Asymptotic behavior of BV functions and sets of finite perimeter in metric measure spaces

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Abstract: In this paper, we study the asymptotic behavior of BV functions in complete metric measure spaces equipped with a doubling measure supporting a 1-Poincaré inequality. We show that at almost every point x outside the Cantor and jump parts of a BV function, the asymptotic limit of the function is a Lipschitz continuous function of least gradient on a tangent space to the metric space based at x. We also show that, at co-dimension 1 Hausdorff measure almost every measure-theoretic boundary point of a set E of finite perimeter, there is an asymptotic limit set $(E)_\infty$ corresponding to the asymptotic expansion of E and that every such asymptotic limit $(E)_\infty$ is a quasiminimal set of finite perimeter. We also show that the perimeter measure of $(E)_\infty$ is Ahlfors co-dimension 1 regular.

Key words and phrases: Bounded variation, finite perimeter, asymptotic limit, doubling measure, Poincaré inequality, least gradient function.

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1 Introduction

The classical notion of differentiability for a function f on a subset of Euclidean space $\Omega \subset \mathbb{R}^n$ at a point $x \in \Omega$ is that the graph of f should, at $(x, f(x)) \in \mathbb{R}^n \times \mathbb{R}$, asymptotically approach an n-dimensional hyperplane in $\mathbb{R}^{n+1}$. In other words, the function f behaves asymptotically like an affine function. This notion has been extended to mappings between domains in Riemannian manifolds in the study of differential geometry.

The seminal work of Cheeger [11] extended the above notion of affine approximation to the realm of metric measure spaces, with *generalized linear functions* defined on measured Gromov-Hausdorff tangent spaces playing the role of affine functions. It was shown there that, if the space is complete, the measure is doubling, and the space supports a p-Poincaré inequality for some $1 \le p < \infty$, then

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every Lipschitz function f on the metric space is asymptotically generalized linear at almost every point x in the space. More specifically, we have the following: let X_∞ be obtained as a pointed measured Gromov-Hausdorff limit of scaled versions (X_n, d_n, x, μ_n) of the metric measure space (X, d, μ) with $x \in X$. In considering corresponding scaled versions $f_n \colon X_n \to \mathbb{R}$ of $f \colon X \to \mathbb{R}$, where

$$f_n(y) := \frac{f(y) - f(x)}{r_n}$$

with $\{r_n\}_{n\in\mathbb{N}}$ a sequence of positive numbers decreasing to zero which form the scales associated with the metric $d_n := r_n^{-1} d$ in the Gromov-Hausdorff limit, the sequence of functions f_n converges to a limit function $f_\infty \colon X_\infty \to \mathbb{R}$ (after passing to a subsequence if necessary). Cheeger proved that this asymptotic limit function f_∞ is a generalized linear function on X_∞. Here by a generalized linear function, the paper [11] means a function that is p-harmonic on X_∞ with a constant function as its minimal p-weak upper gradient.

In this paper, we extend the study of asymptotic behavior of Lipschitz functions in [11] to functions of bounded variation in complete metric measure spaces equipped with a doubling measure supporting a 1-Poincaré inequality. The following theorems give a summary of the principal results of this paper; the precise versions can be found in the statements of the corresponding theorems in Sections 4–6.

Theorem A (Theorems 4.8 and 4.9)**.** *Let u be a function of bounded variation on X. For μ-a.e. $x \in X$ and any tangent space $(X_\infty, d_\infty, x_\infty, \mu_\infty)$ of X based at x, any limit function u_∞ as described above is 1-harmonic (also known as function of least gradient) and has quasi-constant minimal 1-weak upper gradient.*

The most fundamental of BV functions are characteristic functions of sets of finite perimeter. For these functions, the most interesting behavior happens solely at their jump points. Here the study of asymptotics is different, see e.g. [12, Theorem 5.13] in the Euclidean setting. Similarly, for general BV functions u, the approach of scaling the function as described above works well when considering points in X that asymptotically see neither the Cantor nor the jump parts of the variation measure $\|Du\|$ of u, but it is not helpful in the study of asymptotic behavior of u at points in its jump set S_u. Instead, an approach based on weak* limits of measures, which can also be used to define the limit function u_∞ as in Theorem 5.5, is more in line with studying the behavior of u at points in the jump set S_u of u and gives an alternative approach to Theorem A. This measure-theoretic approach is applied to characteristic functions of sets of finite perimeter in Sections 5 and 6 and the main conclusions are described in Theorems B and C below.

Theorem B (Theorem 5.5)**.** *Let $E \subset X$ be of finite perimeter $P(E, \cdot)$. Then for $P(E, \cdot)$-a.e. point, appropriately scaled versions of $P(E, \cdot)$ converge to a measure on X_∞ that is comparable to the co-dimension 1 Hausdorff measure.*

In $\mathbb{R}^n$, the corresponding limits are not just $(n-1)$-dimensional, but are hyperplanes that are boundaries of sets whose characteristic functions are functions of least gradient, that is, of minimal boundary surface. In the metric setting, we obtain an analogue with quasiminimal sets playing the role of hyperplanes and minimal boundary surfaces.

Theorem C (Theorem 6.3)**.** *Let $E \subset X$ be of finite perimeter. Then, with respect to the co-dimension 1 Hausdorff measure, almost every point x on the measure-theoretic boundary of E satisfies the following properties: fixing a (pointed) tangent space $(X_\infty, d_\infty, x_\infty, \mu_\infty)$ arising as a*

Gromov-Hausdorff limit of the scaled sequence (X_n, d_n, x, μ_n), and by passing to a subsequence if necessary we obtain that

- *the sequence of measures $\chi_E\, d\mu_n$ on X_n converges weakly* to a measure μ_∞^E on X_∞,*
- *this limit measure is absolutely continuous with respect to μ_∞,*
- *there is a set $(E)_\infty \subset X_\infty$ of finite perimeter such that $d\mu_\infty^E = \chi_{(E)_\infty}\, d\mu_\infty$,*
- *the set $(E)_\infty$ is of quasiminimal boundary surface (see Definition 6.2 or [25]), and*
- *the measure described in Theorem B is supported on the boundary of $(E)_\infty$, and is comparable to the perimeter measure $P((E)_\infty, \cdot)$ of $(E)_\infty$.*

Thus, beginning with extensions of the Cheeger's Rademacher theorem for doubling metric spaces with a Poincaré inequality to the functions of bounded variation, we recover important aspects of the classical theory of the boundaries of finite perimeter sets in $\mathbb{R}^n$.

The class of BV functions considered here is based on the notion first proposed by Miranda Jr. [32], and was further developed in [1, 5, 2]. The corresponding notion of a function of least gradient was studied in [25, 18, 28]. Just as [11] related asymptotic limits of Lipschitz functions to generalized linear functions (which are a priori p-harmonic for the indices $p > 1$ for which X supports a p-Poincaré inequality), we relate asymptotic limits of BV functions to functions of least gradient when the point of asymptoticity does not lie in the set where the jump and Cantor parts of the variation measure live. Additionally, at almost every point with respect to the co-dimension 1 Hausdorff measure in the measure-theoretic boundary of the set of finite perimeter, we relate the asymptotic limit of that set to sets of finite perimeter that have a quasiminimal boundary surface as in [25].

In the setting of Heisenberg groups (perhaps the simplest non-Riemannian example of the type of metric measure spaces studied here), more is known of the asymptotic behavior of BV functions; the key papers to study this setting are those of Magnani [31], Franchi, Serapioni and Serra-Cassano [13], and Ambrosio, Ghezzi and Magnani [4]. It is shown in [13, Theorem 4.1] that asymptotic limits of sets of finite perimeter in a Heisenberg group, based at a reduced boundary point of that set, is a Euclidean (vertical) half-space with the boundary plane parallel to the non-horizontal direction. Studies of asymptotic limits of sets of finite perimeter in more general step-2 Carnot groups can be found in [14], and for more general Carnot groups in [15]. While the Heisenberg groups are topologically Euclidean, there are more sets of finite perimeter in the Heisenberg sense than in the Euclidean sense, see [13, Proposition 2.15]. The papers [13, 14, 15] rely on the group structure on the Carnot groups, and so they do not address the case of more general Carnot-Carathéodory spaces.

Carnot-Carathéodory spaces are (locally) doubling metric measure spaces supporting a 1-Poincaré inequality, and hence the results of the present paper also apply there. Note that tangent spaces of Carnot-Carathéodory spaces are topological groups equipped with dilation operations, and if the tangent space is based at a regular point of the Carnot-Carathéodory space, then it is a nilpotent group equipped with a dilation, see [33, 7, 30]. Under further assumptions on the Carnot-Carathéodory space (which lead to knowing that the tangent spaces are all Carnot groups), a similar asymptoticity study is undertaken in [4]. We point out here that the results in the current paper are applicable to all Carnot-Carathéodory spaces of topological dimension at least 2.

If ν is a Radon measure on X and $x \in X$, then for almost every $r > 0$ we know that $\nu(\overline{B}(x,r) \setminus B(x,r)) = 0$. If X is a geodesic space and μ is a doubling measure, then $\mu(\overline{B}(x,r) \setminus B(x,r)) = 0$ for *each* $r > 0$ and $x \in X$, see [9]. In this paper, we will assume that X is geodesic in order to simplify many of the proofs (by avoiding the discussion of having to slightly adjust the radius r in order to ensure that $\mu(\overline{B}(x,r) \setminus B(x,r)) = 0$), but our results hold also in spaces that are *not* geodesic by an easy (but notationally cumbersome) modification discussed in Section 2 below.

The structure of this paper is as follows. In Section 2 we give the basic definitions necessary for the study of sets of finite perimeter and functions of bounded variation on metric measure spaces. In Section 3 we discuss pointed measured Gromov-Hausdorff limits. In Section 4 we show the results stated above regarding that asymptotic limits of BV functions converge to a function of least gradient (1-harmonic) in the tangent space, see Theorem 4.9. In Section 5 we discuss asymptotic limits of a set of finite perimeter, and show that for co-dimension 1 almost every point on the measure-theoretic boundary of that set we have a tangential behavior of the set; more specifically, there is a Gromov-Hausdorff type limit $(E)_\infty$ of the set E at such a point, and this limit is a set of (locally) finite perimeter; this is the content of Theorem 5.5. We also verify certain geometric structural regularity of these limit sets, see Theorem 5.4. The final section of this paper is devoted to the discussion on asymptotic minimality for sets of finite perimeter. In Theorem 6.3 we show that these limit sets $(E)_\infty$ are sets of quasiminimal boundary surfaces.

2 Notation and definitions

Here we lay out the main definitions and assumptions for this paper. Much of the terminology will be similar to that used in [1, 5, 32].

We assume that (X, d, μ) is a complete metric measure space with $\operatorname{diam} X > 0$, that is, X consists of at least two points. We use the notation $B(x, r)$ for the open ball centered at $x \in X$ and of radius $r > 0$. If we wish to be specific that the ball is in the metric space X, we write $B_X(x, r)$. Given a ball $B = B(x, r)$, we sometimes denote $\operatorname{rad} B := r$; note that in metric spaces, a ball (as a set) does not necessarily have a unique center and radius, but we understand these to be prescribed for all balls that we consider. We will always assume that μ is *doubling*: there is a constant $C_d \geq 1$ such that for all $x \in X$ and $r > 0$,

$$0 < \mu(B(x, 2r)) \leq C_d \mu(B(x, r)) < \infty.$$

By iterating the doubling condition, we obtain for any $x \in X$ and any $y \in B(x, R)$ with $0 < r \leq R < \infty$ that

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \geq \frac{1}{C_d^2} \left(\frac{r}{R}\right)^Q, \tag{2.1}$$

where $Q > 1$ only depends on the doubling constant C_d.

When a property holds outside a set of μ-measure zero, we say that it holds for μ-a.e. $x \in X$. As complete doubling metric spaces are proper, every closed and bounded set is compact. Given an open set $W \subset X$, we take $\mathrm{Lip}_{\mathrm{loc}}(W)$ to be the space of functions on W that are Lipschitz on every closed and bounded subset of W, and $L^1_{\mathrm{loc}}(W)$ to be the space of functions integrable with respect to μ on every closed and bounded subset of W.

Given a rectifiable curve $\gamma\colon [0,1] \to X$, we define the length of γ to be

$$\ell(\gamma) := \sup \sum_i d(\gamma(t_i), \gamma(t_{i+1}))$$

where the supremum is taken over all finite partitions $\{t_i\}$ of $[0,1]$. We will always assume that X is a *geodesic space*: for all x, y in X,

$$d(x,y) = \min \ell(\gamma)$$

where the minimum is taken over all curves γ joining x to y and is achieved.

Given a function $u\colon X \to \mathbb{R}$, an *upper gradient* g of u is a nonnegative Borel function such that for every $x, y \in X$ and every rectifiable curve γ containing x and y, we have the inequality

$$|u(x) - u(y)| \le \int_\gamma g\, ds, \tag{2.2}$$

where ds is arc length (see [21] for more information and standard results about upper gradients).

We say that a family of rectifiable curves Γ is of zero p-modulus, for $1 \le p < \infty$, if there is a nonnegative Borel function $\rho \in L^p(X)$ such that for all curves $\gamma \in \Gamma$, the curve integral $\int_\gamma \rho\, ds$ is infinite. If g is a nonnegative μ-measurable function on X and (2.2) holds for all curves apart from a family with zero p-modulus, we say that g is a p-weak upper gradient of u. It is known that if a function u on X has an upper gradient in $L^p(X)$, then there exists a minimal p-weak upper gradient of u, denoted by g_u, satisfying $g_u \le g$ a.e. for any p-weak upper gradient $g \in L^p(X)$ of u, see [8, Theorem 2.25].

We always assume that the space X supports a 1-*Poincaré inequality*. We say that X supports a p-Poincaré inequality, for $1 \le p < \infty$, if there is a constant $C_P > 0$ so that for every $u \in \mathrm{Lip}_{\mathrm{loc}}(X)$, every upper gradient g of u, and every ball $B = B(x,r)$,

$$\fint_B |u - u_B|\, d\mu \le C_P r \left(\fint_B g^p\, d\mu \right)^{1/p},$$

where

$$u_B := \fint_B u\, d\mu := \frac{1}{\mu(B)} \int_B u\, d\mu.$$

We will sometimes suppress the “1-” when discussing the inequality. We will denote by $C \ge 1$ a generic constant that only depends on the doubling and Poincaré constants C_d, C_P, and whose precise value may change even in the same line.

It can be noted that we could only assume that X supports a *weak* 1-Poincaré inequality (involving a ball dilated by a constant factor $\lambda > 1$ on the right-hand side), and drop the assumption that the space is geodesic. The reason for this is as follows: a weak 1-Poincaré inequality implies that the space is quasiconvex, and then a bi-Lipschitz change in the metric will allow the space to become geodesic. In geodesic spaces, a weak Poincaré inequality can be improved to become strong. This is discussed in [19]. The class of functions of bounded variation, to be defined shortly, is invariant under a bi-Lipschitz metric change. Thus the assumptions of geodecicity and the strong version of the Poincaré inequality are not restrictions, only conveniences. This bi-Lipschitz change in the metric on X would induce a bi-Lipschitz change in the tangent space X_∞, with a bi-Lipschitz equivalent geodesic limit metric on X_∞ obtained as a limit of re-scaled geodesics metrics on X. We obtain that the asymptotic limit function u_∞ as in Theorem 4.9 is of least gradient with respect to this length metric on X_∞, and therefore is of quasi-least gradient with respect to the original metric on the tangent space X_∞.

We now wish to discuss functions of bounded variation and sets of finite perimeter in the metric space (X, d, μ). The definitions are quite different than those typically used for $X = \mathbb{R}^n$; see [32] for discussion relating these to the classical definitions. Many results from [32] and [1] will be used (and cited) in what follows. For $u \in \mathrm{Lip}_{\mathrm{loc}}(X)$, we define

$$\mathrm{lip}\, u(x) := \liminf_{r\to 0} \frac{\sup_{y\in B(x,r)} |u(y)-u(x)|}{r},$$

often known as the *lower Lipschitz constant* of u at x. It is well known that $\mathrm{lip}\, u$ is an upper gradient of u (see [32, Section 2], for example). We also define

$$\mathrm{Lip}\, u(x) := \limsup_{r\to 0} \frac{\sup_{y\in B(x,r)} |u(y)-u(x)|}{r},$$

the *upper Lipschitz constant* of u at x.

Since $\mathrm{Lip}_{\mathrm{loc}}(X)$ is dense in $L^1_{\mathrm{loc}}(X)$, we define the *total variation* of $u \in L^1_{\mathrm{loc}}(X)$ on an open set $W \subset X$ as

$$V(u,W) := \inf\left\{ \liminf_{i\to\infty} \int_W \mathrm{lip}\, u_i \, d\mu :\ u_i \in \mathrm{Lip}(W),\ u_i \to u \text{ in } L^1_{\mathrm{loc}}(W) \right\}.$$

A function $u \in L^1_{\mathrm{loc}}(X)$ is said to be of *locally bounded variation* if $V(u,W)$ is finite for all bounded open $W \subset X$. A function u is said to be of *bounded variation* if $V(u,X)$ is finite. Let $BV(X)$ denote the set of functions of bounded variation. For an arbitrary set $A \subset X$, we define

$$V(u,A) := \inf\{V(u,W) :\ A \subset W,\ W \subset X \text{ is open}\}.$$

If $V(u,X) < \infty$, then $V(u,\cdot)$ is a Radon measure on X by [32, Theorem 3.4], called the variation measure. In much of current literature on BV functions in metric setting, $V(u,A)$ is also denoted $\|Du\|(A)$. In a significant part of the current literature on BV functions in metric spaces a slightly different notion of $V(u,W)$ is used, where instead of infimum over $\int_W \mathrm{lip}\, u_i \, d\mu$ the infimum of the integrals $\int_W g_{u_i} \, d\mu$ is considered, where g_{u_i} is the minimal 1-weak upper gradient of u_i, see for example [25]. It follows from [2] that these notions all give the same BV class as well as the *same* BV energy $V(u,W)$ for open sets W (and hence all Borel sets). Thus we can equivalently define

$$V(u,W) := \inf\left\{ \liminf_{i\to\infty} \int_W g_{u_i} \, d\mu :\ u_i \in \mathrm{Lip}_{\mathrm{loc}}(W),\ u_i \to u \text{ in } L^1(W) \right\}.$$

Let $E \subset X$ and let χ_E denote the characteristic function of E. If χ_E is of locally bounded variation we say that E is of *locally finite perimeter* and if χ_E is of bounded variation, we say that E is of *finite perimeter*. We use $P(E,\cdot) := V(\chi_E,\cdot)$ for the *perimeter measure*.

The following coarea formula is proven in [32, Proposition 4.2]: if $u \in BV(X)$ and $W \subset X$ is a Borel set, then

$$V(u,W) = \int_{-\infty}^{\infty} P(\{u>t\},W)\, dt. \tag{2.3}$$

Applying the 1-Poincaré inequality to approximating functions, we get for any μ-measurable set $E \subset X$ and any ball $B = B(x,r)$ the *relative isoperimetric inequality*

$$\min\{\mu(B(x,r)\cap E), \mu(B(x,r)\setminus E)\} \le 2C_P r P(E, B(x,r)), \tag{2.4}$$

see e.g. [27, Theorem 3.3].

The 1-Poincaré inequality implies the so-called Sobolev-Poincaré inequality, see e.g. [8, Theorem 4.21], from which we get the following BV version: for every ball $B(x,r)$ and every $u \in L^1_{\mathrm{loc}}(X)$, we have

$$\left(\fint_{B(x,r)} |u - u_{B(x,r)}|^{Q/(Q-1)}\, d\mu\right)^{(Q-1)/Q} \le Cr\frac{V(u,B(x,2r))}{\mu(B(x,2r))}, \tag{2.5}$$

where Q is the exponent from (2.1).

Moreover, we have the following Poincaré inequality for functions vanishing outside a ball. For any ball $B(x,r)$ with $0 < r < \frac{1}{4}\operatorname{diam} X$ and any $u \in L^1(B(x,r))$ with compact support in $B(x,r)$, we have

$$\int_{B(x,r)} |u|\, d\mu \le CrV(u,B(x,r)); \tag{2.6}$$

this again follows by applying the analogous inequality for Lipschitz functions (see [8, Theorem 4.21, Theorem 5.51]) to an approximating sequence.

