ENERGY EQUALITIES FOR COMPRESSIBLE NAVIER-STOKES EQUATIONS

QUOC-HUNG NGUYEN, PHUOC-TAI NGUYEN, AND BAO QUOC TANG

ABSTRACT. The energy equalities of compressible and inhomogeneous incompressible Navier-Stokes equations are studied. By using a unified approach, we give sufficient conditions on the regularity of weak solutions for these equalities to hold. The method of proof is suitable for the case of periodic as well as homogeneous Dirichlet boundary conditions. In particular, by a careful analysis using homogeneous Dirichlet boundary condition, no boundary layer assumptions are required when dealing with bounded domains with boundary.

CONTENTS

1. INTRODUCTION AND MAIN RESULTS

Let $d = 2, 3$ and Ω be either the torus $\mathbb{T}^d$ or a bounded domain in $\mathbb{R}^d$ with C^2 boundary $\partial\Omega$. This paper studies the energy equalities for compressible Navier-Stokes equation with degenerate viscosity

$$\begin{cases} \partial_t \varrho + \nabla \cdot (\varrho \mathbf{u}) = 0, & \text{in } \Omega \times (0,T), \\ \partial_t(\varrho \mathbf{u}) + \nabla \cdot (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla(\varrho^\gamma) - 2\nu \nabla \cdot (\varrho \mathbb{D}\mathbf{u}) = 0, & \text{in } \Omega \times (0,T), \end{cases} \tag{cNSd}$$

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and the equation

$$\begin{cases} \partial_t \varrho + \nabla \cdot (\varrho \mathbf{u}) = 0, & \text{in } \Omega \times (0,T), \\ \partial_t(\varrho \mathbf{u}) + \nabla \cdot (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla(\varrho^\gamma) - 2\nu\Delta \mathbf{u} - \lambda \nabla(\nabla \cdot \mathbf{u}) = 0, & \text{in } \Omega \times (0,T), \end{cases} \tag{cNS}$$

as well as the inhomogeneous incompressible Navier-Stokes equation

$$\begin{cases} \partial_t \varrho + \nabla \cdot (\varrho \mathbf{u}) = 0, & \text{in } \Omega \times (0,T), \\ \partial_t(\varrho \mathbf{u}) + \nabla \cdot (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla P - 2\nu \nabla \cdot (\varrho \mathbb{D}\mathbf{u}) = 0, & \text{in } \Omega \times (0,T), \\ \nabla \cdot \mathbf{u} = 0, & \text{in } \Omega \times (0,T), \end{cases} \tag{icNS}$$

with initial data

$$\begin{cases} (\varrho \mathbf{u})(x,0) = \varrho_0(x)\mathbf{u}_0(x), \quad x \in \Omega, \\ \varrho(x,0) = \varrho_0(x), \quad x \in \Omega, \end{cases} \tag{1}$$

and homogeneous Dirichlet boundary condition

$$\mathbf{u} = 0 \quad \text{on} \quad \partial\Omega \times (0,T). \tag{2}$$

Here $T > 0$, $\gamma > 1$, $\nu, \lambda > 0$, ϱ denotes the density of the fluid, $\mathbf{u}$ is the velocity of fluid and $\mathbb{D} = \frac{1}{2}[\nabla \mathbf{u} + \nabla^\top \mathbf{u}]$ stands for the strain tensor, and in the case of (icNS), P is the scalar pressure. Naturally, the boundary condition $\mathbf{u} = 0$ on $\partial\Omega \times (0,T)$ is only imposed in the case of bounded domain.

Energy conservation is one aspect of the Onsager's conjecture which was announced in his celebrated paper on statistical hydrodynamics [Ons49]. More precisely, Onsager [Ons49] conjectured that, in the context of homogeneous incompressible Euler equations, kinetic energy is globally conserved for Hölder continuous solutions with the exponent greater than 1/3, while energy dissipation phenomenon occurs for Hölder continuous solutions with the exponent less than 1/3. The 'positive' part of the conjecture was first proved by by Eyink [Eyi94] and Constantin-E-Titi [CET94] for the whole space $\mathbb{R}^d$ or periodic boundary conditions, i.e. $\mathbb{T}^d$. Significant contributions in the case of domains with boundary were recently achieved in [BT18, BTW, DN18, NN18] for homogeneous incompressible Euler equations and in [NNT18] for inhomogeneous incompressible and compressible isentropic Euler equations. The other direction of the Onsager's conjecture was initiated in the groundbreaking paper of Scheffer [Sch93], then has reached to its full flowering with a work of Shnirelman [Shn97], a series of celebrated works of De Lellis and Székelyhidi [DS12, DS13, DS14, BDIS15], and recently was settled by Isett in [Ise18a, Ise18] and by Buckmaster et al. in [BDSV18].

Roughly speaking, Onsager's conjecture (for Euler equations) addresses the question how much regularities needed for a weak solution to conserve energy. In the context of classical incompressible Navier-Stokes equations, since global regularity in three dimensions has been a long standing open problem, it is natural to ask how much regularities needed for a weak solution to satisfy the *energy equality* (rather than energy conservation in the context of Euler equations). This topic of research has been studied already in the sixties starting with the work of Serrin [Ser62] where he asserted that the energy equality must hold if $\mathbf{u} \in L^p(0,T;L^q(\mathbb{T}^d))$ with $\frac{2}{p} + \frac{d}{q} \le 1$. Later Shinbrot [Shi74] assumed a condition which is independent of dimensions, more precisely $\mathbf{u} \in L^p(0,T;L^q(\mathbb{T}^d))$ with $\frac{1}{p} + \frac{1}{q} \le \frac{1}{2}$ and $q \ge 4$. By using a new approach based on a lemma introduced by Lions, C. Yu [Yu16] obtained the same result as in [Shi74]. All these results deal with either $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{T}^d$. Recently, C. Yu [Yu17a] considered the case where Ω is a bounded domain with C^2 boundary and proved the energy equality under additional condition $\mathbf{u} \in L^s(0,T;B_s^{\alpha,\infty}(\Omega))$ (with $s > 2$ and $\frac{1}{2} + \frac{1}{s} < \alpha < 1$) which enables to deal with the boundary effects. Here $B_s^{\alpha,\infty}(\Omega)$ denotes the Besov space.

Much less is known concerning energy equalities for compressible or inhomogeneous incompressible Navier-Stokes equations. Recent results for (cNSd) and (cNS) in $\mathbb{T}^d$ were carried out by C. Yu [Yu17]. More precisely, he gave sufficient conditions on the regularity of the density ϱ and the velocity $\mathbf{u}$ for the validity of the energy equality. The framework employed in [Yu17] is remarkably different from that in the incompressible case (see [Yu16, Yu17a]). In particular, solutions defined in [Yu17, Definition 1] are required to satisfy a set of regularities which allow to deduce the continuity of $(\sqrt{\varrho}\mathbf{u})(t)$ in the strong topology at $t = 0$. The existence of this kind of solutions is guaranteed by [VY16] and [LV18].

Motivated by the aforementioned works, we present in this paper a unified approach to show energy equalities for the compressible and inhomogeneous incompressible Navier-Stokes equations (cNSd), (cNS) and (icNS). The main idea is inspired by our recent work for compressible Euler equations [NNT18], which is different from the method employed in e.g. [Yu16, Yu17, Yu17a]. In particular, we use the test function $(\varrho^\varepsilon)^{-1}(\varrho\mathbf{u})^\varepsilon$ instead of $\mathbf{u}^\varepsilon$, where *the convolution is only taken in spatial variables*. The choice of this test function allows to obtain mild conditions on the density ϱ and the velocity $\mathbf{u}$ which are in fact weaker than previous works[1]. For instance, when $d = 3$ we assume only $\mathbf{u} \in L^4(\mathbb{T}^d \times (0,T))$ while [Yu17] required $\mathbf{u} \in L^p(0,T; L^q(\mathbb{T}^d))$ where $\frac{1}{p} + \frac{1}{q} \le \frac{5}{12}$ and $q \ge 6$ (see more discussion after Theorem 1.2). Moreover, by carefully using the Dirichlet boundary conditions (in the case of bounded domains), we show that no additional regularities near the boundary on the velocity $\mathbf{u}$ are required.

Before stating the main results, we introduce the definition of weak solutions.

Definition 1.1. *A couple $(\varrho, \mathbf{u})$ is called a weak solution to (cNSd) with initial data (1) if*

(i)

$$\int_0^T \int_\Omega (\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla\varphi)dxdt = 0 \tag{3}$$

for every test function $\varphi \in C_0^\infty(\Omega \times (0,T))$.

(ii)

$$\int_0^T \int_\Omega (\varrho\mathbf{u}\cdot\partial_t\psi + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla\psi + \varrho^\gamma\nabla\cdot\psi - 2\nu\varrho\mathbb{D}\mathbf{u} : \nabla\psi)dxdt = 0 \tag{4}$$

for every test vector field $\psi \in C_0^\infty(\Omega \times (0,T))^d$.

(iii) $\varrho(\cdot,t) \rightharpoonup \varrho_0$ in $\mathcal{D}'(\Omega)$ as $t \to 0$, i.e.

$$\lim_{t\to 0}\int_\Omega \varrho(x,t)\varphi(x)dx = \int_\Omega \varrho_0(x)\varphi(x)dx \tag{5}$$

for every test function $\varphi \in C_0^\infty(\Omega)$.

(iv) $(\varrho\mathbf{u})(\cdot,t) \rightharpoonup \varrho_0\mathbf{u}_0$ in $\mathcal{D}'(\Omega)$ as $t \to 0$, i.e.

$$\lim_{t\to 0}\int_\Omega (\varrho\mathbf{u})(x,t)\psi(x)dx = \int_\Omega (\varrho_0\mathbf{u}_0)(x)\psi(x)dx \tag{6}$$

for every test vector field $\psi \in C_0^\infty(\Omega)^d$.

Weak solutions to (cNS) and (icNS) can be defined similarly.

Remark 1.1. *The notion of weak solutions defined in Definition 1.1 requires only modest regularities of the density and the velocity, e.g. for (3) and (4) to make sense, one only needs $\varrho, \varrho\mathbf{u}, \varrho\mathbf{u}\otimes\mathbf{u}, \varrho^\gamma, \varrho\mathbb{D}\mathbf{u} \in L^1_{loc}(\Omega \times (0,T))$. If one were to show energy equality for weak solutions*

[1]Admittedly, our approach seems not directly applicable to the case with vacuum.

defined in [VY16] *(or* [LV18]*), as it was done in* [Yu17]*, more regularities on these solutions are already given, see* [VY16, Theorem 1.2] *or* [Yu17, Definition 1.1].

Theorem 1.2. *Let $\Omega = \mathbb{T}^d$ and $(\varrho, \mathbf{u})$ be a weak solution of* (cNSd) *with initial data* (1)*. Assume that*

$$0 < c_1 \le \varrho(t,x) \le c_2 < \infty \quad \text{and} \quad \mathbf{u} \in L^\infty(0,T;L^2(\mathbb{T}^d)) \cap L^2(0,T;H^1(\mathbb{T}^d)).$$

Define

$$\alpha := \min\{\gamma; 2\}. \tag{7}$$

We assume in the case $d = 2$ that

$$\sup_{t\in(0,T)} \sup_{|h|<\varepsilon} |h|^{-\frac{1}{\alpha}} \|\varrho(\cdot + h, t) - \varrho(\cdot, t)\|_{L^\alpha(\mathbb{T}^2)} < \infty, \tag{8}$$

and in the case $d = 3$ that

$$\mathbf{u} \in L^4(\mathbb{T}^3 \times (0,T)) \quad \text{and} \quad \sup_{t\in(0,T)} \sup_{|h|<\varepsilon} |h|^{-\frac{1}{\alpha}} \|\varrho(\cdot + h, t) - \varrho(\cdot, t)\|_{L^{\frac{6}{5}\alpha}(\mathbb{T}^3)} < \infty. \tag{9}$$

Then the energy equality holds, i.e.

$$\begin{aligned}\int_{\mathbb{T}^d} \left(\frac{1}{2}(\varrho|\mathbf{u}|^2)(x,t) + \frac{\varrho(x,t)^\gamma}{\gamma-1}\right) dx + \nu \int_0^t \int_{\mathbb{T}^d} \varrho |\mathbb{D}\mathbf{u}|^2 dx dt \\ = \int_{\mathbb{T}^d} \left(\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \frac{\varrho_0^\gamma}{\gamma-1}\right) dx \quad \forall t \in (0,T).\end{aligned} \tag{10}$$

Remark 1.3. *The conditions* (8) *and* (9) *are satisfied if, for instance, $\varrho \in L^\infty(0,T; B_\alpha^{\frac{1}{\alpha},\infty}(\mathbb{T}^2))$ and $\varrho \in L^\infty(0,T; B_{\frac{6}{5}\alpha}^{\frac{1}{\alpha},\infty}(\mathbb{T}^3))$ respectively, where $B_p^{\alpha,\infty}(\mathbb{T}^d)$ is the classical Besov space.*

Remark 1.4. *If we have*

$$\mathbf{u} \in L^p(0,T;L^q(\mathbb{T}^d)) \quad \text{with} \quad \frac{1}{p} + \frac{1}{q} \le \frac{1}{2} \ \text{ and } \ q \ge 4 \tag{11}$$

then by interpolation it follows

$$\|\mathbf{u}\|_{L^4(\mathbb{T}^d\times(0,T))} \le C\|\mathbf{u}\|^a_{L^\infty(0,T;L^2(\mathbb{T}^d))} \|\mathbf{u}\|^{1-a}_{L^p(0,T;L^q(\mathbb{T}^d))}$$

for some $a \in (0,1)$. Therefore, the results of Theorem 1.2 (and the subsequent theorems) are also valid with the classical assumption (11) *on the velocity.*

Our results in Theorem 1.2 improve those in [Yu17] particularly in the following sense:

- We allow in the case $d = 3$ the condition on the velocity $\mathbf{u} \in L^4(\mathbb{T}^3 \times (0,T))$ (and thus (11)) while [Yu17] required that $\mathbf{u} \in L^p(0,T;L^q(\mathbb{T}^3))$ with $1/p + 1/q \le 5/12$ and $q \ge 6$. It's worth emphasizing that our conditions can be implied from classical conditions for homogeneous incompressible Navier-Stokes equations as in e.g. [Shi74] (see Remark 1.4).
- At first sight, conditions (8) and (9) seem stronger than what assumed in [Yu17]. However, since [Yu17] used the notion of weak solutions constructed in [VY16] and [LV18] which requires several regularity properties on the density ϱ and the velocity $\mathbf{u}$. For instance, in [Yu17], the condition $\nabla\sqrt{\varrho} \in L^\infty(0,T;L^2(\mathbb{T}^d))$ is imposed. This, in combination with $0 < \underline{\varrho} \le \varrho(x,t) \le \overline{\varrho} < \infty$, implies that $\varrho \in L^\infty(0,T;H^1(\mathbb{T}^d))$ which is clearly stronger than (8) or (9).

