CRITICAL WEAK-L^p DIFFERENTIABILITY OF SINGULAR INTEGRALS

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ABSTRACT. We establish that for every function $u \in L^1_{\mathrm{loc}}(\Omega)$ whose distributional Laplacian Δu is a signed Borel measure in an open set Ω in $\mathbb{R}^N$, the distributional gradient ∇u is differentiable almost everywhere in Ω with respect to the weak-$L^{\frac{N}{N-1}}$ Marcinkiewicz norm. We show in addition that the absolutely continuous part of Δu with respect to the Lebesgue measure equals zero almost everywhere on the level sets $\{u = \alpha\}$ and $\{\nabla u = e\}$, for every $\alpha \in \mathbb{R}$ and $e \in \mathbb{R}^N$. Our proofs rely on an adaptation of Calderón and Zygmund's singular-integral estimates inspired by subsequent work by Hajłasz.

1. INTRODUCTION AND MAIN RESULTS

Let Ω be an open set in $\mathbb{R}^N$ with $N \geq 2$. This paper was originally motivated by the following question of H. Brezis's: Given $u \in L^1_{\mathrm{loc}}(\Omega)$ whose distributional Laplacian satisfies $\Delta u \in L^1_{\mathrm{loc}}(\Omega)$, is it true that for any $\alpha \in \mathbb{R}$ one has

$$\Delta u = 0 \quad \text{almost everywhere on } \{u = \alpha\}\ ? \tag{1.1}$$

The answer is straightforward when $\Delta u \in L^p_{\mathrm{loc}}(\Omega)$ for some $1 < p < \infty$, since in this case u belongs to the Sobolev space $W^{2,p}_{\mathrm{loc}}(\Omega)$. One then has $D^2 u = 0$ almost everywhere on the level set $\{\nabla u = 0\}$ and the latter contains $\{u = \alpha\}$ except for a negligible set; see Theorem 4.4 in [12]. As $\Delta u = \mathrm{tr}\,(D^2 u)$ is the trace of $D^2 u$, assertion (1.1) is satisfied.

When one merely has $\Delta u \in L^1_{\mathrm{loc}}(\Omega)$, it need not be true that u belongs to $W^{2,1}_{\mathrm{loc}}(\Omega)$. The question above has nevertheless a positive answer that includes its generalization when Δu is merely a measure. By the latter, we mean that there exists a locally finite Borel measure λ in Ω, possibly signed, such that

$$\int_\Omega u \Delta \varphi = \int_\Omega \varphi \,\mathrm{d}\lambda \quad \text{for every } \varphi \in C_c^\infty(\Omega)$$

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and *we always identify* Δu *with* λ. One shows in this case that the distributional gradient ∇u belongs to $L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ and has an approximate derivative at almost every point $y \in \Omega$, denoted

$$D^2_{\mathrm{ap}} u(y) := D_{\mathrm{ap}}(\nabla u)(y).$$

From the definition of the approximate derivative which we recall in Section 2 below, $D^2_{\mathrm{ap}} u(y)$ is a linear transformation from $\mathbb{R}^N$ to $\mathbb{R}^N$. The fact that u belongs to the Sobolev space $W^{1,1}_{\mathrm{loc}}(\Omega)$ follows from standard elliptic regularity theory, see Theorem 5.1 and Proposition 6.11 in [23], while the existence of the approximate derivative of ∇u is a consequence of the Remark on p. 129 by Calderón and Zygmund [7].

The answer to H. Brezis's question can be seen as a consequence of an identification of Δu in terms of $D^2_{\mathrm{ap}} u$:

Theorem 1.1. *If* $u \in L^1_{\mathrm{loc}}(\Omega)$ *is such that* Δu *is a locally finite Borel measure in* Ω*, then the approximate derivative* $D^2_{\mathrm{ap}} u$ *satisfies*

$$(\Delta u)_{\mathrm{a}} = \mathrm{tr}\,(D^2_{\mathrm{ap}} u)\,\mathrm{d}x. \tag{1.2}$$

Hence, for every $\alpha \in \mathbb{R}$ *and* $e \in \mathbb{R}^N$*,*

$$(\Delta u)_{\mathrm{a}} = 0 \quad \text{almost everywhere on } \{u = \alpha\} \cup \{\nabla u = e\}. \tag{1.3}$$

Here, $(\Delta u)_{\mathrm{a}}$ is the absolutely continuous part of Δu with respect to the Lebesgue measure $\mathrm{d}x$. In particular, when $\Delta u \in L^1_{\mathrm{loc}}(\Omega)$, one has

$$\Delta u = \mathrm{tr}\,(D^2_{\mathrm{ap}} u) \quad \text{almost everywhere in } \Omega$$

and (1.1) holds.

Assertion (1.3) follows from (1.2) and a standard property of the approximate derivative on level sets; see (2.1). Marano and Mosconi [22] have established (1.3) using alternatively the L^1-differentiability of ∇u by Alberti, Bianchini and Crippa [2] in the sense of (1.5) below. Identity (1.2) has been proved independently by Raita [24]. His proof also relies on [2] and includes a generalization to elliptic operators of any integer order; see also [14].

One might wonder whether Theorem 1.1 is a consequence of a stronger locality property of the divergence of vector fields, namely the absolutely continuous part of the divergence of a vector field being 0 almost everywhere on level sets of the vector field itself. A simple application of Alberti's Lusin-type theorem [1] shows that such a property is not true for general vector fields, despite the fact that it holds for distributional gradients of Sobolev functions:

Example 1.1. Let Ω be a bounded open set in $\mathbb{R}^2$ and let us consider the vector field $V(x_1, x_2) = (x_2, -x_1)$. For every $\phi \in C^1_c(\Omega)$, the continuous vector field

$$W = (V - \nabla\phi)^{\perp},$$

where $(a_1, a_2)^\perp := (-a_2, a_1)$, satisfies

$$\operatorname{div} W = 2\,\mathrm{d}x \quad \text{in the sense of distributions in } \Omega.$$

This is a consequence of Schwarz's theorem which implies that $\operatorname{div}(\nabla\phi)^\perp = 0$ in the sense of distributions in Ω for any ϕ. By Alberti's theorem [1], for any $0 < \epsilon < |\Omega|$ we can find some $\phi \in C^1_c(\Omega)$ such that the Lebesgue measure of $\{\nabla\phi \neq V\}$ is less than ϵ. With such a choice of ϕ, the set $\{V - \nabla\phi = 0\}$ where the vector field W equals 0 has positive Lebesgue measure.

Our strategy to prove identity (1.2) has been inspired by the work [16] of Hajłasz's that has some analogy with the pointwise estimates by Calderón and Zygmund [8] and De Vore and Sharpley [11]; see also [19]. It provides some reinvigorating insight concerning existence of the approximate derivative in connection with the theory of singular integrals. To this end, we first observe that for a smooth homogeneous kernel K of order $-(N-1)$ in $\mathbb{R}^N \setminus \{0\}$, that is

$$K(x) = \frac{1}{|x|^{N-1}} K\Big(\frac{x}{|x|}\Big) \quad \text{for every } x \in \mathbb{R}^N \setminus \{0\},$$

the convolution $K * \mu$ is defined almost everywhere in $\mathbb{R}^N$ for every finite Borel measure μ in $\mathbb{R}^N$. More precisely, the complement of the set

$$\operatorname{dom}(K * \mu) := \bigg\{ x \in \mathbb{R}^N : \int_{\mathbb{R}^N} |K(x-y)| \,\mathrm{d}|\mu|(y) < \infty \bigg\}$$

is negligible with respect to the Lebesgue measure.

Note that $K * \mu$ belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^N)$, but need not have a distributional gradient in $L^1_{\mathrm{loc}}(\mathbb{R}^N; \mathbb{R}^N)$. Hajłasz made the observation that a singular-integral estimate of $(\nabla K) * \mu$ can be formulated in terms of a Lipschitz-type estimate of $K * \mu$ with variable coefficient. Existence almost everywhere of the approximate derivative $D_{\mathrm{ap}}(K * \mu)$ can then be straightforwardly obtained using Rademacher's theorem.

This approach applies more generally to kernels K that satisfy

(1.4)
$$|K(x)| \le \frac{A}{|x|^{N-1}} \quad \text{and} \quad |D^2 K(x)| \le \frac{B}{|x|^{N+1}} \quad \text{for every } x \in \mathbb{R}^N \setminus \{0\},$$

where $A, B > 0$ are constants. We prove a quantitative version of Calderón-Zygmund's approximate-differentiability property, which can be written as

Theorem 1.2. *Let $K \in C^2(\mathbb{R}^N \setminus \{0\})$ be any function that satisfies* (1.4)*. If μ is a finite Borel measure in $\mathbb{R}^N$, then there exists a measurable function $I : \mathbb{R}^N \to [0, \infty]$ in the Marcinkiewicz space $L^1_{\mathrm{w}}(\mathbb{R}^N)$ of weak-L^1 functions such that*

$$|K * \mu(x) - K * \mu(y)| \le (I(x) + I(y))|x - y| \quad \textit{for every } x, y \in \operatorname{dom}(K * \mu)$$

and

$$[I]_{L^1_{\mathrm{w}}(\mathbb{R}^N)} \le C\|\mu\|_{\mathcal{M}(\mathbb{R}^N)},$$

for some constant $C > 0$ depending on A, B and N. Hence, $K*\mu$ is approximately differentiable almost everywhere in $\mathbb{R}^N$ and

$$|D_{\mathrm{ap}}(K*\mu)| \le 2I \quad \textit{almost everywhere in } \mathbb{R}^N.$$

We recall that the total-variation norm of a finite Borel measure μ in an open set Ω is

$$\|\mu\|_{\mathcal{M}(\Omega)} := |\mu|(\Omega) = \int_\Omega \mathrm{d}|\mu|$$

and, for every $1 \le p < \infty$, the weak-L^p Marcinkiewicz quasinorm of a measurable function f in Ω is

$$[f]_{L^p_{\mathrm{w}}(\Omega)} := \sup\left\{ t\left|\{|f| > t\}\right|^{\frac{1}{p}} : t > 0\right\},$$

where $|E|$ denotes the Lebesgue measure of E.

The choice of I is far from being unique or canonical. Compared to [16, Lemma 9], our function I satisfies a true weak-L^1 estimate in the entire space $\mathbb{R}^N$ that comes from a uniformization principle in Section 6 below. Such a global property was not stated nor needed in [16], whose focus was on the existence of the approximate derivative of $K*\mu$. In our case, the identification of $\operatorname{tr}(D^2_{\mathrm{ap}}u)$ as the absolutely continuous part of Δu relies on an approximation argument based on the weak-L^1 estimate of $D^2_{\mathrm{ap}}u$.

Under the additional assumption that μ belongs to $L^p(\mathbb{R}^N)$ for some $1 < p < \infty$, by standard singular-integral estimate of $(\nabla K)*\mu$ the distributional gradient $\nabla(K*\mu)$ belongs to $L^p(\mathbb{R}^N;\mathbb{R}^N)$. In such a case, $K*\mu$ satisfies a Lipschitz-type estimate as above with an explicit coefficient I in $L^p(\mathbb{R}^N)$ that involves the maximal function $\mathcal{M}|\nabla(K*\mu)|$, see (6.6), with

$$\|I\|_{L^p(\mathbb{R}^N)} \le C'\|\mu\|_{L^p(\mathbb{R}^N)}.$$

Although Theorem 1.2 is enough for proving Theorem 1.1, there is a notion which is stronger than approximate differentiability and is adapted to L^p_{loc} functions, namely L^p differentiability. The goal is to determine whether at points $y \in \mathbb{R}^N$ one has

$$\lim_{r\to 0} \frac{1}{r}\,\frac{\|v - T^1_y v\|_{L^p(B_r(y))}}{\|1\|_{L^p(B_r(y))}} = \lim_{r\to 0}\frac{1}{r}\left(\fint_{B_r(y)} |v - T^1_y v|^p\right)^{\frac{1}{p}} = 0, \tag{1.5}$$

where $\fint_{B_r(y)}$ denotes the average integral over the ball and $T^1_y v$ is the first-order Taylor approximation of v at y defined by

$$T^1_y v(x) := v(y) + D_{\mathrm{ap}}v(y)[x-y] \quad \text{for every } x \in \mathbb{R}^N,$$

provided that the approximate derivative $D_{\mathrm{ap}}v(y)$ exists.

A fundamental result by Calderón and Zygmund [8, Theorem 12], see also [12, Theorem 6.2], asserts that every $v \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$ is $L^{\frac{N}{N-1}}$-differentiable at almost every point $y \in \mathbb{R}^N$ and

$$D_{\mathrm{ap}}v(y)[h] = \nabla v(y)\cdot h \quad \text{for every } h \in \mathbb{R}^N. \tag{1.6}$$

Under the assumptions of Theorem 1.2, it may happen that $K * \mu$ does not belong to $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$ and just misses the imbedding in $L^{\frac{N}{N-1}}_{\mathrm{loc}}(\mathbb{R}^N)$. Such an example is given by $K(z) = 1/|z|^{N-1}$ and $\mu = \delta_a$ with $a \in \mathbb{R}^N$. Nonetheless, Alberti, Bianchini and Crippa prove in [2] that $K * \mu$ is always L^p-differentiable almost everywhere in $\mathbb{R}^N$ in the range $1 \le p < \frac{N}{N-1}$.

Concerning the critical exponent $p = \frac{N}{N-1}$, a variant of Young's inequality easily implies that $K * \mu$ belongs to the Marcinkiewicz space $L^{\frac{N}{N-1}}_{\mathrm{w}}(\mathbb{R}^N)$, which is locally contained in all L^p spaces for $1 \le p < \frac{N}{N-1}$. One may thus wonder whether there is some notion of differentiability that could handle such a critical imbedding. The answer is affirmative and is our next

Theorem 1.3. *If $K \in C^2(\mathbb{R}^N \setminus \{0\})$ satisfies* (1.4) *and μ is a finite Borel measure in $\mathbb{R}^N$, then $K * \mu$ is weak-$L^{\frac{N}{N-1}}$ differentiable at almost every point $y \in \mathbb{R}^N$ in the sense that*

$$\lim_{r \to 0} \frac{1}{r} \frac{[K * \mu - T^1_y(K * \mu)]_{L^{\frac{N}{N-1}}_{\mathrm{w}}(B_r(y))}}{[1]_{L^{\frac{N}{N-1}}_{\mathrm{w}}(B_r(y))}} = 0.$$

The normalization factor in the denominator satisfies

$$[1]_{L^{\frac{N}{N-1}}_{\mathrm{w}}(B_r(y))} = |B_r(y)|^{\frac{N-1}{N}} = d_N r^{N-1},$$

for some constant $d_N > 0$. Our proof of this weak-$L^{\frac{N}{N-1}}$ differentiability property of $K * \mu$ relies on the Calderón-Zygmund decomposition of μ and the $L^{\frac{N}{N-1}}$ differentiability of Sobolev functions. After completing this paper we have been informed by J. Verdera that the methods used in his joint work with Cufí [10], where they investigate new fine properties of functions u such that Δu is a measure, can be adapted to yield an alternative proof of Theorem 1.3; see their comments on p. 1087 in [10] and also [30]. While there is no hope of having $L^{\frac{N}{N-1}}$ differentiability in full generality, under additional ellipticity assumptions on the differential operator associated to K, Gmeineder and Raita prove in [14] that $K * \mu$ belongs to $L^{\frac{N}{N-1}}(\mathbb{R}^N)$ and is $L^{\frac{N}{N-1}}$-differentiable in the usual sense.

