DIMENSION REDUCTION FOR THIN FILMS WITH TRANSVERSALLY VARYING PRESTRAIN: THE OSCILLATORY AND THE NON-OSCILLATORY CASE

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Abstract. We study the non-Euclidean (incompatible) elastic energy functionals in the description of prestressed thin films, at their singular limits (Γ-limits) as $h \to 0$ in the film's thickness h. Firstly, we extend the prior results [12, 38, 39] to arbitrary incompatibility metrics that depend on both the midplate and the transversal variables (the "non-oscillatory" case). Secondly, we analyze a more general class of incompatibilities, where the transversal dependence of the lower order terms is not necessarily linear (the "oscillatory" case), extending the results of [3, 46] to arbitrary metrics and higher order scalings. We exhibit connections between the two cases via projections of appropriate curvature forms on the polynomial tensor spaces. We also show the effective energy quantisation in terms of scalings as a power of h and discuss the scaling regimes h^2 (Kirchhoff), h^4 (von Kármán), h^6 in the general case, and all possible (even power) regimes for conformal metrics. Thirdly, we prove the coercivity inequalities for the singular limits at h^2- and h^4- scaling orders, while disproving the full coercivity of the classical von Kármán energy functional at scaling h^4.

1. Introduction

The purpose of this paper is to further develop the analytical tools for understanding the mechanisms through which the local properties of a material lead to changes in its mechanical responses.

Motivated by the idea of imposing and controlling the *prestrain* (or *"misfit"*) field in order to cause the plate to achieve a desired shape, our work is concerned with the analysis of thin elastic films exhibiting residual stress at free equilibria. Examples of this type of structures and their actuations include: plastically strained sheets, swelling or shrinking gels, growing tissues such as leaves, flowers or marine invertebrates, nanotubes, atomically thin graphene layers, etc. In the same vein, advancements in the construction of novel materials in thin film format require an analytical insight how the parameters affect the product and how to mimic the architectures found in nature.

In this paper, we will be concerned with the forward problem associated to the mentioned structures, based on the minimisation of the elastic energy with incorporated inelastic effects.

1.1. **The set-up of the problem.**

Let $\omega \subset \mathbb{R}^2$ be an open, bounded, connected set with Lipschitz boundary. We consider a family of thin hyperelastic sheets occupying the reference domains:

$$\Omega^h = \omega \times \Big(-\frac{h}{2}, \frac{h}{2}\Big) \subset \mathbb{R}^3, \qquad 0 < h \ll 1.$$

A typical point in Ω^h is denoted by $x = (x_1, x_2, x_3) = (x', x_3)$. For $h = 1$ we use the notation $\Omega = \Omega^1$ and view Ω as the referential rescaling of each Ω^h via: $\Omega^h \ni (x', x_3) \mapsto (x', x_3/h) \in \Omega$.

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In this paper we study the singular limit behaviour, as $h \to 0$, of the energy functionals:

$$\mathcal{E}^h(u^h) = \frac{1}{h}\int_{\Omega^h} W\big(\nabla u^h(x) G^h(x)^{-1/2}\big)\,\mathrm{d}x, \tag{1.1}$$

defined on vector fields $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$, that are interpreted as deformations of Ω^h. The films are characterized by the smooth incompatibility (Riemann metric) tensors $G^h \in \mathcal{C}^\infty(\bar\Omega^h, \mathbb{R}^{3\times 3}_{\mathrm{sym,pos}})$, satisfying the following structure assumption, referred to as the *"oscillatory"* assumption:

$$\text{(O)}\quad \left[\begin{array}{c} \textsc{Oscillatory case:} \\ G^h(x) = \mathcal{G}^h(x', \frac{x_3}{h}) \quad \text{for all } \ x = (x', x_3) \in \Omega^h, \\ \mathcal{G}^h(x', t) = \bar{\mathcal{G}}(x') + h\mathcal{G}_1(x', t) + \frac{h^2}{2}\mathcal{G}_2(x', t) + o(h^2) \in \mathcal{C}^\infty(\bar\Omega, \mathbb{R}^{3\times 3}_{\mathrm{sym,pos}}), \\ \text{where } \bar{\mathcal{G}} \in \mathcal{C}^\infty(\bar\omega, \mathbb{R}^{3\times 3}_{\mathrm{sym,pos}}),\ \mathcal{G}_1, \mathcal{G}_2 \in \mathcal{C}^\infty(\bar\Omega, \mathbb{R}^{3\times 3}_{\mathrm{sym}}) \text{ and } \int_{-1/2}^{1/2} \mathcal{G}_1(x', t)\,\mathrm{d}t = 0 \text{ for all } x' \in \bar\omega. \end{array}\right.$$

The requirement of $\bar{\mathcal{G}}$ being independent of the transversal variable $t \in (-1/2, 1/2)$ is essential for the energy scaling order: $\inf \mathcal{E}^h \leq Ch^2$. The zero mean requirement on $\mathcal{G}_1$ can be relaxed to requesting that $\int_{-1/2}^{1/2} \mathcal{G}_1(x', t)_{2\times 2}\,\mathrm{d}t$ be a linear strain with respect to the leading order midplate metric $(\bar{\mathcal{G}}_1)_{2\times 2}$ (in case $(\bar{\mathcal{G}})_{2\times 2} = Id_2$ the sufficient and necessary condition for this to happen is $\mathrm{curl}^{\mathsf{T}}\mathrm{curl} \int_{-1/2}^{1/2} \mathcal{G}_1(x', t)_{2\times 2}\,\mathrm{d}t = 0$; this case has been studied in [3] where $\bar{\mathcal{G}} = Id_3$), and we also conjecture that it can be removed altogether, which will be the content of future work. In the present work, we assume the said condition in light of the special case (NO) below.

We refer to the family of films Ω^h prestrained by metrics in (O):

$$G^h(x) = \bar{\mathcal{G}}(x') + h\mathcal{G}_1(x', \frac{x_3}{h}) + \frac{h^2}{2}\mathcal{G}_2(x', \frac{x_3}{h}) + o(h^2) \quad \text{for all } \ x = (x', x_3) \in \Omega^h, \tag{1.2}$$

as "oscillatory", and note that this set-up includes a subcase of a single metric $G^h = G$, upon taking:

$$\mathcal{G}_1(x', t) = t\bar{\mathcal{G}}_1(x'), \qquad \mathcal{G}_2(x', t) = t^2\bar{\mathcal{G}}_2(x').$$

We refer to this special case as *"non-oscillatory"*; formula (1.2) becomes then Taylor's expansion in:

$$\text{(NO)}\quad \left[\begin{array}{c} \textsc{Non-oscillatory case:} \\ G^h = G_{|\bar\Omega^h} \quad \text{for some } \ G \in \mathcal{C}^\infty(\bar\Omega, \mathbb{R}^{3\times 3}_{\mathrm{sym,pos}}), \\ G^h(x) = \bar{\mathcal{G}}(x') + x_3\partial_3 G(x', 0) + \frac{x_3^2}{2}\partial_{33}G(x', 0) + o(x_3^2) \quad \text{for all } \ x = (x', x_3) \in \Omega^h. \end{array}\right.$$

Mechanically, the assumption (NO) describes thin sheets that have been cut out of a single specimen block Ω, prestrained according to a fixed (though arbitrary) tensor G. As we shall see, the general case (O) can be reduced to (NO) via the following *effective metric*:

$$\text{(EF)}\quad \left[\begin{array}{l} \qquad\qquad\qquad \textsc{Effective non-oscillatory case:} \\ \bar G^h(x) = \bar G(x) = \bar{\mathcal{G}}(x') + x_3\bar{\mathcal{G}}_1(x') + \frac{x_3^2}{2}\bar{\mathcal{G}}_2(x') \quad \text{for all } \ x = (x', x_3) \in \Omega^h, \\ \text{where: } \ \bar{\mathcal{G}}_1(x')_{2\times 2} = 12\int_{-1/2}^{1/2} t\mathcal{G}_1(x', t)_{2\times 2}\,\mathrm{d}t, \ \ \bar{\mathcal{G}}_1(x')e_3 = -60\int_{-1/2}^{1/2}(2t^3 - \frac{1}{2}t)\mathcal{G}_1(x', t)e_3\,\mathrm{d}t, \\ \text{and: } \quad \bar{\mathcal{G}}_2(x')_{2\times 2} = 30\int_{-1/2}^{1/2}(6t^2 - \frac{1}{2})\mathcal{G}_2(x', t)_{2\times 2}\,\mathrm{d}t. \end{array}\right.$$

In (1.1), the homogeneous elastic energy density $W : \mathbb{R}^{3\times 3} \to [0,\infty]$ is a Borel measurable function, assumed to satisfy the following properties:

(i) $W(RF) = W(F)$ for all $R \in SO(3)$ and $F \in \mathbb{R}^{3\times 3}$,
(ii) $W(F) = 0$ for all $F \in SO(3)$,
(iii) $W(F) \geq C\,\mathrm{dist}^2\big(F, SO(3)\big)$ for all $F \in \mathbb{R}^{3\times 3}$, with some uniform constant $C > 0$,
(iv) there exists a neighbourhood $\mathcal{U}$ of $SO(3)$ such that W is finite and $\mathcal{C}^2$ regular on $\mathcal{U}$.

We will be concerned with the regimes of curvatures of G^h in (O) which yield the incompatibility rate, quantified by $\inf \mathcal{E}^h$, of order higher than h^2 in the plate's thickness h. With respect to the prior works in this context, the present paper proposes the following three new contributions.

1.2. **New results of this work: Singular energies in the non-oscillatory case.**

1.2.1. *Kirchhoff scaling regime.* We begin by deriving (in section 2), the Γ-limit of the rescaled energies $\frac{1}{h^2}\mathcal{E}^h$. In the setting of (NO), we obtain:

$$\begin{aligned}\mathcal{I}_2(y) &= \frac{1}{2}\big\|Tensor_2\big\|^2_{\mathcal{Q}_2} = \frac{1}{2}\big\|x_3\big((\nabla y)^\intercal \nabla \vec{b}\big)_{\mathrm{sym}} - \frac{1}{2}x_3\partial_3 G(x',0)_{2\times 2}\big\|^2_{\mathcal{Q}_2} \\ &= \frac{1}{24}\big\|\big((\nabla y)^\intercal \nabla \vec{b}\big)_{\mathrm{sym}} - \frac{1}{2}\partial_3 G(x',0)_{2\times 2}\big\|^2_{\mathcal{Q}_2}.\end{aligned}$$

We now explain the notation above. Firstly, $\|\cdot\|_{\mathcal{Q}_2}$ is a weighted L^2 norm in (2.8) on the space $\mathbb{E}$ of $\mathbb{R}^{2\times 2}_{\mathrm{sym}}$-valued tensor fields on Ω. The weights in (2.6) are determined by the elastic energy W together with the leading order metric coefficient $\bar{\mathcal{G}}$. The functional $\mathcal{I}_2$ is defined on the set of isometric immersions $\mathcal{Y}_{\bar{\mathcal{G}}_{2\times 2}} = \{y \in W^{2,2}(\omega,\mathbb{R}^3);\ (\nabla y)^\intercal \nabla y = \bar{\mathcal{G}}_{2\times 2}\}$; each such immersion generates the corresponding Cosserat vector $\vec{b}$, uniquely given by requesting: $\big[\partial_1 y,\ \partial_2 y,\ \vec{b}\big] \in SO(3)\bar{\mathcal{G}}^{1/2}$ on ω. The family of energies obtained in this manner is parametrised by all matrix fields $S \in \mathcal{C}^\infty(\bar{\omega},\mathbb{R}^{2\times 2}_{\mathrm{sym}})$ and $T \in \mathcal{C}^\infty(\bar{\omega},\mathbb{R}^{3\times 3}_{\mathrm{sym,pos}})$, namely: $\mathcal{I}_2^{T,S}(y) = \frac{1}{24}\|((\nabla y)^\intercal \nabla \vec{b}(y))_{\mathrm{sym}} - S\|^2_{\mathcal{Q}_2}$ defined on the set $\mathcal{Y}_{T_{2\times 2}}$.

The energy $\mathcal{I}_2$ measures the bending quantity $Tensor_2$ which is linear in x_3, resulting in its reduction to the single nonlinear bending term, that equals the difference of the curvature form $\big((\nabla y)^\intercal \nabla \vec{b}\big)_{\mathrm{sym}}$ from the preferred curvature $\frac{1}{2}\partial_3 G(x',0)_{2\times 2}$.The same energy has been derived in [12, 38] under the assumption that G is independent of x_3 and in [30] for a general manifold (M^n, g) with any codimension submanifold $(N^k, g_{|N})$ replacing the midplate $\omega \times \{0\}$. Since our derivation of $\mathcal{I}_2$ is a particular case of the result in case (O), we still state it here for completeness.

In section 3 we identify the necessary and sufficient conditions for $\min \mathcal{I}_2 = 0$ (when ω is simply connected), in terms of the vanishing of the Riemann curvatures $R_{1212}, R_{1213}, R_{1223}$ of G at $x_3 = 0$. In this case, it follows that $\inf \mathcal{E}^h \leq Ch^4$. For the discussed case (NO), the recent work [41] generalized the same statements for arbitrary dimension and codimension.

1.2.2. *Von Kármán scaling regime.* In section 6 we derive the Γ-limit of $\frac{1}{h^4}\mathcal{E}^h$, which is given by:

$$\mathcal{I}_4(V,\mathbb{S}) = \frac{1}{2}\big\|Tensor_4\big\|^2_{\mathcal{Q}_2},$$

defined on the spaces of: finite strains $\mathscr{S}_{y_0} = \{\mathbb{S} = \lim_{n\to\infty, L^2}\big((\nabla y_0)^\intercal \nabla w_n\big)_{\mathrm{sym}};\ w_n \in W^{1,2}(\omega,\mathbb{R}^3)\}$ and first order infinitesimal isometries $\mathscr{V}_{y_0} = \{V \in W^{2,2}(\omega,\mathbb{R}^3);\ \big((\nabla y_0)^\intercal \nabla V\big)_{\mathrm{sym}} = 0\}$ on the deformed midplate $y_0(\omega) \subset \mathbb{R}^3$. Here, y_0 is the unique smooth isometric immersion of $\bar{\mathcal{G}}_{2\times 2}$ for which $\mathcal{I}_2(y_0) = 0$; recall that it generates the corresponding Cosserat's vector $\vec{b}_0$.

The expression in $Tensor_4$ is quite complicated but it has the structure of a quadratic polynomial in x_3. A key tool for identifying this expression, also in the general case (O), involves the subspaces $\{\mathbb{E}_n \subset \mathbb{E}\}_{n\geq 1}$ in (2.9), consisting of the tensorial polynomials in x_3 of order n. The bases of $\{\mathbb{E}_n\}$ are then naturally given in terms of the Legendre polynomials $\{p_n\}_{n\geq 0}$ on $(-\frac{1}{2}, \frac{1}{2})$. Since $Tensor_4 \in \mathbb{E}_2$, we write the decomposition:

$$Tensor_4 = p_0(x_3)Stretching_4 + p_1(x_3)Bending_4 + p_2(x_3)Curvature_4,$$

which, as shown in section 7, results in:

$$\begin{split}\mathcal{I}_4(V,\mathbb{S}) = {} & \frac{1}{2}\Big(\big\|Stretching_4\big\|^2_{\mathcal{Q}_2} + \big\|Bending_4\big\|^2_{\mathcal{Q}_2} + \big\|Curvature_4\big\|^2_{\mathcal{Q}_2}\Big) \\ = {} & \frac{1}{2}\big\|\mathbb{S} + \frac{1}{2}(\nabla V)^\mathsf{T}\nabla V + \frac{1}{24}(\nabla\vec{b}_0)^\mathsf{T}\nabla\vec{b}_0 - \frac{1}{48}\partial_{33}G(x',0)_{2\times 2}\big\|^2_{\mathcal{Q}_2} \\ & + \frac{1}{24}\big\|\big[\langle\nabla_i\nabla_j V, \vec{b}_0\rangle\big]_{i,j=1,2}\big\|^2_{\mathcal{Q}_2} + \frac{1}{1440}\big\|\big[R_{i3j3}(x',0)\big]_{i,j=1,2}\big\|^2_{\mathcal{Q}_2}.\end{split}$$

Above, ∇_i denotes the covariant differentiation with respect to the metric $\bar{\mathcal{G}}$ and R_{i3j3} are the potentially non-zero curvatures of G on ω at $x_3 = 0$.

The family of energies obtained in this manner is parametrised by all quadruples: vector fields $y_0, \vec{b}_0 \in \mathcal{C}^\infty(\bar{\omega}, \mathbb{R}^3)$ satisfying $\det\big[\partial_1 y_0,\ \partial_2 y_0,\ \vec{b}_0\big] > 0$, matrix fields $T \in \mathcal{C}^\infty(\bar{\omega}, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$, and numbers $r \in \mathbb{R}$ in the range (the left parentheses in the last interval below may be open or closed):

$$r \in \frac{2}{5}\Big\{\|T - \big((\nabla y_0)^\mathsf{T}\nabla\vec{d}_0\big)_{\mathrm{sym}}\|^2_{\mathcal{Q}_2};\ \vec{d}_0 \in \mathcal{C}^\infty(\bar{\omega}, \mathbb{R}^{2\times 2}_{\mathrm{sym}})\Big\} = \Big[\frac{2}{5}\mathrm{dist}^2_{\mathcal{Q}_2}(T, \mathscr{S}_{y_0}), +\infty\Big). \tag{1.3}$$

The functionals are then: $\mathcal{I}_4^{y_0,\vec{b}_0,T,r}(V,\mathbb{S}) = \frac{1}{2}\|\mathbb{S} + \frac{1}{2}(\nabla V)^\mathsf{T}\nabla V - T\|^2_{\mathcal{Q}_2} + \frac{1}{24}\|\big[\langle\nabla_i\nabla_j V, \vec{b}_0\rangle\big]_{ij}\|^2_{\mathcal{Q}_2} + r$, defined on the linear space $\mathcal{V}_{y_0} \times \mathcal{S}_{y_0}$. Particular cases where the range of r may be identified are:

(i) $y_0 = id_2$. Then $\mathscr{S}_{y_0} = \{\mathbb{S} \in L^2(\omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}});\ \mathrm{curl}^\mathsf{T}\mathrm{curl}\,\mathbb{S} = 0\}$ and the range of r is defined by the appropriate norm of: $\mathrm{curl}^\mathsf{T}\mathrm{curl}\,T$.

(ii) Gauss curvature $\kappa((\nabla y_0)^\mathsf{T}\nabla y_0) > 0$ in $\bar{\omega}$. Then in [36] it is shown that $\mathscr{S}_{y_0} = L^2(\omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$. The range of possible r (for any T) is then: $[0, +\infty)$.

When $y_0 = id_2$ (which occurs automatically when $\bar{\mathcal{G}} = Id_3$), then $\vec{b}_0 = e_3$ and the first two terms in $\mathcal{I}_4$ reduce to the stretching and the linear bending contents of the classical von Kármán energy. The third term is purely metric-related and measures the non-immersability of G relative to the present quartic scaling. These findings generalize the results of [12] valid for x_3-independent G in (NO). We also point out that, following the same general principle in the h^2- scaling regime, one may readily decompose:

$$Tensor_2 = p_0(x_3)Stretching_2 + p_1(x_3)Bending_2;$$

since $Tensor_2$ is already a multiple of x_3, then $Stretching_2 = 0$ in the ultimate form of $\mathcal{I}_2$.

It is not hard to deduce (see section 8) that the necessary and sufficient conditions for having $\min \mathcal{I}_4 = 0$ are precisely that $R_{ijkl} \equiv 0$ on $\omega \times \{0\}$, for all $i,j,k,l = 1\ldots 3$. In that case, we show in section 10 that $\inf \mathcal{E}^h \leq Ch^6$. We also identify the curvature term that will be present in the corresponding decomposition of $Tensor_6$. It turns out to be $\big[\partial_3 R_{i3j3}(x',0)\big]_{i,j=1,2} = \big[\nabla_3 R_{i3j3}(x',0)\big]_{i,j=1,2}$ which in view of the second Bianchi identity carries the only potentially non-vanishing components of the covariant gradient $\nabla Riem(x',0)$. This finding is consistent with results of section 9, analyzing the conformal non-oscillatory metric $G = e^{2\phi(x_3)}Id_3$. Namely, we show that different orders of vanishing of ϕ at $x_3 = 0$ correspond to different even orders of scaling of $\mathcal{E}^h$ as $h \to 0$:

$$\phi^{(k)}(0) = 0 \quad \text{for } k = 1 \dots n-1 \quad \text{ and } \quad \phi^{(n)}(0) \neq 0 \quad \Leftrightarrow \quad ch^{2n} \leq \inf \mathcal{E}^h \leq Ch^{2n}$$

with the lower bound: $\inf \mathcal{E}^h \geq c_n h^n \big\| \big[\partial_3^{(n-2)} R_{i3j3}(x', 0)\big]_{i,j=1,2}\big\|^2_{\mathcal{Q}_2}$.

1.3. **New results of this work: Singular energies in the oscillatory case.** We show that the analysis in the general case (O) may follow a similar procedure, where we first project the limiting quantity $Tensor^O$ on an appropriate polynomial space and then decompose the projection along the respective Legendre basis. For the Γ-limit of $\frac{1}{h^2}\mathcal{E}^h$ in section 2, we show that:

$$\begin{aligned} Tensor^O_2 = x_3\big((\nabla y)^{\mathsf{T}}\nabla \vec{b}\big)_{\mathrm{sym}} - \frac{1}{2}(\mathcal{G}_1)_{2\times 2} = p_0(x_3) Stretching^O_2 + p_1(x_3) Bending^O_2 + Excess_2, \\ \text{with } \ Excess_2 = Tensor^O_2 - \mathbb{P}_1(Tensor^O_2). \end{aligned}$$

Consequently:

$$\begin{aligned} \mathcal{I}^O_2(y) &= \frac{1}{2}\Big(\big\|Stretching^O_2\big\|^2_{\mathcal{Q}_2} + \big\|Bending^O_2\big\|^2_{\mathcal{Q}_2} + \big\|Excess_2\big\|^2_{\mathcal{Q}_2}\Big) \\ &= \frac{1}{24}\Big\|\big((\nabla y)^{\mathsf{T}}\nabla \vec{b}\big)_{\mathrm{sym}} - \frac{1}{2}(\bar{\mathcal{G}}_1)_{2\times 2}\Big\|^2_{\mathcal{Q}_2} + \frac{1}{8}\mathrm{dist}^2_{\mathcal{Q}_2}\big((\mathcal{G}_1)_{2\times 2}, \mathbb{E}_1\big), \end{aligned}$$

where again $Stretching^O_2 = 0$ in view of the assumed $\int_{-1/2}^{1/2} \mathcal{G}_1 \, \mathrm{d}x_3 = 0$. For the same reason:

$$Excess_2 = -\frac{1}{2}\Big((\mathcal{G}_1)_{2\times 2} - \mathbb{P}_1\big((\mathcal{G}_1)_{2\times 2}\big)\Big) = -\frac{1}{2}\Big((\mathcal{G}_1)_{2\times 2} - 12\int_{-1/2}^{1/2} x_3 (\mathcal{G}_1)_{2\times 2}\, \mathrm{d}x_3\Big)$$

and also: $\mathbb{P}_1\big((\mathcal{G}_1)_{2\times 2}\big) = x_3(\bar{\mathcal{G}}_1)_{2\times 2}$ with $(\bar{\mathcal{G}}_1)_{2\times 2}$ defined in (EF). The limiting oscillatory energy $\mathcal{I}^O_2$ consists thus of the bending term that coincides with $\mathcal{I}_2$ for the effective metric $\bar{G}$, plus the purely metric-related excess term. A special case of $\mathcal{I}^O_2$ when $\bar{\mathcal{G}} = Id_3$ and without analyzing the excess term, has been derived in [3], following the case with $G^h = Id_3 + h\mathcal{G}_1(\frac{x_3}{h})$ considered in [46]. An excess term has also been present in the work [27] on rods with misfit; we do not attempt to compare our results with studies of dimension reduction for rods; the literature there is abundant.

It is easy to observe that: $\min \mathcal{I}^O_2 = 0$ if and only if $(\mathcal{G}_1)_{2\times 2} = x_3(\bar{\mathcal{G}}_1)_{2\times 2}$ on $\omega \times \{0\}$. We show in section 5 that this automatically implies: $\inf \mathcal{E}^h \leq Ch^4$. The Γ-limit of $\frac{1}{h^4}\mathcal{E}^h$ is further derived in sections 6 and 7, by considering the decomposition:

$$\begin{aligned} Tensor^O_4 = p_0(x_3) Stretching^O_4 + p_1(x_3) Bending^O_4 + p_2(x_3) Curvature^O_4 + Excess_4, \\ \text{with } \ Excess_4 = Tensor^O_4 - \mathbb{P}_2(Tensor^O_4). \end{aligned}$$

It follows that:

$$\begin{aligned} \mathcal{I}^O_4(V, \mathbb{S}) &= \frac{1}{2}\Big(\big\|Stretching^O_4\big\|^2_{\mathcal{Q}_2} + \big\|Bending^O_4\big\|^2_{\mathcal{Q}_2} + \big\|Curvature^O_4\big\|^2_{\mathcal{Q}_2} + \big\|Excess_4\big\|^2_{\mathcal{Q}_2}\Big) \\ &= \frac{1}{2}\Big\|\mathbb{S} + \frac{1}{2}(\nabla V)^{\mathsf{T}}\nabla V + B_0\Big\|^2_{\mathcal{Q}_2} + \frac{1}{24}\Big\|\big[\langle \nabla_i \nabla_j V, \vec{b}_0\rangle\big]_{i,j=1,2} + B_1\Big\|^2_{\mathcal{Q}_2} \\ &\quad + \frac{1}{1440}\Big\|\big[R_{i3j3}(x', 0)\big]_{i,j=1,2}\Big\|^2_{\mathcal{Q}_2} \\ &\quad + \frac{1}{2}\mathrm{dist}^2_{\mathcal{Q}_2}\Big(\frac{1}{4}(\mathcal{G}_2)_{2\times 2} - \int_0^{x_3} \big[\nabla_i\big((\mathcal{G}_1 e_3) - \frac{1}{2}(\mathcal{G}_1)_{33} e_3\big)\big]_{i,j=1,2,\mathrm{sym}} \, \mathrm{d}t, \mathbb{E}_2\Big), \end{aligned}$$

where $R_{1313}, R_{1323}, R_{2323}$ are the respective Riemann curvatures of the effective metric $\bar{G}$ in (EF) at $x_3 = 0$. The corrections B_0 and B_1 coincide with the same expressions written for $\bar{G}$ under two extra constraints (see Theorem 7.5), that can be seen as the h^4-order counterparts of the h^2-order condition $\int_{-1/2}^{1/2} \mathcal{G}_1 \,\mathrm{d}x_3 = 0$ assumed throughout. In case these conditions are valid, the functional $\mathcal{I}_4^O$ is the sum of the effective stretching, bending and curvature in $\mathcal{I}_4$ for $\bar{G}$, plus the additional purely metric-related excess term.