Next we deal with the case where Ω is a bounded domain with C^2 boundary. For $\delta > 0$, put

$$\Omega_\delta := \{x \in \Omega : d(x, \partial\Omega) > \delta\}.$$

Since Ω is a bounded, connected domain with C^2 boundary, we find $r_0 > 0$ and a unique C^1_b-vector function $n : \Omega\backslash\Omega_{r_0} \to S^{d-1}$ such that the following holds true: for any $r \in [0, r_0)$, $x \in \Omega_r\backslash\Omega_{r_0}$ there exists a unique $x_r \in \partial\Omega_r$ such that $d(x, \partial\Omega_r) = |x - x_r|$ and $n(x)$ is the outward unit normal vector field to the boundary $\partial\Omega_r$ at x_r.

The energy equality for (cNSd) in a domain with boundary is stated in the following theorem.

Theorem 1.5. *Let Ω be a bounded domain with C^2 boundary $\partial\Omega$ and $(\varrho, \mathbf{u})$ be a weak solution of (cNSd) with initial data (1) and Dirichlet boundary condition (2). Assume that*

$$0 < c_1 \le \varrho(x,t) \le c_2 < \infty, \qquad \text{and} \qquad \mathbf{u} \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega)).$$

We assume in the case $d = 2$ that, for each $0 < \delta < 1$,

$$\sup_{t\in(0,T)} \sup_{|h|\le\delta} |h|^{-\frac{1}{\alpha}} \|\varrho(\cdot + h, t) - \varrho(\cdot, t)\|_{L^\alpha(\Omega_\delta)} < \infty, \tag{12}$$

and in case $d = 3$ that, for each $0 < \delta < 1$,

$$\mathbf{u} \in L^4(\Omega \times (0,T)) \quad \text{and} \quad \sup_{t\in(0,T)} \sup_{|h|\le\delta} |h|^{-\frac{1}{\alpha}} \|\varrho(\cdot + h, t) - \varrho(\cdot, t)\|_{L^{\frac{6}{5}\alpha}(\Omega_\delta)} < \infty. \tag{13}$$

Then the energy equality holds, i.e.

$$\begin{aligned} \int_\Omega \left(\frac{1}{2}(\varrho|\mathbf{u}|^2)(x,t) + \frac{\varrho(x,t)^\gamma}{\gamma - 1} \right) dx + \nu \int_0^t \int_\Omega \varrho |\mathbb{D}\mathbf{u}|^2 dx dt \\ = \int_\Omega \left(\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \frac{\varrho_0^\gamma}{\gamma - 1} \right) dx \quad \forall t \in (0,T). \end{aligned} \tag{14}$$

Remark 1.6. *In contrast to Euler equations, we do not need any additional regularities near the boundary to establish energy equality for Navier-Stokes equations in bounded domains. This is due to a careful use of homogeneous Dirichlet boundary conditions and the assumption $\mathbf{u} \in L^2(0,T; H^1(\Omega))$ (see Lemma 2.6).*

As mentioned above, our method of proof is also suitable to obtain energy equalities for the compressible Navier-Stokes equation (cNS) and the inhomogeneous incompressible Navier-Stokes equation (icNS).

Theorem 1.7. *Let either $\Omega = \mathbb{T}^d$ or $\Omega \subset \mathbb{R}^d$ be a bounded domain with C^2 boundary $\partial\Omega$ where $d = 2, 3$. Let $(\varrho, \mathbf{u})$ be a weak solution to (cNS) with initial data (1) (and with Dirichlet boundary condition (2) in case Ω is a bounded domain). Assume*

$$0 \le c_1 \le \varrho(x,t) \le c_2 < +\infty \qquad \text{and} \qquad \mathbf{u} \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega)).$$

Moreover, if $\Omega = \mathbb{T}^d$ we assume (8) for $d = 2$ and (9) for $d = 3$, and if Ω is a bounded domain we assume (12) for $d = 2$ and (13) for $d = 3$.

Then the energy equality holds, i.e.

$$\begin{aligned} \int_\Omega \left(\frac{1}{2}(\varrho|\mathbf{u}|^2)(x,t) + \frac{\varrho(x,t)^\gamma}{\gamma - 1} \right) dx + 2\nu \int_0^t \int_\Omega \varrho |\nabla\mathbf{u}|^2 dx dt + \lambda \int_0^t \int_\Omega |\nabla \cdot \mathbf{u}|^2 dx ds \\ = \int_\Omega \left(\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \frac{\varrho_0^\gamma}{\gamma - 1} \right) dx \quad \forall t \in (0,T). \end{aligned}$$

Theorem 1.8. *Let either $\Omega = \mathbb{T}^d$ or $\Omega \subset \mathbb{R}^d$ be a bounded domain with C^2 boundary $\partial\Omega$ where $d = 2, 3$. Let $(\varrho, \mathbf{u}, P)$ be a weak solution to (icNS) with initial data (1) (and with Dirichlet boundary condition (2) in case Ω is a bounded domain). Assume*

$$0 \le c_1 \le \varrho(x,t) \le c_2 < +\infty, \quad \mathbf{u} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)), \quad \textit{and} \quad P \in L^2(\Omega \times (0,T)).$$

Moreover, in the case $d = 3$ we assume additionally $\mathbf{u} \in L^4(\Omega \times (0,T))$. Then the energy equality holds, i.e.

$$\frac{1}{2}\int_\Omega \varrho(x,t)|\mathbf{u}(x,t)|^2 dx + \nu \int_0^t \int_\Omega \varrho |\mathbb{D}\mathbf{u}|^2 dxds = \frac{1}{2}\int_\Omega \varrho_0(x)|\mathbf{u}_0(x)|^2 dx \quad \forall\, t \in (0,T).$$

We emphasize again that, in Theorem 1.8, the condition $\mathbf{u} \in L^2(0,T;H^1(\Omega))$ helps to handle the boundary effects without requiring extra conditions of $\mathbf{u}$ near the boundary. As a consequence, we improve the results of [Yu17a] for homogeneous incompressible Navier-Stokes equations by removing the assumption $\mathbf{u} \in L^s(0,T;B^{\alpha,\infty}_s(\Omega))$ with $\frac{1}{2} + \frac{1}{s} < \alpha < 1$.

Theorem 1.9. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with C^2 boundary and $(\mathbf{u}, P)$ be a weak solution to the homogeneous incompressible Navier-Stokes equation*

$$\begin{cases} \partial_t \mathbf{u} - \mu\Delta\mathbf{u} + \nabla\cdot(\mathbf{u}\otimes\mathbf{u}) + \nabla P = 0, & \textit{in } \Omega \times (0,T), \\ \nabla \cdot \mathbf{u} = 0, & \textit{in } \Omega\times(0,T), \\ \mathbf{u} = 0, & \textit{on } \partial\Omega \times (0,T), \\ \mathbf{u}(x,0) = \mathbf{u}_0(x), & \textit{in } \Omega \end{cases} \tag{15}$$

where $\mu > 0$ is the viscosity. Assume $\mathbf{u} \in L^4(\Omega \times (0,T))$ and there exists $\delta_0 > 0$ such that

$$P \in L^2(\Omega \backslash \Omega_{\delta_0} \times (0,T)). \tag{16}$$

Then the energy equality holds

$$\int_\Omega |\mathbf{u}(x,t)|^2 dx + \mu \int_0^t \int_\Omega |\nabla \mathbf{u}(x,s)|^2 dxds = \int_\Omega |\mathbf{u}_0(x)|^2 dx \quad \forall\, t \in (0,T).$$

Remark 1.10. *The assumption (16) is to deal with the boundary layer. We remark that [Yu17a] did not impose any condition on the pressure, but the author used $P = 0$ on the boundary (see* [Yu17a, Proposition 2.3]*) which is neither assumed nor implied from (15). Nevertheless, if somehow $P = 0$ on the boundary (in a weak sense), from the equation $-\Delta P = \sum_{i,j=1}^d \partial_{x_i}\partial_{x_j}(\mathbf{u}_i\mathbf{u}_j)$ we obtain $\|P\|_{L^2(\Omega\times(0,T))} \le C\|\mathbf{u}\|^2_{L^4(\Omega\times(0,T))}$ and therefore (16) is automatically satisfied.*

The organization of this paper is as follows: In the next section, we prove some auxiliary estimates which will play important roles in our proof. The proofs of Theorems 1.2, 1.5, 1.7, 1.8 and 1.9 are presented in the last five sections respectively.

Notation. Throughout the paper, C denotes generic constants which may depend on d, T, $\|\varrho\|_{L^\infty(\Omega\times(0,T))}$, $\|\varrho^{-1}\|_{L^\infty(\Omega\times(0,T))}$ and other scalar parameters. We use the notation $\|f(s)\|_{L^p(\Omega)}$ to denote $\|f(\cdot,s)\|_{L^p(\Omega)}$.

For any Borel set E, we denote by $⨍_E f(x)dx = \frac{1}{\mathcal{L}^d(E)}\int_E f(x)dx$ the average of f over E, where $\mathcal{L}^d(E)$ is the Lebesgue measure of E.

2. Preliminaries

Let $\omega : \mathbb{R}^d \to \mathbb{R}$ be a standard mollifier, i.e. $\omega(x) = c_0 e^{-\frac{1}{1-|x|^2}}$ for $|x| < 1$ and $\omega(x) = 0$ for $|x| \geq 1$, where c_0 is a constant such that $\int_{\mathbb{R}^d} \omega(x)dx = 1$. For any $\varepsilon > 0$, we define the rescaled mollifier $\omega_\varepsilon(x) = \varepsilon^{-d}\omega(\frac{x}{\varepsilon})$. For any function $f \in L^1_{loc}(\Omega)$, its mollified version is defined as

$$f^\varepsilon(x) = (f \star \omega_\varepsilon)(x) = \int_{\mathbb{R}^d} f(x-y)\omega_\varepsilon(y)dy, \quad x \in \Omega_\varepsilon,$$

recalling $\Omega_\varepsilon = \{x \in \Omega : d(x, \partial\Omega) > \varepsilon\}$.