For a set $E \subset X$, the *measure-theoretic boundary* is defined as the set of points of positive upper density for E and $X \setminus E$:

$$\partial^* E := \left\{x \in X : \limsup_{r\to 0} \frac{\mu(B(x,r)\cap E)}{\mu(B(x,r))} > 0 \quad \text{and} \quad \limsup_{r\to 0} \frac{\mu(B(x,r)\setminus E)}{\mu(B(x,r))} > 0\right\}.$$

We will also be interested in co-dimension 1 Hausdorff measures on X. Recall that μ is *Ahlfors s-regular* for $s > 0$ if there is some constant $C_A \ge 1$ such that whenever $x \in X$ and $0 < r < \frac{1}{2}\operatorname{diam} X$,

$$\frac{r^s}{C_A} \le \mu(B(x,r)) \le C_A r^s. \tag{2.7}$$

If μ is Ahlfors s-regular, then the co-dimension 1 Hausdorff measure defined below is just (comparable to) the $(s-1)$-dimensional Hausdorff measure. We do not wish to always assume Ahlfors regularity, however. We define the *co-dimension* 1 *Hausdorff measure* of a set $E \subset X$ by

$$\mathcal{H}(E) := \sup_{\delta>0} \mathcal{H}_\delta(E),$$

where for $\delta > 0$,

$$\mathcal{H}_\delta(E) := \inf\left\{\sum_{i\in I} \frac{\mu(B_i)}{r_i} : B_i = B(x_i,r_i),\, r_i \le \delta,\, E \subset \bigcup_{i\in I} B_i\right\}.$$

The following density results can be proved similarly as in [6, Theorem 2.4.3].

Lemma 2.1. *Let ν be a Radon measure on X, let $A \subset X$, and let $t > 0$. Then the following hold:*

$$\textit{if} \quad \limsup_{r\to 0} r\frac{\nu(B(x,r))}{\mu(B(x,r))} \ge t \quad \textit{for all } x \in A, \textit{ then} \quad \nu(A) \ge t\mathcal{H}(A)$$

and

$$\textit{if} \quad \limsup_{r\to 0} r\frac{\nu(B(x,r))}{\mu(B(x,r))} \le t \quad \textit{for all } x \in A, \textit{ then} \quad \nu(A) \le C_d t\mathcal{H}(A).$$

Let $E \subset X$ be a set of finite perimeter. We know that for any Borel set $A \subset X$,

$$P(E,A) = \int_{\partial^* E \cap A} \theta_E \, d\mathcal{H}, \tag{2.8}$$

where $\theta_E \colon X \to [\alpha, C_d]$ with $\alpha = \alpha(C_d, C_P) > 0$, see [1, Theorem 5.3] and [5, Theorem 4.6]. Furthermore, let

$$\Sigma_\gamma E := \left\{ x \in X : \liminf_{r \to 0} \min \left\{ \frac{\mu(B(x,r) \cap E)}{\mu(B(x,r))}, \frac{\mu(B(x,r) \setminus E)}{\mu(B(x,r))} \right\} \geq \gamma \right\} \tag{2.9}$$

for a constant $\gamma \in (0, 1/2]$ depending only on C_d, C_P. Note that $\Sigma_\gamma E \subset \partial^* E$; by [1, Theorem 5.4] we know that conversely,

$$\mathcal{H}(\partial^* E \setminus \Sigma_\gamma E) = 0. \tag{2.10}$$

Lemma 2.2. *Let $E \subset X$ be a set of finite perimeter. Then for $\mathcal{H}$-a.e. $x \in \partial^* E$ (and thus $P(E, \cdot)$-a.e. $x \in \partial^* E$),*

$$\frac{\gamma}{C_P} \leq \liminf_{r \to 0} \frac{P(E, B(x,r))}{\mu(B(x,r))/r} \leq \limsup_{r \to 0} \frac{P(E, B(x,r))}{\mu(B(x,r))/r} \leq C_d. \tag{2.11}$$

Proof. The first inequality holds for every $x \in \Sigma_\gamma E$ by the relative isoperimetric inequality (2.4). To show the second inequality, note that if $A \subset \partial^* E$ and $\varepsilon > 0$ are such that

$$\limsup_{r \to 0} r \frac{P(E, B(x,r))}{\mu(B(x,r))} \geq C_d + \varepsilon$$

for all $x \in A$, then by the first part of Lemma 2.1 and by (2.8), we have $P(E, A) \geq (C_d + \varepsilon)\mathcal{H}(A)$. However, according to (2.8), we have $P(E, A) \leq C_d \mathcal{H}(A)$. Thus we must have $\mathcal{H}(A) = 0$. □

The lower and upper approximate limits of a function u on X are defined respectively by

$$u^\wedge(x) := \sup \left\{ t \in \mathbb{R} : \lim_{r \to 0} \frac{\mu(B(x,r) \cap \{u < t\})}{\mu(B(x,r))} = 0 \right\}$$

and

$$u^\vee(x) := \inf \left\{ t \in \mathbb{R} : \lim_{r \to 0} \frac{\mu(B(x,r) \cap \{u > t\})}{\mu(B(x,r))} = 0 \right\}.$$

The *jump set* S_u is defined to be the set where $u^\wedge < u^\vee$.

By [5, Theorem 5.3], the variation measure of a BV function can be decomposed into the absolutely continuous and singular part, and the latter into the Cantor part and jump part, as follows. Given $u \in BV(X)$, we have for any Borel set $A \subset X$

$$\begin{aligned} V(u,A) &= V_a(u,A) + V_s(u,A) \\ &= V_a(u,A) + V_c(u,A) + V_j(u,A) \\ &= \int_A g \, d\mu + V_c(u,A) + \int_{A \cap S_u} \int_{u^\wedge(x)}^{u^\vee(x)} \theta_{\{u>t\}}(x) \, dt \, d\mathcal{H}(x), \end{aligned}$$

where $g \in L^1(X)$ is the density of the absolutely continuous part and the functions $\theta_{\{u>t\}}$ are as in (2.8).

We denote by $BV_c(X)$ the class of BV functions with compact support in X.

Definition 2.3. We say that $u \in BV(X)$ is a *function of least gradient* if for all $\varphi \in BV_c(X)$,

$$V(u, \operatorname{supp} \varphi) \leq V(u + \varphi, \operatorname{supp} \varphi). \tag{2.12}$$

3 Pointed measured Gromov-Hausdorff limits

In this section we consider tangent spaces of a metric space at a given point. For this, we first need to specify what is meant by the convergence of metric spaces. Existing literature has some slightly different definitions and diverging terminology; here we describe them and provide brief explanation on how these are equivalent.

Definition 3.1. We say that the sequence of pointed metric spaces (Y_n, d_n, y_n) *converges in the pointed Gromov-Hausdorff distance* to the space $(Y_\infty, d_\infty, y_\infty)$ if for each positive integer n there is a map $\phi_n : Y_\infty \to Y_n$ so that $\phi_n(y_\infty) = y_n$, and for each $R > 0$ and $\epsilon > 0$ there is a positive integer $N_{\epsilon,R}$ such that whenever $k \geq N_{\epsilon,R}$, we have

1. $\sup_{x,y\in B_{Y_\infty}(y_\infty,R)} |d_{Y_k}(\phi_k(x), \phi_k(y)) - d_{Y_\infty}(x, y)| < \epsilon$,
2. $B_{Y_k}(y_k, R - \epsilon) \subset \bigcup_{y\in\phi_k(B_{Y_\infty}(y_\infty,R))} B_{Y_k}(y, \epsilon)$.

Note that these maps are not required to be continuous, or even measurable. It is possible to modify ϕ_n to be measurable, but this is technical, and not necessary for our presentation below.

Remark 3.2. The above definition is compatible with those of [10, 24]. In [10, Definition 8.1.1] and [21, Chapter 11] the following definition of pointed Gromov-Hausdorff convergence was considered: For all $r > 0$ and all $0 < \epsilon < r$ there exists an $n_0 = n_0(r, \epsilon)$ such that for all $n \geq n_0$ there exist functions $\phi^\epsilon_n \colon B_{Y_\infty}(y_\infty, r) \to Y_n$ with

1. $\phi^\epsilon_n(y_\infty) = y_n$,
2. $|d_n(\phi^\epsilon_n(x), \phi^\epsilon_n(y)) - d_\infty(x, y)| < \epsilon$ for all $x, y \in B_{Y_\infty}(y_\infty, r)$,
3. $B_{Y_n}(y_n, r - \epsilon) \subset \bigcup_{y\in\phi^\epsilon_n(B_{Y_\infty}(y_\infty,r))} B_{Y_n}(y, \epsilon)$.

See [22] for more on pointed Gromov-Hausdorff convergence. To see the compatibility between these two definitions we note that the scales R and ϵ play the role of localizing the convergence of the tangent spaces. Thus, the second notion is implied by the first, as seen by the choice $\phi^\epsilon_n := \phi_n$. Conversely, given ϕ^ϵ_n, choosing a sequence of R_n monotonically increasing to ∞ and ϵ_n monotonically decreasing to 0, we can even choose ϕ^ϵ_n to be independent of ϵ and r; hence the equivalence of the notion of [10] with ours. However, in proofs it is often easier to work with the localized versions ϕ^ϵ_n, since it avoids this additional diagonal argument. Where we wish to use globally defined functions, we use ϕ_n. These are interchangeable.

The notion considered in [24] is also equivalent to the above. Since this notion of [24, Definition 2 and Definition 7] is also useful in this paper, especially in defining notions of weak convergence of measures to tangent spaces, we now provide that definition as well. According to [24], the sequence (Y_n, d_n, y_n) converges to a proper space $(Y_\infty, d_\infty, y_\infty)$ if there is a proper metric space (Z, d_Z) and a point $z_0 \in Z$, an isometric embedding $\iota : Y_\infty \to Z$, and for each $n \in \mathbb{N}$ there is an isometric embedding $\iota_n : Y_n \to Z$, such that $\iota(y_\infty) = z_0 = \iota_n(y_n)$ and for each $R > 0$,

1. $\lim_{n\to\infty} \sup_{y\in B_{Y_n}(y_n,R)} \mathrm{dist}_Z(\iota_n(y), \iota(Y_\infty)) = 0$,
2. $\lim_{n\to\infty} \sup_{z\in B_{Y_\infty}(y_\infty,R)} \mathrm{dist}_Z(\iota(z), \iota(Y_n)) = 0$.

From this definition we see that whenever $R, \varepsilon > 0$ there is some positive integer $N_{\varepsilon,R}$ such that whenever $n > N_{\varepsilon,R}$, for each $x, y \in B_{Y_n}(y_n, R)$ we can find $\widehat{x}, \widehat{y} \in B_{Y_\infty}(y_\infty, R+\varepsilon)$ such that

$$\max\{d_Z(\iota_n(x), \iota(\widehat{x})), d_Z(\iota_n(y), \iota(\widehat{y}))\} < \varepsilon, \quad |d_{Y_n}(x,y) - d_{Y_\infty}(\widehat{x}, \widehat{y})| < 3\varepsilon.$$

We also have that for $R > 0$ and $\varepsilon > 0$ there is some positive integer $N_{\varepsilon,R}$ such that for $n > N_{\varepsilon,R}$, whenever $x, y \in B_{Y_\infty}(y_\infty, R)$ there exist $x_n, y_n \in B_{Y_n}(y_n, R+\varepsilon)$ such that

$$\max\{d_Z(\iota_n(x_n), \iota(x)), d_Z(\iota_n(y_n), \iota(y))\} < \varepsilon, \quad |d_{Y_n}(x_n, y_n) - d_{Y_\infty}(x,y)| < 3\varepsilon.$$

This shows that the definition of [24] implies our definition above. The fact that our definition implies the one of [24] comes from the construction of the ambient space Z found in [22], where the space Z should be considered to be the completion of the "disjoint union" space Y found in [22, Section 4.1.1].

Indeed, we can construct the maps ι_n and ι from the maps ϕ_n and vice versa so that the following compatibility condition between these two classes of maps is satisfied: For all $r > 0$,

$$\lim_{n\to\infty} \sup_{y \in B_{Y_\infty}(y_\infty, r)} d_Z(\iota_n \circ \phi_n(y), \iota(y)) = 0. \tag{3.1}$$

For simplicity, and avoiding modifying the space Z, as well as the approximating maps ϕ_n, we will generally fix them throughout the exposition below. In order to define other notions, such as convergence of points, curves and functions, passing to a subsequence in n may be necessary. However, this subsequence will be of n and will not require coming up with new ϕ_n or embedding space Z.

In the light of the above discussion, we can say that a sequence, $z_n \in Y_n$, converges to $z \in Y_\infty$ if $\lim_{n\to\infty} d_Z(\iota_n(z_n), \iota(z)) = 0$, and see that every $z \in Y_\infty$ is a limit of a sequence $z_n \in Y_n$ as here. By a not-terrible abuse of notation we denote this by

$$\lim_{n\to\infty} z_n = z.$$

Next, we define pointed *measured* Gromov-Hausdorff convergence. For this, we use the embeddings described in the above remark. First consider a sequence of Borel measures ν_n on a metric space Z. The measures ν_n converge weakly* to a Borel measure ν on Z if

$$\int_Z \phi \, d\nu_n \to \int_Z \phi \, d\nu$$

as $n \to \infty$ for all boundedly supported continuous functions ϕ on Z. We denote this convergence by $\nu_n \overset{*}{\rightharpoonup} \nu$.

To define measured Gromov-Hausdorff convergence, we consider the push-forward measures

$$\iota_{n,*}\nu_n(A) := \nu_n(\iota_n^{-1}(A)).$$

We say that the sequence of Radon measures ν_n on Y_n converges to a Radon measure ν_∞ on Y_∞, denoted $\nu_n \overset{*}{\rightharpoonup} \nu_\infty$, if $\iota_{n,*}\nu_n \overset{*}{\rightharpoonup} \iota_*\nu_\infty$ on Z.

Definition 3.3. We say that a sequence of pointed metric measure spaces (Y_n, d_n, y_n, ν_n) converges pointed measured Gromov-Hausdorff to a space $(Y_\infty, d_\infty, y_\infty, \nu_\infty)$, if the sequence converges in the pointed Gromov-Hausdorff sense, and

$$\nu_n \overset{*}{\rightharpoonup} \nu_\infty.$$

Since Z is a proper metric space, it follows that whenever $\sup_n \nu_n(\iota_n^{-1}(Z)) < \infty$, there is a subsequence ν_{n_k} and a Radon measure $\widehat{\nu_\infty}$ on Z such that $\iota_{n_k,*}\nu_{n_k} \overset{*}{\rightharpoonup} \widehat{\nu_\infty}$ in Z. This limit measure must have support in $\iota(Y_\infty)$, since the support of ν_∞ is contained in the limit of the supports of ν_{n_k}. Indeed, given $\varepsilon > 0$ and a radius $R > 0$, we know that for large n the set $\iota_n(B_{Y_n}(y_n, R))$ is in an 3ε-neighborhood of $\iota(Y_\infty)$. Recall that Y_∞ is a proper metric space. We call such measures ν_∞ limit measures of the sequence ν_{n_k}, and they may depend on the choice of the subsequence; the full sequence ν_n may not converge to ν_∞. In the proofs below, we will always pass to the subsequence where this limit holds.

Lemma 3.4. *In the above situation, $\widehat{\nu_\infty}(Z \setminus \iota(Y_\infty)) = 0$, and hence there is a Radon measure ν_∞ on Y_∞ such that $\widehat{\nu_\infty} = \iota_*\nu_\infty$.*

Definition 3.5. Let $x \in X$ and let $r_n > 0$ with $r_n \to 0$. Define the sequence of scaled metrics d_n on X by

$$d_n(y, z) := \frac{d(y, z)}{r_n},$$

and the scaled measures

$$\mu_n := \frac{1}{\mu(B(x, r_n))}\, \mu.$$

If the sequence $(X_n, d_n, x, \mu_n) := (X, d_n, x, \mu_n)$ converges to $(X_\infty, d_\infty, x_\infty, \mu_\infty)$ in the pointed measured Gromov-Hausdorff sense, then we say that X_∞ is a *tangent space to X at x, with tangent measure μ_∞.*

We know that if μ is doubling and X is a complete geodesic space, then by passing to a subsequence of (X, d_n, x, μ_n) if necessary, we will always have a tangent metric measure space as above, which is also geodesic, see [16, Section 6] or the discussion in [21, Section 11]. Note that points of distance less than r_n from x in (X, d) are, in the space X_n, at distance less than 1 from x, and that the ball $B(x, r_n)$ has μ_n-measure 1. The tangent space may be non-unique, and depends on the subsequence chosen.

From the work of [24] we know that if μ is doubling and supports a 1-Poincaré inequality, then for every $x \in X$, all the corresponding tangent spaces have the tangent measure be doubling and support a 1-Poincaré inequality, with the doubling and Poincaré constants depending quantitatively only on the corresponding constants for X, see also [21]. A proof of this first appeared in the work [24] of Keith, but he reports in [24] that it was independently found by himself, Koskela, and Cheeger.

We will fix the following notion of a limit of functions.

Definition 3.6. We say that a function u_∞ on X_∞ *is a limit of* u_n (with u_n a function on X_n) if there exists some subsequence n_k and $\epsilon_k \searrow 0$ such that for all $r > 0$

$$\lim_{k\to\infty} \|u_\infty - u_{n_k} \circ \phi_{n_k}^{\epsilon_k}\|_{L^\infty(B_{X_\infty}(x_\infty, r))} = 0. \tag{3.2}$$

This is equivalent to the following definition of limits using globally defined maps ϕ_k:

$$\lim_{k\to\infty} \|u_\infty - u_{n_k} \circ \phi_k\|_{L^\infty(B_{X_\infty}(x_\infty, r))} = 0. \tag{3.3}$$

Given a sequence of functions u_n that are uniformly Lipschitz and locally uniformly bounded, one can extract some subsequence and find a limit function u_∞. For example, by considering first the convergence of the values of u_n along some sequence of ever-denser nets of X_n converging to nets in X_∞ and diagonalizing. This is easier to see in terms of the globally defined ϕ_n, but can also be done via a detailed diagonal argument with ϕ_n^ϵ. This idea can also be seen from the point of view of the definition of [24]. For each n we can find a Lipschitz extension $\widehat{u_n}$ of $u_n \circ \iota_n^{-1}$ from $\iota_n(X_n)$ to Z. This sequence forms an equibounded and equicontinuous sequence of functions in Z which, being a proper space, lends itself to an application of the Arzelà-Ascoli theorem. Thus we may find a subsequence of $\widehat{u_n}$ that converges locally uniformly to a Lipschitz function $\widehat{u_\infty}$ on Z. We can now choose $u_\infty = \widehat{u_\infty} \circ \iota$. That this choice of u_∞ is a limit of u_n follows from the compatibility condition 3.1.

The notion of limit of functions as given above is concordant with the notion of limit of measures. If u_n and u_∞ are uniformly Lipschitz and u_∞ is a limit of u_{n_k}, then along the same subsequence $\nu_{n_k} := u_{n_k}\, d\mu_{n_k} \overset{*}{\rightharpoonup} \nu_\infty := u_\infty\, d\mu_\infty$.

As before, these functions depend on the subsequence chosen. In fact, there are several dependencies. The very tangent spaces depend on the subsequence of blow ups $r_n \searrow 0$, as well as on the mappings $\iota_n, \phi_n^{\epsilon_k}$ which are not canonical. We will always assume that all these choices are given to us.

4 Asymptotics at approximate continuity points and generalized linear functions

The goal of this section is to study the asymptotic behavior of BV functions at points outside the jump and Cantor parts of their variation measures. We start with the following handy lemma.

Lemma 4.1. *Let $u, v \in BV(X)$. Suppose $E \subset X$ is a Borel set such that for each $x \in E$ we have*

$$\lim_{r\to 0^+} \frac{\mu(B(x,r)\cap E)}{\mu(B(x,r))} = 1,$$

and $u(x) = v(x)$. Then $V(u-v, E) = 0$ and so for each $A \subset E$ we have $V(u, A) = V(v, A)$.

Proof. From the above, we know that for each $t \in \mathbb{R}$ the set $E \cap \partial^* E_t$ is empty, where $E_t = \{x \in X : u(x) - v(x) > t\}$. Therefore by the coarea formula (2.3) and by (2.8), the claim follows. □

In the Euclidean setting we know that for $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$ (where $\mathcal{L}^n$ denotes the n-dimensional Lebesgue measure) a BV function converges under blow-up to a linear function (see e.g. [3, Theorem 3.83]). In the metric setting, for $p > 1$ the notion of linear function is interpreted as a function that is constant or else satisfies the following two properties: (a) the image of X under the function is $\mathbb{R}$, and (b) the minimal p-weak upper gradient of the function is constant (and given that we have a Poincaré inequality, this constant should be non-zero if the function is not the constant function); see for example [11]. It was shown in [11] that given a Lipschitz function, any asymptotic limit of that function at almost every point yields such a linear function on the corresponding tangent space, which we defined in Section 3. In the case $p = 1$, which is the natural setting for BV functions, we will prove that the asymptotic limits are the so-called *generalized linear functions* on the tangent spaces.

