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FINITE-TIME BLOW-UP AND GLOBAL SOLUTIONS FOR SOME NONLINEAR PARABOLIC EQUATIONS

Filippo Gazzola
Dipartimento di Matematica del Politecnico
Piazza L. da Vinci - 20133 Milano, Italy

(Submitted by: Takashi Suzuki)

Abstract. For a class of semilinear parabolic equations, we prove both global existence and finite-time blow-up depending on the initial datum. The proofs involve tools from the potential-well theory, from the critical-point theory, and from classical comparison principles.

1. Introduction

Let Ω be an open, bounded domain of $\mathbb{R}^n$ $(n \geq 3)$ with smooth boundary $\partial\Omega$. Depending on suitable properties of the initial datum u_0, we are interested in existence of both finite-time blow-up solutions and of solutions which exist globally in time of the following parabolic problem:

$$\begin{cases} u_t - \Delta u = \lambda|1+u|^{p-1}(1+u) & \text{in } \Omega \times (0,T) \\ u(0) = u_0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{1.1}$$

where $T \in (0,\infty]$, $\lambda > 0$ and $p > 1$. Problems like (1.1) take origin in the pioneering works by Gelfand [18], Keller-Cohen [25] and Fujita [13, 14], where semilinear parabolic equations like

$$u_t - \Delta u = f(u) \ , \tag{1.2}$$

with f increasing, convex, and satisfying $f(0) > 0$ are considered. Many subsequent developments followed: with no hope of being complete, let us just mention [7, 10, 24, 27, 28, 29]. This kind of equation arises from physical models [25] for which one is usually interested in positive solutions.

On the other hand, when $f(0) = 0$ and f satisfies suitable growth conditions, Payne-Sattinger [31] developed the so-called *potential-well theory* in

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order to study (1.2). Depending on the initial datum u_0, they show the existence of both solutions of (1.2) which blow up in finite time and of solutions of (1.2) which exist globally in time and converge to 0 as time tends to infinity (since $f(0) = 0$, the null function $u \equiv 0$ is a stationary solution of (1.2)). We also refer to [21, 22, 23, 32], where the case $f(u) = |u|^{p-1}u$ is extensively studied.

Our paper is in some sense a compromise between these two classes of problems. We study (1.1) (a problem with $f(0) > 0$, also called a "positone" problem [25]) with the tools of the potential-well theory by taking advantage of the power-like behavior of the reaction term. The potential-well theory strongly relies on critical-point theory which, in turn, enables us to refine classical results about (1.1) obtained with comparison principles and for bounded initial data.

In equation (1.1), the null function *is not* a stationary solution, and this gives several complications in the definition of the potential well and in the characterization of the Nehari manifold. Firstly, the corresponding action functional must be translated; secondly, if $1 < p < 2$ it is not clear whether half lines starting from the (unique) stable stationary solution cross exactly once the Nehari manifold; see Problem 2 in Section 12. Therefore, we are led to give a new characterization of the potential well. In Theorem 4 we prove that if p is subcritical and the initial datum u_0 is inside the (generalized) potential well then the unique solution of (1.1) is global and converges in the Dirichlet norm towards the stable stationary solution. Due to a better characterization of the potential well, this statement slightly simplifies in the case $p \geq 2$; see Theorem 5 below. As far as the blow-up is concerned, we prove two different kinds of statements: with the test function method by Mitidieri-Pohožaev [30] we prove in Theorems 6–7 that the solution of (1.1) blows up in finite time for nonnegative initial data u_0 with sufficiently large norm, whereas in Theorem 8 we prove that the same occurs if u_0 is outside the potential well at low energy.

In Section 3, we consider the critical problem ($p = \frac{n+2}{n-2}$) in the unit ball where positive stationary solutions of (1.1) are explicitly known. This enables us to test our results and to show that in some cases the potential-well theory gives better results than the classical comparison methods.

The remaining of this paper is organized as follows. In the next section we first overview the state of the art, then we give the precise statement of our main results. Sections 4–11 are devoted to the proofs of the results, while in Section 12 we quote some further remarks and open problems.

2. OLD AND NEW RESULTS

Throughout this paper we assume that $\Omega \subset \mathbb{R}^n$ is an open, bounded domain with smooth boundary $\partial\Omega$. We denote by $\|\cdot\|_q$ the $L^q(\Omega)$ norm for $1 \le q \le \infty$, by $\|\cdot\|$ the Dirichlet norm in $H_0^1(\Omega)$, and by $\|\cdot\|_{H^s}$ the norm in the Sobolev space H^s $(s > 1)$. Let $2^* = \frac{2n}{n-2}$ be the critical Sobolev exponent. We will also need the cone of nonnegative functions

$$\mathbb{K} = \{u \in L^1_{\mathrm{loc}}(\Omega);\, u \ge 0 \text{ a.e. in } \Omega\}\ .$$

In order to explain our results, we first recall some well-known statements.

2.1. Some well-known facts. Positive stationary solutions of (1.1) solve the elliptic problem

$$\begin{cases} -\Delta u = \lambda(1+u)^p & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

By solutions of (2.1) we mean here *weak* solutions in $H_0^1 \cap L^p(\Omega)$. Weak solutions may be either bounded (in $L^\infty(\Omega)$) or unbounded. By elliptic regularity, it follows that any bounded solution belongs to $C^\infty(\Omega)$ and, up to the boundary, it is as smooth as $\partial\Omega$ permits. Finally, if $p \le \frac{n+2}{n-2}$, by the Moser iteration scheme any solution u of (2.1) is bounded.

Problem (2.1) may admit several different kinds of solutions, but one of them is particularly important because of its stability properties:

Definition 1. *A solution u_λ of* (2.1) *is called* ***minimal****, if $u_\lambda \le v$ almost everywhere in Ω for any further solution v of* (2.1)*.*

Let us recall that if $p \le \frac{n+2}{n-2}$, then solutions of (2.1) may be found by using critical-point theory. We assume that the minimax variational characterization of mountain-pass solutions given by Ambrosetti-Rabinowitz [1] is familiar to the reader. Then, in the next statement we collect a number of known facts:

Theorem 1. [7, 8, 9, 10, 24, 27, 29] *Let $p > 1$. Then, there exists $\lambda^* = \lambda^*(\Omega, p) > 0$ such that*

(i) if $\lambda > \lambda^*$ there are no solutions of (2.1) *even in the distributional sense.*

(ii) if $\lambda = \lambda^*$ there exists a unique (possibly unbounded) solution U_* of (2.1)*.*

(iii) if $0 < \lambda < \lambda^*$, (2.1) *admits a minimal bounded solution* u_λ*; moreover, if* $p \leq \frac{n+2}{n-2}$ *problem* (2.1) *also admits a (bounded) mountain-pass solution* U_λ *which satisfies* $U_\lambda > u_\lambda$ *in* Ω.

Results in the case $0 < p \leq 1$ may be found in [25], while further details about (2.1) and some properties of the map $\lambda^* = \lambda^*(\Omega, p)$ may be found in [17] and references therein.

From now on we denote by u_λ and U_* the functions defined in Theorem 1. Any mountain-pass solution of (2.1) will be denoted by U_λ: such a solution may not be unique (see [11]), but also in this case its energy level is uniquely determined by its minimax characterization [1].

Concerning the evolution equation (1.1), we need to recall its local solvability in the at most critical case. The next statement follows from some results by Lunardi [26] and Arrieta-Carvalho [2]. In Section 4 we sketch the basic ideas of its proof:

Theorem 2. [2, 26] *Let* $1 < p \leq \frac{n+2}{n-2}$ *and* $\lambda > 0$*. For all* $u_0 \in H_0^1(\Omega)$ *there exists* $T \in (0, \infty]$ *such that* (1.1) *admits a unique solution*

$$u \in C^0([0,T); H_0^1(\Omega)) \cap C^1([0,T); H^{-1}(\Omega)),$$

which becomes a classical solution for $t > 0$*. Moreover, if* $[0, T^*)$ *denotes the maximal interval of continuation of* u *and* $T^* < \infty$ *then, if* $p < \frac{n+2}{n-2}$,

$$\lim_{t \to T^*} \|u(t)\| = +\infty \ , \tag{2.2}$$

whereas, if $p = \frac{n+2}{n-2}$,

$$\lim_{t \to T^*} \|u(t)\|_{H^{1+\varepsilon}} = +\infty \qquad \textit{for all } \varepsilon > 0 \ . \tag{2.3}$$

Finally, if $u_0 \geq 0$*, then* $u(x,t) \geq 0$ *in* $\Omega \times [0, T^*)$.

Remark 1. Theorem 2 has a long story, and it looks difficult to give precise references. Apart from the already mentioned [2, 26], let us also refer the reader to [6, 20, 33, 34].

Remark 2. If $p = \frac{n+2}{n-2}$, it is not clear in which way the blow-up occurs; see [23, Remark 2.5]. Nevertheless, it is reasonable to expect that (2.3) cannot be improved with (2.2); see [3, Theorem 1.1] where it is shown that critical-growth parabolic problems are not uniformly well posed.

Theorem 2 also defines $T^* \in (0, +\infty]$; if $T^* < \infty$ we say that T^* is the **blow-up time**, whereas if $T^* = +\infty$ we say that u is a **global solution**. Finally, let us recall that the existence of global solutions of (1.1) is related to the existence of weak solutions of (2.1):

Theorem 3. [7, 14, 28] *Assume that $p > 1$, and let λ^* be as in Theorem* 1.

(i) *If there exists a global classical solution of* (1.1) *for some $u_0 \in L^\infty(\Omega) \cap \mathbb{K}$, then $\lambda \le \lambda^*$.*

(ii) *Conversely, if $\lambda \le \lambda^*$ and w denotes any weak solution of* (2.1), *then for any $u_0 \in L^\infty(\Omega)$ such that $0 \le u_0 \le w$, the solution $u = u(t)$ of* (1.1) *is global; moreover, if $u_0 \not\equiv w$, then $u(t) \to u_\lambda$ uniformly as $t \to \infty$ (with the convention that $u_{\lambda^*} = U_*$).*

The first statement in Theorem 3 is just [7, Theorem 1], while the second statement was proved in [14] under the additional assumption that $w \in L^\infty(\Omega)$ and in its general form in [7, Theorem 2] and [28, Corollary 7].