Lemma 2.1. *Let $2 \leq d \in \mathbb{N}$, $1 \leq p, q \leq \infty$ and $f : \mathbb{T}^d \times (0,T) \to \mathbb{R}$.*

(i) Assume $f \in L^p(0,T; L^q(\mathbb{T}^d))$. Then for any $\varepsilon > 0$, there holds

$$\|f^\varepsilon\|_{L^p(0,T;L^\infty(\mathbb{T}^d))} \leq C\varepsilon^{-\frac{d}{q}}\|f\|_{L^p(0,T;L^q(\mathbb{T}^d))}, \tag{17}$$

$$\|\nabla f^\varepsilon\|_{L^p(0,T;L^\infty(\mathbb{T}^d))} \leq C\varepsilon^{-1-\frac{d}{q}}\|f\|_{L^p(0,T;L^q(\mathbb{T}^d))}. \tag{18}$$

(ii) Assume $f \in L^p(0,T; L^q(\mathbb{T}^d))$. Then for any $\varepsilon > 0$, there holds

$$\|\nabla f^\varepsilon\|_{L^p(0,T;L^q(\mathbb{T}^d))} \leq C\varepsilon^{-1}\|f\|_{L^p(0,T;L^q(\mathbb{T}^d))}. \tag{19}$$

Moreover, if $p, q < \infty$ then

$$\limsup_{\varepsilon\to 0} \varepsilon\|\nabla f^\varepsilon\|_{L^p(0,T;L^q(\mathbb{T}^d))} = 0. \tag{20}$$

(iii) Assume $f \in L^p(0,T; L^q(\mathbb{T}^d))$ and $g : \mathbb{T}^d \times (0,T) \to \mathbb{R}$ with $0 < c_1 \leq g \leq c_2 < \infty$. Then for any $\varepsilon > 0$, there holds

$$\left\|\nabla \frac{f^\varepsilon}{g^\varepsilon}\right\|_{L^p(0,T;L^q(\mathbb{T}^d))} \leq C(c_1, c_2)\varepsilon^{-1}\|f\|_{L^p(0,T;L^q(\Omega))}. \tag{21}$$

Moreover, if $p, q < \infty$ then

$$\limsup_{\varepsilon\to 0} \varepsilon \left\|\nabla \frac{f^\varepsilon}{g^\varepsilon}\right\|_{L^p(0,T;L^q(\mathbb{T}^d))} = 0. \tag{22}$$

(iv) Assume $f \in L^2(0,T; H^1(\mathbb{T}^2))$. Then for any $\varepsilon > 0$, there holds

$$\|\nabla f^\varepsilon\|_{L^2(0,T;L^\infty(\mathbb{T}^2))} \leq C\varepsilon^{-1}\|f\|_{L^2(0,T;H^1(\mathbb{T}^2))}. \tag{23}$$

Moreover, for any $r \in [1,2]$, there holds

$$\limsup_{\varepsilon\to 0} \varepsilon\|\nabla f^\varepsilon\|_{L^r(0,T;L^\infty(\mathbb{T}^2))} = 0. \tag{24}$$

Proof. (i) By the definition of f^ε and Hölder's inequality, for a.e. $x \in \mathbb{T}^d$ and $s \in (0,T)$, we have

$$\begin{aligned}|f^\varepsilon(x,s)| &\leq \int_{\mathbb{T}^d} |f(x-y,s)\omega_\varepsilon(y,s)|dy \leq \left(\int_{\mathbb{T}^d} |f(x-y,s)|^q dy\right)^{\frac{1}{q}} \left(\int_{\mathbb{T}^d} |\omega_\varepsilon(y)|^{\frac{q}{q-1}} dy\right)^{\frac{q-1}{q}} \\ &\leq C\varepsilon^{-\frac{d}{q}}\|f(s)\|_{L^q(\mathbb{T}^d)},\end{aligned}$$

where we have used $\omega_\varepsilon(x) = \varepsilon^{-d}\omega(x/\varepsilon)$ at the last step. This implies

$$\|f^\varepsilon(s)\|_{L^\infty(\mathbb{T}^d)} \leq C\varepsilon^{-\frac{d}{q}}\|f(s)\|_{L^q(\mathbb{T}^d)},$$

which in turn yields (17).

Next we use Hölder's inequality again to estimate

$$|\nabla f^\varepsilon(x,s)| \leq \int_{\mathbb{T}^d} |f(x-y,s)||\nabla\omega_\varepsilon(y)|dy \leq C\varepsilon^{-1-\frac{d}{q}}\|f(s)\|_{L^q(\mathbb{T}^d)}.$$

This leads to (18).

(ii) From the fact that $\int_{\mathbb{R}^d} \nabla\omega_\varepsilon(y)dy = 0$ and Hölder's inequality, we obtain, for a.e. $x \in \mathbb{T}^d$ and $s \in (0,T)$,

$$|\nabla f^\varepsilon(x,s)| \le \left| \int_{|y|<\varepsilon} [f(x-y,s) - f(x,s)]\nabla\omega_\varepsilon(y)dy \right| \le C\varepsilon^{\frac{d}{q}} \left(⨍_{|y|<\varepsilon} |f(x-y,s)-f(x,s)|^q dy \right)^{\frac{1}{q}} \varepsilon^{-1-\frac{d}{q}} \|\nabla\omega\|_{L^{\frac{q}{q-1}}(\mathbb{T}^d)}.$$

It follows that

$$\|\nabla f^\varepsilon(s)\|_{L^q(\mathbb{T}^d)} \le C\varepsilon^{-1} \left(⨍_{|y|<\varepsilon} \int_{\mathbb{T}^d} |f(x-y,s)-f(x,s)|^q \, dxdy \right)^{\frac{1}{q}}. \tag{25}$$

This implies (19). Thus, for any $g \in C^\infty(\mathbb{T}^d \times (0,T))$, we have

$$\varepsilon\|\nabla f^\varepsilon\|_{L^p(0,T;L^q(\mathbb{T}^d))} \le C\varepsilon\|g\|_{C^1_b(\mathbb{T}^d\times(0,T))} + C\|f-g\|_{L^p(0,T;L^q(\mathbb{T}^d))}$$

Therefore, for any $g \in C^\infty(\mathbb{T}^d \times (0,T))$,

$$\limsup_{\varepsilon\to 0} \varepsilon\|\nabla f^\varepsilon\|_{L^p(0,T;L^q(\mathbb{T}^d))} \le C\|f-g\|_{L^p(0,T;L^q(\mathbb{T}^d))}$$

and since $C^\infty(\mathbb{T}^d \times (0,T))$ is dense in $L^p(0,T;L^q(\mathbb{T}^d))$ (for $p,q<\infty$), we obtain (20).

(iii) We have, for any $g_1, f_1 \in C^\infty_b(\mathbb{T}^d \times (0,T))$,

$$\begin{aligned}\left\|\nabla\frac{f^\varepsilon}{g^\varepsilon}\right\|_{L^p(0,T;L^q(\mathbb{T}^d))} &\le C\varepsilon^{-1}\|f-f_1\|_{L^p(0,T;L^q(\Omega))} + C\|\nabla g^\varepsilon\|_{L^p(0,T;L^q(\mathbb{T}^d))}\|f_1\|_{L^\infty(\mathbb{T}^d\times(0,T))} \\ &\le C\varepsilon^{-1}\|f-f_1\|_{L^p(0,T;L^q(\Omega))} + C\varepsilon^{-1}\|g-g_1\|_{L^p(0,T;L^q(\mathbb{T}^d))}\|f_1\|_{L^\infty(\mathbb{T}^d\times(0,T))} \\ &\quad + C\|g_1\|_{C^1_b(\mathbb{T}^d\times(0,T))}\|f_1\|_{L^\infty(\mathbb{T}^d\times(0,T))}\end{aligned}$$

Thus, by density this implies (21) and (22).

(iv) From the fact that $\int_{\mathbb{R}^2} \nabla\omega_\varepsilon(y)dy = 0$ and Hölder's inequality, we obtain, for a.e. $x \in \mathbb{T}^d$ and $s \in (0,T)$,

$$\begin{aligned}|\nabla f^\varepsilon(x,s)| &\le \left(\int_{|y|<\varepsilon} |f(x-y,s)-f(x,s)|^2 \, dy\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} |\nabla\omega_\varepsilon(y)|^2 dy\right)^{\frac{1}{2}} \\ &\le C\varepsilon^{-1}\left(\int_0^1\int_{|y|<\varepsilon} |\nabla f(x+\rho y,s)|^2 dyd\rho\right)^{\frac{1}{2}} \|\nabla\omega\|_{L^2(\mathbb{R}^2)}.\end{aligned} \tag{26}$$

This implies (23). Thus, as (ii), we have, for any $g \in C^\infty(\mathbb{T}^d \times (0,T))$,

$$\limsup_{\varepsilon\to 0} \varepsilon\|\nabla f^\varepsilon\|_{L^2(0,T;L^\infty(\mathbb{T}^2))} \le C\|f-g\|_{L^2(0,T;H^1(\mathbb{T}^2))}. \tag{27}$$

By using the fact that $C^\infty(\mathbb{T}^d \times (0,T))$ is dense in $L^2(0,T;H^1(\mathbb{T}^d))$, we derive

$$\limsup_{\varepsilon\to 0} \varepsilon\|\nabla f^\varepsilon\|_{L^2(0,T;L^\infty(\mathbb{T}^2))} = 0,$$

which implies (24). The proof is complete. □

The following variant of Lemma 2.1 in bounded domains can be proved similarly, so we only state the results. We recall that, for each $\delta > 0$, $\Omega_\delta = \{x \in \Omega : d(x,\partial\Omega) > \delta\}$.

Lemma 2.2. *Let $2 \le d \in \mathbb{N}$, $\Omega \subset \mathbb{R}^d$ be a bounded domain with C^2 boundary $\partial\Omega$, $1 \le p,q \le \infty$ and $f : \Omega \times (0,T) \to \mathbb{R}$.*

(i) Assume $f \in L^p(0,T;L^q(\Omega))$. *Then for any* $0<\varepsilon<\delta$, *there holds*

$$\|f^\varepsilon\|_{L^p(0,T;L^\infty(\Omega_\delta))} \le C\varepsilon^{-\frac{d}{q}}\|f\|_{L^p(0,T;L^q(\Omega))}, \tag{28}$$

$$\|\nabla f^\varepsilon\|_{L^p(0,T;L^\infty(\Omega_\delta))} \le C\varepsilon^{-1-\frac{d}{q}}\|f\|_{L^p(0,T;L^q(\Omega))}. \tag{29}$$

(ii) Assume $f \in L^p(0,T;L^q(\Omega))$. *Then for any* $0<\varepsilon<\delta$, *there holds*

$$\|\nabla f^\varepsilon\|_{L^p(0,T;L^q(\Omega_\delta))} \le C\varepsilon^{-1}\|f\|_{L^p(0,T;L^q(\Omega))}. \tag{30}$$

Moreover, if $p,q<\infty$ *then*

$$\limsup_{\varepsilon\to 0}\varepsilon\|\nabla f^\varepsilon\|_{L^p(0,T;L^q(\Omega_\delta))} = 0. \tag{31}$$

(iii) Assume $f \in L^p(0,T;L^q(\Omega))$ *with* $p,q<\infty$ *and* $g:\Omega\times(0,T)\to\mathbb{R}$ *with* $0<c_1\le g\le c_2<\infty$. *Then for any* $0<\varepsilon<\delta$, *there holds*

$$\left\|\nabla\frac{f^\varepsilon}{g^\varepsilon}\right\|_{L^p(0,T;L^q(\Omega_\delta))} \le C(c_1,c_2)\varepsilon^{-1}\|f\|_{L^p(0,T;L^q(\Omega))}. \tag{32}$$

(iv) Assume $d=2$ *and* $f\in L^2(0,T;H^1(\Omega))$. *Then for any* $0<\varepsilon<\delta$, *there holds*

$$\|\nabla f^\varepsilon\|_{L^2(0,T;L^\infty(\Omega_\delta))} \le C\varepsilon^{-1}\|f\|_{L^2(0,T;H^1(\Omega))}. \tag{33}$$

Moreover, for any $r\in[1,2]$, *there holds*

$$\limsup_{\varepsilon\to 0}\varepsilon\|\nabla f^\varepsilon\|_{L^r(0,T;L^\infty(\Omega_\delta))} = 0. \tag{34}$$

Lemma 2.3. [2] *Let* $p,p_1\in[1,\infty)$ *and* $p_2\in(1,\infty]$ *with* $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$. *Assume* $f\in L^{p_1}(0,T;W^{1,p_1}(\mathbb{T}^d))$ *and* $g\in L^{p_2}(\mathbb{T}^d\times(0,T))$. *Then for any* $\varepsilon>0$, *there holds*

$$\|(fg)^\varepsilon - f^\varepsilon g^\varepsilon\|_{L^p(\mathbb{T}^d\times(0,T))} \le C\varepsilon\|f\|_{L^{p_1}(0,T;W^{1,p_1}(\mathbb{T}^d))}\|g\|_{L^{p_2}(\mathbb{T}^d\times(0,T))}. \tag{35}$$

Moreover, if $p_2<\infty$ *then*

$$\limsup_{\varepsilon\to 0}\varepsilon^{-1}\|(fg)^\varepsilon - f^\varepsilon g^\varepsilon\|_{L^p(\mathbb{T}^d\times(0,T))} = 0. \tag{36}$$

Proof. We note that (see e.g. [CET94])

$$(fg)^\varepsilon - f^\varepsilon g^\varepsilon = R^\varepsilon - (f^\varepsilon - f)(g^\varepsilon - g) \tag{37}$$

where

$$R^\varepsilon(x,s) := \int_{\mathbb{R}^d}(f(x-y,s)-f(x,s))(g(x-y,s)-g(x,s))\omega_\varepsilon(y)dy.$$

This yields

$$\|(fg)^\varepsilon - f^\varepsilon g^\varepsilon\|_{L^p(\mathbb{T}^d\times(0,T))} \le \|R^\varepsilon\|_{L^p(\mathbb{T}^d\times(0,T))} + \|(f-f^\varepsilon)(g-g^\varepsilon)\|_{L^p(\mathbb{T}^d\times(0,T))}. \tag{38}$$

The first term on the right hand-side of (38) can be estimated using Hölder's inequality as

$$\|R^\varepsilon\|_{L^p(\mathbb{T}^d\times(0,T))}$$
$$\le C\varepsilon\left[\int_0^T\int_0^1 \fint_{|y|<\varepsilon}\int_{\mathbb{T}^d}|\nabla f(x+\rho y,s)|^{p_1}dxdyd\rho ds\right]^{\frac{1}{p_1}}\left[\int_0^T \fint_{|y|<\varepsilon}\int_{\mathbb{T}^d}|g(x-y,s)-g(x,s)|^{p_2}dxdyds\right]^{\frac{1}{p_2}}. \tag{39}$$

[2] Similar estimates were obtained in [CET94] in the context of Hölder spaces.