For $g \in L^1_{\mathrm{loc}}(X)$ nonnegative and $R > 0$, we define the restricted maximal function of g at $x \in X$ by

$$\mathcal{M}_R g(x) := \sup_{0<r\le R} \fint_{B(x,r)} g\, d\mu.$$

The maximal function of a Radon measure ν is defined similarly by

$$\mathcal{M}_R\nu(x) := \sup_{0<r\le R} \frac{\nu(B(x,r))}{\mu(B(x,r))}.$$

Recall that if $u \in BV(X)$, then

$$dV(u,\cdot) = g\,d\mu + dV_s(u,\cdot),$$

where $g \in L^1(X)$ is the Radon-Nikodym derivative of $V(u,\cdot)$ with respect to μ and $V_s(u,\cdot)$ is the singular part.

Proposition 4.2. *Let $u \in BV(X)$. Then for μ-a.e. $x \in X$ for which $g(x) > 0$ there exists $R > 0$ and a set $A_x \not\ni x$ with density 0 at the point x such that $u|_{B(x,R)\setminus A_x}$ is Lipschitz with constant $Cg(x)$. For μ-a.e. $x \in X$ for which $g(x) = 0$, for each $\delta > 0$ there is a set $A_{x,\delta} \not\ni x$ with density 0 at x such that $u|_{B(x,R)\setminus A_{x,\delta}}$ is Lipschitz with constant δ.*

Proof. We follow the proof of [21, Proposition 13.5.2]. For μ-a.e. $x \in X$, we have

$$\lim_{r\to 0} \fint_{B(x,r)} |g - g(x)|\,d\mu = 0 \quad \text{and} \quad \lim_{r\to 0} \frac{V_s(u, B(x,r))}{\mu(B(x,r))} = 0 \tag{4.1}$$

by the Lebesgue-Radon-Nikodym theorem, see e.g. [21, Section 3.4]. Fix such $x \in X$ for which also $g(x) > 0$.

Let $R > 0$ and let $y, z \in B(x,R)$ be Lebesgue points of u. For nonnegative integers i we set $B_i := B(y, 2^{-i}d(y,z))$, and for negative integers i we set $B_i := B(z, 2^{i}d(y,z))$. Then, by the doubling property of μ and the Poincaré inequality,

$$\begin{aligned}
|u(y) - u(z)| &\le \sum_{i\in\mathbb{Z}} |u_{B_i} - u_{B_{i+1}}| \\
&\le C \sum_{i\in\mathbb{Z}} \fint_{2B_i} |u - u_{2B_i}|\,d\mu \\
&\le Cd(y,z) \sum_{i\in\mathbb{Z}} 2^{-|i|} \frac{V(u,2B_i)}{\mu(2B_i)} \\
&= Cd(y,z) \sum_{i\in\mathbb{Z}} 2^{-|i|} \left(\fint_{2B_i} g\,d\mu + \frac{V_s(u,2B_i)}{\mu(2B_i)} \right) \\
&\le Cd(y,z) \sum_{i\in\mathbb{Z}} 2^{-|i|} \left(\fint_{2B_i} |g - g(x)|\,d\mu + \frac{V_s(u,2B_i)}{\mu(2B_i)} \right) + Cd(y,z)g(x).
\end{aligned}$$

For $s > 0$, set

$$\tau_s := \sup_{0<r\le s} \left(\fint_{B(x,r)} |g - g(x)|\,d\mu + \frac{V_s(u,B(x,r))}{\mu(B(x,r))} \right).$$

Note that since μ is doubling, for any $s > 0$ and any Radon measure ν we have

$$\mathcal{M}_s\nu(y) \le C\mathcal{M}_{2d(x,y)}\nu(y) + C\mathcal{M}_{2s}\nu(x).$$

Applying this with $s = 2d(y,z) < 4R$ in the second inequality below, we get

$$\begin{aligned}
|u(y)-u(z)| &\le Cd(y,z)[g(x) + \mathcal{M}_{2d(y,z)}|g-g(x)|(y) + \mathcal{M}_{2d(y,z)}|g-g(x)|(z) \\
&\quad + \mathcal{M}_{2d(y,z)}V_s(u,\cdot)(y) + \mathcal{M}_{2d(y,z)}V_s(u,\cdot)(z)] \\
&\le Cd(y,z)[g(x) + \mathcal{M}_{2d(x,y)}|g-g(x)|(y) + \mathcal{M}_{2d(x,z)}|g-g(x)|(z) \\
&\quad + \mathcal{M}_{2d(x,y)}V_s(u,\cdot)(y) + \mathcal{M}_{2d(x,z)}V_s(u,\cdot)(z) + \mathcal{M}_{8R}[g-g(x)](x) + \mathcal{M}_{8R}V_s(u,\cdot)(x)] \\
&\le Cd(y,z)[g(x) + \mathcal{M}_{2d(x,y)}|g-g(x)|(y) + \mathcal{M}_{2d(x,z)}|g-g(x)|(z) \\
&\quad + \mathcal{M}_{2d(x,y)}V_s(u,\cdot)(y) + \mathcal{M}_{2d(x,z)}V_s(u,\cdot)(z) + \tau_{8R}]. \qquad (4.2)
\end{aligned}$$

We only consider $R > 0$ to be small enough so that $\tau_{8R} < g(x)$ (here we need the fact that $g(x) > 0$). We choose a sequence of radii $R_M \searrow 0$ as $M \to \infty$ such that $2^M \tau_{8R_M} < g(x)$ for each $M \in \mathbb{N}$. Next let A_M be the set of all points $y \in B(x, R_M)$ such that for some $0 < r \le 2d(x,y)$,

$$\fint_{B(y,r)} |g-g(x)|\, d\mu + \frac{V_s(u, B(y,r))}{\mu(B(y,r))} > 2^M \tau_{8R_M}.$$

For each $y \in A_M$ there is a ball $B(y, r_y)$ with $0 < r_y \le 2d(x,y) < 2R_M$ such that the above inequality holds, and so the family $\{B(y,r_y)\}_{y\in A_M}$ is a cover of A_M. By the 5-covering theorem we can extract a countable, pairwise disjoint subfamily $\mathcal{G}$ of the above family such that $A_M \subset \bigcup_{B\in\mathcal{G}} 5B$. If $\tau_{8R_M} = 0$, then $\mu(A_M) = 0$; else we see by the doubling property of μ that

$$\begin{aligned}
\mu(A_M) &\le C \sum_{B\in\mathcal{G}} \mu(B) \\
&\le \frac{C}{2^M \tau_{8R_M}} \sum_{B\in\mathcal{G}} \left(\int_B |g-g(x)|\, d\mu + V_s(u,B) \right) \\
&\le \frac{C}{2^M \tau_{8R_M}} \left(\int_{B(x,4R_M)} |g-g(x)|\, d\mu + V_s(u, B(x,4R_M)) \right) \\
&\le \frac{C\mu(B(x,4R_M))}{2^M \tau_{8R_M}} \left(\fint_{B(x,4R_M)} |g-g(x)|\, d\mu + \frac{V_s(u,B(x,4R_M))}{\mu(B(x,4R_M))} \right) \\
&\le \frac{C\mu(B(x,R_M))}{2^M \tau_{8R_M}} \tau_{4R_M} \le \frac{C\mu(B(x,R_M))}{2^M}.
\end{aligned}$$

We can add to each A_M all the non-Lebesgue points of u different from x, without adding measure. Now note that $x \notin A_M$ and that by (4.2), u is $Cg(x)$-Lipschitz in $B(x,R_M) \setminus A_M$. By choosing

$$A_x := \bigcup_{M=1}^{\infty} A_M \setminus B(x, R_{M+1}),$$

we see that u is $Cg(x)$-Lipschitz in $B(x,R_1)\setminus A_x$. Indeed, if $y, z \in B(x,R_1)\setminus A_x$ such that $y \ne x \ne z$, then there are positive integers M_1 and M_2 such that $y \in B(x, R_{M_1}) \setminus B(x, R_{M_1+1})$ with $y \notin A_{M_1}$ and $z \in B(x, R_{M_2}) \setminus B(x, R_{M_2+1})$ with $z \notin A_{M_2}$. We can assume that $M_2 > M_1$. It then follows from (4.2) that

$$|u(y)-u(z)| \le C\, d(y,z)[g(x) + 2^{M_1}\tau_{8R_{M_1}} + 2^{M_2}\tau_{8R_{M_2}} + \tau_{8R_{M_1}}] \le 4C\, g(x)\, d(y,z). \qquad (4.3)$$

Therefore u is $Cg(x)$-Lipschitz continuous in $B(x,R_1)\setminus(A_x\cup\{x\})$. The fact that u is approximately continuous at x, together with the fact that A_x has lower density zero at x (see the argument below), tells us that u is $Cg(x)$-Lipschitz continuous in $B(x,R_1)\setminus A_x$.

Moreover,

$$\frac{\mu(A_x\cap B(x,R_{M_0}))}{\mu(B(x,R_{M_0}))}\le C\sum_{M=M_0}^{\infty}\frac{\mu(A_M)}{\mu(B(x,R_M))}\le C\sum_{M=M_0}^{\infty}2^{-M}\to 0\quad\text{as } M_0\to\infty.$$

This guarantees that A_x has *lower* density 0 at x. On the other hand, by the choice of the covering of A_M by balls $B(y,r_y)$ with radius $r_y\le 2d(x,y)$, in the estimate for $\mu(A_M)$ obtained above we can in fact obtain for any $0<r\le R_M$ that

$$\mu(A_M\cap B(x,r))\le\frac{C}{2^M}\mu(B(x,4r))\le\frac{CC_d^2}{2^M}\mu(B(x,r)).$$

This guarantees that A_x has density 0 at x. Choosing $R=R_1$, we have proved the first claim.

Finally, if $\{x\in X:\ g(x)=0\}$ has positive measure, then the above argument gives that for μ-a.e. x in this set, for every $\delta>0$ there exists $A_{x,\delta}$ with $x\notin A_{x,\delta}$ and $A_{x,\delta}$ is of density 0 at x such that $u|_{B(x,R)\setminus A_{x,\delta}}$ is δ-Lipschitz. □

Lemma 4.3. *Let $u\in BV(X)$. For μ-a.e. $x\in X$ the following holds: if $A\subset X$ has density 0 at x, then*

$$\lim_{r\to 0}\frac{1}{r}\frac{1}{\mu(B(x,r))}\int_{A\cap B(x,r)}|u-u(x)|\,d\mu=0.$$

Proof. Excluding a μ-negligible set, we can take a Lebesgue point x of u such that (just as in (4.1))

$$\lim_{r\to 0}\frac{V(u,B(x,r))}{\mu(B(x,r))}=g(x). \tag{4.4}$$

By Hölder's inequality,

$$\frac{1}{r}\frac{1}{\mu(B(x,r))}\int_{A\cap B(x,r)}|u-u(x)|\,d\mu\le\frac{1}{r}\left(\fint_{B(x,r)}|u-u(x)|^{Q/(Q-1)}\,d\mu\right)^{(Q-1)/Q}\left(\frac{\mu(A\cap B(x,r))}{\mu(B(x,r))}\right)^{1/Q}.$$

Since x is a Lebesgue point of u, by the Sobolev-Poincaré inequality (2.5),

$$\begin{aligned}\left(\fint_{B(x,r)}|u-u(x)|^{Q/(Q-1)}\,d\mu\right)^{\frac{Q-1}{Q}}&\le\left(\fint_{B(x,r)}|u-u_{B(x,r)}|^{Q/(Q-1)}\,d\mu\right)^{\frac{Q-1}{Q}}\\&\quad+\sum_{j=1}^{\infty}\left(\fint_{B(x,r)}|u_{B(x,2^{-j+1}r)}-u_{B(x,2^{-j}r)}|^{Q/(Q-1)}\,d\mu\right)^{\frac{Q-1}{Q}}\\&\le Cr\frac{V(u,B(x,r))}{\mu(B(x,r))}+\sum_{j=1}^{\infty}|u_{B(x,2^{-j+1}r)}-u_{B(x,2^{-j}r)}|\\&\le Cr\frac{V(u,B(x,r))}{\mu(B(x,r))}+C\sum_{j=1}^{\infty}2^{-j+1}r\frac{V(u,B(x,2^{-j+1}r))}{\mu(B(x,2^{-j+1}r))}\\&\le Cr\mathcal{M}_rV(u,\cdot)(x).\end{aligned}$$

Thus we get

$$\frac{1}{r}\frac{1}{\mu(B(x,r))}\int_{A\cap B(x,r)}|u-u(x)|\,d\mu \le C\mathcal{M}_r V(u,\cdot)(x)\left(\frac{\mu(A\cap B(x,r))}{\mu(B(x,r))}\right)^{1/Q}.$$

Note that by (4.4), $\lim_{r\to 0}\mathcal{M}_r V(u,\cdot)(x) = g(x) < \infty$, and so by the fact that A has density 0 at x, we get the conclusion. □

Definition 4.4. Let v be a real-valued function on a metric space (Z,d). The *oscillation* of v in a ball $B(x,r)$ is

$$\operatorname*{osc}_{(x,r)} v := \sup_{y\in B(x,r)} \frac{|v(y)-v(x)|}{r}.$$

We also set

$$\mathrm{LIP}\, v := \sup_{y,z\in Z\,:\, y\neq z} \frac{|v(y)-v(z)|}{d(y,z)}.$$

Observe that $\operatorname*{osc}_{(x,r)} v \le \mathrm{LIP}\, v$.

We now return to the sequence X_n of zoomed-in versions of X, as defined in Section 3. For $u \in BV(X)$, we wish to study the limit of the functions

$$u_n(y) := \frac{u(y)-u(x)}{r_n}$$

that are defined on X_n. Suppose the point x satisfies the conclusion of Proposition 4.2. Zooming in and defining u_n as above, we note that u is only known to be Lipschitz continuous on $B(x,R)\setminus A_x$. This poses a problem for studying the supposed limit function u_∞. Though the set A_x from Proposition 4.2 has zero density at point x, it could still be very much in the image of the functions ϕ_n^ϵ used for comparing u_n with u_∞ (note that the ϕ_n^ϵ are not necessarily continuous). This could have the effect of a limit function u_∞ having little to do with the values of u outside A_x, a set which is quantitatively marginal as we have shown. It seems prudent to search for a limit function u_∞ that reflects the values of u on $B(x,R)\setminus A_x$ if we want to explore any properties of this limit function. With that in mind, we make the following definition:

Definition 4.5. For $x \in X$ for which the conclusion of Proposition 4.2 holds, we say that the functions ϕ_n in the definition of pointed measured Gromov-Hausdorff convergence are *adapted to* u if

$$\phi_n(B_{X_\infty}(x_\infty,R))\cap A_x = \emptyset$$

for all $n \ge N_{\epsilon,R}$.

Thanks to the following lemma we know that whenever $(X_\infty, d_\infty, x_\infty, \mu_\infty)$ is a tangent space to X at x as in Definition 3.5, we can always find a subsequence of the sequence (X_n, d_n, x, μ_n) such that the corresponding maps ϕ_n are adapted to u.

Lemma 4.6. *Suppose $u \in BV(X)$ and $x \in X$ is a point for which the conclusion of Proposition 4.2 holds. If $(X_\infty, d_\infty, x_\infty, \mu_\infty)$ is a pointed measured Gromov-Hausdorff limit of (X, d_n, x, μ_n) for some positive sequence $r_n \to 0$, then there exist functions ϕ_n that are adapted to u at x.*

Proof. Assume for simplicity that $R = 1/2$ and fix $0 < \epsilon \le 1$. By the doubling condition, we have that

$$\frac{\mu(B(x, 2s) \cap A_x)}{\mu(B(x, s))} \to 0 \quad \text{as } s \to 0. \tag{4.5}$$

By (2.1), there exists $Q > 1$ such that whenever $y \in B(x, s)$ and $0 < t \le s$,

$$\frac{1}{C}\left(\frac{t}{s}\right)^Q \le \frac{\mu(B(y, t))}{\mu(B(x, s))}.$$

It follows that if $y \in B(x, r_n)$ (i.e. $y \in B_n(x, 1)$), then

$$\frac{1}{C}\epsilon^Q \le \frac{\mu(B(y, \epsilon r_n))}{\mu(B(x, r_n))} = \mu_n(B_n(y, \epsilon)),$$

where B_n is the ball in the metric $d_n = r_n^{-1} d$. On the other hand, (4.5) implies that for sufficiently small r_n,

$$\mu_n(B_n(x, 2) \cap A_x) = \frac{\mu(B(x, 2r_n) \cap A_x)}{\mu(B(x, r_n))} < \frac{1}{C}\epsilon^Q.$$

It follows that for such n, the set $B_n(y, \epsilon) \setminus A_x$ has positive measure and therefore cannot be empty. That is, $X \setminus A_x$ is ϵ-dense in $B_n(x, 1)$.

The points in A_x can be easily avoided by redefining the approximating isometries ϕ_n such that points 1 and 2 of Definition 3.1 still hold, but with 3ϵ rather than ϵ. □

Lemma 4.7. *Let v be a Lipschitz function on a metric measure space (Z, d_Z, μ_Z), where μ_Z is doubling. Suppose that $K \subset Z$ and $z \in Z$ such that*

$$\lim_{r \to 0^+} \frac{\mu_Z(B(z, r) \cap K)}{\mu_Z(B(z, r))} = 0.$$

Then

$$\operatorname{Lip} v(z) := \limsup_{Z \ni y \to z} \frac{|v(z) - v(y)|}{d_Z(y, z)} = \limsup_{Z \setminus K \ni y \to z} \frac{|v(z) - v(y)|}{d_Z(y, z)}.$$

Proof. Clearly

$$\limsup_{Z \ni y \to z} \frac{|v(z) - v(y)|}{d_Z(y, z)} \ge \limsup_{Z \setminus K \ni y \to z} \frac{|v(z) - v(y)|}{d_Z(y, z)}.$$

Let $y_i \in Z$ be a sequence converging to z such that

$$\frac{|v(z) - v(y_i)|}{d_Z(y_i, z)} \to \limsup_{Z \ni y \to z} \frac{|v(z) - v(y)|}{d_Z(y, z)}.$$

If we have a subsequence of this sequence that lies in $Z \setminus K$, then we have the desired equality. So suppose without loss of generality that each $y_i \in K$. We claim that for each $\varepsilon > 0$ there is some positive integer N_ε such that when $i \ge N_\varepsilon$, we have $d_Z(w, y_i) \le \varepsilon d_Z(z, y_i)$ for some $w \in Z \setminus K$. Indeed, if this is not the case, then there is a positive number ε_0 and a subsequence i_k such that $B(y_{i_k}, \varepsilon_0 d_Z(z, y_{i_k})) \subset K$, in which case by the doubling property of μ_Z we have

$$\limsup_{r \to 0^+} \frac{\mu_Z(B(z, r) \cap K)}{\mu_Z(B(z, r))} \ge \frac{1}{C_d^\alpha} > 0$$

where α is the real number for which $2^\alpha \varepsilon_0 \geq 4$. This would violate the assumption on the density of K at z.

