The quadruple planar bubble enclosing equal areas is symmetric

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Abstract

In this paper we make the final step in finding the optimal way to enclose and separate four planar regions with equal area. In [10] the graph-topology of the optimal cluster was found reducing the set of candidates to a one-parameter family of different clusters. With a simple argument we show that the minimal set has a further symmetry and hence is uniquely determined up to isometries.

1 Introduction

The problem of enclosing and separating N regions of $\mathbb{R}^2$ with prescribed area and with the minimal possible interface length has been widely analyzed.

The case $N = 1$ corresponds to the celebrated isoperimetric problem whose solution, the circle, was known since antiquity.

For $N \geq 1$ first existence and partial regularity in $\mathbb{R}^n$ was given by Almgren [1] while Taylor [13] describes the singularities for minimizers in $\mathbb{R}^3$. Existence and regularity of minimizers in $\mathbb{R}^2$ was proved by Morgan [7] (see also [5]): the regions of a minimizer in $\mathbb{R}^2$ are delimited by a finite number of circular arcs, or line segments, which meet in "Steiner-triples" at their end-points, with angles of $\frac{2}{3}\pi$.

Foisy et al. [2] proved that for $N = 2$ in $\mathbb{R}^2$ the two regions of any minimizer are delimited by three circular arcs joining in two points (standard double bubble) and are uniquely determined by their enclosed areas. Wichiramala [15] proved that for $N = 3$ in $\mathbb{R}^2$ the three regions of any minimizer are delimited by six circular arcs joining in four points. Lawlor [4] recently proposed a new simpler proof, which is also valid in the sphere. The minimizer (standard triple bubble) is uniquely determined by the given enclosed areas, as shown by Montesinos [6].

Recently, in [10], the case of four regions with equal areas in $\mathbb{R}^2$ has been considered and in this case the graph-topology of minimal clusters has been

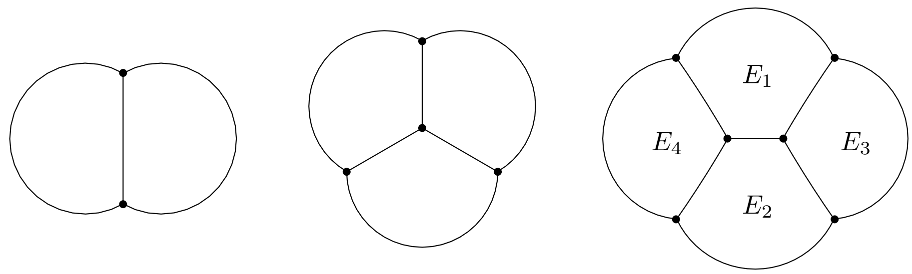

Figure 1: The optimal sets enclosing two, three and four equal areas. The cluster with four regions has two axes of symmetry. The edge between regions E_1 and E_2 is a straight segment while the other four internal edges have a small curvature so that regions E_3 and E_4 are actually strictly convex.

determined: the cluster is composed by four connected regions, two among them, E_1 and E_2 are quadrangular and are adjacent to each other, whilst the remaining, E_3 and E_4, are triangular and are adjacent to both the quadrangular ones (see Figure 1). It was conjectured that, with this structure, there is a unique minimizer (up to isometries) which is determined by two orthogonal axes of symmetry: one symmetry should map E_3 and E_4 each onto itself whilst swapping E_1 and E_2, the other should map E_1 and E_2 each onto itself whilst swapping E_3 and E_4. The conjecture was backed by numerical evidence [11]. In this paper we present a simple proof of such conjecture thus finally obtaining a proof of the following Theorem (see next section for notation).