1.4. **New results of this work: coercivity of $\mathcal{I}_2$ and $\mathcal{I}_4$.** We additionally analyze the derived limiting functionals by identifying their kernels, when nonempty. In section 4 we show that the kernel of $\mathcal{I}_2$ consists of the rigid motions of a single smooth deformation y_0 that solves:

$$(\nabla y_0)^{\mathsf{T}}\nabla y_0 = \bar{\mathcal{G}}_{2\times 2}, \qquad \big((\nabla y_0)^{\mathsf{T}}\nabla \vec{b}_0\big)_{\mathrm{sym}} = \frac{1}{2}\partial_3 G(x',0)_{2\times 2}.$$

Further, $\mathcal{I}_2(y)$ bounds from above the squared distance of an arbitrary $W^{2,2}$ isometric immersion y of the midplate metric $\bar{\mathcal{G}}_{2\times 2}$, from the indicated kernel of $\mathcal{I}_2$.

In section 8 we consider the case of $\mathcal{I}_4$. We first identify (see Theorem 8.2) the zero-energy displacement-strain couples $(V,\mathbb{S})$. In particular, we show that the minimizing displacements are exactly the linearised rigid motions of the referential y_0. We then prove that the bending term in $\mathcal{I}_4$, which is solely a function of V, bounds from above the squared distance of an arbitrary $W^{2,2}$ displacement obeying $\big((\nabla y_0)^{\mathsf{T}}\nabla V\big)_{\mathrm{sym}} = 0$, from the indicated minimizing set in V. On the other hand, the full coercivity result involving minimization in both V and $\mathbb{S}$ is false. In Remark 8.4 we exhibit an example in the setting of the classical von Kármán functional, where $\mathcal{I}_4(V_n,\mathbb{S}_n)\to 0$ as $n\to\infty$, but the distance of $(V_n,\mathbb{S}_n)$ from the kernel of $\mathcal{I}_4$ remains uniformly bounded away from 0. We note that this lack of coercivity is not prevented by the fact that the kernel is finite dimensional.

1.5. **Other related works.** Recently, there has been a sustained interest in studying shape formation driven by internal prestrain, through the experimental, modelling via formal methods, numerics, and analytical arguments [15, 22, 26, 29]. General results have been derived in the abstract setting of Riemannian manifolds: in [29, 30] Γ-convergence statements were proved for any dimension ambient manifold and codimension midplate, in the scaling regimes $\mathcal{O}(h^2)$ and $\mathcal{O}(1)$, respectively. In a work parallel to ours [41], the authors analyze scaling orders $o(h^2)$, $\mathcal{O}(h^4)$ and $o(h^4)$, extending condition (3.6), Lemma 5.1 in (NO) case, and condition (8.2) to arbitrary manifolds. Although they do not identify the Γ-limits of the rescaled energies $\mathcal{E}^h$, they are able to provide the revealing lower bounds for $\inf\mathcal{E}^h$ in terms of the appropriate curvatures.

Higher energy scalings $\inf\mathcal{E}^h \sim h^\beta$ than the ones analyzed in the present paper may result from the interaction of the metric with boundary conditions or external forces, leading to the "wrinkling-like" effects. Indeed, our setting pertains to the "no wrinkling" regime where $\beta\geq 2$ and the prestrain metric admits a $W^{2,2}$ isometric immersion. While the systematic description of the singular limits at scalings $\beta<2$ is not yet available, the following studies are examples of the variety of emerging patterns. In [9, 10, 23], energies leading to the buckling- or compression- driven *blistering* in a thin film breaking away from its substrate and under clamped boundary conditions, are discussed ($\beta=1$). Paper [6] displays dependence of the energy minimization on boundary conditions and classes of admissible deformations, while [7] discusses coarsening of *fold singularities* in hanging drapes, where the energy identifies the number of generations of coarsening. In [48], *wrinkling* patterns are obtained, reproducing the experimental observations when placing a thin cup on liquid bath ($\beta=1$), while [28] analyses wrinkling in the center of a stretched and twisted ribbon ($\beta=4/3$). In [11, 49], energy levels of the *origami patterns* in paper crumpling are studied ($\beta=5/3$). See also

[42, 43, 44] for an analysis of the *conical singularities* ($\mathcal{E}^h \sim h^2 \log(1/h)$). We remark that the mentioned papers do not address the dimension reduction, but rather analyze the chosen actual configuration of the prestrained sheet. Closely related is also the literature on shape selection in non-Euclidean plates, exhibiting hierarchical *buckling patterns* in zero-strain plates ($\beta = 2$), where the complex morphology is due to the non-smooth energy minimization [16, 17, 18].

Alternative geometrically nonlinear thin plate theories can be used to analyze the self-similar structures with metric asymptotically flat at infinity [5], a disk with edge-localized growth [15], the shape of a long leaf [40], or torn plastic sheets [47]. In [13, 14] a variant of the Föppl-von Kármán equilibrium equations has been formally derived from finite incompressible elasticity, via the multiplicative decomposition of deformation gradient [45] similar to ours. See also models related to cockling of paper, grass blades, sympatelous flowers and movement of euglenids [4, 8, 14]. The forward and inverse problems in the study of self-folding in thin sheets of patterned hydrogel bilayers are discussed in the forthcoming paper [2].

On the frontiers of experimental modeling of shape formation, we mention the *halftone gel lithography* method for polymeric materials that can swell by imbibing fluids [24, 25, 31, 50]. By blocking the ability of portions of plate to swell or causing them to swell inhomogeneously, it is possible to have the plate assume a variety of deformed shapes. Even more sophisticated techniques of *biomimetic 4d printing* allow for engineering of the 3d shape-morphing systems that mimic nastic plant motions where organs such as tendrils, leaves and flowers respond to the environmental stimuli [19]. Optimal control in such systems has been studied in [22], see also [1].

In [34, 35, 37], derivations similar to the results of the present paper were carried out under a different assumption on the asymptotic behavior of the prestrain, which also implied energy scaling h^β in non-even regimes of $\beta > 2$. In [34] it was shown that the resulting Euler-Lagrange equations of the residual energy are identical to those describing the effects of growth in elastic plates [40]. In [37], a model with a Monge-Ampère constraint was derived and analysed from various aspects.

1.6. **Notation.** Given a matrix $F \in \mathbb{R}^{n\times n}$, we denote its transpose by F^{T} and its symmetric part by $F_{\rm sym} = \frac{1}{2}(F + F^{\mathsf{T}})$. The space of symmetric $n \times n$ matrices is denoted by $\mathbb{R}^{n\times n}_{\rm sym}$, whereas $\mathbb{R}^{n\times n}_{\rm sym,pos}$ stands for the space of symmetric, positive definite $n \times n$ matrices. By $SO(n) = \{R \in \mathbb{R}^{n\times n};\ R^{\mathsf{T}} = R^{-1} \text{ and } \det R = 1\}$ we mean the group of special rotations; its tangent space at Id_n consists of skew-symmetric matrices: $T_{Id_n}SO(n) = so(n) = \{F \in \mathbb{R}^{n\times n};\ F_{\rm sym} = 0\}$. We use the matrix norm $|F| = (\mathrm{trace}(F^{\mathsf{T}}F))^{1/2}$, which is induced by the inner product $\langle F_1 : F_2\rangle = \mathrm{trace}(F_1^{\mathsf{T}}F_2)$. The 2×2 principal minor of $F \in \mathbb{R}^{3\times 3}$ is denoted by $F_{2\times 2}$. Conversely, for a given $F_{2\times 2} \in \mathbb{R}^{2\times 2}$, the 3×3 matrix with principal minor equal $F_{2\times 2}$ and all other entries equal to 0, is denoted by $F^*_{2\times 2}$. Unless specified otherwise, all limits are taken as the thickness parameter h vanishes: $h \to 0$. By C we denote any universal positive constant, independent of h. We use the Einstein summation convention over repeated lower and upper indices running from 1 to 3.

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2. Compactness and Γ-limit under Ch^2 energy bound

Define the matrix fields $\bar A \in \mathcal{C}^\infty(\bar\omega, \mathbb{R}^{3\times 3}_{\mathrm{sym,pos}})$ and $A^h, A_1, A_2 \in \mathcal{C}^\infty(\bar\Omega, \mathbb{R}^{3\times 3}_{\mathrm{sym}})$ so that, uniformly for all $(x', x_3)\in\Omega^h$ there holds:

$$A^h(x',x_3) = \mathcal{G}^h(x',x_3)^{1/2} = \bar A(x') + hA_1(x', \frac{x_3}{h}) + \frac{h^2}{2}A_2(x', \frac{x_3}{h}) + o(h^2).$$

Equivalently, $\bar A, A_1, A_2$ solve the following system of equations:

$$\bar A^2 = \bar{\mathcal{G}}, \qquad 2(\bar A A_1)_{\mathrm{sym}} = \mathcal{G}_1, \qquad 2A_1^2 + 2(\bar A A_2)_{\mathrm{sym}} = \mathcal{G}_2 \qquad \text{in } \bar\Omega. \tag{2.1}$$

Under the assumption (O), condition (iii) on W easily implies:

$$\begin{aligned}\frac{1}{h}\int_{\Omega^h} \mathrm{dist}^2\big(\nabla u^h(x)\bar A(x')^{-1}, SO(3)\big)\,\mathrm{d}x &\leq \frac{C}{h}\int_{\Omega^h}\mathrm{dist}^2\big(\nabla u^h(x)A^h(x)^{-1}, SO(3)\big) + h^2\,\mathrm{d}x \\ &\leq C\big(\mathcal{E}^h(u^h) + h^2\big).\end{aligned}$$

Consequently, the results of [12] automatically give the following compactness properties of any sequence of deformations with the quadratic energy scaling:

Theorem 2.1. *Assume (O). Let $\{u^h\in W^{1,2}(\Omega^h,\mathbb{R}^3)\}_{h\to 0}$ be a sequence of deformations satisfying:*

$$\mathcal{E}^h(u^h) \leq Ch^2. \tag{2.2}$$

Then the following properties hold for the rescalings $y^h\in W^{1,2}(\Omega,\mathbb{R}^3)$ given by:

$$y^h(x',x_3) = u^h(x', hx_3) - \fint_{\Omega^h} u^h\,\mathrm{d}x.$$

(i) There exist $y\in W^{2,2}(\omega,\mathbb{R}^3)$ and $\vec b\in W^{1,2}\cap L^\infty(\omega,\mathbb{R}^3)$ such that, up to a subsequence:

$$y^h\to y \quad \text{strongly in } W^{1,2}(\Omega,\mathbb{R}^3) \quad \text{and} \quad \frac{1}{h}\partial_3 y^h \to \vec b \quad \text{strongly in } L^2(\Omega,\mathbb{R}^3), \qquad \text{as } h\to 0.$$

(ii) The limit deformation y realizes the reduced midplate metric on ω:

$$(\nabla y)^{\mathsf{T}}\nabla y = \bar{\mathcal{G}}_{2\times 2}. \tag{2.3}$$

In particular $\partial_1 y, \partial_2 y\in L^\infty(\omega,\mathbb{R}^3)$ and the unit normal $\vec\nu = \frac{\partial_1 y\times\partial_2 y}{|\partial_1 y\times \partial_2 y|}$ to the surface $y(\omega)$ satisfies: $\vec\nu\in W^{1,2}\cap L^\infty(\omega,\mathbb{R}^3)$. The limit displacement $\vec b$ is the Cosserat field defined via:

$$\vec b = (\nabla y)(\bar{\mathcal{G}}_{2\times 2})^{-1}\begin{bmatrix}\bar{\mathcal{G}}_{13}\\ \bar{\mathcal{G}}_{23}\end{bmatrix} + \frac{\sqrt{\det\bar{\mathcal{G}}}}{\sqrt{\det\bar{\mathcal{G}}_{2\times 2}}}\,\vec\nu. \tag{2.4}$$

Recall that the results in [12] also give:

$$\liminf_{h\to 0}\frac{1}{h^2}\frac{1}{h}\int_{\Omega^h} W\big(\nabla u^h\bar{\mathcal{G}}^{-1/2}\big) \geq \frac{1}{24}\int_\omega \mathcal{Q}_2\Big(x', \nabla y(x')^{\mathsf{T}}\nabla\vec b(x')\Big)\,\mathrm{d}x', \tag{2.5}$$

with the curvature integrand $(\nabla y)^{\mathsf{T}}\nabla\vec b$ quantified by the quadratic forms:

$$\begin{aligned}&\mathcal{Q}_2(x', F_{2\times 2}) = \min\Big\{\mathcal{Q}_3\big(\bar A(x')^{-1}\tilde F\bar A(x')^{-1}\big);\ \tilde F\in\mathbb{R}^{3\times 3} \text{ with } \tilde F_{2\times 2} = F_{2\times 2}\Big\},\\ &\mathcal{Q}_3(F) = D^2W(Id_3)(F,F).\end{aligned} \tag{2.6}$$

The form $\mathcal{Q}_3$ is defined for all $F\in\mathbb{R}^{3\times 3}$, while $\mathcal{Q}_2(x',\cdot)$ are defined on $F_{2\times 2}\in\mathbb{R}^{2\times 2}$. Both forms $\mathcal{Q}_3$ and all $\mathcal{Q}_2$ are nonnegative definite and depend only on the symmetric parts of their arguments, in view of the assumptions on the elastic energy density W. Clearly, the minimization problem in

(2.6) has a unique solution among symmetric matrices $\tilde{F}$ which for each $x' \in \omega$ is described by the linear function $F_{2\times 2} \mapsto c(x', F_{2\times 2}) \in \mathbb{R}^3$ in:

$$\mathcal{Q}_2(x', F_{2\times 2}) = \min\Big\{\mathcal{Q}_3\big(\bar{A}(x')^{-1}(F^*_{2\times 2} + c\otimes e_3)\bar{A}(x')^{-1}\big);\ c\in\mathbb{R}^3\Big\}. \tag{2.7}$$

The energy in the right hand side of (2.5) is a Kirchhoff-like fully nonlinear bending, which in case of $\bar{A}e_3 = e_3$ reduces to the classical bending content relative to the second fundamental form $(\nabla y)^{\mathsf{T}}\nabla\vec{b} = (\nabla y)^{\mathsf{T}}\nabla\vec{\nu}$ on the deformed surface $y(\omega)$.

In the present setting, we start with an observation about projections on polynomial subspaces of L^2. Consider the following Hilbert space, with its norm:

$$\mathbb{E} \doteq \big(L^2(\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}}), \|\cdot\|_{\mathcal{Q}_2}\big), \qquad \|F\|_{\mathcal{Q}_2} = \Big(\int_\Omega \mathcal{Q}_2(x', F(x))\,\mathrm{d}x\Big)^{1/2}, \tag{2.8}$$

associated to the scalar product (with obvious notation):

$$\langle F_1, F_2\rangle_{\mathcal{Q}_2} = \int_\Omega \mathcal{L}_{2,x'}\big(F_1(x), F_2(x)\big)\,\mathrm{d}x.$$

We define $\mathbb{P}_1$ and $\mathbb{P}_2$, respectively, as the orthogonal projections onto the following subspaces of $\mathbb{E}$:

$$\begin{aligned} \mathbb{E}_1 &= \Big\{x_3\mathcal{F}_1(x') + \mathcal{F}_0(x');\ \mathcal{F}_1, \mathcal{F}_0 \in L^2(\omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}})\Big\},\\ \mathbb{E}_2 &= \Big\{x_3^2\mathcal{F}_2(x') + x_3\mathcal{F}_1(x') + \mathcal{F}_0(x');\ \mathcal{F}_2, \mathcal{F}_1, \mathcal{F}_0 \in L^2(\omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}})\Big\}, \end{aligned} \tag{2.9}$$

obtained by projecting each $F(x', \cdot)$ on the appropriate polynomial subspaces of $L^2(-1/2, 1/2)$ whose orthonormal bases consist of the Legendre polynomials $\{p_i\}_{i=0}^\infty$. The first three polynomials are:

$$p_0(t) = 1, \qquad p_1(t) = \sqrt{12}t, \qquad p_2(t) = \sqrt{5}\big(6t^2 - \frac{1}{2}\big).$$

Lemma 2.2. *For every $F \in \mathbb{E}$, we have:*

$$\begin{aligned} &\mathbb{P}_1(F) = 12\Big(\int_{-1/2}^{1/2} x_3F\,\mathrm{d}x_3\Big)x_3 + \Big(\int_{-1/2}^{1/2} F\,\mathrm{d}x_3\Big),\\ &\mathbb{P}_2(F) = \Big(\int_{-1/2}^{1/2} (180x_3^2 - 15)F\,\mathrm{d}x_3\Big)x_3^2 + 12\Big(\int_{-1/2}^{1/2} x_3F\,\mathrm{d}x_3\Big)x_3 + \Big(\int_{-1/2}^{1/2} (-15x_3^2 + \frac{9}{4})F\,\mathrm{d}x_3\Big) \end{aligned}$$

Moreover, the distances from spaces $\mathbb{E}_1$ and $\mathbb{E}_2$ are given by:

$$\begin{aligned} &\mathrm{dist}^2_{\mathcal{Q}_2}(F, \mathbb{E}_1) = \int_\omega \bigg(\int_{-1/2}^{1/2} \mathcal{Q}_2\big(x', F\big)\,\mathrm{d}x_3 - 12\mathcal{Q}_2\big(x', \int_{-1/2}^{1/2} x_3F\,\mathrm{d}x_3\big) - \mathcal{Q}_2\big(x', \int_{-1/2}^{1/2} F\,\mathrm{d}x_3\big)\bigg)\,\mathrm{d}x',\\ &\mathrm{dist}^2_{\mathcal{Q}_2}(F, \mathbb{E}_2) = \int_\omega \bigg(\int_{-1/2}^{1/2} \mathcal{Q}_2\big(x', F\big)\,\mathrm{d}x_3 - 180\mathcal{Q}_2\big(x', \int_{-1/2}^{1/2} \big(x_3^2 - \frac{1}{12}\big)F\,\mathrm{d}x_3\big)\\ &\qquad\qquad - 12\mathcal{Q}_2\big(x', \int_{-1/2}^{1/2} x_3F\,\mathrm{d}x_3\big) - \mathcal{Q}_2\big(x', \int_{-1/2}^{1/2} F\,\mathrm{d}x_3\big)\bigg)\,\mathrm{d}x'. \end{aligned}$$

Proof. The Lemma results by a straightforward calculation:

$$\begin{aligned}\mathrm{dist}^2_{\mathcal{Q}_2}(F,\mathbb{E}_1) &= \|F\|^2_{\mathcal{Q}_2} - \|\mathbb{P}_1(F)\|^2_{\mathcal{Q}_2} = \|F\|^2_{\mathcal{Q}_2} - \Big(\|\int_{-1/2}^{1/2} p_1 F\,\mathrm{d}x_3\|^2_{\mathcal{Q}_2} + \|\int_{-1/2}^{1/2} p_0 F\,\mathrm{d}x_3\|^2_{\mathcal{Q}_2}\Big),\\ \mathrm{dist}^2_{\mathcal{Q}_2}(F,\mathbb{E}_2) &= \|F\|^2_{\mathcal{Q}_2} - \|\mathbb{P}_2(F)\|^2_{\mathcal{Q}_2}\\ &= \|F\|^2_{\mathcal{Q}_2} - \Big(\|\int_{-1/2}^{1/2} p_2 F\,\mathrm{d}x_3\|^2_{\mathcal{Q}_2} + \|\int_{-1/2}^{1/2} p_1 F\,\mathrm{d}x_3\|^2_{\mathcal{Q}_2} + \|\int_{-1/2}^{1/2} p_0 F\,\mathrm{d}x_3\|^2_{\mathcal{Q}_2}\Big),\end{aligned}$$

where we have used that: $\mathbb{P}_1(F) = p_1\int_{-1/2}^{1/2} p_1 F\,\mathrm{d}x_3 + p_0\int_{-1/2}^{1/2} p_0 F\,\mathrm{d}x_3$ and similarly: $\mathbb{P}_2(F) = p_2\int_{-1/2}^{1/2} p_2 F\,\mathrm{d}x_3 + p_1\int_{-1/2}^{1/2} p_1 F\,\mathrm{d}x_3 + p_0\int_{-1/2}^{1/2} p_0 F\,\mathrm{d}x_3$. □

Theorem 2.3. *In the setting of Theorem 2.1,* $\liminf_{h\to 0}\frac{1}{h^2}\mathcal{E}^h(u^h)$ *is bounded from below by:*

$$\begin{aligned}\mathcal{I}_2^O(y) &= \frac{1}{2}\int_\Omega \mathcal{Q}_2\Big(x', x_3\nabla y(x')^{\mathsf{T}}\nabla\vec b(x') - \frac{1}{2}\mathcal{G}_1(x)_{2\times 2}\Big)\,\mathrm{d}x\\ &= \frac{1}{24}\int_\omega \mathcal{Q}_2\Big(x', \big(\nabla y(x')^{\mathsf{T}}\nabla\vec b(x')\big)_{\mathrm{sym}} - \frac{1}{2}\bar{\mathcal{G}}_1(x')_{2\times 2}\Big)\,\mathrm{d}x' + \frac{1}{8}\mathrm{dist}^2_{\mathcal{Q}_2}\Big((\mathcal{G}_1)_{2\times 2},\mathbb{E}_1\Big),\end{aligned}$$

where $\bar{\mathcal{G}}_1$ *is as in (EF). In the non-oscillatory case (NO) this formula becomes:*

$$\mathcal{I}_2(y) = \frac{1}{24}\int_\omega \mathcal{Q}_2\Big(x', \big(\nabla y(x')^{\mathsf{T}}\nabla\vec b(x')\big)_{\mathrm{sym}} - \frac{1}{2}\partial_3 G(x',0)_{2\times 2}\Big)\,\mathrm{d}x'.$$

The first term in $\mathcal{I}_2^O$ *coincides with* $\mathcal{I}_2$ *for the effective metric* $\bar G$ *in (EF).*

Proof. The argument follows the proof of [12, Theorem 2.1] and thus we only indicate its new ingredients. Applying the compactness analysis for the x_3-independent metric $\bar G$, one obtains the sequence $\{R^h \in L^2(\omega, SO(3))\}_{h\to 0}$ of approximating rotation-valued fields, satisfying:

$$\frac{1}{h}\int_{\Omega^h} |\nabla u^h(x)\bar A(x')^{-1} - R^h(x')|^2\,\mathrm{d}x \le Ch^2. \tag{2.10}$$

Define now the family $\{S^h \in L^2(\Omega,\mathbb{R}^{3\times 3})\}_{h\to 0}$ by:

$$S^h(x',x_3) = \frac{1}{h}\Big(R^h(x')^{\mathsf{T}}\nabla u^h(x',hx_3)A^h(x',hx_3)^{-1} - Id_3\Big).$$

According to [12], the same quantities, written for the metric $\bar G$ rather than G^h:

$$\bar S^h(x',x_3) = \frac{1}{h}\Big(R^h(x')^{\mathsf{T}}\nabla u^h(x',hx_3)\bar A(x')^{-1} - Id_3\Big),$$

converge weakly in $L^2(\Omega,\mathbb{R}^{3\times 3})$ to $\bar S$, such that:

$$\big(\bar A(x')\bar S(x',x_3)\bar A(x')\big)_{2\times 2} = \bar s(x') + x_3\nabla y(x')^{\mathsf{T}}\nabla\vec b(x'), \tag{2.11}$$

with some appropriate $\bar s \in L^2(\omega,\mathbb{R}^{2\times 2})$. Observe that:

$$S^h(x',x_3) = \bar S^h(x',x_3) + \frac{1}{h}R^h(x')^{\mathsf{T}}\nabla u(x',hx_3)\big(A^h(x',hx_3)^{-1} - \bar A(x')^{-1}\big)$$

and that the term $R^h(x')^{\mathsf{T}}\nabla u(x',hx_3)$ converges strongly in $L^2(\Omega,\mathbb{R}^{3\times 3})$ to $\bar A(x')$. On the other hand, the remaining factor converges uniformly on Ω as $h\to 0$, because:

$$\frac{1}{h}\big(A^h(x',hx_3)^{-1} - \bar A(x')^{-1}\big) = -\bar A(x')^{-1}A_1(x',x_3)\bar A(x')^{-1} + \mathcal{O}(h) \tag{2.12}$$

Concluding, S^h converge weakly in $L^2(\Omega,\mathbb{R}^{3\times3})$ to S, satisfying by (2.11):

$$\big(\bar{A}(x')S(x',x_3)\bar{A}(x')\big)_{2\times2} = \bar{s}(x') + x_3\nabla y(x')^{\mathsf{T}}\nabla\vec{b}(x') - \bar{A}(x')A_1(x',x_3). \tag{2.13}$$

Consequently, using the definition of S^h and frame invariance of W and Taylor expanding W at Id_3 on the set $\{|S^h|^2\leq 1/h\}$, we obtain:

$$\begin{aligned}\liminf_{h\to0}\mathcal{E}^h(u^h) &= \liminf_{h\to0}\frac{1}{h^2}\int_\Omega W\big(Id_3+hS^h(x)\big)\,\mathrm{d}x\\ &\geq \liminf_{h\to0}\frac{1}{2}\int_{\{|S^h|^2\leq1/h\}}\mathcal{Q}_3(S^h(x))+o(|S^h|^2)\,\mathrm{d}x\\ &\geq \frac{1}{2}\int_\Omega\mathcal{Q}_3\big(S(x)\big)\,\mathrm{d}x = \frac{1}{2}\int_\Omega\mathcal{Q}_2\Big(x',\big(\bar{A}(x')S(x)\bar{A}(x')\big)_{2\times2}\Big)\,\mathrm{d}x.\end{aligned}$$

Further, recalling (2.13) and (2.1) we get:

$$\begin{aligned}\liminf_{h\to0}\mathcal{E}^h(u^h) &\geq \frac{1}{2}\big\|\bar{s}_{\mathrm{sym}}+x_3\big((\nabla y)^{\mathsf{T}}\nabla\vec{b}\big)_{\mathrm{sym}}-\frac{1}{2}(\mathcal{G}_1)_{2\times2}\big\|^2_{\mathcal{Q}_2}\\ &= \frac{1}{2}\big\|\mathbb{P}_1\big(\bar{s}_{\mathrm{sym}}+x_3\big((\nabla y)^{\mathsf{T}}\nabla\vec{b}\big)_{\mathrm{sym}}-\frac{1}{2}(\mathcal{G}_1)_{2\times2}\big)\big\|^2_{\mathcal{Q}_2}+\frac{1}{8}\big\|(\mathcal{G}_1)_{2\times2}-\mathbb{P}_1((\mathcal{G}_1)_{2\times2})\big\|^2_{\mathcal{Q}_2}\\ &= \frac{1}{2}\big\|\bar{s}_{\mathrm{sym}}\big\|^2_{\mathcal{Q}_2}+\frac{1}{2}\Big\|x_3\Big(\big((\nabla y)^{\mathsf{T}}\nabla\vec{b}\big)_{\mathrm{sym}}-6\int_{-1/2}^{1/2}t(\mathcal{G}_1)_{2\times2}\,\mathrm{d}t\Big)\Big\|^2_{\mathcal{Q}_2}+\frac{1}{8}\mathrm{dist}^2_{\mathcal{Q}_2}\big((\mathcal{G}_1)_{2\times2},\mathbb{E}_1\big)\\ &\geq \frac{1}{24}\Big\|\big((\nabla y)^{\mathsf{T}}\nabla\vec{b}\big)_{\mathrm{sym}}-6\int_{-1/2}^{1/2}t(\mathcal{G}_1)_{2\times2}\,\mathrm{d}t\Big\|^2_{\mathcal{Q}_2}+\frac{1}{8}\mathrm{dist}^2_{\mathcal{Q}_2}\big((\mathcal{G}_1)_{2\times2},\mathbb{E}_1\big)=\mathcal{I}_2^O(y),\end{aligned}$$

where we have used the fact that $\mathcal{Q}_2(x',\cdot)$ is a function of its symmetrized argument and Lemma 2.2. The formula for $\mathcal{I}_2$ in case (NO) is immediate. ☐

Our next result is the upper bound, parallel to the lower bound in Theorem 2.3:

Theorem 2.4. *Assume (O). For every isometric immersion $y\in W^{2,2}(\omega,\mathbb{R}^3)$ of the reduced midplate metric $\bar{\mathcal{G}}_{2\times2}$ as in (2.3), there exists a sequence $\{u^h\in W^{1,2}(\Omega^h,\mathbb{R}^3)\}_{h\to0}$ such that the sequence $\{y^h(x',x_3)=u^h(x',hx_3)\}_{h\to0}$ converges in $W^{1,2}(\Omega,\mathbb{R}^3)$ to y and:*

$$\lim_{h\to0}\frac{1}{h^2}\mathcal{E}^h(u^h)=\mathcal{I}_2^O(y) \tag{2.14}$$

Automatically, $\frac{1}{h}\partial_3y^h$ converges in $L^2(\Omega,\mathbb{R}^3)$ to $\vec{b}\in W^{1,2}\cap L^\infty(\omega,\mathbb{R}^3)$ as in (2.4).