Now fixing $\varepsilon > 0$, with $w_i \in Z \setminus K$ such that $d_Z(y_i, w_i) \leq \varepsilon d_Z(z, y_i)$, we have

$$\begin{aligned}\frac{|v(z)-v(y_i)|}{d_Z(y_i,z)} \leq \frac{|v(z)-v(w_i)|}{d_Z(z,w_i)}\frac{d_Z(z,w_i)}{d_Z(y_i,z)} + \frac{|v(w_i)-v(y_i)|}{d_Z(y_i,z)} &\leq \frac{|v(z)-v(w_i)|}{d_Z(z,w_i)}\frac{d_Z(z,w_i)}{d_Z(y_i,z)} + L\varepsilon \\ &\leq \frac{|v(z)-v(w_i)|}{d_Z(z,w_i)}[1+\varepsilon] + L\varepsilon\end{aligned}$$

where L is a Lipschitz constant of v. Letting $i \to \infty$ followed by $\varepsilon \to 0^+$ gives the desired identity. □

Now we wish to speak about functions u_∞ that are limits of u_n according to (3.2), with the functions ϕ_n adapted to u. Note that the values of u_n on A_x are not tested in (3.2) by the maps ϕ_n. So u_∞ will also be the limit of functions $(\widetilde{u})_n$, where $\tilde{u}$ is any McShane extension of $u|_{B(x,R)\setminus A_x}$. By Hölder's inequality, the 1-Poincaré inequality implies the p-Poincaré inequality for all $1 < p < \infty$. By [26, Proposition 4.3], for each $k \in \mathbb{N}$ there is a *Lipschitz* function $v_k \in BV(X)$ such that

$$\mu(\{y \in X \,:\, u(y) \neq v_k(y)\}) < 1/k.$$

Since for any measurable set $K \subset X$ we have that the *upper* density of K at almost every point in $X \setminus K$ is zero, by modifying the set $K_k = \{y \in X \,:\, u(y) \neq v_k(y)\}$ on a set of measure zero we can assume that $\mu(K_k) < 1/k$ and that for every $x \in X \setminus K_k$ we have

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r)\cap K_k)}{\mu(B(x,r))} = 0$$

and that v_k is asymptotically generalized linear in the sense of [11, Theorem 3.7] and that the analysis of Proposition 4.2 holds for x. By further enlarging K_k if necessary (without increasing its measure), we can also assume that x is a Lebesgue point of $\operatorname{Lip} v_k$. Since both K_k and A_x have upper density zero at x, from Lemma 4.7 we know that x is a Lebesgue point also for $\operatorname{Lip} \widetilde{u}$ of any Lipschitz extension $\widetilde{u}$ of $u|_{B(x,R)\setminus[A_x\cup K_k]}$ to $B(x,R)$ (of course, this extension could depend on x as well, but in this case we can choose v_k itself to be that extension). Note that we then have by Lemma 4.7 that

$$\operatorname{Lip} u|_{B(x,R)\setminus A_x}(x) = \operatorname{Lip} \widetilde{u}(x) = \operatorname{Lip} v_k(x).$$

Thus the theory developed in [11, Section 10] is applicable for $x \in X \setminus K_k$. Let $N = \bigcap_{k\in\mathbb{N}} K_k$. Then $\mu(N) = 0$, and for each $x \in X \setminus N$ the theory developed in [11, Section 10] is applicable. Therefore by [11, Section 10], this immediately implies the equality of upper and lower Lipschitz constants for $\widetilde{u}$ and that

$$\operatorname{Lip} u_\infty \equiv \operatorname{Lip} \widetilde{u}(x) = \operatorname{Lip} u|_{B(x,R)\setminus A_x}(x),$$

and this constant minimal p-weak upper gradient, for any $1 < p < \infty$, of u_∞ is bounded above by $Cg(x)$ thanks to Proposition 4.2. On the other hand, by the lower semicontinuity of BV energy, we know that $dV(v_k, \cdot) \leq \operatorname{Lip} v_k \, d\mu$, and so the Radon-Nikodym derivative of $V(v_k, \cdot)$ with respect to μ, which by Lemma 4.1 is also equal to the Radon-Nikodym derivative g of $V(u, \cdot)$ with respect to μ in $X \setminus K_k$, is bounded above by $\operatorname{Lip} v_k$. Therefore we have

$$g(x) \leq \operatorname{Lip} u_\infty \leq C\, g(x).$$

We collect these observations below.

Theorem 4.8. *Let $u \in BV(X)$. Then for μ-a.e. $x \in X$ and any tangent space $(X_\infty, d_\infty, x_\infty, \mu_\infty)$, any function u_∞ that arises as a limit adapted to u at x has a constant minimal p-weak upper gradient for each $p > 1$ and that constant is less than $Cg(x)$, where C is as in Proposition 4.2. Furthermore, with h the minimal 1-weak upper gradient of u_∞, we have that $L/(4C_0) \le h \le L$ where L is the constant minimal p-weak upper gradient, and C_0 depends solely on the doubling and the 1-Poincaré constants of X_∞.*

Proof. The proof of the first part of the theorem follows from the discussions above. Thus it now suffices to prove the last statement of the theorem. From a telescoping argument for the Lipschitz function u_∞ on X_∞, we see that whenever $\varepsilon > 0$, for $z, w \in X_\infty$ with $d(z, w) < \varepsilon$ we have

$$|u(z) - u(w)| \le C_0\, d_{X_\infty}(z, w)[\mathcal{M}_{4\varepsilon} h(z) + \mathcal{M}_{4\varepsilon} h(w)],$$

where $\mathcal{M}_r h(o) := \sup_{0<\rho\le r} \fint_{B(o,\rho)} h\, d\mu_\infty$ for $o \in X_\infty$. Thus it follows from the local version of [21, Theorem 10.2.8] that $4C_0\, \mathcal{M}_{4\varepsilon} h$ is an upper gradient of u. Therefore by the minimality of the constant function L as a p-weak upper gradient of u_∞, we see that $L \le 4C_0\, \mathcal{M}_{4\varepsilon} h$ for each $\varepsilon > 0$. Letting $\varepsilon \to 0$ and invoking the Lebesgue differentiation theorem, we see that $L \le 4C_0 h$. Finally, as u_∞ is Lipschitz, the constant function L is also equal to $\operatorname{Lip} u_\infty$ which is also an upper gradient of u_∞, and so by the minimality of h as a 1-weak upper gradient, we see that $h \le L$, completing the proof. □

Theorem 4.9. *Let $u \in BV(X)$. Then for μ-a.e. $x \in X$ for which $g(x) > 0$, and for every tangent space $(X_\infty, d_\infty, x_\infty, \mu_\infty)$, any function u_∞ that arises as a limit adapted to u at x satisfies $u_\infty(x_\infty) = 0$ and*

$$\frac{g(x)}{C} \le \operatorname*{osc}_{B(y,s)} u_\infty \le \operatorname{LIP} u_\infty \le Cg(x)$$

for every $y \in X_\infty$ and $s > 0$. Furthermore, u_∞ is a function of least gradient. For μ-a.e. $x \in X$ for which $g(x) = 0$, u_∞ is a constant function.

Proof. The inequality $\operatorname*{osc}_{B(y,s)} u_\infty \le \operatorname{LIP} u_\infty$ is true by definition, and the inequality $\operatorname{LIP} u_\infty \le Cg(x)$ follows from Proposition 4.2 and [11, Section 10]. For the inequality $\frac{g(x)}{C} \le \operatorname*{osc}_{(y,s)} u_\infty$ we first note that by [23, Theorem 6.2.1],

$$\operatorname{lip} \widetilde{u}(x) \le \operatorname*{osc}_{B(y,s)} u_\infty,$$

where $\widetilde{u}$ is a McShane extension of $u|_{B(x,1)\setminus A_x}$ to $B(x, 1)$. Now note by Lemma 4.1 that

$$\frac{g(x)}{C} \le \operatorname{lip} \widetilde{u}(x).$$

By [11] we know that u_∞ is p-harmonic for each $p > 1$. Letting $p \to 1^+$, it follows from [28, Theorem 3.3] that u_∞ is a function of least gradient in X_∞.

Finally, if $g(x) = 0$ and x is a point of density 1 for the set $\{y \in X : g(y) = 0\}$, then we can choose for each $n \in \mathbb{N}$ a set $B(x, r_n) \setminus A_{x,1/n}$ as in Proposition 4.2 such that u is $1/n$-Lipschitz on $B(x, r_n) \setminus A_{x,1/n}$. Thus the limit function u_∞ is $1/n$-Lipschitz continuous for each $n \in \mathbb{N}$, and so is 0-Lipschitz, that is, u_∞ is constant. □

The focus of the next section will be to study asymptotic behavior of the characteristic function χ_E of a set E of finite perimeter at a boundary point. In considering such behavior, it is not possible to obtain a fruitful notion of the asymptotic limit of χ_E in a manner analogous to the above. Instead of considering a sequence of scaled versions of χ_E, as with the scaled versions $u_n = [u - u(x)]/r_n$ above, we consider the scaled versions of the *measures* μ_E given by $d\mu_{E,n} := \mu(B(x, r_n))^{-1}\chi_E\, d\mu$, and study weak* limits of such measures. The rest of this section discusses how the two notions, one dealing with a scaled version of the function and the other with a scaled version of the measure, are related.

We fix a sequence $X_n = (X, d_n, x, \mu_n)$ that converges in the pointed measured Gromov-Hausdorff sense to a tangent space $X_\infty = (X_\infty, d_\infty, x_\infty, \mu_\infty)$ as discussed above, and for such a sequence we let ν_n be the measure on X_n given by

$$d\nu_n := \mu(B(x, r_n))^{-1}(u - u(x))/r_n\, d\mu.$$

We wish to show that the sequence of measures ν_n has a subsequence that converges to the measure $u_\infty\, d\mu_\infty$.

Theorem 4.10. *Let $u \in BV(X)$. Then for μ-a.e. $x \in X$ we have the following: if $(X_\infty, d_\infty, x_\infty, \mu_\infty)$ is any tangent space to X at x, and u_∞ is a function that arises as a limit adapted to u at x, then also*

$$d\nu_n = \mu(B(x, r_n))^{-1}(u - u(x))/r_n\, d\mu \overset{*}{\rightharpoonup} u_\infty\, d\mu_\infty \quad \text{as } n \to \infty.$$

Naturally, this weak limit is attained along the same subsequence as u_∞ is. In addition to the connection that the above theorem makes between the way the limit function u_∞ was obtained above and the tangent-space analysis of sets of finite perimeter in the next section, the theorem also gives an elegant way of constructing the limit function u_∞ *without* having to modify the functions ϕ_n of Definition 3.1 to avoid the sets A_x.

Proof. Assume that x is a Lebesgue point of u such that

$$\lim_{r\to 0} \fint_{B(x,r)} |g - g(x)|\, d\mu = 0 \quad \text{and} \quad \lim_{r\to 0} \frac{V_s(u, B(x, r))}{\mu(B(x, r))} = 0,$$

and such that the conclusion of Lemma 4.3 holds. Let A be the set of all points $y \in X$ such that for some $0 < r \le 2d(x, y)$,

$$\fint_{B(y,r)} |g - g(x)|\, d\mu + \frac{V_s(u, B(y, r))}{\mu(B(y, r))} > 1.$$

Just as in the proof of Proposition 4.2, we get

$$\lim_{r\to 0} \frac{\mu(A \cap B(x, r))}{\mu(B(x, r))} = 0. \tag{4.6}$$

By Lemma 4.3 we obtain

$$\lim_{r\to 0} \frac{1}{\mu(B(x, r))} \int_{A\cap B(x,r)} \frac{|u - u(x)|}{r}\, d\mu = 0. \tag{4.7}$$

Fix $R > 0$ and consider the embeddings $\iota\colon X_\infty \to Z$ and $\iota_n\colon X_n \to Z$. To prove the theorem, we need to show that whenever ϕ is a continuous function supported in $B_Z(\iota(x_\infty), R)$, we have

$$\lim_{n\to\infty} \int_Z \phi\, \iota_{n,*}(d\nu_n) = \int_Z \phi\, \iota_*(u_\infty\, d\mu_\infty).$$

Just as in Proposition 4.2, we have that for all sufficiently large $n \in \mathbb{N}$, $u|_{B(x,2Rr_n)\setminus A}$ is $C(g(x)+1)$-Lipschitz. Then, define $\widetilde{u}$ to be the McShane extension of $u|_{B(x,2Rr_n)\setminus A}$. Just as in Theorem 4.8, we see that u_∞ is a limit function of $(\widetilde{u}-\widetilde{u}(x))/r_n$ which are all $C[g(x)+1]$-Lipschitz, and the sequence of measures $\mu(B(x,r_n))^{-1} r_n^{-1}(\widetilde{u}-\widetilde{u}(x))\, d\mu$ also converges weakly* to $u_\infty\, d\mu_\infty$. See Definition 3.6 and the discussion following it. It thus suffices to show that if $f_n := (\widetilde{u}-\widetilde{u}(x))/r_n$ and $g_n := (u - u(x))/r_n$, then we have that $\mu(B(x,r_n))^{-1}(f_n - g_n)\, d\mu \overset{*}{\rightharpoonup} 0$.

Let ϕ be a continuous function supported in the ball $B_Z(\iota(x_\infty), R)$. Then for all sufficiently large n,

$$\begin{aligned}
\left| \int_Z \phi\, \iota_{n,*}\left((f_n - g_n)\, d\mu_n\right) \right| &= \left| \int_{X_n} \phi(\iota_n(y))(f_n-g_n)(y)\, d\mu_n(y) \right| \\
&\le \frac{\|\phi\|_\infty}{\mu(B(x,r_n))} \int_{B(x,2Rr_n)} \left| \frac{\widetilde{u}-\widetilde{u}(x)}{r_n} - \frac{u-u(x)}{r_n} \right| d\mu \\
&\le \frac{\|\phi\|_\infty}{\mu(B(x,r_n))} \int_{B(x,2Rr_n)\cap A} \left| \frac{\widetilde{u}-\widetilde{u}(x)}{r_n} \right| + \left| \frac{u-u(x)}{r_n} \right| d\mu.
\end{aligned}$$

The last line follows since $\widetilde{u}(x) = u(x)$ and $\widetilde{u} = u$ on $B(x, 2Rr_n) \setminus A$. The first term converges to zero since $|\widetilde{u}-\widetilde{u}(x)|/r_n \le C(g(x)+1)R$ on $B(x,2Rr_n)$ by the Lipschitz bound for $\widetilde{u}$ and (4.6). Finally, the second term converges to zero by (4.7). □

5 Asymptotic limits of sets of finite perimeter

Let $E \subset X$ be a set of finite perimeter, and fix a point $x \in \partial^* E$ such that Lemma 2.2 holds. We will zoom in at x to study the asymptotic properties of E. Let $r_n > 0$ with $r_n \to 0$. In this section, we always consider the sequence

$$(X_n, d_n, x, \mu_n) := \left(X, \frac{1}{r_n}\cdot d, x, \frac{1}{\mu(B(x,r_n))}\cdot \mu \right)$$

under pointed measured Gromov-Hausdorff convergence. We also wish to study the behavior of the measure $P(E,\cdot)$ as we zoom in, so let

$$(X_n, d_n, x, P_n(E,\cdot)) := \left(X, \frac{1}{r_n}\cdot d, x, \frac{r_n}{\mu(B(x,r_n))} P(E,\cdot) \right).$$

Taking subsequences as necessary (not relabeled), we find the following measures on the limit space $(X_\infty, d_\infty, x_\infty)$:

$$\begin{aligned}
\mu_n &\overset{*}{\rightharpoonup} \mu_\infty, \\
\mu_n(\cdot \cap E) &\overset{*}{\rightharpoonup} \mu_\infty^E, \\
\mu_n(\cdot \cap E^c) &\overset{*}{\rightharpoonup} \mu_\infty^{E^c},
\end{aligned}$$

$$\text{and } P_n(E,\cdot) \overset{*}{\rightharpoonup} \pi_\infty.$$

Here, P_n is the scaled perimeter measure given by

$$P_n(E, B(z,\rho)) = r_n \frac{P(E, B(z, r_n\rho))}{\mu(B(x, r_n))}.$$

In Section 3 it was noted that a tangent space $(X_\infty, d_\infty, x, \mu_\infty)$ always exists, is geodesic, and μ_∞ is doubling and supports a 1-Poincaré inequality. Note that $\mu_n(\cdot \cap E) + \mu_n(\cdot \cap E^c) = \mu_n$, and so $\mu_\infty^E + \mu_\infty^{E^c} = \mu_\infty$. The existence of π_∞ follows from Lemma 2.2: by (2.11), for every $k \in \mathbb{N}$,

$$\limsup_{n\to\infty} P_n(E, B_n(x,k)) = \limsup_{n\to\infty} \frac{r_n}{\mu(B(x,r_n))} P(E, B(x, kr_n)) \le C_d^{1+\lceil \log_2 k \rceil}. \tag{5.1}$$

At various points in this section, we specify additional conditions on $x \in \partial^* E$ by excluding $\mathcal{H}$-negligible parts of $\partial^* E$.

Since X is geodesic and μ is doubling, the space satisfies the following *annular decay property*: there exists $\delta = \delta(C_d) \in (0,1]$ such that for all $y \in X$, $r > 0$, and $0 < \varepsilon < 1$, we have

$$\mu(B(y,r) \setminus B(y, r(1-\varepsilon))) \le C\varepsilon^\delta \mu(B(y,r)), \tag{5.2}$$

see [9, Corollary 2.2]. In particular, this property implies that all spheres have zero μ-measure.

We now define two sets in X_∞. Let $(E)_\infty$ be the collection of all points $z \in X_\infty$ for which

$$\lim_{r\to 0} \frac{\mu_\infty^E(B_{X_\infty}(z,r))}{\mu_\infty(B_{X_\infty}(z,r))} = 1,$$

and let $(E^c)_\infty$ be the analogous collection of all points $z \in X_\infty$ for which

$$\lim_{r\to 0} \frac{\mu_\infty^{E^c}(B_{X_\infty}(z,r))}{\mu_\infty(B_{X_\infty}(z,r))} = 1.$$

Lemma 5.1. *If $z \in X_\infty$ and $z_n \in X_n$ with $z_n \to z$ (or, more precisely, $\iota_n(z_n) \to \iota(z_\infty)$ in Z), then for every $r > 0$,*

$$\mu_\infty(B_{X_\infty}(z,r)) = \lim_{n\to\infty} \mu_n(B_n(z_n, r)).$$

The analogous result holds for the measures $\mu_n(\cdot \cap E)$ and μ_∞^E, and for $\mu_n(\cdot \cap E^c)$ and $\mu_\infty^{E^c}$.

Proof. We prove the result for the measures $\mu_n(\cdot \cap E)$ and μ_∞^E; the proofs for the other two pairs are analogous. Fix $\eta > 0$. By the lower semicontinuity of measure in open sets under weak* convergence (see e.g. [3, Proposition 1.62]),

$$\mu_\infty^E(B_{X_\infty}(z, r-\eta)) = \mu_\infty^E(B_Z(z, r-\eta)) \le \liminf_{n\to\infty} \left[\iota_{n,*}\mu_n(E \cap \cdot)\right](B_Z(z, r-\eta)).$$

Here $B_Z(z, r-\eta)$ is the ball in Z whose center is the image of z under the isometric embedding ι. Letting $\epsilon_n := d_Z(z_n, z)$, we have $\epsilon_n \to 0$ and for large n,

$$B_Z(z, r-\eta) \subset B_Z(z_n, r-\eta+\epsilon_n) \subset B_Z(z_n, r),$$

where again we label the image of z_n under the isometric embedding ι_n also by z_n. Now we can conclude

$$\mu_\infty^E(B_{X_\infty}(z, r-\eta)) \le \liminf_{n\to\infty} \mu_n(B_n(z_n, r) \cap E).$$

Thus letting $\eta \to 0$,

$$\mu_\infty^E(B_{X_\infty}(z,r)) \le \liminf_{n\to\infty} \mu_n(B_n(z_n,r)\cap E). \tag{5.3}$$

Again fix $\eta > 0$. By upper semicontinuity of measure in compact sets under weak* convergence,

$$\mu_\infty^E(B_{X_\infty}(z,r+2\eta)) \ge \mu_\infty^E(\overline{B_{X_\infty}(z,r+\eta)}) \ge \limsup_{n\to\infty} \left[\iota_{n,*}\mu_n(E\cap\cdot)\right](\overline{B_Z(z,r+\eta)}).$$

Again for large n,

$$\overline{B_Z(z,r+\eta))} \supset B_Z(z_n, r+\eta-\epsilon_n) \supset B_Z(z_n,r),$$

and we conclude

$$\mu_\infty^E(B_{X_\infty}(z,r+2\eta)) \ge \limsup_{n\to\infty} \mu_n(B_n(z_n,r)\cap E).$$

Note that the measure μ_∞ also satisfies the annular decay property, and that $\mu_\infty^E \le \mu_\infty$, so spheres in X_∞ do not carry positive μ_∞^E-weight; therefore letting $\eta \to 0$, we get

$$\mu_\infty^E(B_{X_\infty}(z,r)) \ge \limsup_{n\to\infty} \mu_n(B_n(z_n,r)\cap E).$$

Combining this with (5.3) completes the proof. □

Proposition 5.2. *The sets $(E)_\infty$ and $(E^c)_\infty$ are disjoint. Moreover,*

$$\mu_\infty(X_\infty \setminus [(E)_\infty \cup (E^c)_\infty]) = 0.$$

Proof. Suppose $z \in (E)_\infty \cap (E^c)_\infty$. Then

$$\lim_{r\to 0} \frac{\mu_\infty^E(B_{X_\infty}(z,r))}{\mu_\infty(B_{X_\infty}(z,r))} = 1 = \lim_{r\to 0} \frac{\mu_\infty^{E^c}(B_{X_\infty}(z,r))}{\mu_\infty(B_{X_\infty}(z,r))}.$$

By the Gromov-Hausdorff convergence, there is a sequence $x_n \in X_n$ with $x_n \to z$. Thus for any small enough $r > 0$, Lemma 5.1 gives

$$2/3 < \frac{\mu_\infty^E(B_{X_\infty}(z,r))}{\mu_\infty(B_{X_\infty}(z,r))} = \lim_{n\to\infty} \frac{\mu_n(B_n(x_n,r)\cap E)}{\mu_n(B_n(x_n,r))} = \lim_{n\to\infty} \frac{\mu(B(x_n,rr_n)\cap E)}{\mu(B(x_n,rr_n))}$$

and similarly

$$2/3 < \frac{\mu_\infty^{E^c}(B_{X_\infty}(z,r))}{\mu_\infty(B_{X_\infty}(z,r))} = \lim_{n\to\infty} \frac{\mu(B(x_n,rr_n)\cap E^c)}{\mu(B(x_n,rr_n))}.$$

Adding together, we find for all small enough $r > 0$ and large enough n

$$4/3 < \frac{\mu(B(x_n,rr_n))}{\mu(B(x_n,rr_n))},$$

which is not possible. Therefore $(E)_\infty$ and $(E^c)_\infty$ are disjoint.