Similarly, one can show that the second term on the right hand-side of (38) is bounded from above by the term on the right hand-side of (39). Therefore

$$\|(fg)^\varepsilon - f^\varepsilon g^\varepsilon\|_{L^p(\mathbb{T}^d\times(0,T))}$$
$$\le C\varepsilon^{1-\frac{d}{p}}\left[\int_0^T\int_0^1\int_{|y|<\varepsilon}\int_{\mathbb{T}^d}|\nabla f(x+\rho y,s)|^{p_1}dxdyd\rho ds\right]^{\frac{1}{p_1}}\left[\int_0^T\int_{|y|<\varepsilon}\int_{\mathbb{T}^d}|g(x-y,s)-g(x,s)|^{p_2}dxdyds\right]^{\frac{1}{p_2}}. \tag{40}$$

This implies (35). By using (40) and the density argument, we get (36). □

Similar results can be obtained for bounded domains using only minor modifications.

Lemma 2.4. *Let $\Omega\subset\mathbb{R}^d$ be a bounded domain with C^2 boundary $\partial\Omega$, $p,p_1\in[1,\infty)$ and $p_2\in(1,\infty]$ with $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$. Assume $f\in L^{p_1}(0,T;W^{1,p_1}(\Omega))$ and $g\in L^{p_2}(\Omega\times(0,T))$. Then for any $0<\varepsilon<\delta$ small, there holds*

$$\|(fg)^\varepsilon - f^\varepsilon g^\varepsilon\|_{L^p(\Omega_{2\delta}\times(0,T))}\le C\varepsilon\|f\|_{L^{p_1}(0,T;W^{1,p_1}(\Omega_\delta))}\|g\|_{L^{p_2}(\Omega_\delta\times(0,T))}. \tag{41}$$

Moreover, if $p_2<\infty$ then

$$\limsup_{\varepsilon\to0}\varepsilon^{-1}\|(fg)^\varepsilon - f^\varepsilon g^\varepsilon\|_{L^p(\Omega_{2\delta}\times(0,T))}=0. \tag{42}$$

The following interpolation lemma shows that the additional assumption $\mathbf{u}\in L^4(\Omega\times(0,T))$ is only needed in three dimensions.

Lemma 2.5. *Let either $\Omega=\mathbb{T}^2$ or $\Omega\subset\mathbb{R}^2$ be a bounded domain with C^2 boundary. Assume $f\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))$. Then $f\in L^4(\Omega\times(0,T))$ and*

$$\|f\|_{L^4(\Omega\times(0,T))}\le C\|f\|^{\frac12}_{L^\infty(0,T;L^2(\Omega))}\|f\|^{\frac12}_{L^2(0,T;H^1(\Omega))}. \tag{43}$$

Proof. The proof follows directly from Gargliardo-Nirenberg's inequality so we omit it. □

To deal with domains with boundary, we need the following coarea formula for $0<\kappa_1<\kappa_2$

$$\int_{\kappa_1}^{\kappa_2}\int_{\partial\Omega_\kappa}g(\theta)d\mathcal{H}^{d-1}(\theta)d\kappa=\int_{\Omega_{\kappa_1}\backslash\Omega_{\kappa_2}}g(x)dx. \tag{44}$$

Lemma 2.6. *Let Ω be a bounded domain with C^2 boundary. Assume that $f\in L^2(0,T;H^1_0(\Omega))$. Then, for $\delta>0$ small,*

$$\|f\|_{L^2((\Omega\backslash\Omega_\delta)\times(0,T))}\le C\delta\|\nabla f\|_{L^2((\Omega\backslash\Omega_{2\delta})\times(0,T))}. \tag{45}$$

Proof. Let $\delta>0$ be small. For any $x\in\Omega\backslash\Omega_\delta$, there exists a unique $x_{\partial\Omega}$ such that $|x_{\partial\Omega}-x|=d(x,\partial\Omega)$. Let $\mathcal{T}$ be the projection mapping defined by $\mathcal{T}(x):=x_{\partial\Omega}$. Then we have

$$||\nabla\mathcal{T}-\mathbf{id}||_{L^\infty(\Omega\backslash\Omega_\delta)}=\circ(1)\text{ as }\delta\to0. \tag{46}$$

Since $f = 0$ on $\partial\Omega \times (0,T)$, by the coarea formula, we have

$$\begin{aligned}
\int_{\Omega\setminus\Omega_\delta} |f(x,s)|^2 dx &= \int_0^\delta \int_{\partial\Omega_\kappa} |f(\theta,s)|^2 d\mathcal{H}^{d-1}(\theta) d\kappa \\
&= \int_0^\delta \int_{\partial\Omega_\kappa} |f(\theta,s) - f(\mathcal{T}(\theta),s)|^2 d\mathcal{H}^{d-1}(\theta) d\kappa \\
&\le \int_0^\delta \int_{\partial\Omega_\kappa} \int_0^1 |\nabla f(\theta + \rho(\mathcal{T}(\theta)-\theta), s)|^2 |\mathcal{T}(\theta)-\theta|^2 d\rho d\mathcal{H}^{d-1}(\theta) d\kappa \\
&\le \delta^2 \int_0^\delta \int_{\partial\Omega_\kappa} \int_0^1 |\nabla f(\theta + \rho(\mathcal{T}(\theta)-\theta), s)|^2 d\rho d\mathcal{H}^{d-1}(\theta) d\kappa \\
&= \delta^2 \int_0^1 \int_{\Omega\setminus\Omega_\delta} |\nabla f(x + \rho(\mathcal{T}(x)-x), s)|^2 dx d\rho.
\end{aligned} \tag{47}$$

Set $\mathcal{T}_\rho(x) = x + \rho(\mathcal{T}(x) - x)$. From (46), we deduce, for $\delta > 0$ small and $\rho \in (0,1)$, that

$$\frac{1}{2} \le |\det(\nabla \mathcal{T}_\rho(x))| \le \frac{3}{2} \quad \text{and} \quad \mathcal{T}_\rho(\Omega\setminus\Omega_\delta) \subset \Omega\setminus\Omega_{2\delta}$$

Therefore

$$\begin{aligned}
\int_0^1 \int_{\Omega\setminus\Omega_\delta} |\nabla f(x + \rho(\mathcal{T}(x)-x), s)|^2 dx d\rho &= \int_0^1 \int_{\mathcal{T}_\rho(\Omega\setminus\Omega_\delta)} |\nabla f(x,s)|^2 \frac{dx}{|\det(\nabla \mathcal{T}_\rho)(\mathcal{T}_\rho^{-1}(x))|} d\rho \\
&\le 2 \int_{\Omega\setminus\Omega_{2\delta}} |\nabla f(x,s)|^2 dx.
\end{aligned} \tag{48}$$

Therefore, from (47) and (48) we derive (45). □

3. Proof of Theorem 1.2

By smoothing (cNSd), we obtain

$$\partial_t \varrho^\varepsilon + \nabla \cdot (\varrho \mathbf{u})^\varepsilon = 0 \tag{49}$$

and

$$\partial_t (\varrho \mathbf{u})^\varepsilon + \nabla \cdot (\varrho \mathbf{u} \otimes \mathbf{u})^\varepsilon + \nabla (\varrho^\gamma)^\varepsilon - 2\nu \nabla \cdot (\varrho \mathbb{D}\mathbf{u})^\varepsilon = 0 \tag{50}$$

for any $0 < \varepsilon < 1$.

Multiplying (50) by $(\varrho^\varepsilon)^{-1}(\varrho\mathbf{u})^\varepsilon$ then integrating on $(\tau,t)\times\mathbb{T}^d$, for $0 < \tau < t < T$, we get

$$\begin{aligned}
\int_\tau^t \int_{\mathbb{T}^d} \frac{(\varrho\mathbf{u})^\varepsilon}{\varrho^\varepsilon} \partial_t (\varrho\mathbf{u})^\varepsilon dx ds + \int_\tau^t \int_{\mathbb{T}^d} \frac{(\varrho\mathbf{u})^\varepsilon}{\varrho^\varepsilon} \nabla \cdot (\varrho \mathbf{u} \otimes \mathbf{u})^\varepsilon dx ds + \int_\tau^t \int_{\mathbb{T}^d} \frac{(\varrho\mathbf{u})^\varepsilon}{\varrho^\varepsilon} \nabla (\varrho^\gamma)^\varepsilon dx ds \\
- 2\nu \int_\tau^t \int_{\mathbb{T}^d} \nabla \cdot (\varrho \mathbb{D}\mathbf{u})^\varepsilon \frac{(\varrho\mathbf{u})^\varepsilon}{\varrho^\varepsilon} dx ds = 0.
\end{aligned} \tag{51}$$

Denote by $(A), (B), (C)$ and (D) the terms on the left-hand side of (51) respectively. We will estimate them below.

3.1. **Estimate of** (A).

By using (49), we can compute

$$\begin{aligned}
(A) &= \frac{1}{2} \int_\tau^t \int_{\mathbb{T}^d} \partial_t \left(\frac{|(\varrho\mathbf{u})^\varepsilon|^2}{\varrho^\varepsilon} \right) dx ds + \frac{1}{2} \int_\tau^t \int_{\mathbb{T}^d} \partial_t \varrho^\varepsilon \frac{|(\varrho\mathbf{u})^\varepsilon|^2}{(\varrho^\varepsilon)^2} dx ds \\
&= \frac{1}{2} \int_\tau^t \int_{\mathbb{T}^d} \partial_t \left(\frac{|(\varrho\mathbf{u})^\varepsilon|^2}{\varrho^\varepsilon} \right) dx ds - \frac{1}{2} \int_\tau^t \int_{\mathbb{T}^d} \nabla \cdot (\varrho\mathbf{u})^\varepsilon \frac{|(\varrho\mathbf{u})^\varepsilon|^2}{(\varrho^\varepsilon)^2} dx ds \\
&=: (A1) + (A2).
\end{aligned}$$

We see that $(A1)$ is the desired term while $(A2)$ will be canceled with the term $(B3)$ later.

3.2. **Estimate of** (B)**.** By integration by parts,

$$
\begin{aligned}
(B) &= -\int_\tau^t\int_{\mathbb{T}^d}\nabla\frac{(\varrho\mathbf{u})^\varepsilon}{\varrho^\varepsilon}(\varrho\mathbf{u}\otimes\mathbf{u})^\varepsilon dxds\\
&= -\int_\tau^t\int_{\mathbb{T}^d}\nabla\frac{(\varrho\mathbf{u})^\varepsilon}{\varrho^\varepsilon}[(\varrho\mathbf{u}\otimes\mathbf{u})^\varepsilon-(\varrho\mathbf{u})^\varepsilon\otimes\mathbf{u}^\varepsilon]dxds-\int_\tau^t\int_{\mathbb{T}^d}\nabla\frac{(\varrho\mathbf{u})^\varepsilon}{\varrho^\varepsilon}((\varrho\mathbf{u})^\varepsilon\otimes\mathbf{u}^\varepsilon)dxds\\
&=: (B1)+\int_\tau^t\int_{\mathbb{T}^d}\frac{(\varrho\mathbf{u})^\varepsilon}{\varrho^\varepsilon}\nabla\cdot((\varrho\mathbf{u})^\varepsilon\otimes\mathbf{u}^\varepsilon)dxds\\
&= (B1)+\int_\tau^t\int_{\mathbb{T}^d}\nabla\cdot\mathbf{u}^\varepsilon\frac{|(\varrho\mathbf{u})^\varepsilon|^2}{\varrho^\varepsilon}dxds+\frac{1}{2}\int_\tau^t\int_{\mathbb{T}^d}\frac{\mathbf{u}^\varepsilon}{\varrho^\varepsilon}\nabla|(\varrho\mathbf{u})^\varepsilon|^2dxds\\
&= (B1)+\frac{1}{2}\int_\tau^t\int_{\mathbb{T}^d}\nabla\cdot\mathbf{u}^\varepsilon\frac{|(\varrho\mathbf{u})^\varepsilon|^2}{\varrho^\varepsilon}dxds-\frac{1}{2}\int_\tau^t\int_{\mathbb{T}^d}\mathbf{u}^\varepsilon\nabla(\frac{1}{\varrho^\varepsilon})|(\varrho\mathbf{u})^\varepsilon|^2dxds\\
&= (B1)+\frac{1}{2}\int_\tau^t\int_{\mathbb{T}^d}\nabla\cdot(\varrho^\varepsilon\mathbf{u}^\varepsilon)\frac{|(\varrho\mathbf{u})^\varepsilon|^2}{(\varrho^\varepsilon)^2}dxds\\
&= (B1)+\frac{1}{2}\int_\tau^t\int_{\mathbb{T}^d}\nabla\cdot[(\varrho^\varepsilon\mathbf{u}^\varepsilon)-(\varrho\mathbf{u})^\varepsilon]\frac{|(\varrho\mathbf{u})^\varepsilon|^2}{(\varrho^\varepsilon)^2}dxds+\frac{1}{2}\int_\tau^t\int_{\mathbb{T}^d}\nabla\cdot(\varrho\mathbf{u})^\varepsilon\frac{|(\varrho\mathbf{u})^\varepsilon|^2}{(\varrho^\varepsilon)^2}dxds\\
&=: (B1)+(B2)+(B3).
\end{aligned}
$$

