

ON THE ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS TO PARABOLIC VARIATIONAL INEQUALITIES

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ABSTRACT. We consider various versions of the obstacle and thin-obstacle problems, we interpret them as variational inequalities, with non-smooth constraint, and prove that they satisfy a new *constrained Lojasiewicz inequality*. The difficulty lies in the fact that, since the constraint is non-analytic, the pioneering method of L. Simon ([25]) does not apply and we have to exploit a better understanding on the constraint itself. We then apply this inequality to two associated problems. First we combine it with an abstract result on parabolic variational inequalities, to prove the convergence at infinity of the strong global solutions to the parabolic obstacle and thin-obstacle problems to a unique stationary solution with a rate. Secondly, we give an abstract proof, based on a parabolic approach, of the epiperimetric inequality, which we then apply to the singular points of the obstacle and thin-obstacle problems.

1. INTRODUCTION

In this paper we consider parabolic variational inequalities of the form

$$\begin{cases} (u'(t) + \nabla \mathcal{F}(u(t))) \cdot (v - u(t)) \geq 0, & \text{for every } v \in \mathcal{K} \text{ and } t > 0, \\ u(0) = u_0 \in \mathcal{K}, \end{cases} \quad (1.1)$$

where \mathcal{F} is a given analytic integral functional, \mathcal{K} is a convex subset of $L^2(\Omega)$ and the dot stands for the scalar product in $L^2(\Omega)$; Ω being a smooth domain in \mathbb{R}^d or a d -dimensional manifold. We provide a new method for the study of the asymptotic behavior of the solution at infinity and we apply it to the parabolic obstacle and thin-obstacle problems, which are related to several relevant physical models (for more details and an extensive reference list we refer to the books [18, 13]).

In the absence of the constraint \mathcal{K} , the parabolic problem (1.1) reduces to the infinite dimensional gradient flow of \mathcal{F} , which is given by

$$u'(t) = -\nabla \mathcal{F}(u(t)) \quad \text{for every } t > 0, \quad u(0) = u_0.$$

In this case, it is well known (for more details we refer to Subsection 1.1) that the asymptotic behavior of the solution can be deduced from the so-called Lojasiewicz inequality, that is for every stationary point φ of \mathcal{F} there are constants $\gamma \in]0, 1/2]$, $C > 0$ such that

$$(\mathcal{F}(u) - \mathcal{F}(\varphi))_+^{1-\gamma} \leq C \|\nabla \mathcal{F}(u)\|_{L^2} \quad \text{for every } u \text{ in a neighborhood of } \varphi. \quad (1.2)$$

Precisely, (1.2) implies that:

There is a neighborhood \mathcal{U} of φ such that: if $u_0 \in \mathcal{U}$ and $\mathcal{F}(u(t)) \geq \mathcal{F}(\varphi)$, for every $t \geq 0$, then $u(t)$ converges, as $t \rightarrow \infty$, to a critical point u_∞ of \mathcal{F} , with a rate depending on γ . (1.3)

In the seminal paper [25], Leon Simon proved (1.3) for the flow associated to harmonic maps between two analytic manifolds. Notice that, also in this case, there is a geometric constraint given by the target manifold, but a change of coordinates allows to trivialize this constraint, while transforming the Dirichlet energy into an analytic functional \mathcal{F} . In [25], Simon showed that,

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thanks to the analyticity of this functional, (1.2) reduces to the classical Łojasiewicz inequality (1.6) for analytic functions on \mathbb{R}^n .

Suppose now that \mathcal{K} is a non-analytic convex set of L^2 , as in the case of the obstacle and the thin-obstacle problems. From one side, the non-smooth nature of \mathcal{K} does not allow the use of a Lagrange multiplier argument in order to replace the constraint by an additional analytic term in the energy \mathcal{F} . On the other hand, just the analyticity of \mathcal{F} , and the consequent (1.2), are not sufficient to obtain the convergence result (1.3) (see Example 1.4) since the geometry of the constraint may affect the flow.

In this paper, we introduce the following quantitative estimate that, slightly abusing the terminology, we call *constrained Łojasiewicz inequality*: for every stationary point $\varphi \in \mathcal{K}$ of \mathcal{F} there are constants $\gamma \in]0, 1/2]$, $C > 0$ such that

$$(\mathcal{F}(u) - \mathcal{F}(\varphi))_+^{1-\gamma} \leq C \|\nabla \mathcal{F}(u)\|_{\mathcal{K}} \quad \text{for every } u \in \mathcal{K} \text{ in a neighborhood of } \varphi, \quad (1.4)$$

where $\|\nabla \mathcal{F}(u)\|_{\mathcal{K}}$ is defined as

$$\|\nabla \mathcal{F}(u)\|_{\mathcal{K}} := \sup \left\{ 0, \sup_{v \in \mathcal{K} \setminus \{u\}} \frac{-(v-u) \cdot \nabla \mathcal{F}(u)}{\|v-u\|} \right\}.$$

We show that (1.4) can be used to determine the asymptotic behavior of the solutions of constrained gradient flows (1.1). Precisely, we prove that, under some mild natural assumptions on the functional \mathcal{F} and the constraint \mathcal{K} , the constrained Łojasiewicz inequality (1.4) still gives (1.3) (see Proposition 2.10). Thus, all the information, on the presence of the constraint and its properties, is now contained in this new constrained Łojasiewicz inequality. In particular, the estimate (1.4) becomes an intrinsic property of \mathcal{K} and \mathcal{F} , whose proof needs to be adapted to each specific situation. In particular we verify (1.4) for the obstacle and thin obstacle problems (see Section 4) thus leading to our main results on the associated parabolic flows (see Theorems 1.7 and 1.8).

Finally, let us remark that the constrained Łojasiewicz inequality (1.4), combined with a new construction based on the parabolic flow, also implies a logarithmic epi-perimetric inequality at the singular points of the (time-independent) obstacle and thin-obstacle problems (see Theorem 1.10), which was previously obtained in [9, 10] with a different proof. In particular, this implies the uniqueness of the blow-up limits and the logarithmic rate of convergence of the blow-up sequences at the singular free boundary points for these problems.

The paper is organized as follows. In Section 2 we prove that (1.4) implies the claim (1.3) and in Section 3 we show that (1.4) implies a logarithmic epi-perimetric inequality. In Section 4 we prove the constrained Łojasiewicz inequality for the obstacle and thin-obstacle problems, while in Section 5 we prove our main results on the parabolic obstacle and thin-obstacle problems.

In the rest of the present section, we introduce the obstacle and the thin-obstacle problems (Subsections 1.2 and 1.3); we state our main results in Subsection 1.4, while the next subsection is dedicated to the classical Łojasiewicz inequality for analytic functions, its applications and the relation to our results.

1.1. Łojasiewicz inequality on constrained domains. In this subsection we use several examples of constrained and unconstrained problems in order to illustrate the main novelty of this paper: *the constrained Łojasiewicz inequality*, which is a new Łojasiewicz-type estimate for constrained functionals. We go through the classical finite dimensional approach of Łojasiewicz and we argue on the effects of the geometric constraint \mathcal{K} on it. At the end of the Subsection, we make a connection with the infinite dimensional setting, by a simple model case.

Let $\mathcal{F} : \mathbb{R}^N \rightarrow \mathbb{R}$ be a given function. For any $\xi_0 \in \mathbb{R}^N$, consider the ODE

$$\xi'(t) = -\nabla \mathcal{F}(\xi(t)) \quad \text{for } t > 0, \quad \xi(0) = \xi_0. \quad (1.5)$$

The asymptotic behavior of the global solutions starting from a point ξ_0 in a neighborhood of x_0 is a problem of major interest in several fields. One of the conditions on the function \mathcal{F} , which implies that $\xi(t)$ admits a unique limit, as $t \rightarrow \infty$, is the so-called *Łojasiewicz inequality*, that is:

for every $\bar{\xi} \in \mathbb{R}^N$ critical point of \mathcal{F} , there exist an open neighborhood $U(\bar{\xi})$ of $\bar{\xi}$,
and constants $C > 0$ and $\gamma \in]0, 1/2]$, such that:

$$|\mathcal{F}(\xi) - \mathcal{F}(\bar{\xi})|^{1-\gamma} \leq C|\nabla\mathcal{F}(\xi)| \quad \text{for every } \xi \in U(\bar{\xi}). \quad (1.6)$$

The following result is essentially due to Łojasiewicz (see [22]). We sketch the proof below and for more details, we refer to the proof of Proposition 2.10.

Proposition 1.1 (Łojasiewicz decay-rate condition). *If (1.6) holds and $\xi : \mathbb{R}^+ \rightarrow \mathbb{R}^N$ is a bounded solution of (1.5), then the limit $\xi_\infty := \lim_{t \rightarrow \infty} \xi(t)$ exists and*

$$|\xi(t) - \xi_\infty| \leq \begin{cases} Ct^{-\frac{\gamma}{1-2\gamma}} & \text{if } \gamma < 1/2; \\ Ce^{-t} & \text{if } \gamma = 1/2. \end{cases} \quad (1.7)$$

Proof. Let K be a compact set such that $\xi(t) \in K$, for every $t > 0$. Furthermore, let us suppose that K is contained in the neighborhood of a critical point $\bar{\xi}$, where the Łojasiewicz inequality (1.6) does hold; this assumption is not necessary (see Proposition 2.10), but it simplifies the proof and allows us to concentrate on the main idea. First, using (1.5), we calculate

$$\mathcal{F}(\xi(t)) - \min_K \mathcal{F} \geq \mathcal{F}(\xi(t)) - \mathcal{F}(\xi(T)) = - \int_t^T \xi'(s) \cdot \nabla\mathcal{F}(\xi(s)) ds = \int_t^T |\nabla\mathcal{F}(\xi(s))|^2 ds,$$

for every $0 \leq t < T < \infty$. In particular, $t \mapsto \mathcal{F}(\xi(t))$ is non-increasing and the limit $\lim_{t \rightarrow \infty} \mathcal{F}(\xi(t))$ exists and is finite. Let y be any limit point of $\xi(t)$, as $t \rightarrow \infty$. Then, we have

$$\lim_{t \rightarrow \infty} \mathcal{F}(\xi(t)) = \mathcal{F}(y) \quad \text{and} \quad \mathcal{F}(\xi(t)) > \mathcal{F}(y) \quad \text{for every } t > 0.$$

On the other hand, $t \mapsto |\nabla\mathcal{F}(\xi(t))|^2$ is integrable at infinity and so, there is a sequence $t_n \rightarrow \infty$ such that $|\nabla\mathcal{F}(\xi(t_n))| \rightarrow 0$. This, together with (1.6), implies that $\lim_{n \rightarrow \infty} \mathcal{F}(\xi(t_n)) = \mathcal{F}(\bar{\xi}) = \mathcal{F}(y)$.