Next, we show that $\mu_\infty(X_\infty \setminus [(E)_\infty \cup (E^c)_\infty]) = 0$. To this end, we will show that the Radon-Nikodym derivative of μ_∞^E with respect to μ_∞ is μ_∞-a.e. either 1 or 0. Let this Radon-Nikodym derivative be denoted by φ. Let $A_0 := \{z \in X_\infty : 0 < \varphi(z) < 1\}$, and suppose that $\mu_\infty(A_0) > 0$. Then there is some $R > 0$ and $0 < \delta < 1$ such that the set

$$A := \{z \in B_{X_\infty}(x_\infty, R) : \delta < \varphi(z) < 1-\delta \text{ and } z \text{ is a Lebesgue point of } \varphi\}$$

satisfies

$$\mu_\infty(A) > \delta \mu_\infty(B_{X_\infty}(x_\infty, R)).$$

We fix $0 < \varepsilon < R$ and consider the family of balls $B_{X_\infty}(z, \rho)$, $z \in A$ and $0 < \rho < \varepsilon$, such that

$$\delta < \frac{1}{\mu_\infty(B_{X_\infty}(z,\rho))} \int_{B_{X_\infty}(z,\rho)} \varphi \, d\mu_\infty < 1 - \delta. \tag{5.4}$$

As every $z \in A$ is a Lebesgue point of φ, the corresponding family of closed balls is a fine cover of A, and hence there is a pairwise disjoint subfamily $\{\overline{B_i}\}_{i=1}^\infty$ such that $\mu_\infty(A \setminus \bigcup_{i=1}^\infty \overline{B_i}) = 0$ and then also $\mu_\infty(A \setminus \bigcup_{i=1}^\infty B_i) = 0$, since spheres have μ_∞-measure zero. Now, observe that

$$\delta \mu_\infty(B_{X_\infty}(x_\infty, R)) < \mu_\infty(A) \le \sum_{i=1}^\infty \mu_\infty(B_i),$$

and so we can find $N \in \mathbb{N}$ such that

$$\delta \mu_\infty(B_{X_\infty}(x_\infty, R)) < \sum_{i=1}^N \mu_\infty(B_i). \tag{5.5}$$

Denote the center of each B_i by x^i. By Lemma 5.1 we can find $j_\varepsilon \in \mathbb{N}$ such that whenever $n \ge j_\varepsilon$, there are points $x_n^1, \cdots, x_n^N \in X = X_n$ converging to $x^1, \ldots, x^n$ respectively, and a number $\eta > 0$ such that

$$\mu_n(B_n(x_n^i, \operatorname{rad} B_i)) \le (1+\delta^2)\mu_\infty(B_i).$$

for all $i = 1, \ldots, N$. This gives the second inequality of (5.7) below; analogously we obtain for all $i = 1, \ldots, N$

$$(1-\delta^2)\mu_\infty(B_{X_\infty}(x_\infty, R)) \le \mu_n(B_n(x, R)) \le (1+\delta^2)\mu_\infty(B_{X_\infty}(x_\infty, R)), \tag{5.6}$$

$$(1-\delta^2)\mu_\infty(B_i) \le \mu_n(B_n(x_n^i, \operatorname{rad} B_i)) \le (1+\delta^2)\mu_\infty(B_i), \tag{5.7}$$

$$\begin{aligned} \delta\frac{1-\delta^2}{1+\delta^2} \le \frac{1-\delta^2}{1+\delta^2}\frac{1}{\mu_\infty(B_i)}\int_{B_i}\varphi\, d\mu_\infty \le \frac{\mu_n(B_n(x_n^i, \operatorname{rad} B_i) \cap E)}{\mu_n(B_n(x_n^i, \operatorname{rad} B_i))} \\ \le \frac{1+\delta^2}{1-\delta^2}\frac{1}{\mu_\infty(B_i)}\int_{B_i}\varphi\, d\mu_\infty \le \frac{1+\delta^2}{1+\delta} \le 1-\delta/2; \end{aligned} \tag{5.8}$$

here, to obtain (5.8) we also used (5.4). We can also ensure that the collection of balls ("lifts" of B_i to X_n) $\{B(x_n^i, r_n \operatorname{rad} B_i)\}_{i=1}^N$ are pairwise disjoint. Inequality (5.8) tells us that if $\delta > 0$ was chosen small enough, then

$$\delta/2 \le \frac{\mu(B(x_n^i, r_n \operatorname{rad} B_i) \cap E)}{\mu(B(x_n^i, r_n \operatorname{rad} B_i))} \le 1 - \delta/2.$$

Now, applying the relative isoperimetric inequality (2.4) to these balls gives

$$\delta/2 \le 2C_P \frac{r_n \operatorname{rad} B_i}{\mu(B(x_n^i, r_n \operatorname{rad} B_i))} P(E, B(x_n^i, r_n \operatorname{rad} B_i)).$$

Thus we obtain, recalling that $\operatorname{rad} B_i < \varepsilon < R$,

$$\delta \sum_{i=1}^{N} \mu(B(x_n^i, r_n \operatorname{rad} B_i)) \le 4C_P \varepsilon r_n \sum_{i=1}^{N} P(E, B(x_n^i, r_n \operatorname{rad} B_i)) \le 4C_P \varepsilon r_n P(E, B(x, 2r_n R)).$$

By (5.5) and (5.7), we now have

$$\begin{aligned}
\delta^2(1-\delta^2)\mu_\infty(B_{X_\infty}(x_\infty, R)) &\le \delta(1-\delta^2)\sum_{i=1}^{N} \mu_\infty(B_i) \\
&\le \delta \sum_{i=1}^{N} \frac{\mu(B_n(x_n^i, r_n \operatorname{rad} B_i))}{\mu(B(x, r_n))} \\
&\le \frac{4C_P \varepsilon r_n}{\mu(B(x, r_n))} P(E, B(x, 2r_n R)).
\end{aligned}$$

Applying (5.6) now gives

$$\frac{\delta^2(1-\delta^2)}{1+\delta^2} \frac{\mu(B(x, r_n R))}{\mu(B(x, r_n))} \le \frac{4C_P \varepsilon r_n}{\mu(B(x, r_n))} P(E, B(x, 2r_n R)).$$

By the doubling property of μ we obtain

$$0 < \frac{\delta^2(1-\delta^2)}{1+\delta^2} \frac{1}{4C_P C_d \varepsilon} \le \frac{r_n}{\mu(B(x, 2r_n R))} P(E, B(x, 2r_n R)).$$

Now letting $n \to \infty$, by (2.11) we get

$$0 < \frac{\delta^2(1-\delta^2)}{1+\delta^2} \frac{1}{4C_P C_d \varepsilon} \le C_d$$

for every $0 < \varepsilon < 1$, which is not possible. Thus $\mu(A_0) = 0$.

Now the claim $\mu_\infty(X_\infty \setminus [(E)_\infty \cup (E^c)_\infty]) = 0$ follows from the fact that $\mu_\infty^E + \mu_\infty^{E^c} = \mu_\infty$. □

Note that by the above proposition and by the Radon-Nikodym theorem, we now have

$$\mu_\infty^E(A) = \mu_\infty((E)_\infty \cap A) \quad \text{and} \quad \mu_\infty^{E^c}(A) = \mu_\infty((E^c)_\infty \cap A)$$

for Borel sets $A \subset X_\infty$, and moreover $\partial^*(E)_\infty = X_\infty \setminus ((E)_\infty \cup (E^c)_\infty)$.

Now we wish to study the support of the asymptotic perimeter measure π_∞ in $(X_\infty, d_\infty, x_\infty)$. We first prove a proposition that states that if a point $z \in X_\infty$ is in the support of π_∞, then it can be seen as the limit of special points in $\partial^* E$. Recall from (2.9) that

$$\Sigma_\gamma E := \left\{ y \in X : \liminf_{r \to 0} \min \left\{ \frac{\mu(B(y,r) \cap E)}{\mu(B(y,r))}, \frac{\mu(B(y,r) \cap E^c)}{\mu(B(y,r))} \right\} \ge \gamma \right\} \subset \partial^* E$$

for some $\gamma = \gamma(C_d, C_P) \in (0, 1/2]$. By (2.8) and (2.10) we know that $P(E, \cdot)$ is concentrated on $\Sigma_\gamma E$.

For each $m \in \mathbb{N}$, let

$$G_m := \left\{ z \in \Sigma_\gamma E : \frac{\gamma}{2C_P} \le r \frac{P(E, B(z,r))}{\mu(B(z,r))} \le 2C_d \quad \text{for all } 0 < r < \frac{1}{m} \right\}.$$

By Lemma 2.2 we know that

$$\mathcal{H}\left(\partial^* E \setminus \bigcup_{m \in \mathbb{N}} G_m \right) = 0,$$

and $G_m \subset G_{m+1}$ for all $m \in \mathbb{N}$. Note that for every $r > 0$ the map $z \mapsto P(E, B(z,r))$ is lower semicontinuous, and so G_m is a Borel set. Combining the definitions of $\Sigma_\gamma E$ and G_m, for every $z \in \Sigma_\gamma E \cap G_m$ we find $r_z > 0$ such that

$$rP(E, B(z,r)) \le K \min\{\mu(B(z,r) \cap E), \mu(B(z,r) \cap E^c)\} \tag{5.9}$$

for all $0 < r \le r_z$, where $K = K(C_d, C_P)$. Hence we can refine G_m further by considering the set

$$G_m^* := \left\{ z \in G_m : \text{(5.9) holds for all } 0 < r < \frac{1}{m} \right\}.$$

Note then that G_m^* is a Borel set, and that $G_m^* \subset G_{m+1}^*$ for $m \in \mathbb{N}$ with

$$\mathcal{H}\left(\partial^* E \setminus \bigcup_{m \in \mathbb{N}} G_m^* \right) = 0.$$

As $P(E, \cdot)$ is asymptotically doubling by (2.11), we know that the Lebesgue differentiation theorem holds for the measure $P(E, \cdot)$. Hence for any fixed $m \in \mathbb{N}$, by the Lebesgue differentiation theorem, G_m^* is of density 1 (with respect to the measure $P(E, \cdot)$) at $P(E, \cdot)$-a.e. $x \in G_m^*$. It is at such a point that we will zoom in and take our limiting measures.

Proposition 5.3. *Let (X_n, d_n, x, μ_n) be a pointed measured Gromov-Hausdorff convergent sequence such that the base point x is a point of $P(E, \cdot)$-density 1 for G_m^* for some $m \in \mathbb{N}$. Suppose that $z \in X_\infty$ is such that*

$$\pi_\infty(B_{X_\infty}(z, R)) > 0$$

for all $R > 0$. Then there is a sequence $z_n \in X_n$ that converges to z (in Z) such that each z_n is in G_m^*.

Proof. Fix $R > 0$. By the Gromov-Hausdorff convergence there exist sequences $\epsilon_n \to 0$ and $x_n \in X_n = X$ such that $d_Z(x_n, z) < \epsilon_n$, and then by lower semicontinuity under weak* convergence,

$$\liminf_{n\to\infty} r_n \frac{P(E, B(x_n, (R+\epsilon_n) r_n))}{\mu(B(x, r_n))} \ge \liminf_{n\to\infty} r_n \frac{P(E, \iota_n^{-1}(B_Z(z,R)))}{\mu(B(x, r_n))} \ge \pi_\infty(B_{X_\infty}(z,R)) > 0. \tag{5.10}$$

We would like to know that there exists $\widetilde{x}_n \in G_m^* \cap B(x_n, (R+\epsilon_n)r_n)$ for all sufficiently large $n \in \mathbb{N}$. Suppose that this is not the case. Then there is a subsequence n_k such that $B(x_{n_k}, (R+\epsilon_{n_k})r_{n_k})$ is disjoint from G_m^*. Choose $M > 0$ large enough so that for all $k \in \mathbb{N}$,

$$B(x_{n_k}, (R+\epsilon_{n_k})r_{n_k}) \subset B(x, M r_{n_k}).$$

As $B(x_{n_k},(R+\epsilon_{n_k})r_{n_k})$ and $B(x,Mr_{n_k})\cap G_m^*$ are disjoint, we have

$$P(E,B(x_{n_k},(R+\epsilon_{n_k})r_{n_k}))+P(E,B(x,Mr_{n_k})\cap G_m^*)\le P(E,B(x,Mr_{n_k})),$$

which is equivalent to

$$r_{n_k}\frac{P(E,B(x_{n_k},(R+\epsilon_{n_k})r_{n_k}))}{\mu(B(x,r_{n_k}))}+r_{n_k}\frac{P(E,B(x,Mr_{n_k})\cap G_m^*)}{\mu(B(x,r_{n_k}))}\le r_{n_k}\frac{P(E,B(x,Mr_{n_k}))}{\mu(B(x,r_{n_k}))}.$$

Call the left-hand side of this inequality A_k+B_k, and the right-hand side C_k. The assumption that x is a point of density 1 in G_m^* implies that $B_k/C_k\to 1$ as $k\to\infty$. Thus $A_k/C_k\to 0$ as $k\to\infty$. On the other hand, by the definition of G_m, we must have $C_k\le 2M^{-1}C_d^{2+\log_2(M)}<\infty$ for large $k\in\mathbb{N}$. Therefore $A_k\to 0$ as $k\to\infty$, which contradicts (5.10). Thus, there is some $N_1\in\mathbb{N}$ such that there is a point $\widetilde{x}_n\in G_m^*\cap B(x_n,(R+\epsilon_n)r_n)$ for all $n\ge N_1$. We rename this sequence $\widetilde{x}_n^1$. Similarly, there exists a sequence

$$\widetilde{x}_n^2\in G_m^*\cap B(x_n,(2^{-1}R+\epsilon_n)r_n)$$

for all $n\ge N_2>N_1$. We continue inductively in this fashion to find for each $k\in\mathbb{N}$,

$$\widetilde{x}_n^k\in G_m^*\cap B(x_n,(2^{-k}R+\epsilon_n)r_n)$$

for all $n\ge N_k>N_{k-1}$. Now

$$d_Z(\widetilde{x}_n^k,z)\le d_Z(\widetilde{x}_n^k,x_n)+d_Z(x_n,z)\le 2^{-k}R+\epsilon_n+\epsilon_n.$$

For $n\in[N_k,N_{k+1})$, set $z_n:=\widetilde{x}_n^k$. Then z_n has the desired properties. □

Note also that the support of π_∞ is contained in X_∞; this can be seen as follows. If $z\in Z$ such that $\pi_\infty(B_Z(z,R))>0$ for all $R>0$, then there exists a sequence $x_n\in X_n$ such that $x_n\to z$ in Z. It follows that $z\in X_\infty$.

We now provide growth estimates for the measure π_∞.

Theorem 5.4. *Consider the sequence (X_n,d_n,x,μ_n) such that x is a point of $P(E,\cdot)$-density 1 for G_m^* for some $m\in\mathbb{N}$. Suppose that $z\in X_\infty$ is such that*

$$\pi_\infty(B_{X_\infty}(z,R))>0$$

for all $R>0$. Then

$$\frac{1}{C}\frac{\mu_\infty(B_{X_\infty}(z,r))}{r}\le\pi_\infty(B_{X_\infty}(z,r))\le C\frac{\mu_\infty(B_{X_\infty}(z,r))}{r}\quad\text{for all } r>0,\tag{5.11}$$

where $C=C(C_d,C_P)$, and

$$\pi_\infty((E)_\infty\cup(E^c)_\infty)=0.\tag{5.12}$$

Proof. By Proposition 5.3, there is a sequence $z_n\in G_m^*$ that converges to z in Z. Fix $r>0$ and $0<\eta<r/2$. Using the basic properties of weak* convergence as in the proof of Proposition 5.3, we find a sequence of positive numbers ϵ_n with $\lim_{n\to\infty}\epsilon_n=0$ such that

$$\liminf_{n\to\infty}r_n\frac{P(E,B(z_n,(r+\epsilon_n)r_n))}{\mu(B(x,r_n))}\ge\pi_\infty(B_{X_\infty}(z,r))\ge\limsup_{n\to\infty}r_n\frac{P(E,B(z_n,(r-\eta)r_n))}{\mu(B(x,r_n))}.\tag{5.13}$$

We can rewrite the term on the right-most side of (5.13) as

$$\limsup_{n\to\infty}(r-\eta)r_n\frac{P(E,B(z_n,(r-\eta)r_n))}{\mu(B(z_n,(r-\eta)r_n))}\cdot\frac{\mu(B(z_n,(r-\eta)r_n))}{(r-\eta)\mu(B(x,r_n))}.$$

Since $z_n\in G_m^*$, we know that

$$(r-\eta)r_n\frac{P(E,B(z_n,(r-\eta)r_n))}{\mu(B(z_n,(r-\eta)r_n))}\geq\frac{\gamma}{2C_P}$$

for all large enough n (that is, when $(r-\eta)r_n<1/m$). Additionally, by Lemma 5.1,

$$\begin{aligned}\limsup_{n\to\infty}\frac{\mu(B(z_n,(r-\eta)r_n))}{(r-\eta)\mu(B(x,r_n))}&=\frac{1}{(r-\eta)}\limsup_{n\to\infty}\frac{\mu(B(z_n,(r-\eta)r_n))}{\mu(B(x,r_n))}\\&=\frac{\mu_\infty(B_{X_\infty}(z,r-\eta))}{r-\eta}\\&\geq\frac{1}{C_d^2}\,\frac{\mu_\infty(B_{X_\infty}(z,r))}{r},\end{aligned}$$

since μ_∞ is doubling with constant C_d^2. Thus by (5.13), we get

$$\pi_\infty(B_{X_\infty}(z,r))\geq\frac{\gamma}{2C_d^2C_P}\frac{\mu_\infty(B_{X_\infty}(z,r))}{r}.$$

Next we rewrite the term on the left-most side of (5.13) as

$$\liminf_{n\to\infty}(r+\epsilon_n)r_n\frac{P(E,B(z_n,(r+\epsilon_n)r_n))}{\mu(B(z_n,(r+\epsilon_n)r_n))}\cdot\frac{\mu(B(z_n,(r+\epsilon_n)r_n))}{(r+\epsilon_n)\mu(B(x,r_n))}.$$

Since $z\in G_m^*$, we know that

$$(r+\epsilon_n)r_n\frac{P(E,B(z_n,(r+\epsilon_n)r_n))}{\mu(B(z_n,(r+\epsilon_n)r_n))}\leq 2C_d$$

for all large enough n. Similarly to above, we obtain

$$\lim_{n\to\infty}\frac{\mu(B(z_n,(r+\epsilon_n)r_n))}{(r+\epsilon_n)\mu(B(x,r_n))}=\frac{\mu_\infty(B_{X_\infty}(z,r))}{r},$$

whence from (5.13) we obtain

$$\pi_\infty(B_{X_\infty}(z,r))\leq 2C_d\,\frac{\mu_\infty(B_{X_\infty}(z,r))}{r}.$$

This proves (5.11).