Proof. Given an admissible y, we define $\vec{b}$ by (2.4) and also define the matrix field:

$$Q=\big[\partial_1y,\ \partial_2y,\ \vec{b}\big]\in W^{1,2}\cap L^\infty(\omega,\mathbb{R}^{3\times3}). \tag{2.15}$$

It follows that $Q(x')\bar{A}(x')^{-1}\in SO(3)$ on ω. The recovery sequence u^h satisfying (2.14) is then constructed via a diagonal argument, applied to the explicit deformation fields below. Again, we only indicate the new ingredients with respect to the proof in [12, Theorem 3.1].

We define the vector field $\vec{d}\in L^2(\Omega,\mathbb{R}^3)$ by:

$$
\begin{aligned}
\vec{d}(x',x_3) = Q(x')^{\mathsf{T},-1}\Big(\frac{x_3^2}{2}\Big(c\big(x',\nabla y(x')^{\mathsf{T}}\nabla\vec{b}(x')\big) - \frac{1}{2}\left[\begin{array}{c}\nabla|\vec{b}|^2(x')\\ 0\end{array}\right]\Big)\\
- \frac{1}{2}c\Big(x',\int_0^{x_3}\mathcal{G}_1(x',t)_{2\times 2}\,\mathrm{d}t\Big)\\
+ \int_0^{x_3}\mathcal{G}_1(x',t)\,\mathrm{d}t\,e_3 - \frac{1}{2}\int_0^{x_3}\mathcal{G}_1(x',t)_{33}\,\mathrm{d}t\,e_3\Big).
\end{aligned}
\tag{2.16}
$$

In view of definition (2.7), the formula in (2.16) is equivalent to the vector field $\partial_3\vec{d}\in L^2(\Omega,\mathbb{R}^3)$ being, for each $(x',x_3)\in\Omega$, the unique solution to:

$$
\begin{aligned}
&\mathcal{Q}_2\Big(x', x_3\nabla y(x')^{\mathsf{T}}\nabla\vec{b}(x') - \frac{1}{2}\mathcal{G}_1(x',x_3)_{2\times 2}\Big)\\
&\quad = \mathcal{Q}_3\Big(\bar{A}(x')^{-1}\Big(Q(x')^{\mathsf{T}}\Big[x_3\partial_1\vec{b}(x'),\ x_3\partial_2\vec{b}(x'),\ \partial_3\vec{d}(x',x_3)\Big] - \frac{1}{2}\mathcal{G}_1(x',x_3)\Big)\bar{A}(x')^{-1}\Big).
\end{aligned}
$$

One then approximates $y,\vec{b}$ by sequences $\{y^h\in W^{2,\infty}(\omega,\mathbb{R}^3)\}_{h\to 0}$, $\{b^h\in W^{1,\infty}(\omega,\mathbb{R}^3)\}_{h\to 0}$ respectively, and request them to satisfy conditions exactly as in the proof of [12, Theorem 3.1]. The warping field $\vec{d}$ is approximated by $d^h(x',x_3)=\int_0^{x_3}\bar{d}^h(x',t)\,\mathrm{d}t$, where:

$$
\bar{d}^h\to\bar{d}=\partial_3\vec{d}\quad\text{strongly in } L^2(\Omega,\mathbb{R}^3)\qquad\text{and}\qquad h\|\bar{d}^h\|_{W^{1,\infty}(\Omega,\mathbb{R}^3)}\to 0\qquad\text{as } h\to 0.
$$

Finally, we define:

$$
u^h(x',x_3) = y^h(x') + x_3 b^h(x') + h^2 d^h\big(x',\frac{x_3}{h}\big),
\tag{2.17}
$$

so that, with the right approximation error, there holds:

$$
\nabla u^h(x',x_3)\approx Q(x') + h\Big[\frac{x_3}{h}\partial_1\vec{b}(x'),\ \frac{x_3}{h}\partial_2\vec{b}(x'),\ \partial_3\vec{d}\big(x',\frac{x_3}{h}\big)\Big].
$$

Using Taylor's expansion of W, the definition (2.16) and the controlled blow-up rates of the approximating sequences, the construction is done. □

We conclude this section by noting the following easy direct consequence of Theorems 2.3 and 2.4:

Corollary 2.5. *If the set of $W^{2,2}(\omega,\mathbb{R}^3)$ isometric immersions of $\bar{\mathcal{G}}_{2\times 2}$ is nonempty, then the functional $\mathcal{I}_2^O$ attains its infimum and:*

$$
\lim_{h\to 0}\frac{1}{h^2}\inf\mathcal{E}^h = \min\ \mathcal{I}_2^0.
$$

The infima in the left hand side are taken over $W^{1,2}(\Omega^h,\mathbb{R}^3)$ deformations u^h, whereas the minima in the right hand side are taken over $W^{2,2}(\omega,\mathbb{R}^3)$ isometric immersions y of $\bar{\mathcal{G}}_{2\times 2}$.

3. Identification of the Ch^2 scaling regime

In this section, we identify the equivalent conditions for $\inf\mathcal{E}^h\sim h^2$ in terms of curvatures of the metric tensor $\bar{G}$ in case (NO). We begin by expressing the integrand tensor in the residual energy $\mathcal{I}_2$ in terms of the shape operator on the deformed midplate. Recall that we always use the Einstein summation convention over repeated indices running from 1 to 3.

Lemma 3.1. *In the the non-oscillatory setting (NO), let $y \in W^{2,2}(\omega, \mathbb{R}^3)$ be an isometric immersion of the metric $\bar{\mathcal{G}}_{2\times 2}$, so that (2.3) holds on ω. Define the Cosserat vector $\vec{b}$ according to (2.4). Then:*

$$\big((\nabla y)^{\mathsf{T}}\nabla\vec{b}\big)_{\mathrm{sym}} - \frac{1}{2}\partial_3 G(x',0)_{2\times 2} = \frac{1}{\sqrt{\bar{\mathcal{G}}^{33}}}\Pi_y + \frac{1}{\bar{\mathcal{G}}^{33}}\begin{bmatrix}\Gamma^3_{11} & \Gamma^3_{12}\\ \Gamma^3_{12} & \Gamma^3_{22}\end{bmatrix}(x',0), \tag{3.1}$$

for all $x' \in \omega$. Above, $\bar{\mathcal{G}}^{33} = \langle \bar{\mathcal{G}}^{-1}e_3, e_3\rangle$, whereas $\Pi_y = (\nabla y)^{\mathsf{T}}\nabla\vec{\nu} \in W^{1,2}(\omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ is the second fundamental form of the surface $y(\omega) \subset \mathbb{R}^3$, and $\{\Gamma^i_{kl}\}_{i,k,l=1\dots 3}$ are the Christoffel symbols of G:

$$\Gamma^i_{kl} = \frac{1}{2}G^{im}\big(\partial_l G_{mk} + \partial_k G_{ml} - \partial_m G_{kl}\big).$$

Proof. The proof is an extension of the arguments in [12, Theorem 5.3], which we modify for the case of x_3-dependent metric G. Firstly, the fact that $Q^{\mathsf{T}}Q = \bar{\mathcal{G}}$ with Q defined in (2.15), yields:

$$\big((\nabla y)^{\mathsf{T}}\nabla\vec{b}\big)_{\mathrm{sym}} = \Big(\big[\partial_i\bar{\mathcal{G}}_{j3}\big]_{i,j=1,2}\Big)_{\mathrm{sym}} - \big[\langle\partial_{ij}y, \vec{b}\rangle\big]_{i,j=1,2}. \tag{3.2}$$

Also, $\partial_i\bar{\mathcal{G}} = 2\big((\partial_i Q)^{\mathsf{T}}Q\big)_{\mathrm{sym}}$ for $i = 1, 2$, results in:

$$\langle\partial_{ij}y, \partial_k y\rangle = \frac{1}{2}\big(\partial_i G_{kj} - \partial_j G_{kl} - \partial_k G_{ij}\big) \tag{3.3}$$

and:

$$(\nabla y)^{\mathsf{T}}\partial_{ij}y = \Gamma^m_{ij}(x',0)\begin{bmatrix}\bar{\mathcal{G}}_{m1}\\ \bar{\mathcal{G}}_{m2}\end{bmatrix} \qquad \text{for } \ i,j = 1,2.$$

Consequently, we obtain the formula:

$$\begin{aligned}\big[\bar{\mathcal{G}}_{13},\ \bar{\mathcal{G}}_{23}\big]\,(\bar{\mathcal{G}}_{2\times 2})^{-1}(\nabla y)^{\mathsf{T}}\partial_{ij}y &= \Bigg[\bar{\mathcal{G}}_{13},\ \bar{\mathcal{G}}_{23},\ \big[\bar{\mathcal{G}}_{13},\ \bar{\mathcal{G}}_{23}\big]\,(\bar{\mathcal{G}}_{2\times 2})^{-1}\begin{bmatrix}\bar{\mathcal{G}}_{13}\\ \bar{\mathcal{G}}_{23}\end{bmatrix}\Bigg]\begin{bmatrix}\Gamma^1_{ij}\\ \Gamma^2_{ij}\\ \Gamma^3_{ij}\end{bmatrix}(x',0)\\ &= \bar{\mathcal{G}}_{m3}\Gamma^m_{ij}(x',0) - \frac{1}{\bar{\mathcal{G}}^{33}}\Gamma^3_{ij}(x',0).\end{aligned}$$

Computing the normal vector $\vec{\nu}$ from (2.4) and noting that $\det\bar{\mathcal{G}}_{2\times 2}/\det\bar{\mathcal{G}} = \bar{\mathcal{G}}^{33}$, we get:

$$\begin{aligned}\Pi_{ij} &= -\langle\partial_{ij}y, \vec{\nu}\rangle = -\sqrt{\bar{\mathcal{G}}^{33}}\Big(\langle\partial_{ij}y, \vec{b}\rangle - \big[\bar{\mathcal{G}}_{13},\ \bar{\mathcal{G}}_{23}\big]\,(\bar{\mathcal{G}}_{2\times 2})^{-1}(\nabla y)^{\mathsf{T}}\partial_{ij}y\Big)\\ &= \sqrt{\bar{\mathcal{G}}^{33}}\big((\nabla y)^{\mathsf{T}}\nabla\vec{b}\big)_{\mathrm{sym}\,,ij} - \frac{1}{\sqrt{\bar{\mathcal{G}}^{33}}}\Gamma^3_{ij}(x',0) - \frac{\sqrt{\bar{\mathcal{G}}^{33}}}{2}\partial_3 G_{ij}(x',0), \qquad \text{for } \ i,j = 1,2,\end{aligned}$$

which completes the proof of (3.1). □

The key result of this section is the following:

Theorem 3.2. *The energy scaling beyond the Kirchhoff regime:*

$$\lim_{h\to 0}\frac{1}{h^2}\inf\mathcal{E}^h = 0$$

is equivalent to the following conditions:

(i) in the oscillatory case (O)

$$
(3.4)\qquad \left[\begin{array}{l} (\mathcal{G}_1)_{2\times 2}\in \mathbb{E}_1 \text{ or equivalently there holds:} \\ \qquad \mathcal{G}_1(x',x_3)_{2\times 2} = x_3\bar{\mathcal{G}}_1(x')_{2\times 2} \qquad \text{for all } (x',x_3)\in\bar{\Omega}. \\ \text{Moreover, condition (3.5) below must be satisfied with } G \text{ replaced by the effective metric} \\ \bar{G} \text{ in (EF). This condition involves only } \bar{\mathcal{G}} \text{ and } (\bar{\mathcal{G}}_1)_{2\times 2} \text{ terms of } \bar{G}. \end{array}\right.
$$

(ii) in the non-oscillatory case (NO)

$$
(3.5)\qquad \left[\begin{array}{l} \text{There exists } y_0\in W^{2,2}(\omega,\mathbb{R}^3) \text{ satisfying (2.3) and such that:} \\ \qquad \Pi_{y_0}(x') = -\dfrac{1}{\sqrt{\bar{\mathcal{G}}^{33}}}\begin{bmatrix} \Gamma^3_{11} & \Gamma^3_{12} \\ \Gamma^3_{12} & \Gamma^3_{22}\end{bmatrix}(x',0) \qquad \text{for all } x'\in\omega, \\ \text{where } \Pi_{y_0} \text{ is the second fundamental form of the surface } y_0(\omega) \text{ and } \{\Gamma^i_{jk}\} \text{ are the Christoffel} \\ \text{symbols of the metric } G. \end{array}\right.
$$

The isometric immersion y_0 in (3.5) is automatically smooth (up to the boundary) and it is unique up to rigid motions. Further, on a simply connected midplate ω, condition (3.5) is equivalent to:

$$
(3.6)\qquad \left[\begin{array}{l} \text{The following Riemann curvatures of the metric } G \text{ vanish on } \omega\times\{0\}: \\ \qquad R_{1212}(x',0) = R_{1213}(x',0) = R_{1223}(x',0) = 0 \qquad \text{for all } x'\in\omega. \end{array}\right.
$$

Above, the Riemann curvatures of a given metric G are:

$$
R_{iklm} = \frac{1}{2}\Big(\partial_{kl}G_{im} + \partial_{im}G_{kl} - \partial_{km}G_{il} - \partial_{il}G_{km}\Big) + G_{np}\big(\Gamma^n_{kl}\Gamma^p_{im} - \Gamma^n_{km}\Gamma^p_{il}\big).
$$

Proof. By Corollary 2.5, it suffices to determine the equivalent conditions for $\min \mathcal{I}_2^O = 0$ and $\min \mathcal{I}_2 = 0$. In case (O), the linearity of $x_3 \mapsto \mathcal{G}_1(x',x_3)_{2\times 2}$ is immediate, while condition (3.5) follows in both cases (O) and (NO) by Lemma 3.1. Note that the Christoffel symbols $\{\Gamma^i_{jk}\}$ depend only on $\bar{\mathcal{G}}$ and $\partial_3 G(x',0)_{2\times 2}$ in the Taylor expansion of G. This completes the proof of (i) and (ii).

Regularity of y_0 is an easy consequence, via the bootstrap argument, of the continuity equation:

$$
(3.7)\qquad \partial_{ij}y_0 = \sum_{m=1}^{2}\gamma^m_{ij}\partial_m y_0 - (\Pi_{y_0})_{ij}\vec{\nu}_0 \qquad \text{for } i,j=1,2,
$$

where $\{\gamma^m_{ij}\}_{i,j,m=1\ldots 2}$ denote the Christoffel symbols of $\bar{\mathcal{G}}_{2\times 2}$ on ω. Uniqueness of y_0 is a consequence of (3.5), due to uniqueness of isometric immersion with prescribed second fundamental form.

To show (iv), we argue as in the proof of [12, Theorem 5.5]. The compatibility of $\bar{G}_{2\times 2}$ and Π_{y_0} is equivalent to the satisfaction of the related Gauss-Codazzi-Meinardi equations. By an explicit calculation, we see that the two Codazzi-Meinardi equations become:

$$\begin{aligned}
&\big(\partial_2\Gamma_{11}^3 - \partial_1\Gamma_{12}^3\big) - \frac{1}{2}\left(\frac{\partial_2 G^{33}}{G^{33}}\Gamma_{11}^3 - \frac{\partial_1 G^{33}}{G^{33}}\Gamma_{12}^3\right) + \frac{1}{G^{33}}G^{m3}\big(\Gamma_{2m}^3\Gamma_{11}^3 - \Gamma_{1m}^3\Gamma_{12}^3\big)\\
&\qquad = \left(\sum_{m=1}^{2}\Gamma_{1m}^3\Gamma_{12}^m - \sum_{m=1}^{2}\Gamma_{2m}^3\Gamma_{11}^m\right) + \frac{G^{32}}{G^{33}}\big(\Gamma_{11}^3\Gamma_{22}^3 - (\Gamma_{12}^3)^2\big),\\
&\big(\partial_2\Gamma_{12}^3 - \partial_1\Gamma_{22}^3\big) - \frac{1}{2}\left(\frac{\partial_2 G^{33}}{G^{33}}\Gamma_{12}^3 - \frac{\partial_1 G^{33}}{G^{33}}\Gamma_{22}^3\right) + \frac{1}{G^{33}}G^{m3}\big(\Gamma_{2m}^3\Gamma_{12}^3 - \Gamma_{1m}^3\Gamma_{22}^3\big)\\
&\qquad = \left(\sum_{m=1}^{2}\Gamma_{1m}^3\Gamma_{22}^m - \sum_{m=1}^{2}\Gamma_{2m}^3\Gamma_{12}^m\right) - \frac{G^{31}}{G^{33}}\big(\Gamma_{11}^3\Gamma_{22}^3 - (\Gamma_{12}^3)^2\big),
\end{aligned}$$

and are equivalent to $R_{121}^3 = R_{221}^3 = 0$ on $\omega\times\{0\}$. The Gauss equation is, in turn, equivalent to $R_{1212} = 0$ exactly as in [12]. The simultaneous vanishing of $R_{121}^3, R_{221}^3, R_{1212}$ is equivalent with the vanishing of R_{1212}, R_{1213} and R_{1223}, which proves the claim in (3.6). □

4. Coercivity of the limiting energy $\mathcal{I}_2$

In this section we quantify the statement in Theorem 3.2 and prove that when either of $\mathcal{I}_2$ or $\mathcal{I}_2^O$ can be minimized to zero, the effective energy $\mathcal{I}_2(y)$ measures then the distance of a given isometric immersion y from the kernel: $\ker \mathcal{I}_2 = \big\{Qy_0 + d;\ Q\in SO(3),\ d\in\mathbb{R}^3\big\}$.

Assume that the set of $W^{2,2}(\omega,\mathbb{R}^3)$ isometric immersions y of $\bar{\mathcal{G}}_{2\times2}$ is nonempty, which in view of Theorems 2.3 and 2.4 is equivalent to: $\inf \mathcal{E}^h \leq Ch^2$. For each such y, the continuity equation (3.7) combined with Lemma 3.1 gives the following formula, valid for all $i,j=1,2$:

$$\partial_{ij}y = \sum_{m=1,2}\gamma_{ij}^m\partial_m y - \sqrt{\bar{\mathcal{G}}^{33}}\Big(\big((\nabla y)^{\mathsf{T}}\nabla\vec{b}\big)_{\rm sym} - \frac{1}{2}\partial_3 G(x',0)_{2\times2}\Big)_{ij}\vec{\nu} + \frac{\Gamma_{ij}^3}{\sqrt{\bar{\mathcal{G}}^{33}}}\vec{\nu}\quad \text{on } \omega, \tag{4.1}$$

and resulting in:

$$|\nabla^2 y|^2 = |\Pi_y|^2 + \sum_{i,j=1,2}\big\langle \bar{\mathcal{G}}_{2\times2} : [\gamma_{ij}^1,\ \gamma_{ij}^2]^{\otimes 2}\big\rangle \quad \text{on } \omega.$$

By Lemma 3.1 and since $|\nabla y|^2 = \operatorname{trace}\bar{\mathcal{G}}_{2\times2}$, this yields the bound:

$$\|y - \fint_\omega y\|_{W^{2,2}(\omega,\mathbb{R}^3)}^2 \leq C\big(\mathcal{I}_2(y) + 1\big), \tag{4.2}$$

where C is a constant independent of y. Clearly, when condition (3.6) does not hold, so that $\min\mathcal{I}_2 > 0$, the right hand side $C\big(\mathcal{I}_2(y)+1\big)$ above may be replaced by $C\mathcal{I}_2(y)$. On the other hand, in presence of (3.6), the bound (4.2) can be refined to the following coercivity result:

Theorem 4.1. *Assume the curvature condition (3.6) on a metric G as in (NO), and let y_0 be the unique (up to rigid motions in $\mathbb{R}^3$) isometric immersion of $\bar{\mathcal{G}}_{2\times2}$ satisfying (3.5). Then, for all $y\in W^{2,2}(\omega,\mathbb{R}^3)$ such that $(\nabla y)^{\mathsf{T}}\nabla y = \bar{\mathcal{G}}_{2\times2}$, there holds:*

$$\operatorname{dist}^2_{W^{2,2}(\omega,\mathbb{R}^3)}\Big(y,\ \big\{Ry_0 + c;\ R\in SO(3),\ c\in\mathbb{R}^3\big\}\Big) \leq C\mathcal{I}_2(y), \tag{4.3}$$

with a constant $C>0$ that depends on G,ω and W but it is independent of y.

Proof. Without loss of generality, we set $\fint_\omega y = \fint_\omega y_0 = 0$. For any $R \in SO(3)$, identity (4.1) implies:

$$\begin{aligned}\int_\omega \big|\nabla^2 y - \nabla^2(Ry_0)\big|^2 \,\mathrm{d}x' \leq C\Big(\int_\omega \big|\nabla y - \nabla(Ry_0)\big|^2 \,\mathrm{d}x' \\ + \int_\omega \big|\big((\nabla y)^\intercal \nabla\vec{b}\big)_{\mathrm{sym}} - \frac{1}{2}\partial_3 G(x',0)_{2\times 2}\big|^2 \,\mathrm{d}x' + \int_\omega |\vec{\nu} - R\vec{\nu}_0|^2 \,\mathrm{d}x'\Big) \\ \leq C\Big(\int_\omega \big|\nabla y - \nabla(Ry)\big|^2 \,\mathrm{d}x' + \int_\omega \big|\big((\nabla y_0)^\intercal \nabla\vec{b}\big)_{\mathrm{sym}} - \frac{1}{2}\partial_3 G(x',0)_{2\times 2}\big|^2 \,\mathrm{d}x'\Big),\end{aligned}$$

where we used $\mathcal{I}_2(Ry_0) = 0$ and the fact that $\int_\omega |\vec{\nu} - R\vec{\nu}_0|^2 \,\mathrm{d}x' \leq C\int_\omega \big|\nabla y - \nabla(Ry_0)\big|^2 \,\mathrm{d}x'$ following, in particular, from $|\partial_1 y \times \partial_2 y| = |\partial_1(Ry_0) \times \partial_2(Ry_0)| = \sqrt{\det \bar{\mathcal{G}}_{2\times 2}}$. Also, the non-degeneracy of quadratic forms $\mathcal{Q}_2(x', \cdot)$ in (2.6), implies the uniform bound:

$$\int_\omega \big|\big((\nabla y)^\intercal \nabla\vec{b}\big)_{\mathrm{sym}} - \frac{1}{2}\partial_3 G(x',0)_{2\times 2}\big|^2 \,\mathrm{d}x' \leq C\mathcal{I}_2(y).$$

Taking $R \in SO(3)$ as in Lemma 4.2 below, (4.3) directly follows in view of (4.4). □

The next weak coercivity estimate has been the essential part of Theorem 4.1:

Lemma 4.2. *Let y and y_0 be as in Theorem 4.1. Then there exists $R \in SO(3)$ such that:*

$$\int_\omega |\nabla y - R\nabla y_0|^2 \,\mathrm{d}x' \leq C\int_\omega \big|\big((\nabla y)^\intercal \nabla\vec{b}\big)_{\mathrm{sym}} - \frac{1}{2}\partial_3 G(x',0)_{2\times 2}\big|^2 \,\mathrm{d}x', \tag{4.4}$$

with a constant $C > 0$ that depends on G, ω but it is independent of y.

Proof. Consider the natural extensions u and u_0 of y and y_0, namely:

$$u(x', x_3) = y(x') + x_3\vec{b}(x'), \quad u_0(x', x_3) = y_0(x') + x_3\vec{b}_0(x') \qquad \text{for all } (x', x_3) \in \Omega^h.$$

Clearly, $u \in W^{1,2}(\Omega^h, \mathbb{R}^3)$ and $u_0 \in \mathcal{C}^1(\bar{\Omega}^h, \mathbb{R}^3)$ satisfies $\det \nabla u_0 > 0$ for h sufficiently small. Write:

$$\omega = \bigcup_{k=1}^N \omega_k, \qquad \Omega^h = \bigcup_{k=1}^N \Omega_k^h$$

as the union of $N \geq 1$ open, bounded, connected domains with Lipschitz boundary, such that on each $\{\Omega_k^h = \omega_k \times (-\frac{h}{2}, \frac{h}{2})\}_{k=1}^N$, the deformation $u_{0|\Omega_k^h}$ is a $\mathcal{C}^1$ diffeomorphism onto its image $\mathcal{U}_k^h \subset \mathbb{R}^3$.