We now set

$$f(t) := \mathcal{F}(\xi(t)) - \mathcal{F}(\bar{\xi}) = \int_t^\infty |\nabla\mathcal{F}(\xi(s))|^2 ds < \infty, \quad \text{thus} \quad f'(t) = -|\nabla\mathcal{F}(\xi(t))|^2.$$

By (1.6), we obtain the differential inequality

$$-f'(t) \geq C(\mathcal{F}(\xi(t)) - \mathcal{F}(\bar{\xi}))^{2(1-\gamma)} \geq Cf(t)^{2(1-\gamma)},$$

which provides a decay rate for f at infinity. Precisely, $f(t) \leq Ct^{-\frac{1}{1-2\gamma}}$. On the other hand, using again the equation and the Cauchy-Schwarz inequality, we get

$$|\xi(t) - \xi(T)| \leq \int_t^T |\xi'(s)| ds = \int_t^T |\nabla\mathcal{F}(\xi(s))| ds \leq f(t)^{1/2}(T-t)^{1/2}. \quad (1.8)$$

Applying this inequality first to $t = 2^n$ and $T = 2^{n+1}$, we get that the limit $\xi_\infty := \lim_{n \rightarrow \infty} \xi(2^n)$ exists and that the rate of convergence is given precisely by (1.7). The inequality for any $t > 0$ follows again by (1.8) and implies that $\xi_\infty = y$. \square

Remark 1.2. It was proved by Łojasiewicz (see [22]) that the inequality (1.6) holds whenever the function $\mathcal{F} : \mathbb{R}^N \rightarrow \mathbb{R}$ is analytic, while it is well known that the previous proposition is false in general if \mathcal{F} is only C^∞ .

Suppose next that $\mathcal{F} : \mathbb{R}^N \rightarrow \mathbb{R}$ is analytic, so that the Łojasiewicz inequality holds for \mathcal{F} at every critical point, and that $\mathcal{K} \subset \mathbb{R}^N$ is a (smooth) open convex set. Then the gradient flow $\xi : \mathbb{R}^+ \rightarrow \mathbb{R}^N$ of \mathcal{F} in \mathcal{K} exists and satisfies

$$\xi'(t) := \begin{cases} -\nabla\mathcal{F}(\xi(t)) & \text{if } \xi(t) \in \mathcal{K}, \\ -P_{\xi(t)}(\nabla\mathcal{F}(\xi(t))) & \text{if } \xi(t) \in \partial\mathcal{K}, \end{cases} \quad (1.9)$$

where $P_{\xi(t)}$ is the projection on the tangent space to ∂K at the point $\xi(t)$. This can be equivalently formulated as a variational inequality, that is

$$\langle \xi'(t) + \nabla \mathcal{F}(\xi(t)), \zeta - \xi(t) \rangle \geq 0 \quad \text{for every } \zeta \in \mathcal{K} \quad \text{and } t > 0.$$

In this setting, the inequality (1.6) is not sufficient to conclude the convergence of the flow with a rate, since it corresponds only to the first regime of (1.9), when $\xi(t) \in \mathcal{K}$. Thus, we replace (1.6) with the following *constrained Lojasiewicz inequality*, which takes into account both regimes:

$$|\mathcal{F}(\xi) - \mathcal{F}(\bar{\xi})|^{1-\gamma} \leq C_L \|\nabla \mathcal{F}(\xi)\|_{\mathcal{K}} \quad \text{for every } \xi \in U \cap \mathcal{K}, \quad (1.10)$$

where we define

$$\|\nabla \mathcal{F}(\xi)\|_{\mathcal{K}} := \sup \left\{ 0, \sup_{\zeta \in \mathcal{K} \setminus \{\xi\}} \frac{-(\zeta - \xi) \cdot \nabla \mathcal{F}(\xi)}{\|\zeta - \xi\|} \right\}.$$

Using the inequality (1.10) as in Proposition 1.1, one can prove the convergence of the flow (1.9), under the proper assumptions (see Proposition 2.10). However, this inequality is more difficult to prove as $\|\cdot\|_{\mathcal{K}}$ is smaller than the usual Euclidean norm. In particular, it vanishes on the boundary of \mathcal{K} , whenever the gradient points outside the constraint. This means that the constraint itself generates new critical points and makes the proof more challenging.

Let $\bar{\xi} \in \partial \mathcal{K}$ be a critical point for \mathcal{F} and let the constraint \mathcal{K} be (locally) the graph of a convex function $\eta : \mathbb{R}^{N-1} \rightarrow \mathbb{R}$. We consider two examples: in the first one, the decay estimate (1.10) does hold, while in the second one, it fails.

Example 1.3 (Constraint with an analytic boundary). If $\eta : \mathbb{R}^{N-1} \rightarrow \mathbb{R}$ is an analytic function in a neighborhood of $\bar{\xi}$, then both $\mathcal{F} : \mathbb{R}^N \rightarrow \mathbb{R}$ and $x \mapsto \mathcal{F}(x, \eta(x))$ are analytic. Thus, the Lojasiewicz inequality holds for both of them (with possibly different exponents). Taking $\gamma \in (0, 1/2]$ to be the smallest of the two exponents, we get that (1.10) holds with such γ and so (1.7) also holds.

Example 1.4 (Constraint with a non-analytic boundary). We now consider the two-dimensional problem with $\mathcal{K} = \{(x, y) \in \mathbb{R}^2 : y \geq \eta(x)\}$,

$$\mathcal{F}(x, y) = y^2, \quad \eta(x) = e^{-1/x^2} \quad \text{and} \quad \xi(0) = (x_0, \eta(x_0)) \in \partial \mathcal{K}.$$

Then, the solution ξ in \mathcal{K} exists for every $t \geq 0$ and is of the form $\xi(t) = (x(t), \eta(x(t)))$, where $x(t)$ is a solution of the ODE

$$x'(t) = -2\eta(x)\eta'(x) (1 + (\eta'(x))^2)^{-1/2} = -4x^{-3}e^{-2/x^2} \left(1 + 4x^{-6}e^{-2/x^2}\right)^{-1/2} =: f(x),$$

with initial datum $x(0) = x_0$ with $0 < x_0 \ll 1$. Thus, $\xi(t)$ converges to zero, but the decay rate is only logarithmic. Indeed, for $C > 1$ large, the function $\tilde{x}(t) := (\ln(t + C))^{-1/2}$ is such that

$$\tilde{x}'(t) = -\frac{1}{2}\tilde{x}^3 e^{-1/\tilde{x}^2} \leq f(\tilde{x}) \quad \text{and} \quad \tilde{x}(0) \leq x_0,$$

which means that $x(t) \geq \tilde{x}(t) = (\ln(t + C))^{-1/2}$ and so, (1.7) fails.

The above examples show that if the constraint is analytic, then as expected (1.10) holds; on the other hand *the non-analyticity of the constraint might make the inequality fail even if the functional itself is analytic!*

Let us now briefly consider the infinite dimensional case. If a Lojasiewicz-type inequality holds, then the same argument of Proposition 1.1 still works in this setting. As an example, let $u = u(t, x) \in C^1(\mathbb{R}^+; L^2(B_1)) \cap C(\mathbb{R}^+; H^2(B_1) \cap H_g^1(B_1))^1$ be the solution of the heat equation

$$\partial_t u = \Delta u \quad \text{in } \mathbb{R}^+ \times B_1, \quad u(0, \cdot) = u_0 \in L^2(B_1), \quad u(t, x) = g(x) \quad \text{for } (t, x) \in \mathbb{R}^+ \times \partial B_1,$$

¹For a given function $g \in H^1(B_1)$, we use the notation $H_g^1(B_1) := \{u \in H^1(B_1) : u - g \in H_0^1(B_1)\}$.

then the above decay-rate condition still holds (in $L^2(B_1)$ instead of \mathbb{R}^N) with

$$\mathcal{F}(u) = \frac{1}{2} \int_{B_1} |\nabla u|^2 dx, \quad \nabla \mathcal{F}(u) = \Delta u \quad \text{and} \quad |\nabla \mathcal{F}(u)| = \|\Delta u\|_{L^2(B_1)}.$$

In this case, (1.6) holds with $\gamma = 1/2$ and reads as

$$\int_{B_1} |\nabla u|^2 dx - \int_{B_1} |\nabla h|^2 dx \leq C \int_{B_1} |\Delta u|^2 dx, \quad \text{for every } u \in H^2(B_1) \cap H_g^1(B_1), \quad (1.11)$$

where h is the harmonic function in B_1 with boundary datum g on ∂B_1 . Indeed,

$$\begin{aligned} \|\Delta u\|_{L^2} &\geq -\frac{1}{\|u-h\|_{L^2}} \int_{B_1} (u-h)\Delta u dx = -\frac{1}{\|u-h\|_{L^2}} \int_{B_1} (u-h)\Delta(u-h) dx \\ &= \frac{1}{\|u-h\|_{L^2}} \int_{B_1} |\nabla(u-h)|^2 dx \geq C \left(\int_{B_1} |\nabla(u-h)|^2 dx \right)^{1/2}, \end{aligned}$$

where in the last line we used the Poincaré inequality for $u-h \in H_0^1(B_1)$. Next, since h is harmonic in B_1 and $u=h$ on ∂B_1 , we have

$$\int_{B_1} |\nabla(u-h)|^2 dx = \int_{B_1} |\nabla u|^2 dx - \int_{B_1} |\nabla h|^2 dx,$$

which gives precisely (1.11).

Let now \mathcal{F} be a functional defined on a vector space \mathcal{H} of \mathbb{R}^n -valued functions and let (the constraint) \mathcal{K} be a subset of this family. As we mentioned above, the key to the understanding of the behavior of the flow of \mathcal{F} in the set \mathcal{K} is the constrained Łojasiewicz inequality (1.6). In the seminal paper [25], L. Simon proved that the estimate (1.6) holds for harmonic maps with values in analytic manifolds (this corresponds to the case when \mathcal{F} is the Dirichlet energy and \mathcal{K} are the functions with values in an analytic manifold (M, g) embedded in \mathbb{R}^n). The core of the Simon's approach is to reduce the infinite dimensional inequality to the finite dimensional one through the so-called Lyapunov-Schmidt reduction and then use the classical finite dimensional Łojasiewicz inequality. Notice that the analyticity of the target manifold (M, g) is crucial as it allows to remove the constraint by replacing \mathcal{F} with a new functional, which is still analytic. More in general, this approach may be used when the constraint \mathcal{K} (which is infinite dimensional) is analytic, in the following sense.

For every function $u_0 \in \mathcal{K}$, there exists a map $\Phi : \mathcal{H} \rightarrow \mathcal{K}$ defined in a neighborhood \mathcal{N} of u_0 in \mathcal{H} such that:

- Φ is the identity on \mathcal{K} ;
- for every $n \in \mathbb{N}$ and every $u_1, \dots, u_n \in \mathcal{H}$, the function

$$\mathbb{R}^n \ni (t_1, \dots, t_n) \mapsto \Phi(u_0 + t_1 u_1 + \dots + t_n u_n) \in \mathcal{H}$$

is analytic.

In fact, if both the functional \mathcal{F} and the constraint \mathcal{K} are analytic, then we can remove the constraint by taking directly the functional $\mathcal{F} \circ \Phi$, which is still analytic. Moreover, it is clearly the case of the harmonic maps; in fact, if \mathcal{K} is the set of function with values in the analytic manifold (M, g) , then the functional Φ is simply the projection on M .