It is obvious that $(A2)+(B3)=0$. We now estimate the remainders $(B1)$ and $(B2)$. By Hölder's inequality

$$
\begin{aligned}
|(B1)| &= \left|\int_\tau^t\int_{\mathbb{T}^d}\nabla\frac{(\varrho\mathbf{u})^\varepsilon}{\varrho^\varepsilon}[(\varrho\mathbf{u}\otimes\mathbf{u})^\varepsilon-(\varrho\mathbf{u})^\varepsilon\otimes\mathbf{u}^\varepsilon]dxds\right|\\
&\le \left\|\nabla\frac{(\varrho\mathbf{u})^\varepsilon}{\varrho^\varepsilon}\right\|_{L^4(\mathbb{T}^d\times(0,T))}\|(\varrho\mathbf{u}\otimes\mathbf{u})^\varepsilon-(\varrho\mathbf{u})^\varepsilon\otimes\mathbf{u}^\varepsilon\|_{L^{\frac{4}{3}}(\mathbb{T}^d\times(0,T))}.
\end{aligned}
\tag{52}
$$

From Lemma 2.5 (for $d=2$) and the assumption (9) (for $d=3$), we see that $\mathbf{u}\in L^4(\mathbb{T}^d\times(0,T))$. Moreover, by assumption, $\mathbf{u}\in L^2(0,T;H^1(\mathbb{T}^d))$. Therefore, we can apply Lemmas 2.1 (iii) and 2.3 to obtain

$$
\limsup_{\varepsilon\to0}\varepsilon\left\|\nabla\cdot\frac{(\varrho\mathbf{u})^\varepsilon}{\varrho^\varepsilon}\right\|_{L^4(\mathbb{T}^d\times(0,T))}=0,\tag{53}
$$

$$
\limsup_{\varepsilon\to0}\varepsilon^{-1}\|(\varrho\mathbf{u}\otimes\mathbf{u})^\varepsilon-(\varrho\mathbf{u})^\varepsilon\otimes\mathbf{u}^\varepsilon\|_{L^{\frac{4}{3}}(\mathbb{T}^d\times(0,T))}=0,\tag{54}
$$

which yields

$$
\limsup_{\varepsilon\to0}\limsup_{\tau\to0}|(B1)|=0.\tag{55}
$$

For $(B2)$ we first use integration by parts to have

$$
\begin{aligned}
|(B2)| &= \frac{1}{2}\left|\int_\tau^t\int_{\mathbb{T}^d}[(\varrho^\varepsilon\mathbf{u}^\varepsilon)-(\varrho\mathbf{u})^\varepsilon]\nabla\cdot\frac{|(\varrho\mathbf{u})^\varepsilon|^2}{(\varrho^\varepsilon)^2}dxds\right|\\
&\le C\|(\varrho\mathbf{u})^\varepsilon-\varrho^\varepsilon\mathbf{u}^\varepsilon\|_{L^2(\mathbb{T}^d\times(0,T))}\left\|\nabla\cdot\frac{|(\varrho\mathbf{u})^\varepsilon|^2}{(\varrho^\varepsilon)^2}\right\|_{L^2(\mathbb{T}^d\times(0,T))}.
\end{aligned}
\tag{56}
$$

One can justify from Hölder's inequality and Lemma 2.1 that

$$
\left\|\nabla\cdot\frac{|(\varrho\mathbf{u})^\varepsilon|^2}{(\varrho^\varepsilon)^2}\right\|_{L^2(\mathbb{T}^d\times(0,T))}\le C\varepsilon^{-1}\|\mathbf{u}\|^2_{L^4(\mathbb{T}^d\times(0,T))}.\tag{57}
$$

On the other hand, it follows from Lemma 2.3 that

$$\limsup_{\varepsilon\to 0} \varepsilon^{-1}\|(\varrho\mathbf{u})^\varepsilon - \varrho^\varepsilon\mathbf{u}^\varepsilon\|_{L^2(\mathbb{T}^d\times(0,T))} = 0. \tag{58}$$

From (56) – (58), we obtain

$$\limsup_{\varepsilon\to 0}\limsup_{\tau\to 0} |(B2)| = 0. \tag{59}$$

3.3. **Estimate of** (C)**.** Integrating by parts leads to

$$\begin{aligned}
(C) &= \int_\tau^t\int_{\mathbb{T}^d} \frac{(\varrho\mathbf{u})^\varepsilon}{\varrho^\varepsilon}\nabla[(\varrho^\gamma)^\varepsilon - (\varrho^\varepsilon)^\gamma]dxds + \int_\tau^t\int_{\mathbb{T}^d} \frac{(\varrho\mathbf{u})^\varepsilon}{\varrho^\varepsilon}\nabla(\varrho^\varepsilon)^\gamma dxds\\
&=: (C1) - \frac{\gamma}{\gamma-1}\int_\tau^t\int_{\mathbb{T}^d} \nabla\cdot(\varrho\mathbf{u})^\varepsilon(\varrho^\varepsilon)^{\gamma-1}dxds\\
&= (C1) + \frac{\gamma}{\gamma-1}\int_\tau^t\int_{\mathbb{T}^d} \partial_t\varrho^\varepsilon(\varrho^\varepsilon)^{\gamma-1}dxds\\
&= (C1) + \frac{1}{\gamma-1}\int_\tau^t\int_{\mathbb{T}^d} \partial_t(\varrho^\varepsilon)^{\gamma}dxds\\
&=: (C1) + (C2).
\end{aligned}$$

Since $(C2)$ is a desired term, we only need to estimate $(C1)$.

We first consider the case $d = 2$. We see that

$$\begin{aligned}
|(C1)| &= \left|\int_\tau^t\int_{\mathbb{T}^2} \frac{(\varrho\mathbf{u})^\varepsilon}{\varrho^\varepsilon}\nabla[(\varrho^\gamma)^\varepsilon - (\varrho^\varepsilon)^\gamma]dxds\right|\\
&\le \left|\int_\tau^t\int_{\mathbb{T}^2} \frac{(\varrho\mathbf{u})^\varepsilon - \varrho^\varepsilon\mathbf{u}^\varepsilon}{\varrho^\varepsilon}\nabla[(\varrho^\gamma)^\varepsilon - (\varrho^\varepsilon)^\gamma]dxds\right| + \left|\int_\tau^t\int_{\mathbb{T}^2} \nabla\cdot\mathbf{u}^\varepsilon[(\varrho^\gamma)^\varepsilon - (\varrho^\varepsilon)^\gamma]dxds\right|\\
&\le \left\|\frac{(\varrho\mathbf{u})^\varepsilon - \varrho^\varepsilon\mathbf{u}^\varepsilon}{\varrho^\varepsilon}\right\|_{L^1(\mathbb{T}^2\times(0,T))} \|\nabla[(\varrho^\gamma)^\varepsilon - (\varrho^\varepsilon)^\gamma]\|_{L^\infty(\mathbb{T}^2\times(0,T))}\\
&\quad + \|\nabla\cdot\mathbf{u}^\varepsilon\|_{L^1(0,T;L^\infty(\mathbb{T}^2))}\|(\varrho^\varepsilon)^\gamma - (\varrho^\gamma)^\varepsilon\|_{L^\infty(0,T;L^1(\mathbb{T}^2))}.
\end{aligned} \tag{60}$$

We now deal with the first term on the right-hand side of (60). By Lemma 2.1 (i), we derive

$$\|\nabla[(\varrho^\gamma)^\varepsilon - (\varrho^\varepsilon)^\gamma]\|_{L^\infty(\mathbb{T}^2\times(0,T))} \le C\varepsilon^{-1}$$

thanks to $\varrho \in L^\infty(\mathbb{T}^2\times(0,T))$. Since $\mathbf{u} \in L^2(0,T;H^1(\mathbb{T}^2))$, by Lemma 2.3,

$$\limsup_{\varepsilon\to 0}\varepsilon^{-1}\left\|\frac{(\varrho\mathbf{u})^\varepsilon - \varrho^\varepsilon\mathbf{u}^\varepsilon}{\varrho^\varepsilon}\right\|_{L^1(\mathbb{T}^2\times(0,T))} \le C\limsup_{\varepsilon\to 0}\varepsilon^{-1}\|(\varrho\mathbf{u})^\varepsilon - \varrho^\varepsilon\mathbf{u}^\varepsilon\|_{L^1(\mathbb{T}^2\times(0,T))} = 0.$$

It follows that

$$\limsup_{\varepsilon\to 0}\left\|\frac{(\varrho\mathbf{u})^\varepsilon - \varrho^\varepsilon\mathbf{u}^\varepsilon}{\varrho^\varepsilon}\right\|_{L^1(\mathbb{T}^2\times(0,T))}\|\nabla[(\varrho^\gamma)^\varepsilon - (\varrho^\varepsilon)^\gamma]\|_{L^\infty(\mathbb{T}^2\times(0,T))} = 0. \tag{61}$$

For the second term on the right hand side of (60), we first show that

$$\|(\varrho^\gamma)^\varepsilon - (\varrho^\varepsilon)^\gamma\|_{L^\infty(0,T;L^1(\mathbb{T}^2))} \le C\varepsilon. \tag{62}$$

Indeed, we infer from [NNT18, estimate (54)] that

$$|(\varrho^\gamma)^\varepsilon(x,s) - (\varrho^\varepsilon)^\gamma(x,s)| \le C\int_{\mathbb{T}^2}|\varrho(x-y,s) - \varrho(x,s)|^\alpha\omega_\varepsilon(y)dy \tag{63}$$

recalling $\alpha = \min\{\gamma, 2\}$. Consequently, for any $\kappa \in (0,1)$, we obtain

$$\|(\varrho^\gamma)^\varepsilon - (\varrho^\varepsilon)^\gamma\|_{L^1(\mathbb{T}^2)} \le C\varepsilon^{\kappa\alpha}\left[\sup_{|h|<\varepsilon}|h|^{-\kappa}\|\varrho(.+h,s)-\varrho(\cdot,s)\|_{L^\alpha(\mathbb{T}^2)}\right]^\alpha. \tag{64}$$

We use (64) with $\kappa = \frac{1}{\alpha}$ and assumption (8) to deduce (62). Therefore, the second term on the right hand side of (60) is estimated as

$$\|\nabla\cdot\mathbf{u}^\varepsilon\|_{L^1(0,T;L^\infty(\mathbb{T}^2))}\|(\varrho^\varepsilon)^\gamma - (\varrho^\gamma)^\varepsilon\|_{L^\infty(0,T;L^1(\mathbb{T}^2))} \le C\varepsilon\|\nabla\cdot\mathbf{u}^\varepsilon\|_{L^1(0,T;L^\infty(\mathbb{T}^2))}. \tag{65}$$

By Lemma 2.1 (iv),

$$\lim_{\varepsilon\to 0}\varepsilon\|\nabla\cdot\mathbf{u}^\varepsilon\|_{L^1(0,T;L^\infty(\mathbb{T}^2))} = 0.$$

This and (65) ensure

$$\limsup_{\varepsilon\to 0}\|\nabla\cdot\mathbf{u}^\varepsilon\|_{L^1(0,T;L^\infty(\mathbb{T}^2))}\|(\varrho^\varepsilon)^\gamma - (\varrho^\gamma)^\varepsilon\|_{L^\infty(0,T;L^1(\mathbb{T}^2))} = 0. \tag{66}$$

We conclude from (60), (61) and (66) that in the case $d = 2$,

$$\limsup_{\varepsilon\to 0}\limsup_{\tau\to 0}|(C1)| = 0. \tag{67}$$

For the case $d = 3$ we estimate by using integration by parts and Hölder's inequality that

$$\begin{aligned}|(C1)| &= \left|\int_0^t\int_{\mathbb{T}^3}\nabla\frac{(\varrho\mathbf{u})^\varepsilon}{\varrho^\varepsilon}[(\varrho^\gamma)^\varepsilon - (\varrho^\varepsilon)^\gamma]dxds\right| \\ &\le \left\|\nabla\cdot\frac{(\varrho\mathbf{u})^\varepsilon}{\varrho^\varepsilon}\right\|_{L^1(0,T;L^6(\mathbb{T}^3))}\|[(\varrho^\varepsilon)^\gamma - (\varrho^\gamma)^\varepsilon]\|_{L^\infty(0,T;L^{\frac{6}{5}}(\mathbb{T}^3))}.\end{aligned} \tag{68}$$

From (63) (note that this estimate is independent of dimensions) and Hölder's inequality, for any $1 \le p < \infty$ and $\kappa \in (0,1)$, we obtain

$$\|(\varrho^\gamma)^\varepsilon - (\varrho^\varepsilon)^\gamma\|_{L^p(\mathbb{T}^3)} \le C\varepsilon^{\kappa\alpha}\left[\sup_{|h|<\varepsilon}|h|^{-\kappa}\|\varrho(.+h)-\varrho\|_{L^{p\alpha}(\mathbb{T}^3)}\right]^\alpha. \tag{69}$$