1. We first prove (4.4) under the assumption $N = 1$. Call $v = u \circ u_0^{-1} \in W^{1,2}(\mathcal{U}^h, \mathbb{R}^3)$ and apply the geometric rigidity estimate [20] for the existence of $R \in SO(3)$ satisfying:

$$\int_{\mathcal{U}^h} |\nabla v - R|^2 \,\mathrm{d}z \leq C\int_{\mathcal{U}^h} \mathrm{dist}^2\big(\nabla v, SO(3)\big) \,\mathrm{d}z, \tag{4.5}$$

with a constant C depending on a particular choice of h (and ultimately k, when $N > 1$), but independent of v. Since $\nabla v(u_0(x)) = \nabla u(x)\big(\nabla u_0(x)\big)^{-1}$ for all $x \in \Omega^h$, we get:

$$\begin{aligned}\int_{\mathcal{U}^h} |\nabla v - R|^2 \,\mathrm{d}z &= \int_{\Omega^h} (\det \nabla u_0)\big|(\nabla u - R\nabla u_0)(\nabla u_0)^{-1}\big|^2 \,\mathrm{d}x \geq C\int_{\Omega^h} |\nabla u - R\nabla u_0|^2 \,\mathrm{d}x \\ &= C\int_{\Omega^h} \Big|\big[\partial_1 y,\ \partial_2 y,\ \vec{b}\big] - R\big[\partial_1 y_0,\ \partial_2 y_0,\ \vec{b}_0\big]\Big|^2 + x_3^2|\nabla\vec{b} - R\nabla\vec{b}_0|^2 \,\mathrm{d}x \\ &\geq Ch\int_\omega |\nabla y - R\nabla y_0|^2 \,\mathrm{d}x'.\end{aligned} \tag{4.6}$$

Likewise, the change of variables in the right hand side of (4.5) gives:

$$\int_{\mathcal{U}^h} \mathrm{dist}^2\big(\nabla v, SO(3)\big)\,\mathrm{d}z \leq C\int_{\Omega^h} \mathrm{dist}^2\big((\nabla u)(\nabla u_0)^{-1}, SO(3)\big)\,\mathrm{d}x. \tag{4.7}$$

Since $(\nabla u)^{\mathsf{T}}\nabla u(x',0) = (\nabla u_0)^{\mathsf{T}}\nabla u_0(x',0) = \bar{\mathcal{G}}(x')$, by polar decomposition it follows that: $\nabla u(x',0) = Q(x') = \bar{R}\bar{\mathcal{G}}^{1/2}$ and $\nabla u_0(x',0) = Q_0(x') = \bar{R}_0\bar{\mathcal{G}}^{1/2}$ for some $\bar{R}, \bar{R}_0 \in SO(3)$. The notation Q, Q_0 is consistent with that introduced in (2.15). Observe further:

$$\begin{aligned}\nabla u(x',x_3) &= Q + x_3\big[\partial_1\vec{b},\ \partial_2\vec{b},\ 0\big] = \bar{R}\bar{\mathcal{G}}^{1/2}\Big(Id_3 + x_3\bar{\mathcal{G}}^{-1}Q^{\mathsf{T}}\big[\partial_1\vec{b},\ \partial_2\vec{b},\ 0\big]\Big)\\ &= \bar{R}\bar{\mathcal{G}}^{1/2}\Big(Id_3 + x_3\bar{\mathcal{G}}^{-1}\Big(\big((\nabla y)^{\mathsf{T}}\nabla\vec{b}\big)^* + e_3\otimes\big[\nabla\vec{b},\ 0\big]^{\mathsf{T}}\vec{b}\Big)\Big),\end{aligned}$$

and similarly:

$$\nabla u_0(x',x_3) = \bar{R}_0\bar{\mathcal{G}}^{1/2}\Big(Id_3 + x_3\bar{\mathcal{G}}^{-1}\Big(\big((\nabla y_0)^{\mathsf{T}}\nabla\vec{b}_0\big)^* + e_3\otimes\big[\nabla\vec{b}_0,\ 0\big]^{\mathsf{T}}\vec{b}_0\Big)\Big).$$

Consequently, the integrand in the right hand side of (4.7) becomes:

$$\begin{aligned}&(\nabla u)(\nabla u_0)^{-1}\\ &= \bar{R}\bar{\mathcal{G}}^{1/2}\bigg(Id_3 + x_3\bar{\mathcal{G}}^{-1}S\Big(Id_3 + x_3\bar{\mathcal{G}}^{-1}\big((\nabla y_0)^{\mathsf{T}}\nabla\vec{b}_0\big)^* + e_3\otimes\big[\nabla\vec{b}_0,\ 0\big]^{\mathsf{T}}\vec{b}_0\Big)^{-1}\bigg)\bar{\mathcal{G}}^{-1/2}\bar{R}_0^{\mathsf{T}},\end{aligned} \tag{4.8}$$

where:

$$S = \Big((\nabla y)^{\mathsf{T}}\nabla\vec{b} - (\nabla y_0)^{\mathsf{T}}\nabla\vec{b}_0\Big)^* + e_3\otimes\big[\nabla\vec{b},\ 0\big]^{\mathsf{T}}\vec{b} - e_3\otimes\big[\nabla\vec{b}_0,\ 0\big]^{\mathsf{T}}\vec{b}_0 = \Big((\nabla y)^{\mathsf{T}}\nabla\vec{b} - (\nabla y_0)^{\mathsf{T}}\nabla\vec{b}_0\Big)^*_{\mathrm{sym}}.$$

The last equality follows from the easy facts that, for $i,j = 1,2$, we have:

$$\begin{aligned}&\langle\partial_i\vec{b},\vec{b}\rangle = \langle\partial_i\vec{b}_0,\vec{b}_0\rangle = \frac{1}{2}\partial_i\bar{\mathcal{G}}_{33}\\ &\langle\partial_i y,\partial_j\vec{b}\rangle - \langle\partial_j y,\partial_i\vec{b}\rangle = \langle\partial_i y_0,\partial_j\vec{b}_0\rangle - \langle\partial_j y_0,\partial_i\vec{b}_0\rangle = \partial_j\bar{\mathcal{G}}_{i3} - \partial_i\bar{\mathcal{G}}_{j3}.\end{aligned}$$

Thus, (4.7) and (4.8) imply:

$$\begin{aligned}\int_{\mathcal{U}^h}\mathrm{dist}^2\big(\nabla v, SO(3)\big)\,\mathrm{d}z &\leq C\int_{\Omega^h}\big|(\nabla u)(\nabla u_0)^{-1} - \bar{R}\bar{R}_0^{\mathsf{T}}\big|^2\,\mathrm{d}x \leq C\int_{\Omega^h}\big|x_3S(x',x_3)\big|^2\,\mathrm{d}x\\ &\leq C\int_\omega\Big|\big((\nabla y)^{\mathsf{T}}\nabla\vec{b}\big)_{\mathrm{sym}} - \big((\nabla y_0)^{\mathsf{T}}\nabla\vec{b}_0\big)_{\mathrm{sym}}\Big|^2\,\mathrm{d}x'\\ &= C\int_\omega\Big|\big((\nabla y)^{\mathsf{T}}\nabla\vec{b}\big)_{\mathrm{sym}} - \frac{1}{2}\partial_3 G(x',0)_{2\times 2}\Big|^2\,\mathrm{d}x'\end{aligned} \tag{4.9}$$

with a constant C that depends on G,ω and h, but not on y. We conclude (4.4) in view of (4.5), (4.6) and (4.9).

2. To prove (4.4) in case $N > 1$, let $k, s : 1\ldots N$ be such that $\omega_k\cap\omega_s \neq \emptyset$. Define:

$$F = \Big(\int_{\Omega_k^h\cap\Omega_s^h}\det\nabla u_0\,\mathrm{d}x\Big)^{-1}\int_{\Omega_k^h\cap\Omega_s^h}(\det\nabla u_0)(\nabla u)(\nabla u_0)^{-1}\,\mathrm{d}x \in \mathbb{R}^{3\times 3}.$$

Denote by $R_k, R_s \in SO(3)$ the corresponding rotations in (4.4) on ω_k, ω_s. For $i \in \{k, s\}$ we have:

$$\begin{aligned}|F - R_i|^2 &= \Big|\Big(\int_{\Omega_k^h\cap\Omega_s^h} \det\nabla u_0\,\mathrm{d}x\Big)^{-1}\int_{\Omega_k^h\cap\Omega_s^h}(\det\nabla u_0)\big(\nabla u - R_i\nabla u_0\big)(\nabla u_0)^{-1}\,\mathrm{d}x\Big|^2\\ &\leq C\int_{\Omega_k^h\cap\Omega_s^h}|\nabla u - R_i\nabla u_0|^2\,\mathrm{d}x \leq C\int_{\Omega_i^h}|\nabla u - R_i\nabla u_0|^2\,\mathrm{d}x\\ &\leq \int_{\omega_i}\Big|\big((\nabla y)^{\mathsf{T}}\nabla\vec b\big)_{\mathrm{sym}} - \frac{1}{2}\partial_3 G(x',0)_{2\times 2}\Big|^2\,\mathrm{d}x',\end{aligned}$$

where for the sake of the last bound we applied the intermediate estimate in (4.6) to the left hand side of (4.5), as discussed in the previous step. Consequently:

$$|R_k - R_s|^2 \leq C\int_{\omega}\Big|\big((\nabla y)^{\mathsf{T}}\nabla\vec b\big)_{\mathrm{sym}} - \frac{1}{2}\partial_3 G(x',0)_{2\times 2}\Big|^2\,\mathrm{d}x',$$

and thus:

$$\begin{aligned}\int_{\omega_k}|\nabla y - R_s\nabla y_0|^2\,\mathrm{d}x' &\leq 2\Big(\int_{\omega_k}|\nabla y - R_k\nabla y_0|^2\,\mathrm{d}x' + \int_{\omega_k}|R_k - R_s|^2|\nabla y_0|^2\,\mathrm{d}x'\Big)\\ &\leq C\int_{\omega}\Big|\big((\nabla y)^{\mathsf{T}}\nabla\vec b\big)_{\mathrm{sym}} - \frac{1}{2}\partial_3 G(x',0)_{2\times 2}\Big|^2\,\mathrm{d}x'.\end{aligned}$$

This shows that one can take one and the same $R = R_1$ on each $\{\omega_k\}_{k=1}^N$, at the expense of possibly increasing the constant C by a controlled factor depending only on N. The proof of (4.4) is done. □

Remark 4.3. A similar reasoning as in the proof of Lemma 4.2, yields a quantitative version of the uniqueness of isometric immersion with a prescribed second fundamental form compatible to the metric by the Gauss-Codazzi-Meinardi equations. More precisely, given a smooth metric g in $\omega \subset \mathbb{R}^2$, for every two isometric immersions $y_1, y_2 \in W^{2,2}(\omega, \mathbb{R}^3)$ of g, there holds:

$$\min_{R\in SO(3)}\int_{\omega}|\nabla y_1 - R\nabla y_2|^2\,\mathrm{d}x' \leq C\int_{\omega}|\Pi_{y_1} - \Pi_{y_2}|^2\,\mathrm{d}x',$$

with a constant $C > 0$, depending on g and ω but independent of y_1 and y_2. ■

5. Higher order energy scalings

In this and the next sections we assume that:

$$\lim_{h\to 0}\frac{1}{h^2}\inf\mathcal{E}^h = 0. \tag{5.1}$$

Recall that by Theorem 3.2 this condition is equivalent to the existence of a (automatically smooth and unique up to rigid motions) vector field $y_0 : \bar\omega \to \mathbb{R}^3$ satisfying:

$$(\nabla y_0)^{\mathsf{T}}\nabla y_0 = \bar{\mathcal{G}}_{2\times 2} \qquad \text{and} \qquad \big((\nabla y_0)^{\mathsf{T}}\nabla\vec b_0\big)_{\mathrm{sym}} = \frac{1}{2}(\bar{\mathcal{G}}_1)_{2\times 2} \quad \text{on } \omega, \tag{5.2}$$

where in the oscillatory case (O) the symmetric x'-dependent matrix $\mathcal{G}_1$ is given in (EF) and there must be $(\mathcal{G}_1)_{2\times 2} = x_3(\bar{\mathcal{G}}_1)_{2\times 2}$, whereas in the non-oscillatory (NO) case $\bar{\mathcal{G}}_1(x')$ is simply $\partial_3 G(x',0)$. The (smooth) Cosserat field $\vec b_0 : \bar\omega \to \mathbb{R}^3$ in (2.4) is uniquely given by requesting that:

$$Q_0 \doteq \Big[\partial_1 y_0,\ \partial_2 y_0,\ \vec b_0\Big] \qquad \text{satisfies:} \qquad Q_0^{\mathsf{T}}Q_0 = \bar{\mathcal{G}}, \quad \det Q_0 > 0 \quad \text{on } \omega,$$

with notation similar to (2.15). We now introduce the new vector field $\vec{d}_0 : \bar{\Omega} \to \mathbb{R}^3$ through:

$$Q_0^{\mathsf{T}} \Big[x_3\partial_1\vec{b}_0(x'),\ x_3\partial_2\vec{b}_0(x'),\ \partial_3\vec{d}_0(x',x_3)\Big] - \frac{1}{2}\mathcal{G}_1(x',x_3) \in so(3), \tag{5.3}$$

justified by (5.2) and in agreement with the construction (2.16) of second order terms in the recovery sequence for the Kirchhoff limiting energies. Explicitly, we have:

$$\vec{d}_0(x',x_3) = Q_0(x')^{\mathsf{T},-1}\Big(\int_0^{x_3} \mathcal{G}_1(x',t)\,\mathrm{d}t\, e_3 - \frac{1}{2}\int_0^{x_3} \mathcal{G}_1(x',t)_{33}\,\mathrm{d}t\, e_3 - \frac{x_3^2}{2}\begin{bmatrix} (\nabla\vec{b}_0)^{\mathsf{T}}\vec{b}_0(x') \\ 0 \end{bmatrix}\Big).$$

In what follows, the smooth matrix field in (5.3) will be referred to as $P_0 : \bar{\Omega} \to \mathbb{R}^{3\times 3}$, namely:

$$P_0(x',x_3) = \Big[x_3\partial_1\vec{b}_0(x'),\ x_3\partial_2\vec{b}_0(x'),\ \partial_3\vec{d}_0(x',x_3)\Big]. \tag{5.4}$$

In the non-oscillatory case (NO), the above formulas become:

$$\begin{aligned} &\vec{d}_0 = \frac{x_3^2}{2}\tilde{d}_0(x'), \qquad P_0(x',x_3) = x_3\Big[\partial_1\vec{b}_0,\ \partial_2\vec{b}_0,\ \tilde{d}_0\Big](x'), \\ &\text{where:}\quad \tilde{d}_0(x') = Q_0(x')^{\mathsf{T},-1}\Big(\partial_3 G(x',0)e_3 - \frac{1}{2}\partial_3 G(x',0)_{33}e_3 - \begin{bmatrix} (\nabla\vec{b}_0)^{\mathsf{T}}\vec{b}_0(x') \\ 0 \end{bmatrix}\Big). \end{aligned} \tag{5.5}$$

We also note that the assumption $\int_{-1/2}^{1/2} \mathcal{G}_1(x',t)\,\mathrm{d}t = 0$ implies:

$$\int_{-1/2}^{1/2} P_0(x',x_3)\,\mathrm{d}x_3 = 0 \qquad \text{for all } x' \in \bar{\omega}. \tag{5.6}$$

With the aid of $\vec{d}_0$ we now construct the sequence of deformations with low energy:

Lemma 5.1. *Assume (O). Then (5.1) implies:*

$$\inf \mathcal{E}^h \leq Ch^4.$$

Proof. Define the sequence of smooth maps $\{u^h : \bar{\Omega}^h \to \mathbb{R}^3\}_{h\to 0}$ by:

$$u^h(x',x_3) = y_0(x') + x_3\vec{b}_0(x') + h^2\vec{d}_0\big(x', \frac{x_3}{h}\big) \tag{5.7}$$

In order to compute $\nabla u^h (A^h)^{-1}$, recall the expansion of $(A^h)^{-1}$, so that:

$$\nabla u^h(x)A^h(x)^{-1} = Q_0(x')\bar{A}(x')^{-1}\big(Id_3 + hS^h(x) + \mathcal{O}(h^2)\big), \tag{5.8}$$

where for every $x = (x',x_3) \in \Omega^h$:

$$S^h(x) = \bar{A}(x')^{-1}\Big(Q_0(x')^{\mathsf{T}}P_0\big(x', \frac{x_3}{h}\big) - \bar{A}(x')A_1\big(x', \frac{x_3}{h}\big)\Big)\bar{A}(x')^{-1}.$$

By frame invariance of the energy density W and since $Q_0(x')\bar{A}(x')^{-1} \in SO(3)$, we obtain:

$$\begin{aligned} W\big(\nabla u^h(x)A^h(x)^{-1}\big) &= W\big(Id_3 + hS^h(x) + \mathcal{O}(h^2)\big) = W\big(Id_3 + hS^h(x)_{\mathrm{sym}} + \mathcal{O}(h^2)\big) \\ &= W\big(Id_3 + \mathcal{O}(h^2)\big) = \mathcal{O}(h^4), \end{aligned}$$

where we also used the fact that $S^h(x)_{\mathrm{sym}} = 0$ following directly from the definition (5.3). This implies that $\mathcal{E}^h(u^h) = \mathcal{O}(h^4)$ as well, proving the claim. □

Lemma 5.2. *Assume (O) and (5.1). For an open, Lipschitz subset $\mathcal{V}\subset\omega$, denote:*

$$\mathcal{V}^h=\mathcal{V}\times\big(-\frac{h}{2},\frac{h}{2}\big),\qquad \mathcal{E}^h(u^h,\mathcal{V}^h)=\frac{1}{h}\int_{\mathcal{V}^h}W\big(\nabla u^h(A^h)^{-1}\big)\,\mathrm{d}x.$$

If y_0 is injective on $\mathcal{V}$, then for every $u^h\in W^{1,2}(\mathcal{V}^h,\mathbb{R}^3)$ there exists $\bar R^h\in SO(3)$ such that:

$$\frac{1}{h}\int_{\mathcal{V}^h}\Big|\nabla u^h(x)-\bar R^h\Big(Q_0(x')+hP_0\big(x',\frac{x_3}{h}\big)\Big)\Big|^2\,\mathrm{d}x\leq C\big(\mathcal{E}^h(u^h,\mathcal{V}^h)+h^3|\mathcal{V}^h|\big),\tag{5.9}$$

with the smooth correction matrix field P_0 in (5.4). The constant C in (5.9) is uniform for all subdomains $\mathcal{V}^h\subset\Omega^h$ which are bi-Lipschitz equivalent with controlled Lipschitz constants.

Proof. The proof, similar to [39, Lemma 2.2], is a combination of the change of variable argument in Lemma 4.2 and the low energy deformation construction in Lemma 5.1. Observe first that:

$$Q_0(x')+hP_0\big(x',\frac{x_3}{h}\big)=\nabla Y^h(x',x_3)+\mathcal{O}(h^2),$$

where by $Y^h:\bar\Omega^h\to\mathbb{R}^3$ we denote the smooth vector fields in (5.7). It is clear that for sufficiently small $h>0$, each $Y^h_{|\mathcal{V}^h}$ is a smooth diffeomorphism onto its image $\mathcal{U}^h\subset\mathbb{R}^3$, satisfying uniformly: $\det\nabla Y^h>c>0$. We now consider $v^h=u^h\circ(Y^h)^{-1}\in W^{1,2}(\mathcal{U}^h,\mathbb{R}^3)$. By the rigidity estimate [**?**]:

$$\int_{\mathcal{U}^h}|\nabla v^h-\bar R^h|^2\,\mathrm{d}z\leq C\int_{\mathcal{U}^h}\mathrm{dist}^2\big(\nabla v^h,SO(3)\big)\,\mathrm{d}z,\tag{5.10}$$

for some rotation $\bar R^h\in SO(3)$. Noting that: $(\nabla v^h)\circ Y^h=(\nabla u^h)(\nabla Y^h)^{-1}$ in the set $\mathcal{V}^h$, the change of variable formula yields for the left hand side in (5.10):

$$\begin{aligned}\int_{\mathcal{U}^h}|\nabla v^h-\bar R^h|^2\,\mathrm{d}z&=\int_{\mathcal{V}^h}(\det\nabla Y^h)\big|(\nabla u^h)(\nabla Y^h)^{-1}-\bar R^h\big|^2\,\mathrm{d}x\\&\geq c\int_{\mathcal{V}^h}\Big|\nabla u^h-\bar R^h\Big(Q_0(x')+hP_0\big(x',\frac{x_3}{h}\big)+\mathcal{O}(h^2)\Big)\Big|^2\,\mathrm{d}x\\&\geq c\int_{\mathcal{V}^h}\Big|\nabla u^h-\bar R^h\Big(Q_0(x')+hP_0\big(x',\frac{x_3}{h}\big)\Big)\Big|^2\,\mathrm{d}x-c\int_{\mathcal{V}^h}\mathcal{O}(h^4)\,\mathrm{d}x.\end{aligned}$$

Similarly, the right hand side in (5.10) can be estimated by:

$$\begin{aligned}\int_{\mathcal{U}^h}\mathrm{dist}^2\big(\nabla v^h,SO(3)\big)\,\mathrm{d}z&=\int_{\mathcal{V}^h}(\det\nabla Y^h)\,\mathrm{dist}^2\big((\nabla u^h)(\nabla Y^h)^{-1},SO(3)\big)\,\mathrm{d}x\\&\leq C\int_{\mathcal{V}^h}\mathrm{dist}^2\big((\nabla u^h)(A^h)^{-1}\cdot A^h(\nabla Y^h)^{-1},SO(3)\big)\,\mathrm{d}x\\&\leq C\int_{\mathcal{V}^h}\mathrm{dist}^2\Big((\nabla u^h)(A^h)^{-1},SO(3)(\nabla Y^h)(A^h)^{-1}\Big)\,\mathrm{d}x.\end{aligned}$$

Recall that from (5.8) we have: $(\nabla Y^h)(A^h)^{-1}\in SO(3)\big(Id_3+hS^h+\mathcal{O}(h^2)\big)\subset SO(3)\big(Id_3+\mathcal{O}(h^2)\big)$, since $S^h\in so(3)$. Consequently, the above bound becomes:

$$\begin{aligned}\int_{\mathcal{U}^h}\mathrm{dist}^2\big(\nabla v^h,SO(3)\big)\,\mathrm{d}z&\leq C\int_{\mathcal{V}^h}\mathrm{dist}^2\Big((\nabla u^h)(A^h)^{-1},SO(3)(Id_3+\mathcal{O}(h^2))\Big)\,\mathrm{d}x\\&\leq C\int_{\mathcal{V}^h}\mathrm{dist}^2\Big((\nabla u^h)(A^h)^{-1},SO(3)\Big)+\mathcal{O}(h^4)\,\mathrm{d}x.\end{aligned}$$

The estimate (5.9) follows now in view of (5.10) and by the lower bound on energy density W. □

The well-known approximation technique [20] combined with the arguments in [39, Corollary 2.3], yield the following approximation result that can be seen as a higher order counterpart of (2.10):

Corollary 5.3. *Assume (O) and (5.1). Then, for any sequence* $\{u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)\}_{h\to 0}$ *satisfying:* $\mathcal{E}^h(u^h) \leq Ch^4$, *there exists a sequence of rotation-valued maps* $R^h \in W^{1,2}(\omega, SO(3))$, *such that with* P_0 *defined in (5.4) we have:*

$$\begin{aligned} &\frac{1}{h}\int_{\Omega^h} \Big|\nabla u^h(x) - R^h(x')\Big(Q_0(x') + hP_0\big(x', \frac{x_3}{h}\big)\Big)\Big|^2 \,\mathrm{d}x \leq Ch^4, \\ &\int_\omega |\nabla R^h(x')|^2 \,\mathrm{d}x' \leq Ch^2. \end{aligned} \tag{5.11}$$

6. Compactness and Γ-limit under Ch^4 energy bound

In this section, we derive the Γ-convergence result for the energy functionals $\mathcal{E}^h$ in the von Kármán scaling regime.The general form of the limiting energy $\mathcal{I}_4^O$ will be further discussed and split into the stretching, bending, curvature and excess components in section 7. We begin by stating the compactness result, that is the higher order version of Theorem 2.1.

Theorem 6.1. *Assume (O) and (5.1). Fix* y_0 *solving (5.2) and normalize it to have:* $\int_\omega y_0 \,\mathrm{d}x' = 0$. *Then, for any sequence of deformations* $\{u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)\}_{h\to 0}$ *satisfying:*

$$\mathcal{E}^h(u^h) \leq Ch^4, \tag{6.1}$$

there exists a sequence $\{\bar{R}^h \in SO(3)\}_{h\to 0}$ *such that the following convergences (up to a subsequence) below, hold for* $y^h \in W^{1,2}(\Omega, \mathbb{R}^3)$:

$$y^h(x', x_3) = (\bar{R}^h)^{\mathsf{T}}\Big(u^h(x', hx_3) - \fint_{\Omega^h} u^h \,\mathrm{d}x\Big).$$

(i) $y^h \to y_0$ *strongly in* $W^{1,2}(\Omega, \mathbb{R}^3)$ *and* $\frac{1}{h}\partial_3 y^h \to \vec{b}_0$ *strongly in* $L^2(\Omega, \mathbb{R}^3)$, *as* $h \to 0$.

(ii) There exists $V \in W^{2,2}(\omega, \mathbb{R}^3)$ *and* $\mathbb{S} \in L^2(\omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ *such that, as* $h \to 0$:

$$V^h(x') = \frac{1}{h}\int_{-1/2}^{1/2} y^h(x', x_3) - \big(y_0(x') + hx_3\vec{b}_0(x')\big)\,\mathrm{d}x_3 \to V \quad \textit{strongly in } W^{1,2}(\omega, \mathbb{R}^3)$$

$$\frac{1}{h}\big((\nabla y_0)^{\mathsf{T}}\nabla V^h\big)_{\mathrm{sym}} \rightharpoonup \mathbb{S} \quad \textit{weakly in } L^2(\omega, \mathbb{R}^{2\times 2}).$$

(iii) The limiting displacement V *satisfies:* $\big((\nabla y_0)^{\mathsf{T}}\nabla V\big)_{\mathrm{sym}} = 0$ *in* ω.