Theorem 1.1. *Up to rotation, translation, rescaling, reordering and modification by zero measure sets, there is a unique planar minimal cluster* (E_1, E_2, E_3, E_4) *composed by four regions of equal area. The cluster has two orthogonal axes of symmetry. Regions* E_1 *and* E_2 *are quadrangular regions: each one is the mirror-image of the other through one axis. They are adjacent, and the common edge is a line segment of the other axis. Regions* E_3 *and* E_4 *are strictly convex triangular regions, and each one is the mirror-image of the other through the remaining axis (see Figure 1).*

The idea of the proof is the following. The minimal cluster is known to be composed by two triangular regions and two quadrangular regions. If we remove the two triangular regions we obtain a double bubble which obviously has a line of symmetry. Since the two triangular regions have the same area the whole minimal cluster has the same line of symmetry which we suppose is vertical (Corollary 3.4). To find the horizontal line of symmetry it is enough to prove that the edge between the two quadrangular regions is a straight segment. Suppose by contradiction that instead this edge is curved. By means of a circle inversion we are able to transform the two quadrangular regions into two congruent regions: the area of the original regions can be obtained by integrating the jacobian of the transformation on the transformed regions. It turns out that the jacobian, in one region, is pointwise larger than the jacobian on the other region, hence we obtain a contradiction.

2 Notation and Tools

We follow the notation introduced in [10]. We denote the outer Lebesgue measure of a subset A of $\mathbb{R}^2$ by $|A|$ (the area of A) and by $P(A)$ its Caccioppoli perimeter (which is the length of the boundary ∂A if A is sufficiently regular). We say that two subsets A, B of $\mathbb{R}^2$ are adjacent if $P(A \cup B) < P(A) + P(B)$ (the common boundary has positive length). If E has finite perimeter we say that a Lebesgue measurable set C is a component of E if $|C| > 0$, $|C \setminus E| = 0$ and $P(E) = P(C) + P(E \setminus C)$. Moreover we say that E is connected (in the measure theoretic sense) if it has no component C of measure $|C| < |E|$. Let us denote with $\mathbf{E} = (E_1, \dots, E_N)$ an N-uple of measurable subsets of $\mathbb{R}^2$ such that $|E_i \cap E_j| = 0$ for $i \neq j$. We will say that $\mathbf{E}$ is a *cluster* and that $E_1, \dots, E_N$ are its *regions*. We define the *external region* E_0 as

$$E_0 = \mathbb{R}^2 \setminus \bigcup_{i=1}^{N} E_i.$$

The sets $E_0, E_1, \dots, E_N$ are hence a partition of the whole plane $\mathbb{R}^2$.

We define the *perimeter* of the cluster as

$$P(\mathbf{E}) = \frac{1}{2} \sum_{i=0}^{N} P(E_i)$$

The perimeter of the cluster would represent the total length of the interfaces between the regions. In fact, up to a set of zero length (in the sense of $\mathcal{H}^1$ Hausdorff measure) every point in the union of the reduced boundaries of the regions (the reduced boundary is the measure theoretic boundary of a Caccioppoli set) belongs to exactly two different boundaries (see [5]), hence the factor $\frac{1}{2}$ in the previous definition.

We are interested in the problem of finding the clusters with minimal perimeter among all clusters with prescribed areas. Such clusters will be called *minimal clusters*.

The following result states the existence of minimal clusters (see [1], [7] and [5]).

Theorem 2.1 (existence of minimal clusters)**.** *Given* $(a_1, \dots, a_N) \in \mathbb{R}^N_+$ *there exists a cluster* $\mathbf{E}$ *in* $\mathbb{R}^n$ *such that* $|E_i| = a_i$, $1 \le i \le N$, *and such that*

$$P(\mathbf{E}) \le P(\mathbf{F})$$

for all $\mathbf{F}$ *such that* $|F_i| = a_i$, $1 \le i \le N$.

Minimal clusters have very good regularity properties. In particular the structure of minimal clusters has been widely studied when the ambient space is $\mathbb{R}^2$ (see [7]) or $\mathbb{R}^3$ (see [13]). We recall the regularity result for the planar case.