We aim at a self-contained presentation, reproducing for the benefit of the reader also some intermediate results already present in the literature. The paper is then organized as follows. We explain in Section 2 the connection between approximate differentiability and the Lipschitz-type condition used by Hajłasz in [16]. In Section 3 we recall the singular estimates for $K * \mu$ when $\mu \in L^2(\mathbb{R}^N)$, based on the Fourier transform. In Sections 4 and 5 we obtain a weak-L^1 estimate for the approximate derivative of $K * \mu$ when μ is a measure, following the approach of Calderón and Zygmund's. We prove Theorems 1.2 and 1.3 in Sections 6 and 7, respectively.

In Section 8, we apply the singular-integral estimates to identify the second-order approximate derivative of the Newtonian potential and the solution of the Dirichlet problem with measure data. We then prove identity (1.2) in Section 9. In Section 10 we give two applications of Theorem 1.1. The first one is a new proof of a property of level sets of subharmonic functions by Frank and Lieb [13]. The second one concerns the description of limiting vorticities of the Ginzburg-Landau model with magnetic field in bounded open subsets Ω of $\mathbb{R}^2$ that extends previous result by Sandier and Serfaty [26, 27] in the L^p setting. Based on regularity results by Caffarelli and Salazar [5], we deduce that a limiting vorticity $\mu \in L^1_{\mathrm{loc}}(\Omega)$ can be written as

$$\mu = \sum_{j\in J} m_j \chi_{U_j} \quad \text{almost everywhere in } \Omega,$$

for some disjoint family of open subsets $U_j \subset \Omega$, where m_j is the constant value of the limiting induced magnetic field on U_j.

2. Approximate differentiability via Lipschitz-type estimates

We recall that the approximate limit $c := \operatorname*{ap\,lim}_{x\to y} v$ of a measurable function $v : \mathbb{R}^N \to \mathbb{R}^m$ at y is defined by the property

$$\lim_{r\to 0} \frac{|A_\epsilon(v,c) \cap B_r(y)|}{|B_r(y)|} = 0 \quad \text{for every } \epsilon > 0,$$

with

$$A_\epsilon(v,c) := \bigl\{x \in \mathbb{R}^N : |v(x) - c| > \epsilon\bigr\}.$$

In the terminology of Measure theory, y must be a density point of the set $\mathbb{R}^N \setminus A_\epsilon(v,c)$ for every $\epsilon > 0$. By Lebesgue's density theorem, one has $\operatorname*{ap\,lim}_{x\to y} v = v(y)$ almost everywhere in $\mathbb{R}^N$ and in particular, for all $\alpha \in \mathbb{R}^m$,

$$\operatorname*{ap\,lim}_{x\to y} v = \alpha \quad \text{almost everywhere on } \{v = \alpha\}.$$

Accordingly, the approximate derivative of v at y is a linear transformation $D_{\mathrm{ap}}v(y) : \mathbb{R}^N \to \mathbb{R}^m$ such that

$$\operatorname*{ap\,lim}_{x\to y} \frac{v(x) - v(y) - D_{\mathrm{ap}}v(y)[x-y]}{|x-y|} = 0.$$

Notice that the existence of the approximate derivative at y implies the existence of the approximate limit of v at y, with

$$\operatorname*{ap\,lim}_{x\to y} v = v(y).$$

Lebesgue's density theorem can be invoked again, see for instance [12, Theorem 6.3], to get for all $\alpha \in \mathbb{R}^m$,

$$D_{\mathrm{ap}}v = 0 \quad \text{almost everywhere on } \{v = \alpha\}. \tag{2.1}$$

These concepts extend to functions defined on measurable subsets of $\mathbb{R}^N$ and both $\operatorname*{ap\,lim}_{x\to y} v$ and $D_{\mathrm{ap}}v(y)$ are uniquely defined at each density point of the domain.

Hajłasz's strategy to prove approximate differentiability from a Lipschitz-type estimate with variable coefficient is based on the following

Proposition 2.1. *Let $E \subset \mathbb{R}^N$ and $v : E \to \mathbb{R}$ be a measurable function. If there exists a measurable function $I : E \to [0, \infty)$ such that*

$$|v(x) - v(y)| \le (I(x) + I(y))|x - y| \quad \textit{for every } x, y \in E,$$

then v is approximately differentiable almost everywhere in E and its approximate derivative satisfies

$$|D_{\mathrm{ap}}v| \le 2I \quad \textit{almost everywhere in } E.$$

Proof. Given $\epsilon > 0$ and $\alpha > 0$, the set

$$M = \{\alpha \le I \le \alpha + \epsilon\}.$$

is a measurable subset of E. Since v is Lipschitz-continuous on M with Lipschitz constant $2(\alpha + \epsilon)$, there exists a Lipschitz-continuous function $h : \mathbb{R}^N \to \mathbb{R}$ with the same Lipschitz constant and such that $h = v$ on M. Observe that if $y \in M$ is a density point of M and h is differentiable at y, then v has an approximate derivative at y and $D_{\mathrm{ap}}v(y) = Dh(y)$. Thus,

$$|D_{\mathrm{ap}}v(y)| = |Dh(y)| \le 2(\alpha + \epsilon) \le 2(I(y) + \epsilon).$$

Since by Lebesgue's density theorem almost every point of M is a density point of M and by Rademacher's theorem h is differentiable almost everywhere in $\mathbb{R}^N$, we deduce from the observation above that v is approximately differentiable at almost every point of M and

$$|D_{\mathrm{ap}}v| \le 2(I + \epsilon) \quad \text{almost everywhere in } M.$$

Since E can be covered by a countable union of such sets M, we thus have

$$|D_{\mathrm{ap}}v| \le 2(I + \epsilon) \quad \text{almost everywhere in } E,$$

and the estimate follows since $\epsilon > 0$ is arbitrary. □

The Lipschitz-type estimate of Proposition 2.1 is satisfied for example by Sobolev functions:

Proposition 2.2. *If $v : \mathbb{R}^N \to \mathbb{R}$ is a measurable function such that $v \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$, then*

$$|\widetilde{v}(x) - \widetilde{v}(y)| \le 2^N \big(\mathcal{M}|\nabla v|(x) + \mathcal{M}|\nabla v|(y)\big)|x - y|$$

for every Lebesgue points x and y of v, where $\widetilde{v}$ denotes the precise representative of v.

The Hardy-Littlewood maximal function $\mathcal{M}f : \mathbb{R}^N \to [0, \infty]$ of a locally summable function f is defined by

$$\mathcal{M}f(x) := \sup \left\{ \fint_{B_r(x)} |f| : r > 0 \right\}.$$

We also recall that $x \in \mathbb{R}^N$ is a Lebesgue point of f and $\widetilde{f}(x)$ is the value of the precise representative of f at x whenever

$$\lim_{r \to 0} \fint_{B_r(x)} |f - \widetilde{f}(x)| = 0.$$

The classical Lebesgue differentiation theorem implies that almost every point in $\mathbb{R}^N$ is a Lebesgue point of f and

$$\widetilde{f} = f \quad \text{almost everywhere in } \mathbb{R}^N. \tag{2.2}$$

Notice also that, as a simple consequence of the Markov-Chebyshev inequality, for all $x \in \mathbb{R}^N$ one has the implication

$$\lim_{r \to 0} \fint_{B_r(x)} |f - \widetilde{f}(x)| = 0 \quad \Longrightarrow \quad \operatorname*{ap\,lim}_{y \to x} f(y) = \widetilde{f}(x),$$

but the converse implication does not hold in general.

A natural problem is to find sufficient conditions that identify Lebesgue points of a given function. For example, when $v \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$ as above, it follows from Remark 3.82 of [3] that every $x \in \mathbb{R}^N$ such that

$$\mathcal{M}|\nabla v|(x) < \infty$$

is a Lebesgue point of v. Hence, the inequality in Proposition 2.2 is valid for every $x, y \in \{\mathcal{M}|\nabla v| < \infty\}$. Observe however that it does not provide the identification (1.6) between $D_{\mathrm{ap}}v$ and ∇v, which goes back to the work by Calderón and Zygmund [9] that includes the case of functions with bounded variation (BV); see also [3, Theorem 3.83].

We sketch the proof of Proposition 2.2 for the convenience of the reader; see also [3, Theorem 5.34] for a different argument. A variant of Proposition 2.2 based on the sharp maximal function can be found for example in [11, Theorem 2.5]. We also refer the reader to [19] for applications of pointwise inequalities of this type in the setting on nonlinear Potential theory.

Proof of Proposition 2.2. Assume temporarily that v is smooth. By the Fundamental theorem of Calculus and Fubini's theorem,

$$\left| v(x) - \fint_{B_r(\frac{x+y}{2})} v \right| \le \int_0^1 \fint_{B_r(\frac{x+y}{2})} |\nabla v(x + t(z - x))| |x - z| \,\mathrm{d}z \,\mathrm{d}t,$$

for any $r > 0$. Taking $r = |x - y|/2$, one has $B_r(\frac{x+y}{2}) \subset B_{2r}(x)$. Moreover, for every $z \in B_r(\frac{x+y}{2})$, $|x - z| \le |x - y|$. Hence,

$$\left| v(x) - \fint_{B_r(\frac{x+y}{2})} v \right| \le \frac{1}{|B_1| r^N} \int_0^1 \int_{B_{2r}(x)} |\nabla v(x + t(z - x))| \,\mathrm{d}z \,\mathrm{d}t \; |x - y|,$$

where $B_1 := B_1(0)$ is the unit ball centered at 0. Making the change of variables $\xi = x + t(z-x)$ between z and ξ in the right-hand side and using the definition of $\mathcal{M}|\nabla v|(x)$, one gets

$$\begin{aligned}\left|v(x) - \fint_{B_r(\frac{x+y}{2})} v\right| &\le \frac{1}{|B_1|r^N}\int_0^1 \int_{B_{2tr}(x)} |\nabla v(\xi)| \frac{\mathrm{d}\xi}{t^N}\,\mathrm{d}t\,|x-y| \\ &\le 2^N \int_0^1 \mathcal{M}|\nabla v|(x)\,\mathrm{d}t\,|x-y| = 2^N \mathcal{M}|\nabla v|(x)\,|x-y|.\end{aligned}$$

A similar estimate holds for the point y and one concludes using the triangle inequality.

For a measurable function v as in the statement, one can apply the conclusion to the smooth function $\rho_n * v$, where $(\rho_n)_{n\in\mathbb{N}}$ is a sequence of mollifiers of the form $\rho_n(z) := \frac{1}{\epsilon_n^N}\rho(z/\epsilon_n)$ for some fixed nonnegative function $\rho \in C_c^\infty(B_1)$ such that $\int_{B_1}\rho = 1$ and $(\epsilon_n)_{n\in\mathbb{N}}$ is a sequence of positive numbers that converges to zero. As $n \to \infty$, one has

$$\rho_n * v(x) \to \widetilde{v}(x) \quad \text{for every Lebesgue point } x \in \mathbb{R}^N.$$

Moreover, the pointwise estimate $|\nabla(\rho_n * v)| \le \rho_n * |\nabla v|$ implies that

$$\mathcal{M}|\nabla(\rho_n * v)|(x) \le \mathcal{M}(\rho_n * |\nabla v|)(x) \le \mathcal{M}|\nabla v|(x) \quad \text{for every } x \in \mathbb{R}^N. \quad \square$$

3. WEAK DIFFERENTIABILITY OF $K * \mu$ IN THE L^2 CASE

By the first estimate in (1.4), one can write K as a sum of L^1 and L^∞ functions, for instance:

$$K = K\chi_{B_1} + K\chi_{\mathbb{R}^N\setminus B_1}. \tag{3.1}$$

In particular, the convolution $K * \mu$ is defined almost everywhere in $\mathbb{R}^N$ for every $\mu \in L^1(\mathbb{R}^N)$ and, more generally, for finite Borel measures, and belongs to $(L^1 + L^\infty)(\mathbb{R}^N)$. The fact that the distributional derivative $\nabla(K*\mu)$ belongs to $L^2(\mathbb{R}^N;\mathbb{R}^N)$ when $\mu \in L^2(\mathbb{R}^N)$ is a known property in Harmonic analysis that can be proved using the Fourier transform.

Proposition 3.1. *If $K \in C^2(\mathbb{R}^N \setminus \{0\})$ satisfies (1.4) and $\mu \in (L^1 \cap L^2)(\mathbb{R}^N)$, then the distributional gradient of $K * \mu$ belongs to $L^2(\mathbb{R}^N;\mathbb{R}^N)$ and we have*

$$\|\nabla(K*\mu)\|_{L^2(\mathbb{R}^N)} \le C\|\mu\|_{L^2(\mathbb{R}^N)},$$

for some constant $C > 0$ depending on A, B and N.

We recall that the Fourier transform of a function $f \in L^1(\mathbb{R}^N)$ is defined for every $\xi \in \mathbb{R}^N$ by

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^N} \mathrm{e}^{-2\pi\imath x\cdot\xi} f(x)\,\mathrm{d}x,$$

and has a unique continuous extension to every function in $L^2(\mathbb{R}^N)$. We give a proof of the proposition above for the sake of completeness. It relies on the following property of the Fourier transform of K; see Theorem 2.4 in Chapter 3 of [29]:

Lemma 3.2. *If $K \in C^2(\mathbb{R}^N \setminus \{0\})$ satisfies* (1.4)*, then its Fourier transform $\mathcal{F}K$ is continuous in $\mathbb{R}^N \setminus \{0\}$ and there exists $C' > 0$ such that*

$$|\mathcal{F}K(\xi)| \leq \frac{C'}{|\xi|} \quad \textit{for every } \xi \in \mathbb{R}^N \setminus \{0\}.$$

Proof of Lemma 3.2. To prove the continuity of $\mathcal{F}K$ in $\mathbb{R}^N \setminus \{0\}$, one relies on a smooth counterpart of the decomposition (3.1). For this purpose, take $\psi \in C_c^\infty(B_2)$ such that $0 \leq \psi \leq 1$ in $\mathbb{R}^N$ and $\psi = 1$ in B_1, and write

$$K = K\psi_r + K(1 - \psi_r),$$

where $\psi_r(x) = \psi(x/r)$ for $r > 0$. Since $K\psi_r \in L^1(\mathbb{R}^N)$, the Fourier transform $\mathcal{F}(K\psi_r)$ is a bounded continuous function in $\mathbb{R}^N$.

Interpolation in (1.4) gives one the first-order counterpart

$$|DK(x)| \leq \frac{C_1}{|x|^N} \quad \text{for every } x \in \mathbb{R}^N \setminus \{0\}. \tag{3.2}$$

Since the function $K_{\infty,r} := K(1 - \psi_r)$ vanishes in B_r, we thus have

$$|\Delta K_{\infty,r}| \leq \frac{C_2}{|x|^{N+1}} \chi_{\mathbb{R}^N \setminus B_r}(x) \quad \text{for every } x \in \mathbb{R}^N,$$

for some constant $C_2 > 0$ independent of r. In particular, $\Delta K_{\infty,r}$ belongs to $L^1(\mathbb{R}^N)$, hence its Fourier transform is also a bounded continuous function in $\mathbb{R}^N$. From the identity

$$\mathcal{F}(\Delta K_{\infty,r}) = -4\pi^2 |\xi|^2 \mathcal{F}(K_{\infty,r}),$$

we deduce that $\mathcal{F}(K_{\infty,r})$ is continuous in $\mathbb{R}^N \setminus \{0\}$, hence so is $\mathcal{F}(K)$.