We omit the proof because it follows as in [39, Theorem 3.1] in view of condition (5.6). We only recall the definitions used in the sequel. The rotations $\bar{R}^h$ are given by:

$$\bar{R}^h = \mathbb{P}_{SO(3)} \fint_{\Omega^h} \nabla u^h(x) Q_0(x')^{-1} \,\mathrm{d}x$$

and (5.11) implies that they satisfy, for some limiting rotation $\bar{R}$:

$$\int_\omega |R^h(x') - \bar{R}^h|^2 \,\mathrm{d}x' \leq Ch^2 \qquad \text{and} \qquad \bar{R}^h \to \bar{R} \in SO(3). \tag{6.2}$$

Consequently:

$$S^h = \frac{1}{h}\big((\bar{R}^h)^{\mathsf{T}} R^h(x') - Id_3\big) \rightharpoonup S \qquad \text{weakly in } \; W^{1,2}(\omega, \mathbb{R}^{3\times 3}) \tag{6.3}$$

The field $S \in W^{1,2}(\omega, so(3))$ is such that $(\nabla y_0)^\mathsf{T}\nabla V = \big(Q_0^\mathsf{T} S Q_0\big)_{2\times 2} \in so(2)$, which allows for defining a new vector field $\vec p \in W^{1,2}(\omega, \mathbb{R}^3)$ through:

$$(6.4)\quad [\nabla V,\ \vec p] = SQ_0 \quad \text{or equivalently: } \ \vec p(x') = -Q_0(x')^{\mathsf{T},-1}\left[\begin{array}{c} \nabla V(x')^\mathsf{T}\vec b_0(x') \\ 0 \end{array}\right] \quad \text{for all } x' \in \omega.$$

Finally, by (5.11) we note the uniform boundedness of the fields $\{Z^h \in L^2(\Omega, \mathbb{R}^{3\times 3})\}_{h\to 0}$ below, together with their convergence (up to as subsequence) as $h \to 0$:

$$(6.5)\quad Z^h(x) = \frac{1}{h^2}\Big(\nabla u^h(x', hx_3) - R^h(x')\big(Q_0(x') + hP_0(x', x_3)\big)\Big) \rightharpoonup Z \qquad \text{weakly in } L^2(\Omega, \mathbb{R}^{3\times 3}).$$

Rearranging terms and using the previously established convergences, it can be shown that:

$$(6.6)\qquad \mathbb{S}(x') = \left(Q_0^\mathsf{T}(x')\bar R^\mathsf{T}\int_{-1/2}^{1/2} Z(x', x_3)\,\mathrm{d}x_3\right)_{2\times 2,\mathrm{sym}} - \frac{1}{2}\nabla V(x')^\mathsf{T}\nabla V(x') \qquad \text{for all } x' \in \omega.$$

Theorem 6.2. *In the setting of Theorem 6.1,* $\liminf_{h\to 0}\frac{1}{h^4}\mathcal{E}^h(u^h)$ *is bounded below by:*

$$\mathcal{I}_4^O(V, \mathbb{S}) = \frac{1}{2}\int_\Omega \mathcal{Q}_2\Big(x', I(x') + x_3 III(x') + II(x)\Big)\,\mathrm{d}x = \frac{1}{2}\|I + x_3 III + II\|^2_{\mathcal{Q}_2},$$

where:

$$(6.7)\qquad \begin{aligned} I(x') &= \mathbb{S}(x') + \frac{1}{2}\nabla V(x')^\mathsf{T}\nabla V(x') - \nabla y_0(x')^\mathsf{T}\nabla\int_{-1/2}^{1/2}\vec d_0(x)\,\mathrm{d}x_3, \\ III(x') &= \nabla y_0(x')^\mathsf{T}\nabla\vec p(x') + \nabla V(x')^\mathsf{T}\nabla\vec b_0, \\ II(x) &= \frac{x_3^2}{2}\nabla\vec b_0(x')^\mathsf{T}\nabla\vec b_0(x') + \nabla y_0(x')^\mathsf{T}\nabla_{\tan}\vec d_0(x) - \frac{1}{4}\mathcal{G}_2(x)_{2\times 2}. \end{aligned}$$

Proof. **1.** Towards estimating the energy $\mathcal{E}^h(u^h)$, we replace the argument $\nabla u^h(x)A^h(x)^{-1}$ of the frame invariant density W by:

$$(6.8)\qquad \begin{aligned} &\big(Q_0\bar A^{-1}\big)(x')^\mathsf{T}R^h(x')^\mathsf{T}\nabla u^h(x)A^h(x)^{-1} \\ &\qquad = \big(Q_0\bar A^{-1}\big)(x')^\mathsf{T}Q_0(x')A^h(x)^{-1} + h\bar A(x')^{-1}Q_0(x')^\mathsf{T}P_0\big(x', \frac{x_3}{h}\big)A^h(x)^{-1} \\ &\qquad\quad + h^2 I_3^h\big(x', \frac{x_3}{h}\big) \qquad \text{for all } x \in \Omega^h, \end{aligned}$$

where I_3^h is given in (6.11). Calculating the higher order expansion of (2.12):

$$(6.9)\qquad \begin{aligned} A^h(x)^{-1} = \bar A(x')^{-1} + \bar A(x')^{-1}\Big(&-hA_1\big(x', \frac{x_3}{h}\big) + h^2 A_1\big(x', \frac{x_3}{h}\big)\bar A(x')^{-1}A_1\big(x', \frac{x_3}{h}\big) \\ &- \frac{h^2}{2}A_2\big(x', \frac{x_3}{h}\big)\Big)\bar A(x')^{-1} + o(h^2), \end{aligned}$$

the expressions in (6.8) can be written as:

$$(6.10)\qquad \begin{aligned} &\big(Q_0\bar A^{-1}\big)(x')^\mathsf{T}R^h(x')^\mathsf{T}\nabla u^h(x', hx_3)A^h(x', hx_3)^{-1} \\ &\qquad = Id_3 + hI_1(x', x_3) + h^2\Big(I_2(x', x_3) + I_3^h(x', x_3)\Big) + o(h^2) \qquad \text{for all } x \in \Omega, \end{aligned}$$

where $I_1 : \Omega \to so(3)$ and $I_2 : \Omega \to \mathbb{R}^{3\times 3}$ are smooth matrix fields, given by:

$$I_1(x) = \bar{A}(x')^{-1}\Big(Q_0(x')^{\mathsf{T}}P_0(x) - \bar{A}(x')A_1(x)\Big)\bar{A}(x')^{-1}$$
$$I_2(x) = \bar{A}(x')^{-1}\Big(\bar{A}(x')A_1(x)\bar{A}(x')^{-1}A_1(x) - \frac{1}{2}\bar{A}(x')A_2(x) - Q_0(x')^{\mathsf{T}}P_0(x)\bar{A}(x')^{-1}A_1(x)\Big)\bar{A}(x')^{-1}.$$

The fact that $I_1(x) \in so(3)$ follows from (5.3). Also, we have:

$$(6.11)\qquad \begin{aligned} I_3^h(x) = {}& \bar{A}(x')^{-1}Q_0(x')^{\mathsf{T}}R^h(x')^{\mathsf{T}}Z^h(x)A^h(x', hx_3)^{-1} \\ & \rightharpoonup I_3(x) = \bar{A}(x')^{-1}Q_0(x')^{\mathsf{T}}\bar{R}^{\mathsf{T}}Z(x)\bar{A}(x)^{-1} \qquad \text{weakly in } L^2(\Omega, \mathbb{R}^{3\times 3}), \end{aligned}$$

where we used (6.8) and (6.2) to pass to the limit with $(R^h)^{\mathsf{T}}$. As in the proof of Theorem 2.3, we now identify the "good" sets $\{|I_3^h|^2 \leq 1/h\} \subset \Omega$ and employ (6.10) to write there the following Taylor's expansion of $W(\nabla u^h (A^h)^{-1})$:

$$(6.12)\qquad \begin{aligned} W\Big(\nabla u^h(x', hx_3)A^h(x', hx_3)^{-1}\Big) &= W\Big(Id_3 + hI_1(x) + h^2(I_2(x) + I_3^h(x)) + o(h^2)\Big) \\ &= W\Big(e^{-hI_1(x)}\big(Id_3 + hI_1(x) + h^2(I_2(x) + I_3^h(x))\big) + o(h^2)\Big) \\ &= W\Big(Id_3 + h^2\big(I_2 - \frac{1}{2}I_1^2 + I_3^h\big) + o(h^2)\Big) \\ &= \frac{h^4}{2}\mathcal{Q}_3\Big(\big(I_2 - \frac{1}{2}I_1^2 + I_3^h\big)_{\mathrm{sym}}\Big) + o(h^4). \end{aligned}$$

Above, we repeatedly used the frame invariance of W and the exponential formula:

$$e^{-hI_1} = Id_3 - hI_1 + \frac{h^2}{2}I_1^2 + \mathcal{O}(h^3).$$

Since the weak convergence in (6.11) implies convergence of measures $\big||I_3^h|^2 \leq 1/h\big| \to 0$ as $h \to 0$, with the help of (6.12) we finally arrive at:

$$(6.13)\qquad \begin{aligned} \liminf_{h\to 0} \frac{1}{h^4}\frac{1}{h}\int_{\Omega^h} W\big(\nabla u^h(x)A^h(x)^{-1}\big)\,\mathrm{d}x &\geq \liminf_{h\to 0} \frac{1}{2}\int_{|I_3^h|^2\leq 1/h} \mathcal{Q}_3\big(I_2 - \frac{1}{2}I_1^2 + I_3^h\big)\,\mathrm{d}x \\ \geq \frac{1}{2}\int_{\Omega} \mathcal{Q}_3\big(I_2 - \frac{1}{2}I_1^2 + I_3\big)\,\mathrm{d}x &= \frac{1}{2}\int_{\Omega} \mathcal{Q}_2\Big(x', \big(\bar{A}(I_2 - \frac{1}{2}I_1^2 + I_3)\bar{A}\big)_{2\times 2}\Big)\,\mathrm{d}x. \end{aligned}$$

2. We now compute the effective integrand in (6.13). Firstly, by (2.1) a direct calculation yields:

$$(6.14)\qquad \big(I_2(x) - \frac{1}{2}I_1(x)^2\big)_{\mathrm{sym}} = \big(I_2\big)_{\mathrm{sym}} + \frac{1}{2}I_1^{\mathsf{T}}I_1 = \frac{1}{2}\bar{A}(x')^{-1}\Big(P_0(x)^{\mathsf{T}}P_0(x) - \frac{1}{2}\mathcal{G}_2(x)\Big)\bar{A}(x')^{-1}$$

Secondly, to address the symmetric part of the limit I_3 in (6.11), consider functions $f^{s,h} : \Omega \to \mathbb{R}^3$:

$$f^{s,h}(x) = \fint_0^s \Big(h(\bar{R}^h)^{\mathsf{T}}Z^h(x', x_3 + t) + S^h(x')\big(Q_0(x') + hP_0(x', x_3 + t)\big)\Big)e_3\,\mathrm{d}t.$$

By (6.3) it easily follows that:

$$(6.15)\qquad f^{s,h} \to S\vec{b}_0 = \vec{p} \quad \text{and} \quad \partial_3 f^{s,h} \to 0 \qquad \text{strongly in } L^2(\Omega, \mathbb{R}^3), \quad \text{as } h \to 0.$$

On the other hand, we write an equivalent form of $f^{s,h}$ and compute the tangential derivatives:

$$\begin{aligned} f^{s,h}(x) &= \frac{1}{h^2 s}\big(y^h(x',x_3+s) - y^h(x',x_3)\big) - \frac{1}{h}\vec{b}_0(x') - \frac{1}{s}\Big(\vec{d}_0(x',x_3+s) - \vec{d}_0(x',x_3)\Big), \\ \partial_i f^{s,h}(x) &= \ \frac{1}{s}(\bar{R}^h)^{\mathsf{T}}\big(Z^h(x',x_3+s) - Z^h(x',x_3)\big)e_i + S^h(x')\partial_i\vec{b}_0(x') \\ & \qquad\qquad\qquad\qquad\qquad\qquad - \frac{1}{s}\Big(\partial_i\vec{d}_0(x',x_3+s) - \partial_i\vec{d}_0(x',x_3)\Big) \end{aligned}$$

for $i = 1,2$. In view of (6.2) and (6.3), convergence in (6.15) can thus be improved to: $f^{sh} \rightharpoonup \vec{p}$ weakly in $W^{1,2}(\Omega,\mathbb{R}^3)$ as $h \to 0$. Equating the tangential derivatives ∂_1, ∂_2, results in:

$$\bar{R}^{\mathsf{T}}\big(Z(x',x_3) - Z(x',0)\big)e_i = x_3\big(\partial_i\vec{p}(x') - S(x')\partial_i\vec{b}_0(x')\big) + \partial_i\vec{d}_0(x',x_3) - \partial_i\vec{d}_0(x',0).$$

Further, by (6.11), (6.4) and since $S \in so(3)$, it follows that:

$$\begin{aligned} \big(\bar{A}(x')I_3(x)\bar{A}(x')\big)_{2\times2,\mathrm{sym}} &= \Big(Q_0(x')^{\mathsf{T}}\bar{R}^{\mathsf{T}}Z(x)\Big)_{2\times2,\mathrm{sym}} \\ &= \Big(Q_0(x')^{\mathsf{T}}\bar{R}^{\mathsf{T}}Z(x',0)\Big)_{2\times2,\mathrm{sym}} + x_3\Big(\nabla y_0(x')^{\mathsf{T}}\nabla\vec{p}(x') + \nabla V(x')^{\mathsf{T}}\nabla\vec{b}_0\Big)_{\mathrm{sym}} \\ & \qquad + \Big(Q_0(x')^{\mathsf{T}}\nabla\vec{d}_0(x)\Big)_{2\times2,\mathrm{sym}} \end{aligned} \tag{6.16}$$

On the other hand, taking the x_3-average and recalling (6.6), we get:

$$\Big(Q_0(x')^{\mathsf{T}}\bar{R}^{\mathsf{T}}Z(x',0)\Big)_{2\times2,\mathrm{sym}} = \mathbb{S}(x') + \frac{1}{2}\nabla V(x')^{\mathsf{T}}\nabla V(x') - \Big(\nabla y_0(x')^{\mathsf{T}}\nabla\int_{-1/2}^{1/2}\vec{d}_0(x)\,\mathrm{d}x_3\Big)_{\mathrm{sym}} \tag{6.17}$$

3. We now finish the proof of Theorem 6.2. Combining (6.14), (6.16) and (6.17), we see that:

$$\Big(\bar{A}(x')(I_2 - \frac{1}{2}I_1^2 + I_3)\bar{A}(x')\Big)_{2\times2,\mathrm{sym}} = \Big(I(x') + x_3 III(x') + II(x)\Big)_{\mathrm{sym}} \qquad \text{on } \ \Omega,$$

where I, II, III are as in (6.7). In virtue of (6.13), we obtain:

$$\liminf_{h\to0}\frac{1}{h^4}\frac{1}{h}\int_{\Omega^h} W\big(\nabla u^h(x)A^h(x)^{-1}\big)\,\mathrm{d}x \geq \frac{1}{2}\int_\Omega \mathcal{Q}_2\Big(x', I(x') + x_3 III(x') + II(x)\Big)\,\mathrm{d}x.$$

This yields the claimed lower bound by $\mathcal{I}_4^O(V,\mathbb{S})$. □

For the upper bound statement, define the linear spaces:

$$\begin{aligned} \mathscr{V} &= \Big\{V \in W^{2,2}(\omega,\mathbb{R}^3);\ \big(\nabla y_0(x')^{\mathsf{T}}\nabla V(x')\big)_{\mathrm{sym}} = 0 \ \text{ for all } x' \in \omega\Big\}, \\ \mathscr{S} &= \mathrm{cl}_{L^2(\omega,\mathbb{R}^{2\times2})}\Big\{\big((\nabla y_0)^{\mathsf{T}}\nabla w\big)_{\mathrm{sym}};\ w \in W^{1,2}(\omega,\mathbb{R}^3)\Big\}. \end{aligned} \tag{6.18}$$

We see that the limiting quantities V and $\mathbb{S}$ in Theorem 6.2 satisfy: $V \in \mathscr{V}$, $\mathbb{S} \in \mathscr{S}$. The space $\mathscr{V}$ consists of the first order infinitesimal isometries on the smooth minimizing immersion surface $y_0(\omega)$, i.e. those Sobolev-regular displacements V that preserve the metric on $y_0(\omega)$ up to first order. The tensor fields $\mathbb{S} \in \mathscr{S}$ are the finite strains on $y_0(\omega)$, eventually forcing the stretching term in the von Kármán energy $\mathcal{I}_4^O$ to be of second order.

Theorem 6.3. *Assume that y_0 solves (5.2). Then, for every $(V,\mathbb{S})\in\mathscr{V}\times\mathscr{S}$ there exists a sequence $\{u^h\in W^{1,2}(\Omega^h,\mathbb{R}^3)\}_{h\to 0}$ such that the rescaled sequence $\{y^h(x',x_3)=u^h(x',hx_3)\}_{h\to 0}$ satisfies (i) and (ii) of Theorem 6.1, together with:*

$$\lim_{h\to 0}\frac{1}{h^4}\mathcal{E}^h(u^h)=\mathcal{I}_4^O(V,\mathbb{S}). \tag{6.19}$$

Proof. **1.** Given admissible V and $\mathbb{S}$, we first define the ε-recovery sequence $\{u^h\in W^{1,\infty}(\Omega^h,\mathbb{R}^3)\}_{h\to 0}$. The ultimate argument for (6.19) will be obtained via a diagonal argument. We set:

$$\begin{split}u^h(x',x_3)=\,&y_0(x')+hv^h(x')+h^2w^h(x')+x_3\vec b_0(x')+h^2\vec d_0\big(x',\frac{x_3}{h}\big)\\&+h^3\vec k_0\big(x',\frac{x_3}{h}\big)+hx_3\vec p^{\,h}(x')+h^2x_3\vec q^{\,h}(x')+h^3\vec r^{\,h}\big(x',\frac{x_3}{h}\big)\qquad\text{for all }\ (x',x_3)\in\Omega^h.\end{split}$$

The smooth vector fields $\vec b_0$ and $\vec d_0$ are as in (5.2), (5.3). We now introduce other terms in the above expansion. The sequence $\{w^h\in\mathcal{C}^\infty(\overline{\omega},\mathbb{R}^3)\}_{h\to 0}$ is such that:

$$\begin{split}&\Big((\nabla y_0)^{\mathsf T}\nabla\big(w^h+\int_{-1/2}^{1/2}\vec d_0(\cdot,t)\,\mathrm{d}t\big)\Big)_{\rm sym}\to\mathbb{S}\quad\text{strongly in }\ L^2(\omega,\mathbb{R}^{2\times 2})\quad\text{as }h\to 0,\\&\lim_{h\to 0}\sqrt{h}\|w^h\|_{W^{2,\infty}(\omega,\mathbb{R}^3)}=0.\end{split}\tag{6.20}$$

Existence of such a sequence is guaranteed by the fact that $\mathbb{S}\in\mathscr{S}$, where we "slow down" the approximations $\{w^h\}$ to guarantee the blow-up rate of order less that $h^{-1/2}$. Further, for a fixed small $\varepsilon>0$, the truncated sequence $\{v^h\in W^{2,\infty}(\omega,\mathbb{R}^3)\}_{h\to 0}$ is chosen according to the standard construction in [20] (see also references therein), in a way that:

$$\begin{split}&v^h\to V\quad\text{strongly in }\ W^{2,2}(\omega,\mathbb{R}^3)\quad\text{as }\ h\to 0,\\&h\|v^h\|_{W^{2,\infty}(\omega,\mathbb{R}^3)}\leq\varepsilon\quad\text{ and }\quad\lim_{h\to 0}\frac{1}{h^2}\big|\{x'\in\omega;\ v^h(x')\neq V(x')\}\big|=0.\end{split}\tag{6.21}$$

The vector field $\vec k_0\in\mathcal{C}^\infty(\bar\Omega,\mathbb{R}^3)$ and sequences $\{\vec p^{\,h},\vec q^{\,h}\in W^{1,\infty}(\omega,\mathbb{R}^3)\}_{h\to 0}$, $\{r^h\in L^\infty(\Omega,\mathbb{R}^3)\}_{h\to 0}$ are defined by:

$$\begin{split}Q_0^{\mathsf T}\vec p^{\,h}=&\begin{bmatrix}-(\nabla v^h)^{\mathsf T}\vec b_0\\0\end{bmatrix},\\Q_0^{\mathsf T}\vec q^{\,h}=&\,c\Big(x',\big((\nabla y_0)^{\mathsf T}\nabla w^h\big)_{\rm sym}+\frac12(\nabla v^h)^{\mathsf T}\nabla v^h\Big)-\begin{bmatrix}(\nabla v^h)^{\mathsf T}\vec p^{\,h}\\\frac12|\vec p^{\,h}|^2\end{bmatrix}-\begin{bmatrix}(\nabla w^h)^{\mathsf T}\vec b_0\\0\end{bmatrix},\\Q_0^{\mathsf T}\partial_3\vec k_0=&\,c\Big(x',\big((\nabla y_0)^{\mathsf T}\nabla_{\tan}\vec d_0\big)_{\rm sym}+\frac{x_3^2}{2}(\nabla\vec b_0)^{\mathsf T}\nabla\vec b_0-\frac14(\mathcal{G}_2)_{2\times 2}\Big)\\&-\begin{bmatrix}x_3(\nabla\vec b_0)^{\mathsf T}\partial_3\vec d_0\\\frac12|\partial_3\vec d_0|^2\end{bmatrix}+\begin{bmatrix}(\nabla_{\tan}\vec d_0)^{\mathsf T}\vec b_0\\0\end{bmatrix}+\frac12\mathcal{G}_2e_3-\frac14(\mathcal{G}_2)_{33}e_3,\\Q_0^{\mathsf T}\tilde r^h=&\,x_3c\Big(x',\big((\nabla y_0)^{\mathsf T}\nabla\vec p^{\,h}+(\nabla v^h)^{\mathsf T}\nabla\vec b_0\big)_{\rm sym}\Big)-\begin{bmatrix}(\nabla v^h)^{\mathsf T}\partial_3\vec d_0\\\langle\vec p^{\,h},\partial_3\vec d_0\rangle\end{bmatrix}.\end{split}\tag{6.22}$$

Finally, we choose $\{\vec r^{\,h}\in W^{1,\infty}(\Omega,\mathbb{R}^3)\}_{h\to 0}$ to satisfy:

$$\lim_{h\to 0}\|\partial_3\vec r^{\,h}-\tilde r^h\|_{L^2(\Omega,\mathbb{R}^3)}=0\quad\text{ and }\quad\lim_{h\to 0}\sqrt{h}\|\vec r^{\,h}\|_{W^{1,\infty}(\Omega,\mathbb{R}^3)}=0.\tag{6.23}$$

2. Observe that for all $(x', x_3)\in\Omega$ there holds::

$$\nabla u^h(x',hx_3) = Q_0 + h\Big(\big[\nabla v^h,\ \vec p^h\big] + P_0\Big) + h^2\Big(\big[\nabla w^h,\ \vec q^h\big] + \big[x_3\nabla \vec p^h,\ \partial_3 r^h\big] + \big(\nabla_{\tan}\vec d_0,\ \partial_3\vec k_0\big]\Big) + \mathcal{O}(h^3)\Big(|\nabla\vec k_0| + |\nabla\vec q^h| + |\nabla \vec r^h|\Big).$$

Consequently, by (6.9) it follows that:

$$\big((\nabla u^h)(A^h)^{-1}\big)(x',hx_3) = Q_0\bar A^{-1}\Big(Id_3 + h\bar A^{-1}J_1^h\bar A^{-1} + h^2\bar A^{-1}J_2^h\bar A^{-1} + J_h^3\Big),$$

where:

$$\begin{aligned} J_1^h &= Q_0^{\mathsf T}\Big(\big[\nabla v^h,\ \vec p^h\big] + P_0\Big) - \bar A A_1,\\ J_2^h &= Q_0^{\mathsf T}\Big(\big[\nabla w^h,\ \vec q^h\big] + \big[x_3\nabla\vec p^h,\ \partial_3\vec r^h\big] + \big[\nabla\vec d_0,\ \partial_3\vec k_0\big]\Big) - J_1^h\bar A^{-1}A_1 - \frac12\bar A A_2,\end{aligned}$$

and where J_1^h, J_2^h, J_3^h satisfy the uniform bounds (independent of ε):

$$\begin{aligned} &|J_1^h| \le C\big(1+|\nabla v^h|\big),\\ &|J_2^h| \le C\big(1+|\nabla w^h| + |\nabla v^h|^2 + |\nabla^2 v^h| + |\nabla\vec r^h|\big),\\ &|J_3^h| \le Ch^3\big(1+|\nabla w^h| + |\nabla^2 w^h| + |\nabla v^h|^2 + |\nabla^2 v^h| + |\nabla v^h|\cdot|\nabla^2 v^h| + |\nabla\vec r^h|\big) + o(h^2).\end{aligned}$$

In particular, the distance $\mathrm{dist}\big((\nabla u^h)(A^h)^{-1}, SO(3)\big) \le |(\nabla u^h)(A^h)^{-1} - Q_0\bar A^{-1}|$ is as small as one wishes, uniformly in $x\in\Omega$, for h sufficiently small. Thus, the argument $(\nabla u^h)(A^h)^{-1}$ of the frame invariant density W in $\mathcal{E}^h(u^h)$ may be replaced by its polar decomposition factor:

$$\begin{aligned}\Big(\big(\nabla u^h(A^h)^{-1}\big)^{\mathsf T}\big(\nabla u^h(A^h)^{-1}\big)\Big)^{1/2} &= \Big(Id_3 + 2h^2\bar A^{-1}\big((J_2^h)_{\rm sym} + \frac12 (J_1^h)^{\mathsf T}\bar A^{-2}J_1^h\big)\bar A^{-1} + \mathcal{R}^h\Big)^{1/2}\\ &= Id_3 + h^2\bar A^{-1}\big((J_2^h)_{\rm sym} + \frac12(J_1^h)^{\mathsf T}\bar A^{-2}J_1^h\big)\bar A^{-1} + \mathcal{R}^h,\end{aligned}$$

where $\mathcal{R}^h$ stands for any quantity obeying the following bound:

$$\mathcal{R}^h = \mathcal{O}(h)|(J_1^h)_{\rm sym}| + \mathcal{O}(h^3)\big(1+|\nabla v^h|\big)\big(1+|\nabla w^h| + |\nabla v^h|^2 + |\nabla^2 v^h| + |\nabla\vec r^h|\big) + \mathcal{O}(h^3)|\nabla^2 w^h| + o(h^2).$$

In conclusion, Taylor's expansion of W at Id_3 gives:

$$\begin{aligned}&\frac{1}{h^4}\int_\Omega W\Big((\nabla u^h)(A^h)^{-1}(x',hx_3)\Big)\,\mathrm{d}x\\ &\quad= \frac{1}{h^4}\int_\Omega W\Big(\big((\nabla u^h(A^h)^{-1})^{\mathsf T}\big(\nabla u^h(A^h)^{-1}\big)\big)^{1/2}(x',hx_3)\Big)\,\mathrm{d}x\\ &\quad\le \frac12\int_\Omega \mathcal{Q}_3\Big(\bar A^{-1}\big((J_2^h)_{\rm sym} + \frac12(J_1^h)^{\mathsf T}\bar A^{-2}J_1^h\big)\bar A^{-1} + \frac{1}{h^2}\mathcal{R}^h\Big)\,\mathrm{d}x\\ &\qquad + \mathcal{O}(h^2)\int_\Omega |J_2^h|^3 + |J_1^h|^6\,\mathrm{d}x + \frac{\mathcal{O}(1)}{h^4}\int_\Omega|\mathcal{R}^h|^3\,\mathrm{d}x.\end{aligned}\tag{6.24}$$