Theorem 2.2 (regularity of planar minimal clusters)**.** *Let* $\mathbf{E}$ *be a minimal cluster in* $\mathbb{R}^2$*. Then, up to redefining each region on a zero measure set, each region* E_k *of* $\mathbf{E}$ *is composed by a finite number of connected components. Each connected component is delimited by a finite number of circular arcs or straight line segments. Each arc separates two components of different regions. The arcs meet in triples at their end points (which we call vertices) with equal angles of* $\frac{2}{3}\pi$*. The sum of the signed curvatures*[1] *of the three arcs joining in a vertex is zero.*

A cluster $\mathbf{E}$ satisfying the regularity properties stated in the previous theorem will be called *stationary.*

Lemma 2.3. *Stationarity is preserved under isometries and rescalings of the plane. Stationarity is also preserved by circle inversion.*

The first part of the previous Lemma is trivial, for the second part see [14], [6], [16], [12]. Recall that circle inversion has the well known property of being a conformal mapping transforming circles and straight lines into circles or straight lines.

To further investigate the geometry of minimal clusters we point out a general result which can be stated for a triangular region of any stationary cluster (see [16]).

Theorem 2.4 (removal of triangular components)**.** *Let* T *be a triangular component of a region of a stationary cluster* $\mathbf{E}$*. Consider the three oriented arcs not edges of* T *and each concurrent to one among the three vertices of* T*. The sum of the signed curvatures of these three arcs is zero. Moreover, if prolonged inside* T*, these arcs meet in a point* P *inside* T *with equal angles of* $\frac{2}{3}\pi$*. Hence, if we remove the triangle* T *and prolong the three arcs, we obtain a new stationary cluster.*

Currently it is not known if the regions of every minimal cluster are connected (see [7, introduction]). In the case of four equal areas this was proved in [10] where the topology of the minimal clusters is determined.

Theorem 2.5. *Let* $\mathbf{E} = (E_1, E_2, E_3, E_4)$ *be a cluster with* $N = 4$ *regions in the plane which is minimal with prescribed equal areas* (a, a, a, a)*,* $a > 0$*. Then all the four regions are connected. Moreover two of them (say* E_1 *and* E_2*) have four edges and two of them (say* E_3 *and* E_4*) have three edges (see Figure 2).*

3 Proof of Theorem 1.1

We are going to prove that up to isometries, rescalings, reorderings and modifications by zero measure sets there is at most one stationary cluster $\mathbf{E}$ with the

[1]If the three arcs are oriented so that the vertex is the end point of each of the three arcs, then the orientation defines a normal vector ν (for example by rotating the tangent vector counter-clockwise) on the three arcs and the signed curvature k is defined by $k = \mathbf{k} \cdot \nu$ where $\mathbf{k}$ is the curvature vector. Clearly k is constant on each circular arc or line segment.