To obtain the pointwise estimate of $\mathcal{F}(K)$, observe that for every $\xi \neq 0$ and $r > 0$ we have

$$|\mathcal{F}K(\xi)| \leq \|\mathcal{F}(K\psi_r)\|_{L^\infty(\mathbb{R}^N)} + \frac{1}{4\pi^2|\xi|^2} \|\mathcal{F}(\Delta K_{\infty,r})\|_{L^\infty(\mathbb{R}^N)}.$$

Since

$$\|\mathcal{F}(K\psi_r)\|_{L^\infty(\mathbb{R}^N)} \leq \|K\psi_r\|_{L^1(\mathbb{R}^N)} \leq A \int_{B_{2r}} \frac{\mathrm{d}x}{|x|^{N-1}} = C_3 r$$

and

$$\|\mathcal{F}(\Delta K_{\infty,r})\|_{L^\infty(\mathbb{R}^N)} \leq \|\Delta K_{\infty,r}\|_{L^1(\mathbb{R}^N)} \leq C_2 \int_{\mathbb{R}^N \setminus B_r} \frac{\mathrm{d}x}{|x|^{N+1}} = \frac{C_4}{r},$$

we get

$$|\mathcal{F}K(\xi)| \leq C_3 r + \frac{C_4}{4\pi^2 |\xi|^2 r}.$$

To conclude it thus suffices to take $r = 1/|\xi|$. □

Proof of Proposition 3.1. By a standard property of the Fourier transform of the convolution,

$$\mathcal{F}(K * \mu) = (\mathcal{F}K)(\mathcal{F}\mu).$$

For $j \in \{1, \ldots, N\}$ and $\varphi \in C_c^\infty(\mathbb{R}^N)$, by the Plancherel theorem we have

$$\int_{\mathbb{R}^N} K * \mu \, \frac{\partial \varphi}{\partial x_j} = \int_{\mathbb{R}^N} \mathcal{F}(K * \mu) \, \overline{\mathcal{F}\Big(\frac{\partial \varphi}{\partial x_j}\Big)} = -\int_{\mathbb{R}^N} (\mathcal{F}K)(\mathcal{F}\mu) \, 2\pi \imath \xi_j \overline{\mathcal{F}\varphi}.$$

By Lemma 3.2, the function $\xi \mapsto \xi_j \mathcal{F}K(\xi)$ is bounded in $\mathbb{R}^N$. It thus follows from the Cauchy-Schwarz inequality and another application of the Plancherel theorem that

$$\left| \int_{\mathbb{R}^N} K * \mu \, \frac{\partial \varphi}{\partial x_j} \right| \le 2\pi C' \|\mathcal{F}\mu\|_{L^2(\mathbb{R}^N)} \|\mathcal{F}\varphi\|_{L^2(\mathbb{R}^N)} = 2\pi C' \|\mu\|_{L^2(\mathbb{R}^N)} \|\varphi\|_{L^2(\mathbb{R}^N)}.$$

It now suffices to apply the Riesz representation theorem in $L^2(\mathbb{R}^N)$ to conclude. □

4. WEAK-L^1 ESTIMATE OF THE APPROXIMATE DERIVATIVE

The fundamental tool to prove Theorem 1.1 is the weak-L^1 estimate from the Calderón-Zygmund theory of singular integrals. We revisit their approach to reformulate and make more transparent the role of the approximate differentiability as follows:

Theorem 4.1. *If $K \in C^2(\mathbb{R}^N \setminus \{0\})$ satisfies* (1.4) *and μ is a finite Borel measure in $\mathbb{R}^N$, then $K * \mu$ is approximately differentiable almost everywhere in $\mathbb{R}^N$ and we have*

$$[D_{\mathrm{ap}}(K * \mu)]_{L^1_{\mathrm{w}}(\mathbb{R}^N)} \le C \|\mu\|_{\mathcal{M}(\mathbb{R}^N)},$$

where $C > 0$ depends on A, B and N.

Theorem 4.1 relies on the following estimate, whose proof we postpone to the next section:

Proposition 4.2. *Let $K \in C^2(\mathbb{R}^N \setminus \{0\})$ be a function that satisfies* (1.4) *and let $Q \subset \mathbb{R}^N$ be a cube. If ν is a finite Borel measure in $\mathbb{R}^N$ with*

$$\nu(Q) = 0 \quad \textit{and} \quad |\nu|(\mathbb{R}^N \setminus Q) = 0,$$

then for every $\theta > 1$ there exists a bounded continuous function $I : \mathbb{R}^N \setminus \theta Q \to [0, \infty)$ such that

$$|K * \nu(x) - K * \nu(y)| \le (I(x) + I(y)) \, |x - y| \quad \textit{for every } x, y \in \mathbb{R}^N \setminus \theta Q,$$

and

$$\|I\|_{L^1(\mathbb{R}^N \setminus \theta Q)} \le C' \|\nu\|_{\mathcal{M}(\mathbb{R}^N)},$$

where $C' > 0$ depends on A, B, N and θ, but not on Q.

We denote by θQ the rescaled cube with the same center as Q and side-length θ times the side-length of Q. Proposition 4.2 has been proved by Hajłasz without an explicit L^1-estimate of I; see Lemma 9 in [16]. We rely on a variant of his proof that keeps track of the quantity $\|I\|_{L^1(\mathbb{R}^N \setminus \theta Q)}$.

Proof of Theorem 4.1. Given $t > 0$, we explain below that, thanks to Proposition 2.1, it is sufficient to prove the existence of a measurable function $I : \mathbb{R}^N \to [0, \infty]$, *possibly depending on* t, such that

(4.1)
$$|K * \mu(x) - K * \mu(y)| \leq (I(x) + I(y))|x - y| \quad \text{for every } x, y \in \operatorname{dom}(K * \mu)$$

and

$$|\{I > t\}| \leq \frac{C_1}{t} \|\mu\|_{\mathcal{M}(\mathbb{R}^N)}, \tag{4.2}$$

where $C_1 > 0$ is independent of t. In contrast with Theorem 1.2, the set $\{I = \infty\}$ need not be Lebesgue-negligible. We later show in Section 6 that I can be chosen independently of t and then in this case $\{I = \infty\}$ is negligible. For the sake of the proof of Theorem 4.1 such an independence of t is not needed.

By analogy with the definition of the maximal function in the L^1 setting, we first define the maximal function $\mathcal{M}\mu(x)$ of the Borel measure μ in $\mathbb{R}^N$ by computing the supremum of $|\mu|(B_r(x))/|B_r(x)|$ over $r > 0$. The set

$$F := \{\mathcal{M}\mu \leq t\}$$

is closed and its complement verifies the weak maximal inequality, see p. 19 in [28]:

$$|\mathbb{R}^N \setminus F| = |\{\mathcal{M}\mu > t\}| \leq \frac{C_2}{t} \|\mu\|_{\mathcal{M}(\mathbb{R}^N)}. \tag{4.3}$$

To construct a function I that satisfies (4.1) and (4.2), we take a Whitney covering of the open set $\mathbb{R}^N \setminus F$ in terms of cubes $(Q_n)_{n \in \mathbb{N}}$. We assume that each cube Q_n is *half-closed*, by this we mean that Q_n is a Cartesian product of intervals of the form $[a_i, b_i)$ with $a_i < b_i$. Such a choice does not change Whitney's construction and yields a *disjoint* family $(Q_n)_{n \in \mathbb{N}}$. Each cube Q_n satisfies

$$2Q_n \subset \mathbb{R}^N \setminus F \quad \text{and} \quad \alpha Q_n \cap F \neq \emptyset,$$

for some fixed $\alpha > 2$, depending on N. Hence,

$$F \subset \bigcap_{n=0}^{\infty} \big(\mathbb{R}^N \setminus (2Q_n)\big) \tag{4.4}$$

and there exist $0 < c_1 < c_2$ such that

$$c_1 \operatorname{diam}(Q_n) \leq d(Q_n, F) \leq c_2 \operatorname{diam}(Q_n).$$

By the choice of F, we also have

$$\frac{|\mu|(Q_n)}{|Q_n|} \leq C_3 t, \tag{4.5}$$

for some constant $C_3 > 0$ depending on N. Indeed, taking $z \in \alpha Q_n \cap F$, we have $\alpha Q_n \subset B_{r_n}(z)$, where $r_n = \sqrt{N}\alpha\ell_n$ and ℓ_n is the side-length of Q_n. Since $\mathcal{M}\mu(z) \leq t$ and the volumes of αQ_n and $B_{r_n}(z)$ are comparable, we thus have by monotonicity of $|\mu|$,

$$\frac{|\mu|(Q_n)}{c|Q_n|} \leq \frac{|\mu|(B_{r_n}(z))}{|B_{r_n}(z)|} \leq \mathcal{M}\mu(z) \leq t,$$

for a constant $c > 0$ depending on α and N.

Using the Whitney covering of $\mathbb{R}^N \setminus F$, we now decompose the measure μ as

$$\mu = \mu\lfloor_F + \mu\lfloor_{\mathbb{R}^N\setminus F} = \mu\lfloor_F + \sum_{n=0}^{\infty} \mu\lfloor_{Q_n},$$

which we further write as

$$\mu = \underbrace{\mu\lfloor_F + \sum_{n=0}^{\infty} \frac{\mu(Q_n)}{|Q_n|}\chi_{Q_n}\,\mathrm{d}x}_{g\,\mathrm{d}x} + \underbrace{\sum_{n=0}^{\infty} b_n}_{b}, \tag{4.6}$$

where

$$b_n := \mu\lfloor_{Q_n} - \frac{\mu(Q_n)}{|Q_n|}\chi_{Q_n}\,\mathrm{d}x,$$

so that $b_n(Q_n) = 0$ and $|b_n|(\mathbb{R}^N \setminus Q_n) = 0$ for every $n \in \mathbb{N}$. We refer the reader to the excellent introductions [15, 28] for a detailed explanation of the Whitney covering of a set and the subsequent Calderón-Zygmund decomposition of a function or a measure.

The term denoted by $g\,\mathrm{d}x$ in (4.6) is the good part of the measure and is absolutely continuous with respect to the Lebesgue measure. The absolute continuity of $\mu\lfloor_F$ follows from the definition of F, which implies that $|\mu|\lfloor_F(A) \leq t|A|$ for every Borel subset $A \subset \mathbb{R}^N$. By the Radon-Nikodym theorem, $\mu\lfloor_F$ can be written as $\mu\lfloor_F = f\,\mathrm{d}x$ with $f \in L^1(\mathbb{R}^N)$ such that $|f| \leq t$.

We now observe that the density g belongs to $(L^1 \cap L^\infty)(\mathbb{R}^N)$ and satisfies

$$\|g\|_{L^1(\mathbb{R}^N)} \leq \|\mu\|_{\mathcal{M}(\mathbb{R}^N)} \quad \text{and} \quad \|g\|_{L^\infty(\mathbb{R}^N)} \leq C_4 t. \tag{4.7}$$

Indeed, we have $|g| = |f| \leq t$ on F and, by (4.5), $|g| \leq C_3 t$ on each Q_n. Thus, the L^∞ bound of g holds with $C_4 := \max\{1, C_3\}$. Since the cubes Q_n are disjoint, by additivity of $|\mu|$ we also have

$$\|g\|_{L^1(\mathbb{R}^N)} = |\mu|(F) + \sum_{n=0}^{\infty} |\mu(Q_n)| \leq |\mu|(\mathbb{R}^N) = \|\mu\|_{\mathcal{M}(\mathbb{R}^N)}.$$

Hence, (4.7) is satisfied. Since $K \in (L^1 + L^\infty)(\mathbb{R}^N)$ and $g \in (L^1 \cap L^\infty)(\mathbb{R}^N)$, we have

$$\mathrm{dom}\,(K * g) = \mathbb{R}^N$$

and the convolution $K * g$ is continuous in $\mathbb{R}^N$.

As $g \in (L^1 \cap L^\infty)(\mathbb{R}^N)$, by interpolation between L^1 and L^∞, we have $g \in L^2(\mathbb{R}^N)$. We deduce from Proposition 3.1 that $\nabla(K * g) \in L^2(\mathbb{R}^N)$. By Proposition 2.2 and continuity of $K * g$, we then have

$$|K * g(x) - K * g(y)| \le (J(x) + J(y))|x - y| \quad \text{for every } x, y \in \mathbb{R}^N, \tag{4.8}$$

with $J = 2^N \mathcal{M}|\nabla(K * g)|$.

We now focus on b, which is the bad part of the measure μ:

Claim. There exists a Lebesgue-negligible set $S \subset \mathbb{R}^N$ such that $\mathrm{dom}\,(K * b) \supset \mathbb{R}^N \setminus S$ and

$$K * b(x) = \sum_{n=0}^{\infty} K * b_n(x) \quad \text{for every } x \in \mathbb{R}^N \setminus S. \tag{4.9}$$

Proof of the Claim. Let

$$S := \Big\{ z \in \mathbb{R}^N : \sum_{n=0}^{\infty} |K| * |b_n|(z) = \infty \Big\}. \tag{4.10}$$

For $x \in \mathbb{R}^N \setminus S$, we have $\sum_{n=0}^{\infty} |K| * |b_n|(x) < \infty$. Then, by Fatou's lemma, $|K| * |b|(x) < \infty$ and so

$$\mathrm{dom}\,(K * b) \supset \mathbb{R}^N \setminus S.$$

The Dominated convergence theorem implies (4.9). To prove that S is negligible, we proceed as follows. By Fubini's theorem and assumption (1.4), for every $r > 0$ and $n \in \mathbb{N}$ we have

$$\begin{aligned}\int_{B_r(0)} |K| * |b_n| &= \int_{\mathbb{R}^N} \left(\int_{B_r(0)} |K(x - y)| \,\mathrm{d}x \right) \mathrm{d}|b_n|(y) \\ &\le A \int_{\mathbb{R}^N} \left(\int_{B_r(0)} \frac{\mathrm{d}x}{|x - y|^{N-1}} \right) \mathrm{d}|b_n|(y).\end{aligned}$$

For every $y \in \mathbb{R}^N$,

$$\int_{B_r(0)} \frac{\mathrm{d}x}{|x - y|^{N-1}} \le \int_{B_r(y)} \frac{\mathrm{d}x}{|x - y|^{N-1}} = \int_{B_r(0)} \frac{\mathrm{d}z}{|z|^{N-1}} = C_5 r.$$

Thus,

$$\int_{B_r(0)} |K| * |b_n| \le A C_5 r \int_{\mathbb{R}^N} \mathrm{d}|b_n| = C_6 |b_n|(\mathbb{R}^N) \le 2C_6 |\mu|(Q_n),$$

for some constant $C_6 > 0$ depending on r. Since the cubes Q_n are disjoint, by Fatou's lemma and by additivity of the measure $|\mu|$ we then get

$$\int_{B_r(0)} \left(\sum_{n=0}^{\infty} |K| * |b_n| \right) \le \sum_{n=0}^{\infty} \int_{B_r(0)} |K| * |b_n| \le 2C_6 \sum_{n=0}^{\infty} |\mu|(Q_n) = 2C_6 |\mu|(\mathbb{R}^N \setminus F) < \infty.$$