The situation we deal with in this paper is more similar to Example 1.4. For instance, in the case of the parabolic obstacle problem, the gradient flow is governed by the functional

$$\mathcal{F}(u) = \int_{B_1} \left(\frac{1}{2} |\nabla u|^2 + u \right) dx \quad \text{and} \quad \nabla \mathcal{F}(u) = -\Delta u + 1,$$

while the constraint \mathcal{K} is given by the (convex) set of non-negative functions in $L^2(B_1)$. In this case, \mathcal{K} has a boundary; arguing as in Example 1.3, we consider two cases: when the function u is in the interior of \mathcal{K} (this means that $u : B_1 \rightarrow \mathbb{R}$ is strictly positive) and when u is on the boundary of \mathcal{K} (that is, when u vanishes somewhere in B_1). When u is in the interior of \mathcal{K} , a similar reasoning as for the Dirichlet energy shows that the functional \mathcal{F} satisfies the Łojasiewicz

inequality (1.6). Unfortunately, at the boundary, the Simon's approach cannot be applied, since $\partial\mathcal{K}$ is not analytic. Indeed, suppose that there exists an analytic function $\Phi : L^2(B_1) \rightarrow L^2(B_1)$ with values in $\partial\mathcal{K}$ such that $\Phi = Id$ on $\partial\mathcal{K}$ and such that $(s, t) \mapsto \Phi(su + tv)$ is analytic, where u and v are nonnegative and have disjoint compact supports. In this case, $su + tv \in \partial\mathcal{K}$, for every $s \geq 0$ and $t \geq 0$. Now, the analyticity implies that $\Phi(su + tv) = su + tv$ for every $s, t \in \mathbb{R}$. But this is impossible since $su + tv \notin \partial\mathcal{K}$ when $s < 0$ or $t < 0$.

Summarizing, in the case of the obstacle problem, the functional \mathcal{F} satisfies the Łojasiewicz inequality (1.6), however (1.10) *cannot be deduced from (1.6) since the constraint has non-analytic boundary!* Simon's technique, therefore, does not apply, and we have to *heavily use the structure of the constraint* to conclude (1.10).

1.2. Obstacle and parabolic obstacle problems. Let B_1 be the unit ball in \mathbb{R}^d and let $g \in H^1(B_1)$ be a given non-negative function. We consider the functional

$$\mathcal{F}_{\text{OB}}(u) := \frac{1}{2} \int_{B_1} |\nabla u|^2 dx + \int_{B_1} u dx$$

and the set of admissible functions

$$\mathcal{K}_{\text{OB}}^g := \{u \in H^1(B_1) : u - g \in H_0^1(B_1), u \geq 0 \text{ in } B_1\}.$$

The classical obstacle problem can be written as

$$\min_{u \in \mathcal{K}_{\text{OB}}^g} \mathcal{F}_{\text{OB}}(u) \tag{1.12}$$

and admits a unique minimizer $\phi \in \mathcal{K}_{\text{OB}}^g$, which is also a solution of the variational inequality

$$(v - \phi) \cdot \nabla \mathcal{F}_{\text{OB}}(\phi) \geq 0 \quad \text{for every } v \in \mathcal{K}_{\text{OB}}^g. \tag{1.13}$$

The parabolic obstacle problem is the time-dependent counterpart of (1.13). We say that the function $u \in H^1([0, +\infty[; L^2(B_1)) \cap L^2([0, +\infty[; H^2(B_1) \cap \mathcal{K}_{\text{OB}}^g)$ is a (global in time) solution of the parabolic obstacle problem if

$$\begin{cases} (u'(t) + \nabla \mathcal{F}_{\text{OB}}(u(t))) \cdot (v - u(t)) \geq 0 & \text{for every } v \in \mathcal{K}_{\text{OB}}^g, t > 0, \\ u(0) = u_0 \in \mathcal{K}_{\text{OB}}^g, \end{cases} \tag{1.14}$$

where u_0 is a given initial datum. The existence of a (strong) solution was proved in [4], while for the regularity we refer to the recent paper [7]. In Theorem 1.7 we prove that $u(t)$ converges in $L^2(B_1)$ to the stationary solution ϕ with an exponential rate, while Theorem 1.10 is a result on the fine structure of the (singular part of the) free boundary $\partial\{\phi > 0\}$.

Let now (\mathcal{M}, g) be a compact connected oriented Riemannian manifold of dimension $d \geq 1$ and let $\lambda > 0$ be an eigenfunction of the Laplace-Beltrami operator on \mathcal{M} . Consider the functional

$$\mathcal{F}_{\text{OB}}^\lambda(u) = \frac{1}{2} \int_{\mathcal{M}} (|\nabla u|^2 - \lambda u^2) dV_g + \int_{\mathcal{M}} u dV_g \tag{1.15}$$

and the admissible set $\mathcal{K}_{\text{OB}}^{\mathcal{M}} := \{u \in L^2(\mathcal{M}) : u \geq 0\}$.

We say that $u \in H^1([0, +\infty[; L^2(\mathcal{M})) \cap L^2([0, +\infty[; H^2(\mathcal{M}) \cap \mathcal{K}_{\text{OB}}^{\mathcal{M}})$ is a (global in time) solution of the parabolic obstacle problem on \mathcal{M} if

$$\begin{cases} (u'(t) + \nabla \mathcal{F}_{\text{OB}}^\lambda(u(t))) \cdot (v - u(t)) \geq 0 & \text{for every } v \in \mathcal{K}_{\text{OB}}^{\mathcal{M}}, t > 0, \\ u(0) = u_0 \in \mathcal{K}_{\text{OB}}^{\mathcal{M}}. \end{cases} \tag{1.16}$$

In Theorem 1.8 and Remark 1.9, we will show that if the energy $\mathcal{F}_{\text{OB}}^\lambda(u(t))$ remains above certain critical threshold, then the solution $u(t)$ converges to a critical point of the functional $\mathcal{F}_{\text{OB}}^\lambda$ restricted to the convex set $\mathcal{K}_{\text{OB}}^{\mathcal{M}}$. We notice that, contrary to (1.14), there might be numerous critical points of $\mathcal{F}_{\text{OB}}^\lambda$ in $\mathcal{K}_{\text{OB}}^{\mathcal{M}}$ and thus, numerous candidates for the limit of $u(t)$. A priori, in such a situation the asymptotic behavior of the solution might be more complex and a limit at infinity might fail to exist. In Theorem 1.8, using a constrained Łojasiewicz inequality argument,

we show that the solution of (1.16) admits a unique limit at infinity and that the presence of a whole manifold of stationary points only affects the decay rate, which is only of power type.

Remark 1.5 (On the critical points of $\mathcal{F}_{\text{OB}}^\lambda$ in $\mathcal{K}_{\text{OB}}^M$). We notice that there is more than one critical point of the functional $\mathcal{F}_{\text{OB}}^\lambda$ in $\mathcal{K}_{\text{OB}}^M$. For example, the stationary points of the unconstrained functional $\mathcal{F}_{\text{OB}}^\lambda$ are precisely the functions of the form

$$u = \frac{1}{\lambda} + \phi_\lambda, \tag{1.17}$$

where ϕ_λ is a λ -eigenvalue of the Laplace-Beltrami operator $\Delta_{\mathcal{M}}$, that is,

$$-\Delta_{\mathcal{M}}\phi = \lambda\phi \quad \text{on } \mathcal{M}.$$

Thus, all the positive functions u of the form (1.17) are stationary solutions of (1.16), but for a generic \mathcal{M} and λ , there may also exist other stationary points. On the other hand, if \mathcal{M} is the $(d-1)$ -dimensional sphere and $\lambda = 2d$, then all the stationary solutions of (1.16) are of the form (1.17) or there is a vector $\nu \in \mathbb{R}^d$ such that $u(x) = (x \cdot \nu)_+^2$. This is due to the fact that the 2-homogeneous extension (in the unit ball $B_1 \subset \mathbb{R}^d$) of a stationary solution is a solution of the obstacle problem (1.12) in B_1 and the 2-homogeneous solutions of the obstacle problem are classified (see [5]).

1.3. Thin-obstacle and parabolic thin-obstacle problems. Let $d \geq 2$ and B_1 be the unit ball in \mathbb{R}^d . For $x \in \mathbb{R}^d$, we will write $x = (x', x_d)$, where $x' \in \mathbb{R}^{d-1}$ and $x_d \in \mathbb{R}$. Let $g \in H^1(B_1)$ be a given function, which is:

- non-negative on $B_1 \cap \{x_d = 0\}$;
- even with respect to the hyperplane $\{x_d = 0\}$, where we say that a function $f : B_1 \rightarrow \mathbb{R}$ is even with respect to the hyperplane $\{x_d = 0\}$ if $f(x', x_d) = f(x', -x_d)$, for every $x = (x', x_d) \in B_1$.

We consider the functional

$$\mathcal{F}_{\text{TH}}(u) = \frac{1}{2} \int_{B_1} |\nabla u|^2 dx,$$

and the admissible set

$$\mathcal{K}_{\text{TH}}^g = \{u \in H^1(B_1) : u - g \in H_0^1(B_1), u \geq 0 \text{ on } B_1 \cap \{x_d = 0\}, u \text{ is even w.r.t. } \{x_d = 0\}\}.$$

There is a unique solution $\phi \in \mathcal{K}_{\text{TH}}^g$ to the thin-obstacle problem

$$\min_{u \in \mathcal{K}_{\text{TH}}^g} \mathcal{F}_{\text{TH}}(u), \tag{1.18}$$

and it satisfies the variational inequality

$$(v - \phi) \cdot \nabla \mathcal{F}_{\text{TH}}(\phi) \geq 0 \quad \text{for every } v \in \mathcal{K}_{\text{TH}}^g. \tag{1.19}$$

Moreover, ϕ satisfies the optimality conditions

$$\Delta \phi = 0 \quad \text{in } B_1 \setminus \{x_d = 0\}, \quad \frac{\partial \phi}{\partial x_d} \leq 0 \quad \text{and} \quad \phi \frac{\partial \phi}{\partial x_d} = 0 \quad \text{on } B_1 \cap \{x_d = 0\}.$$

We will use the notation $B_1^+ := B_1 \cap \{x_d > 0\}$ and we identify the even functions with their restriction on B_1^+ . We say that $u \in H^1(]0, +\infty[; L^2(B_1^+)) \cap C(]0, +\infty[; H^2(B_1^+) \cap \mathcal{K}_{\text{TH}}^g)$ is a (global in time) solution of the parabolic thin-obstacle problem if

$$\begin{cases} (u'(t) + \nabla \mathcal{F}_{\text{TH}}(u(t))) \cdot (v - u(t)) \geq 0 & \text{for every } v \in \mathcal{K}_{\text{TH}}^g, t > 0, \\ u(0) = u_0 \in \mathcal{K}_{\text{TH}}^g. \end{cases} \tag{1.20}$$

We recall that, for every $t > 0$, $u(t)$ satisfies the optimality condition

$$u(t) \geq 0, \quad \frac{\partial u(t)}{\partial x_d} \leq 0 \quad \text{and} \quad u(t) \frac{\partial u(t)}{\partial x_d} = 0 \quad \text{on } B_1 \cap \{x_d = 0\}.$$

The existence of a solution was first addressed in [21] (see also [4]), while the latest regularity results can be found in [11]. As for the obstacle problem, we will prove in Theorem 1.7 that the solution $u(t)$ converges exponentially to the stationary limit ϕ , while in Theorem 1.10 we will

prove a logarithmic epiperimetric inequality at the singular points of the stationary free boundary $\partial\{\phi > 0\} \subset \{x_d = 0\}$.