By choosing $p = \frac{6}{5}$ and $\kappa = \frac{1}{\alpha}$ in (69) and using assumption (9), we assert that

$$\|(\varrho^\varepsilon)^\gamma - (\varrho^\gamma)^\varepsilon\|_{L^\infty(0,T;L^{\frac{6}{5}}(\mathbb{T}^3))} \le C\varepsilon. \tag{70}$$

Inserting (70) into (68) one has

$$|(C1)| \le C\varepsilon\left\|\nabla\cdot\frac{(\varrho\mathbf{u})^\varepsilon}{\varrho^\varepsilon}\right\|_{L^1(0,T;L^6(\mathbb{T}^3))}. \tag{71}$$

Since $\varrho, \varrho^{-1} \in L^\infty(\mathbb{T}^3\times(0,T))$ and $\mathbf{u} \in L^2(0,T;H^1(\mathbb{T}^3)) \subset L^1(0,T;L^6(\mathbb{T}^3))$, by using Lemma 2.1 (iii), we deduce

$$\lim_{\varepsilon\to 0}\varepsilon\left\|\nabla\cdot\frac{(\varrho\mathbf{u})^\varepsilon}{\varrho^\varepsilon}\right\|_{L^1(0,T;L^6(\mathbb{T}^3))} = 0.$$

Therefore, in the case $d = 3$,

$$\limsup_{\varepsilon\to 0}\limsup_{\tau\to 0}|(C1)| = 0. \tag{72}$$

3.4. **Estimate of** (D)**.** It is easy to see that

$$(D) = -2\nu \int_\tau^t \int_{\mathbb{T}^d} \nabla \cdot (\varrho \mathbb{D}\mathbf{u})^\varepsilon \mathbf{u}^\varepsilon dxds - 2\nu \int_\tau^t \int_{\mathbb{T}^d} \nabla \cdot (\varrho \mathbb{D}\mathbf{u})^\varepsilon \frac{(\varrho \mathbf{u})^\varepsilon - \varrho^\varepsilon \mathbf{u}^\varepsilon}{\varrho^\varepsilon} dxds.$$

By Hölder's inequality and Lemma 2.1 (ii), we deduce

$$\begin{aligned}\left| \int_\tau^t \int_{\mathbb{T}^d} \nabla \cdot (\varrho \mathbb{D}\mathbf{u})^\varepsilon \frac{(\varrho \mathbf{u})^\varepsilon - \varrho^\varepsilon \mathbf{u}^\varepsilon}{\varrho^\varepsilon} dxds \right| &\le \|\nabla \cdot (\varrho \mathbb{D}\mathbf{u})^\varepsilon\|_{L^2(\mathbb{T}^d\times(0,T))} \|\frac{(\varrho \mathbf{u})^\varepsilon - \varrho^\varepsilon \mathbf{u}^\varepsilon}{\varrho^\varepsilon}\|_{L^2(\mathbb{T}^d\times(0,T))} \\ &\le C\varepsilon^{-1} \|\mathbb{D}\mathbf{u}\|_{L^2(\mathbb{T}^d\times(0,T))} \|(\varrho \mathbf{u})^\varepsilon - \varrho^\varepsilon \mathbf{u}^\varepsilon\|_{L^2(\mathbb{T}^d\times(0,T))} \\ &\le C\varepsilon^{-1} \|\nabla\mathbf{u}\|_{L^2(\mathbb{T}^d\times(0,T))} \|(\varrho \mathbf{u})^\varepsilon - \varrho^\varepsilon \mathbf{u}^\varepsilon\|_{L^2(\mathbb{T}^d\times(0,T))}.\end{aligned} \tag{73}$$

Since $\nabla \mathbf{u} \in L^2(\mathbb{T}^d \times (0,T))$ and $\varrho \in L^\infty(\mathbb{T}^d \times (0,T))$, by Lemma 2.3 we get

$$\limsup_{\varepsilon\to 0} \varepsilon^{-1} \|(\varrho \mathbf{u})^\varepsilon - \varrho^\varepsilon \mathbf{u}^\varepsilon\|_{L^2(\mathbb{T}^d\times(0,T))} = 0. \tag{74}$$

Therefore,

$$\limsup_{\varepsilon\to 0} \limsup_{\tau\to 0} \left| \int_\tau^t \int_{\mathbb{T}^d} \nabla \cdot (\varrho \mathbb{D}\mathbf{u})^\varepsilon \frac{(\varrho \mathbf{u})^\varepsilon - \varrho^\varepsilon \mathbf{u}^\varepsilon}{\varrho^\varepsilon} dxds \right| = 0. \tag{75}$$

3.5. **Conclusion of the Proof of Theorem 1.2.** Collecting all the above estimates and putting them into (51) we obtain

$$\limsup_{\varepsilon\to 0} \limsup_{\tau\to 0} \left| \int_\tau^t \int_{\mathbb{T}^d} \partial_t \left[\frac{1}{2} \frac{|(\varrho \mathbf{u})^\varepsilon|^2}{\varrho^\varepsilon} + \frac{(\varrho^\varepsilon)^\gamma}{\gamma - 1} \right] dxds - 2\nu \int_\tau^t \int_{\mathbb{T}^d} \nabla \cdot (\varrho \mathbb{D}\mathbf{u})^\varepsilon \mathbf{u}^\varepsilon dxds \right| = 0. \tag{76}$$

Using the weak continuity of ϱ and $\varrho\mathbf{u}$ in (5) and (6), and the limit

$$\limsup_{\varepsilon\to 0} \limsup_{\tau\to 0} \left| 2\nu \int_\tau^t \int_{\mathbb{T}^d} \nabla \cdot (\varrho \mathbb{D}\mathbf{u})^\varepsilon \mathbf{u}^\varepsilon dxds + \nu \int_0^t \int_{\mathbb{T}^d} \varrho |\mathbb{D}\mathbf{u}|^2 dxds \right| = 0,$$

we can finally conclude the proof of Theorem 1.2. □

4. Proof of Theorem 1.5

The proof of Theorem 1.5 is similar to that of Theorem 1.2, except for the fact that we have to take care of the boundary layers when integrating by parts. More precisely, by smoothing (cNSd) we obtain

$$\partial_t \varrho^\varepsilon + \nabla \cdot (\varrho \mathbf{u})^\varepsilon = 0, \qquad \text{in } \Omega_\varepsilon, \tag{77}$$

and

$$\partial_t (\varrho \mathbf{u})^\varepsilon + \nabla \cdot (\varrho \mathbf{u} \otimes \mathbf{u})^\varepsilon + \nabla (\varrho^\gamma)^\varepsilon - 2\nu \nabla \cdot (\varrho \mathbb{D}\mathbf{u})^\varepsilon = 0, \qquad \text{in } \Omega_\varepsilon, \tag{78}$$

recalling $\Omega_\varepsilon = \{x \in \Omega : d(x, \partial\Omega) > \varepsilon\}$. Take $0 < \varepsilon < \varepsilon_1/10 < \varepsilon_2/10 < r_0/100$ we obtain by multiplying (78) by $(\varrho \mathbf{u})^\varepsilon / \varrho^\varepsilon$ and integrating on $(\tau, t) \times \Omega_{\varepsilon_2}$ with $0 < \tau < t < T$

$$\begin{aligned}\int_\tau^t \int_{\Omega_{\varepsilon_2}} \frac{(\varrho \mathbf{u})^\varepsilon}{\varrho^\varepsilon} \partial_t (\varrho \mathbf{u})^\varepsilon dxds + \int_\tau^t \int_{\Omega_{\varepsilon_2}} \frac{(\varrho \mathbf{u})^\varepsilon}{\varrho^\varepsilon} \nabla \cdot (\varrho \mathbf{u} \otimes \mathbf{u})^\varepsilon dxds + \int_\tau^t \int_{\Omega_{\varepsilon_2}} \frac{(\varrho \mathbf{u})^\varepsilon}{\varrho^\varepsilon} \nabla (\varrho^\gamma)^\varepsilon dxds \\ - 2\nu \int_\tau^t \int_{\Omega_{\varepsilon_2}} \nabla \cdot (\varrho \mathbb{D}\mathbf{u})^\varepsilon \frac{(\varrho \mathbf{u})^\varepsilon}{\varrho^\varepsilon} dxds = 0.\end{aligned} \tag{79}$$

Taking $\varepsilon_3 > 0$ to be small, we integrate (79) with respect to ε_2 on $(\varepsilon_1, \varepsilon_1+\varepsilon_3)$ to get

$$\begin{aligned}&\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\frac{(\varrho\mathbf{u})^\varepsilon}{\varrho^\varepsilon}\partial_t(\varrho\mathbf{u})^\varepsilon dxdsd\varepsilon_2+\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\frac{(\varrho\mathbf{u})^\varepsilon}{\varrho^\varepsilon}\nabla\cdot(\varrho\mathbf{u}\otimes\mathbf{u})^\varepsilon dxdsd\varepsilon_2\\&+\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\frac{(\varrho\mathbf{u})^\varepsilon}{\varrho^\varepsilon}\nabla(\varrho^\gamma)^\varepsilon dxdsd\varepsilon_2-\frac{2\nu}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\nabla\cdot(\varrho\mathbb{D}\mathbf{u})^\varepsilon\frac{(\varrho\mathbf{u})^\varepsilon}{\varrho^\varepsilon}dxdsd\varepsilon_2=0.\end{aligned} \tag{80}$$

We denote by (E), (F), (G) and (H) the first, second, third and forth term on the left hand side of (80) respectively. We will estimate them separately in the following subsections.

4.1. **Estimate of** (E)**.** This term is estimated similarly to (A),

$$\begin{aligned}(E)&=\frac{1}{2}\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\partial_t\left(\frac{|(\varrho\mathbf{u})^\varepsilon|^2}{\varrho^\varepsilon}\right)dxdsd\varepsilon_2+\frac{1}{2}\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\partial_t\varrho^\varepsilon\frac{|(\varrho\mathbf{u})^\varepsilon|^2}{(\varrho^\varepsilon)^2}dxdsd\varepsilon_2\\&=\frac{1}{2}\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\partial_t\left(\frac{|(\varrho\mathbf{u})^\varepsilon|^2}{\varrho^\varepsilon}\right)dxdsd\varepsilon_2-\frac{1}{2}\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\nabla\cdot(\varrho\mathbf{u})^\varepsilon\frac{|(\varrho\mathbf{u})^\varepsilon|^2}{(\varrho^\varepsilon)^2}dxdsd\varepsilon_2\\&=:(E1)+(E2).\end{aligned}$$

The term $(E1)$ is desired, while $(E2)$ will be canceled by $(F4)$ later.

4.2. **Estimate of** (F)**.** We estimate (F) similarly to (B) by using integration by parts

$$\begin{aligned}(F)&=\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\frac{(\varrho\mathbf{u})^\varepsilon}{\varrho^\varepsilon}\nabla\cdot[(\varrho\mathbf{u}\otimes\mathbf{u})^\varepsilon-(\varrho\mathbf{u})^\varepsilon\otimes\mathbf{u}^\varepsilon]dxdsd\varepsilon_2\\&\quad+\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\frac{(\varrho\mathbf{u})^\varepsilon}{\varrho^\varepsilon}\nabla\cdot((\varrho\mathbf{u})^\varepsilon\otimes\mathbf{u}^\varepsilon)dxdsd\varepsilon_2\\&=:(F1)+\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\nabla\cdot\mathbf{u}^\varepsilon\frac{|(\varrho\mathbf{u})^\varepsilon|^2}{\varrho^\varepsilon}dxdsd\varepsilon_2\\&\quad+\frac{1}{2}\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\frac{\mathbf{u}^\varepsilon}{\varrho^\varepsilon}\nabla|(\varrho\mathbf{u})^\varepsilon|^2dxdsd\varepsilon_2\\&=:(F1)+\frac{1}{2}\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\partial\Omega_{\varepsilon_2}}|(\varrho\mathbf{u})^\varepsilon|^2\frac{\mathbf{u}^\varepsilon}{\varrho^\varepsilon}n(\theta)d\mathcal{H}^{d-1}(\theta)dsd\varepsilon_2\\&\quad+\frac{1}{2}\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\nabla\cdot[\varrho^\varepsilon\mathbf{u}^\varepsilon-(\varrho\mathbf{u})^\varepsilon]\frac{|(\varrho\mathbf{u})^\varepsilon|^2}{(\varrho^\varepsilon)^2}dxdsd\varepsilon_2\\&\quad+\frac{1}{2}\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\nabla\cdot(\varrho\mathbf{u})^\varepsilon\frac{|(\varrho\mathbf{u})^\varepsilon|^2}{(\varrho^\varepsilon)^2}dxdsd\varepsilon_2\\&=:(F1)+(F2)^{\mathrm{bdr}}+(F3)+(F4).\end{aligned}$$

The superscript "bdr" in $(F2)^{\mathrm{bdr}}$ means that this term contains a boundary layer. It's obvious that $(F4)+(E2)=0$. The term $(F1)$ is estimated using integration by parts as

$$\begin{aligned}(F1)&=-\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\nabla\cdot\frac{(\varrho\mathbf{u})^\varepsilon}{\varrho^\varepsilon}[(\varrho\mathbf{u}\otimes\mathbf{u})^\varepsilon-(\varrho\mathbf{u})^\varepsilon\otimes\mathbf{u}^\varepsilon]dxdsd\varepsilon_2\\&\quad+\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\partial\Omega_{\varepsilon_2}}[(\varrho\mathbf{u}\otimes\mathbf{u})^\varepsilon-(\varrho\mathbf{u})^\varepsilon\otimes\mathbf{u}^\varepsilon]\frac{(\varrho\mathbf{u})^\varepsilon}{\varrho^\varepsilon}n(\theta)d\mathcal{H}^{d-1}(\theta)dsd\varepsilon_2\\&=:(F11)+(F12)^{\mathrm{bdr}}.\end{aligned}$$

We estimate $(F11)$ similarly to $(B1)$ in (52)–(54) and therefore

$$\limsup_{\varepsilon\to 0}\limsup_{\tau\to 0}|(F11)|=0. \tag{81}$$

The term $(F12)^{\text{bdr}}$ will be treated later on, together with other boundary terms.