Hence, $\sum\limits_{n=0}^{\infty} |K| * |b_n| < \infty$ almost everywhere in $B_r(0)$, for every $r > 0$. We conclude that S is Lebesgue-negligible. □

As a consequence of the Claim,

$$\operatorname{dom}(K * \mu) \supset \operatorname{dom}(K * g) \cap \operatorname{dom}(K * b) \supset \mathbb{R}^N \setminus S$$

and, from linearity of the convolution,

$$K * \mu(x) = K * g(x) + \sum_{n=0}^{\infty} K * b_n(x) \quad \text{for every } x \in \mathbb{R}^N \setminus S. \tag{4.11}$$

By Proposition 4.2 with $\theta = 2$, each measure b_n satisfies

$$|K * b_n(x) - K * b_n(y)| \le (I_n(x) + I_n(y))|x - y| \quad \text{for every } x, y \in \mathbb{R}^N \setminus 2Q_n,$$

where $I_n : \mathbb{R}^N \setminus 2Q_n \to [0, \infty)$ is a bounded continuous function such that

$$\|I_n\|_{L^1(\mathbb{R}^N \setminus 2Q_n)} \le C' |b_n|(Q_n) \le 2C' |\mu|(Q_n). \tag{4.12}$$

By (4.4) and (4.9), we thus have

$$|K * b(x) - K * b(y)| \le \left(\sum_{n=0}^{\infty} I_n(x) + \sum_{n=0}^{\infty} I_n(y) \right) |x - y| \quad \text{for every } x, y \in F \setminus S. \tag{4.13}$$

Combining (4.8) and (4.13), we get (4.1) with

$$I := \begin{cases} J + \sum\limits_{n=0}^{\infty} I_n & \text{in } F \setminus S, \\ \infty & \text{in } (\mathbb{R}^N \setminus F) \cup S. \end{cases}$$

To prove (4.2), we first observe that by subadditivity of the Lebesgue measure,

$$|\{I > t\}| \le |\{J > t/2\}| + \left| \left\{ \sum_{n=0}^{\infty} I_n > t/2 \right\} \cap F \right| + |\mathbb{R}^N \setminus F| + |S|. \tag{4.14}$$

As S is Lebesgue-negligible, $|S| = 0$. Since $J = 2^N \mathcal{M} |\nabla (K * g)|$, by the L^2-maximal inequality and Proposition 3.1 we have

$$\|J\|_{L^2(\mathbb{R}^N)} \le C_7 \|\nabla (K * g)\|_{L^2(\mathbb{R}^N)} \le C_8 \|g\|_{L^2(\mathbb{R}^N)}.$$

We also have by (4.7) and interpolation between Lebesgue spaces,

$$\|g\|_{L^2(\mathbb{R}^N)}^2 \le \|g\|_{L^1(\mathbb{R}^N)} \|g\|_{L^\infty(\mathbb{R}^N)} \le C_4 \, t \|\mu\|_{\mathcal{M}(\mathbb{R}^N)}.$$

By the Markov-Chebyshev inequality at height $t/2$, we thus get

$$|\{J > t/2\}| \le \Big(\frac{2}{t}\Big)^2 \|J\|_{L^2(\mathbb{R}^N)}^2 \le \frac{C_9}{t}\|\mu\|_{\mathcal{M}(\mathbb{R}^N)}, \tag{4.15}$$

which gives the estimate of the first term in the right-hand side of (4.14).

By (4.12) and the fact that the cubes Q_n are disjoint and contained in $\mathbb{R}^N \setminus F$,

$$\Big\|\sum_{n=0}^{\infty} I_n\Big\|_{L^1(F)} \le \sum_{n=0}^{\infty} \|I_n\|_{L^1(\mathbb{R}^N\setminus 2Q_n)} \le 2C' \sum_{n=0}^{\infty} |\mu|(Q_n) \le 2C'|\mu|(\mathbb{R}^N \setminus F), \tag{4.16}$$

which can also be deduced from a classical inequality in Harmonic analysis; see Remark 4.3 below. By the Markov-Chebyshev inequality at height $t/2$, we then have

$$\Big|\Big\{\sum_{n=0}^{\infty} I_n > t/2\Big\} \cap F\Big| \le \frac{4C'}{t}|\mu|(\mathbb{R}^N \setminus F) \le \frac{4C'}{t}\|\mu\|_{\mathcal{M}(\mathbb{R}^N)}. \tag{4.17}$$

Inequality (4.2) then follows from (4.14), (4.15), (4.17) and (4.3).

Observe that $K * \mu$ is defined in

$$\{I_t < \infty\} \subset \mathbb{R}^N \setminus S,$$

where for the rest of proof we make explicit the dependence of $I = I_t$ with respect to the parameter t. We conclude from (4.1) and Proposition 2.1 that $K * \mu$ is approximately differentiable almost everywhere in $\{I_t < \infty\}$. Taking a sequence $(t_j)_{j\in\mathbb{N}}$ of positive numbers with $t_j \to \infty$, we have

$$\mathbb{R}^N \setminus \bigcup_{j=0}^{\infty} \{I_{t_j} < \infty\} \subset \bigcap_{j=0}^{\infty} \{I_{t_j} > t_j\}$$

As a consequence of (4.2), the set in the right-hand side is Lebesgue-negligible and we deduce that $K * \mu$ is approximately differentiable almost everywhere in $\mathbb{R}^N$. Since the approximate derivative $D_{\mathrm{ap}}(K * \mu)$ satisfies

$$|D_{\mathrm{ap}}(K * \mu)| \le 2t \quad \text{almost everywhere in } \{I_t \le t\},$$

we have

$$\big|\{|D_{\mathrm{ap}}(K * \mu)| > 2t\}\big| \le |\{I_t > t\}| \le \frac{C_1}{t}\|\mu\|_{\mathcal{M}(\mathbb{R}^N)} \quad \text{for every } t > 0.$$

This implies the weak-L^1 estimate of $D_{\mathrm{ap}}(K * \mu)$. □

Remark 4.3. Estimate (4.16) also follows from the classical inequality satisfied by the Marcinkiewicz integral, see p. 15 of [28]:

$$\int_F \int_{\mathbb{R}^N\setminus F} \frac{d(z, F)}{|y - z|^{N+1}} \,\mathrm{d}|\mu|(z)\,\mathrm{d}y \le C''|\mu|(\mathbb{R}^N \setminus F). \tag{4.18}$$

The reason is that in the proof of Proposition 4.2, see (5.6), we choose

$$I_n(y) = \frac{C_1\ell_n}{|y - \bar{z}_n|^{N+1}}\|b_n\|_{\mathcal{M}(\mathbb{R}^N)} \quad \text{for every } y \in \mathbb{R}^N \setminus 2Q_n,$$

where $\bar{z}_n$ is the center of the cube Q_n, and this function I_n is controlled by the Marcinkiewicz integral of the measure $|\mu|\lfloor_{Q_n}$, namely

$$\int_{Q_n} \frac{d(z, F)}{|y - z|^{N+1}} d|\mu|(z).$$

Indeed, by construction of the Whitney covering, for every $z \in Q_n$ we have $\ell_n/2 \leq d(z, F)$ and, for every $y \in \mathbb{R}^N \setminus 2Q_n$,

$$|y - z| \leq |y - \bar{z}_n| + |z - \bar{z}_n| \leq |y - \bar{z}_n| + \sqrt{N}\,\frac{\ell_n}{2} \leq \Big(1 + \frac{\sqrt{N}}{2}\Big)|y - \bar{z}_n|.$$

Thus,

$$I_n(y) \leq \frac{C_1 \ell_n}{|y - \bar{z}_n|^{N+1}}\, 2 \int_{Q_n} \mathrm{d}|\mu|(z) \leq C_2 \int_{Q_n} \frac{d(z, F)}{|y - z|^{N+1}} d|\mu|(z). \tag{4.19}$$

Therefore, for every $y \in F$,

$$\sum_{n=0}^{\infty} I_n(y) \leq C_2 \int_{\bigcup\limits_{n=0}^{\infty} Q_n} \frac{d(z, F)}{|y - z|^{N+1}} d|\mu|(z) = C_2 \int_{\mathbb{R}^N \setminus F} \frac{d(z, F)}{|y - z|^{N+1}} d|\mu|(z)$$

and then (4.16) is implied by (4.18). We shall return to this observation in the proof of Theorem 1.3.

5. Proof of Proposition 4.2

In the next two lemmas we rely on the notation of Proposition 4.2 and, in particular, ν is a finite Borel measure in $\mathbb{R}^N$ with

$$\nu(Q) = 0 \quad \text{and} \quad |\nu|(\mathbb{R}^N \setminus Q) = 0.$$

We also write $\bar{z}$ for the center of the cube Q and ℓ for its side-length. The first lemma gives an estimate of the decay of $K * \nu(x)$ as $|x| \to \infty$ and is used in the proof of Proposition 4.2 when x and y are far from each other relatively to their distances to Q.

Lemma 5.1. *For every $x \in \mathbb{R}^N \setminus \theta Q$,*

$$|K * \nu(x)| \leq \frac{C'' \ell}{|x - \bar{z}|^N} \|\nu\|_{\mathcal{M}(\mathbb{R}^N)}.$$

Proof. Since $\nu(Q) = 0$, we can write

$$K * \nu(x) = \int_Q K(x - z)\, \mathrm{d}\nu(z) = \int_Q [K(x - z) - K(x - \bar{z})]\, \mathrm{d}\nu(z). \tag{5.1}$$

The estimate of the integrand that we require is given by

Claim. For every $z \in Q$ and $x \in \mathbb{R}^N \setminus \theta Q$,

$$|K(x - z) - K(x - \bar{z})| \leq C_1 \frac{|z - \bar{z}|}{|x - \bar{z}|^N},$$

where $C_1 > 0$ depends on θ, A, B and N.

Proof of the Claim. By the Mean value theorem, there exists a point ζ in the line segment $[z, \bar{z}]$ that joins z and $\bar{z}$ such that

$$K(x-z) - K(x-\bar{z}) = \nabla K(x-\zeta) \cdot (z-\bar{z}).$$

By (3.2), we thus have

$$|K(x-z) - K(x-\bar{z})| \le C_2 \frac{|z-\bar{z}|}{|x-\zeta|^N}. \tag{5.2}$$

Since $\zeta \in Q$, we have $|\zeta - \bar{z}|_\infty \le \ell/2$, where $|y|_\infty$ denotes the max-norm of a vector $y = (y_1, \dots, y_N)$ in $\mathbb{R}^N$, i.e.

$$|y|_\infty := \max\{|y_1|, \dots, |y_N|\}.$$

For $x \in \mathbb{R}^N \setminus \theta Q$, we also have $|x - \bar{z}|_\infty \ge \theta\ell/2$. Thus, by the triangle inequality,

$$|x-\bar{z}|_\infty \le |x-\zeta|_\infty + |\zeta-\bar{z}|_\infty \le |x-\zeta|_\infty + \frac{\ell}{2} \le |x-\zeta|_\infty + \frac{1}{\theta}|x-\bar{z}|_\infty.$$

This yields

$$\Big(1 - \frac{1}{\theta}\Big)|x-\bar{z}|_\infty \le |x-\zeta|_\infty$$

and then a similar estimate is satisfied by the Euclidean norm. The Claim thus follows from such an estimate and (5.2). □

From (5.1) and the Claim, we then get

$$|K * \nu(x)| \le \frac{C_3 \ell}{|x-\bar{z}|^N} \int_Q \mathrm{d}|\nu|(z) = \frac{C_3\ell}{|x-\bar{z}|^N} \|\nu\|_{\mathcal{M}(\mathbb{R}^N)}. \qquad \square$$

The next lemma deals with the case where x and y are close to each other relatively to their distances to Q:

Lemma 5.2. *There exists $\epsilon > 0$, depending on θ and N, such that for every $x, y \in \mathbb{R}^N \setminus \theta Q$ with $|x-y| \le \epsilon |x-\bar{z}|$, we have*

$$|K * \nu(x) - K * \nu(y)| \le \frac{C'''\ell}{|x-\bar{z}|^{N+1}} \|\nu\|_{\mathcal{M}(\mathbb{R}^N)} |x-y|.$$

In particular, for every $x \in \mathbb{R}^N \setminus \theta Q$,

$$|\nabla(K * \nu)(x)| \le \frac{C'''\ell}{|x-\bar{z}|^{N+1}} \|\nu\|_{\mathcal{M}(\mathbb{R}^N)}.$$

Proof. Applying the Fundamental theorem of Calculus and Fubini's theorem, we have

$$\begin{aligned} K * \nu(x) - K * \nu(y) &= \int_0^1 \nabla(K * \nu)(\xi_t) \cdot (x-y)\,\mathrm{d}t \\ &= \int_0^1 \int_Q \nabla K(\xi_t - z) \cdot (x-y)\,\mathrm{d}\nu(z)\,\mathrm{d}t, \end{aligned}$$

where $\xi_t := tx+(1-t)y$ belongs to the line segment $[x,y]$ for every $t \in [0,1]$. We do not explicit its dependence on x and y, but one should keep in mind that ξ_t *is independent of* z. Since $\nu(Q) = 0$, we can thus write

(5.3)
$$K * \nu(x) - K * \nu(y) = \int_0^1 \int_Q \big[\nabla K(\xi_t - z) - \nabla K(\xi_t - \bar{z})\big] \cdot (x-y)\,\mathrm{d}\nu(z)\,\mathrm{d}t.$$

Claim. Let $0 < \beta < \frac{\theta-1}{\theta}$ and $x \in \mathbb{R}^N \setminus \theta Q$. For every $\xi \in \mathbb{R}^N$ such that $|x-\xi|_\infty \le \beta|x-\bar{z}|_\infty$ and every $z \in Q$, we have

$$\big|\nabla K(\xi - z) - \nabla K(\xi - \bar{z})\big| \le C_1 \frac{|z-\bar{z}|}{|x-\bar{z}|^{N+1}},$$

where $C_1 > 0$ depends on β, θ, A, B and N.