2.2. **New results.** Our first purpose is to modify Theorem 3 in two aspects. We allow initial data u_0 in $H_0^1(\Omega)$, and we avoid the use of the comparison principle in the proof of global existence results (pointwise inequalities which require $u_0 \ge 0$). The price we must pay is to require an upper bound for p.

Assume that $p \le \frac{n+2}{n-2}$ and $0 < \lambda < \lambda^*$; then, the minimal solution u_λ and the mountain-pass solution U_λ enable us to define the set

$$\mathcal{P} := \Big\{ u \in H_0^1(\Omega); \int_\Omega \Big(\frac{|\nabla u|^2}{2} - \frac{\lambda |1+u|^{p+1}}{p+1} \Big) < \int_\Omega \Big(\frac{|\nabla U_\lambda|^2}{2} - \frac{\lambda (1+U_\lambda)^{p+1}}{p+1} \Big),$$
$$\int_\Omega \Big(|\nabla (u-u_\lambda)|^2 + \lambda [(1+u_\lambda)^p - (1+u_\lambda + |u-u_\lambda|)^p] |u-u_\lambda| \Big) > 0 \Big\} \cup \{u_\lambda\}.$$

In spite of its unpleasant form $\mathcal{P}$ is not a complicated object. It is just the *potential well* relative to (2.1) as $\int_\Omega \frac{|\nabla U_\lambda|^2}{2} - \frac{\lambda(1+U_\lambda)^{p+1}}{p+1}$ is the energy of the mountain-pass solution; for this reason, $\mathcal{P}$ is well-defined even if (2.1) admits more than one mountain-pass solution. Clearly, $\mathcal{P} = \mathcal{P}_\lambda$; recall that, roughly speaking, $\mathcal{P}_\lambda \to \{U_*\}$ as $\lambda \to \lambda^*$, whereas $\mathcal{P}_\lambda \to H_0^1(\Omega)$ as $\lambda \to 0$; see [17, Theorem 4]. Let us also mention that $\mathcal{P}$ contains a neighborhood of u_λ:

Proposition 1. *Assume that $1 < p \le \frac{n+2}{n-2}$ and $0 < \lambda < \lambda^*$. There exists $R > 0$ such that if $\|u_0 - u_\lambda\| < R$, then $u_0 \in \mathcal{P}$.*

We point out that Proposition 1 is not a perturbation result. Together with its proof, in Section 7 we give an explicit formula to determine R.

Taking advantage of the particular form of the reaction term, we prove the following result:

Theorem 4. *Assume that $1 < p \le \frac{n+2}{n-2}$, that $0 < \lambda < \lambda^*$, and that $u_0 \in \mathcal{P}$.*

If $1 < p < \frac{n+2}{n-2}$, then the (local) solution $u = u(t)$ of (1.1) *is global and $u(t) \to u_\lambda$ in the $H_0^1(\Omega)$ norm topology as $t \to \infty$.*

If $p = \frac{n+2}{n-2}$ *and the solution* $u = u(t)$ *of* (1.1) *is global, then* $u(t) \to u_\lambda$ *in the* $H_0^1(\Omega)$ *norm topology as* $t \to \infty$.

In the critical case, it is not clear if for $u_0 \in \mathcal{P}$ the solution of (1.1) is global; this is strictly related to the blow-up statement (2.3), which may not occur if $\varepsilon = 0$. On the other hand, if $\|u_0 - u_\lambda\|$ is sufficiently small then (1.1) admits a global solution also in the critical case; see [20, Theorem 3].

Results similar to Theorem 4 were obtained in [22, 23] for (1.2) when $f(u) = |u|^{p-1}u$. The spirit of our proof is more similar to [21, 32] since it is based on critical-point theory methods and compactness is gained by treating the flow somehow as a Palais-Smale sequence.

As a trivial consequence, the previous results (and [20, Theorem 3]) enable us to find *unbounded* initial data which generate global solutions of (1.1) and which converge to the minimal solution:

Corollary 1. *Assume that* $1 < p \leq \frac{n+2}{n-2}$ *and* $0 < \lambda < \lambda^*$. *Then, there exist initial data* $u_0 \in H_0^1 \setminus L^\infty(\Omega)$ *such that the (local) solution* $u = u(t)$ *of* (1.1) *is global and* $u(t) \to u_\lambda$ *in the* $H_0^1(\Omega)$ *norm topology as* $t \to \infty$.

Furthermore, by combining Theorem 4 with the blow-up statement by Fujita [14] we obtain a striking statement linking pointwise and integral inequalities:

Corollary 2. *Assume that* $1 < p < \frac{n+2}{n-2}$ *and that* $0 < \lambda < \lambda^*$. *Let* U_λ *be any mountain-pass solution of* (2.1), *and let* $u \in H_0^1(\Omega)$ *satisfy* $u \geq U_\lambda$ *almost everywhere in* Ω. *Then, either*

$$\int_\Omega \Big(\frac{|\nabla u|^2}{2} - \frac{\lambda(1+u)^{p+1}}{p+1}\Big) \geq \int_\Omega \Big(\frac{|\nabla U_\lambda|^2}{2} - \frac{\lambda(1+U_\lambda)^{p+1}}{p+1}\Big)$$

or

$$\int_\Omega |\nabla(u-u_\lambda)|^2 \leq \lambda \int_\Omega [(1+u)^p - (1+u_\lambda)^p](u-u_\lambda)\ .$$

If the Nehari manifold has a particular structure, the definition of the potential well may be simplified. We are able to show that this is the case when $p \geq 2$, which is possible only in low dimensions, namely $n \leq 6$. Let

$$\widehat{\mathcal{P}} := \Big\{u \in H_0^1(\Omega); \int_\Omega \Big(\frac{|\nabla u|^2}{2} - \frac{\lambda|1+u|^{p+1}}{p+1}\Big) < \int_\Omega \Big(\frac{|\nabla U_\lambda|^2}{2} - \frac{\lambda(1+U_\lambda)^{p+1}}{p+1}\Big),$$
$$\int_\Omega \Big(|\nabla(u-u_\lambda)|^2 + \lambda[(1+u_\lambda)^p - |1+u|^{p-1}(1+u)](u-u_\lambda)\Big) > 0\Big\} \cup \{u_\lambda\}.$$

In Section 8 we sketch the proof of the following result:

Theorem 5. *Assume that* $2 \le p \le \frac{n+2}{n-2}$, *that* $0 < \lambda < \lambda^*$, *and that* $u_0 \in \widehat{\mathcal{P}}$.

If $n \le 5$ *and* $p < \frac{n+2}{n-2}$, *then the* (*local*) *solution* $u = u(t)$ *of* (1.1) *is global and* $u(t) \to u_\lambda$ *in the* $H^1_0(\Omega)$ *norm topology as* $t \to \infty$.

If $n \le 6$, $p = \frac{n+2}{n-2}$, *and the solution* $u = u(t)$ *of* (1.1) *is global, then* $u(t) \to u_\lambda$ *in the* $H^1_0(\Omega)$ *norm topology as* $t \to \infty$.

We now state that the solution of (1.1) may blow up in finite time in a stronger sense (in a weaker topology when compared to (2.2)–(2.3)): we consider as a possible set of initial data the whole of $\mathbb{K}$.

Definition 2. *Let* $p > 1$, $\lambda > 0$, $T > 0$, *and* $u_0 \in \mathbb{K}$; *we call a nonnegative function* $u \in L^p_{\mathrm{loc}}([0,T); L^p_{\mathrm{loc}}(\Omega) \cap \mathbb{K})$ *a generalized solution of* (1.1) *over* $(0,T)$ *if*

$$\int_0^T \int_\Omega (\phi_t + \Delta\phi)u + \lambda \int_0^T \int_\Omega (1+u)^p \phi + \int_\Omega u_0 \phi(0) = 0 \quad \forall \phi \in C_c^\infty(\Omega \times [0,T)). \tag{2.4}$$

If u *satisfies* (2.4) *we also say that* u_0 *is the generalized initial value of* u. *Finally, we call generalized blow-up time the supremum* T^* *of* $T > 0$ *for which a generalized solution of* (1.1) *over* $(0,T)$ *exists; if no such* T *exists, then we set* $T^* = 0$.

Generalized solutions should be compared with *weak solutions* as defined in [28, Definition 1] (see also the definition of *integral solution* in [4, 5]); they are slightly different in several aspects. In the above definition there is no upper bound for p since we do not wonder about the existence of generalized solutions. On the contrary, we are interested in nonexistence results. By using the test-function method developed by Mitidieri-Pohožaev [30], we prove

Theorem 6. *Assume that* $p > 1$ *and* $\lambda > 0$. *For all* $u_0 \in \mathbb{K} \setminus \{0\}$ *and all* $T > 0$ *there exists* $\overline{\alpha} = \overline{\alpha}(u_0, T) > 0$ *such that for any* $\alpha > \overline{\alpha}$ *the problem* (1.1) *with generalized initial value* $u(0) = \alpha u_0$ *admits no generalized solution over* $(0,T)$. *Moreover, if* T_α *denotes the generalized blow-up time of* (1.1) *with generalized initial value* $u(0) = \alpha u_0$, *then* $T_\alpha \to 0$ *as* $\alpha \to \infty$ (*possibly,* $T_\alpha \equiv 0$).

Not only does Theorem 6 extend the blow-up statement of [28, Theorem 4] to the case of generalized solutions (with generalized initial value $u_0 \in \mathbb{K}$), but it also gives a link between α and T_α. Let us also mention that a similar statement for a different kind of solution may be found in [34, Corollary 5.1]. The proof of Theorem 6 is given in Section 9, where one can also find a constructive method to determine the value of $\overline{\alpha}$ for which the nonexistence statement holds true.