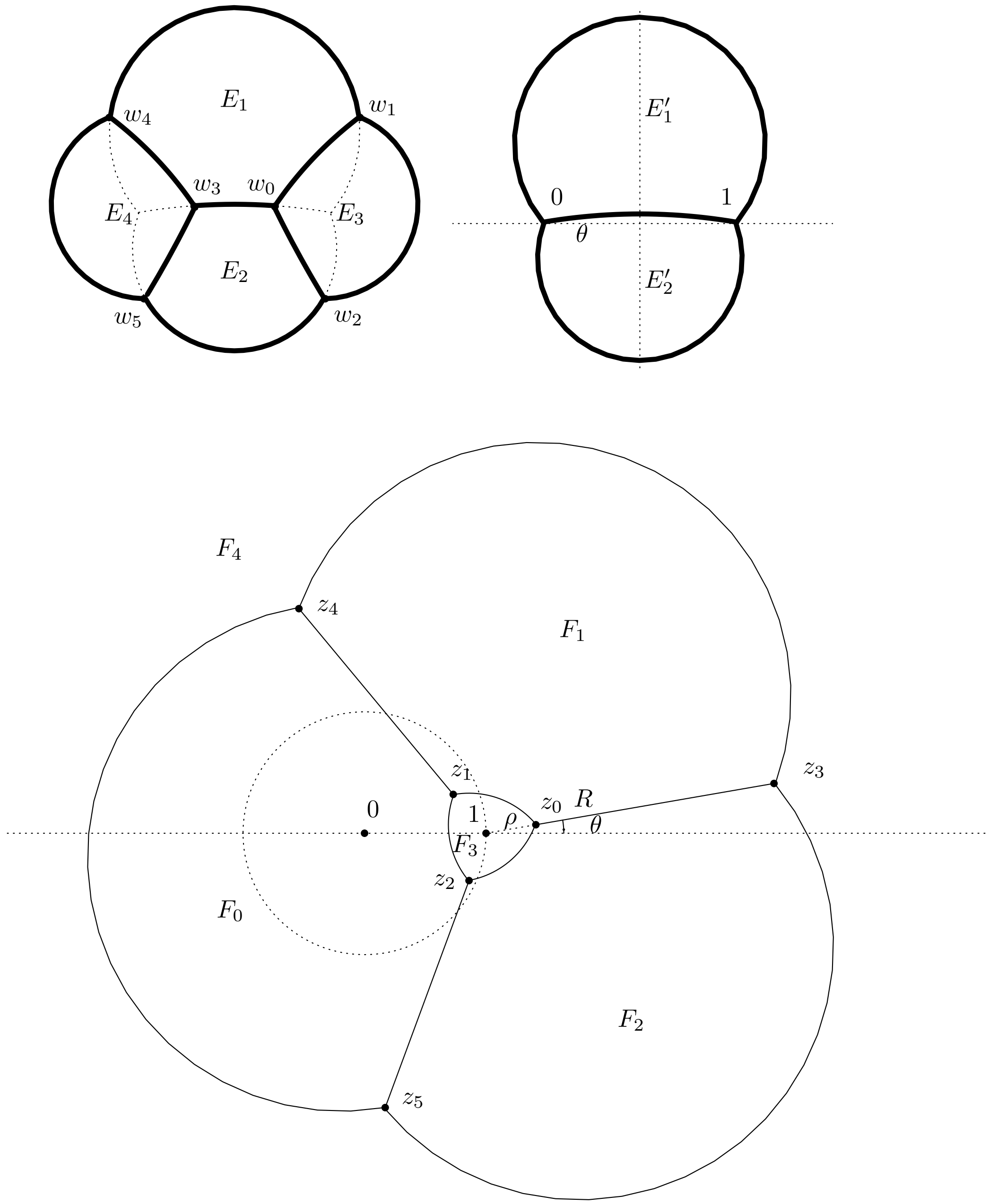

Figure 2: In bold lines the edges of the cluster $\mathbf{E}$, and $\mathbf{E}'$. In solid lines the edges of the cluster $\mathbf{F}$. The dotted circle is used for circle inversion. The two dotted lines have equations $y = 0$ and $x = 1/2$.

topology described in Theorem 2.5 and with equal areas $|E_1| = |E_2| = |E_3| = |E_4| = a > 0$ and that such cluster $\mathbf{E}$ has all the properties stated in the claim of Theorem 1.1. In view of Theorems 2.2 and 2.5 this is enough to obtain the main Theorem.

Let $\mathbf{E} = (E_1, E_2, E_3, E_4)$ be a any stationary cluster with the topology described in Theorem 2.5. Up to relabeling we may suppose that E_1 and E_2 are the quadrangular regions while E_3 and E_4 are the triangular ones. The cluster contains six vertices: let w_0, w_1, w_2 be the three vertices of E_3 where the first is the vertex in common with both E_1 and E_2, the second the vertex in common only with E_1, and the third in common only with E_2. Similarly label w_3, w_4, w_5 the vertices of E_4 (see Figure 2).

By Theorem 2.4 if we remove the two triangular regions and extend the remaining three edges we obtain a stationary cluster $\mathbf{E}' = (E_1', E_2')$ (a double bubble) with $E_1' \supset E_1$, $E_2' \supset E_2$. So every stationary cluster $\mathbf{E}$ with the topology given by Theorem 2.5 can be obtained by a double bubble. Notice that $\mathbf{E}'$ is symmetric with respect to the axis of the common chord of the three arcs.