The residual terms above are estimated as in [39], using (6.20), (6.21), (6.23). We have:

$$h^2\int_\Omega |J_2^h|^3 + |J_1^h|^6\,\mathrm{d}x \le h^2\int_\Omega 1 + |\nabla w^h|^3 + |\nabla v^h|^6 + |\nabla^2 v^h|^3 + |\nabla\vec r^h|^3\,\mathrm{d}x \le o(1),$$

as $h^2\int_\Omega |\nabla v^h|^6\,\mathrm{d}x \leq Ch^2\|\nabla v^h\|^6_{W^{1,2}} = o(1)$ and $h^2\int_\Omega |\nabla^2 v^h|^3\,\mathrm{d}x \leq \epsilon h\int_\Omega |\nabla^2 v^h|^2\,\mathrm{d}x = o(1)$. Further:

$$\begin{aligned}\frac{1}{h^4}\int_\Omega |\mathcal{R}^h|^2\,\mathrm{d}x \leq \frac{1}{h^2}\int_\Omega \big|\big((\nabla y_0)^\mathsf{T}\nabla v^h\big)_{\rm sym}\big|^2\,\mathrm{d}x \\ & + \mathcal{O}(h^2)\int_\Omega \big(1+|\nabla v^h|^2\big)\big(1+|\nabla w^h|^2+|\nabla v^h|^4+|\nabla^2 v^h|^2+|\nabla \vec{r}^h|^2\big)\,\mathrm{d}x \\ & + \mathcal{O}(h^2)\int_\Omega |\nabla^2 w^h|^2\,\mathrm{d}x + o(1) \\ & = o(1) + \mathcal{O}(h^2)\int_\Omega |\nabla v^h|\cdot|\nabla^2 v^h|^2 \leq C\varepsilon,\end{aligned}$$

because the last condition in (6.21) implies:

$$\begin{aligned}\frac{1}{h^2}\int_\Omega \big|\big((\nabla y_0)^\mathsf{T}\nabla v^h\big)_{\rm sym}\big|^2\,\mathrm{d}x &\leq \frac{C}{h^2}\|\nabla^2 v^h\|_{L^\infty}\int_{\{v^h\neq V\}} \mathrm{dist}^2(x',\{v^h=V\})\,\mathrm{d}x' \\ &\leq \frac{C\varepsilon^2}{h^4}\int_{\{v^h\neq V\}} \mathrm{dist}^2(x',\{v^h=V\})\,\mathrm{d}x' \leq C\varepsilon^2\frac{1}{h^2}\big|\{v^h\neq V\}\big| = o(1).\end{aligned} \tag{6.25}$$

From the two estimates above it also follows that $\frac{1}{h^4}\int_\Omega |\mathcal{R}^h|^3\,\mathrm{d}x = o(1)$. Consequently, (6.24) yields:

$$\limsup_{h\to 0}\frac{1}{h^4}\mathcal{E}^h(u^h) \leq C\varepsilon + \limsup_{h\to 0}\frac{1}{2}\int_\Omega \mathcal{Q}_3\Big(\bar{A}^{-1}\big((J_2^h)_{\rm sym}+\frac{1}{2}(J_1^h)^\mathsf{T}\big)\bar{A}^{-2}J_1^h\big)\bar{A}^{-1}\Big)\,\mathrm{d}x. \tag{6.26}$$

3. Observe now that:

$$\begin{aligned}(J_2^h)_{\rm sym} + \frac{1}{2}(J_1^h)^\mathsf{T}\bar{A}^{-2}J_1^h = &-\Big(\big((\nabla y_0)^\mathsf{T}\nabla v^h\big)^*_{\rm sym}\bar{A}^{-1}A_1\Big)_{\rm sym} \\ &+ \Big(Q_0^\mathsf{T}\big[\nabla w^h,\ \vec{q}^h\big] + Q_0^\mathsf{T}\big[x_3\nabla\vec{p}^h,\ \partial_3\vec{r}^h\big] + Q_0^\mathsf{T}\big[\nabla\vec{d}_0,\ \partial_3\vec{k}_0\big]\Big)_{\rm sym} \\ &+ \frac{1}{2}\big[\nabla v^h,\ \vec{p}^h\big]^\mathsf{T}\big[\nabla v^h,\ \vec{p}^h\big] + \Big(\big[\nabla v^h,\ \vec{p}^h\big]^\mathsf{T}P_0\Big)_{\rm sym} + \frac{1}{2}P_0^\mathsf{T}P_0 - \frac{1}{4}\mathcal{G}_2.\end{aligned}$$

Replacing $\partial_3\vec{r}^h$ by $\tilde{r}^h$ and using (6.22), it follows that:

$$\begin{aligned}&\Big(\int_\Omega \mathcal{Q}_3\Big(\bar{A}^{-1}\big((J_2^h)_{\rm sym}+\frac{1}{2}(J_1^h)^\mathsf{T}\bar{A}^{-2}J_1^h\big)\bar{A}^{-1}\Big)\,\mathrm{d}x\Big)^{1/2} \\ &\quad\leq \Big(\int_\Omega \mathcal{Q}_2\Big(x', \big((\nabla y_0)^\mathsf{T}\nabla w^h\big)_{\rm sym} + \frac{1}{2}(\nabla v^h)^\mathsf{T}\nabla v^h + x_3\big((\nabla y_0)^\mathsf{T}\nabla\vec{p}^h + (\nabla v^h)^\mathsf{T}\nabla\vec{b}_0\big)_{\rm sym} \\ &\qquad\qquad + \frac{x_3^2}{2}(\nabla\vec{b}_0)^\mathsf{T}\nabla\vec{b}_0 - \frac{1}{4}(\mathcal{G}_2)_{2\times 2}\Big)\,\mathrm{d}x\Big)^{1/2} \\ &\qquad + \|\big((\nabla y_0)^\mathsf{T}\nabla v^h\big)_{\rm sym}\|_{L^2(\Omega)} + \|\partial_3\vec{r}^h - \tilde{r}^h\|_{L^2(\Omega)}.\end{aligned}$$

The second term above converges to 0 by (6.25) and the third term also converges to 0, by (6.23). On the other hand, the first term can be split into the integral on the set $\{v^h = V\}$, whose limit as $h\to 0$ is estimated by $\mathcal{I}_4^O(V,\mathbb{S})$, and the remaining integral that is bounded by:

$$\begin{aligned}&C\int_{\{v^h\neq V\}\times(-\frac{1}{2},\frac{1}{2})} 1+|\nabla w^h|^2+|\nabla v^h|^4+|\nabla^2 v^h|^2+|\nabla\vec{r}^h|^3\,\mathrm{d}x \\ &\quad\leq C\epsilon^2\frac{1}{h^2}|\{v^h\neq V\}| + C\int_{\{v^h\neq V\}}|\nabla v^h|^4\,\mathrm{d}x' \leq o(1) + C|\{v^h\neq V\}|^{1/2}\|\nabla v^h\|^4_{L^8} = o(1).\end{aligned}$$

In conclusion, (6.26) becomes (with a uniform constant C that does not depend on ε):

$$\limsup_{h\to 0}\frac{1}{h^4}\mathcal{E}^h(u^h) \leq C\varepsilon + \mathcal{I}_4^O(V,\mathbb{S}).$$

A diagonal argument applied to the indicated ε-recovery sequence $\{u^h\}_{h\to 0}$ completes the proof. □

Corollary 6.4. *The functional $\mathcal{I}_4^O$ attains its infimum and there holds:*

$$\lim_{h\to 0}\frac{1}{h^4}\inf \mathcal{E}^h = \min\ \mathcal{I}_4^O.$$

The infima in the left hand side are taken over $W^{1,2}(\Omega,\mathbb{R}^3)$ deformations u^h, whereas the minimum in the right hand side is taken over admissible displacement-strain couples $(V,\mathbb{S})\in\mathscr{V}\times\mathscr{S}$ in (6.18).

7. Further discussion of $\mathcal{I}_4^O$ and reduction to the non-oscillatory case (NO)

In this section, we identify the appropriate components of the integrand in the energy $\mathcal{I}_4^O$ as: stretching, bending, curvature and the order-4 excess, the latter quantity being the projection of the entire integrand on the orthogonal complement of $\mathbb{E}_2$ in $\mathbb{E}$. This superposition is in the same spirit, as the integrand of $\mathcal{I}_2^O$ in Theorem 2.3 decoupling into bending and the order-2 excess, defined as the projection on the orthogonal complement of $\mathbb{E}_1$. There, the assumed condition $\int_{-1/2}^{1/2}\mathcal{G}_1\,\mathrm{d}x_3 = 0$ served as the compatibility criterion, assuring that the 2-excess being null results in $\mathcal{I}_4^O$ coinciding with the non-oscillatory limiting energy $\mathcal{I}_4$, written for the effective metric $\bar G$ in (EF). Below, we likewise derive the parallel version $\mathcal{I}_4$ of $\mathcal{I}_4^O$, corresponding to the non-oscillatory case, and show that the vanishing of the 4-excess reduces $\mathcal{I}_4^O$ to $\mathcal{I}_4$ (for the effective metric (EF)), under two new further compatibility conditions (7.9) on $(\mathcal{G}_2)_{2\times 2}$.

The following formulas will be useful in the sequel:

Lemma 7.1. *In the non-oscillatory setting (NO), let y_0, $\vec b_0$ be as in (5.2) and $\tilde d_0$ as in (5.5). Then:*

$$\big[\partial_{ij}y_0,\ \partial_i\vec b_0,\ \tilde d_0\big](x') = \big[\partial_1 y_0,\ \partial_2\vec y_0,\ \vec b_0\big](x')\cdot\begin{bmatrix}\Gamma_{ij}^1 & \Gamma_{i3}^1 & \Gamma_{33}^1\\ \Gamma_{ij}^2 & \Gamma_{i3}^2 & \Gamma_{33}^2\\ \Gamma_{ij}^3 & \Gamma_{i3}^3 & \Gamma_{33}^3\end{bmatrix}(x',0) \qquad \textit{for}\ \ i,j=1,2, \tag{7.1}$$

for all $x'\in\omega$. Consequently, for any smooth vector field $\vec q:\omega\to\mathbb{R}^3$ there holds:

$$\Big[\nabla y_0(x')^{\mathsf T}\nabla\big(Q_0(x')^{\mathsf T,-1}\vec q(x')\big)\Big]_{i,j=1,2} = \nabla\vec q(x')_{2\times 2} - \Big[\Big\langle \vec q(x'),[\Gamma_{ij}^1,\ \Gamma_{ij}^2,\ \Gamma_{ij}^3](x',0)\Big\rangle\Big]_{i,j=1,2}.$$

Above, $\{\Gamma_{ij}^k\}$ are the Christoffel symbols of the metric G and the expression in the right and side represents the tangential part of the covariant derivative of the $(0,1)$ tensor field $\vec q$ with respect to G.

Proof. In view of $\big((\nabla y_0)^{\mathsf T}\nabla\vec b_0\big)_{\mathrm{sym}} = \frac12\partial_3 G(x',0)_{2\times 2}$ in (5.2) and recalling (3.2), we get:

$$\langle\partial_{ij}y_0,\vec b_0\rangle = \frac12\big(\partial_i G_{j3}+\partial_j G_{i3}-\partial_3 G_{ij}\big)(x',0) \qquad \text{for all } i,j=1,2,$$

which easily results in:

$$\langle\partial_i\vec b_0,\partial_j y_0\rangle = \frac12\big(\partial_i G_{j3}-\partial_j G_{i3}+\partial_3 G_{ij}\big)(x',0) \quad \text{and} \quad \langle\partial_i\vec b_0,\vec b_0\rangle = \frac12\partial_i G_{33}(x',0).$$

Thus (3.3) and the above allow for computing the coordinates in the basis $\partial_1 y_0, \partial_2 y_0, \vec{b}_0$ as claimed in (7.1); see also [39, Theorem 6.2] for more details. The second formula results from:

$$\begin{aligned}\big\langle \partial_i y_0, \partial_j\big(Q_0^{\mathsf{T},-1}\vec{q}\big)\big\rangle &= \big\langle \partial_i y_0, \partial_j\big(Q_0^{\mathsf{T},-1}\big)\vec{q}\big\rangle + \big\langle \partial_i y_0, Q_0^{\mathsf{T},-1}\partial_j\vec{q}\big\rangle \\ &= -\big\langle \partial_i y_0, Q_0^{\mathsf{T},-1}\partial_j\big(Q_0^{\mathsf{T}}\big)Q_0^{\mathsf{T},-1}\vec{q}\big\rangle + \big\langle Q_0^{-1}\partial_i y_0, \partial_j\vec{q}\big\rangle \\ &= -\big\langle Q_0^{-1}\partial_j\big(Q_0\big)e_i, \vec{q}\big\rangle + \big\langle e_i, \partial_j\vec{q}\big\rangle,\end{aligned}$$

which together with (7.1) yields the Lemma. □

Lemma 7.2. *In the non-oscillatory setting (NO), let y_0, $\vec{b}_0$ be as in (5.2) and $\tilde{d}_0$ as in (5.5). Then the metric-related term II in (6.7) has the form $II = \frac{x_3^2}{2}\bar{II}(x')$ and for all $x' \in \omega$ we have:*

$$\bar{II}_{\mathrm{sym}} = (\nabla\vec{b}_0)^{\mathsf{T}}\nabla\vec{b}_0 + \big((\nabla y_0)^{\mathsf{T}}\nabla\tilde{d}_0\big)_{\mathrm{sym}} - \frac{1}{2}\partial_{33}G(x',0)_{2\times 2} = \begin{bmatrix} R_{1313} & R_{1323} \\ R_{1323} & R_{2323}\end{bmatrix}(x',0). \tag{7.2}$$

Above, R_{ijkl} are the Riemann curvatures of the metric G, evaluated at the midplate points $x' \in \omega$.

Proof. We argue as in the proof of [39, Theorem 6.2]. Using (5.3) we arrive at:

$$\begin{aligned}\big((\nabla y_0)^{\mathsf{T}}\nabla\tilde{d}_0\big)_{\mathrm{sym}} &= -\,\big[\langle\partial_{ij}y_0, \vec{d}_0\rangle\big]_{i,j=1,2} + \frac{1}{2}\partial_{33}G(x',0)_{2\times 2} \\ &\qquad + \Big[R_{i2j3} - G_{np}\big(\Gamma^n_{i3}\Gamma^p_{j3} - \Gamma^n_{ij}\Gamma^p_{33}\big)\Big]_{i,j=1,2}(x',0).\end{aligned} \tag{7.3}$$

Directly from (7.1) we hence obtain:

$$\langle\partial_{ij}\vec{y}_0, \tilde{d}_0\rangle = G_{np}\Gamma^n_{ij}\Gamma^p_{33}, \qquad \langle\partial_i\vec{b}_0, \partial_j\vec{b}_0\rangle = G_{np}\Gamma^n_{i3}\Gamma^p_{j3}, \tag{7.4}$$

which together with (7.3) yields (7.2). □

With the use of Lemma 7.2, it is quite straightforward to derive the ultimate form of the energy $\mathcal{I}_4^O$ in the non-oscillatory setting. In particular, the proof of the following result is a special case of the proof of Theorem 7.5 below.

Theorem 7.3. *Assume (NO) and (5.1). The expression (7.6) becomes:*

$$\begin{aligned}\mathcal{I}_4(V,\mathbb{S}) = \frac{1}{2}\int_\omega \mathcal{Q}_2\Big(x', \mathbb{S}(x') + \frac{1}{2}\nabla V(x')^{\mathsf{T}}\nabla V(x') + \frac{1}{24}\nabla\vec{b}_0(x')^T\nabla\vec{b}_0(x') - \frac{1}{48}\partial_{33}G(x',0)_{2\times 2}\Big)\,\mathrm{d}x' \\ + \frac{1}{24}\int_\omega \mathcal{Q}_2\Big(x', \nabla y_0(x')^{\mathsf{T}}\nabla\vec{p}(x') + \nabla V(x')^{\mathsf{T}}\nabla\vec{b}_0(x')\Big)\,\mathrm{d}x' \\ + \frac{1}{1440}\int_\omega \mathcal{Q}_2\Big(x', \begin{bmatrix} R_{1313} & R_{1323} \\ R_{1323} & R_{2323}\end{bmatrix}(x',0)\Big)\,\mathrm{d}x',\end{aligned}$$

where R_{ijkl} stand for the Riemann curvatures of the metric G.

Remark 7.4. In the particular, "flat" case of $G = Id_3$ the functional $\mathcal{I}_4$ reduces to the classical von Kármán energy below. Indeed, the unique solution to (5.2) is: $y_0 = id$, $\vec{b}_0 = e_3$ and further:

$$\begin{aligned}\mathscr{V} &= \big\{V(x) = (\alpha x^\perp + \vec{\beta}, v(x));\ \alpha \in \mathbb{R},\ \vec{\beta} \in \mathbb{R}^2,\ v \in W^{2,2}(\omega)\big\}, \\ \mathscr{S} &= \big\{\mathrm{sym}\,\nabla w;\ w \in W^{1,2}(\omega,\mathbb{R}^2)\big\}.\end{aligned}$$

Given $V \in \mathscr{V}$, we have $\vec{p} = (-\nabla v, 0)$ and thus:

$$\mathcal{I}_4(V,\mathbb{S}) = \frac{1}{2}\int_\omega \mathcal{Q}_2\big(x', \mathrm{sym}\,\nabla w + \frac{1}{2}(\alpha^2 Id_2 + \nabla v \otimes \nabla v)\big)\,\mathrm{d}x' + \frac{1}{24}\int_\omega \mathcal{Q}_2\big(x', \nabla^2 v\big)\,\mathrm{d}x'.$$

Absorbing the stretching $\alpha^2 Id_2$ into $\mathrm{sym}\,\nabla w$, the above energy can be expressed in a familiar form:

$$\mathcal{I}_4(v,w) = \frac{1}{2}\int_\omega \mathcal{Q}_2\big(x', \mathrm{sym}\,\nabla w + \frac{1}{2}\nabla v\otimes\nabla v\big)\,\mathrm{d}x' + \frac{1}{24}\int_\omega \mathcal{Q}_2\big(x', \nabla^2 v\big)\,\mathrm{d}x', \tag{7.5}$$

as a function of the out-of-plane scalar displacement v and the in-plane vector displacement w. ■

As done for the Kirchhoff energy $\mathcal{I}_2^O$ in Theorem 2.3, we now identify conditions allowing $\mathcal{I}_4^O$ to coincide with $\mathcal{I}_4$ of the effective metric $\bar{G}$, modulo the introduced below order-4 excess term.

Theorem 7.5. *In the setting of Theorem 6.2, we have:*

$$\begin{aligned}\mathcal{I}_4^O(V,\mathbb{S}) = {} & \frac{1}{2}\int_\omega \mathcal{Q}_2\Big(x', \mathbb{S} + \frac{1}{2}(\nabla V)^\mathsf{T}\nabla V + B_0\Big)\,\mathrm{d}x' \\ & + \frac{1}{24}\int_\omega \mathcal{Q}_2\Big(x', (\nabla V)^\mathsf{T}\nabla\vec{b}_0 + (\nabla y_0)^\mathsf{T}\nabla\vec{p} + 12B_1\Big)\,\mathrm{d}x' \\ & + \frac{1}{1440}\int_\omega \mathcal{Q}_2\Big(x', (\nabla\vec{b}_0)^\mathsf{T}\nabla\vec{b}_0 + (\nabla y_0)^\mathsf{T}\nabla\tilde{d}_0 - \frac{1}{2}(\bar{\mathcal{G}}_2)_{2\times2}\Big)\,\mathrm{d}x' \\ & + \frac{1}{2}\mathrm{dist}^2_{\mathcal{Q}_2}\Big(II_{\mathrm{sym}}, \mathbb{E}_2\Big),\end{aligned} \tag{7.6}$$

where $\bar{\mathcal{G}}_1$ and $\bar{\mathcal{G}}_2$ are given in (EF), inducing $\tilde{d}_0$ via (5.5) for $\partial_3 G = \bar{\mathcal{G}}_1$, and where we introduce the following purely metric-related quantities:

$$\begin{aligned}\mathrm{dist}^2_{\mathcal{Q}_2}\big(II_{\mathrm{sym}}, \mathbb{E}_2\big) = \mathrm{dist}^2\Big(& \int_0^{x_3} \nabla(\mathcal{G}_1 e_3)_{2\times2,\mathrm{sym}}\,\mathrm{d}t \\ & - \Big[\Big\langle \int_0^{x_3} \mathcal{G}_1 e_3\,\mathrm{d}t, [\Gamma_{ij}^1,\ \Gamma_{ij}^2,\ \Gamma_{ij}^3](x',0)\Big\rangle\Big]_{i,j=1,2} \\ & + \frac{1}{2}\int_0^{x_3} (\mathcal{G}_1)_{33}\,\mathrm{d}t\,\big[\Gamma_{ij}^3(x',0)\big]_{i,j=1,2} - \frac{1}{4}(\mathcal{G}_2)_{2\times2},\ \mathbb{E}_2\Big),\end{aligned} \tag{7.7}$$

$$\begin{aligned} B_0 & = \frac{1}{24}(\nabla\vec{b}_0)^\mathsf{T}\nabla\vec{b}_0 - \frac{1}{4}\int_{-1/2}^{1/2}(\mathcal{G}_2)_{2\times2}\,\mathrm{d}x_3 = \frac{1}{24}\Big[\bar{\mathcal{G}}_{np}\Gamma_{i3}^n\Gamma_{j3}^p\Big]_{i,j=1,2} - \frac{1}{4}\int_{-1/2}^{1/2}(\mathcal{G}_2)_{2\times2}\,\mathrm{d}x_3, \\ B_1 & = (\nabla y_0)^\mathsf{T}\nabla\big(\int_{-1/2}^{1/2} x_3\tilde{d}_0\,\mathrm{d}x_3\big) - \frac{1}{4}\int_{-1/2}^{1/2} x_3(\mathcal{G}_2)_{2\times2}\,\mathrm{d}x_3 \\ & = -\nabla\big(\int_{-1/2}^{1/2}\frac{x_3^2}{2}\mathcal{G}_1 e_3\,\mathrm{d}x_3\big)_{2\times2} + \Big[\Big\langle\int_{-1/2}^{1/2}\frac{x_3^2}{2}\mathcal{G}_1 e_3\,\mathrm{d}x_3, [\Gamma_{ij}^1,\ \Gamma_{ij}^2,\ \Gamma_{ij}^3](x',0)\Big\rangle\Big]_{i,j=1,2} \\ & \quad - \frac{1}{2}\int_{-1/2}^{1/2}\frac{x_3^2}{2}(\mathcal{G}_1)_{33}\,\mathrm{d}x_3\Big[\Gamma_{ij}^3(x',0)\Big]_{i,j=1,2} - \frac{1}{4}\int_{-1/2}^{1/2} x_3(\mathcal{G}_2)_{2\times2}\,\mathrm{d}x_3,\end{aligned} \tag{7.8}$$

By $\{\Gamma_{ij}^k\}$ we denote the Christoffel symbols of the metric $\bar{G}$ in (EF). The third term in (7.6) equals the scaled norm of the Riemann curvatures of the effective metric $\bar{G}$:

$$\frac{1}{1440}\int_\omega \mathcal{Q}_2\Big(x', \begin{bmatrix} R_{1313} & R_{1323} \\ R_{1323} & R_{2323}\end{bmatrix}(x',0)\Big)\,\mathrm{d}x'.$$

The first three terms in $\mathcal{I}_4^O$ coincide with $\mathcal{I}_4$ in Theorem 7.3 for the effective metric $\bar G$ in (EF), provided that the following compatibility conditions hold:

$$
\begin{aligned}
& \int_{-1/2}^{1/2} \big(15x_3^2 - \frac{9}{4}\big)\mathcal{G}_2(x', x_3)_{2\times 2}\,\mathrm{d}x_3 = 0,\\
& \frac14 \int_{-1/2}^{1/2} x_3\mathcal{G}_2(x', x_3)_{2\times 2}\,\mathrm{d}x_3 + \nabla\Big(\int_{-1/2}^{1/2}\frac{x_3^2}{2}\mathcal{G}_1 e_3\,\mathrm{d}x_3\Big)_{2\times 2,\mathrm{sym}}\\
& \qquad - \Big[\Big\langle \int_{-1/2}^{1/2}\frac{x_3^2}{2}\mathcal{G}_1 e_3\,\mathrm{d}x_3, [\Gamma_{ij}^1,\ \Gamma_{ij}^2,\ \Gamma_{ij}^3](x',0)\Big\rangle\Big]_{i,j=1,2}\\
& \qquad + \frac12 \int_{-1/2}^{1/2}\frac{x_3^2}{2}(\mathcal{G}_1)_{33}\,\mathrm{d}x_3\Big[\Gamma_{ij}^3(x',0)\Big]_{i,j=1,2} = 0.
\end{aligned}
\tag{7.9}
$$

Proof. We write:

$$
\mathcal{I}_4^O(V,\mathbb{S}) = \frac12\|I + x_3III + II\|_{\mathcal{Q}_2}^2 = \frac12\|I + x_3III + \mathbb{P}_2(II)\|_{\mathcal{Q}_2}^2 + \frac12\mathrm{dist}_{\mathcal{Q}_2}^2\big(II_{\mathrm{sym}}, \mathbb{E}_2\big),
$$

and further decompose the first term above along the Legendre projections:

$$
\begin{aligned}
\|I + x_3III + \mathbb{P}_2(II)\|_{\mathcal{Q}_2}^2 &= \Big\|\int_{-1/2}^{1/2}(I + x_3III + II)p_0(x_3)\,\mathrm{d}x_3\Big\|_{\mathcal{Q}_2}^2\\
&\quad + \Big\|\int_{-1/2}^{1/2}(I + x_3III + II)p_1(x_3)\,\mathrm{d}x_3\Big\|_{\mathcal{Q}_2}^2 + \Big\|\int_{-1/2}^{1/2}(I + x_3III + II)p_2(x_3)\,\mathrm{d}x_3\Big\|_{\mathcal{Q}_2}^2\\
&= \underbrace{\Big\|I + \int_{-1/2}^{1/2} II\,\mathrm{d}x_3\Big\|_{\mathcal{Q}_2}^2}_{\textit{Stretching}} + \frac{1}{12}\underbrace{\Big\|III + 12\int_{-1/2}^{1/2}x_3 II\,\mathrm{d}x_3\Big\|_{\mathcal{Q}_2}^2}_{\textit{Bending}} + \underbrace{\Big\|\int_{-1/2}^{1/2}p_2(x_3)II\,\mathrm{d}x_3\Big\|_{\mathcal{Q}_2}^2}_{\textit{Curvature}}.
\end{aligned}
$$

To identify the four indicated terms in $\mathcal{I}_4^O$, observe that $\int_{-1/2}^{1/2}x_3\int_0^{x_3}\mathcal{G}_1\,\mathrm{d}x_3 = -\int_{-1/2}^{1/2}\frac{x_3^2}{2}\mathcal{G}_1\,\mathrm{d}x_3$ and:

$$
\begin{aligned}
&\mathrm{dist}_{\mathcal{Q}_2}^2\big(II_{\mathrm{sym}}, \mathbb{E}_2\big)\\
&\quad = \mathrm{dist}_{\mathcal{Q}_2}^2\bigg(\Big((\nabla y_0)^{\mathsf{T}}\nabla\big(Q_0^{\mathsf{T},-1}\int_0^{x_3}\mathcal{G}_1 e_3\,\mathrm{d}t - \frac12 Q_0^{\mathsf{T},-1}\int_0^{x_3}(\mathcal{G}_1)_{33}\,\mathrm{d}t\, e_3\big)\Big)_{\mathrm{sym}} - \frac14(\mathcal{G}_2)_{2\times 2}, \mathbb{E}_2\bigg).
\end{aligned}
$$