Proof of the Claim. Applying the Mean value theorem, we deduce from the second estimate in (1.4) that there exists $\zeta \in [z,\bar{z}]$ such that

$$\big|\nabla K(\xi - z) - \nabla K(\xi - \bar{z})\big| \le C_2 \frac{|z-\bar{z}|}{|\xi-\zeta|^{N+1}}. \tag{5.4}$$

We now show that for any β as above, one has

$$|x-\bar{z}|_\infty \le C_3|\xi-\zeta|_\infty. \tag{5.5}$$

To this end, observe that for $\zeta \in Q$,

$$|\zeta-\bar{z}|_\infty \le \frac{\ell}{2} \le \frac{1}{\theta}|x-\bar{z}|_\infty.$$

By the triangle inequality and the assumption on ξ, we thus have

$$|x-\bar{z}|_\infty \le |x-\xi|_\infty + |\xi-\zeta|_\infty + |\zeta-\bar{z}|_\infty \le \Big(\beta + \frac{1}{\theta}\Big)|x-\bar{z}|_\infty + |\xi-\zeta|_\infty$$

and we conclude that

$$\Big(1-\beta-\frac{1}{\theta}\Big)|x-\bar{z}|_\infty \le |\xi-\zeta|_\infty.$$

Since the quantity in parenthesis is positive by the choice of β, this inequality is equivalent to (5.5). The claim thus follows from (5.4) and the counterpart of (5.5) for the Euclidean norm. □

To conclude the proof of the lemma, observe that if $|x-y| \le \epsilon|x-z|$ for some $\epsilon > 0$, then $|x-y|_\infty \le \epsilon\sqrt{N}|x-z|_\infty$. Thus, taking $\epsilon\sqrt{N} = \frac{\theta-1}{2\theta}$, every $\xi \in [x,y]$ satisfies the assumption of the Claim. By (5.3) and the Claim, we thus get

$$\begin{aligned}|K * \nu(x) - K * \nu(y)| &\le \frac{C_4\ell}{|x-\bar{z}|^{N+1}} \int_Q \mathrm{d}|\nu|(z)\,|x-y| \\ &= \frac{C_4\ell}{|x-\bar{z}|^{N+1}} \|\nu\|_{\mathcal{M}(\mathbb{R}^N)}\,|x-y|,\end{aligned}$$

which gives the main estimate of the lemma. From there, one estimates $\nabla(K*\nu)(x)$ by letting $y \to x$ in the direction of the gradient. □

Proof of Proposition 4.2. Let $\epsilon > 0$ be as in Lemma 5.2. Assuming that $x, y \in \mathbb{R}^N \setminus \theta Q$ satisfy

$$|x - y| \leq \epsilon \max\{|x - \bar{z}|, |y - \bar{z}|\},$$

then after relabeling x and y if necessary we have $|x - y| \leq \epsilon |x - \bar{z}|$. By Lemma 5.2, we thus have

$$|K * \nu(x) - K * \nu(y)| \leq C''' \ell \biggl(\frac{1}{|x - \bar{z}|^{N+1}} + \frac{1}{|y - \bar{z}|^{N+1}} \biggr) \|\nu\|_{\mathcal{M}(\mathbb{R}^N)} |x - y|.$$

We now assume instead that $x, y \in \mathbb{R}^N \setminus \theta Q$ satisfy

$$\epsilon \max\{|x - \bar{z}|, |y - \bar{z}|\} < |x - y|.$$

The triangle inequality and Lemma 5.1 imply that

$$|K * \nu(x) - K * \nu(y)| \leq C'' \ell \biggl(\frac{1}{|x - \bar{z}|^{N}} + \frac{1}{|y - \bar{z}|^{N}} \biggr) \|\nu\|_{\mathcal{M}(\mathbb{R}^N)}.$$

In view of the assumption on x and y in this case,

$$|K * \nu(x) - K * \nu(y)| \leq C'' \ell \biggl(\frac{1}{|x - \bar{z}|^{N+1}} + \frac{1}{|y - \bar{z}|^{N+1}} \biggr) \|\nu\|_{\mathcal{M}(\mathbb{R}^N)} \frac{|x - y|}{\epsilon}.$$

We thus have the conclusion with $I : \mathbb{R}^N \setminus \theta Q \to [0, \infty)$ defined by

$$I(y) = \frac{C' \ell}{|y - \bar{z}|^{N+1}} \|\nu\|_{\mathcal{M}(\mathbb{R}^N)}, \tag{5.6}$$

where $C' := \max\{C''', C''/\epsilon\}$. □

6. A UNIFORMIZATION PRINCIPLE

We now establish Theorem 1.2, whose main ingredient is already contained in the proof of Theorem 4.1. There we show that, for every $t > 0$, there exists a measurable function $I = I_t : \mathbb{R}^N \to [0, \infty]$ which satisfies (4.1) and (4.2). The distribution function of I_t only verifies the estimate we seek at height t. The next lemma is a uniformization property that encodes this family of functions $(I_t)_{t>0}$ into a single weak-L^1 function and immediately implies Theorem 1.2 by choosing

$$E = \operatorname{dom}(K * \mu) \quad \text{and} \quad A' = C_1 \|\mu\|_{\mathcal{M}(\mathbb{R}^N)},$$

where C_1 is the constant in (4.2).

Lemma 6.1. *Let $E \subset \mathbb{R}^N$ and $v : E \to \mathbb{R}$ be such that, for every $t > 0$, there exists a measurable function $I_t : \mathbb{R}^N \to [0, \infty]$ with*

$$|v(x) - v(y)| \leq (I_t(x) + I_t(y))|x - y| \quad \textit{for every } x, y \in E$$

and

$$|\{I_t > t\}| \leq \frac{A'}{t},$$

for some constant $A' > 0$ independent of t. Then, there exists a measurable function $H : \mathbb{R}^N \to [0, \infty]$ with

$$|v(x) - v(y)| \le (H(x) + H(y))|x - y| \quad \text{for every } x, y \in E \tag{6.1}$$

and

$$[H]_{L^1_{\mathrm{w}}(\mathbb{R}^N)} \le 8A'.$$

Proof. Given $x, y \in E$ such that $v(x) \neq v(y)$, take $n \in \mathbb{Z}$ such that

$$2^{n+1}|x - y| < |v(x) - v(y)| \le 2^{n+2}|x - y|.$$

By the first inequality, for each $t > 0$ we have that x or y belongs to $\{I_t > 2^n\}$. By the second inequality, we thus have

$$|v(x) - v(y)| \le 2^{n+2}\big(\chi_{\{I_t > 2^n\}}(x) + \chi_{\{I_t > 2^n\}}(y)\big)|x - y|. \tag{6.2}$$

Define $H : \mathbb{R}^N \to [0, \infty]$ by

$$H = \sup\big\{2^{n+2}\,\chi_{\{I_{2^n} > 2^n\}} : n \in \mathbb{Z}\big\}.$$

Then, (6.1) holds. To prove that H satisfies the desired weak-L^1 estimate, we first observe that, from the definition of H,

$$\{H > 2^k\} \subset \bigcup_{l=k-1}^{\infty} \{I_{2^l} > 2^l\} \quad \text{for every } k \in \mathbb{Z}. \tag{6.3}$$

Given $t > 0$, let $k \in \mathbb{Z}$ be such that $2^k \le t < 2^{k+1}$. By monotonicity and subadditivity of the Lebesgue measure, it follows from (6.3) that

$$|\{H > t\}| \le |\{H > 2^k\}| \le \sum_{l=k-1}^{\infty} |\{I_{2^l} > 2^l\}|.$$

From the assumption on the measure of the superlevel sets of I_t,

$$|\{H > t\}| \le \sum_{l=k-1}^{\infty} \frac{A'}{2^l} = \frac{A'}{2^{k-2}} \le \frac{8A'}{t}.$$

Since $t > 0$ is arbitrary, we thus have $[H]_{L^1_{\mathrm{w}}(\mathbb{R}^N)} \le 8A'$. ☐

The classical L^p singular-integral estimates can be also formulated using a Lipschitz-type formalism, whose proof relies on a standard interpolation argument that we sketch for the convenience of the reader:

Proposition 6.2. *Let $K \in C^2(\mathbb{R}^N \setminus \{0\})$ be any function that satisfies* (1.4) *and let $1 < p < \infty$. If $\mu \in (L^1 \cap L^p)(\mathbb{R}^N)$, then $K * \mu$ has a distributional gradient $\nabla(K * \mu)$ in $L^p(\mathbb{R}^N; \mathbb{R}^N)$ and there exists a measurable function $I : \mathbb{R}^N \to [0, \infty]$ in $L^p(\mathbb{R}^N)$ such that*

$$|K*\mu(x) - K*\mu(y)| \le (I(x) + I(y))|x - y| \quad \text{for every } x, y \in \operatorname{dom}(K * \mu) \tag{6.4}$$

and

$$\|I\|_{L^p(\mathbb{R}^N)} \le C\|\mu\|_{L^p(\mathbb{R}^N)},$$

where the constant $C > 0$ depends on A, B, p and N, *Proof.* A combination of Proposition 3.1, Theorem 4.1 and the Marcinkiewicz interpolation theorem implies that the linear functional

$$\mu \in C_c^\infty(\mathbb{R}^N) \longmapsto \nabla(K * \mu) \in L^p(\mathbb{R}^N) \tag{6.5}$$

is continuous with respect to the strong L^p topology on both sides for $1 < p < 2$ and then, by duality, the same conclusion holds for $2 < p < \infty$. By unique continuous extension of (6.5) one deduces that the distributional derivative $\nabla(K * \mu)$ belongs to $L^p(\mathbb{R}^N)$ for every $\mu \in (L^1 \cap L^p)(\mathbb{R}^N)$.

We claim that (6.4) holds with

$$I := 2^N \mathcal{M}|\nabla(K * \mu)| + R * |\mu|, \tag{6.6}$$

where $R(\xi) := \chi_{B_1}(\xi)/|\xi|^{N-1}$. Observe that such an explicit choice of coefficient I behaves sublinearly with respect to μ.

In view of Proposition 2.2, to check the claim it suffices to verify that every $x \in \mathbb{R}^N$ with $I(x) < \infty$ is a Lebesgue point of $K * \mu$ and $K * \mu(x)$ is the precise representative. That x is a Lebesgue point of $K * \mu$ is a consequence of $\mathcal{M}|\nabla(K * \mu)|(x) < \infty$, but without identification of the precise representative. The full property can be obtained instead using $R * |\mu|(x) < \infty$, as it implies that

$$\lim_{r \to 0} \fint_{B_r(x)} |K * \mu - K * \mu(x)| = 0. \tag{6.7}$$

Indeed, by Fubini's theorem, for every $r > 0$ we have

$$\fint_{B_r(x)} |K * \mu - K * \mu(x)| \leq \int_{\mathbb{R}^N} \left(\fint_{B_r(x)} |K(y - z) - K(x - z)| \, \mathrm{d}y \right) \mathrm{d}|\mu|(z).$$

By continuity of K in $\mathbb{R}^N \setminus \{0\}$,

$$\lim_{r \to 0} \fint_{B_r(x)} |K(y - z) - K(x - z)| \, \mathrm{d}y = 0 \quad \text{for every } z \in \mathbb{R}^N \setminus \{x\}.$$

Since $|K(\xi)| \leq A/|\xi|^{N-1}$, we also have

$$\begin{aligned} \fint_{B_r(x)} |K(y - z) - K(x - z)| \, \mathrm{d}y &\leq \fint_{B_r(x)} |K(y - z)| \, \mathrm{d}y + |K(x - z)| \\ &\leq \fint_{B_r(x)} \frac{A}{|y - z|^{N-1}} \, \mathrm{d}y + \frac{A}{|x - z|^{N-1}} \\ &\leq \frac{C_1}{|x - z|^{N-1}} \leq C_1 \big(R(x - z) + 1 \big), \end{aligned}$$

for every $x, z \in \mathbb{R}^N$ and $r > 0$, where the constant $C_1 > 0$ depends on A and N. As $R * |\mu|(x) < \infty$ and $\mu \in L^1(\mathbb{R}^N)$, we can apply the Dominated convergence theorem to deduce (6.7). Hence, $K * \mu(x)$ is the precise representative of $K * \mu$ at x and then (6.4) is satisfied thanks to Proposition 2.2.

To verify the L^p estimate of I, we apply the maximal inequality in $L^p(\mathbb{R}^N)$ and the interpolation argument in the beginning of the proof to get

$$\big\|\mathcal{M}|\nabla(K*\mu)|\big\|_{L^p(\mathbb{R}^N)} \le C_2\|\nabla(K*\mu)\|_{L^p(\mathbb{R}^N)} \le C_3\|\mu\|_{L^p(\mathbb{R}^N)}.$$

The estimate for I thus follows since $R \in L^1(\mathbb{R}^N)$ and then, by Young's inequality,

$$\|R*|\mu|\|_{L^p(\mathbb{R}^N)} \le \|R\|_{L^1(\mathbb{R}^N)}\|\mu\|_{L^p(\mathbb{R}^N)}.$$

□

7. Proof of Theorem 1.3

When $p > 1$, it is convenient to equip the space $L^p_{\mathrm{w}}(\mathbb{R}^N)$ of weak-L^p functions with the norm

$$\|f\|_{L^p_{\mathrm{w}}(\mathbb{R}^N)} := \sup\left\{\frac{1}{|A|^{\frac{p-1}{p}}}\int_A |f| : A \subset \mathbb{R}^N \text{ has finite measure}\right\},$$

which is equivalent to the quasinorm $[f]_{L^p_{\mathrm{w}}(\mathbb{R}^N)}$; see e.g. the proof of Proposition 5.6 in [23]. A straightforward application of Fubini's theorem gives the following counterpart of Young's inequality in weak-L^p spaces for $p > 1$:

$$\|K*\mu\|_{L^p_{\mathrm{w}}(\mathbb{R}^N)} \le \|K\|_{L^p_{\mathrm{w}}(\mathbb{R}^N)}\|\mu\|_{\mathcal{M}(\mathbb{R}^N)}, \tag{7.1}$$

where μ is any finite Borel measure in $\mathbb{R}^N$; see [4, Lemma A.4].

Proof of Theorem 1.3. Given $t > 0$, we rely on the Calderón-Zygmund decomposition (4.6), namely

$$\mu = g\,\mathrm{d}x + \sum_{n=0}^{\infty} b_n,$$

where $g \in (L^1 \cap L^\infty)(\mathbb{R}^N)$ and each measure b_n satisfies $b_n(Q_n) = 0$, $|b_n|(\mathbb{R}^N \setminus Q_n) = 0$ and

$$\|b_n\|_{\mathcal{M}(\mathbb{R}^N)} \le 2|\mu|(Q_n). \tag{7.2}$$

The half-closed cubes Q_n are disjoint and given by the Whitney covering of the open set

$$\mathbb{R}^N \setminus F = \{\mathcal{M}\mu > t\}.$$

Moreover,

$$K*\mu(x) = K*g(x) + \sum_{n=0}^{\infty} K*b_n(x) \quad \text{for every } x \in \mathbb{R}^N \setminus S,$$

where $S \subset \mathbb{R}^N$ is the Lebesgue-negligible set given by (4.10). In particular, this identity holds almost everywhere in $\mathbb{R}^N$.

At a point $y \in \mathbb{R}^N \setminus S$ where $K*g$ and all $K*b_n$ are approximately differentiable, we have

$$|K*\mu - P_y| \le |K*g - T^1_y(K*g)| + \sum_{n=0}^{\infty}|K*b_n - T^1_y(K*b_n)| \tag{7.3}$$

almost everywhere in $\mathbb{R}^N$, where P_y is the affine function

$$P_y := T^1_y(K * g) + \sum_{n=0}^{\infty} T^1_y(K * b_n),$$

provided that the series in the right-hand side converges.