Under further assumptions, and combining it with [28, Theorem 4], we may strengthen Theorem 6 with the following statement:

Theorem 7. *Assume that $1 < p \leq \frac{n+2}{n-2}$ and $0 < \lambda \leq \lambda^*$. For $u_0 \in H^1_0(\Omega) \cap \mathbb{K} \setminus \{0\}$ and $\alpha \geq 0$ let u^α denote the unique local solution u of* (1.1) *with initial value $u(0) = \alpha u_0$. Then there exists $\overline{\alpha} = \overline{\alpha}(u_0) > 0$ such that*

(i) for any $\alpha > \overline{\alpha}$ the solution u^α has finite blow-up time T_α; moreover, $T_\alpha \to 0$ as $\alpha \to \infty$.

(ii) for any $\alpha < \overline{\alpha}$, u^α is global and $u^\alpha(t) \to u_\lambda$ in $H^1_0 \cap L^\infty(\Omega)$ as $t \to \infty$.

We point out that Theorem 7 *is not* completely satisfactory and, perhaps, may be improved. For this reason, we establish a further blow-up statement in the spirit of the potential-well theory. We define the set

$$\mathcal{Q} := \Big\{ u \in H^1_0(\Omega);\ u \geq u_\lambda \text{ a.e.}, \int_\Omega \Big(\frac{|\nabla u|^2}{2} - \frac{\lambda(1+u)^{p+1}}{p+1} \Big) < \int_\Omega \Big(\frac{|\nabla U_\lambda|^2}{2} - \frac{\lambda(1+U_\lambda)^{p+1}}{p+1} \Big), \int_\Omega \Big(|\nabla(u-u_\lambda)|^2 + \lambda[(1+u_\lambda)^p - (1+u)^p](u-u_\lambda) \Big) < 0 \Big\}.$$

We are here forced to require $u_0 \geq u_\lambda$ because we do not know the shape of the Nehari manifold; see Problem 2 in Section 12. When $p \geq 2$, this restriction may be removed by arguing as in the proof of Theorem 5. We have

Theorem 8. *Assume that $1 < p \leq \frac{n+2}{n-2}$, that $0 < \lambda < \lambda^*$, and that $u_0 \in \mathcal{Q}$; then the (local) solution $u = u(t)$ of* (1.1) *blows up in finite time.*

The proof of Theorem 8 is given in Section 11. Due to the condition $f(0) > 0$, it is significantly different from that in [22, 23] for (1.2) in the case $f(u) = |u|^{p-1}u$.

3. The critical case in the unit ball of $\mathbb{R}^4$

When Ω is a ball, the mountain-pass solution U_λ is unique and (2.1) admits no other solutions:

Theorem 9. [19, 24] *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ be a ball, and let $1 < p \leq \frac{n+2}{n-2}$. Then, for all $\lambda < \lambda^*$ problem* (2.1) *admits exactly two (bounded) solutions which are radially symmetric and radially decreasing.*

In this section we consider the particular case where $n = 4$, $\Omega = B_1$ (the unit ball), and $p = 3$ (the critical exponent). Then, it is known [24] that $\lambda^* = 2$. In view of Theorem 9, problem (2.1) admits exactly two solutions for any $0 < \lambda < 2$. These solutions may also be explicitly written using Theorem 7 in [17]. This fact enables us to try to answer some natural questions.

(I) Question: is it true that $0 \in \mathcal{P}$**?** This question is motivated by an attempt to compare the strength of the comparison principle (used, for instance, in Theorem 3) and the potential-well tools as used in Theorem 4. As we will see, it may have different answers according to the value of λ.

• Take first $\lambda = 1$ so that the two positive solutions of (2.1) are (in radial coordinates $r = |x|$)

$$u_1(r) = \frac{(3-2\sqrt{2})(1-r^2)}{1+(3-2\sqrt{2})r^2}\ , \qquad U_1(r) = \frac{(3+2\sqrt{2})(1-r^2)}{1+(3+2\sqrt{2})r^2}\ . \tag{3.1}$$

Consider the energy functional

$$J(u) = \int_{B_1} \left(\frac{|\nabla u|^2}{2} - \frac{|1+u|^4}{4} \right)\ .$$

If we restrict ourselves to radial functions $u = u(r)$, up to a multiplicative constant it may be rewritten as

$$J_r(u) = \int_0^1 [2|u'(r)|^2 - |1+u(r)|^4]\, r^3\, dr\ .$$

By (3.1) and direct computations we obtain

$$\begin{aligned} J_r(U_1) &= \frac{4(99+70\sqrt{2})^2(3-2\sqrt{2})^3}{3(2+\sqrt{2})^3} - \frac{(3-2\sqrt{2})^2(2+\sqrt{2})(75+53\sqrt{2})}{3} \\ &\approx 1.6095 > -\frac{1}{4} = J_r(0) \end{aligned}$$

and

$$\int_{B_1} \left(|\nabla u_1|^2 + [(1+u_1)^3 - (1+2u_1)^3]u_1\right) = \int_{B_1} [2(1+u_1)^3 - (1+2u_1)^3]u_1 > 0\ ;$$

these inequalities prove that $0 \in \mathcal{P}$ when $\lambda = 1$.

• Take now $\lambda = \frac{576}{289} \approx 1.993$ so that $d_- = \frac{8}{9}$ and $d_+ = \frac{9}{8}$. Then,

$$u_\lambda(r) = \frac{8(1-r^2)}{8r^2+9}\ , \qquad U_\lambda(r) = \frac{9(1-r^2)}{9r^2+8}\ .$$

We consider the "rescaled radial" energy functional

$$J_r(u) = \int_0^1 [289|u'(r)|^2 - 288|1+u(r)|^4]\, r^3\, dr \ .$$

Direct calculations show that $J_r(U_\lambda) \approx -95.625 < -72 = J_r(0)$. Hence, if $\lambda = \frac{576}{289}$, $0 \notin \mathcal{P}$.

(II) Remarks on the map $t \mapsto u(t)$. Take again $\lambda = 1$, choose as initial datum

$$v^\alpha(r) = \alpha(1-r^2)\ , \quad \alpha \in \mathbb{R}, \tag{3.2}$$

and consider the evolution problem

$$\begin{cases} u_t - \Delta u = (1+u)^3 & \text{in } B_1 \times (0,T) \\ u(0) = v^\alpha & \text{in } B_1 \\ u = 0 & \text{on } \partial B_1 \times (0,T)\ . \end{cases} \tag{3.3}$$

We denote by u^α the corresponding local solution (see Theorem 2); since $v^\alpha \in C^\infty(\overline{\Omega})$, by [26, Proposition 7.1.10] u^α is a strict solution so that, in particular, $u^\alpha \in C([0,T); C^2(\Omega)) \cap C^1([0,T); C(\Omega))$. Therefore, thanks to some tedious but simple calculations, we obtain the following striking facts:

• Take $\alpha \in (\frac{1}{8}, \frac{2-\sqrt{2}}{4}]$. Then $v^\alpha(r) < u_1(r)$ for all $r \in [0,1)$ so that by Theorem 3 we have $u^\alpha(t) \to u_1$ uniformly as $t \to \infty$. Moreover,

$$\Delta v^\alpha(r) + (1+v^\alpha(r))^3 < 0 \qquad \forall r > r_\alpha := \sqrt{\frac{\alpha - 2\alpha^{1/3}+1}{\alpha}}\ ; \tag{3.4}$$

note that $r_\alpha < 1$. By (3.4) we infer $u_t^\alpha(r,t)|_{t=0} < 0$ for all $r > r_\alpha$ which shows that at these points r the map $t \mapsto u^\alpha(r,t)$ initially decreases; on the other hand, we recall that

$$\lim_{t\to\infty} u^\alpha(r,t) = u_1(r) > v^\alpha(r) = u^\alpha(r,0) \qquad \forall r \in [0,1)\ .$$

• Take $\alpha \in (3-2\sqrt{2}, \sqrt{5}-2)$. Then $v^\alpha(r) < U_1(r)$ for all $r \in [0,1)$ so that Theorem 3 entails $u^\alpha(t) \to u_1$ uniformly as $t \to \infty$. Therefore, as $u_1(0) < v^\alpha(0)$, we have

$$\lim_{t\to\infty} \|u^\alpha(t)\|_\infty = \|u_1\|_\infty < \|v^\alpha\|_\infty = \|u^\alpha(0)\|_\infty\ .$$

On the other hand, $u_t^\alpha(0,t)|_{t=0} = \Delta v^\alpha(0) + (1+v^\alpha(0))^3 > 0$, which shows that the map $t \mapsto \|u^\alpha(t)\|_\infty$ initially increases.

(III) A case where the potential well theory improves classical results. Take again $\lambda = 1$ so that u_1 and U_1 are given by (3.1) and $J_r(U_1) \approx 1.6095$. Take also v^α as in (3.2) with $\alpha > 0$, and consider problem (3.3).

First note that if $\alpha \le \frac{2+\sqrt{2}}{4} \approx 0.854$ then $v^\alpha \le U_1$ in B_1 so that the solution u^α of (3.3) is global and converges to u_1 as $t \to \infty$ in view of Theorem 3. On the other hand, if $\alpha \ge 3 + 2\sqrt{2} \approx 5.828$ then $v^\alpha \ge U_1$ in B_1, and therefore u^α blows up according to [14]. Hence, it remains to establish what happens to u^α for

$$\frac{2+\sqrt{2}}{4} < \alpha < 3 + 2\sqrt{2}\ .$$

Theorem 8 gives partial answers. After some calculations we find

$$J_r(v^\alpha) = -\frac{\alpha^4}{60} - \frac{\alpha^3}{10} + \frac{13}{12}\alpha^2 - \frac{\alpha}{3} - \frac{1}{4}$$

so that $J_r(v^\alpha) < J_r(U_1)$ for $\alpha \in [0, 1.977) \cup (5.113, +\infty)$, the numerical approximating values being obtained with Mathematica. Moreover, using again Mathematica we see that the second integral inequality, which characterizes $\mathcal{Q}$, is satisfied for $\alpha > 3.774$. In conclusion, Theorem 8 enables us to obtain blow-up results also in the interval $\alpha \in (5.113, 5.828)$.

