is not lower semicontinuous on $(\mathscr{P}_2(\mathbb{H}^2), W_2)$. Yet, the Bishop-Gromov comparison result available in $CD(K, N)$ spaces (see [12]) ensures that bounded sets have finite mass. Hence Theorem 2.7 remains true also in this case: at the beginning of the argument by contradiction it is sufficient to restrict the plan $\boldsymbol{\pi}$ to a bounded set of geodesics where it is still not induced by map. Then everything takes place on a bounded set and the proof goes on as in the case discussed.

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