From Proposition 6.2, since $g \in (L^1 \cap L^\infty)(\mathbb{R}^N)$ the function $K * g$ has a distributional gradient in $L^p(\mathbb{R}^N; \mathbb{R}^N)$ and then is L^p-differentiable almost everywhere in $\mathbb{R}^N$ for every $1 < p < \infty$. In particular, for $p = \frac{N}{N-1}$,

$$\lim_{r\to 0} \frac{\|K * g - T^1_y(K * g)\|_{L^{\frac{N}{N-1}}(B_r(y))}}{r^N} = 0 \quad \text{for almost every } y \in \mathbb{R}^N. \tag{7.4}$$

Observe that, by smoothness of the functions $K * b_n$ in a neighborhood of $y \in F$, each term in the series in the right-hand side of (7.3) evaluated at a point x behaves like $o(|x-y|)$ as $x \to y$. Thus, for every $n \in \mathbb{N}$ we also have

$$\lim_{r\to 0} \frac{\|K * b_n - T^1_y(K * b_n)\|_{L^{\frac{N}{N-1}}(B_r(y))}}{r^N} = 0 \quad \text{for every } y \in F. \tag{7.5}$$

To handle the fact that we are dealing with infinitely many terms in (7.3), we need a uniform estimate of the tail of the series. To this end, we take $y \in F$. For every $n \in \mathbb{N}$, we have $y \in \mathbb{R}^N \setminus 2Q_n$ and then, by Lemma 5.2 applied at y,

$$|\nabla(K * b_n)(y)| \le \frac{C''' \ell_n}{|y - \bar{z}_n|^{N+1}} \|b_n\|_{\mathcal{M}(\mathbb{R}^N)},$$

where ℓ_n is the side-length and $\bar{z}_n$ is the center of Q_n. By (4.19) in Remark 4.3, we also have

$$\frac{\ell_n}{|y - \bar{z}_n|^{N+1}} \|b_n\|_{\mathcal{M}(\mathbb{R}^N)} \le C_1 \int_{Q_n} \frac{d(z, F)}{|y - z|^{N+1}} \, \mathrm{d}|\mu|(z). \tag{7.6}$$

Hence,

$$|\nabla(K * b_n)(y)| \le C_2 \int_{Q_n} \frac{d(z, F)}{|y - z|^{N+1}} \, \mathrm{d}|\mu|(z). \tag{7.7}$$

Next, let $J \in \mathbb{N}$ and $r > 0$, and let $\epsilon > 0$ be given by Lemma 5.2. Observe that ϵ only depends on the dimension N. We divide the cubes Q_n with indices $n \ge J$ in two disjoint classes, according to their distances from the point y as follows: $\mathcal{F}_{J,r}$ is the subset of indices $n \ge J$ such that $\epsilon|y - \bar{z}_n| > r$ and $\mathcal{C}_{J,r}$ is the subset of indices $n \ge J$ such that $\epsilon|y - \bar{z}_n| \le r$.

The class $\mathcal{F}_{J,r}$ keeps track of the cubes which are *far from* y. We claim that, for every $n \in \mathcal{F}_{J,r}$,

$$\|K * b_n - T^1_y(K * b_n)\|_{L^\infty(B_r(y))} \le C_3\, r \int_{Q_n} \frac{d(z, F)}{|y - z|^{N+1}} \, \mathrm{d}|\mu|(z). \tag{7.8}$$

Indeed, for every $x \in B_r(y)$ and $n \in \mathcal{F}_{J,r}$,

$$|x - y| < r < \epsilon|y - \bar{z}_n|.$$

By Lemma 5.2 (reversing the roles of x and y) and (7.6), we then have

$$
\begin{aligned}
|K * b_n(x) - K * b_n(y)| &\le \frac{C'''\ell_n}{|y-\bar{z}_n|^{N+1}}\|b_n\|_{\mathcal{M}(\mathbb{R}^N)}|x-y| \\
&\le C_4\, r \int_{Q_n} \frac{d(z,F)}{|y-z|^{N+1}}\,\mathrm{d}|\mu|(z).
\end{aligned}
\tag{7.9}
$$

Combining (7.9) and (7.7), we deduce (7.8). The latter implies that, for every $n \in \mathcal{F}_{J,r}$,

$$
\|K * b_n - T_y^1(K * b_n)\|_{L_{\mathrm{w}}^{\frac{N}{N-1}}(B_r(y))} \le C_5\, r^N \int_{Q_n} \frac{d(z,F)}{|y-z|^{N+1}}\,\mathrm{d}|\mu|(z).
\tag{7.10}
$$

The class $\mathcal{C}_{J,r}$ gathers the cubes which are *close to* y. We claim in this case that, for every $n \in \mathcal{C}_{J,r}$,

$$
\|K*b_n - T_y^1(K*b_n)\|_{L_{\mathrm{w}}^{\frac{N}{N-1}}(B_r(y))} \le C_6\bigg(|\mu|(Q_n) + r^N \int_{Q_n} \frac{d(z,F)}{|y-z|^{N+1}}\,\mathrm{d}|\mu|(z)\bigg).
\tag{7.11}
$$

Indeed, the decay assumption (1.4) on K implies that $K \in L_{\mathrm{w}}^{\frac{N}{N-1}}(\mathbb{R}^N)$. Then, by Young's inequality (7.1) and (7.2) we get

$$
\|K * b_n\|_{L_{\mathrm{w}}^{\frac{N}{N-1}}(B_r(y))} \le \|K\|_{L_{\mathrm{w}}^{\frac{N}{N-1}}(\mathbb{R}^N)}\|b_n\|_{\mathcal{M}(\mathbb{R}^N)} \le C_7|\mu|(Q_n).
\tag{7.12}
$$

Using Lemma 5.1 and (7.6), we also have

$$
|K * b_n(y)| \le \frac{C''\ell_n}{|y-\bar{z}_n|^N}\|b_n\|_{\mathcal{M}(\mathbb{R}^N)} \le C_8|y-\bar{z}_n| \int_{Q_n} \frac{d(z,F)}{|y-z|^{N+1}}\,\mathrm{d}|\mu|(z).
$$

Since $\epsilon|y-\bar{z}_n| \le r$, we thus have

$$
|K * b_n(y)| \le \frac{C_8 r}{\epsilon} \int_{Q_n} \frac{d(z,F)}{|y-z|^{N+1}}\,\mathrm{d}|\mu|(z).
\tag{7.13}
$$

Estimate (7.11) now follows from the combination of (7.12), (7.13) and (7.7).

Note that for every $n \in \mathcal{C}_{J,r}$ we have $Q_n \subset B_{\gamma r}(y)$, for some constant $\gamma > 0$ depending on N. Since $\bigcup_{n\in\mathcal{C}_{J,r}} Q_n \subset B_{\gamma r}(y) \setminus F$ and the cubes Q_n are disjoint, we deduce from (7.10) for $n \in \mathcal{F}_{J,r}$ and (7.11) for $n \in \mathcal{C}_{J,r}$ that

$$
\begin{aligned}
&\sum_{n=J}^{\infty} \|K * b_n - T_y^1(K * b_n)\|_{L_{\mathrm{w}}^{\frac{N}{N-1}}(B_r(y))} \\
&\qquad \le C_9\bigg(|\mu|(B_{\gamma r}(y)\setminus F) + r^N \int_{\bigcup_{n=J}^{\infty} Q_n} \frac{d(z,F)}{|y-z|^{N+1}}\,\mathrm{d}|\mu|(z)\bigg).
\end{aligned}
$$

Recall that for almost every $y \in F$ we have

$$
\lim_{r\to 0} \frac{|\mu|(B_{\gamma r}(y)\setminus F)}{r^N} = 0 \quad \text{and} \quad \int_{\mathbb{R}^N\setminus F} \frac{d(z,F)}{|y-z|^{N+1}}\,\mathrm{d}|\mu|(z) < \infty.
\tag{7.14}
$$

The first assertion follows from the Besicovitch differentiation theorem, while the second one is a consequence of inequality (4.18) in Remark 4.3

satisfied by the Marcinkiewicz integral. At a point y where the first property in (7.14) holds, we have

$$\limsup_{r\to 0} \frac{\sum\limits_{n=J}^{\infty} \|K * b_n - T_y^1(K * b_n)\|_{L_{\mathrm{w}}^{\frac{N}{N-1}}(B_r(y))}}{r^N} \le C_9 \int_{\bigcup\limits_{n=J}^{\infty} Q_n} \frac{d(z,F)}{|y-z|^{N+1}} \,\mathrm{d}|\mu|(z).$$

It thus follows from (7.3), (7.4) and (7.5) that

$$\limsup_{r\to 0} \frac{\|K * \mu - P_y\|_{L_{\mathrm{w}}^{\frac{N}{N-1}}(B_r(y))}}{r^N} \le C_9 \int_{\bigcup\limits_{n=J}^{\infty} Q_n} \frac{d(z,F)}{|y-z|^{N+1}} \,\mathrm{d}|\mu|(z).$$

At a point y where the second property in (7.14) holds, the integral in the right-hand side converges to zero as $J \to \infty$ and we deduce that the limsup in the left-hand side vanishes. Hence, by uniqueness of the Taylor approximation we have

$$P_y = T_y^1(K * \mu)$$

and $K * \mu$ is $L_{\mathrm{w}}^{\frac{N}{N-1}}$-differentiable at almost every $y \in F$. Since $F = F_t$ satisfies (4.3), the conclusion of the theorem then follows by letting $t \to \infty$. □

8. Existence of $D^2_{\mathrm{ap}} u$ when Δu is a mesure

Let $E : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ be the fundamental solution of $-\Delta$ defined in dimension $N \ge 3$ by

$$E(x) = \frac{1}{(N-2)\sigma_N} \frac{1}{|x|^{N-2}},$$

where σ_N denotes the area of the unit sphere in $\mathbb{R}^N$. Since $E \in (L^1 + L^\infty)(\mathbb{R}^N)$, the Newtonian potential $E * \mu$ is defined almost everywhere for a finite Borel measure μ in $\mathbb{R}^N$ and belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^N)$. In addition, $E * \mu \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)$ for every $1 \le p < \frac{N}{N-1}$ and one has

$$-\Delta(E * \mu) = \mu \quad \text{in the sense of distributions in } \mathbb{R}^N;$$

see Example 2.12 in [23]. Since $\nabla(E * \mu) = (\nabla E) * \mu$ and

$$\nabla E(x) = -\frac{1}{\sigma_N} \frac{x}{|x|^N}$$

satisfies (1.4), we have by Theorem 4.1 that $\nabla(E*\mu)$ is approximately differentiable almost everywhere and its approximate derivative $D^2_{\mathrm{ap}}(E * \mu) = D_{\mathrm{ap}}((\nabla E) * \mu)$ satisfies

$$[D^2_{\mathrm{ap}}(E * \mu)]_{L^1_{\mathrm{w}}(\mathbb{R}^N)} \le C\|\mu\|_{\mathcal{M}(\mathbb{R}^N)}. \tag{8.1}$$

We apply this estimate to identify the trace of $D^2_{\mathrm{ap}}(E * \mu)$ in terms of μ:

Proposition 8.1. *Let $N \geq 3$. For every finite Borel measure μ in $\mathbb{R}^N$, the absolutely continuous part of μ with respect to the Lebesgue measure satisfies*

$$\mu_{\mathrm{a}} = -\operatorname{tr}\left(D^2_{\mathrm{ap}}(E * \mu)\right) \mathrm{d}x.$$

Observe that for $f \in C^\infty_c(\mathbb{R}^N)$, the Newtonian potential $E * f$ is a smooth function. Thus, $D^2_{\mathrm{ap}}(E * f)$ is the classical second-order derivative of $E * f$. Since $E * f$ solves the Poisson equation with density f, we get

$$f = -\Delta(E * f) = -\operatorname{tr}\left(D^2_{\mathrm{ap}}(E * f)\right) \quad \text{in } \mathbb{R}^N,$$

which is Proposition 8.1 for smooth functions.

Next, for a finite Borel measure μ, the Newtonian potential $E * \mu$ is smooth and harmonic in $\mathbb{R}^N \setminus \operatorname{supp}\mu$. Thus,

$$\operatorname{tr}\left(D^2_{\mathrm{ap}}(E * \mu)\right) = \Delta(E * \mu) = 0 \quad \text{in } \mathbb{R}^N \setminus \operatorname{supp}\mu.$$

In particular, when the support $\operatorname{supp}\mu$ is negligible with respect to the Lebesgue measure, one has

$$\operatorname{tr}\left(D^2_{\mathrm{ap}}(E * \mu)\right) = 0 \quad \text{almost everywhere in } \mathbb{R}^N.$$

The proof of Proposition 8.1 is based on an approximation argument that relies on estimate (8.1) and these two cases.

Proof of Proposition 8.1. We first assume that μ is absolutely continuous with respect to the Lebesgue measure. Thus, $\mu = \mu_{\mathrm{a}} = f \,\mathrm{d}x$ for some $f \in L^1(\mathbb{R}^N)$. Take a sequence $(f_n)_{n\in\mathbb{N}}$ in $C^\infty_c(\mathbb{R}^N)$ that converges to f in $L^1(\mathbb{R}^N)$. For every $n \in \mathbb{N}$,

$$f_n = -\operatorname{tr}\left(D^2_{\mathrm{ap}}(E * f_n)\right) \quad \text{in } \mathbb{R}^N.$$

Thus, by the triangle inequality and linearity of the approximate derivative,

$$\left|\operatorname{tr}\left(D^2_{\mathrm{ap}}(E * f)\right) + f_n\right| \leq \left|D^2_{\mathrm{ap}}(E * f) - D^2_{\mathrm{ap}}(E * f_n)\right| = \left|D^2_{\mathrm{ap}}(E * (f - f_n))\right|.$$

Applying estimate (8.1) to $f - f_n$, we thus have

$$\begin{aligned}\left[\operatorname{tr}\left(D^2_{\mathrm{ap}}(E * f)\right) + f_n\right]_{L^1_{\mathrm{w}}(\mathbb{R}^N)} &\leq \left[D^2_{\mathrm{ap}}(E * (f - f_n))\right]_{L^1_{\mathrm{w}}(\mathbb{R}^N)} \\ &\leq C\|(f - f_n)\,\mathrm{d}x\|_{\mathcal{M}(\mathbb{R}^N)} = C\|f - f_n\|_{L^1(\mathbb{R}^N)}.\end{aligned}$$

Hence, the sequence $(f_n)_{n\in\mathbb{N}}$ converges in measure simultaneously to f and $-\operatorname{tr}\left(D^2_{\mathrm{ap}}(E * f)\right)$. By uniqueness of the limit, we deduce that

$$-\operatorname{tr}\left(D^2_{\mathrm{ap}}(E * f)\right) = f \quad \text{almost everywhere in } \mathbb{R}^N,$$

when $\mu = f \,\mathrm{d}x$ is absolutely continuous with respect to the Lebesgue measure.

We now assume that μ is singular with respect to the Lebesgue measure. Let $S \subset \mathbb{R}^N$ be a negligible Borel set such that $|\mu|(\mathbb{R}^N \setminus S) = 0$. By inner regularity of $|\mu|$, there exists a sequence of compact sets $K_n \subset S$ such that

$$\lim_{n\to\infty} |\mu|(S \setminus K_n) = 0.$$

Each measure $\mu_n = \mu\lfloor_{K_n}$ is supported in the negligible set K_n. Since $E * \mu_n$ is harmonic in $\mathbb{R}^N \setminus K_n$, we thus have

$$\operatorname{tr}(D^2_{\mathrm{ap}}(E * \mu_n)) = 0 \quad \text{almost everywhere in } \mathbb{R}^N.$$

Again, by linearity of the approximate derivative,

$$\left|\operatorname{tr}(D^2_{\mathrm{ap}}(E * \mu))\right| \le \left|D^2_{\mathrm{ap}}(E * \mu) - D^2_{\mathrm{ap}}(E * \mu_n)\right| = \left|D^2_{\mathrm{ap}}(E * (\mu - \mu_n))\right|$$

almost everywhere in $\mathbb{R}^N$. Estimate (8.1) applied to $\mu - \mu_n$ then implies

$$\left[\operatorname{tr}(D^2_{\mathrm{ap}}(E * \mu))\right]_{L^1_{\mathrm{w}}(\mathbb{R}^N)} \le C\|\mu - \mu_n\|_{\mathcal{M}(\mathbb{R}^N)} = |\mu|(\mathbb{R}^N \setminus K_n) = |\mu|(S \setminus K_n).$$

As $n \to \infty$, the right-hand side converges to zero. Hence,

$$\operatorname{tr}(D^2_{\mathrm{ap}}(E * \mu)) = 0 \quad \text{almost everywhere in } \mathbb{R}^N,$$

when μ is singular with respect to the Lebesgue measure.