The same calculations show that $v^\alpha \in \mathcal{P}$ if and only if $\alpha < 1.977$. But since we are in the critical case, Theorem 4 does not allow us to conclude that the solution u^α is global; if it were indeed global, then necessarily it would converge to u_1 as $t \to \infty$.

(IV) An estimate of the blow-up time. Let $\phi^\gamma(r,t) = \psi(\gamma t)(1 - r^2)$, where $\psi(t) = (1-t)^+$; even if ϕ^γ is only Lipschitz continuous we may argue as in Section 9 below. More precisely, we first refine (9.2) by taking "optimal" constants C, and then we compute the constants in (9.6). Using also Remark 3, numerical calculations performed with Mathematica show that (9.6) is violated (so that we have blow-up of the solution u^α of (3.3)) for $\alpha > 14.215$. Unfortunately, this does not bring any further information when compared to (III) above where the blow-up was obtained for $\alpha > 5.113$. The same functions ϕ^γ with a different choice of the constants in (9.2) enable us to prove that the blow-up time T_α of u^α satisfies

$$T_\alpha \le \frac{64}{27\alpha^2} \quad \text{for large } \alpha\ .$$

4. Hints for the proof of Theorem 2

We first rewrite (1.1) in the standard abstract way,

$$u' + Au = f(u),$$

where $f(u) = \lambda|1+u|^{p-1}(1+u)$, $A = -\Delta$, and prime $'$ denotes differentiation with respect to time. It is well-known [26] that $-A$ generates an analytic semigroup of bounded linear operators $\{e^{-tA}; t \geq 0\}$ in $X := L^2(\Omega)$. Moreover, the fractional powers A^α of A are well defined for $0 \leq \alpha \leq 1$; since we are here interested in the cases $\alpha \geq \frac{1}{2}$, we may characterize their domains by means of interpolation theory; namely, $D(A^\alpha) = H^{2\alpha} \cap H_0^1(\Omega)$.

By arguing as for (B.1) in [23], we obtain, $\forall u, v \in D(A^\alpha)$,

$$\|f(u) - f(v)\|_2 \leq C\big(1 + \|A^\alpha u\|_2 + \|A^\alpha v\|_2\big)^{4/(n-2)} \|A^\alpha u - A^\alpha v\|_2, \quad (4.1)$$

where $C > 0$ and, from now on, $\alpha = \frac{n}{n+2}$. Note that in (4.1) there is an additional "1" when compared with [23, (B.1)] and that we may restrict our attention to the critical case $p = \frac{n+2}{n-2}$ thanks to Hölder's inequality. The growth estimate (4.1) shows that condition (7.1.15) in [26] is fulfilled with $\gamma = \frac{n+2}{n-2}$ and $\alpha = \frac{n}{n+2}$.

For $\delta > 0$, consider the space

$$C_*(\delta;\alpha) = \Big\{u \in C((0,\delta]; D(A^\alpha));\ \sup_{0<t\leq\delta} t^{\frac{n-2}{2(n+2)}} \|A^\alpha\big(u(t)\big)\|_2 < \infty,$$

$$\lim_{t\to 0} t^{\frac{n-2}{2(n+2)}} \|A^\alpha\big(u(t)\big)\|_2 = 0\Big\}\ .$$

Now take $\beta = \frac{1}{2}$ so that $D(A^\beta) = H_0^1(\Omega)$; then, by Theorem 7.1.5 (*iii*) in [26] there exist $\delta > 0$ and a unique function $u \in C([0,\delta]; H_0^1(\Omega)) \cap C_*(\delta,\alpha)$ which satisfies the variation-of-constants formula

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A} f(u(s))\, ds \qquad \forall t \in [0,\delta]\ .$$

By [26, Proposition 7.1.10], the just-found solution u is also a classical solution of (1.1). Moreover, since $u \in C([0,\delta]; H_0^1(\Omega))$, we have $\Delta u \in C([0,\delta]; H^{-1}(\Omega))$ and $|1+u|^{p-1}(1+u) \in C([0,\delta]; H^{-1}(\Omega))$ (recall $p \leq \frac{n+2}{n-2}$); hence, $u_t \in C([0,\delta]; H^{-1}(\Omega))$ and $u \in C^1([0,\delta]; H^{-1}(\Omega))$.

Let T^* be the supremum of the δ's for which the solution satisfies the above conditions. According to [2, Proposition 3], we know that if $1 < p < \frac{n+2}{n-2}$ then f is a subcritical map relative to $(H_0^1(\Omega), H^{-1}(\Omega))$, while if $p = \frac{n+2}{n-2}$ then f is a critical and ε-regular map relative to $(H_0^1(\Omega), H^{-1}(\Omega))$ for all

$0 \le \varepsilon < \frac{n-2}{2(n+2)}$. Therefore, [2, Proposition 1] applies and we deduce (2.2) and (2.3).

Finally, if $u_0 \ge 0$, it suffices to repeat the fixed-point argument of [26] in the complete metric space $C([0,\delta]; H_0^1(\Omega) \cap \mathbb{K})$, and we obtain a solution $u(x,t) \ge 0$. □

5. Preliminaries about the stationary problem

Let $\lambda \in (0, \lambda^*)$. Since $p \le \frac{n+2}{n-2}$, the functional

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{p+1}\int_\Omega |1+u|^{p+1}$$

is of class C^1 over the space $H_0^1(\Omega)$ and its critical points are (weak) solutions of the problem

$$\begin{cases} -\Delta u = \lambda |1+u|^{p-1}(1+u) & \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Clearly, if u is positive, the above problem is just (2.1). In order to have the minimal solution u_λ coinciding with the origin we introduce the "translated functional"

$$I(w) := J(w + u_\lambda) - J(u_\lambda) \qquad \forall w \in H_0^1(\Omega)$$

so that, writing explicitly I and recalling that u_λ solves (2.1), we have

$$I(w) = \frac{1}{2}\int_\Omega |\nabla w|^2 + \lambda \int_\Omega \left[(1+u_\lambda)^p w - \frac{|1+u_\lambda+w|^{p+1}}{p+1} + \frac{(1+u_\lambda)^{p+1}}{p+1}\right] . \tag{5.1}$$

Note also that

$$I'(w)[w] = \int_\Omega |\nabla w|^2 + \lambda \int_\Omega \left[(1+u_\lambda)^p w - |1+u_\lambda+w|^{p-1}(1+u_\lambda+w)w\right] . \tag{5.2}$$

If $w \in H_0^1(\Omega)$, then also $|w| \in H_0^1(\Omega)$, and we can prove

Lemma 1. *For any $w \in H_0^1(\Omega)$ we have $I(w) \ge I(|w|)$ and $I'(w)[w] \ge I'(|w|)[|w|]$.*

Proof. We first claim that

$$2x^p y + \frac{|x-y|^{p+1}}{p+1} - \frac{(x+y)^{p+1}}{p+1} \le 0 \qquad \forall x \ge 1,\ y \ge 0 . \tag{5.3}$$

Alternatively, since $x \geq 1$, we may divide (5.3) by x^{p+1} and prove that

$$g(t) := 2t + \frac{|t-1|^{p+1}}{p+1} - \frac{(t+1)^{p+1}}{p+1} \leq 0 \qquad \forall t \geq 0 \ . \tag{5.4}$$

Simple calculations show that $g(0) = g'(0) = 0$ and $g''(t) = p[|t-1|^{p-1} - (t+1)^{p-1}] \leq 0$ for all $t \geq 0$, the latter inequality following from the assumption $p > 1$. Therefore, (5.4) and (5.3) follow.

By (5.1), the first inequality in the statement follows if we show that

$$(1+u_\lambda)^p w - \frac{|1+u_\lambda+w|^{p+1}}{p+1} \geq (1+u_\lambda)^p |w| - \frac{\big(1+u_\lambda+|w|\big)^{p+1}}{p+1} \ \text{ a.e. in } \Omega \ ;$$

but this is trivial if $w \geq 0$ (equality holds!), whereas it follows at once from (5.3) (with $x = 1 + u_\lambda$ and $y = |w|$) if $w \leq 0$.

For the second inequality we proceed similarly. We first claim that

$$2x^p y - |x-y|^{p-1}(x-y)y - (x+y)^p y \leq 0 \qquad \forall x \geq 1, \, y \geq 0 \ . \tag{5.5}$$

If $y = 0$, there is nothing to prove. Otherwise, we may divide by yx^p and prove that

$$h(t) := 2 + |t-1|^{p-1}(t-1) - (t+1)^p \leq 0 \qquad \forall t \geq 0 \ . \tag{5.6}$$

But $h(0) = 0$ and $h'(t) = g''(t) \leq 0$ for all $t \geq 0$, so that (5.6) and (5.5) follow.