Up to translation, rotation and rescaling we might and shall suppose that the two vertices of $\mathbf{E}'$ are the points $(0,0)$ and $(1,0)$ with $(0,0) \in E_4$ and $(1,0) \in E_3$. From now on we will identify $\mathbb{R}^2$ with $\mathbb{C}$ so that the two vertices of $\mathbf{E}'$ are represented by the complex numbers 0 and 1 (see Figure 2). The known axis of symmetry of $\mathbf{E}'$ is $\operatorname{Re} w = \frac{1}{2}$. Either E_1' or E_2' is convex: without loss of generality we assume that such region is E_2'. Up to reflection we can also suppose that E_1' (and hence E_1) is contained in the upper half-space $\operatorname{Im} w \geq 0$.

Let $\theta \geq 0$ be the angle between the circular edge separating E_1' and E_2' and the line segment joining 0 and 1. By construction of $\mathbf{E}'$ and the Steiner angle condition we have $\theta < \pi/3$. On the other hand, given any such angle, up to isometries and rescaling, there is just one double bubble with θ equal to the given angle.

We will consider the cluster $\mathbf{F}$ obtained from $\mathbf{E}$ by the circle inversion $T(w) = w/|w|^2 = 1/\overline{w}$. We know by Lemma 2.2 that stationarity is preserved under inversion so $\mathbf{F} = (F_1, F_2, F_3, F_4)$ is also stationary. The region E_4 contains 0 so the corresponding triangular region F_4 is unbounded. On the other hand the exterior of the cluster $\mathbf{E}$ goes into a quadrangular bounded region F_0 which is the exterior of $\mathbf{F}$. Let $z_0, \dots, z_5$ be the six vertices of $\mathbf{F}$: $z_j = T(w_j)$.

Under circle inversion the three circular edges of the double bubble $\mathbf{E}'$ joining 0 and 1 become three half lines meeting in the point 1 with equal angles of $\frac{2}{3}\pi$. In particular the edge of $\mathbf{E}$ joining w_0 with w_3 corresponds to a straight segment $[z_0, z_3]$ which is contained in the half line starting from 1 with an angle θ with respect to the real axis (the angle is preserved because circle inversion is a conformal mapping). With the previous definitions and notations, we state the following.

Lemma 3.1. *The points z_0, z_1 and z_2 have the same distance $\rho > 0$ from the point* 1*. The points z_3, z_4, z_5 have the same distance $R > \rho$ from the point* 1.

Proof. The stationarity condition in the vertex z_0 ensures that the arcs z_0z_1 and z_0z_2 are symmetric with respect to the line z_0z_3 (the angles are equal to

$2\pi/3$, and the radii are equal because the sum of the three signed curvatures is zero but the line has zero curvature). Since 1 is on the line containing $z_0 z_3$ we obtain $|z_1 - 1| = |z_2 - 1|$.

The same holds in the vertices z_1 and z_2 hence the points z_0, z_1, z_2 are the vertices of an equilater triangle with center in the point 1.

The same reasoning can be applied to the arcs joining the vertices z_3, z_4 and z_5 to find that also these points have equal distance from the point 1. □

Thus we can write $z_j = 1 + \rho \cdot e^{i(\theta + 2j\pi/3)}$ and $z_{j+3} = 1 + R \cdot e^{i(\theta+2j\pi/3)}$ for some $R > \rho > 0$ with $j = 0, 1, 2$. Incidentally the arcs joining the vertices z_0, z_1, z_2 are each centered in the opposite vertex (they form a so called Reuleaux triangle) while the arcs joining the vertices z_3, z_4, z_5 are half circles.