Thus the formulas in (7.7) and (7.8) follow directly from Lemma 7.1 and (7.4). There also holds:

$$
\begin{aligned}
\textit{Stretching} &= \int_\omega \mathcal{Q}_2\Big(x', \mathbb{S} + \frac12(\nabla V)^{\mathsf{T}}\nabla V + \frac{1}{24}(\nabla\vec b_0)^{\mathsf{T}}\nabla\vec b_0 - \frac14\int_{-1/2}^{1/2}(\mathcal{G}_2)_{2\times 2}\,\mathrm{d}x_3\Big)\,\mathrm{d}x',\\
\textit{Bending} &= \frac{1}{12}\int_\omega \mathcal{Q}_2\Big(x', (\nabla V)^{\mathsf{T}}\nabla\vec b_0 + (\nabla y_0)^{\mathsf{T}}\nabla\vec p\\
&\qquad + 12(\nabla y_0)^{\mathsf{T}}\nabla\big(\int_{-1/2}^{1/2}x_3\vec d_0\,\mathrm{d}x_3\big) - 3\int_{-1/2}^{1/2}x_3(\mathcal{G}_2)_{2\times 2}\,\mathrm{d}x_3\Big),\ \mathrm{d}x'\\
\textit{Curvature} &= \frac{1}{720}\int_\omega \mathcal{Q}_2\Big(x', (\nabla\vec b_0)^{\mathsf{T}}\nabla\vec b_0 + 60(\nabla y_0)^{\mathsf{T}}\nabla\big(\int_{-1/2}^{1/2}(6x_3^2 - \frac12)\vec d_0\,\mathrm{d}x_3\big)\\
&\qquad - 15\int_{-1/2}^{1/2}(6x_3^2 - \frac12)(\mathcal{G}_2)_{2\times 2}\,\mathrm{d}x_3\Big)\,\mathrm{d}x'.
\end{aligned}
$$

It is easy to check that with the choice of the effective metric components $\bar{\mathcal{G}}_1 e_3$ and $(\bar{\mathcal{G}}_2)_{2\times 2}$ and denoting $\tilde{d}_0$ the corresponding vector in (5.5), we have:

$$\mathit{Curvature} = \frac{1}{720}\int_\omega \mathcal{Q}_2\Big(x', (\nabla\vec{b}_0)^{\mathsf{T}}\nabla\vec{b}_0 + (\nabla y_0)^{\mathsf{T}}\nabla\tilde{d}_0 - \frac{1}{2}(\bar{\mathcal{G}}_2)_{2\times 2}\Big)\,\mathrm{d}x'.$$

This proves (7.6). Equivalence of the constraints (7.9) with:

$$\int_{-1/2}^{1/2}(\mathcal{G}_2)_{2\times 2}\,\mathrm{d}x_3 = \frac{1}{12}(\bar{\mathcal{G}}_2)_{2\times 2} \quad \text{and} \quad (B_1)_{\mathrm{sym}} = 0 \qquad \text{in } \omega,$$

follows by a direct inspection. We now invoke Lemma 7.2 to complete the proof. □

Remark 7.6. Observe that the vanishing of the 4-excess and curvature terms in $\mathcal{I}_4^O$:

$$II_{\mathrm{sym}} \in \mathbb{E}_2 \quad \text{and} \quad \mathit{Curvature} = 0,$$

are the necessary conditions for $\min \mathcal{I}_4^O = 0$ and they are equivalent to $II_{\mathrm{sym}} \in \mathbb{E}_1$. Consider now a particular case scenario of $\bar{\mathcal{G}} = Id_3$ and $\mathcal{G}_1 = 0$, where the spaces $\mathscr{V}$ and $\mathscr{S}$ are given in Remark 7.4, together with $\vec{d}_0 = 0$. Then, the above necessary condition reduces to: $(\mathcal{G}_2)_{2\times 2} \in \mathbb{E}_1$, namely:

$$(\mathcal{G}_2)_{2\times 2}(x', x_3) = x_3\mathcal{F}_1(x') + \mathcal{F}_0(x') \qquad \text{for all } x = (x', x_3) \in \bar{\Omega}.$$

It is straightforward that, on a simply connected midplate ω, both terms:

$$\mathit{Stretching} = \int_\omega \mathcal{Q}_2\Big(x', \operatorname{sym}\nabla w + \frac{1}{2}\nabla v\otimes\nabla v - \frac{1}{4}\mathcal{F}_0\Big)\,\mathrm{d}x', \quad \mathit{Bending} = \int_\omega \mathcal{Q}_2\Big(x', \nabla^2 v + \frac{1}{4}\mathcal{F}_1\Big)\,\mathrm{d}x',$$

can be equated to 0 by choosing appropriate displacements v and w, if and only if there holds:

$$\operatorname{curl}\mathcal{F}_1 = 0, \qquad \operatorname{curl}^{\mathsf{T}}\operatorname{curl}\mathcal{F}_0 + \frac{1}{4}\det\mathcal{F}_1 = 0 \quad \text{in } \omega. \tag{7.10}$$

Note that these are precisely the linearised Gauss-Codazzi-Meinardi equations corresponding to the metric $Id_2 + 2h^2\mathcal{F}_0$ and shape operator $\frac{1}{2}h\mathcal{F}_1$ on ω. We see that these conditions are automatically satisfied in presence of (7.9), when $(\mathcal{G}_2)_{2\times 2} \in \mathbb{E}_1$ actually results in $(\mathcal{G}_2)_{2\times 2} = 0$. An integrability criterion similar to (7.10) can be derived also in the general case, under $II_{\mathrm{sym}} \in \mathbb{E}_1$ and again it automatically holds with (7.9). This last statement will be pursued in the next section. ■

8. Identification of the Ch^4 scaling regime and coercivity of the limiting energy $\mathcal{I}_4$

Theorem 8.1. *The energy scaling beyond the von Kármán regime:*

$$\lim_{h\to 0}\frac{1}{h^4}\inf\mathcal{E}^h = 0$$

is equivalent to the following condition, on a simply connected ω:

(i) in the oscillatory case (O), in presence of the compatibility conditions (7.9)

(8.1) *$II_{\mathrm{sym}} \in \mathbb{E}_2$ and (8.2) holds with G replaced by the effective metric $\bar{G}$ in (EF). This condition involves $\bar{\mathcal{G}}$, $\bar{\mathcal{G}}_1$ and $(\bar{\mathcal{G}}_2)_{2\times 2}$ terms of $\bar{G}$.*

(ii) in the non-oscillatory case (NO)

(8.2) *All the Riemann curvatures of the metric G vanish on $\omega\times\{0\}$:*

$$R_{ijkl}(x', 0) = 0 \qquad \text{for all } x' \in \omega \quad \text{and all } i,j,k,l = 1\dots 3.$$

Proof. By Corollary 6.4, it suffices to determine the equivalent conditions for $\min \mathcal{I}_4 = 0$. Clearly, $\min \mathcal{I}_4 = 0$ implies (8.2). Vice versa, if (8.2) holds, then:

$$\frac{1}{24}(\nabla\vec{b}_0)^{\mathsf{T}}\nabla\vec{b}_0 - \frac{1}{48}\partial_{33}G(x',0) = -\frac{1}{24}\big((\nabla y_0)^{\mathsf{T}}\nabla\tilde{d}_0\big)_{\mathrm{sym}},$$

by Lemma 7.2. Taking $V = \vec{p} = 0$ and $\mathbb{S} = \frac{1}{24}\big((\nabla y_0)^{\mathsf{T}}\nabla\tilde{d}_0\big)_{\mathrm{sym}} \in \mathscr{S}$, we get $\mathcal{I}_4(V,\mathbb{S}) = 0$. □

We further have the following counterpart of the essential uniqueness of the minimizing isometric immersion y_0 statement in Theorem 3.2:

Theorem 8.2. *In the non-oscillatory setting (NO), assume (8.2). Then $\mathcal{I}_4(V,\mathbb{S}) = 0$ if and only if:*

$$V = Sy_0 + c \quad \textit{and} \quad \mathbb{S} = \frac{1}{2}\Big((\nabla y_0)^{\mathsf{T}}\nabla\big(S^2 y_0 + \frac{1}{12}\tilde{d}_0\big)\Big)_{\mathrm{sym}} \qquad \textit{on } \omega, \tag{8.3}$$

for some skew-symmetric matrix $S \in so(3)$ and a vector $c \in \mathbb{R}^3$.

Proof. We first observe that the bending term III in (6.7) is already symmetric, because:

$$\begin{aligned}\Big[\langle\partial_i y_0, \partial_j\vec{p}\rangle + \langle\partial_i V, \partial_j\vec{b}_0\rangle\Big]_{i,j=1,2} &= \Big[\partial_j\big(\langle\partial_i y_0, \vec{p}\rangle + \langle\partial_i V, \vec{b}_0\rangle\big)\Big]_{i,j=1,2} - \Big[\langle\partial_{ij}y_0, \vec{p}\rangle + \langle\partial_{ij}V, \vec{b}_0\rangle\Big]_{i,j=1,2} \\ &= -\Big[\langle\partial_{ij}y_0, \vec{p}\rangle + \langle\partial_{ij}V, \vec{b}_0\rangle\Big]_{i,j=1,2} \in \mathbb{R}^{2\times 2}_{\mathrm{sym}},\end{aligned}$$

where we used the definition of $\vec{p}$ in (6.4). Recalling (7.2), we see that $\mathcal{I}_4(V,\mathbb{S}) = 0$ if and only if:

$$\begin{aligned}&\mathbb{S} + \frac{1}{2}(\nabla V)^{\mathsf{T}}\nabla V - \frac{1}{24}\big((\nabla y_0)^{\mathsf{T}}\nabla\tilde{d}_0\big)_{\mathrm{sym}} = 0, \\ &(\nabla y_0)^{\mathsf{T}}\nabla\vec{p} + (\nabla V)^{\mathsf{T}}\nabla\vec{b}_0 = 0.\end{aligned} \tag{8.4}$$

Consider the matrix field $S = \big[\nabla V,\ \vec{p}\big]Q_0^{-1} \in W^{1,2}(\omega, so(3))$ as in (6.4). Note that:

$$\begin{aligned}&\partial_i S = \big[\nabla\partial_i V,\ \partial_i\vec{p}\big]Q_0^{-1} - \big[\nabla V,\ \vec{p}\big]Q_0^{-1}(\partial_i Q_0)Q_0^{-1} = Q_0^{-1,\mathsf{T}}\bar{S}^i Q_0^{-1} \qquad \text{for } i = 1,2 \\ &\qquad \text{where} \quad \bar{S}^i = Q_0^{\mathsf{T}}\big[\nabla\partial_i V,\ \partial_i\vec{p}\big] + \big[\nabla V,\ \vec{p}\big]^{\mathsf{T}}(\partial_i Q_0) \in L^2(\omega, so(3)).\end{aligned} \tag{8.5}$$

Then we have:

$$\langle\bar{S}^i e_1, e_2\rangle = \partial_i\big(\langle\partial_2 y_0, \partial_i V\rangle + \langle\partial_2 V, \partial_i y_0\rangle\big) - \big(\langle\partial_{12}y_0, \partial_i V\rangle + \langle\partial_{12}V, \partial_i y_0\rangle\big) = 0,$$

because the first term in the right hand side above equals 0 in view of $V \in \mathscr{V}$, whereas the second term equals $\partial_2\langle\partial_1 y_0, \partial_1 V\rangle$ for $i = 1$ and $\partial_1\langle\partial_2 y_0, \partial_2 V\rangle$ for $i = 2$, both expression being null again in view of $V \in \mathscr{V}$. We now claim that $\{\bar{S}^i\}_{i=1,2} = 0$ is actually equivalent to the second condition in (8.4). It suffices to examine the only possibly nonzero components:

$$\langle\bar{S}^i e_3, e_j\rangle = \langle\partial_j y_0, \partial_i\vec{p}\rangle + \langle\partial_j V, \partial_i\vec{b}_0\rangle = \big((\nabla y_0)^{\mathsf{T}}\nabla\vec{p} + (\nabla V)^{\mathsf{T}}\nabla\vec{b}_0\big)_{ij} \qquad \text{for all } i,j = 1,2, \tag{8.6}$$

proving the claim.

Consequently, the second condition in (8.4) is equivalent to S being constant, to the effect that $\nabla V = \nabla(Sy_0)$, or equivalently that $V - Sy_0$ is a constant vector. In this case:

$$\mathbb{S} = \frac{1}{2}(\nabla y_0)^{\mathsf{T}}\nabla\big(S^2 y_0\big) + \frac{1}{24}\big((\nabla y_0)^{\mathsf{T}}\nabla\tilde{d}_0\big)_{\mathrm{sym}} = \frac{1}{2}\Big((\nabla y_0)^{\mathsf{T}}\nabla\big(S^2 y_0 + \frac{1}{12}\tilde{d}_0\big)\Big)_{\mathrm{sym}}$$

is equivalent to the first condition in (8.4), as $(\nabla V)^{\mathsf{T}}\nabla V = -(\nabla y_0)^{\mathsf{T}}S^2\nabla y_0$. The proof is done. □

From Theorem 8.2 we deduce its quantitative version, that is a counterpart of Theorem 4.1 in the present von Kármán regime:

Theorem 8.3. *In the non-oscillatory setting (NO), assume (8.2). Then for all $V \in \mathscr{V}$ there holds:*

$$\mathrm{dist}^2_{W^{2,2}(\omega,\mathbb{R}^3)}\Big(V,\ \big\{Sy_0 + c;\ S \in so(3),\ c \in \mathbb{R}^3\big\}\Big) \leq C \int_\omega \mathcal{Q}_2\big(x', (\nabla y_0)^{\mathsf{T}}\nabla \vec{p} + (\nabla V)^{\mathsf{T}}\nabla \vec{b}_0\big)\,\mathrm{d}x' \tag{8.7}$$

with a constant $C > 0$ that depends on G, ω and W but it is independent of V.

Proof. We argue by contradiction. Since $\mathscr{V}_{lin} \doteq \{Sy_0 + c;\ S \in so(3),\ c \in \mathbb{R}^3\}$ is a linear subspace of $\mathscr{V}$ and likewise the expression III in (6.7) is linear in V, with its kernel equal to $\mathscr{V}_{lin}$ in virtue of Theorem 8.2, it suffices to take a sequence $\{V_n \in \mathscr{V}\}_{n\to\infty}$ such that:

$$\begin{aligned} &\|V_n\|_{W^{2,2}(\omega,\mathbb{R}^3)} = 1, \qquad V_n \perp_{W^{2,2}(\omega,\mathbb{R}^3)} \mathscr{V}_{lin} \qquad \text{for all } n,\\ &\text{and:}\quad (\nabla y_0)^{\mathsf{T}}\nabla \vec{p}_n + (\nabla V_n)^{\mathsf{T}}\nabla \vec{b}_0 \to 0 \quad \text{strongly in } L^2(\omega, \mathbb{R}^{2\times 2}), \quad \text{as } n \to \infty. \end{aligned} \tag{8.8}$$

Passing to a subsequence if necessary and using the definition of $\vec{p}$ in (6.4), it follows that:

$$V_n \rightharpoonup V \quad \text{weakly in } W^{2,2}(\omega,\mathbb{R}^3), \qquad \vec{p}_n \rightharpoonup \vec{p} \quad \text{weakly in } W^{1,2}(\omega,\mathbb{R}^3). \tag{8.9}$$

Clearly, $Q_0^{\mathsf{T}}\big[\nabla V,\ \vec{p}\big] \in L^2(\omega, so(3))$ so that $V \in \mathscr{V}$, but also $(\nabla y_0)^{\mathsf{T}}\nabla \vec{p} + (\nabla V)^{\mathsf{T}}\nabla \vec{b}_0 = 0$. Thus, Theorem 8.2 and the perpendicularity assumption in (8.8) imply: $V = \vec{p} = 0$. We will now prove:

$$V_n \to 0 \quad \text{strongly in } W^{2,2}(\omega,\mathbb{R}^3), \tag{8.10}$$

which will contradict the first (normalisation) condition in (8.7).

As in (8.5), the assumption $V_n \in \mathscr{V}$ implies that for each $x' \in \omega$ and $i = 1,2$, the following matrix (denoted previously by $\bar{S}^i$) is skew-symmetric:

$$\bar{Q}_0^{\mathsf{T}}\big[\nabla \partial_i V_n,\ \partial \vec{p}_n\big] + \big[\nabla V_n,\ \vec{p}_n\big]^{\mathsf{T}}(\partial_i Q_0) \in so(3).$$

Equating tangential entries and observing (8.8), yields for every $i, j, k = 1, 2$:

$$\langle \partial_j y_0, \partial_{ik} V_n\rangle + \langle \partial_k y_0, \partial_{ij} V_n\rangle = -\Big(\langle \partial_j V_n, \partial_{ik} y_0\rangle + \langle \partial_k V_n, \partial_{ij} y_0\rangle\Big) \to 0 \quad \text{strongly in } L^2(\omega).$$

Permuting i, j, k we eventually get:

$$\langle \partial_j y_0, \partial_{ik} V_n\rangle \to 0 \quad \text{strongly in } L^2(\omega) \qquad \text{for all } \ i, j, k = 1, 2.$$

On the other hand, equating off-tangential entries, we get by (8.8) and (8.9) that for each $i = 1, 2$:

$$\langle \vec{b}_0, \partial_{ij} V_n\rangle = -\big((\nabla y_0)^{\mathsf{T}}\nabla \vec{p}_n + (\nabla V_n)^{\mathsf{T}}\nabla \vec{b}_0\big)_{ij} - \langle \vec{p}_n, \partial_{ij} y_0\rangle \to 0 \quad \text{strongly in } L^2(\omega).$$

Consequently, $\{Q_0^{\mathsf{T}}\partial_{ij} V_n \to 0\}_{i,j=1,2}$ in $L^2(\omega, \mathbb{R}^3)$, which implies convergence (8.10) as claimed. This ends the proof of (8.7). □

Remark 8.4. Although the kernel of the (nonlinear) energy $\mathcal{I}_4$, displayed in Theorem 8.2, is finite dimensional, the full coercivity estimate of the form below is *false*:

$$\begin{aligned} \min_{S\in so(3), c\in\mathbb{R}^3} &\Big(\|V - (Sy_0 + c)\|^2_{W^{2,2}(\omega,\mathbb{R}^3)} + \|\mathbb{S} - \frac{1}{2}\big((\nabla y_0)^{\mathsf{T}}\nabla\big(S^2 y_0 - \frac{1}{12}\tilde{d}_0\big)\big)_{\mathrm{sym}}\|^2_{L^2(\omega,\mathbb{R}^{2\times 2})}\Big)\\ &\leq C\mathcal{I}_4(V, \mathbb{S}) \qquad \text{for all } \ (V, \mathbb{S}) \in \mathscr{V} \times \mathscr{S}. \end{aligned} \tag{8.11}$$

For a counterexample, consider the particular case of classical von Kármán functional (7.5), specified in Remark 7.4. Clearly, $\mathcal{I}_4(v, w) = 0$ if an only if $v(x) = \langle a, x\rangle + \alpha$ and $w(x) = \beta x^\perp - \frac{1}{2}\langle a, x\rangle a + \gamma$, for some $a \in \mathbb{R}^2$ and $\alpha, \beta, \gamma \in \mathbb{R}$. Note that (8.7) reflects then the Poincaré inequality: $\int_\omega |\nabla v - \fint_\omega \nabla v|^2 \,\mathrm{d}x' \leq C \int_\omega |\nabla^2 v|^2 \,\mathrm{d}x'$, whereas (8.11) takes the form:

$$\min_{a\in\mathbb{R}^2} \Big(\int_\omega |\nabla v - a|^2 \,\mathrm{d}x' + \int_\omega |\mathrm{sym}\,\nabla w + \frac{1}{2} a \otimes a|^2 \,\mathrm{d}x' \Big) \leq C\mathcal{I}_4(v, w). \tag{8.12}$$

Let $\omega = B_1(0)$. Given $v \in W^{2,2}(\omega)$ such that $\det \nabla^2 v = 0$, let w satisfy: $\mathrm{sym}\,\nabla w = -\frac{1}{2}\nabla v \otimes \nabla v$, which results in vanishing of the first term in (7.5). Neglecting the first term in the left hand side of (8.12), leads in this context to the following weaker form, which we below disprove:

$$\min_{a\in\mathbb{R}^2} \int_\omega |\nabla v \otimes \nabla v - a \otimes a|^2 \,\mathrm{d}x' \leq C \int_\omega |\nabla^2 v|^2 \,\mathrm{d}x'. \tag{8.13}$$

Define $v_n(x) = n(x_1+x_2)+\frac{1}{2}(x_1+x_2)^2$ for all $x = (x_1, x_2) \in \omega$. Then $\nabla v_n = (n+x_1+x_2)(1, 1)$ and $\det \nabla^2 v_n = 0$. Minimization in (8.13) becomes: $\min_{a\in\mathbb{R}^2} \int_\omega |(n + x_1 + x_2)^2(1, 1) \otimes (1, 1) - a \otimes a|^2 \,\mathrm{d}x'$ and an easy explicit calculation yields the necessary form of the minimizer: $a = \delta(1, 1)$. Thus, the same minimization can be equivalently written and estimated in:

$$4 \cdot \min_{\delta\in\mathbb{R}} \int_\omega \big|(n + x_1 + x_2)^2 - \delta^2\big|^2 \,\mathrm{d}x' \sim 4n^2 \to \infty \qquad \text{as } n \to \infty.$$

On the other hand, $|\nabla^2 v_n|^2 = 4$ at each $x' \in \omega$. Therefore, the estimate (8.13) cannot hold. ■

9. Beyond the von Kármán regime: an example

Given a function $\phi \in \mathcal{C}^\infty((-\frac{1}{2}, \frac{1}{2}), \mathbb{R})$, consider the conformal metric:

$$G(x', x_3) = e^{2\phi(x_3)} Id_3 \qquad \text{for all } x = (x', x_3) \in \Omega^h.$$

The midplate metric $\bar{\mathcal{G}}_{2\times2} = e^{2\phi(0)} Id_2$ has a smooth isometric immersion $y_0 = e^{\phi(0)} id_2 : \omega \to \mathbb{R}^2$ and thus by Theorem 2.4 there must be:

$$\inf \mathcal{E}^h \leq Ch^2.$$

By a computation, we get that the only possibly non-zero Christoffel symbols of G are: $\Gamma^3_{11} = \Gamma^3_{22} = -\phi'(x_3)$ and $\Gamma^1_{13} = \Gamma^2_{23} = \Gamma^3_{33} = \phi'(x_3)$, while the only possibly nonzero Riemann curvatures are:

$$R_{1212} = -\phi'(x_3)^2 e^{2\phi(x_3)}, \qquad R_{1313} = R_{2323} = -\phi''(x_3) e^{2\phi(x_3)}. \tag{9.1}$$

Consequently, the results of this paper provide the following hierarchy of possible energy scalings:

(a) $\{ch^2 \leq \inf \mathcal{E}^h \leq Ch^2\}_{h\to0}$ with $c, C > 0$. This scenario is equivalent to $\phi'(0) \neq 0$. The functionals $\frac{1}{h^2}\mathcal{E}^h$ as in Theorems 2.1, 2.3 and 2.4 exhibit the indicated compactness properties and Γ-converge to the following energy $\mathcal{I}_2$ defined on the set of deformations: $\{y \in W^{2,2}(\omega, \mathbb{R}^3);\ (\nabla y)^{\mathsf{T}}\nabla y = Id_2\}$:

$$\mathcal{I}_2(y) = \frac{1}{24} \int_\omega \mathcal{Q}_2\big(\Pi_y - \phi'(0) Id_2\big) \,\mathrm{d}x'.$$

Here $\mathcal{Q}_2(F_{2\times2}) = \min\big\{D^2W(Id_3)(\tilde{F} \otimes \tilde{F});\ \tilde{F} \in \mathbb{R}^{3\times3} \text{ with } \tilde{F}_{2\times2} = F_{2\times2}\big\}$.

(b) $\{ch^4 \leq \inf \mathcal{E}^h \leq Ch^4\}_{h\to0}$ with $c, C > 0$. This scenario is equivalent to $\phi'(0) = 0$ and $\phi''(0) \neq 0$. The unique (up to rigid motions) minimizing isometric immersion is then $id_2 : \omega \to \mathbb{R}^2$ and the functionals $\frac{1}{h^4}\mathcal{E}^h$ have the compactness and Γ-convergence properties as in Theorems

6.1, 6.2 and 6.3. The following limiting functional $\mathcal{I}_4$ is defined on the set of displacements $\{(v,w)\in W^{2,2}(\omega,\mathbb{R})\times W^{1,2}(\omega,\mathbb{R}^2)\}$ as in Remark 7.4:

$$\mathcal{I}_4(v,w)=\frac{1}{2}\int_\omega \mathcal{Q}_2\big(\operatorname{sym}\nabla w+\frac{1}{2}\nabla v\otimes\nabla v-\frac{1}{24}\phi''(0)Id_2\big)\,\mathrm{d}x'$$
$$+\frac{1}{24}\int_\omega \mathcal{Q}_2\big(\nabla^2 v\big)\,\mathrm{d}x'+\frac{1}{1440}\phi''(0)^2|\omega|\mathcal{Q}_2\big(Id_2\big).$$

(c) $\{\inf\mathcal{E}^h\leq Ch^6\}_{h\to 0}$ with $C>0$. This scenario is equivalent to $\phi'(0)=0$ and $\phi''(0)=0$ (in agreement with Lemma 10.1) and in fact we have the following more precise result below.

Theorem 9.1. *Let $G(x',x_3)=e^{2\phi(x_3)}Id_3$, where $\phi^{(k)}(0)=0$ for $k=1\ldots n-1$ up to some $n>2$. Then:* $\inf\mathcal{E}^h\leq Ch^{2n}$ *and:*

$$\lim_{h\to 0}\frac{1}{h^{2n}}\inf\mathcal{E}^h\geq c_n\,\phi^{(n)}(0)^2|\omega|\mathcal{Q}_2(Id_2),\tag{9.2}$$

where $c_n>0$. In particular, if $\phi^{(n)}(0)\neq 0$ then we have: $ch^{2n}\leq\inf\mathcal{E}^h\leq Ch^{2n}$ with $c,C>0$.