For $(F3)$ it follows from integration by parts that

$$\begin{aligned}(F3)=&-\frac{1}{2}\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}[\varrho^\varepsilon\mathbf{u}^\varepsilon-(\varrho\mathbf{u})^\varepsilon]\nabla\frac{|(\varrho\mathbf{u})^\varepsilon|^2}{(\varrho^\varepsilon)^2}dxdsd\varepsilon_2\\&+\frac{1}{2}\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\partial\Omega_{\varepsilon_2}}[\varrho^\varepsilon\mathbf{u}^\varepsilon-(\varrho\mathbf{u})^\varepsilon]n(\theta)\frac{|(\varrho\mathbf{u})^\varepsilon|^2}{(\varrho^\varepsilon)^2}d\mathcal{H}^{d-1}(\theta)dsd\varepsilon_2\\&=:(F31)+(F32)^{\text{bdr}}.\end{aligned}$$

The term $(F31)$ is estimated similarly to $(B2)$ in (56)–(58) and therefore

$$\limsup_{\varepsilon\to 0}\limsup_{\tau\to 0}|(F31)|=0. \tag{82}$$

It remains to estimate the boundary terms $(F12)^{\text{bdr}}$, $(F2)^{\text{bdr}}$ and $(F32)^{\text{bdr}}$. As for the term $(F12)^{\text{bdr}}$, we use the coarea formula and then letting successively $\tau\to 0$ and $\varepsilon\to 0$ to obtain

$$\limsup_{\varepsilon\to 0}\limsup_{\tau\to 0}|(F12)^{\text{bdr}}|=0. \tag{83}$$

Next, we deal with the term $(F2)^{\text{bdr}}$. By using the coarea formula, Holder's inequality and Lemma 2.6, we get

$$\begin{aligned}\limsup_{\varepsilon_1,\varepsilon\to 0}\limsup_{\tau\to 0}|(F2)^{\text{bdr}}|&=\limsup_{\varepsilon_1,\varepsilon\to 0}\limsup_{\tau\to 0}\left|\frac{1}{2\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_1}\setminus\Omega_{\varepsilon_1+\varepsilon_3}}|(\varrho\mathbf{u})^\varepsilon|^2\frac{\mathbf{u}^\varepsilon}{\varrho^\varepsilon}n(x)dxds\right|\\&\le\frac{C}{2\varepsilon_3}\int_0^T\int_{\Omega\setminus\Omega_{\varepsilon_3}}|\mathbf{u}|^3dxds\\&\le\frac{C}{2\varepsilon_3}\|\mathbf{u}\|^2_{L^4((\Omega\setminus\Omega_{\varepsilon_3})\times(0,T))}\|\mathbf{u}\|_{L^2((\Omega\setminus\Omega_{\varepsilon_3})\times(0,T))}\\&\le C\|\mathbf{u}\|^2_{L^4((\Omega\setminus\Omega_{\varepsilon_3})\times(0,T))}\|\nabla\mathbf{u}\|_{L^2((\Omega\setminus\Omega_{2\varepsilon_3}\times(0,T))}.\end{aligned}$$

Since $\mathbf{u}\in L^4(\Omega\times(0,T))$ and $\mathbf{u}\in L^2(0,T;H^1(\Omega))$, by letting $\varepsilon_3\to 0$, we obtain

$$\limsup_{\varepsilon_3\to 0}\limsup_{\varepsilon_1,\varepsilon\to 0}\limsup_{\tau\to 0}|(F2)^{\text{bdr}}|=0.$$

Next we treat the term $(F32)^{\text{bdr}}$. Again, by employing the coarea formula and letting successively $\tau\to 0$ and $\varepsilon\to 0$, we derive

$$\limsup_{\varepsilon\to 0}\limsup_{\tau\to 0}|(F32)^{\text{bdr}}|=0.$$

4.3. **Estimate of (G).** By integration by parts, we have

$$\begin{aligned}(G)=&\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\frac{(\varrho\mathbf{u})^\varepsilon}{\varrho^\varepsilon}\nabla[(\varrho^\gamma)^\varepsilon-(\varrho^\varepsilon)^\gamma]dxdsd\varepsilon_2+\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\frac{(\varrho\mathbf{u})^\varepsilon}{\varrho^\varepsilon}\nabla(\varrho^\varepsilon)^\gamma dxdsd\varepsilon_2\\=&-\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\nabla\cdot\frac{(\varrho\mathbf{u})^\varepsilon}{\varrho^\varepsilon}[(\varrho^\gamma)^\varepsilon-(\varrho^\varepsilon)^\gamma]dxdsd\varepsilon_2\\&+\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\partial\Omega_{\varepsilon_2}}\frac{(\varrho\mathbf{u})^\varepsilon}{\varrho^\varepsilon}n(\theta)[(\varrho^\gamma)^\varepsilon-(\varrho^\varepsilon)^\gamma]d\mathcal{H}^{d-1}(\theta)dsd\varepsilon_2\end{aligned}$$

$$
\begin{aligned}
&- \frac{\gamma}{\gamma-1}\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}} \nabla\cdot(\varrho\mathbf{u})^\varepsilon(\varrho^\varepsilon)^{\gamma-1}dxdsd\varepsilon_2\\
&+ \frac{\gamma}{\gamma-1}\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\partial\Omega_{\varepsilon_2}} (\varrho\mathbf{u})^\varepsilon n(\theta)(\varrho^\varepsilon)^{\gamma-1}d\mathcal{H}^{d-1}(\theta)dsd\varepsilon_2\\
&=: (G1) + (G2)^{\text{bdr}} + (G3) + (G4)^{\text{bdr}}
\end{aligned}
$$

where

$$
(G3) = -\frac{\gamma}{\gamma-1}\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}} \nabla\cdot(\varrho\mathbf{u})^\varepsilon(\varrho^\varepsilon)^{\gamma-1}dxdsd\varepsilon_2 = \frac{1}{\gamma-1}\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}} \partial_t(\varrho^\varepsilon)^\gamma dxdsd\varepsilon_2
$$

is a desired term. The term $(G1)$ is rewritten as

$$
\begin{aligned}
(G1) &= \frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}} \frac{(\varrho\mathbf{u})^\varepsilon-\varrho^\varepsilon\mathbf{u}^\varepsilon}{\varrho^\varepsilon}\nabla[(\varrho^\gamma)^\varepsilon-(\varrho^\varepsilon)^\gamma]dxdsd\varepsilon_2\\
&+ \frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}} (\nabla\cdot\mathbf{u}^\varepsilon)[(\varrho^\gamma)^\varepsilon-(\varrho^\varepsilon)^\gamma]dxdsd\varepsilon_2\\
&+ \frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\partial\Omega_{\varepsilon_2}} [(\varrho^\gamma)^\varepsilon-(\varrho^\varepsilon)^\gamma]\mathbf{u}^\varepsilon n(\theta)d\mathcal{H}^{d-1}(\theta)dsd\varepsilon_2\\
&=: (G12) + (G13) + (G14)^{\text{bdr}}.
\end{aligned}
$$

We can handle $(G12)$ and $(G13)$ using similar arguments to estimate of $(C1)$ in (60) and (68), and thus

$$
\limsup_{\varepsilon\to0}\limsup_{\tau\to0}|(G12)| = 0 \quad \text{and} \quad \limsup_{\varepsilon\to0}\limsup_{\tau\to0}|(G13)| = 0.
$$

It remains to estimate the terms involving the boundary $(G14)^{\text{bdr}}$, $(G2)^{\text{bdr}}$ and $(G4)^{\text{bdr}}$. By using the coarea formula and letting successively $\tau\to0$ and $\varepsilon\to0$, we assert that

$$
\limsup_{\varepsilon\to0}\limsup_{\tau\to0}|(G14)^{\text{bdr}}| = 0 \quad \text{and} \quad \limsup_{\varepsilon\to0}\limsup_{\tau\to0}|(G2)^{\text{bdr}}| = 0.
$$

To deal with the term $(G4)^{\text{bdr}}$, we use the coarea formula (44), the fact $\mathcal{L}^d(\Omega\backslash\Omega_{\varepsilon_3}) \approx \varepsilon_3$, Holder's inequality and Lemma 2.6 to get

$$
\begin{aligned}
\limsup_{\varepsilon_1,\varepsilon\to0}\limsup_{\tau\to0}|(G4)^{\text{bdr}}| &= \limsup_{\varepsilon_1,\varepsilon\to0}\limsup_{\tau\to0}\frac{\gamma}{\gamma-1}\frac{1}{\varepsilon_3}\left|\int_\tau^t\int_{\Omega_{\varepsilon_1}\backslash\Omega_{\varepsilon_1+\varepsilon_3}} (\varrho\mathbf{u})^\varepsilon n(x)(\varrho^\varepsilon)^{\gamma-1}dxds\right|\\
&\le \frac{C}{\varepsilon_3}\int_0^T\int_{\Omega\backslash\Omega_{\varepsilon_3}}|\mathbf{u}|dxds\\
&\le \frac{C}{\varepsilon_3}(T\mathcal{L}^d(\Omega\backslash\Omega_{\varepsilon_3}))^{1/2}\left(\int_0^T\int_{\Omega\backslash\Omega_{\varepsilon_3}}|\mathbf{u}|^2dxds\right)^{1/2}\\
&\le C\varepsilon_3^{\frac{1}{2}}\|\nabla\mathbf{u}\|_{L^2((\Omega\backslash\Omega_{2\varepsilon_3})\times(0,T))}.
\end{aligned}
$$

Since $\mathbf{u}\in L^2(0,T;H^1(\Omega))$, it follows that

$$
\limsup_{\varepsilon_3\to0}\limsup_{\varepsilon_1,\varepsilon\to0}\limsup_{\tau\to0}|(G4)^{\text{bdr}}| = 0.
$$

4.4. **Estimate of (H).** By rewriting (H) as

$$
\begin{aligned}
(H) = &-\frac{2\nu}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}} \nabla\cdot(\varrho\mathbb{D}\mathbf{u})^\varepsilon \frac{(\varrho\mathbf{u})^\varepsilon-\varrho^\varepsilon\mathbf{u}^\varepsilon}{\varrho^\varepsilon}dxdsd\varepsilon_2\\
&-2\nu\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}} \nabla\cdot(\varrho\mathbb{D}\mathbf{u})^\varepsilon\mathbf{u}^\varepsilon dxdsd\varepsilon_2\\
=:&\ (H1) - 2\nu\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\partial\Omega_{\varepsilon_2}}(\varrho\mathbb{D}\mathbf{u})^\varepsilon\mathbf{u}^\varepsilon n(\theta)d\mathcal{H}^{d-1}(\theta)dsd\varepsilon_2\\
&+2\nu\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}(\varrho\mathbb{D}\mathbf{u})^\varepsilon\nabla\mathbf{u}^\varepsilon dxdsd\varepsilon_2\\
=:&\ (H1)+(H2)^{\mathrm{bdr}}+(H3).
\end{aligned}
$$

Estimate $(H1)$ using arguments similar to (73) and (75) we have

$$\limsup_{\varepsilon\to0}\limsup_{\tau\to0}|(H1)|=0.$$

The boundary term is computed using coarea formula (44) and Lemma 2.6 as follows

$$
\begin{aligned}
\limsup_{\varepsilon_1,\varepsilon\to0}\limsup_{\tau\to0}|(H2)^{\mathrm{bdr}}| &= \limsup_{\varepsilon_1,\varepsilon\to0}\limsup_{\tau\to0} 2\nu\left|\frac{1}{\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_1}\setminus\Omega_{\varepsilon_1+\varepsilon_3}}(\varrho\mathbb{D}\mathbf{u})\mathbf{u}\,n(x)dxds\right|\\
&\le C\frac{1}{\varepsilon_3}\|\nabla\mathbf{u}\|_{L^2((\Omega\setminus\Omega_{\varepsilon_3})\times(0,T))}\|\mathbf{u}\|_{L^2((\Omega\setminus\Omega_{\varepsilon_3})\times(0,T))}\\
&\le C\|\nabla\mathbf{u}\|^2_{L^2((\Omega\setminus\Omega_{2\varepsilon_3})\times(0,T))}.
\end{aligned}
\tag{84}
$$

Due to the assumption $\mathbf{u}\in L^2(0,T;H^1(\Omega))$, by letting $\varepsilon_3\to0$, we get

$$\limsup_{\varepsilon_3\to0}\limsup_{\varepsilon_1,\varepsilon\to0}\limsup_{\tau\to0}|(H2)^{\mathrm{bdr}}|=0.$$

4.5. **Conclusion of the Proof of Theorem 1.5.** Collecting the estimates for (E), (F), (G) and (H) and using similar arguments to subsection 3.5 we obtain the desired energy equality (14). ☐

5. Proof of Theorem 1.7

The proof of Theorem 1.7 follows closely from that of Theorems 1.2 and 1.5 except we have to take extra care of the terms $-2\nu\Delta\mathbf{u}$ and $-\lambda\nabla(\nabla\cdot\mathbf{u})$.