Let now $k = 2m$, for some $m \in \mathbb{N}$, and let $\lambda = \lambda(k) := k(k + d - 2)$. We consider the functional

$$\mathcal{F}_{\text{TH}}^\lambda(u) = \frac{1}{2} \int_{\partial B_1} (|\nabla u|^2 - \lambda u^2) d\mathcal{H}^{d-1},$$

and the admissible set

$$\mathcal{K}_{\text{TH}}^{\mathbb{S}} := \{u \in L^2(\partial B_1) : u \geq 0 \text{ on } \partial B_1 \cap \{x_d = 0\}, u \text{ is even w.r.t. } \{x_d = 0\}\}.$$

Let $\mathbb{S} = \partial B_1$ and $\mathbb{S}^+ = \partial B_1 \cap \{x_d > 0\}$. We say that u is a (global in time) solution of the parabolic thin-obstacle problem on the sphere, if $u \in H^1(]0, +\infty[; L^2(\mathbb{S}^+)) \cap L^2(]0, +\infty[; H^2(\mathbb{S}^+) \cap \mathcal{K}_{\text{TH}}^{\mathbb{S}})$ and

$$\begin{cases} (u'(t) + \nabla \mathcal{F}_{\text{TH}}^\lambda(u(t))) \cdot (v - u(t)) \geq 0 & \text{for every } v \in \mathcal{K}_{\text{TH}}^{\mathbb{S}}, t > 0 \\ u(0) = u_0 \in \mathcal{K}_{\text{TH}}^{\mathbb{S}}. \end{cases} \quad (1.21)$$

We will study the asymptotic behavior of the solutions to (1.21) in Theorem 1.8.

Remark 1.6 (On the critical points of $\mathcal{F}_{\text{TH}}^\lambda$ in $\mathcal{K}_{\text{TH}}^{\mathbb{S}}$). Let $m \in \mathbb{N}$ be fixed and let $\lambda = \lambda(2m)$. Then, the function $\phi : \partial B_1 \rightarrow \mathbb{R}$ is a critical point of $\mathcal{F}_{\text{TH}}^\lambda$ in $\mathcal{K}_{\text{TH}}^{\mathbb{S}}$ (in sense of (2.3)) or, equivalently, a stationary solution of (1.21), if and only if, the $2m$ -homogeneous extension $\psi : B_1 \rightarrow \mathbb{R}$ (in polar coordinates, $\psi(r, \theta) = r^{2m} \phi(\theta)$) is a solution of the thin-obstacle problem (1.18) in B_1 with trace $g = \phi$ on ∂B_1 . On the other hand, the $2m$ -homogeneous solutions of (1.18) are classified (see [19]) and are given precisely by the $2m$ -homogeneous harmonic functions in B_1 , non-negative on $\{x_d = 0\}$. Thus, the critical points of $\mathcal{F}_{\text{TH}}^\lambda$ in $\mathcal{K}_{\text{TH}}^{\mathbb{S}}$ are the $\lambda(2m)$ -eigenfunctions of the spherical Laplacian, which are non-negative on the equator $\partial B_1 \cap \{x_d = 0\}$.

1.4. Main results. In this subsection, we state our main results on the parabolic (and stationary) obstacle and thin-obstacle problems.

Theorem 1.7 (Asymptotics for the parabolic obstacle and thin-obstacle problems). *Let u be a global (in time) solution to the parabolic obstacle problem (1.14) (resp. the parabolic thin-obstacle problem (1.20)) and let φ be the unique solution of the obstacle problem (1.12) (resp. thin-obstacle problem 1.18) with the same boundary datum. Then, $u(t)$ converges to φ strongly in $H^1(B_1)$, as $t \rightarrow \infty$, and there is a constant $C > 0$ such that, for every $t \geq 1$,*

$$\|u(t) - \varphi\|_{H^1(B_1)} \leq e^{-Ct}. \quad (1.22)$$

On a manifold the situation is more complicated since there is no unique minimizer. As a consequence, we can only conclude that if the flow starts close to a stationary solution and its energy is always above the energy of the solution, then it has to converge (with a rate) to a stationary solution with the same energy.

In the following theorem, \square stands for OB (respectively, TH).

Theorem 1.8 (Asymptotic for the parabolic obstacle and thin-obstacle problems on the sphere). *Let \mathbb{S} be the $(d - 1)$ -dimensional unit sphere in \mathbb{R}^d . Let $\mathcal{S}_\lambda \subset \mathcal{K}_{\square}^{\mathbb{S}}$ be the collection of critical points of the unconstrained functional $\mathcal{F}_{\square}^\lambda$ in the convex set $\mathcal{K}_{\square}^{\mathbb{S}}$, where $\lambda = 2d$, if $\square = \text{OB}$ and $\lambda = \lambda(2m) := 2m(2m + d - 2)$, if $\square = \text{TH}$.*

Then, there are constants $\gamma \in]0, 1/2[$, $\delta > 0$, $E > 0$ and $C > 0$ such that: if u is a solution of (1.16) (resp. of (1.21) when $\square = \text{TH}$) on the sphere \mathbb{S} , satisfying

$$\text{dist}_{L^2}(u_0, \mathcal{S}_\lambda) \leq \delta, \quad \mathcal{F}_{\square}^\lambda(u_0) - \mathcal{F}_{\square}^\lambda(\mathcal{S}_\lambda) \leq E \quad \text{and} \quad \mathcal{F}_{\square}^\lambda(u(t)) > \mathcal{F}_{\square}^\lambda(\mathcal{S}_\lambda), \quad \text{for every } t > 0,$$

then there is a critical point $\varphi \in \mathcal{S}_\lambda \subset \mathcal{K}_{\square}^{\mathbb{S}}$ of $\mathcal{F}_{\square}^\lambda$ such that, for every $t \geq 1$,

$$\|u(t) - \varphi\|_{H^1(\mathbb{S})} \leq Ct^{-\frac{\gamma}{1-2\gamma}}. \quad (1.23)$$

Remark 1.9. In the case of the obstacle problem, the sphere \mathbb{S} can be replaced by a compact Riemannian manifold \mathcal{M} and λ can be taken to be any eigenvalue of the Laplace-Beltrami operator on \mathcal{M} . In this setting, we take the set \mathcal{S} to be the set of critical points of the unconstrained functional $\mathcal{F}_{\text{OB}}^\lambda$, which are positive (so, they lie inside the convex set $\mathcal{K}_{\text{OB}}^{\text{M}}$). In this case, the conclusion is that $u(t)$ converges to a function $u_\infty \in \mathcal{K}_{\text{OB}}^{\text{M}}$, which is a critical point for $\mathcal{F}_{\text{OB}}^\lambda$ in $\mathcal{K}_{\text{OB}}^{\text{M}}$ in the sense of (2.3). Moreover, we have the estimates

$$\|u(t) - \varphi\|_{L^2(\mathcal{M})} \leq Ct^{-\frac{\gamma}{1-2\gamma}} \quad \text{and} \quad \mathcal{F}_{\text{OB}}^\lambda(u(t)) - \mathcal{F}_{\text{OB}}^\lambda(\mathcal{S}) \leq Ct^{-\frac{1}{1-2\gamma}}. \quad (1.24)$$

This is a consequence of Proposition 2.10 and Proposition 4.5.

Our interest in the parabolic problems (1.16) and (1.21) on the sphere is two-fold: on the one hand they are the natural generalizations of the respective parabolic problems in $B_1 \subset \mathbb{R}^d$ studied in Theorem 1.7 above; on the other hand, they are strictly related to the study of uniqueness of blow-ups at singular points for the time-independent problem, where the radial direction is treated as time. This observation goes back to Simon in [25] for stationary varifolds and harmonic maps, and we make it explicit in the context of minimizers of the obstacle and thin-obstacle problems by giving a new proof of the following logarithmic epiperimetric inequality (see [9, 10]). Before stating it, we introduce the notation

$$\mathcal{G}_{\text{OB}}(u) := \mathcal{F}_{\text{OB}}(u) - \int_{\partial B_1} u^2 d\mathcal{H}^{d-1} \quad \text{and} \quad \mathcal{G}_{\text{TH}}(u) := \mathcal{F}_{\text{TH}}(u) - m \int_{\partial B_1} u^2 d\mathcal{H}^{d-1},$$

where $m \in \mathbb{N}$. For the obstacle problem, we define the set of stationary points \mathcal{S}_{OB} on the sphere $\partial B_1 \subset \mathbb{R}^d$ as the traces of all global two-homogeneous non-flat solutions of the obstacle problem in \mathbb{R}^d (see [5]), that is,

$$\mathcal{S}_{\text{OB}} := \{Q_A : \partial B_1 \rightarrow \mathbb{R} : Q_A(x) = x \cdot Ax, \quad A \text{ symmetric non-negative matrix with } \text{tr} A = 1/2\}.$$

We notice that \mathcal{G}_{OB} is constant on \mathcal{S}_{OB} and we set $\Theta := \mathcal{G}_{\text{OB}}(\mathcal{S}_{\text{OB}})$.

Theorem 1.10 (Log-epiperimetric inequality for obstacle and thin-obstacle at singular points). *Let $d \geq 2$. The following logarithmic epiperimetric inequalities hold.*

(OB) *There are dimensional constants $\delta > 0$ and $\varepsilon > 0$ such that for every non-negative function $c \in H^1(\partial B_1)$, with 2-homogeneous extension z on B_1 , satisfying*

$$\text{dist}_{L^2(\partial B_1)}(c, \mathcal{S}_{\text{OB}}) \leq \delta \quad \text{and} \quad \mathcal{G}_{\text{OB}}(z) - \Theta \leq 1,$$

there is a non-negative function $h \in \mathcal{K}_{\text{OB}}^c$ satisfying the inequality

$$\mathcal{G}_{\text{OB}}(h) - \Theta \leq (\mathcal{G}_{\text{OB}}(z) - \Theta)(1 - \varepsilon |\mathcal{G}_{\text{OB}}(z) - \Theta|^\gamma), \quad \text{where} \quad \begin{cases} \gamma = 0 & \text{if } d = 2 \\ \gamma = \frac{d-1}{d+3} & \text{if } d \geq 3 \end{cases}. \quad (1.25)$$

(TH) *Let $d \geq 2$ and $m \in \mathbb{N}$. For every function $c \in H^1(\partial B_1) \cap \mathcal{K}_{\text{TH}}^{\mathbb{S}}$ such that*

$$\int_{\partial B_1} c^2 d\mathcal{H}^{d-1} \leq 1 \quad \text{and} \quad |\mathcal{G}_{\text{TH}}(z)| \leq 1, \quad (1.26)$$

there are a constant $\varepsilon = \varepsilon(d, m) > 0$ and a function $h \in \mathcal{K}_{\text{TH}}^c$ satisfying

$$\mathcal{G}_{\text{TH}}(h) \leq \mathcal{G}_{\text{TH}}(z)(1 - \varepsilon |\mathcal{G}_{\text{TH}}(z)|^\gamma), \quad \text{where} \quad \gamma := \frac{d-2}{d}. \quad (1.27)$$

The epiperimetric and the logarithmic epiperimetric inequalities are part of the same family of quantitative estimates on the energy of the homogeneous functions. They are used to obtain regularity of the free boundaries with modulus of continuity, which is Hölder in the first case and logarithmic in the latter. This homogeneity-improvement argument was pioneered by Reifenberg in [24] in the context of minimal surfaces and several authors used it in the context of minimal surfaces and free boundary problems (see [30, 27, 28, 8, 12, 29, 26, 9, 10, 14, 15]).