The proof now follows from the linearity of $E * \mu$ since any finite Borel measure μ has a decomposition of the form

$$\mu = \mu_{\mathrm{a}} + \mu_{\mathrm{s}} = f\,\mathrm{d}x + \mu_{\mathrm{s}} \quad \text{with } f \in L^1(\mathbb{R}^N),$$

where μ_{s} is the singular part of μ with respect to the Lebesgue measure. By the two cases considered above, we have

$$\operatorname{tr}(D^2_{\mathrm{ap}}(E * \mu)) = \operatorname{tr}(D^2_{\mathrm{ap}}(E * f)) + \operatorname{tr}(D^2_{\mathrm{ap}}(E * \mu_{\mathrm{s}})) = -f$$

almost everywhere in $\mathbb{R}^N$. □

In dimension $N = 2$, the fundamental solution of $-\Delta$ is

$$E(x) = \frac{1}{2\pi} \log \frac{1}{|x|}$$

and the Newtonian potential $E * \mu$ is well-defined for every finite Borel measure with *compact support* in $\mathbb{R}^2$. The counterpart of Proposition 8.1 holds for these measures, with the same proof.

In every dimension $N \ge 2$, Proposition 8.1 has a counterpart for solutions of the Dirichlet problem

$$\begin{cases} -\Delta u = \mu & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases} \tag{8.2}$$

involving a finite Borel measure μ in a smooth bounded open subset $\Omega \subset \mathbb{R}^N$. By a solution of (8.2), we mean that $u \in W^{1,1}_0(\Omega)$ satisfies

$$-\Delta u = \mu \quad \text{in the sense of distributions in } \Omega.$$

Littman, Stampacchia and Weinberger [21] proved that the Dirichlet problem above has a unique solution for every μ. This solution has additional imbedding properties that can be formulated in terms of weak-Lebesgue spaces. For example, using Stampacchia's truncation method one shows in dimension $N \geq 3$ that

$$[u]_{L_{\mathrm{w}}^{\frac{N}{N-2}}(\Omega)} + [\nabla u]_{L_{\mathrm{w}}^{\frac{N}{N-1}}(\Omega)} \leq C\|\Delta u\|_{\mathcal{M}(\Omega)},$$

where $C > 0$ depends on N; see [23, Proposition 5.7]. A second-order counterpart of this inequality is

Proposition 8.2. *Let $N \geq 2$ and let $\Omega \subset \mathbb{R}^N$ be a smooth bounded open set. For every finite Borel measure μ in Ω, the solution u of the Dirichlet problem* (8.2) *has a second-order approximate derivative $D^2_{\mathrm{ap}}u$ almost everywhere in Ω that satisfies*

$$(\Delta u)_{\mathrm{a}} = \operatorname{tr}(D^2_{\mathrm{ap}}u)\,\mathrm{d}x \quad \text{and} \quad [D^2_{\mathrm{ap}}u]_{L^1_{\mathrm{w}}(\Omega)} \leq C'\|\Delta u\|_{\mathcal{M}(\Omega)},$$

for some constant $C' > 0$ depending on N.

Proof. Let $U : \mathbb{R}^N \to \mathbb{R}$ be the function defined by

$$U(x) = \begin{cases} u & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

From Poincaré's balayage method, see [23, Corollary 7.4], one has that ΔU is a finite Borel measure with compact support in $\mathbb{R}^N$ and

$$\|\Delta U\|_{\mathcal{M}(\mathbb{R}^N)} \leq 2\|\Delta u\|_{\mathcal{M}(\Omega)}.$$

Since U has compact support in $\mathbb{R}^N$, we have

$$U = E * (-\Delta U) \quad \text{in } \mathbb{R}^N.$$

This identity is indeed true for functions in $C_c^\infty(\mathbb{R}^N)$ and the general case follows by approximation using $\rho_n * U$, where $(\rho_n)_{n\in\mathbb{N}}$ is a sequence of mollifiers in $C_c^\infty(\mathbb{R}^N)$. We deduce from Proposition 8.1 (and its counterpart in dimension 2) that $u = U$ has a second-order approximate derivative almost everywhere in Ω with

$$(\Delta u)_{\mathrm{a}} = (\Delta U)_{\mathrm{a}} = \operatorname{tr}(D^2_{\mathrm{ap}}U)\,\mathrm{d}x = \operatorname{tr}(D^2_{\mathrm{ap}}u)\,\mathrm{d}x$$

and

$$[D^2_{\mathrm{ap}}u]_{L^1_{\mathrm{w}}(\Omega)} \leq C_1\|\Delta U\|_{\mathcal{M}(\mathbb{R}^N)} \leq 2C_1\|\Delta u\|_{\mathcal{M}(\Omega)}.$$ □

9. Proof of Theorem 1.1

Given a bounded open subset $\omega \Subset \Omega$, let $\varphi \in C_c^\infty(\Omega)$ be such that $\varphi = 1$ on ω. The measure $\mu = -\Delta(u\varphi)$ is finite, has compact support in Ω and can be written as

$$\mu = -\big(\Delta u\,\varphi + 2\nabla u \cdot \nabla\varphi + u\Delta\varphi\big) \quad \text{in the sense of distributions in } \mathbb{R}^N;$$

see [23, Proposition 6.11]. In particular,

$$\mu = -\Delta u \quad \text{in } \omega. \tag{9.1}$$

We extend the measure μ to $\mathbb{R}^N$ as zero on subsets of $\mathbb{R}^N \setminus \Omega$. The Newtonian potential $E * \mu$ and $u\varphi$ satisfy the same Poisson equation in Ω. Thus, by Weyl's lemma we have

$$E * \mu = u\varphi + h \quad \text{almost everywhere in } \Omega,$$

where h is a harmonic function in Ω. By Proposition 8.1 (and its counterpart in dimension 2), we deduce that $\nabla(u\varphi)$ is approximately differentiable almost everywhere in Ω and

$$\mu_{\mathrm{a}} = -\operatorname{tr}(D^2_{\mathrm{ap}}(u\varphi))\,\mathrm{d}x. \tag{9.2}$$

Since $\nabla u = \nabla(u\varphi)$ in ω and the notion of approximate derivative is local, ∇u is approximately differentiable almost everywhere in ω and $D^2_{\mathrm{ap}}u = D^2_{\mathrm{ap}}(u\varphi)$ in ω. It then follows from (9.1) and (9.2) that

$$(\Delta u)_{\mathrm{a}} = -\mu_{\mathrm{a}} = \operatorname{tr}(D^2_{\mathrm{ap}}u)\,\mathrm{d}x \quad \text{in } \omega.$$

Since this is true for every bounded open subset $\omega \Subset \Omega$, we have $(\Delta u)_{\mathrm{a}} = \operatorname{tr}(D^2_{\mathrm{ap}}u)\,\mathrm{d}x$ in Ω.

To conclude the proof we rely on (2.1), that implies

$$D^2_{\mathrm{ap}}u = D_{\mathrm{ap}}(\nabla u) = 0 \quad \text{almost everywhere on } \{\nabla u = e\}$$

for every $e \in \mathbb{R}^N$. Thus,

$$(\Delta u)_{\mathrm{a}} = \operatorname{tr}(D^2_{\mathrm{ap}}u)\,\mathrm{d}x = 0 \quad \text{almost everywhere on } \{\nabla u = e\}. \tag{9.3}$$

Since $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$, we also have $\nabla u = 0$ almost everywhere on every level set $\{u = \alpha\}$ with $\alpha \in \mathbb{R}$. Hence, applying (9.3) with $e = 0$ we also get

$$(\Delta u)_{\mathrm{a}} = 0 \quad \text{almost everywhere on } \{u = \alpha\}.$$

This completes the proof of the theorem. □

10. Two applications

10.1. **Level sets of subharmonic functions.** As a first application of Theorem 1.1 we give a new proof of Theorem B.1 of Frank and Lieb [13] about level sets of subharmonic functions.

Proposition 10.1. *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $u \in L^1_{\mathrm{loc}}(\Omega)$ be such that Δu is a locally finite Borel measure in Ω. If*

$$|\Delta u| \geq \theta\,\mathrm{d}x, \tag{10.1}$$

where θ is a Borel-measurable function in Ω such that

$$\theta > 0 \quad \textit{almost everywhere in } \Omega, \tag{10.2}$$

then the level set $\{u = \alpha\}$ is Lebesgue-negligible for every $\alpha \in \mathbb{R}$.

Proof. Let $\Delta u = (\Delta u)_{\mathrm{a}} + (\Delta u)_{\mathrm{s}}$ be the decomposition of Δu in terms of an absolutely continuous and a singular part with respect to the Lebesgue measure, and let $S \subset \Omega$ be a Borel-measurable set such that $|S| = 0$ and $|(\Delta u)_{\mathrm{s}}|(\Omega \setminus S) = 0$. For every $\alpha \in \mathbb{R}$, we have

$$\{u = \alpha\} \subset \big(\{u = \alpha\} \setminus S\big) \cup S.$$

Since S is negligible, it suffices to prove that $\{u = \alpha\} \setminus S$ is negligible. To this end, we write the estimate

$$\int_{\{u=\alpha\}\setminus S} \theta \, \mathrm{d}x \le \int_{\{u=\alpha\}\setminus S} |\Delta u| \le \int_{\{u=\alpha\}} |(\Delta u)_{\mathrm{a}}| + \int_{\Omega\setminus S} |(\Delta u)_{\mathrm{s}}|.$$

Both integrals in the right-hand side vanish: The first one because of Theorem 1.1 and the second one by the choice of S. Hence, the integral in the left-hand side also vanishes, and then, by assumption on θ, the set $\{u = \alpha\} \setminus S$ must be negligible. □

Assumptions (10.1) and (10.2) are implied by

$$\int_K \Delta u \neq 0 \quad \text{for any compact set } K \subset \Omega \text{ with } |K| > 0, \tag{10.3}$$

which is weaker than the assumption made in [13], namely positivity of $\int_K \Delta u$. Indeed, let $S \subset \Omega$ be a Lebesgue-negligible set where the singular part of $|\Delta u|$ with respect to the Lebesgue measure $\mathrm{d}x$ is concentrated. Condition (10.3) implies, by the Hahn decomposition and inner approximation of Δu,

$$\int_B |\Delta u| > 0 \quad \text{for any Borel set } B \subset \Omega \text{ with } |B| > 0.$$

We now take θ as the density of the absolutely continuous part of $|\Delta u|$ with respect to $\mathrm{d}x$. Since the integral above vanishes with $B = \{\theta = 0\} \setminus S$, we see that $\{\theta = 0\} \setminus S$ and then $\{\theta = 0\}$ must be Lebesgue-negligible, which gives (10.1) and (10.2).

As a consequence of Proposition 10.1 above, we deduce Proposition A.1 from [13]:

Corollary 10.2. *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $u : \Omega \to \mathbb{R}$ be a continuous function. If there exists $\epsilon > 0$ such that*

$$\Delta u \ge \epsilon \quad \textit{in the viscosity sense in } \Omega, \tag{10.4}$$

then $\{u = \alpha\}$ is Lebesgue-negligible for every $\alpha \in \mathbb{R}$.

Proof. We recall that a continuous function u satisfies (10.4) whenever one has $\Delta\varphi(x) \ge \epsilon$ for every $x \in \Omega$ and every $\varphi \in C^2(\Omega)$ such that $\varphi \ge u$ in a neighborhood of x and $\varphi(x) = u(x)$. By the relation between viscosity and distributional solutions, see [17, Theorem 1] and also [18, 20], we have

$$\Delta u \ge \epsilon \quad \text{in the sense of distributions in } \Omega.$$

This implies that Δu is a locally finite Borel measure in Ω with $\Delta u \geq \epsilon\, \mathrm{d}x$, so that Proposition 10.1 with $\theta = \epsilon$ gives the result. □

As Frank and Lieb point out in their paper, the conclusion of Corollary 10.2 is false under the assumption $\Delta u \geq 0$, without strict inequality, as shown by the example $u(x) = \max\{x_1, 0\}$ in $\mathbb{R}^N$.

10.2. **Limiting vorticities of the Ginzburg-Landau system.** The Ginzburg-Laudau model can be used to describe the phenomenon of superconductivity in some materials at low temperature subject to a magnetic field; see [27]. The state of a superconducting sample in a domain $\Omega \subset \mathbb{R}^2$ is then described by an order parameter $u : \Omega \to \mathbb{C}$ and a magnetic potential $A : \Omega \to \mathbb{R}$ that are local minimizers or merely critical points of the energy functional

$$G_\epsilon(u, A) = \frac{1}{2}\int_\Omega |\nabla^A u|^2 + \int_\Omega (h - h_{\text{ex}})^2 + \frac{1}{4\epsilon^2}\int_\Omega (1 - |u|^2)^2. \tag{10.5}$$

Here, $\epsilon > 0$ is a small parameter, the constant $h_{\text{ext}} > 0$ is the intensity of the applied magnetic field,

$$h := \operatorname{curl} A = -\operatorname{div} A^\perp$$

is the induced magnetic field and

$$\nabla^A u := \nabla u - iAu$$

is the covariant gradient. Regions where the order parameter u satisfies $|u| \approx 1$ are in superconductor phase, while the material behaves as a normal conductor in places where $|u| \approx 0$. The Euler-Lagrange equation satisfied by a critical point (u_ϵ, A_ϵ) of (10.5) gives (see Proposition 3.6 in [27])

$$-\nabla^\perp h_\epsilon = (iu_\epsilon | \nabla^A u_\epsilon) \quad \text{in the sense of distributions in } \Omega, \tag{10.6}$$

where

$$\nabla^\perp h_\epsilon := \Big(-\frac{\partial h_\epsilon}{\partial x_2}, \frac{\partial h_\epsilon}{\partial x_1}\Big)$$

and

$$(iu_\epsilon | \nabla^{A_\epsilon} u_\epsilon) := \frac{iu_\epsilon\, \overline{\nabla^{A_\epsilon} u_\epsilon} + \overline{iu_\epsilon}\, \nabla^{A_\epsilon} u_\epsilon}{2} \in \mathbb{R}^2$$

is the superconductivity current. By computing the curl on both sides of (10.6), one obtains

$$-\Delta h_\epsilon + h_\epsilon = \mu_\epsilon \quad \text{in the sense of distributions in } \Omega, \tag{10.7}$$

where the intrinsic vorticity μ_ϵ associated to (u_ϵ, A_ϵ) is given by

$$\mu_\epsilon := \operatorname{curl}(iu_\epsilon | \nabla^{A_\epsilon} u_\epsilon) + \operatorname{curl} A_\epsilon.$$

This quantity is an analogue of the distributional Jacobian that is invariant under the gauge transformation

$$(u, A) \longmapsto (u\mathrm{e}^{if}, A + \nabla f),$$

for any smooth function f.