By (5.2), in order to prove the second inequality, it suffices to show that

$$[(1+u_\lambda)^p - |1+u_\lambda+w|^{p-1}(1+u_\lambda+w)]w \geq [(1+u_\lambda)^p - (1+u_\lambda+|w|)^p]|w|$$

almost everywhere in Ω ; this follows at once from (5.5). $\square$

We now introduce the *Nehari manifold* relative to I, namely

$$\mathcal{N} = \{w \in H^1_0(\Omega) \setminus \{0\};\ I'(w)[w] = 0\} \ . \tag{5.7}$$

This manifold is particularly useful and meaningful for homogeneous elliptic equations such as $-\Delta u = |u|^{p-1}u$. Indeed, in such a case it is easy to show that each half line starting from the origin intersects exactly once the manifold $\mathcal{N}$. For equation (2.1) the situation is not yet completely clear; see Problem 2 in Section 12. Nevertheless, for our scope it is enough to show that half lines inside the cone of nonnegative functions have exactly one intersection with $\mathcal{N}$:

Lemma 2. *Let $w \in \mathcal{N} \cap \mathbb{K}$, and consider the (smooth) map $\Psi_w : [0, +\infty) \to \mathbb{R}$ defined by $\Psi_w(t) = I(tw)$. Then, $\Psi'_w(t) > 0$ for all $t \in (0,1)$ and $\Psi'_w(t) < 0$ for all $t > 1$.*

Proof. In view of (5.1), we have

$$\Psi_w(t) = \frac{t^2}{2}\int_\Omega |\nabla w|^2 + \lambda \int_\Omega \Big[(1+u_\lambda)^p tw - \frac{(1+u_\lambda+tw)^{p+1}}{p+1} + \frac{(1+u_\lambda)^{p+1}}{p+1}\Big].$$

Therefore,

$$\Psi'_w(t) = t\int_\Omega |\nabla w|^2 + \lambda \int_\Omega [(1+u_\lambda)^p - (1+u_\lambda+tw)^p]\, w$$

and

$$\Psi''_w(t) = \int_\Omega |\nabla w|^2 - \lambda p \int_\Omega (1+u_\lambda+tw)^{p-1} w^2 \ .$$

By [9, Lemma 2.1], u_λ is a nondegenerate local minimum of J, so that $\Psi''_w(0) > 0$; moreover, Ψ''_w is decreasing and tends to $-\infty$ as $t \to \infty$. Hence, there exists a unique $t'' > 0$ such that $\Psi''_w(t'') = 0$. Consequently, there exists a unique $t' > 0$ such that $\Psi'_w(t') = 0$; since $I'(w)[w] = 0$, we have $\Psi'_w(1) = 0$, and the statement follows. □

It is now quite simple to characterize the mountain-pass level (the potential-well depth):

Lemma 3. *We have $U_\lambda - u_\lambda \in \mathcal{N} \cap \mathbb{K}$ and*

$$d := I(U_\lambda - u_\lambda) = \min_{\mathcal{N}\cap\mathbb{K}} I(w) \ .$$

Proof. From [1] we know that

$$d \, = \, \min_{\gamma\in\Gamma} \, \max_{0\le s\le 1} \, I(\gamma(s)) \ , \tag{5.8}$$

where $\Gamma = \{\gamma \in C([0,1]; H^1_0(\Omega)),\ \gamma(0) = 0,\ I(\gamma(1)) < 0\}$. For contradiction, assume that there exists $v \in \mathcal{N} \cap \mathbb{K}$ such that $J(v) < d$. Then, by Lemma 2 we have $\max_{t\ge 0} \Psi_v(t) < d$, which contradicts (5.8). □

We now introduce the set corresponding to the potential well $\mathcal{P}$. Let

$$B := \{w \in H^1_0(\Omega),\ I'(|w|)[|w|] > 0\, ,\ I(w) < d\} \ .$$

We first prove

Lemma 4. *The set B is bounded in $H^1_0(\Omega)$.*

Proof. Throughout this proof we denote by C_λ positive constants which depend on λ and u_λ and which may vary from line to line. Since $p > 1$, we may fix $\varepsilon > 0$ such that

$$\delta := \frac{1}{2} - \Big(\frac{1}{p+1} + \varepsilon\Big)\frac{1}{1-\varepsilon} > 0 \ . \tag{5.9}$$

Take first any $w \in B \cap \mathbb{K}$. Then, the condition $I'(w)[w] > 0$ reads

$$\frac{1}{p+1}\int_\Omega |\nabla w|^2 + \frac{1}{p+1}\int_\Omega \nabla w \nabla u_\lambda > \frac{\lambda}{p+1}\int_\Omega (1+u_\lambda+w)^p w, \tag{5.10}$$

which, together with Young's inequality, yields

$$C_\lambda + \Big(\frac{1}{p+1}+\varepsilon\Big)\int_\Omega |\nabla w|^2 \geq \frac{\lambda}{p+1}\int_\Omega (1+u_\lambda+w)^p w\ . \tag{5.11}$$

Using again Young's inequality, we obtain

$$\begin{aligned}\int_\Omega (1+u_\lambda+w)^p w &= \int_\Omega (1+u_\lambda+w)^{p+1} - \int_\Omega (1+u_\lambda+w)^p(1+u_\lambda) \\ &\geq (1-\varepsilon)\int_\Omega (1+u_\lambda+w)^{p+1} - C_\lambda\ .\end{aligned}$$

Inserting this inequality into (5.11) entails

$$C_\lambda + \Big(\frac{1}{p+1}+\varepsilon\Big)\int_\Omega |\nabla w|^2 \geq \frac{(1-\varepsilon)\lambda}{p+1}\int_\Omega (1+u_\lambda+w)^{p+1}\ . \tag{5.12}$$

On the other hand, the condition $I(w) < d$ reads

$$\frac{1}{2}\int_\Omega |\nabla w|^2 + \lambda\int_\Omega (1+u_\lambda)^p w - \frac{\lambda}{p+1}\int_\Omega (1+u_\lambda+w)^{p+1} < C_\lambda\ ;$$

hence,

$$\frac{1}{2}\int_\Omega |\nabla w|^2 \leq \frac{\lambda}{p+1}\int_\Omega (1+u_\lambda+w)^{p+1} + C_\lambda\ . \tag{5.13}$$

Inserting this inequality into (5.12) and recalling (5.9) gives

$$\delta\int_\Omega |\nabla w|^2 \leq C_\lambda\ , \tag{5.14}$$

which, by arbitrariness of u, proves that $B \cap \mathbb{K}$ is bounded.

Take now $w \in B \setminus \mathbb{K}$, and let $v = |w|$. Then, $I'(v)[v] > 0$ and $I(v) \leq I(w) < d$ in view of Lemma 1. Therefore, $v \in B \cap \mathbb{K}$ so that v satisfies (5.14). Since w and v have the same norm, this completes the proof. □

By exploiting the continuity of the maps $w \mapsto I(w)$ and $w \mapsto I'(|w|)[|w|]$, it is straightforward to show that

$$\overline{B} = \{w \in H^1_0(\Omega),\ I'(|w|)[|w|] \geq 0\,,\ I(w) \leq d\}\ . \tag{5.15}$$

We conclude this section by showing that this set has several important properties:

Lemma 5. *Assume that* $w \in \overline{B}$. *Then*
(*i*) $|w| \in \overline{B}$;
(*ii*) $I(w) \geq 0$;
(*iii*) *if* $I'(w)[w] = 0$, *then either* $w \equiv 0$ *or* $I(w) = d$.

Proof. Lemma 1 gives (*i*) at once. Lemmas 1 and 2 show that $I(w) > 0$ for all $w \in B$; this proves (*ii*). In order to prove (*iii*), note that if $w \in B$, by Lemma 1 we have $I'(w)[w] \geq I'(|w|)[|w|] > 0$. Therefore, if $I'(w)[w] = 0$ then $w \notin B$ and (*iii*) follows. □

6. Proof of Theorem 4

6.1. **The subcritical case.** Assume that $1 < p < \frac{n+2}{n-2}$. Let $u = u(t)$ be the local solution of (1.1) as given by Theorem 2. We set

$$w(t) = u(t) - u_\lambda \ . \tag{6.1}$$

Then, $w = w(t)$ is the unique local solution of the problem

$$\begin{cases} w_t - \Delta w = \lambda |1 + u_\lambda + w|^{p-1}(1 + u_\lambda + w) - \lambda(1 + u_\lambda)^p & \text{in } \Omega \times (0, T) \\ w(0) = u_0 - u_\lambda & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \times (0, T). \end{cases} \tag{6.2}$$

In order to prove Theorem 4, we first construct subsets of $\overline{B}$ which are invariant under the flow of (6.2). For any $\varepsilon \in (0, d)$ let

$$B_\varepsilon = \{w \in \overline{B};\ I(w) \leq d - \varepsilon\} \ .$$

Then, by Lemma 4 we get at once that

$$\text{for any } \varepsilon \in (0, d) \text{ the set } B_\varepsilon \text{ is closed and bounded,} \tag{6.3}$$

and we can prove

Lemma 6. *Let* $\varepsilon \in (0, d)$, *and assume that* $w_0 \in B_\varepsilon$; *then the* (*local*) *solution* $w = w(t)$ *of* (6.2) *is global and satisfies* $w(t) \in B_\varepsilon$ *for all* $t \geq 0$.