Even if the epiperimetric inequalities, and the methods to deduce regularity from them, might seem quite similar, the methods to prove them are very different. The first epiperimetric inequality for a free boundary (obstacle) problem was proved by Weiss in [29], where he used an argument by

contradiction, which was then applied to different obstacle problems in [17, 20]; this is a powerful method, which allows to prove the regularity of the flat free boundaries and, in some cases, can be applied to the set of singular points in the highest stratum; however, this method cannot be applied at singular points, where the energy decay and the rate of convergence of the blow-up sequences are in general weaker, mainly as a consequence of the fact that the *integrability* may fail. In [26], we used a different method, the so-called *direct approach*, inspired by the idea of Reifenberg, which consists in the explicit construction of a competitor starting from the Fourier expansion of the trace; we later used this idea in [9] and [10] to prove the logarithmic epiperimetric inequalities from Theorem 1.10 (OB) and (TH), which are able to detect very weak (logarithmic) energy decay rates and can be applied to general singularities.

In this paper we give a new, different type of proof, which is constructive (that is, we construct a competitor), but not direct (the competitor is not explicit). We use stopped parabolic flows for a functional on the unit sphere ∂B_1 and, identifying the time with the radial direction, we reparametrize it over the spheres ∂B_r to obtain a function defined on the unit ball, which lives in the constraint domain and has smaller energy. This is a general abstract procedure, and we describe it in detail in Section 3. We point out that, although inspired by the work of the last two authors with M. Engelstein in [14, 15], the approach of the present paper has two major novelties. First, we stress that the flow that we use here is directly the gradient flow of the energy associated to the problem, while in [14, 15] we constructed by hands a flow that simulated the global qualitative behavior of the real gradient flow. The second major difference is in the proof of the energy decay along the flow. In fact, the functionals in [14, 15] are both analytic; this allows to reduce the analysis to the case of a finite dimensional space, where one can use the classical (finite dimensional) Łojasiewicz inequality and then conclude by an abstract ODE argument. In the present paper, however, the constraint is not analytic, thus there is no way to reduce the problem to the classical finite dimensional case; instead, we prove a new Łojasiewicz-type inequality in this non-analytic setting.

Finally, we recall that the structure of the singular part of the free boundary of minimizers of the obstacle and thin-obstacle problems was studied by several authors; we refer to [5], [6], [23], [16] and [9], for the obstacle problem, and [19], [10] for the thin-obstacle problem. We also note that, the uniqueness of the blow-up and the logarithmic modulus of continuity follow directly by the logarithmic epiperimetric inequality (Theorem 1.10), exactly as in [9, 10].

2. ASYMPTOTIC BEHAVIOR FOR PARABOLIC VARIATIONAL INEQUALITIES

In this section we prove that solutions of parabolic variational inequalities of energies satisfying a *constrained Łojasiewicz inequality* converge at infinity to a stationary solution of the energy. In order to state the main result we need to introduce some notation.

Let \mathcal{H} be a Hilbert space with scalar product

$$u \cdot v = \langle u, v \rangle = \langle u, v \rangle_{\mathcal{H}} \quad \text{for every } u, v \in \mathcal{H},$$

and induced norm $\|u\| = \|u\|_{\mathcal{H}} = \sqrt{u \cdot u}$. Let $\mathcal{W} \subset \mathcal{H}$ be a linear subspace, $\mathcal{F} : \mathcal{W} \rightarrow \mathbb{R}$ and $\nabla \mathcal{F} : \mathcal{W} \rightarrow \mathcal{H}$ be continuous (possibly non-linear) functionals such that

$$\mathcal{F}(u + tv) = \mathcal{F}(u) + t v \cdot \nabla \mathcal{F}(u) + o(t), \quad \text{for every } u, v \in \mathcal{W}. \quad (2.1)$$

Let $\mathcal{K} \subset \mathcal{H}$ be a convex subset of \mathcal{H} . We will suppose that $\mathcal{K} \cap \mathcal{W}$ is dense in \mathcal{K} . For every $u \in \mathcal{K} \cap \mathcal{W}$, we define

$$\|\nabla \mathcal{F}(u)\|_{\mathcal{K}} := \sup \left\{ 0, \sup_{v \in \mathcal{K} \setminus \{u\}} \frac{-(v - u) \cdot \nabla \mathcal{F}(u)}{\|v - u\|} \right\}. \quad (2.2)$$

We say that $u \in \mathcal{K} \cap \mathcal{W}$ is a *critical point of the functional \mathcal{F} in \mathcal{K}* , if we have

$$(v - u) \cdot \nabla \mathcal{F}(u) \geq 0 \quad \text{for every } v \in \mathcal{K}. \quad (2.3)$$

The following is a simple exercise left to the reader.

Lemma 2.1 (Critical points). *Let $u \in \mathcal{K} \cap \mathcal{W}$. Then, the following are equivalent:*

- (i) u is a critical point for \mathcal{F} in \mathcal{K} ;
- (ii) for every $v \in \mathcal{K}$, the function $t \mapsto \mathcal{F}((1-t)u + tv)$ is differentiable at zero and

$$\left. \frac{d}{dt} \right|_{t=0^+} \mathcal{F}((1-t)u + tv) \geq 0;$$

- (iii) $\|\nabla \mathcal{F}(u)\|_{\mathcal{K}} = 0$.

The reader may keep in mind the following guiding example:

Example 2.2. Let $\mathcal{H} = L^2(B_1)$, $\mathcal{K} = \{u \in L^2(B_1) : u \geq 0 \text{ on } B_1\}$, $\mathcal{W} = H^2(B_1) \cap H_g^1(B_1)$, where $H_g^1(B_1) := \{u \in H_g^1(B_1) : u = g \in H^1(B_1) \text{ on } \partial B_1\}$; and let

$$\mathcal{F}(u) = \frac{1}{2} \int_{B_1} |\nabla u|^2 dx + \int_{B_1} u dx.$$

Thus, the critical points of \mathcal{F} in \mathcal{K} are precisely the solutions of the obstacle problem in B_1 with boundary datum g on ∂B_1 .

Definition 2.3 (Parabolic variational inequalities). Let $u_0 \in \mathcal{K}$ and $T \in]0, \infty]$. We say that the function $u : [0, T[\rightarrow \mathcal{K}$ is a (*strong*) *solution* (global, if $T = +\infty$.) of the parabolic variational inequality

$$\begin{cases} (u'(t) + \nabla \mathcal{F}(u(t))) \cdot (v - u(t)) \geq 0 & \text{for every } v \in \mathcal{K}, t \in [0, T[, \\ u(0) = u_0, \end{cases} \quad (2.4)$$

if it satisfies the following conditions:

- (i) (*continuity*) $u \in C([0, T[; \mathcal{H})$ and $u(0) = u_0$;
- (ii) (*regularity in time*) $u \in H_{loc}^1(]0, T[; \mathcal{H})$; in particular, $u :]0, +\infty[\rightarrow \mathcal{H}$ is differentiable in almost every $t > 0$;
- (iii) (*regularity in space*) $u \in C(]0, T[; \mathcal{W} \cap \mathcal{K})$;
- (iv) (*variational inequality*) for every $v \in \mathcal{K}$ and almost-every $t > 0$ we have

$$\langle u'(t) + \nabla \mathcal{F}(u(t)), v - u(t) \rangle_{\mathcal{H}} \geq 0. \quad (2.5)$$

Remark 2.4 (Stationary solutions). We notice that if $\varphi \in \mathcal{K} \cap \mathcal{W}$ is a critical point of \mathcal{F} in \mathcal{K} , then $u(t) \equiv \varphi$ is a solution of (2.4). On the other hand, if $u(t) \equiv \varphi$ is a solution of (2.4), then the variational inequality (2.5) implies that φ is a critical point.

We will need the following property of strong global solutions of parabolic variational inequalities..

Lemma 2.5. *Let u be a strong solution of (2.4). The following properties are true.*

- (i) $\|u'(t)\|^2 = -u'(t) \cdot \nabla \mathcal{F}(u(t))$ for almost-every $t \in]0, T[$.
- (ii) The function $t \mapsto \mathcal{F}(u(t))$ is non-increasing.
- (iii) $\|u'(t)\| = \|\nabla \mathcal{F}(u(t))\|_{\mathcal{K}}$.

Proof. To prove (i) we notice that taking $t > 0$, $h > 0$ and $v := u(t+h)$ in (2.5), we get

$$0 \leq \frac{1}{h} (u(t+h) - u(t)) \cdot (u'(t) + \nabla \mathcal{F}(u(t))).$$

Passing to the limit as $h \rightarrow 0$, we obtain

$$\|u'(t)\|^2 \geq -u'(t) \cdot \nabla \mathcal{F}(u(t)).$$

Conversely, taking $h > 0$ and $v := u(t-h)$ in (2.5), we get the opposite inequality. Combining the two estimates, we deduce

$$\|u'(t)\|^2 = -u'(t) \cdot \nabla \mathcal{F}(u(t)) \quad \text{for almost-every } t > 0. \quad (2.6)$$

(ii) is an immediate consequence of (i). Indeed, let us first notice that (2.1) implies

$$\frac{d}{ds} \mathcal{F}(s u(t+h) + (1-s)u(t)) = (u(t+h) - u(t)) \cdot \nabla \mathcal{F}(s u(t+h) + (1-s)u(t)).$$

Thus,

$$\frac{1}{h} (\mathcal{F}(u(t+h)) - \mathcal{F}(u(t))) = \int_0^1 \frac{1}{h} (u(t+h) - u(t)) \cdot \nabla \mathcal{F}(s u(t+h) + (1-s)u(t)) ds.$$

By the continuity of $\nabla \mathcal{F}$ (in \mathcal{W}), the continuity of u (in \mathcal{W}) and the differentiability of u (in \mathcal{H}), we can pass to the limit, as $h \rightarrow 0$, in all the points t , where $u(t)$ is differentiable:

$$\frac{d}{dt} \mathcal{F}(u(t)) = u'(t) \cdot \nabla \mathcal{F}(u(t)) = -\|u'(t)\|^2 \leq 0. \quad (2.7)$$

We finally come to (iii). We consider two cases:

Case 1. Suppose that $\|u'(t)\| = 0$. Then (2.5) and (2.2) imply that $\|\nabla \mathcal{F}(u(t))\|_{\mathcal{K}} = 0$.