Under certain conditions in the regime where $\epsilon \to 0$, a suitable renormalization of h_ϵ and μ_ϵ yields a nontrivial solution of the equation

$$-\Delta h + h = \mu \quad \text{in the sense of distributions in } \Omega \tag{10.8}$$

where $h \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is called the limiting induced magnetic field and the locally finite Borel measure μ is the limiting vorticity. The limiting magnetic field h satisfies in addition

$$\operatorname{div} T_h = 0 \quad \text{in the sense of distributions in } \Omega, \tag{10.9}$$

where $\operatorname{div} T_h = (\operatorname{div} T_{h,1}, \operatorname{div} T_{h,2})$ and $T_{h,i} : \Omega \to \mathbb{R}^2$ is defined by

$$T_{h,i,j} := -\frac{\partial h}{\partial x_i}\frac{\partial h}{\partial x_j} + \frac{1}{2}(|\nabla h|^2 + h^2)\delta_{ij} \quad \text{for } i, j = 1, 2.$$

The equation $\operatorname{div} T_h = 0$ means that h is a critical point of the functional

$$v \in W^{1,2}(\Omega) \longmapsto \int_\Omega (|\nabla v|^2 + v^2)$$

with respect to inner variations of the domain, i.e. variations of the form $v_t(x) = v(x + tX(x))$ around $t = 0$ for every vector field $X \in C_c^\infty(\Omega; \mathbb{R}^2)$. When h is a smooth function in Ω, one finds that

$$\operatorname{div} T_h = (-\Delta h + h)\nabla h. \tag{10.10}$$

For h that merely belongs to $W^{1,2}_{\mathrm{loc}}(\Omega)$, T_h belongs to $L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^2 \times \mathbb{R}^2)$. In this case, $\operatorname{div} T_h$ is well-defined in the sense of distributions, but the distributional meaning of the right-hand side in (10.10) becomes unclear.

As an application of Theorem 1.1, we identify the absolutely continuous part of the limiting vorticity μ, in connection with (10.10), as follows:

Proposition 10.3. *Let $\Omega \subset \mathbb{R}^2$ be an open set and let μ be a locally finite Borel measure in Ω. If $h \in W^{1,2}_{\mathrm{loc}}(\Omega)$ satisfies* (10.8) *and* (10.9)*, then its precise representative $\widetilde{h}$ is locally Lipschitz-continuous in Ω and*

$$\mu_{\mathrm{a}} = h\chi_{\{\nabla h = 0\}}\,\mathrm{d}x.$$

When $\mu \in L^1_{\mathrm{loc}}(\Omega)$, one thus has

$$\mu = h\chi_{\{\nabla h = 0\}} \quad \text{almost everywhere in } \Omega. \tag{10.11}$$

Such a representation of μ was only known for $\mu \in L^p_{\mathrm{loc}}(\Omega)$ with $p > 1$; see [27, Theorem 13.1]. As a consequence of (10.11), h satisfies the Schrödinger equation

$$-\Delta h + Vh = 0 \quad \text{in the sense of distributions in } \Omega,$$

where the potential $V := \chi_{\{\nabla h \neq 0\}}$ takes its values in $\{0, 1\}$. A description of the singular part of the limiting vorticity when μ is merely a measure has been investigated in the paper [25] by the third author.

The Lipschitz continuity of h was established by Sandier and Serfaty in [27, pp. 278–279], including the case where μ is a measure. We focus on the new property concerning the identification of μ_{a} , but we also present a sketch of their regularity result that is needed in our proof.

Proof of Proposition 10.3. Using the vector field

$$X_h := \frac{1}{2}\biggl(\Bigl(\frac{\partial h}{\partial x_2}\Bigr)^2 - \Bigl(\frac{\partial h}{\partial x_1}\Bigr)^2, -2\frac{\partial h}{\partial x_1}\frac{\partial h}{\partial x_2}\biggr), \tag{10.12}$$

one can write the components $T_{h,1}$ and $T_{h,2}$ of T as

$$T_{h,1} = X_h + \frac{1}{2}(h^2, 0) \quad \text{and} \quad T_{h,2} = -X_h^{\perp} + \frac{1}{2}(0, h^2). \tag{10.13}$$

In addition,

$$|\nabla h|^4 = 4|X_h|^2. \tag{10.14}$$

By (10.13) and the assumption on $\operatorname{div} T_h$, we have that X_h satisfies the div-curl system

$$\begin{cases} \operatorname{div} X_h = -h\,\dfrac{\partial h}{\partial x_1}, \\ \operatorname{curl} X_h = -h\,\dfrac{\partial h}{\partial x_2}, \end{cases}$$

where $\operatorname{curl} X_h = -\operatorname{div} X_h^{\perp}$. Since $h \in W^{1,2}_{\mathrm{loc}}(\Omega)$, one can apply elliptic L^p estimates and a bootstrap argument based on (10.14) to deduce that $X_h \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^2)$ for every $1 < p < \infty$. Hence, $|\nabla h| \in L^{\infty}_{\mathrm{loc}}(\Omega)$ and then $\widetilde{h}$ is locally Lipschitz-continuous in Ω.

We now identify μ_{a} in terms of h. To this end, take open subsets $\omega \Subset O \Subset \Omega$ and a sequence of mollifiers $(\rho_n)_{n\in\mathbb{N}}$ in $C_c^{\infty}(\mathbb{R}^2)$ such that $\omega - \operatorname{supp} \rho_n \subset O$ for every $n \in \mathbb{N}$. Denoting $h_n = \rho_n * h$ and $\mu_n = \rho_n * \mu$, by linearity of the equation in (10.8) we then have

$$-\Delta h_n + h_n = \mu_n \quad \text{in } \omega.$$

In this case, (10.10) can be written as

$$\operatorname{div}(T_{h_n}) = \mu_n \nabla h_n \quad \text{in } \omega.$$

Since $(h_n)_{n\in\mathbb{N}}$ converges to h in $W^{1,2}(\omega)$, the sequence $(T_{h_n})_{n\in\mathbb{N}}$ converges to T_h in $L^1(\omega; \mathbb{R}^2 \times \mathbb{R}^2)$. We thus have

$$\mu_n \nabla h_n = \operatorname{div} T_{h_n} \overset{*}{\rightharpoonup} \operatorname{div} T_h = 0 \quad \text{in the sense of distributions in } \omega. \tag{10.15}$$

To give an alternative identification the limit of the sequence $(\mu_n \nabla h_n)_{n\in\mathbb{N}}$ in terms of h and μ, we write

$$\mu_n = \rho_n * (\mu_{\mathrm{a}}) + \rho_n * (\mu_{\mathrm{s}}).$$

The sequence $(\rho_n * (\mu_{\mathrm{a}}))_{n\in\mathbb{N}}$ converges to f in $L^1(\omega)$, where f is the density of μ_{a} with respect to the Lebesgue measure, i.e. $\mu_{\mathrm{a}} = f\,\mathrm{d}x$. From the Lipschitz-regularity part of the proof, the sequence $(\nabla h_n)_{n\in\mathbb{N}}$ is uniformly bounded in ω and converges to ∇h in $L^1(\omega)$. Passing to a subsequence if necessary, we may assume that $(\nabla h_n)_{n\in\mathbb{N}}$ converges almost everywhere to ∇h in ω. We then write

$$\rho_n * (\mu_{\mathrm{a}})\nabla h_n = (\rho_n * (\mu_{\mathrm{a}}) - f)\nabla h_n + f\nabla h_n. \tag{10.16}$$

Since $(\nabla h_n)_{n\in\mathbb{N}}$ is uniformly bounded, the Dominated convergence theorem thus implies that

$$\rho_n * (\mu_{\mathrm{a}})\,\nabla h_n \to f\,\nabla h \quad \text{in } L^1(\omega).$$

By uniform boundedness of the sequence $(\nabla h_n)_{n\in\mathbb{N}}$ in ω,

$$|\rho_n * (\mu_{\mathrm{s}})\,\nabla h_n| \le C_1\,\rho_n * |\mu_{\mathrm{s}}|. \tag{10.17}$$

Passing to a subsequence if necessary, we may assume that $(\rho_n*(\mu_{\mathrm{s}})\,\nabla h_n)_{n\in\mathbb{N}}$ converges weak* in $(C_0(\overline{\omega}))'$ to a finite measure γ. We thus have

$$\mu_n\nabla h_n \overset{*}{\rightharpoonup} f\,\nabla h\,\mathrm{d}x + \gamma \quad \text{weakly* as measures in } \omega.$$

By uniqueness of the limit in (10.15), we conclude that

$$f\,\nabla h\,\mathrm{d}x + \gamma = 0 \quad \text{in the sense of distributions in } \omega. \tag{10.18}$$

This identity also holds in the sense of measures in ω; see for instance [23, Proposition 6.12]. By (10.17), γ satisfies

$$|\gamma| \le C_1|\mu_{\mathrm{s}}| \quad \text{in } \omega,$$

and in particular is singular with respect to the Lebesgue measure. We deduce from (10.18) that $f\,\nabla h\,\mathrm{d}x = 0$ and $\gamma = 0$. Thus,

$$f = 0 \quad \text{almost everywhere in } \{\nabla h \neq 0\} \cap \omega.$$

On the other hand, by Theorem 1.1 and (10.8),

$$f = h \quad \text{almost everywhere in } \{\nabla h = 0\}.$$

Hence, $f = h\chi_{\{\nabla h=0\}}$ almost everywhere in ω. Since $\omega \Subset \Omega$ is an arbitrary open subset, the conclusion follows. □

We deduce from Proposition 10.3 that a limiting vorticity $\mu \in L^1_{\mathrm{loc}}(\Omega)$ is fully described by at most countably many open sets that confine clouds of vortices where the limiting induced magnetic field h is constant:

Corollary 10.4. *Let $\Omega \subset \mathbb{R}^2$ be an open set. If $h \in W^{1,2}_{\mathrm{loc}}(\Omega)$ satisfies* (10.8) *and* (10.9) *with $\mu \in L^1_{\mathrm{loc}}(\Omega)$, then*

$$\mu = \sum_{j\in J} m_j\chi_{U_j} \quad \textit{almost everywhere in } \Omega,$$

where $(m_j)_{j\in\mathbb{N}}$ is the collection of values in $\mathbb{R}\setminus\{0\}$ such that the level sets $\{\widetilde{h} = m_j\}$ have a non-empty interior U_j for every $j \in J$.

Before proving the corollary, we first recall the meaning used by Caffarelli and Salazar in [5] of a viscosity solution $u : \Omega \to \mathbb{R}$ of the equation

$$\Delta u = u \quad \text{in } \{\nabla u \neq 0\}. \tag{10.19}$$

Definition 10.1. A continuous function $u : \Omega \to \mathbb{R}$ satisfies (10.19) in the viscosity sense whenever both properties hold:

(i) for every $x \in \Omega$ and every polynomial P of degree at most 2 such that $P \geq u$ in a neighborhood of x with $P(x) = u(x)$ and $\nabla P(x) \neq 0$,

$$\Delta P(x) \geq u(x),$$

(ii) for every $x \in \Omega$ and every polynomial P of degree at most 2 such that $P \leq u$ in a neighborhood of x with $P(x) = u(x)$ and $\nabla P(x) \neq 0$,

$$\Delta P(x) \leq u(x).$$

We then apply the regularity theory developed in [5] to prove the decomposition of the limiting vorticity μ. To this end, we need the following lemma that clarifies the connection between (10.19) and the equation satisfied by the limiting induced magnetic field.

Lemma 10.5. *If $u \in (W^{1,1}_{\mathrm{loc}} \cap C^0)(\Omega)$ is such that*

$$-\Delta u + \chi_{\{\nabla u \neq 0\}} u = 0 \quad \textit{in the sense of distributions in } \Omega,$$

then $u : \Omega \to \mathbb{R}$ is a viscosity solution of (10.19) *in the sense of Definition 10.1.*

Proof of Lemma 10.5. Since $\Delta u \in L^\infty_{\mathrm{loc}}(\Omega)$, by elliptic regularity theory we have $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ for every $1 < p < \infty$ and then $u \in C^1(\Omega)$. To prove that (i) in Definition 10.1 is satisfied, take a polynomial P of degree at most 2 such that $P \geq u$ in a neighborhood of $x \in \Omega$, with $P(x) = u(x)$ and $\nabla P(x) \neq 0$. By differentiability of u, we then have

$$\nabla u(x) = \nabla P(x) \neq 0.$$

Since ∇u is continuous, there exists $r > 0$ such that $\nabla u \neq 0$ in $B_r(x)$. It then follows from the equation satisfied by u and elliptic regularity theory that u is smooth in $B_r(x)$ and

$$\Delta u = u \quad \text{in } B_r(x).$$

By local minimality of $P - u$ at x, we then have

$$\Delta P(x) \geq \Delta u(x) = u(x).$$

Hence, u satisfies the first condition in Definition 10.1. The second one is proved in a similar way. □

Proof of Corollary 10.4. The precise representative $\widetilde{h}$ satisfies $\widetilde{h} = h$ and $\nabla\widetilde{h} = \nabla h$ almost everywhere in Ω. Since $\mu \in L^1_{\mathrm{loc}}(\Omega)$, by Proposition 10.3 we then have

$$\mu = \widetilde{h}\chi_{\{\nabla\widetilde{h}=0\}} \quad \text{almost everywhere in } \Omega. \tag{10.20}$$

By the equation satisfied by h, we get

$$-\Delta\widetilde{h} + \chi_{\{\nabla\widetilde{h}\neq 0\}}\widetilde{h} = 0 \quad \text{in the sense of distributions in } \Omega.$$

From elliptic regularity theory, $\widetilde{h}$ thus belongs to $C^1(\Omega)$.

We now decompose the relatively closed set $G := \{\nabla\widetilde{h} = 0\}$ as a disjoint union:

$$G = \operatorname{int} G \cup \big(\partial G \cap \{\widetilde{h} = 0\}\big) \cup \big(\partial G \cap \{\widetilde{h} \neq 0\}\big), \tag{10.21}$$

where the boundary ∂ is computed with respect to the relative topology in Ω, and so yields a subset of Ω. The open set $\operatorname{int} G$ is a finite or countably infinite union of open connected components $(U_j)_{j\in J}$. Since $\nabla\widetilde{h} = 0$ in each U_j, then $\widetilde{h}$ is a constant $m_j \in \mathbb{R}$ in U_j.

By Lemma 10.5, $\widetilde{h} : \Omega \to \mathbb{R}$ is a viscosity solution of (10.19) in the sense of Definition 10.1. One then has by [5, Lemma 9], see also [6], that $\widetilde{h}$ is locally a $C^{1,1}$ function in the open set $\{\widetilde{h} > 0\}$ where $\widetilde{h}$ is positive. As the boundary ∂ is computed in the relative topology in Ω and $\nabla\widetilde{h}$ is continuous, we have $\partial G = \partial\{\nabla\widetilde{h} \neq 0\}$. We can then apply [5, Corollary 14] to deduce that the free boundary

$$B_+ := \partial G \cap \{\widetilde{h} > 0\}$$

has finite Hausdorff measure $\mathcal{H}^{N-1}$. In particular, B_+ is negligible with respect to the Lebesgue measure. The same argument applied to $-\widetilde{h}$ in $\{\widetilde{h} < 0\}$ implies that

$$B_- := \partial G \cap \{\widetilde{h} < 0\}$$

is also negligible and then so is $B_+ \cup B_-$. We now deduce from (10.20) and (10.21) that

$$\mu = \widetilde{h}\chi_{\operatorname{int} G} = \sum_{j\in J} m_j \chi_{U_j} \quad \text{almost everywhere in } \Omega. \qquad \square$$

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