In the case $\Omega=\mathbb{T}^d$, by multiplying the smoothed version of (cNS),

$$\partial_t(\varrho\mathbf{u})^\varepsilon+\nabla\cdot(\varrho\mathbf{u}\otimes\mathbf{u})^\varepsilon+\nabla(\varrho^\gamma)^\varepsilon-2\nu\Delta\mathbf{u}^\varepsilon-\lambda\nabla(\nabla\cdot\mathbf{u})^\varepsilon=0$$

by $(\varrho\mathbf{u})^\varepsilon/\varrho^\varepsilon$ we can proceed similarly to the proof of Theorem 1.2 except for the extra terms

$$(EX1)=-2\nu\int_\tau^t\int_{\mathbb{T}^d}\Delta\mathbf{u}^\varepsilon\frac{(\varrho\mathbf{u})^\varepsilon}{\varrho^\varepsilon}dxds \tag{85}$$

and

$$(EX2)=-\lambda\int_\tau^t\int_{\mathbb{T}^d}\nabla(\nabla\cdot\mathbf{u})^\varepsilon\frac{(\varrho\mathbf{u})^\varepsilon}{\varrho^\varepsilon}dxds. \tag{86}$$

By integration by parts,

$$(EX1)=-2\nu\int_\tau^t\int_{\mathbb{T}^d}\Delta\mathbf{u}^\varepsilon\mathbf{u}^\varepsilon dxds-2\nu\int_\tau^t\int_{\mathbb{T}^d}\Delta\mathbf{u}^\varepsilon\frac{(\varrho\mathbf{u})^\varepsilon-\varrho^\varepsilon\mathbf{u}^\varepsilon}{\varrho^\varepsilon}dxds$$

$$= 2\nu \int_\tau^t \int_{\mathbb{T}^d} |\nabla \mathbf{u}^\varepsilon|^2 dxds - 2\nu \int_\tau^t \int_{\mathbb{T}^d} \nabla\cdot(\nabla \mathbf{u}^\varepsilon) \frac{(\varrho\mathbf{u})^\varepsilon - \varrho^\varepsilon \mathbf{u}^\varepsilon}{\varrho^\varepsilon} dxds.$$

By using an argument similar to that leading to (75) and the assumption $\mathbf{u} \in L^2(0,T;H^1(\mathbb{T}^d))$, we obtain

$$\limsup_{\varepsilon\to 0}\limsup_{\tau\to 0}\left|(EX1) - 2\nu\int_0^t\int_{\mathbb{T}^d} |\nabla\mathbf{u}|^2 dxds\right| = 0.$$

For $(EX2)$ we write similarly

$$\begin{aligned}(EX2) &= -\lambda \int_\tau^t\int_{\mathbb{T}^d} \nabla(\nabla\cdot\mathbf{u})^\varepsilon \mathbf{u}^\varepsilon dxds - \lambda\int_\tau^t\int_{\mathbb{T}^d}\nabla(\nabla\cdot\mathbf{u})^\varepsilon \frac{(\varrho\mathbf{u})^\varepsilon - \varrho^\varepsilon\mathbf{u}^\varepsilon}{\varrho^\varepsilon} dxds \\ &= \lambda\int_\tau^t\int_{\mathbb{T}^d} (\nabla\cdot\mathbf{u})^\varepsilon(\nabla\cdot\mathbf{u}^\varepsilon) dxds - \lambda\int_\tau^t\int_{\mathbb{T}^d}\nabla(\nabla\cdot\mathbf{u})^\varepsilon \frac{(\varrho\mathbf{u})^\varepsilon - \varrho^\varepsilon\mathbf{u}^\varepsilon}{\varrho^\varepsilon} dxds\end{aligned}$$

and therefore obtain

$$\limsup_{\varepsilon\to 0}\limsup_{\tau\to 0}\left|(EX2) - \lambda\int_0^t\int_{\mathbb{T}^d} |\nabla\cdot\mathbf{u}|^2 dxds\right| = 0.$$

In the case $\Omega\subset\mathbb{R}^d$ is a bounded domain with C^2 boundary, by proceeding as in section 4 and using the arguments dealing with (85) and (86), we are left to estimate only the extra boundary terms

$$(BEX1) = -2\nu\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\partial\Omega_{\varepsilon_2}} \nabla\mathbf{u}^\varepsilon\mathbf{u}^\varepsilon n(\theta)d\mathcal{H}^{d-1}(\theta)dsd\varepsilon_2,$$

and

$$(BEX2) = -\lambda\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\partial\Omega_{\varepsilon_2}} (\nabla\cdot\mathbf{u})^\varepsilon\mathbf{u}^\varepsilon n(\theta)d\mathcal{H}^{d-1}(\theta)dsd\varepsilon_2.$$

Both of these terms can be estimated by using an argument similar to the one leading to (84). Therefore we obtain

$$\limsup_{\varepsilon_3\to 0}\limsup_{\varepsilon_1,\varepsilon\to 0}\limsup_{\tau\to 0}|(BEX1)| = 0 \quad\text{and}\quad \limsup_{\varepsilon_3\to 0}\limsup_{\varepsilon_1,\varepsilon\to 0}\limsup_{\tau\to 0}|(BEX2)| = 0.$$

Thus the proof of Theorem 1.7 is complete. □

6. Proof of Theorem 1.8

Thanks to the proof of Theorems 1.2 and 1.5 we only need to take care of the terms concerning the scalar pressure.

In case $\Omega = \mathbb{T}^d$, we multiply

$$\partial_t(\varrho\mathbf{u})^\varepsilon + \nabla\cdot(\varrho\mathbf{u}\otimes\mathbf{u})^\varepsilon + \nabla P^\varepsilon - 2\nu\nabla\cdot(\varrho\mathbb{D}\mathbf{u})^\varepsilon = 0$$

by $(\varrho\mathbf{u})^\varepsilon/\varrho^\varepsilon$. Then we only need to take care of the following term (since other terms can be estimated similarly as in the previous Theorems)

$$\begin{aligned}\int_\tau^t\int_{\mathbb{T}^d}\nabla P^\varepsilon\frac{(\varrho\mathbf{u})^\varepsilon}{\varrho^\varepsilon}dxds &= \int_\tau^t\int_{\mathbb{T}^d}\nabla P^\varepsilon\frac{(\varrho\mathbf{u})^\varepsilon - \varrho^\varepsilon\mathbf{u}^\varepsilon}{\varrho^\varepsilon}dxds + \int_\tau^t\int_{\mathbb{T}^d}\nabla P^\varepsilon\mathbf{u}^\varepsilon dxds \\ &= \int_\tau^t\int_{\mathbb{T}^d}\nabla P^\varepsilon\frac{(\varrho\mathbf{u})^\varepsilon - \varrho^\varepsilon\mathbf{u}^\varepsilon}{\varrho^\varepsilon}dxds.\end{aligned}$$

At the last step we have made used of the free divergence condition. Using Hölder's inequality and Lemma 2.1 (ii), we have

$$\begin{aligned}\left|\int_\tau^t\int_{\mathbb{T}^d}\nabla P^\varepsilon\frac{(\varrho\mathbf{u})^\varepsilon-\varrho^\varepsilon\mathbf{u}^\varepsilon}{\varrho^\varepsilon}dxds\right| &\le \|\nabla P^\varepsilon\|_{L^2(\mathbb{T}^d\times(0,T))}\|\frac{(\varrho\mathbf{u})^\varepsilon-\varrho^\varepsilon\mathbf{u}^\varepsilon}{\varrho^\varepsilon}\|_{L^2(\mathbb{T}^d\times(0,T))}\\ &\le C\|P\|_{L^2(\mathbb{T}^d\times(0,T))}\varepsilon^{-1}\|(\varrho\mathbf{u})^\varepsilon-\varrho^\varepsilon\mathbf{u}^\varepsilon\|_{L^2(\mathbb{T}^d\times(0,T))}.\end{aligned}\tag{87}$$

By assumption $P\in L^2(\mathbb{T}^d\times(0,T))$ and Lemma 2.3 we obtain the desired limit

$$\limsup_{\varepsilon\to0}\limsup_{\tau\to0}\left|\int_\tau^t\int_{\mathbb{T}^d}\nabla P^\varepsilon\frac{(\varrho\mathbf{u})^\varepsilon}{\varrho^\varepsilon}dxds\right|=0$$

and therefore finish the proof in the case $\Omega=\mathbb{T}^d$.

In case Ω is a bounded domain, we need to take care of the boundary. Similar to the case of a torus, we only need to deal with the term

$$\begin{aligned}&\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\nabla P^\varepsilon\frac{(\varrho\mathbf{u})^\varepsilon}{\varrho^\varepsilon}dxds\\ &=\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\nabla P^\varepsilon\frac{(\varrho\mathbf{u})^\varepsilon-\varrho^\varepsilon\mathbf{u}^\varepsilon}{\varrho^\varepsilon}dxds+\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\nabla P^\varepsilon\mathbf{u}^\varepsilon dxds\\ &=\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\nabla P^\varepsilon\frac{(\varrho\mathbf{u})^\varepsilon-\varrho^\varepsilon\mathbf{u}^\varepsilon}{\varrho^\varepsilon}dxds+\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\partial\Omega_{\varepsilon_2}}P^\varepsilon\mathbf{u}^\varepsilon n(\theta)d\mathcal{H}^{d-1}(\theta)dsd\varepsilon_2\\ &=:(K1)+(K2)^{\rm bdr}.\end{aligned}$$

The term $(K1)$ is estimated exactly as in (87) and therefore,

$$\limsup_{\varepsilon\to0}\limsup_{\tau\to0}|(K1)|=0.$$

For $(K2)^{\rm bdr}$ we use the coarea formula, Hölder's inequality and Lemma 2.6 to obtain

$$\begin{aligned}\limsup_{\varepsilon_1,\varepsilon\to0}\limsup_{\tau\to0}|(K2)^{\rm bdr}| &=\limsup_{\varepsilon_1,\varepsilon\to0}\limsup_{\tau\to0}\left|\frac{1}{\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_1}\setminus\Omega_{\varepsilon_1+\varepsilon_3}}P^\varepsilon\mathbf{u}^\varepsilon n(x)dxds\right|\\ &\le\frac{1}{\varepsilon_3}\|P\|_{L^2((\Omega\setminus\Omega_{\varepsilon_3})\times(0,T))}\|\mathbf{u}\|_{L^2((\Omega\setminus\Omega_{\varepsilon_3})\times(0,T))}\\ &\le C\|P\|_{L^2((\Omega\setminus\Omega_{\varepsilon_3})\times(0,T))}\|\nabla\mathbf{u}\|_{L^2((\Omega\setminus\Omega_{2\varepsilon_3})\times(0,T))}.\end{aligned}$$

Since $P\in L^2(\Omega\times(0,T))$ and $\mathbf{u}\in L^2(0,T;H^1(\Omega))$, by letting $\varepsilon_3\to0$, we derive

$$\limsup_{\varepsilon_3\to0}\limsup_{\varepsilon_1,\varepsilon\to0}\limsup_{\tau\to0}|(K2)^{\rm bdr}|=0.$$

Thus the proof of Theorem 1.8 is complete. □

7. Proof of Theorem 1.9

The proof of Theorem 1.9 follows exactly from that of Theorem 1.8 except for the term relating to the pressure. However, since in this case we can take $\varrho\equiv1$ and therefore $(K1)=0$ trivially. From the estimate of $(K2)^{\rm bdr}$

$$\limsup_{\varepsilon_1,\varepsilon\to0}\limsup_{\tau\to0}|(K2)^{\rm bdr}|\le C\|P\|_{L^2(\Omega\setminus\Omega_{\varepsilon_3}\times(0,T))}\|\nabla\mathbf{u}\|_{L^2(\Omega\setminus\Omega_{2\varepsilon_3}\times(0,T))}$$

we use $\mathbf{u} \in L^2(0,T;H^1(\Omega))$ and the assumption (16) to conclude

$$\limsup_{\varepsilon_3 \to 0} \limsup_{\varepsilon_1, \varepsilon \to 0} \limsup_{\tau \to 0} |(K2)^{\mathrm{bdr}}| = 0$$

and therefore finish the proof of Theorem 1.9. □

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