Case 2. Let $\|u'(t)\| > 0$. Then we have

$$\begin{aligned} \sup_{v \in \mathcal{K}} \frac{-(v - u(t)) \cdot \nabla \mathcal{F}(u(t))}{\|v - u(t)\|} &\geq \lim_{h \rightarrow 0^+} \frac{-\frac{1}{h} (u(t+h) - u(t)) \cdot \nabla \mathcal{F}(u(t))}{\frac{1}{h} \|u(t+h) - u(t)\|} \\ &= \frac{-u'(t) \cdot \nabla \mathcal{F}(u(t))}{\|u'(t)\|} = \|u'(t)\|. \end{aligned}$$

On the other hand, the parabolic variational inequality (2.5) implies that

$$\|u'(t)\| \geq \frac{u'(t) \cdot (v - u(t))}{\|v - u(t)\|} \geq \frac{-(v - u(t)) \cdot \nabla \mathcal{F}(u(t))}{\|v - u(t)\|} \quad \text{for every } v \in \mathcal{K}.$$

Taking the supremum over $v \in \mathcal{K}$, we finally get

$$\|u'(t)\| = \sup_{v \in \mathcal{K}} \frac{-(v - u(t)) \cdot \nabla \mathcal{F}(u(t))}{\|v - u(t)\|} = \|\nabla \mathcal{F}(u(t))\|_{\mathcal{K}},$$

where the equality follows by the fact that $\|u'(t)\| \geq 0$. \square

Definition 2.6 (Continuity with respect to the initial datum). We say that the flow (2.4) of $-\nabla \mathcal{F}$ in \mathcal{K} depends continuously on the initial datum, if for every $t > 0$ and every $\varepsilon > 0$ there is $\delta > 0$ such that: if $u_0, v_0 \in \mathcal{K}$ are such that $\|u_0 - v_0\| \leq \delta$, then $\|u(s) - v(s)\| \leq \varepsilon$, for every $s \in [0, t]$ (for which both flows are defined).

We notice that the continuity of the flow of $-\nabla \mathcal{F}$ in \mathcal{K} essentially boils down to the continuity of the flow of $-\nabla \mathcal{F}$ without any constraint. Indeed we have the following simple lemma.

Lemma 2.7. If $\nabla \mathcal{F} : \mathcal{W} \rightarrow \mathcal{H}$ is a linear function and there is a constant $\lambda > 0$ such that

$$-u \cdot \nabla \mathcal{F}(u) \leq \lambda \|u\|^2 \quad \text{for every } u \in \mathcal{W}, \quad (2.8)$$

then the flow (2.4) of $-\nabla \mathcal{F}$ in \mathcal{K} depends continuously on the initial datum.

Proof. If u and v are two solutions of (2.4) on $[0, T[$, then for every $t \in [0, T[$ we have

$$\langle u'(t) + \nabla \mathcal{F}(u(t)), v(t) - u(t) \rangle \geq 0 \quad \text{and} \quad \langle v'(t) + \nabla \mathcal{F}(v(t)), u(t) - v(t) \rangle \geq 0.$$

Thus, by (2.5), we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u(t) - v(t)\|^2 &= \langle u'(t), u(t) - v(t) \rangle + \langle v'(t), v(t) - u(t) \rangle \\ &\leq \langle \nabla \mathcal{F}(u(t)), v(t) - u(t) \rangle + \langle \nabla \mathcal{F}(v(t)), u(t) - v(t) \rangle. \end{aligned}$$

By the linearity of $\nabla \mathcal{F}$, we get

$$\frac{1}{2} \frac{d}{dt} \|u(t) - v(t)\|^2 \leq -\langle u(t) - v(t), \nabla \mathcal{F}(u(t) - v(t)) \rangle. \quad (2.9)$$

Now, (2.9) implies that

$$\frac{d}{dt} \|u(t) - v(t)\|^2 \leq 2\lambda \|u(t) - v(t)\|^2,$$

so, we can take $\delta = e^{2\lambda t} \varepsilon$ in Definition 2.6. \square

In the sequel we will denote by $\mathcal{S} \subset \mathcal{K} \cap \mathcal{W}$ a subset of the set of critical points of \mathcal{F} in \mathcal{K} .

Definition 2.8 (Constrained Łojasiewicz inequality). We say that the functional \mathcal{F} has the *constrained Łojasiewicz property on \mathcal{S}* if there are constants $\gamma \in]0, 1/2]$, $C_L > 0$, $\delta_L > 0$ and $E_L > 0$, depending on \mathcal{S} , such that for every critical point $\varphi \in \mathcal{S}$ the following inequality holds

$$(\mathcal{F}(u) - \mathcal{F}(\varphi))_+^{1-\gamma} \leq C_L \|\nabla \mathcal{F}(u)\|_{\mathcal{K}}, \quad (2.10)$$

for every $u \in \mathcal{K} \cap \mathcal{W}$ such that $\|u - \varphi\| \leq \delta_L$ and $\mathcal{F}(u) - \mathcal{F}(\varphi) \leq E_L$.

The following lemma is an easy exercise left to the reader.

Lemma 2.9 (Stationary points and Łojasiewicz inequality). *If \mathcal{F} has the constrained Łojasiewicz property on the subset of critical points \mathcal{S} , then it is locally constant on \mathcal{S} .*

Given $\varphi \in \mathcal{S}$, we will denote by

$$\mathcal{S}_\varphi := \{\psi \in \mathcal{S} : \mathcal{F}(\psi) = \mathcal{F}(\varphi)\}$$

and write $\mathcal{F}(\mathcal{S}_\varphi) := \mathcal{F}(\varphi)$. If there is only one such energy level, we will drop the index φ . We are now able to state the main result of this section. We will use the notation

$$\text{dist}(u, \mathcal{S}_\varphi) = \inf_{\psi \in \mathcal{S}_\varphi} \|u - \psi\|,$$

and by *neighborhood of \mathcal{S}_φ* (in \mathcal{K}) we will mean a set (containing a set) of the form

$$\{u \in \mathcal{K} : \text{dist}(u, \mathcal{S}_\varphi) < \delta\},$$

for some $\delta > 0$.

Proposition 2.10. *Let \mathcal{H} , \mathcal{W} , \mathcal{K} , $\mathcal{F} : \mathcal{W} \rightarrow \mathbb{R}$ and $\nabla \mathcal{F} : \mathcal{W} \rightarrow \mathcal{H}$ be as above. Let $\mathcal{S} \subset \mathcal{W} \cap \mathcal{K}$ be a subset of the set of critical points of \mathcal{F} in \mathcal{K} . Let $\varphi \in \mathcal{S}$ and \mathcal{S}_φ be as above. Suppose that there is a neighborhood of $\mathcal{S}_\varphi \subset \mathcal{W} \cap \mathcal{K}$, where:*

- (a) *the flow (2.4) of $-\nabla \mathcal{F}$ in \mathcal{K} depends continuously on the initial datum (Definition 2.6);*
- (b) *\mathcal{F} has the constrained Łojasiewicz property on \mathcal{S}_φ (Definition 2.8).*

Then, there are constants $\delta > 0$, $E > 0$ and $C > 0$ such that: if $u_0 \in \mathcal{K} \cap \mathcal{W}$ and $u(t)$ is a global solution (with initial datum u_0) satisfying

$$\text{dist}(u_0, \mathcal{S}_\varphi) \leq \delta, \quad \mathcal{F}(u_0) - \mathcal{F}(\mathcal{S}_\varphi) \leq E \quad \text{and} \quad \mathcal{F}(u(t)) > \mathcal{F}(\mathcal{S}_\varphi), \quad \text{for every } t > 0,$$

then there is a function $u_\infty \in \mathcal{W} \cap \mathcal{K}$ such that $u(t)$ converges to u_∞ and, if $\gamma < 1/2$, then

$$\|u(t) - u_\infty\| \leq Ct^{-\frac{\gamma}{1-2\gamma}} \quad \text{and} \quad \mathcal{F}(u(t)) - \mathcal{F}(\mathcal{S}_\varphi) \leq Ct^{-\frac{1}{1-2\gamma}}, \quad (2.11)$$

for every $t \geq 1$, while if $\gamma = 1/2$, then the decay is exponential:

$$\|u(t) - u_\infty\| \leq e^{-Ct} \quad \text{and} \quad \mathcal{F}(u(t)) - \mathcal{F}(\mathcal{S}_\varphi) \leq e^{-Ct}. \quad (2.12)$$

Proof. Let $\delta_L > 0$ be the constant from Definition 2.8 and set for simplicity $\mathcal{S} = \mathcal{S}_\varphi$. We will prove that there is $\delta \in (0, \delta_L)$ with the following property: for any $u_0 \in \mathcal{K}$ such that $\text{dist}(u_0, \mathcal{S}) < \delta$, the solution $u(t)$ of (2.4) exists for every $t > 0$ and $\text{dist}(u(t), \mathcal{S}) < \delta_L$. The decay rate (2.11) will be then a consequence of the Łojasiewicz inequality. We next suppose that u satisfies the inequality $\mathcal{F}(u(t)) > \mathcal{F}(\mathcal{S})$, for every $t > 0$. Let us first recall that, by Lemma 2.5 (ii), the map $t \mapsto \mathcal{F}(u(t))$ is non-increasing. Thus, the condition $\mathcal{F}(u(t)) - \mathcal{F}(\mathcal{S}) \leq E$ is automatically satisfied for every $t \geq 0$, once it holds for $t = 0$. In particular, we can simply take E to be the constant E_L from the Łojasiewicz inequality (and if $E_L = +\infty$, then also $E = +\infty$).

Let $0 < t < T < +\infty$ be given. Then, we have

$$\int_t^T \|u'(s)\|^2 ds = - \int_t^T u'(s) \cdot \nabla \mathcal{F}(u(s)) ds = \mathcal{F}(u(t)) - \mathcal{F}(u(T)) \leq \mathcal{F}(u(t)) - \mathcal{F}(\mathcal{S}). \quad (2.13)$$

In particular, we obtain that the function $s \mapsto \|u'(s)\|^2$ is square integrable at infinity and

$$\int_t^{+\infty} \|u'(s)\|^2 ds \leq \mathcal{F}(u(t)) - \mathcal{F}(\mathcal{S}) \quad \text{for every } t > 0. \quad (2.14)$$

Let now $T \in]0, +\infty]$ be such that $\text{dist}(u(t), \mathcal{S}) < \delta_L$, for every $t \in]0, T[$. Then, for any $0 < t < T$, we can estimate the right-hand side of (2.14) by the Łojasiewicz inequality (2.10)

$$\int_t^{+\infty} \|u'(s)\|^2 ds \leq \mathcal{F}(u(t)) - \mathcal{F}(\mathcal{S}) \leq C \|\nabla \mathcal{F}(u(t))\|_{\mathcal{K}}^{\frac{1}{1-\gamma}} = C \|u'(t)\|^{\frac{1}{1-\gamma}}, \quad (2.15)$$

where in the last equality we used the identity (iii) of Lemma 2.5. Here and in what follows C will denote *any constant* depending only on the constant C_L from the Łojasiewicz inequality (2.10) and the exponent γ . Setting

$$\xi(t) := \int_t^{+\infty} \|u'(s)\|^2 ds,$$

we get that

$$-\xi'(t) = \|u'(t)\|^2 \geq C \left(\int_t^{+\infty} \|u'(s)\|^2 ds \right)^{2(1-\gamma)} = C \xi(t)^{2(1-\gamma)}. \quad (2.16)$$