Proof. **1.** For the upper bound, we compute:

$$\mathcal{E}^h\big(e^{\phi(0)}id_3\big)=\frac{1}{h}\int_{\Omega^h}W\big(e^{\phi(0)-\phi(x_3)}Id_3\big)\,\mathrm{d}x=\frac{1}{2h}\int_{\Omega^h}\mathcal{Q}_3\big(\phi^{(n)}(0)\frac{x_3^n}{n!}Id_3\big)+\mathcal{O}(h^{2n+2})\,\mathrm{d}x$$
$$=h^{2n}\bigg(\frac{\phi^{(n)}(0)^2}{(n!)^2}\frac{1}{(2n+1)2^{2n+1}}|\omega|\mathcal{Q}_3(Id_3)+o(1)\bigg)\leq Ch^{2n},$$

where we used the fact that $e^{\phi(0)-\phi(x_3)}=1-\phi^{(n)}(0)\frac{x_3^n}{n!}+\mathcal{O}(|x_3|^{n+1})$.

2. To prove the lower bound (9.2), let $\{u^h\in W^{1,2}(\Omega^h,\mathbb{R}^3)\}_{h\to 0}$ satisfy $\mathcal{E}^h(u^h)\leq Ch^{2n}$. Then:

$$\mathcal{E}^h(u^h)\geq\frac{c}{h}\int_{\Omega^h}\operatorname{dist}^2\big(\nabla u^h,e^{\phi(x_3)}SO(3)\big)\,\mathrm{d}x$$
$$\geq\frac{c}{h}\int_{\Omega^h}\operatorname{dist}^2\big(\nabla u^h,e^{\phi(0)}SO(3)\big)\,\mathrm{d}x-\frac{\bar c}{h}\int_{\Omega^h}\big|\phi^{(n)}(0)\frac{x_3^n}{n!}+\mathcal{O}(h^{n+1})\big|^2\,\mathrm{d}x,$$

which results in: $\frac{1}{h}\int_{\Omega^h}\operatorname{dist}^2\big(e^{-\phi(0)}\nabla u^h,SO(3)\big)\,\mathrm{d}x\leq Ch^{2n}$. Similarly as in Lemma 5.2 and Corollary 5.3, it follows that there exist approximating rotation fields $\{R^h\in W^{1,2}(\omega,SO(3))\}_{h\to 0}$ such that:

$$\frac{1}{h}\int_{\Omega^h}|\nabla u^h-e^{\phi(0)}R^h|^2\,\mathrm{d}x\leq Ch^{2n},\qquad \int_\omega|\nabla R^h|^2\,\mathrm{d}x\leq Ch^{2n-2}.\tag{9.3}$$

As in sections 2 and 6, we define the following displacement and deformation fields:

$$y^h(x',x_3)=(\bar R^h)^{\mathsf T}\big(u^h(x',hx_3)-\fint_{\Omega^h}u^h\big)\in W^{1,2}(\Omega,\mathbb{R}^3),\quad\text{where}\quad \bar R^h=\mathbb{P}_{SO(3)}\fint_{\Omega^h}e^{-\phi(0)}\nabla u^h(x)\,\mathrm{d}x,$$
$$V^h(x')=\frac{1}{h^{n-1}}\int_{-1/2}^{1/2}y^h(x',x_3)-e^{\phi(0)}\big(id_2+hx_3e_3\big)\,\mathrm{d}x_3\in W^{1,2}(\omega,\mathbb{R}^3).$$

In view of (9.3), we obtain then the following convergences (up to a not relabelled subsequence):

$$y^h\to e^{\phi(0)}id_2\quad\text{in }W^{1,2}(\omega,\mathbb{R}^3),\qquad \frac{1}{h}\partial_3y^h\to e^{\phi(0)}e_3\ \text{in}\ L^2(\omega,\mathbb{R}^3),$$
$$V^h\to V\in W^{2,2}(\omega,\mathbb{R}^3)\quad\text{in }W^{1,2}(\omega,\mathbb{R}^3),\quad \frac{1}{h}(\nabla V^h)_{2\times 2,\mathrm{sym}}\rightharpoonup\operatorname{sym}\nabla w\quad\text{weakly in }L^2(\omega,\mathbb{R}^{2\times 2}).$$

This allows to conclude the claimed lower bound:

$$\begin{aligned}\liminf_{h\to 0}\frac{1}{h^{2n}}\mathcal{E}^h(u^h) &\geq \frac{1}{2}\Big\|e^{-\phi(0)}\mathrm{sym}\,\nabla w - x_3 e^{-\phi(0)}\nabla^2 V^3 - \phi^{(n)}(0)\frac{x_3^n}{n!}Id_2\Big\|^2_{\mathcal{Q}_2}\\ &\geq \frac{1}{2}\Big\|\phi^{(n)}(0)\frac{x_3^n}{n!}Id_2 - \mathbb{P}_1\Big(\phi^{(n)}(0)\frac{x_3^n}{n!}Id_2\Big)\Big\|^2_{\mathcal{Q}_2}\\ &= \frac{1}{2}\frac{\phi^{(n)}(0)^2}{(n!)^2}\cdot\int_{-1/2}^{1/2}\big(x_3^n - \mathbb{P}_1(x_3^n)\big)^2\,\mathrm{d}x_3\cdot|\omega|\mathcal{Q}_2(Id_2),\end{aligned}$$

as in(9.2), with the following constant c_n:

$$c_n = \frac{1}{2^{2n+1}(n!)^2}\begin{cases}\frac{(n-1)^2}{(2n+1)(n+2)^2} & \text{for } n \text{ odd}\\ \frac{n^2}{(2n+1)(n+1)^2} & \text{for } n \text{ even.}\end{cases}$$

Observe that $c_2 = \frac{1}{1440}$, consistently with the previous direct application of Theorem 6.2. □

10. Beyond the von Kármán regime: preliminary results for Ch^6 scaling regime

In this section, we focus on the non-oscillatory case in the general setting (NO) and derive the equivalent of Lemma 7.2 at the next order scaling, which turns out to be Ch^6. Our findings are consistent with those of the example in section 9. Similarly to Lemma 5.1 we first construct the recovery sequence with energy smaller than that in the von Kármán regime:

Lemma 10.1. *Assume (NO) and write:*

$$G(x) = G(x',0) + x_3\partial_3 G(x',0) + \frac{x_3^2}{2}\partial_{33}G(x',0) + \mathcal{O}(|x_3|^3) \qquad \textit{for all } x = (x',x_3)\in\Omega^h.$$

If there holds: $\lim_{h\to 0}\frac{1}{h^4}\mathcal{E}^h = 0$, *then we automatically have:*

$$\inf\mathcal{E}^h \leq Ch^6.$$

Proof. Under the assumption (8.2) we set the sequence of deformations $\{u^h:\bar\Omega\to\mathbb{R}^3\}_{h\to 0}$ to be:

$$u^h(x',x_3) = y_0(x') + x_3\vec b_0(x') + \frac{x_3^2}{2}\tilde d_0(x') + \frac{x_3^3}{6}\tilde e_0(x'),$$

where $y_0, \vec b_0$ are as in (5.2) and $\tilde d_0$ as in (5.5). The new vector field $\tilde e_0:\bar\omega\to\mathbb{R}^3$ is defined through the last formula below, in view of $\bar{II}=0$ in (7.2):

$$\begin{aligned}&Q_0^{\mathsf T}Q_0 = G(x',0), \qquad \big(Q_0^{\mathsf T}\tilde P_0\big)_{\mathrm{sym}} = \frac{1}{2}\partial_3 G(x',0),\\ &\tilde P_0^{\mathsf T}\tilde P_0 + \big(Q_0^{\mathsf T}\tilde D_0\big)_{\mathrm{sym}} = \frac{1}{2}\partial_{33}G(x',0).\end{aligned}\tag{10.1}$$

We will use the following matrix fields definitions:

$$Q_0 = \big[\partial_1 y_0,\ \partial_2 y_0,\ \vec b_0\big], \qquad \tilde P_0 = \big[\partial_1\vec b_0,\ \partial_2\vec b_0,\ \tilde d_0\big], \qquad \tilde D_0 = \big[\partial_1\tilde d_0,\ \partial_2\tilde d_0,\ \tilde e_0\big].\tag{10.2}$$

Consequently:

$$\nabla u^h = Q_0 + x_3\tilde P_0 + \frac{x_3^2}{2}\tilde D_0 + \frac{x_3^3}{6}\big[\partial_1\tilde e_0,\ \partial_2\tilde e_0,\ 0\big],$$

whereas writing $A = G^{1/2}$, the expansion (6.9) becomes:

$$A(x)^{-1} = \bar{A}(x')^{-1} + \bar{A}(x')^{-1}\Big(- x_3\bar{\partial}_3 A(x',0) + x_3^2\partial_3 A(x',0)\bar{A}(x')^{-1}\partial_3 A(x',0) - \frac{x_3^2}{2}\partial_{33}A(x',0)\Big)\bar{A}(x')^{-1} + \mathcal{O}(|x_3|^3) \qquad \text{for all } \ x=(x',x_3)\in\Omega^h.$$

We thus obtain the following expression:

$$\nabla u^h(x)A(x)^{-1} = (Q_0\bar{A}^{-1})(x')\big(Id_3 + x_3 S_1(x') + \frac{x_3^2}{2}S_2(x')\big) + \mathcal{O}(|x_3|^3), \tag{10.3}$$

with:

$$S_1 = \bar{A}^{-1}\big(Q_0^{\mathsf{T}}\tilde{P}_0 - \bar{A}\partial_3 A\big)\bar{A}^{-1},$$
$$S_2 = \bar{A}^{-1}\big(Q_0^{\mathsf{T}}\tilde{D}_0 - 2Q_0^{\mathsf{T}}\tilde{P}_0\bar{A}^{-1}\partial_3 A + 2\bar{A}(\partial_3 A)\bar{A}^{-1}\partial_3 A - \bar{A}\partial_{33}A\big)\bar{A}^{-1}.$$

As in the proof of Lemma 5.1, we now get:

$$\begin{aligned} W\big(\nabla u^h(x)A(x)^{-1}\big) &= W\Big(Id_3 + x_3 S_1(x')_{\rm sym} + \frac{x_3^2}{2}\big(S_2(x')_{\rm sym} + S_1(x')^{\mathsf{T}}S_1(x')\big) + \mathcal{O}(|x_3|^3)\Big) \\ &= W\big(Id_3 + \mathcal{O}(|x_3|^3)\big) = \mathcal{O}(h^6). \end{aligned} \tag{10.4}$$

The final equality follows from:

$$(S_1)_{\rm sym} = \bar{A}^{-1}\Big((Q_0^{\mathsf{T}}\tilde{P}_0)_{\rm sym} - (\bar{A}\partial_3 A)_{\rm sym}\Big)\bar{A}^{-1} = \bar{A}^{-1}\Big((Q_0^{\mathsf{T}}\tilde{D}_0)_{\rm sym} - \frac{1}{2}\partial_3 G\Big)\bar{A}^{-1} = 0,$$

$$\begin{aligned}(S_2)_{\rm sym} + S_1^{\mathsf{T}}S_1 = \bar{A}^{-1}\Big(&(Q_0^{\mathsf{T}}\tilde{P}_0)_{\rm sym} + 2\big(\bar{A}(\partial_3 A)\bar{A}^{-1}\partial_3 A\big)_{\rm sym} - \big(\bar{A}\partial_{33}A\big)_{\rm sym} - 2\big(Q_0^{\mathsf{T}}\tilde{P}_0\bar{A}^{-1}\partial_3 A\big)_{\rm sym} \\ &+ \big(Q_0^{\mathsf{T}}\tilde{P}_0 - \bar{A}\partial_3 A\big)^{\mathsf{T}}\bar{A}^{-2}\big(Q_0^{\mathsf{T}}\tilde{P}_0 - \bar{A}\partial_3 A\big)\Big)\bar{A}^{-1},\end{aligned}$$

so that:

$$\begin{aligned}(S_2)_{\rm sym} + S_1^{\mathsf{T}}S_1 &= \bar{A}^{-1}\Big((Q_0^{\mathsf{T}}\tilde{D}_0)_{\rm sym} + \tilde{P}_0^{\mathsf{T}}\tilde{P}_0 - \frac{1}{2}\partial_{33}G + (\partial_3 A)^2 - 2\big(Q_0^{\mathsf{T}}\tilde{P}_0\big)_{\rm sym}\bar{A}^{-1}\partial_3 A \\ &\qquad - 2(\partial_3 A)\bar{A}^{-1}\big(Q_0^{\mathsf{T}}\tilde{P}_0\big)_{\rm sym} + 2\big(\bar{A}(\partial_3 A)\bar{A}^{-1}\partial_3 A\big)_{\rm sym}\Big)\bar{A}^{-1} \\ &= \bar{A}^{-1}\Big((\partial_3 A)^2 - 2\big((\partial_3 G)\bar{A}^{-1}\partial_3 A + \bar{A}(\partial_3 A)\bar{A}^{-1}(\partial_3 A)\big)_{\rm sym}\Big)\bar{A}^{-1} = 0,\end{aligned}$$

where we have repeatedly used the assumption (10.1). □

As in section 5, using the change of variable by the smooth deformation $Y = u^h$ in the proof of Theorem 10.1, one obtains the following approximations, by adjusting the proofs of Lemma 5.2 and Corollary 5.3:

Lemma 10.2. *Assume (NO) and (8.2). If y_0 is injective on an open, Lipschitz subset $\mathcal{V}\subset\omega$, then for every $u^h\in W^{1,2}(\mathcal{V}^h,\mathbb{R}^3)$ there exists $\bar{R}^h\in SO(3)$ such that:*

$$\frac{1}{h}\int_{\mathcal{V}^h}\Big|\nabla u^h(x) - \bar{R}^h\Big(Q_0(x') + x_3\tilde{P}_0(x') + \frac{x_3^2}{2}\tilde{D}_0(x')\Big)\Big|^2\,\mathrm{d}x \le C\big(\mathcal{E}^h(u^h,\mathcal{V}^h) + h^5|\mathcal{V}^h|\big),$$

with the smooth correction matrix fields $\tilde{P}_0, \tilde{D}_0$ in (10.2). The constant C above is uniform for all subdomains $\mathcal{V}^h\subset\Omega^h$ which are bi-Lipschitz equivalent with controlled Lipschitz constants.

Corollary 10.3. *Assume (NO). Then, for any sequence $\{u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)\}_{h\to 0}$ with: $\mathcal{E}^h(u^h) \leq Ch^6$, there exists a sequence of rotation-valued maps $R^h \in W^{1,2}(\omega, SO(3))$, such that:*

$$\frac{1}{h}\int_{\Omega^h}\Big|\nabla u^h(x) - R^h(x')\Big(Q_0(x') + x_3\tilde P_0(x') + \frac{x_3^2}{2}\tilde D_0(x')\Big)\Big|^2\,\mathrm{d}x \leq Ch^6,$$
$$\int_\omega |\nabla R^h(x')|^2\,\mathrm{d}x' \leq Ch^4.$$

We now make the following observation. Writing the non-oscillatory metric G in its proper third order Taylor's expansion: $G(x) = G(x',0) + x_3\partial_3 G(x',0) + \frac{x_3^2}{2}\partial_{33}G(x',0) + \frac{x_3^3}{6}\partial_{333}G(x',0) + o(x_3^4)$, one can readily check that the term $\mathcal{O}(|x_3|^3)$ in the right hand side of the formula (10.3) can be explicitly written as: $(Q_0\bar A^{-1})(x')\frac{x_3^3}{6}S_3(x') + o(|x_3|^3)$, where:

$$S_3 = \bar A^{-1}\Big(Q_0^\mathsf{T}\,[\partial_1\tilde e_0,\ \partial_2\tilde e_2,\ 0] - 3Q_0^\mathsf{T}\tilde D_0\bar A^{-1}\partial_3 A - 3Q_0^\mathsf{T}\tilde P_0\bar A^{-1}\partial_{33}A + 6Q_0^\mathsf{T}\tilde P_0\bar A^{-1}(\partial_3 A)\bar A^{-1}\partial_3 A\Big)\bar A^{-1}$$
$$+ \bar A\partial_{333}A^{-1}(x',0).$$

Consequently, (10.4) becomes:

$$W\big(\nabla u^h(x)A(x)^{-1}\big) = W\Big(Id_3 + \frac{x_3^3}{6}\big(S_3(x') + 3S_1(x')^\mathsf{T}S_2(x')\big)_{\mathrm{sym}} + o(|x_3|^3)\Big). \tag{10.5}$$

A tedious but direct inspection shows now that:

$$\big(S_3 + 3(S_1)^\mathsf{T}S_2\big)_{\mathrm{sym}} = \bar A^{-1}\Big(Q_0^\mathsf{T}\,[\partial_1\tilde e_0,\ \partial_2\tilde e_0,\ 0] + 3\tilde P_0^\mathsf{T}\tilde D_0 - \frac{1}{2}\partial_{333}G(x',0)\Big)_{\mathrm{sym}}\bar A^{-1},$$

and we see that the tensor playing the role similar to the curvature term $\bar{II}_{\mathrm{sym}}$, at the present h^6 scaling regime, which equals the 2×2 minor of the right hand side above after discarding the external multiplying factors $\bar A^{-1}$, has the form:

$$\Big((\nabla y_0)^\mathsf{T}\nabla\tilde e_0 + 3(\nabla\vec b_0)^\mathsf{T}\nabla\tilde d_0\Big)_{\mathrm{sym}} - \frac{1}{2}\partial_{333}G(x',0)_{2\times 2}$$

With the eye on future applications, we now identify this tensor in terms of the components R_{ijkl}. Recall that in section 9, the relevant curvature quantity corresponding to $n = 3$ was: $-\phi'''(0)e^{2\phi(0)}Id_2$, equal to $\partial_3\big[R_{i3j3}\big]_{i,j=1,2}(x',0)$ in view of (9.1). We have:

Theorem 10.4. *Assume (NO) and (8.2). Let $y_0, \vec b_0,\ \tilde d_0, \tilde e_0$ be as in (10.1), (10.2). Then for all $x' \in \omega$ we have: $\tilde e_0 = Q_0\big[\partial_3\Gamma_{33}^i + \Gamma_{p3}^i\Gamma_{33}^p\big]_{i=1\ldots 3}(x',0)$ and:*

$$\Big((\nabla y_0)^\mathsf{T}\nabla\tilde e_0 + 3(\nabla\vec b_0)^\mathsf{T}\nabla\tilde d_0\Big)_{\mathrm{sym}}(x') - \frac{1}{2}\partial_{333}G(x',0)_{2\times 2} = \partial_3\left[\begin{array}{cc} R_{1313} & R_{1323} \\ R_{1323} & R_{2323}\end{array}\right](x',0).$$

Proof. **1.** Recall that existence of smooth vector fields $y_0, \vec b_0,\ \tilde d_0, \tilde e_0$ satisfying condition (10.1) is equivalent to vanishing of the entire Riemann curvature tensor of the metric G on $\omega\times\{0\}$. Below, all equalities are valid at points $(x',0)$. Using (7.1) and the third identity in (10.1), we obtain:

$$\begin{aligned}\big(Q_0^\mathsf{T}\tilde e_0\big)_i &= \partial_{33}G_{i3} - \frac{1}{2}\partial_{i3}G_{33} - G_{pq}\Gamma_{33}^p\Gamma_{i3}^q = \partial_3\big(G_{pi}\Gamma_{33}^p\big) - G_{pq}\Gamma_{33}^p\Gamma_{i3}^q \\ &= G_{pi}\partial_3\Gamma_{33}^p + \big(\partial_3 G_{pi} - G_{pq}\Gamma_{i3}^q\big)\Gamma_{33}^p = G_{pi}\partial_3\Gamma_{33}^p + G_{qi}\Gamma_{p3}^q\Gamma_{33}^p \qquad \text{for } i = 1\ldots 3,\end{aligned}$$

by the Levi-Civita connection's compatibility in: $\nabla_3 G_{pi} = 0$. Consequently, it follows that:

$$Q_0^\mathsf{T}\tilde e_0 = G\big[\partial_3\Gamma_{33}^i + \Gamma_{p3}^i\Gamma_{33}^p\big]_{i=1\ldots 3}.$$

By the first equation in (10.1), we deduce the claimed formula for $\tilde e_0$.

2. Similarly, by (7.1), we obtain for all $i,j=1,2$:

$$\begin{aligned}\langle\partial_i y_0,\partial_j\tilde e_0\rangle &= \partial_j\langle\partial_i y_0,\tilde e_0\rangle-\langle\partial_{ij}y_0,\tilde e_0\rangle = \partial_j\big(G_{pi}\partial_3\Gamma^p_{33}+G_{qi}\Gamma^q_{p3}\Gamma^p_{33}\big)-G_{pq}\Gamma^p_{ij}\big(\partial_3\Gamma^q_{33}+\Gamma^q_{t3}\Gamma^t_{33}\big)\\ &= G_{is}\Big(\partial_{j3}\Gamma^s_{33}+\Gamma^s_{jq}\partial_3\Gamma^q_{33}+\partial_j\big(\Gamma^s_{p3}\Gamma^p_{33}\big)+\Gamma^s_{tj}\Gamma^t_{q3}\Gamma^q_{33}\Big)\\ &= G_{is}\Big(\partial_3\big(\partial_j\Gamma^s_{33}+\Gamma^s_{jp}\Gamma^p_{33}\big)+\big(\partial_j\Gamma^p_{33}+\Gamma^p_{qj}\Gamma^q_{33}\big)\Gamma^s_{3p}-\Gamma^q_{33}R^s_{q3j}\Big)\\ &= G_{is}\Big(\partial_3\big(\partial_j\Gamma^s_{33}+\Gamma^s_{jp}\Gamma^p_{33}\big)+\big(\partial_j\Gamma^p_{33}+\Gamma^p_{qj}\Gamma^q_{33}\big)\Gamma^s_{3p}\Big)\end{aligned}$$

where we have used $\nabla_j G_{pi}=0$ and the assumed condition $R^s_{q3j}=0$. Further:

$$\langle\partial_i\vec b_0,\partial_j\tilde d_0\rangle = G_{ps}\Gamma^p_{i3}\big(\partial_j\Gamma^s_{33}+\Gamma^s_{jq}\Gamma^q_{33}\big).$$

Consequently, it follows that:

$$\begin{aligned}\Big((\nabla y_0)^{\mathsf T}\nabla\tilde e_0+3(\nabla\vec b_0)^{\mathsf T}\nabla\tilde d_0\Big)_{ij} &= G_{is}\partial_3\big(\partial_j\Gamma^s_{33}+\Gamma^s_{jq}\Gamma^q_{33}\big)+\big(\partial_j\Gamma^p_{33}+\Gamma^p_{jq}\Gamma^q_{33}\big)\big(G_{is}\Gamma^s_{3p}+3G_{ps}\Gamma^s_{i3}\big)\\ &= G_{is}\partial_3\big(\partial_j\Gamma^s_{33}+\Gamma^s_{jq}\Gamma^q_{33}\big)+\big(\partial_3\Gamma^p_{j3}+\Gamma^q_{j3}\Gamma^p_{q3}\big)\big(2\partial_3 G_{ip}+\partial_i G_{3p}-\partial_p G_{3i}\big),\end{aligned}$$

by $R^p_{3j3}=0$. Observe also that: $\frac12\partial_{333}G_{ij}=\frac12\partial_{33}\big(G_{si}\Gamma^s_{3j}+G_{sj}\Gamma^s_{3i}\big)$. Expanding:

$$\begin{aligned}\partial_{33}\big(G_{si}\Gamma^s_{3j}\big) &= \partial_3\Big(G_{si}\partial_3\Gamma^s_{3j}+G_{si}\Gamma^s_{p3}\Gamma^p_{sj}+G_{sp}\Gamma^p_{i3}\Gamma^s_{3j}\Big)\\ &= G_{si}\partial_3\big(\partial_3\Gamma^s_{3j}+\Gamma^s_{3p}\Gamma^p_{3j}\big)+(\partial_3 G_{si})\big(\partial_3\Gamma^s_{j3}+\Gamma^s_{3p}\Gamma^p_{3j}\big)+\partial_3\big(G_{sp}\Gamma^p_{i3}\Gamma^s_{j3}\big)\end{aligned}$$

we finally obtain:

$$\begin{aligned}&\Big((\nabla y_0)^{\mathsf T}\nabla\tilde e_0+3(\nabla\vec b_0)^{\mathsf T}\nabla\tilde d_0\Big)_{ij}-\partial_{33}\big(G_{si}\Gamma^s_{3j}\big)\\ &\qquad= G_{si}\partial_3\Big(\partial_j\Gamma^s_{33}+\Gamma^s_{jq}\Gamma^q_{33}-\partial_3\Gamma^s_{3j}-\Gamma^s_{3p}\Gamma^p_{3j}\Big)\\ &\qquad\quad+\big(\partial_3\Gamma^p_{j3}+\Gamma^q_{j3}\Gamma^p_{3q}\big)\big(\partial_3 G_{pi}+\partial_i G_{p3}-\partial_p G_{si}\big)-\partial_3\big(G_{sp}\Gamma^p_{i3}\Gamma^s_{j3}\big)\\ &\qquad= G_{is}\partial_3 R^s_{3j3}+S_{ij},\end{aligned}\tag{10.6}$$

It now follows that:

$$\begin{aligned}S_{ij}+S_{ji} &= 2\big(\partial_3\Gamma^s_{j3}+\Gamma^q_{j3}\Gamma^s_{3q}\big)G_{ps}\Gamma^p_{3i}-\partial_3\big(G_{sp}\Gamma^p_{i3}\Gamma^s_{j3}\big)+2\big(\partial_3\Gamma^s_{i3}+\Gamma^q_{i3}\Gamma^s_{3q}\big)G_{ps}\Gamma^p_{3j}-\partial_3\big(G_{sp}\Gamma^p_{i3}\Gamma^s_{j3}\big)\\ &= 2G_{ps}\Big(\partial_3\big(\Gamma^s_{j3}\Gamma^p_{i3}\big)+\big(\Gamma^q_{j3}\Gamma^p_{i3}+\Gamma^q_{i3}\Gamma^p_{j3}\big)\Gamma^s_{3q}\Big)-2\partial_3\big(G_{sp}\Gamma^p_{i3}\Gamma^s_{j3}\big)\\ &= 2G_{ps}\Gamma^s_{3q}\big(\Gamma^q_{j3}\Gamma^p_{i3}+\Gamma^q_{i3}\Gamma^p_{j3}\big)-2\Gamma^p_{i3}\Gamma^s_{j3}\partial_3 G_{sp}=0.\end{aligned}$$

Hence, (10.6) results in:

$$\begin{aligned}&\Big((\nabla y_0)^{\mathsf T}\nabla\tilde e_0+3(\nabla\vec b_0)^{\mathsf T}\nabla\tilde d_0\Big)_{\mathrm{sym}\,,ij}-\frac12\partial_{333}G_{ij}\\ &\qquad= \frac12\Big(G_{is}\partial_3 R^s_{3j3}+G_{js}\partial_3 R^s_{3i3}\Big)=\frac12\partial_3\big(G_{is}R^s_{3j3}+G_{js}R^s_{3i3}\big)\\ &\qquad= \frac12\partial_3\big(R_{i3j3}+R_{j3i3}\big)=\partial_3 R_{i3j3},\end{aligned}$$

by $R^s_{3js}=R^s_{3i3}=0$. This completes the proof. □

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