It now only remains to show (5.12). It suffices to show that when $z\in(E)_\infty\cup(E^c)_\infty$, for each $k\in\mathbb{N}$ there is some $r_z>0$ such that $\pi_\infty(B_{X_\infty}(z,r_z))\leq C\mu_\infty(B_{X_\infty}(z,r_z))/(kr_z)$, from which we will know (via (5.11)) that there must be $\rho_z>0$ with $\pi_\infty(B_{X_\infty}(z,\rho_z))=0$. Fix $k\in\mathbb{N}$. Suppose $z\in(E)_\infty$ is in the support of π_∞. Then there is some $r_z>0$ such that

$$\frac{\mu_\infty(B_{X_\infty}(z,r_z)\cap(E^c)_\infty)}{\mu_\infty(B_{X_\infty}(z,r_z))}<\frac{1}{k}.$$

Let $\epsilon_n \to 0$ and $G_m^* \ni z_n \to z$ as given by the conclusion of Proposition 5.3. By Lemma 5.1 we have

$$\frac{1}{k}\lim_{n\to\infty}\frac{\mu(B(z_n,r_z r_n))}{\mu(B(x,r_n))} = \frac{\mu_\infty(B_{X_\infty}(z,r_z))}{k} > \mu_\infty(B_{X_\infty}(z,r_z)\cap(E^c)_\infty)$$
$$= \lim_{n\to\infty}\frac{\mu(B(z_n,r_z r_n)\cap E^c)}{\mu(B(x,r_n))}.$$

Thus for all large enough $n \in \mathbb{N}$,

$$\frac{1}{k}\mu(B(z_n,r_z r_n)) \geq \mu(B(z_n,r_z r_n)\cap E^c).$$

Since $z_n \in G_m^*$, by (5.9) we have that for all large enough $n \in \mathbb{N}$,

$$r_z r_n \frac{P(E,B(z_n,r_z r_n))}{\mu(B(z_n,r_z r_n))} \leq K\frac{\mu(B(z_n,r_z r_n)\cap E^c)}{\mu(B(z_n,r_z r_n))} \leq \frac{K}{k}.$$

Thus for any $\eta > 0$,

$$\pi_\infty(B_{X_\infty}(z,r_z-\eta)) \leq \liminf_{n\to\infty} r_n \frac{P(E,B(z_n,r_z r_n))}{\mu(B(x,r_n))} \leq \frac{K}{k}\frac{1}{r_z}\liminf_{n\to\infty}\frac{\mu(B(z_n,r_z r_n))}{\mu(B(x,r_n))}$$
$$= \frac{K}{k}\frac{\mu_\infty(B_{X_\infty}(z,r_z))}{r_z}$$

by Lemma 5.1. Since $K = K(C_d, C_P)$, letting $\eta \to 0$ gives

$$\pi_\infty(B_{X_\infty}(z,r_z)) \leq \frac{K}{k}\,\frac{\mu_\infty(B_{X_\infty}(z,r_z))}{r_z}.$$

By choosing $k \in \mathbb{N}$ large enough, the above would violate the left-hand inequality of (5.11), and so z cannot be in the support of π_∞. Thus, there is some $\rho_z > 0$ with $\pi_\infty(B(z,\rho_z)) = 0$. Since this happens for every $z \in (E)_\infty$, we know that π_∞ does not charge $(E)_\infty$. Indeed, with

$$(E)_\infty \subset U := \bigcup_{z\in(E)_\infty} B(z,\rho_z),$$

an open set containing $(E)_\infty$, we have $\pi_\infty(U) = 0$. A similar argument gives the existence of an open set $V \supset (E^c)_\infty$ with $\pi_\infty(V) = 0$. This completes the proof. □

Next we show that the set $(E)_\infty$ is of locally finite perimeter in the space X_∞. Denote by $\mathcal{H}_\infty$ the co-dimension 1 Hausdorff measure in the space $(X_\infty, d_\infty, \mu_\infty)$.

Theorem 5.5. *For all $R > 0$, we have $P((E)_\infty, B_{X_\infty}(x_\infty,R)) < \infty$. The measures $P((E)_\infty,\cdot)$, π_∞, and $\mathcal{H}_\infty(\partial^*(E)_\infty\cap\cdot)$ are comparable. The sets $(E)_\infty$ and $(E^c)_\infty$ are open in X_∞.*

Proof. To prove the first claim, we use a discrete convolution construction. Assume for simplicity that $R = 1$. Fix $0 < \epsilon < 1/9$, and take a maximal ϵ-separated set $\{z_k\}_{k=1}^\infty \subset X_\infty$. Then the balls $B_k := B_{X_\infty}(z_k,\epsilon)$ cover X_∞ and $B_{X_\infty}(z_k, 14\varepsilon)$ have bounded overlap. For each k we can find points

$z_{k,n} \in X_n$ converging to z_k (in Z). Thus, by considering a tail-end of the sequence if necessary, there is a sequence $0 < \delta_n \to 0$ with $\delta_n < \epsilon$ such that $d_Z(\iota_\infty(z_k), \iota_n(z_{k,n})) < \delta_n$, and by Lemma 5.1,

$$(1-\delta_n)\,\mu_\infty^E(B_{X_\infty}(z_k,\epsilon)) \le \mu_n(B_n(z_{k,n},\epsilon)\cap E) \tag{5.14}$$

and

$$\mu_n(B_n(z_{k,n},\epsilon)) \le (1+\delta_n)\mu_\infty(B_{X_\infty}(z_k,\epsilon)) \le (1+\delta_n)^2\,\mu_n(B_n(z_{k,n},\epsilon)). \tag{5.15}$$

Observe that we need only do this for the points $z_k \in B_{X_\infty}(x_\infty, 2)$, of which there are only finitely many, and thus we can choose $\delta_n > 0$ such that the above hold for all corresponding indices k. By the bounded overlap property of the balls $B_{X_\infty}(z_k, 14\epsilon)$, we also have that for each such positive integer n, the collection of balls $B_{X_n}(z_{k,n}, 6\epsilon)$ has a bounded overlap; this will be needed in the computations (5.18).

Now take a partition of unity by means of C/ϵ-Lipschitz functions $\phi_k \in \mathrm{Lip}(X_\infty; [0,1])$ with $\mathrm{supp}(\phi_k) \subset B_{X_\infty}(z_k, 2\epsilon)$ for each $k \in \mathbb{N}$; see e.g. [21, p. 104]. Let $u := \chi_E$, and for each $n \in \mathbb{N}$ we set

$$v_n^\epsilon := \sum_{k=1}^\infty u_{B_n(z_{k,n},\epsilon)}\phi_k, \tag{5.16}$$

where

$$u_{B_n(z_{k,n},\epsilon)} = \fint_{B_n(z_{k,n},\epsilon)} u\,d\mu_n = \fint_{B(z_{k,n},r_n\epsilon)} u\,d\mu = \frac{\mu(B(z_{k,n},r_n\epsilon)\cap E)}{\mu(B(z_{k,n},r_n\epsilon))}.$$

Let $l \in \mathbb{N}$ such that $B_l \cap B_{X_\infty}(x_\infty, 1) \neq \emptyset$. Given $y_1, y_2 \in B_l$, we estimate

$$\begin{aligned}
|v_n^\epsilon(y_1) - v_n^\epsilon(y_2)| &= \left|\sum_{k=1}^\infty u_{B_n(z_{k,n},\epsilon)}\phi_k(y_1) - \sum_{k=1}^\infty u_{B_n(z_{k,n},\epsilon)}\phi_k(y_2)\right| \\
&= \left|\sum_{k=1}^\infty (u_{B_n(z_{k,n},\epsilon)} - u_{B_n(z_{l,n},\epsilon)})(\phi_k(y_1)-\phi_k(y_2))\right| \\
&\le \sum_{k=1}^\infty |u_{B_n(z_{k,n},\epsilon)} - u_{B_n(z_{l,n},\epsilon)}||\phi_k(y_1)-\phi_k(y_2)| \\
&= \sum_{\substack{k\in\mathbb{N}\\ B_{X_\infty}(z_k,2\epsilon)\cap B_l\neq\emptyset}} |u_{B_n(z_{k,n},\epsilon)} - u_{B_n(z_{l,n},\epsilon)}||\phi_k(y_1)-\phi_k(y_2)| \\
&\le C \sum_{\substack{k\in\mathbb{N}\\ B_{X_\infty}(z_k,2\epsilon)\cap B_l\neq\emptyset}} |u_{B_n(z_{k,n},\epsilon)} - u_{B_n(z_{l,n},\epsilon)}|\frac{d_\infty(y_1,y_2)}{\epsilon}.
\end{aligned}$$

Note that for the indices k in the last sum, we have $d_\infty(z_k, z_l) \le 3\epsilon$ and so $d_n(z_{k,n}, z_{l,n}) \le 5\epsilon$. Thus, each ball $B_n(z_{k,n}, \epsilon)$ is contained in $B_n(z_{l,n}, 6\epsilon)$. Thus, we can continue the estimate for via the Poincaré inequality:

$$\begin{aligned}
|v_n^\epsilon(y_1) - v_n^\epsilon(y_2)| &\le C\fint_{B_n(z_{l,n},6\epsilon)} |u - u_{B_n(z_{l,n},6\epsilon)}|\,d\mu_n \frac{d_\infty(y_1,y_2)}{\epsilon} \\
&\le C\epsilon\,\frac{d_\infty(y_1,y_2)}{\epsilon}\,\frac{P_n(E, B_n(z_{l,n},6\epsilon))}{\mu_n(B_n(z_{l,n},6\epsilon))}.
\end{aligned}$$

Thus, we get for $y \in B_l$,

$$\operatorname{Lip} v_n^\epsilon(y) \le C\frac{P_n(E, B_n(z_{l,n}, 6\epsilon))}{\mu_n(B_n(z_{l,n}, 6\epsilon))}.$$

Therefore, in $B_{X_\infty}(x_\infty, 1)$,

$$\operatorname{Lip} v_n^\epsilon \le C\sum_{l=1}^{\infty}\chi_{B_l}\frac{P_n(E, B_n(z_{l,n}, 6\epsilon))}{\mu_n(B_n(z_{l,n}, \epsilon))}. \tag{5.17}$$

By the definition of pointed measured Gromov-Hausdorff convergence, $\lim_{n\to\infty} d_Z(\iota_n(x), \iota(x_\infty)) = 0$, see Definition 3.3 and the discussion preceding it. If $B_l \cap B_{X_\infty}(x_\infty, 1) \neq \emptyset$, then $B_n(z_{l,n}, 2\epsilon) \cap B_n(x, 1+\varepsilon) \neq \emptyset$. Hence by (5.15) and by the bounded overlap of the family $B_n(z_{k,n}, 6\epsilon)$, $k \in \mathbb{N}$, we have

$$\begin{aligned}
\int_{B_{X_\infty}(x_\infty,1)} \operatorname{Lip} v_n^\epsilon \, d\mu_\infty &\le C \sum_{\substack{l\in\mathbb{N}\\ B_n(z_{l,n},2\epsilon)\cap B_n(x,1+\varepsilon)\neq\emptyset}} \mu_\infty(B_l)\frac{P_n(E, B_n(z_{l,n}, 6\epsilon))}{\mu_n(B_n(z_{l,n}, \epsilon))}\\
&\le C(1+\delta_n) \sum_{\substack{l\in\mathbb{N}\\ B_n(z_{l,n},2\epsilon)\cap B_n(x,1+\varepsilon)\neq\emptyset}} P_n(E, B_n(z_{l,n}, 6\epsilon))\\
&\le CP_n(E, B_n(x, 1+9\epsilon))\\
&\le CP_n(E, B_n(x, 2)).
\end{aligned} \tag{5.18}$$

This remains bounded as $n \to \infty$, see (5.1). We can do the above for a sequence $\epsilon_i \to 0$, with $n = n(i) \to \infty$ and $\delta_{n_i} \to 0$, to obtain a sequence of functions $v_i = v_{n(i)}^{\epsilon_i} \in \operatorname{Lip}(B_{X_\infty}(x_\infty, 1))$. Since $V(v_i, (B_{X_\infty}(x_\infty, 1))$ is bounded by (5.18), we find a subsequence, also denoted by v_i, such that $v_i \to w$ in $L^1(B_{X_\infty}(x_\infty, 1))$, see [32, Theorem 3.7]. By lower semicontinuity,

$$V(w, B_{X_\infty}(x_\infty, 1)) \le \liminf_{i\to\infty}\int_{B_{X_\infty}(x_\infty,1)} \operatorname{Lip} v_i \, d\mu_\infty \le C\limsup_{n\to\infty} P_n(B_n(x, 2)) \le \pi_\infty(x_\infty, 3) \tag{5.19}$$

and so $w \in BV(B_{X_\infty}(x_\infty, 1))$. We need to check that $w = \chi_{(E)_\infty}$ in $L^1(B_{X_\infty}(x_\infty, 1))$. To do so, fix $y \in B_{X_\infty}(x_\infty, 1) \cap (E)_\infty$ and fix $\eta \in (0, 1)$. Then by definition of $(E)_\infty$, for large enough $i \in \mathbb{N}$ we have

$$\frac{\mu_\infty^E(B_{X_\infty}(y, 4\epsilon_i))}{\mu_\infty(B_{X_\infty}(y, 4\epsilon_i))} \ge 1 - \eta.$$

We denote the covering of X_∞ corresponding to an index $i \in \mathbb{N}$ by $B_k^i := B(z_k^i, \epsilon_i)$. It follows that for all balls B_k^i with $2B_k^i$ containing y, we have (note that μ_∞ is doubling with constant C_d^2)

$$\frac{\mu_\infty^{E^c}(B_{X_\infty}(z_k^i, \epsilon_i))}{\mu_\infty(B_{X_\infty}(z_k^i, \epsilon_i))} \le C_d^6\frac{\mu_\infty^{E^c}(B_{X_\infty}(y, 4\epsilon_i))}{\mu_\infty(B_{X_\infty}(y, 4\epsilon_i))} \le C_d^6\eta.$$

Thus by (5.14) and (5.15),

$$\frac{\mu_{n_i}(B_{n_i}(z_{k,n_i}, \epsilon_i) \cap E)}{\mu_{n_i}(B_{n_i}(z_{k,n_i}, \epsilon_i))} \ge \frac{1-\delta_{n_i}}{1+\delta_{n_i}}\frac{\mu_\infty^E(B_{X_\infty}(z_k^i, \epsilon_i))}{\mu_\infty(B_{X_\infty}(z_k^i, \epsilon_i))} \ge \frac{1-\delta_{n_i}}{1+\delta_{n_i}}(1 - C_d^6\eta).$$

Now, by definition of the discrete convolutions (5.16), we have

$$v_i(y) \ge \frac{1-\delta_{n_i}}{1+\delta_{n_i}}(1 - C_d^6\eta).$$

Letting $i \to \infty$, we get

$$w(y) \ge 1 - C_d^6\eta.$$

Since $\eta > 0$ was arbitrary, we conclude $w(y) = 1$ (the values taken on by the functions v_i are between 0 and 1, so necessarily $w(y) \le 1$). Similarly, we get $w(y) = 0$ for all $y \in (E^c)_\infty$. Also by Proposition 5.2 we know that $\mu_\infty(X_\infty \setminus ((E)_\infty \cup (E^c)_\infty)) = 0$. Thus $w = \chi_{(E)_\infty}$ as functions in $L^1(B_{X_\infty}(x_\infty, 1))$. Recall that we are assuming $R = 1$ just for convenience; we conclude that $\chi_{(E)_\infty} \in BV(B_{X_\infty}(x_\infty, R))$ for all $R > 0$.

Next, for $z \in X_\infty$ and $r > 0$, by an argument analogous to that leading to (5.19), we obtain

$$P((E)_\infty, B(z,r)) \le \pi_\infty(z, 3r). \tag{5.20}$$

From the final part of the proof of Theorem 5.4, we know that π_∞ is supported inside $X \setminus (U \cup V)$, where U and V are (open) neighborhoods of $(E)_\infty$ and $(E^c)_\infty$, respectively. Conversely, if $z \in X_\infty \setminus [(E)_\infty \cup (E^c)_\infty]$, which we recall is the same set as $\partial^*(E)_\infty$, then z is in the support of $P((E)_\infty, \cdot)$ by the relative isoperimetric inequality (2.4). Thus by (5.20), z is in the support of π_∞. In conclusion, the support of π_∞ is exactly $\partial^*(E)_\infty = X_\infty \setminus [U \cup V]$. Moreover, by (5.11) and Lemma 2.1 we know that π_∞ is comparable to $\mathcal{H}_\infty(\partial^*(E)_\infty \cap \cdot)$. By (2.8) we know that $P((E)_\infty, \cdot)$ is also comparable to $\mathcal{H}_\infty(\partial^*(E)_\infty \cap \cdot)$. Thus the three measures π_∞, $P((E)_\infty, \cdot)$, and $\mathcal{H}_\infty(\partial^*(E)_\infty \cap \cdot)$ are all comparable. Finally, by the relative isoperimetric inequality and the fact that $P((E)_\infty, \cdot)$ does not see the set U, it follows that U cannot intersect $(E^c)_\infty$. Thus $(E)_\infty = U$ and similarly $(E^c)_\infty = V$. □

Remark 5.6. If μ is an Ahlfors s-regular measure for some $s > 1$ (recall (2.7)), then it is straightforward to verify that μ_∞ is also Ahlfors s-regular in X_∞, and then by Theorems 5.4 and 5.5, $P((E)_\infty, \cdot)$ (and π_∞) are Ahlfors $(s-1)$-regular measures in X_∞. This corresponds to what we get in a Euclidean space $\mathbb{R}^n$, for $s = n$.

6 Asymptotic quasi-least gradient property

From Theorem 4.9 we now know that asymptotic limits (μ-a.e.) of a BV function outside of the Cantor and jump parts of the function are of least gradient. We will show in this section that at co-dimension 1 almost every point of the measure-theoretic boundary of a set E of finite perimeter, any limit set $(E)_\infty$ is a set of quasiminimal boundary surface as defined in [25], that is, $\chi_{(E)_\infty}$ is of quasi-least gradient. First we develop some preliminary results that are also of independent interest.

6.1 Asymptotic minimality for sets of finite perimeter

The following theorem shows that given a set E of finite perimeter, at essentially almost every point in ∂^*E the set E is asymptotically a minimal surface; compare this to [1, Proposition 5.7], where a weaker notion of asymptotic quasiminimality is established, where the quasiminimality condition requires to compare (locally) the perimeter of E with the perimeter of modifications of E by balls alone.

Theorem 6.1. *Let $E \subset X$ be a set of finite perimeter. Then*

$$\lim_{r\to 0}\left(\frac{\inf_{u\in BV_c(B(x_0,r))} V(\chi_E+u, B(x_0,r))}{P(E,B(x_0,r))}\right) \geq 1$$

for $P(E,\cdot)$-a.e. $x \in X$.

Proof. Let

$$A := \left\{x \in X : \liminf_{r\to 0}\left(\frac{\inf_{u\in BV_c(B(x,r))} V(\chi_E+u, B(x,r))}{P(E,B(x,r))}\right) < 1\right\}$$

Note that A is the increasing limit of sets A_n where

$$A_n := \left\{x \in X : \liminf_{r\to 0}\left(\frac{\inf_{u\in BV_c(B(x,r))} V(\chi_E+u, B(x,r))}{P(E,B(x,r))}\right) < 1-\frac{1}{n}\right\}, \quad n\in\mathbb{N}.$$

It therefore suffices to show that each A_n satisfies $P(E,A_n)=0$. To this end, fix $n\in\mathbb{N}$. Then for every $x\in A_n$, there exist $r_i^x\to 0$ and $u_i^x\in BV_c(B(x,r_i^x))$ with

$$\frac{V(\chi_E+u_i^x, B(x,r_i^x))}{P(E,B(x,r_i^x))} < 1-n^{-1}. \tag{6.1}$$

Furthermore, as $P(E,X)<\infty$, for every $x\in X$ we have $P(E,\partial B(x,r))=0$ for $\mathcal{H}^1$-almost every $r>0$. We can therefore choose $r_i^x>0$ such that in addition to the above, $P(E,\partial B(x,r_i^x))=0$ for every $x\in A_n$. Fix $k\in\mathbb{N}$ such that $1/k<\frac{1}{4}\operatorname{diam} X$. The collection $\{\overline{B}(x,r_i^x) : 0<r_i^x<1/k\}_{x\in A_n}$ is a fine cover of A_n, that is, for every $x\in A_n$ we have $\inf_i r_i^x=0$. By (2.11) we know that $P(E,\cdot)$ is asymptotically doubling, and so it satisfies the Vitali covering theorem, see [21, Theorem 3.4.3]. So we can pick a countable pairwise disjoint collection $\{B_j^k=B(x_j^k,r_j^k)\}_{j=1}^\infty =: \mathcal{G}_k$ such that, recalling also that $P(E,\partial B)=0$ for each $B\in\mathcal{G}_k$,

$$P\left(E, A_n\setminus \bigcup_{B\in\mathcal{G}_k} B\right) = P\left(E, A_n\setminus \bigcup_{B\in\mathcal{G}_k} \overline{B}\right)=0. \tag{6.2}$$

We use the collection of balls B_j^k to perturb the function χ_E. Recall that for each ball B_j^k there is a function $u_j^k\in BV_c(B_j^k)$ as in (6.1). Set

$$h_k := \chi_E + \sum_{j=1}^\infty u_j^k.$$

By the 1-Poincaré inequality (2.6) for compactly supported functions, for all $j\in\mathbb{N}$

$$\int_{B_j^k}|u_j^k|\,d\mu \le C r_j^k\, V(u_j^k,B_j^k) \le C r_j^k\left[V(\chi_E+u_j^k,B_j^k)+V(\chi_E,B_j^k)\right] \le C\frac{2-n^{-1}}{k}V(\chi_E,B_j^k).$$

Therefore by the pairwise disjointness of the balls in the collection $\mathcal{G}_k$,

$$\int_X|\chi_E-h_k|\,d\mu \le \sum_{j=1}^\infty \int_{B_j^k}|u_j^k|\,d\mu \le \frac{C}{k}V(\chi_E,X).$$

Therefore $h_k \to \chi_E$ in $L^1(X)$ as $k \to \infty$. By the lower semicontinuity of the total variation,

$$V(\chi_E, X) \le \liminf_{k\to\infty} V(h_k, X). \tag{6.3}$$

For ease of notation, for each $j \in \mathbb{N}$ let

$$G_{k,j} := \bigcup_{i=1}^{j} \overline{B}_i^k \quad \text{and} \quad h_{k,j} := u + \sum_{i=1}^{j} \varphi_i^k.$$

Now

$$\begin{aligned} V(h_{k,j}, G_{k,j}) &\le V(h_{k,j}, \bigcup_{i=1}^{j} B_i^k) + V(h_{k,j}, \bigcup_{i=1}^{j} \partial B_i^k) \\ &= V(h_{k,j}, \bigcup_{i=1}^{j} B_i^k) + \sum_{i=1}^{j} V(\chi_E, \partial B_i^k) \\ &= V(h_{k,j}, \bigcup_{i=1}^{j} B_i^k) \end{aligned}$$

and so $V(h_{k,j}, G_{k,j}) = V(h_{k,j}, \bigcup_{i=1}^{j} B_i^k)$. Since $G_{k,j}$ is a closed set, it follows that

$$\begin{aligned} V(h_{k,j}, X) &= V(h_{k,j}, G_{k,j}) + V(h_{k,j}, X \setminus G_{k,j}) \\ &= \sum_{i=1}^{j} V(h_{k,j}, B_i^k) + V(\chi_E, X \setminus G_{k,j}) \\ &< (1 - n^{-1}) \sum_{i=1}^{j} V(\chi_E, B_i^k) + V(\chi_E, X \setminus G_{k,j}) \quad \text{by (6.1)} \\ &= V(\chi_E, X) - n^{-1} V\left(\chi_E, \bigcup_{i=1}^{j} B_i^k\right). \end{aligned}$$

Therefore

$$\begin{aligned} V(h_k, X) \le \liminf_{j\to\infty} V(h_{k,j}, X) &\le V(\chi_E, X) - n^{-1} \lim_{j\to\infty} V\left(\chi_E, \bigcup_{i=1}^{j} B_i^k\right) \\ &= V(\chi_E, X) - n^{-1} V\left(\chi_E, \bigcup_{B \in \mathcal{G}_k} B\right). \end{aligned} \tag{6.4}$$

Set $K_k := \bigcup_{B\in\mathcal{G}_k} B$ and $F_k := A_n \setminus K_k$ for each $k \in \mathbb{N}$. We have $P(E, F_k) = 0$ for each $k \in \mathbb{N}$ by (6.2). For $F := \bigcup_{k=1}^{\infty} F_k$ and $K := \bigcap_{k=1}^{\infty} K_k$ we then have $P(E, F) = 0$. In light of (6.3) and (6.4), we have

$$P(E, X) \le P(E, X) - \liminf_{k\to\infty} n^{-1} P(E, K_k) \le P(E, X) - n^{-1} P(E, K),$$

and so $P(E, K) = 0$. Since $A_n \subset F \cup K$, it follows that $P(E, A_n) = 0$. This completes the proof. $\square$

A similar analysis can be carried out for functions $u \in BV(X)$ with slightly more involved computations to obtain analogous asymptotic minimality results for u; we do not do so here as we have a stronger result for u outside its jump and Cantor sets in Theorem 4.9.