From now on, we consider the case $\gamma < 1/2$. Thus, we get that the function $t \mapsto (\xi(t)^{2\gamma-1} - Ct)$ is non-decreasing on $]0, T[$. Thus, for every $0 < s < t < T$, we have the estimate

$$\xi(t) \leq \left(\xi(s)^{-(1-2\gamma)} + C(t-s) \right)^{-\frac{1}{1-2\gamma}}. \quad (2.17)$$

Now, let $0 < s < t_1 < t_2 < T$. Then

$$\begin{aligned} \|u(t_2) - u(t_1)\| &\leq \left\| \int_{t_1}^{t_2} u'(\tau) d\tau \right\| \leq \int_{t_1}^{t_2} \|u'(\tau)\| d\tau \\ &\leq \left(\int_{t_1}^{t_2} \|u'(\tau)\|^2 d\tau \right)^{1/2} (t_2 - t_1)^{1/2} \leq \xi(t_1)^{1/2} (t_2 - t_1)^{1/2} \\ &\leq \left(\xi(s)^{-(1-2\gamma)} + C(t_1 - s) \right)^{-\frac{1}{2(1-2\gamma)}} (t_2 - t_1)^{1/2} \\ &\leq C(t_1 - s)^{-\frac{1}{2(1-2\gamma)}} (t_2 - t_1)^{1/2}. \end{aligned}$$

Taking $k \geq 1$ and applying the above inequality to $t_2 = 2^{k+1}$, $t_1 = 2^k$ and $s \leq 2^{k-1}$, we get

$$\|u(2^{k+1}) - u(2^k)\| \leq C 2^{-\frac{k\gamma}{1-2\gamma}}.$$

In particular, for every $m > n$ such that $2^m < T$, we obtain

$$\|u(2^m) - u(2^n)\| \leq \sum_{k=n}^{\infty} \|u(2^{k+1}) - u(2^k)\| = \frac{C}{1 - 2^{-\frac{\gamma}{1-2\gamma}}} 2^{-\frac{n\gamma}{1-2\gamma}}. \quad (2.18)$$

On the other hand, if $t < T$ and $2^m \leq t < 2^{m+1}$, then

$$\|u(t) - u(2^m)\| \leq C(2^m)^{-\frac{1}{2(1-2\gamma)}} (t - 2^m)^{1/2} \leq \sqrt{2} C (2^m)^{-\frac{\gamma}{1-2\gamma}}. \quad (2.19)$$

Thus, for every $n \geq 1$ and $2^n \leq t < T$, we obtain

$$\|u(t) - u(2^n)\| \leq C 2^{-\frac{n\gamma}{1-2\gamma}}. \quad (2.20)$$

First of all, we choose n such that $C 2^{-\frac{n\gamma}{1-2\gamma}} < \frac{\delta_L}{2}$. Next, we choose $\delta > 0$ such that

$$\text{dist}(u(t), \mathcal{S}) < \frac{\delta_L}{2}, \quad \text{for every } t \in]0, 2^n[$$

(such a constant exists due to the continuity with respect to the initial datum). In particular, this implies that we can take $T = +\infty$.

We now use again (2.18), this time for every $m > n$, obtaining that the limit

$$u_\infty := \lim_{n \rightarrow \infty} u(2^n),$$

exists and is such that $\|u_\infty - u(2^n)\| \leq C 2^{-\frac{n\gamma}{1-2\gamma}}$. Finally, (2.20) implies that

$$\lim_{t \rightarrow \infty} u(t) = u_\infty \quad \text{and} \quad \|u_\infty - u(t)\| \leq C t^{-\frac{\gamma}{1-2\gamma}}.$$

In order to obtain the energy decay (the second inequality in (2.11)) we notice that, by the monotonicity of $t \mapsto \mathcal{F}(u(t))$, the inequality (2.15) and the integrability of $t \mapsto \|u'(t)\|^2$, we get that $\lim_{t \rightarrow \infty} \mathcal{F}(u(t)) = \mathcal{F}(\mathcal{S})$. Thus, passing to the limit as $T \rightarrow \infty$ in (2.13), we get that (2.14) holds with an equality. Now, the decay rate of the energy is a consequence of (2.17). Finally, let $\psi(t)$ be the solution of (2.4) with initial datum u_∞ . The continuity with respect to the initial datum and (2.11) imply that $\psi(t)$ is stationary and so, u_∞ is a critical point of \mathcal{F} in \mathcal{K} .

The case $\gamma = 1/2$ is analogous. Moreover, since (2.10) with $1/2$ implies (2.10) with any $\gamma < 1/2$ (up to changing C_L), we can actually use what we already know from $\gamma < 1/2$. In particular, (2.10) can be applied all along the flow and $u(t)$ converges to u_∞ . Thus, (2.17) implies that

$$\mathcal{F}(u(t)) - \mathcal{F}(\mathcal{S}_\varphi) = \xi(t) \leq \xi(0)e^{-Ct} = (\mathcal{F}(u_0) - \mathcal{F}(\mathcal{S}_\varphi)) e^{-Ct},$$

which gives the second part of (2.12). In order to get the first part, we notice that, choosing γ sufficiently close to $1/2$, we get from (2.11), that $\int_t^{+\infty} \|u(s) - u_\infty\|^2 ds \leq C$. Thus,

$$\|u(t) - u_\infty\|^2 \leq 2 \int_t^{+\infty} u'(s) \cdot (u_\infty - u(s)) ds \leq 2 \left(\int_t^{+\infty} \|u(s) - u_\infty\|^2 ds \right)^{1/2} \xi(t)^{1/2},$$

which gives the first part of (2.12). □

Remark 2.11. We notice that in Proposition 2.10 it is sufficient to suppose that the Łojasiewicz inequality (2.10) holds for the time-slices of the flow $u(t)$, $t \geq 0$.

To conclude this section we state a simple corollary of Proposition 2.10, whose proof is left to the reader.

Corollary 2.12. *With the same notations of Proposition 2.10, let $\varphi \in \mathcal{S}$ be the unique minimizer of \mathcal{F} and suppose that*

(a) *the flow (2.4) of $-\nabla \mathcal{F}$ in \mathcal{K} depends continuously on the initial datum;*

(b) *\mathcal{F} has the constrained Łojasiewicz property on $\mathcal{S}_\varphi = \{\varphi\}$ with $E_L = \delta_L = +\infty$ and $\gamma = 1/2$.*

Then, any global solution $u :]0, +\infty[\rightarrow \mathcal{K}$ converges to φ and there is a constant $C > 0$ such that

$$\|u(t) - \varphi\| \leq e^{-Ct} \quad \text{and} \quad \mathcal{F}(u(t)) - \mathcal{F}(\varphi) \leq e^{-Ct}, \quad (2.21)$$

for every $t \geq 1$.

3. LOGARITHMIC EPIPERIMETRIC INEQUALITIES

In this section, we show that if a functional \mathcal{G} satisfies a suitable slicing lemma, and the slicing functional \mathcal{F} is of the types considered in the previous section, then with a very general computation we can deduce that the so-called logarithmic epiperimetric inequality holds for \mathcal{G} . The subtlety here is that the competitor is going to live in a constrained subset of the domain of the functional which is not analytic. The link with the previous sections depends on the fact that we will use a parabolic inequality to define such a competitor.

In this section we fix $\mathcal{H} = L^2(\partial B_1)$, $\mathcal{K} \subset \mathcal{H}$ a convex cone, $\mathcal{W} = H^2(\partial B_1)$.

Proposition 3.1. *Let \mathcal{G} be a functional satisfying the following properties.*

(SL) *There exist a constant $k \in \mathbb{N}$ and a functional $\mathcal{F}: H^1(\partial B_1) \rightarrow \mathbb{R}$ such that for every $u: [0, 1] \times \partial B_1 \ni (r, \theta) \mapsto u(r, \theta) \in \mathbb{R}$, with $u \in H^1([0, 1] \times \partial B_1)$, the following slicing inequality holds*

$$\mathcal{G}(r^k u) \leq \int_0^1 \mathcal{F}(u(r, \cdot)) r^{2k+d-3} dr + C_{\text{SL}} \int_0^1 \int_{\partial B_1} |\partial_r u|^2 r^{2k+d-1} d\mathcal{H}^{d-1} dr, \quad (3.1)$$

for a geometric constant $C_{\text{SL}} > 0$, with equality if and only if u is 0-homogeneous (constant in the first variable).

(FL) *There is an open set $\mathcal{U}_{\text{FL}} \subset \mathcal{K}$ and a constant $\varepsilon_{\text{FL}} > 0$ such that for every $u_0 \in \mathcal{U}_{\text{FL}}$ there exists a strong solution $u \in H^1(]0, \varepsilon_{\text{FL}}[, \mathcal{H}) \cap L^2(]0, \varepsilon_{\text{FL}}[, \mathcal{W} \cap \mathcal{K})$ of*

$$\begin{cases} (u'(t) + \nabla \mathcal{F}(u(t))) \cdot (v - u(t)) \geq 0 & \text{for every } v \in \mathcal{K}, \quad 0 < t \leq \varepsilon_{\text{FL}} \\ u(0) = u_0, \end{cases} \quad (3.2)$$

and the flow is continuous with respect to the initial datum (see Definition 2.6).

(LS) \mathcal{F} has the constrained Lojasiewicz property (Definition 2.8) with respect to $\mathcal{S} = \{\psi\}$, where $\psi \in \mathcal{W} \cap \mathcal{K}$ is a critical point of \mathcal{F} in \mathcal{K} (see (2.3)).

Under the conditions (SL), (FL) and (LS), there are constants $\delta_0 > 0$ and $E > 0$, depending only on the dimension and ψ , such that: if $c \in H^1(\partial B_1) \cap \mathcal{K}$ satisfies

$$\|c - \psi\|_{L^2(\partial B_1)} \leq \delta_0 \quad \text{and} \quad \mathcal{F}(c) - \mathcal{F}(\psi) \leq E,$$

then there exists a function $h = h(r, \theta) \in H^1(B_1)$ satisfying $h(r, \cdot) \in \mathcal{K}$, for every $r \in (0, 1]$, and

$$\mathcal{G}(h) - \mathcal{G}(\phi) \leq (1 - \varepsilon |\mathcal{G}(z) - \mathcal{G}(\phi)|^{1-2\gamma}) (\mathcal{G}(z) - \mathcal{G}(\phi)) \quad (3.3)$$

where $\phi(r, \theta) := r^k \psi(\theta)$, $z(r, \theta) := r^k c(\theta)$, $\varepsilon > 0$ is a universal constant and $\gamma > 0$ is the exponent from (LS).