Proof. If $w_0 = 0$, we have $w(t) \equiv 0$ and the result trivially follows. So, assume that $w_0 \neq 0$. As long as $w(t) \in B_\varepsilon$, the solution may be continued in view of (2.2) and (6.3). Consider the energy functional $E(t) := I(w(t))$. By differentiating and by using (6.2) we obtain

$$E'(t) = -\int_\Omega w_t^2 \ , \tag{6.4}$$

so that the energy is strictly decreasing (recall $w_0 \neq 0$). Since $E(0) \leq d - \varepsilon$, this proves that

$$I(w(t)) = E(t) < d - \varepsilon \qquad \forall t > 0. \tag{6.5}$$

For contradiction, assume that $w(t)$ exits B_ε in finite time. Then, by (6.5) there exists a first time $T \geq 0$ when $I'(|w(T)|)[|w(T)|] = 0$; by (6.3) we also know that $w(T) \in B_\varepsilon$. Therefore, Lemma 5 and (6.5) imply that $w(T) = 0$, which is impossible because we would reach a stationary solution in finite time. $\square$

We also need the following result:

Lemma 7. *Let $u_0 \in \mathcal{P}$, and let w be the global solution of* (6.2) *as found in Lemma* 6*. Then, there exists $v \in H_0^1(\Omega)$ which satisfies $I'(v) = 0$ and such that* (*up to a subsequence*) *$w(t) \rightharpoonup v$ as $t \to \infty$ in the weak $H_0^1(\Omega)$ topology.*

Proof. Note that by (6.4) the energy function $E(t)$ is decreasing. By Lemmas 5 and 6, we have $E(t) \geq 0$ for all t, and $E(t)$ admits a finite nonnegative limit as $t \to \infty$. Therefore,

$$\liminf_{t\to\infty} \int_\Omega w_t^2(t) = \limsup_{t\to\infty} E'(t) = 0\ . \tag{6.6}$$

By Lemmas 6 and 4 we deduce that there exists $v \in H_0^1(\Omega)$ such that

$$w(t) \rightharpoonup v \qquad \text{in } H_0^1(\Omega) \qquad \text{as } t \to \infty\ , \tag{6.7}$$

up to a subsequence. By (6.6)–(6.7) we may select an increasing divergent sequence $\{t_m\}$ such that

$$\int_\Omega w_t^2(t_m) \to 0 \qquad \text{and} \qquad w(t_m) \rightharpoonup v \quad \text{in } H_0^1(\Omega) \qquad \text{as } m \to \infty\ . \tag{6.8}$$

Fix any $\phi \in H_0^1(\Omega)$; at each time t_m test (6.2) with ϕ and let $m \to \infty$. Then, by (6.8) and the embedding $H_0^1(\Omega) \subset L^{p+1}(\Omega)$ we infer that $I'(v)[\phi] = 0$. The proof is thus complete. $\square$

We may now prove Theorem 4 when $p < \frac{n+2}{n-2}$. Recall that by [31, Lemma 5.1], the map $t \mapsto \|w(t)\|_2^2$ is Lipschitz continuous (in fact C^1). So, multiply (6.2) by $w(t)$ and integrate by parts to obtain

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_2^2 = -I'(w(t))[w(t)]\ . \tag{6.9}$$

Since $w(0) \in B_0$, by Lemmas 1 and 6 the previous inequality shows that $t \mapsto \|w(t)\|_2^2$ is decreasing, and hence it admits a finite nonnegative limit as

$t \to \infty$. But then we may select another subsequence (still denoted by $\{t_m\}$) satisfying (6.8) and

$$\lim_{m\to\infty} I'(w(t_m))[w(t_m)] = 0\ . \tag{6.10}$$

Lemma 7 gives $I'(v)[v] = 0$, namely

$$\int_\Omega |\nabla v|^2 = \lambda \int_\Omega [|1+u_\lambda+v|^{p-1}(1+u_\lambda+v) - (1+u_\lambda)^p]v\ . \tag{6.11}$$

By (6.8) and compact embedding, we have $w(t_m) \to v$ in $L^{p+1}(\Omega)$ (on a further subsequence). Therefore, (6.10) and (6.11) show that

$$\int_\Omega |\nabla w(t_m)|^2 \to \lambda \int_\Omega [|1+u_\lambda+v|^{p-1}(1+u_\lambda+v) - (1+u_\lambda)^p]v = \int_\Omega |\nabla v|^2\ ; \tag{6.12}$$

this, together with (6.7), proves that $w(t_m) \to v$ in the norm topology of $H_0^1(\Omega)$. Hence, $v \in \overline{B}$ (by Lemma 6) and

$$I(v) = \lim_{m\to\infty} I(w(t_m)) \le I(w(0)) < d\ . \tag{6.13}$$

By (6.11), (6.13), and Lemma 5 we deduce that $v \equiv 0$. We have thus proved that $w(t_m) \to 0$ in the norm topology of $H_0^1(\Omega)$. In particular, by continuity of I we have $I(w(t_m)) \to 0$. But then (6.4) shows that $I(w(t)) \to 0$ as $t \to \infty$. Since $v \equiv 0$ is a strict global minimum of I in B, this shows that $w(t) \to 0$ and that no subsequences have to be extracted. The proof of Theorem 4 is thus complete in the subcritical case $p < \frac{n+2}{n-2}$. □

6.2. **The critical case.** The case $p = \frac{n+2}{n-2}$ is more delicate. Let $u = u(t)$ be the global solution of (1.1) as assumed in Theorem 4. We set again (6.1) so that $w = w(t)$ is the unique global solution of (6.2).

We rewrite the functional I as

$$I(w) = \frac{1}{2}\int_\Omega |\nabla w|^2 - \frac{\lambda}{2^*}\int_\Omega |w|^{2^*} - \int_\Omega F(x,w),$$

where

$$F(x,w) = \lambda\Big[\frac{|1+u_\lambda+w|^{2^*}}{2^*} - \frac{|w|^{2^*}}{2^*} - (1+u_\lambda)^{(n+2)/(n-2)}w - \frac{(1+u_\lambda)^{2^*}}{2^*}\Big].$$

Let $f(x,s) = \frac{\partial}{\partial s}F(x,s)$; clearly, the terms $F(x,\cdot)$ and $f(x,\cdot)$ are subcritical in the sense that

$$\lim_{s\to\infty}\frac{F(x,s)}{|s|^{2^*}} = 0, \quad \lim_{s\to\infty}\frac{f(x,s)}{|s|^{2^*-1}} = 0 \quad \text{uniformly w.r.t. } x \in \Omega. \tag{6.14}$$

The following statement holds:

Lemma 8. *Let S denote the best Sobolev constant for the embedding $H_0^1 \subset L^{2^*}$. Then,*

$$\frac{S^{n/2}}{n}\,\lambda^{(2-n)/2} > I(U_\lambda - u_\lambda) = d > I(w(0)) \geq I(w(t)) \quad \forall t \geq 0\ . \tag{6.15}$$

Proof. Note first that there exists $\overline{v} \in H_0^1(\Omega) \cap \mathbb{K} \setminus \{0\}$ such that

$$\max_{t \geq 0} I(t\overline{v}) \ < \ \frac{S^{n/2}}{n}\,\lambda^{(2-n)/2}\ . \tag{6.16}$$

This follows from Theorem 2.1, Lemma 2.1, and the proofs of Corollaries 2.1, 2.2, and 2.3 in [8]. These statements yield a tool for the study of (2.1) in the critical case $p = \frac{n+2}{n-2}$; see Section 2.6 in [8] after stretching (to this end, one may also refer to [16, Lemma 1]). By (6.4), (6.16), and Lemmas 2–3, we obtain (6.15). □

As already mentioned, it is not clear if in the critical case $w(t)$ may be continued as long as it remains bounded in H^1. But the assumptions of Theorem 4 combined with the arguments of Lemma 6 enable us to establish that $w(t) \in \overline{B}$ for all $t \geq 0$ so that Lemma 7 still holds. Recalling (6.7) we now claim that

$$v \equiv 0\ . \tag{6.17}$$

Proof. The proof of (6.17) is tricky and requires several steps. As for the subcritical case, we argue on a subsequence $\{t_m\}$ which, for simplicity, we just denote by t. Taking into account (6.14), we have

$$\int_\Omega F(x,w(t)) \to \int_\Omega F(x,v), \quad \int_\Omega f(x,w(t))w(t) \to \int_\Omega f(x,v)v \quad \text{as} \quad t \to \infty\ . \tag{6.18}$$

Combining the second limit in (6.18) with (6.10) and (6.11) yields

$$\int_\Omega |\nabla w(t)|^2 - \lambda \int_\Omega |w(t)|^{2^*} = \int_\Omega |\nabla v|^2 - \lambda \int_\Omega |v|^{2^*} + o(1) \qquad \text{as } t \to \infty\ . \tag{6.19}$$

By (6.4), $\lim\limits_{t\to\infty} I(w(t))$ exists, and combining Lemma 6 (in its "weakened" form) with (6.18)–(6.19) yields

$$\begin{aligned}\lim_{t\to\infty} I(w(t)) &= \frac{1}{2^*}\int_\Omega |\nabla v|^2 - \frac{\lambda}{2^*}\int_\Omega |v|^{2^*} - \int_\Omega F(x,v) + \frac{1}{n}\lim_{t\to\infty}\int_\Omega |\nabla w(t)|^2 \\ &\geq I(v),\end{aligned} \tag{6.20}$$

where we used the lower semicontinuity of the norm with respect to weak convergence.

Consider now $z(t) := |w(t)|$ so that $z(t)$ is also bounded. Hence, up to a subsequence, it converges weakly to some $Z \in H_0^1(\Omega)$; by (6.7) and pointwise convergence we infer that

$$z(t) \rightharpoonup Z = |v| \qquad \text{as } t \to \infty \ . \tag{6.21}$$

By definition of B and recalling Lemma 6 we infer that

$$0 \le I'(z(t))[z(t)] \le I'(w(t))[w(t)] = o(1) \qquad \text{as } t \to \infty \ ,$$

where the second inequality follows from Lemma 1 while the last equality is just (6.10). Hence,

$$\lim_{t\to\infty} I'(z(t))[z(t)] = 0 \ . \tag{6.22}$$

On the other hand, by (6.19) and (6.21) we get at once

$$\int_\Omega |\nabla z(t)|^2 - \lambda \int_\Omega |z(t)|^{2^*} = \int_\Omega |\nabla Z|^2 - \lambda \int_\Omega |Z|^{2^*} + o(1) \ \text{ as } t \to \infty \ . \tag{6.23}$$

Summarizing, if we combine (6.14), (6.21), (6.22), and (6.23), we obtain

$$\begin{aligned}\int_\Omega f(x,Z)Z &= \int_\Omega f(x,z(t))z(t) + o(1) = \int_\Omega [|\nabla z(t)|^2 - \lambda |z(t)|^{2^*}] + o(1) \\ &= \int_\Omega [|\nabla Z|^2 - \lambda |Z|^{2^*}] + o(1),\end{aligned}$$

which proves that

$$I'(Z)[Z] = 0 \ . \tag{6.24}$$

Moreover, by (6.20) and Lemma 1 we get $I(Z) < d$, which, in view of Lemma 3, shows that $Z \notin \mathcal{N}$. Together with Lemma 2 and (6.24), this shows that $Z \equiv 0$, which completes the proof of (6.17). □