6.2 Quasiminimality at almost every point

In this subsection we finally prove the quasiminimality property of the asymptotic limit set $(E)_\infty$.

Definition 6.2. A set $E \subset X$ is said to be K-*quasiminimal,* $K \geq 1$, if for every $B(x, R) \subset X$ and every $\phi \in BV_c(B(x, R))$ we have

$$\frac{1}{K} P(E, B(x, R)) \leq V(\chi_E + \phi, B(x, R)).$$

Without loss of generality and applying a truncation, one can restrict attention to ϕ with values in $[-1, 1]$, and such that $\chi_E + \phi$ has values in $[0, 1]$.

The asymptotic minimality of E (Theorem 6.1) can be upgraded to quasiminimality at generic tangents of the limit set $(E)_\infty$. In terms of notation, here we only consider the sequence

$$(X_n, d_n, x, \mu_n) := \left(X, \frac{1}{r_n} \cdot d, x, \frac{1}{\mu(B(x, r_n))} \cdot \mu \right)$$

under the pointed measured Gromov-Hausdorff convergence, with $r_n \searrow 0$, see the discussion in Section 3.

Theorem 6.3. *Let $E \subset X$ be a set of finite perimeter. Then, for $P(E, \cdot)$-almost every $x \in X$ and for any space $(X_\infty, d_\infty, x_\infty, \mu_\infty)$ arising as a pointed measured Gromov-Hausdorff tangent at x, any set $(E)_\infty$ that arises as an asymptotic limit of E along some sequence $r_n \searrow 0$ is a K-quasiminimizer. Here, K depends quantitatively on the data.*

The proof involves lifting Lipschitz functions with small energy to the sequence, and a pasting argument. The desired quasiminimality estimate then follows using Theorem 6.1 for the lifted sequence. We need the following general BV approximation theorem, which is an analog of [11, Lemma 5.2] for $p = 1$. We follow the arguments of Cheeger.

Proposition 6.4. *Let $f \in BV(X)$. Then, there exist Lipschitz continuous f_i with bounded Lipschitz continuous upper gradients v_i such that $f_i \to f$ in $L^1_{loc}(X)$ and $v_i \, d\mu \overset{*}{\rightharpoonup} dV(f, \cdot)$.*

To prove this proposition we define the following auxiliary function. For a nonnegative Borel function g on X we set $\mathcal{F}_g \colon X \times X \to [0, \infty]$ to be

$$\mathcal{F}_g^X(x_1, x_2) := \mathcal{F}_g(x_1, x_2) := \inf_\gamma \int_\gamma g \; ds,$$

whenever $x_1, x_2 \in X$. If $x_1 = x_2$, we set $\mathcal{F}_g(x_1, x_2) = 0$. The infimum is taken over all rectifiable curves γ connecting x_1 to x_2. Note that by the definition of upper gradient (2.2), we have that if g is an upper gradient of a function $f \colon X \to \mathbb{R}$, then for every $x, y \in X$,

$$|f(x) - f(y)| \leq \mathcal{F}_g(x, y).$$

For the proof of the following lemma see [11, Lemma 5.18] or [20, pp. 13–14].

Lemma 6.5. *Fix $\eta > 0$. Let $g \colon X \to [\eta, \infty)$ be a countably valued lower semicontinuous function. Then for $g_n \geq \eta$ an increasing sequence of Lipschitz continuous functions on X converging pointwise $g_n \nearrow g$, we have that for every $x, y \in X$,*

$$\mathcal{F}_g(x, y) = \lim_{n \to \infty} \mathcal{F}_{g_n}(x, y).$$

Moreover, such a sequence g_n exists.

Lemma 6.6. *Let $f \in BV(X)$. Then there is a sequence of Lipschitz functions f_k on X such that $f_k \to f$ in $L^1_{loc}(X)$ and $g_k\, d\mu \overset{*}{\rightharpoonup} dV(f,\cdot)$. Here $g_k = \operatorname{lip} f_k$ is an upper gradient of f_k.*

Proof. By the definition of the total variation we can find a sequence of locally Lipschitz functions f_k and upper gradients $g_k = \operatorname{lip} f_k$ such that $f_k \to f$ in $L^1_{\mathrm{loc}}(X)$ and $\lim_k \int_X g_k\, d\mu = V(f,X)$. Multiplying with suitable cutoff functions if necessary, we can assume that the f_k are Lipschitz. For any open set $U \subset X$, we have by the definition of the total variation that

$$V(f,U) \le \liminf_{k\to\infty} \int_U g_k\, d\mu. \tag{6.5}$$

On the other hand, for any closed set $F \subset X$ we have

$$V(f,X) = \lim_{k\to\infty} \int_X g_k\, d\mu \ge \limsup_{k\to\infty} \int_F g_k\, d\mu + \liminf_{k\to\infty} \int_{X\setminus F} g_k\, d\mu \ge \limsup_{k\to\infty} \int_F g_k\, d\mu + V(f, X\setminus F),$$

where the last inequality again follows by the definition of the total variation. Thus

$$\limsup_{k\to\infty} \int_F g_k\, d\mu \le V(f,F).$$

According to a standard characterization of the weak* convergence of Radon measures, see e.g. [12, p. 54], the above inequality and (6.5) together give $g_k\, d\mu \overset{*}{\rightharpoonup} dV(f,\cdot)$. □

Lemma 6.7. *Let f be a nonnegative Lipschitz function on X and $g \in L^1_{loc}(X)$ a bounded countably valued lower semicontinuous upper gradient of f. Suppose that there is a $\tau > 0$ such that $g \ge \tau$ on X. Then there is a sequence f_k of Lipschitz continuous functions on X with $f_k \to f$ in $L^1_{loc}(X)$ and bounded Lipschitz continuous upper gradients g_k of f_k such that $g_k \to g$ in $L^1_{loc}(X)$ and g_k monotone increases to g everywhere on X, and $g_k \ge \tau$ for each k.*

Proof. Since g is lower semicontinuous, we can find a sequence of Lipschitz continuous functions $g_k \ge \tau$ on X such that $g_k \to g$ in $L^1_{\mathrm{loc}}(X)$ and in addition $g_k \le g_{k+1} \le g$ on X for each $k \in \mathbb{N}$. By Lemma 6.5 we know that $\mathcal{F}_g = \lim_k \mathcal{F}_{g_k}$ pointwise everywhere on $X \times X$.

Next, we fix $x_0 \in X$ and for each positive integer i let $\widehat{A_i}$ be a maximal $1/i$-net of X such that $\widehat{A_i} \subset \widehat{A_{i+1}}$ for each $i \in \mathbb{N}$, and let $A_i = \widehat{A_i} \cap B(x_0, 2i)$. Then $A_i \subset A_{i+1}$, and by the doubling property of μ we know that A_i is a finite set for each i. As g is bounded, we can also ensure that each $g_k \le M$ and $g \le M$ on X for some positive M. Therefore for each $y \in X$ we know that $\mathcal{F}_g$ and $\mathcal{F}_{g_k}$ are MC-Lipschitz where C is the quasiconvexity constant of X. Now, taking inspiration from the McShane extension (see also [11]), we set

$$f_k(x) := \inf\{f(y) + \mathcal{F}_{g_k}(x,y) \,:\, y \in A_k\}.$$

Then f_k is also MC-Lipschitz on X. A standard argument (see e.g. [21, p. 384]) shows that g_k is an upper gradient of f_k.

For $x \in \bigcup_n A_n$, we choose $n \in \mathbb{N}$ such that $x \in A_n$; then for $k \ge n+1$ we see that $x \in A_k$. It follows that $f_k(x) \le f(x)$. If $y \in X \setminus B(x,L)$ for some $L > 0$ then as f is nonnegative, $f(y) + \mathcal{F}_{g_k}(x,y) \ge L\tau$; thus to obtain $f_k(x)$ it suffices to look only at $y \in A_k \cap B(x,L)$ where $L = [1 + f(x)]/\tau$. Let $y_k \in A_k \cap B(x,L)$ such that

$$k^{-1} + f_k(x) \ge f(y_k) + \mathcal{F}_{g_k}(x, y_k).$$

Then the sequence (y_k) lies in the compact set $\overline{B}(x, L)$ and hence has a subsequence y_{k_j} converging to some $y_\infty \in \overline{B}(x, L)$. Thus $f(x) \geq \lim_{k\to\infty} f_k(x) \geq f(y_\infty) + \lim_{k\to\infty} \mathcal{F}_{g_k}(x, y_k)$. Observe that

$$|\mathcal{F}_{g_k}(x, y_k) - \mathcal{F}_{g_k}(x, y_\infty)| \leq MC\, d(y_k, y_\infty).$$

It then follows from Lemma 6.5 that

$$f(x) \geq \lim_{k\to\infty} f_k(x) \geq f(y_\infty) + \lim_{k\to\infty} \mathcal{F}_{g_k}(x, y_\infty) = f(y_\infty) + \mathcal{F}_g(x, y_\infty) \geq f(x),$$

and it then follows that $\lim_{k\to\infty} f_k(x) = f(x)$. Now the uniform Lipschitz continuity of f_k, $k \in \mathbb{N}$ and f shows that $\lim_k f_k = f$ pointwise on X. An appeal to the Lebesgue dominated convergence theorem (and the fact that $f_k \leq \|f\|_{L^\infty(B)} + Mk < \infty$ on the ball $B = B(x_0, k)$) yields the convergence also in $L^1_{\mathrm{loc}}(X)$. □

The above lemmas allow us now to prove Proposition 6.4.

Proof of Proposition 6.4. By Lemma 6.6 we obtain a sequence f_k of Lipschitz functions on X with $f_k \to f$ in $L^1_{\mathrm{loc}}(X)$ and upper gradients $g_k = \operatorname{lip} f_k$ of f_k such that $g_k\, d\mu \overset{*}{\rightharpoonup} dV(f, \cdot)$. Note that each g_k is bounded. By the Vitali-Carathéodory theorem, see e.g. [21, p. 108], for each k we can find a bounded countably valued lower semicontinuous function $g'_k \geq g_k$ such that $\|g'_k - g_k\|_{L^1(X)} \to 0$ as $k \to \infty$. Note that automatically g'_k is also an upper gradient of f_k. Moreover, we now have $g'_k\, d\mu \overset{*}{\rightharpoonup} dV(f, \cdot)$, and so we also have $[g'_k + k^{-1}]d\mu \overset{*}{\rightharpoonup} dV(f, \cdot)$.

Next we apply Lemma 6.7 to obtain bounded Lipschitz functions v_k and Lipschitz functions F_k such that v_k is an upper gradient of F_k, $F_k \to f$ in $L^1_{\mathrm{loc}}(X)$, and $v_k - [g'_k + k^{-1}] \to 0$ in $L^1_{\mathrm{loc}}(X)$ as $k \to \infty$. It follows then also that $v_k\, d\mu \overset{*}{\rightharpoonup} dV(f, \cdot)$, completing the proof of the proposition. □

We will need the following lemma from Keith [24, Proposition 4], see also [20, proof of Proposition 2.17]. This lemma is a simple consequence of the Arzelà-Ascoli theorem together with the lower semicontinuity of g. In the lemmas below we will consider curves to be arc length parametrized.

Lemma 6.8. [24, Proposition 4] *If Z is a proper space, $g\colon Z \to \mathbb{R}$ a nonnegative lower semicontinuous function, $L > 0$, and γ_n a sequence of curves in Z with length at most L and are contained in a fixed compact subset of Z, then there exists a rectifiable curve γ_∞ so that a subsequence of γ_n converges to γ_∞ uniformly. For such γ_∞ we also have that*

$$\int_{\gamma_\infty} g\, ds \leq \liminf_{n\to\infty} \int_{\gamma_n} g\, ds.$$

As a corollary, we obtain the following.

Lemma 6.9. *Let $g\colon Z \to [\tau, \infty)$ be a nonnegative lower semicontinuous function on a proper space Z for some $\tau > 0$, and assume that $x_n \to x$ and $y_n \to y$ are sequences of points in Z. Then,*

$$\mathcal{F}_g(x, y) \leq \liminf_{n\to\infty} \mathcal{F}_g(x_n, y_n). \tag{6.6}$$

Note that we avoid assuming Z has any rectifiable curves, or that it is quasiconvex. This is necessary for our application where Z is the proper metric space into which the sequence of scaled spaces X_i and the tangent space X_∞ embed isometrically as described in the latter part of Remark 3.2.

Proof. If the limit infimum on the right hand side of (6.6) is infinite, there is nothing to prove. So we will assume that it is finite. By passing to a subsequence, we can assume that there is some real number $M > 0$ such that $\mathcal{F}_g(x_n, y_n) \leq M$ for all n. Then for every $0 < \epsilon < M$, there exist curves γ_n connecting x_n and y_n such that

$$\tau\ell(\gamma_n) \leq \int_{\gamma_n} g \; ds \leq \mathcal{F}_g(x_n, y_n) + \epsilon \leq 2M.$$

Since γ_n connects x_n to y_n, and these converge, respectively, to x and y, the curves γ_n lie, for sufficiently large n, in the closed ball $\overline{B(x, M + 2M/\tau)}$ which is compact. Then, by Lemma 6.8, by taking a subsequence if necessary, the sequence γ_n converges to some curve γ_∞, and

$$\mathcal{F}_g(x, y) \leq \int_{\gamma_\infty} g \, ds \leq \liminf_{n\to\infty} \int_{\gamma_n} g \, ds \leq \liminf_{n\to\infty} \mathcal{F}_g(x_n, y_n) + \epsilon.$$

Since this holds for every small $\epsilon > 0$ the claim follows. $\square$

Lemma 6.10. *Let $(X_i, d_i, x_i, \mu_i) \to (X_\infty, d_\infty, x_\infty, \mu_\infty)$ be a sequence of scaled (from X) metric measure spaces converging in the pointed measured Gromov-Hausdorff sense. If f is a nonnegative Lipschitz function on X_∞, with a bounded Lipschitz upper gradient v, then there exists a subsequence, also denoted (X_i, d_i, x_i, μ_i), and uniformly Lipschitz continuous functions f_i with Lipschitz continuous upper gradients v_i on X_i such that*

$$v_i \, d\mu_i \overset{*}{\rightharpoonup} v \, d\mu_\infty,$$

and f is a limit function of f_i in the sense of (3.3).

Proof. Without loss of generality, we can assume that $v \geq \tau$ for some positive τ, since otherwise we can obtain the result by considering $\max\{v, 1/k\}$ instead of v for each positive integer k, and then complete the proof with the help of a diagonalization argument, letting $k \to \infty$.

Let $\hat{v}\colon Z \to \mathbb{R}$ be a McShane extension of the Lipschitz function $v \circ \iota|_{\iota(X_\infty)}^{-1}$ on $\iota(X_\infty)$ to the entirety of Z. Also, such an extension can be chosen to be bounded and so that $\hat{v} \geq \tau$. Let $v_i := \hat{v} \circ \iota_i \colon X_i \to \mathbb{R}$.

Next, let $\hat{f}\colon Z \to \mathbb{R}$ be constructed similarly, by first setting $\hat{f}(z) := f \circ \iota^{-1}(z)$ for $z \in \iota(X_\infty)$, and then taking a McShane extension to Z. We can choose $\hat{f}$ to be nonnegative. Next we construct the functions $f_i\colon X_i \to \mathbb{R}$ so that v_i is an upper gradient of f_i as follows. For $x \in X_i$ we set

$$f_i(x) := \inf_{y\in X_i} [\hat{f}(\iota_i(y)) + \mathcal{F}_{\hat{v}}^{\iota_i(X_i)}(\iota_i(y), \iota_i(x))] = \inf_{y\in X_i} [\hat{f}(\iota_i(y)) + \mathcal{F}_{v_i}^{X_i}(y, x)].$$

For ease of notation, we set $\mathcal{F}_{v_i}(x, y) := \mathcal{F}_{\hat{v}}^{\iota_i(X_i)}(\iota_i(y), \iota_i(x))$ for $x, y \in X_i$. From the definition of f_i it is clear that $f_i(x) \leq \hat{f}(\iota_i(x))$ for each $x \in X_i$. Also, f_i is nonnegative, and has v_i as an upper gradient.