Corollary 3.2. *The cluster* $\mathbf{F}$ *is symmetric with respect to the line throught* z_0 *and* z_3*.*

Clearly the cluster $\mathbf{F} = \mathbf{F}(\theta, \rho, R)$ is uniquely determined by the parameters θ, ρ and R and so is $\mathbf{E} = T(\mathbf{F}) = \mathbf{E}(\theta, \rho, R)$. Moreover the triangular regions only depend on two parameters: $E_3 = E_3(\theta, \rho)$, $E_4 = E_4(\theta, R)$. Clearly to obtain a cluster $\mathbf{E}$ with the given topology it is necessary not only that $0 < \rho < R$, $0 \le \theta \le \frac{\pi}{3}$ but also that 0 is not in the Reuleaux triangle centered in 1 but belongs to the quadrangular regions which doesn't touch the angle θ.

Lemma 3.3. *We have that* $E_3(\theta, \rho)$ *is strictly increasing in* ρ*,* $E_4(\theta, R)$ *is strictly decreasing in* R*,* $E_1(\theta, \rho, R)$ *and* $E_2(\theta, \rho, R)$ *are strictly decreasing in* ρ *and strictly increasing in* R*.*

Proof. If we increase ρ it is clear that the Reuleaux triangle F_3 is strictly increasing (with respect to set inclusion). Hence the $E_3 = T(F_3)$ is also strictly increasing (with respect to set inclusion) and its area is strictly increasing.

Since $E_1 = E_1' \setminus (E_3 \cup E_4)$ if the area of E_3 is increasing the area of E_1 must be decreasing. The same is true for $E_2 = E_2' \setminus (E_3 \cup E_4)$.

The argument can be repeated for the regions F_4 and E_4 when we decrease R. □

In particular given θ and an admissible measure $|E_3|$ then ρ is uniquely determined. Conversely given θ and $|E_4|$ then R is uniquely determined.

Corollary 3.4. *If* $|E_3| = |E_4|$ *the whole cluster* $\mathbf{E}$ *is symmetric with respect to the line* $\operatorname{Re} w = \frac{1}{2}$*.*

Proof. Let σ be the symmetry with respect to the line $\operatorname{Re} w = \frac{1}{2}$. If $\mathbf{E} = \mathbf{E}(\theta, \rho, R)$ the symmetric of $\mathbf{E}$ can be written as $\mathbf{E}(\theta, \rho', R')$. But $|E_3(\theta, \rho')| = |E_4(\theta, R)|$ and if we suppose that $|E_3(\theta, \rho)| = |E_4(\theta, R)|$ we obtain $\rho = \rho'$ in view of Lemma 3.3. Similarly we can state that $R = R'$ and hence $\sigma(E_3) = E_4$. □

Remark 3.5. One can show (even if we don't really need it) that $R' = 1/\rho$, since the symmetry of $\mathbf{E}$ with respect to $\operatorname{Re} w = 1/2$, conjugated with T, becomes the inversion with respect to the unit circle centered in $z = 1$.

Lemma 3.6. *For any given θ there exists at most one value of ρ (and R) such that $|E_1| = |E_3| = |E_4|$. The same for $|E_2| = |E_3| = |E_4|$.*

Proof. Given θ the double bubble $\mathbf{E}'$ is uniquely determined. If there exists $\mathbf{E}$ with the required condition we only need to inflate the two triangular regions E_3 and E_4 by increasing ρ and decreasing R, starting from the given $\mathbf{E}'$: $\rho = 0$, $R = +\infty$. In this process we can mantain the equality $|E_3| = |E_4|$ (namely with the condition $\rho = \frac{1}{R}$, keeping the cluster symmetric with respect to the line $z = \frac{1}{2}$) and by Lemma 3.3 when ρ increases and R decreases the difference $|E_1| - |E_3| = |E_1| - |E_4|$ is strictly decreasing and hence it can be zero for at most a single value of ρ and R. □

Lemma 3.7. *If $\theta > 0$ then we have $|E_1| > |E_2|$.*

Proof. The geometric idea of the proof is that if $\theta > 0$ the region F_1 is "closer" to 0 than the congruent region F_2 and hence by circle inversion the area of E_1 would be greater than the area of E_2.