To conclude the proof it remains to show that $w(t) \to 0$ in the $H_0^1(\Omega)$ norm topology because, as in the subcritical case, this also implies that no subsequences have to be extracted. By (6.17) and (6.18) we infer

$$\int_\Omega F(x, w(t)) \to 0 \ , \qquad \int_\Omega f(x, w(t))w(t) \to 0 \qquad \text{as } t \to \infty \ , \tag{6.25}$$

whereas by (6.19) we deduce

$$\int_\Omega |\nabla w(t)|^2 - \lambda \int_\Omega |w(t)|^{2^*} = o(1) \qquad \text{as } t \to \infty \ . \tag{6.26}$$

Moreover, combining (6.26) with the first of (6.25) entails

$$I(w(t)) = \frac{1}{n} \int_\Omega |\nabla w(t)|^2 + o(1) \qquad \text{as } t \to \infty \ . \tag{6.27}$$

Now, the Sobolev inequality for $w(t)$ together with (6.26) implies

$$o(1) \geq \int_\Omega |\nabla w(t)|^2 \Big[1 - \lambda S^{-n/(n-2)} \Big(\int_\Omega |\nabla w(t)|^2\Big)^{2/(n-2)}\Big].$$

In turn, the latter inequality implies that either $w(t) \to 0$ in the norm topology of $H_0^1(\Omega)$, or

$$\liminf_{t\to\infty} \int_\Omega |\nabla w(t)|^2 \geq S^{n/2} \lambda^{(2-n)/2} \ . \tag{6.28}$$

But (6.28), combined with (6.27), yields

$$\liminf_{t\to\infty} I(w(t)) \geq \ \frac{S^{n/2}}{n} \lambda^{(2-n)/2},$$

which contradicts (6.15). Therefore, (6.28) cannot occur, and $w(t) \to 0$ in the norm topology of $H_0^1(\Omega)$. The proof of Theorem 4 is now complete also in the critical case $p = \frac{n+2}{n-2}$. □

7. Proof of Proposition 1

We use again the change of unknown (6.1) so that the condition $\|u_0 - u_\lambda\| < R$ becomes $\|w(0)\| < R$. Let $\mathcal{N}$ be as in (5.7), and take

$$R := \mathrm{dist}(0, \mathcal{N}) = \inf\{\|w\|;\ w \in \mathcal{N}\} \ .$$

If $\|w(0)\| < R$, then by Lemmas 1, 2, and 3 we infer that $w(0) \in B_0$. This means that $u_0 \in \mathcal{P}$. □

8. Proof of Theorem 5

When $p \geq 2$ we may extend Lemma 2 to all $\mathcal{N}$ and not just to $\mathbb{K} \cap \mathcal{N}$:

Lemma 9. *Let $w \in \mathcal{N}$, and consider the (smooth) map $\Psi_w : [0, +\infty) \to \mathbb{R}$ defined by $\Psi_w(t) = I(tw)$. Then, $\Psi_w'(t) > 0$ for all $t \in (0,1)$ and $\Psi_w'(t) < 0$ for all $t > 1$.*

Proof. Since $p \geq 2$ we have $\Psi_w \in C^2[0, \infty)$ and

$$\Psi_w(t) = \frac{t^2}{2} \int_\Omega |\nabla w|^2 + \lambda \int_\Omega \Big[(1+u_\lambda)^p tw - \frac{|1+u_\lambda+tw|^{p+1}}{p+1} + \frac{(1+u_\lambda)^{p+1}}{p+1}\Big],$$

$$\Psi_w'(t) = t \int_\Omega |\nabla w|^2 + \lambda \int_\Omega \big[(1+u_\lambda)^p - |1+u_\lambda+tw|^{p-1}(1+u_\lambda+tw)\big]\, w \ ,$$

$$\Psi_w''(t) = \int_\Omega |\nabla w|^2 - \lambda p \int_\Omega |1+u_\lambda+tw|^{p-1} w^2 \ .$$

Moreover, the map $t \mapsto \Psi_w''(t)$ is concave since $p \geq 2$. Next, note that $\lim_{t\to\infty} \Psi_w''(t) = -\infty$ and $\Psi_w''(0) > 0$ (in view of [9, Lemma 2.1]), so that there exists a unique $t'' > 0$ such that $\Psi_w''(t'') = 0$. Consequently, there exists a unique $t' > 0$ such that $\Psi_w'(t') = 0$. Finally, since $I'(w)[w] = 0$, we have $\Psi_w'(1) = 0$, and the statement follows. $\square$

The proof of Theorem 5 may now be completed arguing exactly as for Theorem 4.

9. Proof of Theorem 6

Let $u_0 \in \mathbb{K} \setminus \{0\}$ and $T > 0$. Let $u \in L^p_{\rm loc}((0,T); L^p_{\rm loc}(\Omega) \cap \mathbb{K})$ be a generalized solution of (1.1) with generalized initial value $u(0) = \alpha u_0$ ($\alpha > 0$). We have to show that such a solution u cannot exist for large enough α. If the generalized solution u does exist then (2.4) holds, and therefore

$$\lambda \int_0^T \int_\Omega (1+u)^p \phi \leq \int_0^T \int_\Omega (|\Delta\phi|+|\phi_t|)u - \alpha \int_\Omega u_0 \phi(0) \quad \forall \phi \in C_c^\infty(\Omega \times [0,T)). \tag{9.1}$$

Throughout this proof we denote by C_+ the set of functions which are in C_c^∞ and which are strictly positive in the interior of their support. Assume now that $\phi \in C_+(\Omega \times [0,T))$; by Young's inequality there exists $C > 0$ such that in the interior of the support of ϕ we have

$$u|\Delta\phi| = u\phi^{1/p} \frac{|\Delta\phi|}{\phi^{1/p}} \leq \frac{\lambda}{2} u^p \phi + C \frac{|\Delta\phi|^{p/(p-1)}}{\phi^{1/(p-1)}}\ , \quad u|\phi_t| \leq \frac{\lambda}{2} u^p \phi + C \frac{|\phi_t|^{p/(p-1)}}{\phi^{1/(p-1)}}\ . \tag{9.2}$$

As (x,t) approaches the boundary of the support of ϕ, the right-hand sides of (9.2) tend to vanish (see [30]); hence, we set them to 0 outside the support of ϕ. Inserting (9.2) into (9.1) yields

$$\alpha \int_\Omega u_0 \phi(0) \leq C \int_0^T \int_\Omega \frac{|\Delta\phi|^{p/(p-1)} + |\phi_t|^{p/(p-1)}}{\phi^{1/(p-1)}} \tag{9.3}$$

for all $\phi \in C_+(\Omega \times [0,T))$. Take any $v \in C_+(\Omega)$ such that

$$\int_\Omega u_0 v > 0\ . \tag{9.4}$$

Take $\psi \in C_+[0,T)$ such that $\psi(0) = 1$, and use $\phi(x,t) := \psi(t)v(x)$ as test function in (9.3). Then,

$$\alpha \int_\Omega u_0 v \leq C(\phi) \tag{9.5}$$

for some $C(\phi) > 0$ (note that $C(\phi)$ depends on T through ϕ). Let

$$\overline{\alpha} := \frac{C(\phi)}{\int_\Omega u_0 v}$$

so that $\overline{\alpha} > 0$ in view of (9.4); if we take $\alpha > \overline{\alpha}$, (9.5) is violated, showing that no generalized solution over $(0,T)$ exists.

Let us now show that $T_\alpha \to 0$ as $\alpha \to \infty$. Take $\psi \in C_+[0,1)$ such that $\psi(0) = 1$, and for all $\gamma > 1$ let $\phi^\gamma(x,t) = \psi(\gamma t)v(x)$, where $v \in C_+(\Omega)$ satisfies again (9.4). Therefore, $\phi^\gamma \in C_+(\Omega \times [0, 1/\gamma))$, and using ϕ^γ as test function and arguing as for (9.3), we obtain

$$\begin{aligned}
&\alpha \int_\Omega u_0(x)v(x)\,dx \\
&\le C \int_0^{1/\gamma} \int_\Omega \frac{|\psi(\gamma t)\Delta v(x)|^{p/(p-1)} + |\gamma\psi'(\gamma t)v(x)|^{p/(p-1)}}{|\psi(\gamma t)v(x)|^{1/(p-1)}}\,dx\,dt \\
&= \frac{C}{\gamma} \int_0^1 \int_\Omega \frac{|\psi(s)\Delta v(x)|^{p/(p-1)} + |\gamma\psi'(s)v(x)|^{p/(p-1)}}{|\psi(s)v(x)|^{1/(p-1)}}\,dx\,ds,
\end{aligned}$$

where in the last step we used the change of variable $s = \gamma t$. Summarizing, we get

$$C_1\alpha \le \frac{C_2}{\gamma} + C_3\gamma^{1/(p-1)}, \tag{9.6}$$

where C_i $(i = 1,2,3)$ are positive constants depending only on ψ and v. For any sufficiently large $\alpha > 0$ let $\gamma_\alpha > 0$ be the largest value of γ for which equality holds in (9.6). For a given (large) $\alpha > 0$, (9.6) shows that $T_\alpha \le \gamma_\alpha^{-1}$. Since, $\gamma_\alpha \to \infty$ as $\alpha \to \infty$, the proof is complete. □

Remark 3. In the steps from (9.1) to (9.3) the term $\lambda \int_0^T \int_\Omega \phi$ was deleted. If we *do not* delete it and we take $u_0 \equiv 0$, (9.5) becomes

$$\lambda \int_0^T \int_\Omega \phi \le C(\phi)\ .$$

This shows that if λ is sufficiently large, then the solution of (1.1) with initial datum $u_0 \equiv 0$ blows up in finite time. And together with Theorem 3, this gives an upper bound for λ^*. Indeed, not only is the term $\lambda \int_0^T \int_\Omega \phi$ increasing with λ, but also $C(\phi)$ is decreasing with λ; see (9.2).