We will now show that f is a limit function of f_i. To do so, we need to show for every $r > 0$,

$$\lim_{i\to\infty} \|f - f_i \circ \phi_i\|_{L^\infty(B_{X_\infty}(x_\infty, r))} = 0,$$

where ϕ_i are the approximating maps from Definition 3.1. Suppose this is not the case. Then there is some $r > 0$ and some $\delta > 0$ such that, by passing to a subsequence if needed, we have

$$\liminf_{i\to\infty} \|f - f_i \circ \phi_i\|_{L^\infty(B_{X_\infty}(x_\infty, r))} > \delta. \tag{6.7}$$

Thus, for each i there is a point $x_i \in B_{X_\infty}(x_\infty, r)$ such that

$$|f(x_i) - f_i(\phi_i(x_i))| > \delta. \tag{6.8}$$

Since X_∞ is proper and $x_i \in B_{X_\infty}(x_\infty, r)$ for all i, there is a subsequence, also denoted with the index i, such that $x_i \to x \in X_\infty$. Fix $\delta > 0$. Then from the definition of $f_i(\phi_i(x_i))$, we have $y_i \in X_i$ such that

$$|f_i(\phi_i(x_i)) - \hat{f}(\iota_i(y_i)) - \mathcal{F}_{v_i}(y_i, \phi_i(x_i))| \le \delta/4. \tag{6.9}$$

Combining the above with (6.8) we get

$$|f(x_i) - \hat{f}(\iota_i(y_i)) - \mathcal{F}_{v_i}(y_i, \phi_i(x_i))| \ge \delta/2.$$

By (6.9) we have $\hat{f}(\iota_i(y_i)) + \mathcal{F}_{v_i}(y_i, \phi_i(x_i)) \le f_i(\phi_i(x_i)) + \delta/4 \le \hat{f}(\iota_i(\phi_i(x_i))) + \delta/4$, and so the triangle inequality gives

$$|f(x_i) - \hat{f}(\iota_i(\phi_i(x_i)))| + \hat{f}(\iota_i(\phi_i(x_i))) - \hat{f}(\iota_i(y_i)) - \mathcal{F}_{v_i}(y_i, \phi_i(x_i)) \ge \delta/4. \tag{6.10}$$

For the first term, note that $f(x_i) = \hat{f}(\iota(x_i))$, and from (3.1) we get $\lim_{i\to\infty} d_Z(\iota(x_i), \iota_i(\phi_i(x_i))) = 0$, and thus from the Lipschitz continuity of $\hat{f}$,

$$\lim_{i\to\infty} |\hat{f}(\iota(x_i)) - \hat{f}(\iota_i(\phi_i(x_i)))| = 0. \tag{6.11}$$

Since $\lim_i x_i = x$, we also have

$$\begin{aligned} d_Z(\iota_i(\phi_i(x_i)), \iota(x)) &\le d_Z(\iota_i(\phi_i(x_i)), \iota(x_i)) + d_Z(\iota(x_i), \iota(x)) \\ &= d_Z(\iota_i(\phi_i(x_i)), \iota(x_i)) + d_{X_\infty}(x_i, x) \to 0 \quad \text{as } i \to \infty. \end{aligned} \tag{6.12}$$

Therefore the sequence of real numbers $\hat{f}(\iota_i(\phi_i(x_i)))$ is bounded, that is, there is some $M > \delta > 0$ such that $\sup_i \hat{f}(\iota_i(\phi_i(x_i))) \le M$. The functions v_i are bounded from below by τ and f is nonnegative. Therefore, if $d(\phi_i(x_i), y_i) > 2M/\tau$, then

$$\hat{f}(\iota_i(y_i)) + \mathcal{F}_{v_i}(y_i, \phi_i(x_i)) \ge \mathcal{F}_{v_i}^{X_i}(y_i, \phi_i(x_i)) \ge 2M > \hat{f}(\iota_i(\phi_i(x_i))) + \delta \ge f_i(\phi_i(x_i)) + \delta,$$

which would violate the choice of y_i, (6.9). Hence we must have $d(\phi_i(x_i), y_i) \le 2M/\tau$. As the sequence $\iota_i(\phi_i(x_i))$ lies in a ball, in Z, centered at $\iota(x)$ by (6.12), we see then that the sequence $\iota_i(y_i)$ also lies in a ball centered at $\iota(x)$. Therefore, by the properness of Z, there is a subsequence, also denoted with the index i, and a point $\hat{y} \in Z$ such that $\lim_i \iota_i(y_i) = \hat{y}$. As $y_i \in X_i$ and X_i converges to the metric space X_∞, it follows that $\hat{y} = \iota(y)$ for some $y \in X_\infty$. Then by Lemma 6.9 we get $\mathcal{F}_{\hat{v}}^Z(\iota(x), \iota(y)) \le \liminf_{i\to\infty} \mathcal{F}_{\hat{v}}^Z(\iota_i(\phi_i(x_i)), \iota_i(y_i))$. Note that $\mathcal{F}_{v_i}(y_i, \phi_i(x_i)) = \mathcal{F}_{\hat{v}}^{\iota_i(X_i)}(\iota_i(y_i), \iota_i(\phi_i(x_i)))$, which is not the same as $\mathcal{F}_{\hat{v}}^Z(\iota_i(y_i), \iota_i(\phi_i(x_i)))$. However, we have that $\mathcal{F}_{\hat{v}}^Z(\iota_i(y_i), \iota_i(\phi_i(x_i))) \le \mathcal{F}_{\hat{v}}^{\iota_i(X_i)}(\iota_i(y_i), \iota_i(\phi_i(x_i)))$. Now by (6.10) and (6.11), we obtain

$$\begin{aligned} \tfrac{\delta}{4} + \mathcal{F}_{\hat{v}}^Z(\iota(x), \iota(y)) &\le \tfrac{\delta}{4} + \liminf_i \mathcal{F}_{\hat{v}}^Z(\iota_i(y_i), \iota_i(\phi_i(x_i))) \\ &\le \lim_i [\hat{f}(\iota_i(\phi_i(x_i))) - \hat{f}(\iota_i(y_i))] = \hat{f}(\iota(x)) - \hat{f}(\iota(y)) = f(x) - f(y). \end{aligned}$$

We now use the specific structure of Z; by [22], we can choose Z to be the completion of pairwise disjoint union of X_i, $i \in \mathbb{N}$. With such a choice, it follows that if γ is a non-constant rectifiable curve in Z, then either γ lies entirely in $\iota_i(X_i)$ for some positive integer i, or else γ lies entirely in $\iota(X_\infty)$. It follows that

$$\mathcal{F}_{\hat{v}}^{Z}(\iota(x),\iota(y)) = \mathcal{F}_{\hat{v}}^{\iota(X_\infty)}(\iota(x),\iota(y)) = \mathcal{F}_{v}^{X_\infty}(x,y).$$

Hence from the above inequality we obtain

$$\mathcal{F}_{v}^{X_\infty}(x,y) < \tfrac{\delta}{4} + \mathcal{F}_{v}^{X_\infty}(x,y) \le f(x) - f(y) \le |f(x) - f(y)|,$$

which is not possible as v is an upper gradient of f. Thus (6.7) is false, and so $f = \lim_i f_i$ as desired.

Finally, we show that $v_i \, d\mu_i \overset{*}{\rightharpoonup} v \, d\mu_\infty$ as follows. Pick a test function $\phi \in C_c(Z)$. Then also $\phi\hat{v} \in C_c(Z)$. Using this fact and the fact that $\iota_{i,*}\mu_i \overset{*}{\rightharpoonup} \iota_*\mu_\infty$, we get

$$\lim_{i\to\infty} \int_Z \phi \, \iota_{i,*}(v_i \, d\mu_i) = \lim_{i\to\infty} \int_Z \phi \, \hat{v} \, d\iota_{i,*}\mu_i = \int_Z \phi \, \hat{v} \, d\iota_*\mu_\infty = \int_Z \phi \, \iota_*(v \, d\mu_\infty).$$

□

We also need the following lemma, which stitches two given BV functions along an annulus to yield a BV function whose BV energy is controllable.

Lemma 6.11. **[32, Lemma 3.3]** *Let $f \in BV(X)$, $x \in X$, $0 < a < b \le R$, and $g \in BV(B(x,b))$. Then there exists a $2/(b-a)$-Lipschitz function $\eta : X \to [0,1]$ with compact support in $B(x,b)$, and such that $\eta = 1$ on $B(x,a)$ such that $h = \eta g + (1-\eta) f \in BV(X)$ with*

$$V(h, B(x,R)) \le V(f, B(x,R) \setminus \overline{B(x,a)}) + V(g, B(x,b)) + \frac{2}{b-a} \int_{B(x,b)\setminus B(x,a)} |f-g| \, d\mu.$$

Finally, we can conclude the proof of Theorem 6.3.

Proof of Theorem 6.3. Let $x \in \Sigma_\gamma$ be a point where the conclusions of Theorem 6.1, Lemma 2.2 and Theorem 5.5 hold. We will show that the corresponding asymptotic set $(E)_\infty$ is K-quasiminimal for some K, which will be determined at the end of the proof. Since $P(E,\cdot)$-almost every $x \in X$ is such a point, this concludes the proof. Let $R > 0$, $z \in X_\infty$, and $\varphi \in BV_c(B_{X_\infty}(z,R))$. By slightly decreasing R if necessary, we can assume that $P((E)_\infty, \partial B_{X_\infty}(z,R)) = 0$.

From Theorem 6.1, there is some $r_0 > 0$ such that that for every $r_0 > r > 0$ there is some positive ε_r such that $\lim_{r\to 0^+} \varepsilon_r = 0$ and whenever $\psi \in BV_c(B_X(x,r))$, we have

$$P(E, B_X(x,r)) \le (1+\varepsilon_r) \, V(\chi_E + \psi, B_X(x,r)). \tag{6.13}$$

By a standard truncation argument, we can assume without loss of generality that $0 \le \chi_{(E)_\infty} + \varphi \le 1$. By Proposition 6.4 we can find a sequence of Lipschitz function–Lipschitz upper gradient pairs f_i, v_i on X_∞, with each v_i bounded, such that $f_i \to \chi_{(E)_\infty} + \varphi$ in $L^1_{\mathrm{loc}}(X_\infty)$ and $v_i \, d\mu_\infty \overset{*}{\rightharpoonup} dV(\chi_{(E)_\infty} + \varphi, \cdot)$. Next, for each positive integer i we apply Lemma 6.10 to obtain lifts $f_{i,n}, v_{i,n}$ to X_n such that $v_{i,n} \, d\mu_n \overset{*}{\rightharpoonup} v_i \, d\mu_\infty$ and $f_{i,n} \to f_i$. Further, by truncating each f_i and $f_{i,n}$, we can also assume $0 \le f_i, f_{i,n} \le 1$.

By passing to a subsequence of (X_n, d_n, x, μ_n) if necessary, with ρ fixed and chosen in the interval $[2[R + d_{X_\infty}(z, x_\infty)], 3[R + d_{X_\infty}(z, x_\infty)]]$, we have $\rho r_n < r_0$ and that

$$\sup_{y,w \in B_{X_\infty}(x_\infty,\rho)} |d_n(\phi_n(y), \phi_n(w)) - d_{X_\infty}(y, w)| < \frac{1}{n},$$

and

$$B_n(x, \rho) \subset \bigcup_{y \in \phi_n(B_{X_\infty}(x_\infty, \rho+1/n))} B_n(y, 1/n).$$

For each n we set $x_n := \phi_n(z)$. By choosing ρ appropriately, we can also ensure

$$\pi_\infty(\partial B_{X_\infty}(x_\infty, \rho)) = 0. \tag{6.14}$$

Then by the above,

$$\frac{d_X(x, x_n)}{r_n} = d_n(x, x_n) \le d_{X_\infty}(x_\infty, z) + \frac{1}{n}.$$

Fix $\tau \in (0, 1)$. We now use Lemma 6.11 to stitch $f_{i,n}$ on $B_n(x_n, R)$ to χ_E on $B_n(x, \rho) \setminus B_n(x_n, [1+\tau]R)$ using the Lipschitz function η_n to obtain $h_{i,n} := \eta_n f_{i,n} + (1 - \eta_n)\chi_E$. Then, since $\rho r_n < r_0$, we know that

$$P(E, B_X(x, \rho r_n)) \le (1 + \varepsilon_{\rho r_n}) V(h_{i,n}, B_X(x, \rho r_n)).$$

Note by Lemma 6.11 that

$$V(h_{i,n}, B_n(x, \rho)) \le P_n(E, B_n(x, \rho) \setminus B_n(x_n, R)) + V(f_{i,n}, B_n(x_n, [1+\tau]R)) + \frac{2}{\tau R} \int_{B_n(x_n,[1+\tau]R) \setminus B_n(x_n,R)} |f_{i,n} - \chi_E| \, d\mu_n.$$

Note that $h_{i,n} - \chi_E$ has compact support on $B_n(x, \rho)$ for large enough n since we can ensure $B_n(x_n, [1+\tau]R) \subset B_n(x, \rho)$. Combining this with the (asymptotic) minimality of χ_E at x as explained above, we obtain that

$$P_n(E, B_n(x, \rho)) \le [1 + \varepsilon_{\rho r_n}] \Bigg(P_n(E, B_n(x, \rho) \setminus B_n(x_n, R)) + \int_{B_n(x_n,[1+\tau]R)} v_{i,n} \, d\mu_n + \frac{2}{\tau R} \int_{B_n(x_n,[1+\tau]R) \setminus B_n(x_n,R)} |f_{i,n} - \chi_E| \, d\mu_n \Bigg).$$

In the above, we have also used the fact that as $v_{i,n}$ is an upper gradient of $f_{i,n}$, we have $dV(f_{i,n}, \cdot) \le v_{i,n} \, d\mu_n$.

Recall that χ_E is either 0 or 1 on X_∞, and $0 \le f_{i,n}, f_i \le 1$, and so we have

$$|f_{i,n} - \chi_E| = (1 - f_{i,n})\chi_E + (1 - \chi_E) f_{i,n}$$

and

$$|f_i - \chi_{(E)_\infty}| = (1 - f_i)\chi_{(E)_\infty} + (1 - \chi_{(E)_\infty}) f_i.$$

Thus, since $\chi_E\, d\mu_n \overset{*}{\rightharpoonup} \chi_{E_\infty}\, d\mu_\infty$ and since μ_∞ gives measure zero to every sphere due to the geodesic property and doubling of X_∞ (recall (5.2)), we get

$$\begin{aligned}
&\lim_{n\to\infty} \int_{B_n(x_n,[1+\tau]R)\setminus B_n(x_n,R)} |f_{i,n} - \chi_E|\, d\mu_n \\
&\quad = \lim_{n\to\infty} \int_{B_n(x_n,[1+\tau]R)\setminus B_n(x_n,R)} [(1-f_{i,n})\chi_E + (1-\chi_E) f_{i,n}]\, d\mu_n \\
&\quad = \int_{B_{X_\infty}(z,[1+\tau]R)\setminus B_{X_\infty}(z,R)} [(1-f_i)\chi_{(E)_\infty} + (1-\chi_{(E)_\infty}) f_i]\, d\mu_\infty \\
&\quad = \int_{B_{X_\infty}(z,[1+\tau]R)\setminus B_{X_\infty}(z,R)} |f_i - \chi_{(E)_\infty}|\, d\mu_\infty.
\end{aligned}$$

Now letting $n \to \infty$ and using (6.14), we obtain

$$\begin{aligned}
\pi_\infty(B_{X_\infty}(x_\infty,\rho)) \le \pi_\infty(B_{X_\infty}(x_\infty,\rho) \setminus B_{X_\infty}(z,R)) + \int_{B_{X_\infty}(z,[1+\tau]R)} v_i\, d\mu_\infty \\
+ \frac{2}{\tau R} \int_{B_{X_\infty}(z,[1+\tau]R)\setminus B_{X_\infty}(z,R)} |f_i - \chi_{(E)_\infty}|\, d\mu_\infty.
\end{aligned}$$

Thus we get

$$\pi_\infty(B_{X_\infty}(z,R)) \le \int_{B_{X_\infty}(z,[1+\tau]R)} v_i\, d\mu_\infty + \frac{2}{\tau R} \int_{B_{X_\infty}(z,[1+\tau]R)\setminus B_{X_\infty}(z,R)} |f_i - \chi_{(E)_\infty}|\, d\mu_\infty.$$

Now letting $i \to \infty$ gives

$$\pi_\infty(B_{X_\infty}(z,R)) \le V(\chi_{(E)_\infty} + \varphi, B_{X_\infty}(x,[1+2\tau]R)),$$

where we used the fact that $f_i \to \chi_{(E)_\infty}$ in $L^1_{\mathrm{loc}}(X_\infty)$ and $v_i\, d\mu_\infty \overset{*}{\rightharpoonup} dV(\chi_{(E)_\infty} + \varphi, \cdot)$. Now letting $\tau \to 0$ and finally using the assumption $P((E)_\infty, \partial B_{X_\infty}(z,R)) = 0$ we obtain

$$\pi_\infty(B_{X_\infty}(z,R)) \le V(\chi_{(E)_\infty} + \varphi, B_{X_\infty}(z,R)).$$

Now by Theorem 5.5 we have

$$P((E)_\infty, B_{X_\infty}(z,R)) \le C\, V(\chi_{(E)_\infty} + \varphi, B_{X_\infty}(z,R)),$$

where C is the comparison constant that connects π_∞ to $P((E)_\infty, \cdot)$. Thus choosing $K = C$ yields the desired outcome. This completes the proof.

□

6.3 Concluding remarks

In Section 4 we have shown that any asymptotic limit, at μ-almost every point, of a BV function is a function of least gradient on a corresponding tangent space X_∞ and is Lipschitz continuous with a constant minimal p-weak upper gradient. In Section 6 we have shown that given a set E of finite perimeter in X, at $\mathcal{H}$-almost every point of its measure-theoretic boundary we have the existence of an asymptotic limit set $(E)_\infty \subset X_\infty$ such that this asymptotic limit set is of quasiminimal boundary surface (that is, $\chi_{(E)_\infty}$ is of quasi-least gradient).

Remark 6.12. If $u \in BV(X)$, from the co-area formula we know that for almost every $t \in \mathbb{R}$ its super-level set

$$E_t := \{x \in X \,:\, u(x) > t\}$$

is of finite perimeter in X. Let $\mathbb{R}_F$ be the collection of all $t \in \mathbb{R}$ for which E_t is of finite perimeter, and let $A \subset \mathbb{R}_F$ be a countable dense subset of $\mathbb{R}_F$, and for each $t \in A$ let K_t be the collection of all points in X at which the conclusion of Theorem 6.3 fails; then $\mathcal{H}(\bigcup_{t\in A} K_t) = 0$. Let $x \in S_u \setminus \bigcup_{t\in A} K_t$, where S_u is the jump set of u. Note that if $x \in X \setminus \partial^* E_t$, then for every tangent space X_∞ based at that point, the corresponding set $(E_t)_\infty$ is either all of X_∞ or is empty, and hence does satisfy the conclusion of Theorem 6.3. Thus we have here that $K_t \subset \partial^* E_t$. A Cantor diagonalization argument gives us for each $t \in A$ an asymptotic limit $(E_t)_\infty \subset X_\infty$, with X_∞ a tangent space to X based at x, of the set E_t. We know then that each $(E_t)_\infty$ is of quasiminimal boundary in X_∞ in the sense of [25], with the quasiminimality constant K independent of t. Moreover, note that if $t_1, t_2 \in A$ such that $t_1 < t_2$, then $E_{t_2} \subset E_{t_1}$ and so by the construction of $(E_t)_\infty$ we have that $(E_{t_2})_\infty \subset (E_{t_1})_\infty$. Indeed, by the definition of $(E)_\infty$ from the discussion before Lemma 5.1, we have that when $z \in X_\infty$ for which $z \in (E_{t_2})_\infty$, we have $\mu_\infty^{E_{t_2}} \le \mu_\infty^{E_{t_1}}$ on X_∞ (because $E_{t_2} \subset E_{t_1} \subset X$), and so

$$1 = \lim_{r\to 0^+} \frac{\mu_\infty^{E_{t_2}}(B_{X_\infty}(z,r))}{\mu_\infty(B_{X_\infty}(z,r))} \le \lim_{r\to 0^+} \frac{\mu_\infty^{E_{t_1}}(B_{X_\infty}(z,r))}{\mu_\infty(B_{X_\infty}(z,r))} \le 1,$$

and so we must have $z \in (E_{t_1})_\infty$. In this discussion, recall that we have fixed $x \in S_u \setminus \bigcup_{t\in A} K_t$. We can now set

$$u_\infty(z) := \sup\{t \in A :\, z \in (E_t)_\infty\}.$$

An argument as in the proof of [29, Theorem 4.11] tells us that u_∞ is of quasi-least gradient in X_∞. It would be interesting to know in which sense, if any, is this u_∞ an asymptotic limit of u at x.

Remark 6.13. In Definition 6.2 of quasiminimality we used balls $B(x, R)$. The study undertaken in [25] is applicable to functions satisfying this definition; however, the notion of quasiminimality given in [25] is slightly stronger, namely whenever φ is a compactly supported BV function on X, we have

$$V(u, \mathrm{supt}(\varphi)) \le K\, V(u + \varphi, \mathrm{supt}(\varphi)).$$

The proof given in Subsection 6.2 can be easily adapted to prove that $\chi_{(E)_\infty}$ satisfies this stronger version, but the proof gets messy, and hence we gave the relatively more transparent proof showing that $\chi_{(E)_\infty}$ satisfies Definition 6.2. To prove the stronger quasiminimality criterion of [25], one first modifies the stitching lemma (Lemma 6.11) by replacing $B(x, a), B(x, b)$ with open sets U, V with $U \Subset V$ and considering η to be a Lipschitz function with $\eta = 1$ on U, $\eta = 0$ on $X \setminus V$. The term $2/(b - a)$ is then replaced with a constant C that depends solely on U, V. Next, in the proof of quasiminimality, one replaces $B_{X_\infty}(z, [1 + \tau]R)$ with U_τ where U is the support of φ and $U_\tau = \{y \in X_\infty :\, d_{X_\infty}(y, U) < \tau\}$. In this case, $B_n(x_n, [1 + \tau]R)$ is replaced with a suitable approximation of U_τ in X_n, ensuring that this approximating open set is contained within $B_n(x, \rho)$ where $\rho = 2[\mathrm{diam}_{X_\infty}(U) + \mathrm{dist}(U, x_\infty)]$.

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