We are going to compute the area of E_1 and E_2 as integrals over F_1 and F_2. It is easy to check that the Jacobian determinant of the trasformation of $\mathbb{R}^2$ relative to the circle inversion $T(w) = w/|w|^2$ is $1/|w|^4$. Using polar coordinates centered in $w = 1$ can we write:

$$F_1 = \left\{1 + re^{i(\theta+t)} : t \in \left[0, \tfrac{2}{3}\pi\right],\ r \in [\rho \cdot r_1(t), R \cdot r_2(t)]\right\}$$

for suitable continuous functions $r_1(t) \le r_2(t)$. Since, as stated in Corollary 3.2, F_2 is the mirror-symmetric of F_1 with respect to the line of angle θ passing through z_0 and z_3 we have

$$F_2 = \left\{1 + re^{i(\theta-t)} : t \in \left[0, \tfrac{2}{3}\pi\right],\ r \in [\rho \cdot r_1(t), R \cdot r_2(t)]\right\}.$$

So

$$\begin{aligned} |E_1| - |E_2| &= \iint_{F_1} \frac{1}{|x+iy|^4}\, dxdy - \iint_{F_2} \frac{1}{|x+iy|^4}\, dxdy \\ &= \int_0^{\frac{2}{3}\pi} \int_{\rho \cdot r_1(t)}^{R \cdot r_2(t)} \left[\frac{1}{\left|1 + re^{i(\theta+t)}\right|^4} - \frac{1}{\left|1 + re^{i(\theta-t)}\right|^4} \right] r\, dr\, dt. \end{aligned} \tag{1}$$

Considering that for $t \in (0, \frac{2}{3}\pi]$ and $\theta \in (0, \frac{\pi}{3})$ it holds

$$\sin\theta \sin t > 0,$$

by addition formula:

$$\cos(\theta + t) < \cos(\theta - t).$$

Then, since

$$\left|1 + re^{i\alpha}\right|^2 = 1 + 2r\cos\alpha + r^2$$

it follows

$$0 < \left|1 + re^{i(\theta+t)}\right|^2 < \left|1 + re^{i(\theta-t)}\right|^2$$

hence

$$\frac{1}{\left|1+re^{i(\theta+t)}\right|^4} - \frac{1}{\left|1+re^{i(\theta-t)}\right|^4} > 0.$$

We have proven that if $\theta > 0$ then, by (1), $|E_1| > |E_2|$. $\square$

So, the condition $|E_1| = |E_2| = |E_3| = |E_4|$ uniquely determines ρ, $R = \frac{1}{\rho}$ and $\theta = 0$ and hence there is only one possibile stationary cluster $\mathbf{E}$ in the class considered. Moreover such a cluster has the two stated symmetries.

The proof of Theorem 1.1 is concluded by the following.

Lemma 3.8. *If $|E_1| = |E_2| = |E_3| = |E_4|$ the region E_3 (and hence E_4) is strictly convex.*

Proof. Recall that F_3 is a Reuleaux triangle obtained as the intersection of three congruent disks. So $E_3 = T(F_3)$ is the intersection of the inversion of such disks: these are disks themselves when the disks of F_3 do not touch the point $z = 0$. So, if ρ is small enough we know that E_3 is strictly convex. As ρ increases the set E_3 remains strictly convex up to a value $\rho = \rho_0$ when the circle containing the arc z_0z_1 (which is centered in z_2) happens to pass through the point $z = 0$. When $\rho = \rho_0$ the edge w_0w_1 of the cluster $\mathbf{E}$ becomes flat (and, by symmetry, all internal edges are flat) and hence, by stationarity, the curvature of the arc w_2w_1 is equal to the curvature of w_1w_4 (and, by symmetry, all external arcs have the same curvature). In this particular case it is elementary to check that $|E_3| > |E_1|$.

Since we know that $|E_1| - |E_3|$ is strictly decreasing as ρ increases and $R = \frac{1}{\rho}$ decreases (Lemma 3.3) it is apparent that the value of ρ giving the only cluster with $|E_1| = |E_3|$ is smaller than ρ_0 and hence that the region E_3 remains strictly convex. $\square$

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