10. Proof of Theorem 7

The existence of $\overline{\alpha}$ follows at once from (9.5) with a function ϕ as in the proof of Theorem 6. The fact that T_α vanishes at infinity may be obtained as in the previous section.

If $u_0 \in L^\infty(\Omega)$, we may apply [28, Theorem 4] to obtain that $\overline{\alpha}$ satisfies the following: if $\alpha < \overline{\alpha}$, then $u(t) \to u_\lambda$ uniformly as $t \to \infty$. And once we have the uniform convergence, we easily deduce the convergence in $H_0^1(\Omega)$ by using (6.12). If $u_0 \notin L^\infty(\Omega)$, we may use a density argument.

11. Proof of Theorem 8

We use the same notation and similar arguments as in Section 6. Let $u = u(t)$ be the local solution of (1.1) as given by Theorem 2. We make a change of unknown and set

$$w(t) = u(t) - u_\lambda \ . \tag{11.1}$$

Then, $w = w(t)$ is the unique local solution of the problem

$$\begin{cases} w_t - \Delta w = \lambda(1 + u_\lambda + w)^p - \lambda(1 + u_\lambda)^p \quad \text{in } \Omega \times (0, T) \\ w(0) = u_0 - u_\lambda \geq 0 \quad \text{in } \Omega \\ w = 0 \quad \text{on } \partial\Omega \times (0, T) \ . \end{cases} \tag{11.2}$$

Next, we introduce the set corresponding to $\mathcal{Q}$. Let

$$Q := \{w \in H_0^1(\Omega) \cap \mathbb{K};\ I'(w)[w] < 0\, ,\ I(w) < d\} \ .$$

We construct subsets of Q which are invariant under the flow of (11.2). For any $\varepsilon \in (0, d)$ let

$$Q_\varepsilon = \{w \in Q;\ I(w) \leq d - \varepsilon\} \ .$$

Then, we prove

Lemma 10. *Let $\varepsilon \in (0, d)$, and assume that $w_0 \in Q_\varepsilon$; then the (local) solution $w = w(t)$ of (11.2) satisfies $w(t) \in Q_\varepsilon$ for all t in its maximal interval of continuation $[0, T^*)$ ($T^* \in (0, \infty]$).*

Proof. First note that (11.2) is positivity preserving so that $w(t)$ remains nonnegative on its interval of definition. Consider again the energy functional $E(t) := I(w(t))$. By (6.4) we see that E is strictly decreasing. Since $E(0) \leq d - \varepsilon$, this proves that

$$I(w(t)) = E(t) < d - \varepsilon \qquad \forall t \in (0, T^*) \ . \tag{11.3}$$

For contradiction, assume that $w(t)$ exits Q_ε in finite time. Then, by (11.3) there exists a first time $T \geq 0$ when $I'(w(T))[w(T)] = 0$. Hence, Lemma 3 implies $I(w(T)) \geq d$. This contradicts (11.3). $\square$

From now on, we argue by contradiction. We assume that the solution $w(t)$ of (11.2) is global. Then, we have the following:

Lemma 11. *Let $\varepsilon \in (0,d)$, and assume that $w_0 \in Q_\varepsilon$ and that the solution $w = w(t)$ of* (11.2) *exists for all $t \geq 0$. Then, $\lim_{t\to\infty} \|w(t)\|_2 = \infty$.*

Proof. By (6.9), the result follows if we show that

$$\limsup_{t\to\infty} I'(w(t))[w(t)] < 0 \ . \tag{11.4}$$

We argue by contradiction assuming that (11.4) is false. In view of Lemma 10, this means that, up to a subsequence, we have $\lim_{t\to\infty} I'(w(t))[w(t)] = 0$, namely,

$$\frac{1}{p+1}\int_\Omega |\nabla w(t)|^2 + \frac{1}{p+1}\int_\Omega \nabla w(t)\nabla u_\lambda = \frac{\lambda}{p+1}\int_\Omega (1+u_\lambda+w(t))^p w(t) + o(1).$$

By replacing (5.10) with this equality and by repeating exactly the proof of Lemma 4 we obtain that $\{w(t)\}$ is bounded in $H^1_0(\Omega)$. Up to a further subsequence, this fact implies again (6.6) and (6.7). Arguing as in the proof of Lemma 7, we have thus proved that, up to a subsequence,

$$\exists v \in H^1_0(\Omega) \cap \mathbb{K}\ , \quad I'(v) = 0\ , \quad w(t) \rightharpoonup v\ .$$

Therefore, by Lemmas 2 and 3 we know that

$$\text{either} \qquad v \equiv 0 \qquad \text{or} \qquad I(v) \geq d\ . \tag{11.5}$$

If $p < \frac{n+2}{n-2}$, then by compact embedding we obtain both (6.13) and $v \not\equiv 0$ which contradicts (11.5). If $p = \frac{n+2}{n-2}$, by (6.15) and (6.20) we see that (11.5) implies $v \equiv 0$ and we get a contradiction arguing as in the last part of Section 6.2. $\square$

By embedding inequalities, Lemma 11 has the straightforward consequence

$$\lim_{t\to\infty} \|w(t)\|_{p+1} = \lim_{t\to\infty} \|w(t)\| = \infty\ . \tag{11.6}$$

Thanks to (11.6) we can strengthen (11.4) with the following:

Lemma 12. *Let $\varepsilon \in (0,d)$, assume that $w_0 \in Q_\varepsilon$ and that the solution $w = w(t)$ of* (11.2) *exists for all $t \geq 0$. Then,*

$$\lim_{t\to\infty} I'(w(t))[w(t)] = -\infty\ .$$

Proof. Throughout this proof we denote by C_λ positive constants which depend on λ and u_λ and which may vary from line to line. By Young's inequality there exists $C_\lambda > 0$ such that

$$\int_\Omega (1+u_\lambda)^p w(t) + \int_\Omega (1+u_\lambda+w(t))^p(1+u_\lambda) \le C_\lambda + \frac{p-1}{2(p+1)}\int_\Omega (1+u_\lambda+w(t))^{p+1}. \tag{11.7}$$

We may rewrite $I'(w(t))[w(t)]$ as

$$\begin{aligned} &I'(w(t))[w(t)] \\ &= \|w(t)\|^2 - \frac{2\lambda}{p+1}\int_\Omega (1+u_\lambda+w(t))^{p+1} - \frac{(p-1)\lambda}{p+1}\int_\Omega (1+u_\lambda+w(t))^{p+1} \\ &+ \lambda\int_\Omega (1+u_\lambda)^p w(t) + \lambda\int_\Omega (1+u_\lambda+w(t))^p(1+u_\lambda); \end{aligned}$$

therefore, by (5.13) and (11.7) we have

$$I'(w(t))[w(t)] \le C_\lambda - \frac{(p-1)\lambda}{2(p+1)}\int_\Omega (1+u_\lambda+w(t))^{p+1} \to -\infty \tag{11.8}$$

in view of (11.6). □

We can now conclude the proof of Theorem 8. For contradiction we assumed that the solution $w = w(t)$ of (11.2) is global and we obtained Lemmas 11 and 12. We will now show that this leads to a contradiction, namely that $w(t)$ blows up in finite time.

To this end, let $\Phi(t) := \|w(t)\|_2^2$ so that by (6.9) we have

$$\Phi'(t) = -2I'(w(t))[w(t)].$$

In what follows we denote by C_i positive constants. By (11.8) and Hölder's inequality, we have

$$\frac{\Phi'(t)}{\Phi^{\frac{p+1}{2}}(t)} \ge C_1\,\frac{\|w(t)\|_{p+1}^{p+1} - C_2}{\|w(t)\|_2^{p+1}} \ge C_3\ , \qquad \forall t \ge T,$$

where T is a suitably large number and the last inequality follows from (11.6). Integrating this inequality over $[T,t]$ for $t > T$, yields

$$\frac{1}{\Phi^{\frac{p-1}{2}}(t)} \le \frac{1}{\Phi^{\frac{p-1}{2}}(T)} - C_4(t-T) \qquad \forall t \ge T,$$

which shows that $\Phi(t) \to \infty$ in finite time (recall $p > 1$). This completes the proof of Theorem 8. □

12. Some problems

Problem 1. Extend (at least partially) the statements of the present paper to the case where $(1+u)^p$ is replaced by any smooth, convex, nondecreasing function f such that $f(0) > 0$ and

$$\liminf_{s\to\infty} \frac{f'(s)s}{f(s)} > 1 \ ,$$

for instance, $f(u) = e^u$. This does not seem to be straightforward, as in our approach we take advantage of two crucial facts: when $p \leq \frac{n+2}{n-2}$ we make use of critical-point theory, and the power-type behavior of the reaction term enables us to use the test function method as in [30]; see Theorem 6. It is well-known [9, 15, 18] that also for $f(u) = e^u$ equation (1.2) may admit positive stationary solutions different from the minimal one. In such a case is it possible to define a potential well? In which functional space?

Problem 2. Does Lemma 9 also hold in the case $1 < p < 2$? If affirmative, this would allow one to simplify the definition of $\mathcal{P}$ (into $\widehat{\mathcal{P}}$) and of $\mathcal{Q}$ (without requiring $u \geq u_\lambda$). As a consequence, we would have nicer statements for Theorems 4 and 8. On the other hand, to show that Lemma 9 *does not* hold for $1 < p < 2$, the simplest way seems to be to find a counterexample in the critical radial setting (as in Section 3) by taking $n \geq 7$ and $p = \frac{n+2}{n-2}$ and by using the explicit form of u_λ and U_λ.

Problem 3. Which is the behavior of local solutions of (1.1) when $\lambda = \lambda^*$? Partial answers may be found in [28, Theorem 5]. Can one find global existence results (and convergence to U_*) with initial datum $u_0 \notin L^\infty(\Omega)$ and/or without using the comparison principle?

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