Proof. Notice that if $\mathcal{G}(z) - \mathcal{G}(\phi) \leq 0$, then choosing $h := z$ trivially gives the inequality. Therefore we can assume that

$$0 < \mathcal{G}(z) - \mathcal{G}(\phi) = \frac{1}{2k + d - 2} (\mathcal{F}(c) - \mathcal{F}(\psi)).$$

We construct h in the following way. Let $u:]0, \alpha[\rightarrow \mathcal{K}$ be a strong solution of

$$\begin{cases} (u'(t) + \nabla \mathcal{F}(u(t))) \cdot (v - u(t)) \geq 0 & \text{for every } v \in \mathcal{K}, \\ u(0) = c, \end{cases} \quad (3.4)$$

where $\alpha \leq \min\{\varepsilon_{\text{FL}}, \varepsilon_2\}$ and ε_2 is chosen so that

$$\mathcal{F}(u_0) - \mathcal{F}(\psi) \leq 2(\mathcal{F}(u(t)) - \mathcal{F}(\psi)) \quad \text{for every } 0 < t \leq \varepsilon_2, \quad (3.5)$$

with equality exactly at ε_2 (notice that ε_2 is well-defined since $t \mapsto \mathcal{F}(u(t))$ is continuous and non-increasing in t). We then extend u to be constant on $[\alpha, +\infty[$, that is, $u(t, \cdot) \equiv u(\alpha, \cdot)$ for every $t \geq \alpha$. Then we define the competitor h , in polar coordinates, as

$$h(r, \theta) := r^k u(-\alpha \log(r), \theta). \quad (3.6)$$

For the sake of simplicity, we set $\|\cdot\|_2 := \|\cdot\|_{L^2(\partial B_1)}$. By (SL) we have

$$\begin{aligned}
 \mathcal{G}(h) - \mathcal{G}(z) &\stackrel{\text{(SL)}}{\leq} \int_0^1 (\mathcal{F}(r^{-k}h(r, \cdot)) - \mathcal{F}(c)) r^{2k+d-3} dr + C_{\text{SL}} \int_0^1 \int_{\partial B_1} |\partial_r h|^2 r^{2k+d-1} d\mathcal{H}^{d-1} dr \\
 &= \frac{1}{\alpha} \int_0^\infty (\mathcal{F}(u(t)) - \mathcal{F}(u(0))) e^{-\frac{t(2k+d-2)}{\alpha}} dt + C_{\text{SL}} \alpha \int_0^\infty \int_{\partial B_1} |u'|^2 d\mathcal{H}^{d-1} e^{-\frac{t(2k+d-2)}{\alpha}} dt \\
 &= \frac{1}{\alpha} \int_0^\infty \int_0^{\min\{t, \alpha\}} \nabla \mathcal{F}(u(\tau)) \cdot u'(\tau) d\tau e^{-\frac{t(2k+d-2)}{\alpha}} dt + C_{\text{SL}} \alpha \int_0^\infty \|u'(t)\|_2^2 e^{-t(2k+d-2)} dt \\
 &= \frac{1}{\alpha} \int_0^\alpha \nabla \mathcal{F}(u(\tau)) \cdot u'(\tau) \int_\tau^\infty e^{-\frac{t(2k+d-2)}{\alpha}} dt d\tau + C_{\text{SL}} \alpha \int_0^\alpha \|u'(t)\|_2^2 e^{-\frac{t(2k+d-2)}{\alpha}} dt \\
 &= -\frac{1}{\alpha} \int_0^\alpha \left(-\frac{1}{(2k+d-2)} \nabla \mathcal{F}(u(\tau)) \cdot u'(\tau) - C_{\text{SL}} \alpha^2 \|u'(\tau)\|_2^2 \right) e^{-\frac{\tau(2k+d-2)}{\alpha}} d\tau \\
 &= -\frac{1}{\alpha} \int_0^\alpha \left(\frac{1}{(2k+d-2)} \|u'(\tau)\|_2^2 - C_{\text{SL}} \alpha^2 \|u'(\tau)\|_2^2 \right) e^{-\frac{\tau(2k+d-2)}{\alpha}} d\tau \\
 &= -\frac{1}{\alpha} \int_0^\alpha C_{d,k} \|\nabla \mathcal{F}(u(\tau))\|_{\mathcal{K}}^2 e^{-\frac{\tau(2k+d-2)}{\alpha}} d\tau, \tag{3.7}
 \end{aligned}$$

where the next to last equality is due to Lemma 2.5 (i) and the last equality is due to Lemma 2.5 (iii) and a choice of $\alpha > 0$ small enough, depending only on d, k and C_{SL} .

We next fix $\bar{\varepsilon} \in]0, 1[$. Using the property (LS), with C_L being the constant from the constrained Łojasiewicz inequality, and the previous computation we calculate

$$\begin{aligned}
 &(\mathcal{G}(h) - \mathcal{G}(\phi)) - (1 - \bar{\varepsilon})(\mathcal{G}(z) - \mathcal{G}(\phi)) \\
 &= -\frac{C_{d,k}}{\alpha} \int_0^\alpha \|\nabla \mathcal{F}(u(t))\|_{\mathcal{K}}^2 e^{-\frac{t(2k+d-2)}{\alpha}} dt + \frac{\bar{\varepsilon}}{2k+d-2} (\mathcal{F}(u_0) - \mathcal{F}(\psi)) \\
 &\stackrel{\text{(LS)}}{\leq} -\frac{C_{d,k}}{\alpha} \int_0^\alpha C_L^2 (\mathcal{F}(u(t)) - \mathcal{F}(\psi))^{2-2\gamma} e^{-\frac{t(2k+d-2)}{\alpha}} dt + \frac{\bar{\varepsilon}}{2k+d-2} (\mathcal{F}(u_0) - \mathcal{F}(\psi)) \\
 &\stackrel{\text{(3.5)}}{\leq} -\frac{C_{d,k} C_L^2}{\alpha} \int_0^\alpha 2^{2-2\gamma} (\mathcal{F}(u_0) - \mathcal{F}(\psi))^{2-2\gamma} e^{-\frac{t(2k+d-2)}{\alpha}} dt + \frac{\bar{\varepsilon}}{2k+d-2} (\mathcal{F}(u_0) - \mathcal{F}(\psi)) \\
 &\leq -\frac{1}{2k+d-2} \left(C_{d,k,\gamma} C_L^2 (1 - e^{-(2k+d-2)}) - \bar{\varepsilon} \right) (\mathcal{F}(u_0) - \mathcal{F}(\psi))^{2-2\gamma} < 0
 \end{aligned}$$

where in the last inequality we chose $\bar{\varepsilon} := \varepsilon (\mathcal{F}(u_0) - \mathcal{F}(\psi))^{1-2\gamma}$ for some $\varepsilon > 0$ small enough depending on d, k, γ and C_L . Notice that we are allowed to apply (LS) to $u(t)$ for every $0 < t \leq \alpha$, by choosing δ_0 small enough (depending on the dimension and ψ) and using the continuity of the flow with respect to the initial datum. \square

4. CONSTRAINED ŁOJASIEWICZ INEQUALITIES FOR OBSTACLE AND THIN-OBSTACLE PROBLEMS

This section is dedicated to the proofs of the theorems stated in the introduction. In particular we will show that constrained Łojasiewicz-type inequalities hold in all the problems considered there and then conclude using the abstract results of the previous sections.

4.1. The obstacle problem in a ball. Let $B_1 \subset \mathbb{R}^d$ be the unit ball and $g \in H^1(B_1)$ be a given non-negative function. Let $\mathcal{H} = L^2(B_1)$, $\mathcal{W} = H^2(B_1)$ and

$$\mathcal{K}_{\text{OB}} = \{u \in H^1(B_1) : u - g \in H_0^1(B_1), u \geq 0 \text{ in } B_1\}.$$

Recall that the obstacle energy is given by

$$\mathcal{F}_{\text{OB}}(u) = \frac{1}{2} \int_{B_1} |\nabla u|^2 dx + \int_{B_1} u dx \quad \text{and} \quad \nabla \mathcal{F}_{\text{OB}}(u) = \Delta u - 1.$$

Let $\varphi \in \mathcal{K}_{\text{OB}}$ be the unique solution of the obstacle problem

$$\min_{u \in \mathcal{K}_{\text{OB}}} \mathcal{F}_{\text{OB}}(u). \quad (4.1)$$

Then, $\varphi \in H^2(B_1)$ is the only critical point of \mathcal{F}_{OB} in \mathcal{K}_{OB} (in the sense of (2.3)) and is also the unique solution of the problem

$$\Delta\varphi = \mathbb{1}_{\{\varphi > 0\}} \quad \text{in } B_1, \quad \varphi = g \quad \text{on } \partial B_1.$$

Proposition 4.1 (Constrained Lojasiewicz for the obstacle problem). *Let \mathcal{K}_{OB} and \mathcal{F}_{OB} be as above and φ be the solution of the obstacle problem (4.1). Then, there is a dimensional constant $C_d > 0$ such that*

$$(\mathcal{F}_{\text{OB}}(u) - \mathcal{F}_{\text{OB}}(\varphi))^{1/2} \leq C_d \|\nabla \mathcal{F}_{\text{OB}}(u)\|_{\mathcal{K}_{\text{OB}}} \quad \text{for every } u \in H^2(B_1) \cap \mathcal{K}_{\text{OB}}. \quad (4.2)$$

Proof. To simplify the notations we drop the index OB. Let $u \in H^2(B_1) \cap \mathcal{K}$. Then, we have

$$\begin{aligned} \|\Delta u - 1\|_{\mathcal{K}} &\geq -\frac{1}{\|u - \varphi\|_{L^2}} \int_{B_1} (u - \varphi) (\Delta u - 1) \, dx \\ &= -\frac{1}{\|u - \varphi\|_{L^2}} \int_{B_1} (u - \varphi) \Delta(u - \varphi) \, dx + \frac{1}{\|u - \varphi\|_{L^2}} \int_{B_1 \cap \{\varphi=0\}} (u - \varphi) \, dx \\ &\geq \frac{1}{\|u - \varphi\|_{L^2}} \left(\frac{1}{2} \int_{B_1} |\nabla(u - \varphi)|^2 \, dx + \int_{B_1 \cap \{\varphi=0\}} (u - \varphi) \, dx \right). \end{aligned}$$

Next since

$$\begin{aligned} \int_{B_1} |\nabla(u - \varphi)|^2 \, dx &= \int_{B_1} (|\nabla u|^2 - |\nabla \varphi|^2 + 2\nabla \varphi \cdot \nabla(\varphi - u)) \, dx \\ &= \int_{B_1} |\nabla u|^2 \, dx - \int_{B_1} |\nabla \varphi|^2 \, dx + 2 \int_{B_1 \cap \{\varphi > 0\}} (u - \varphi) \, dx, \end{aligned} \quad (4.3)$$

we conclude that

$$\|\Delta u - 1\|_{L^2} \geq \frac{1}{\|u - \varphi\|_{L^2}} (\mathcal{F}(u) - \mathcal{F}(\varphi)). \quad (4.4)$$

Let C_d be the constant of the Poincaré inequality for $u - \varphi \in H_0^1(B_1)$. Using again (4.3) and the fact that $u \geq 0$ in B_1 , we have

$$\|u - \varphi\|_{L^2}^2 \leq C_d \|\nabla(u - \varphi)\|_{L^2}^2 \leq 2C_d (\mathcal{F}(u) - \mathcal{F}(\varphi)).$$

This, together with (4.4) gives (4.2). \square

4.2. The thin-obstacle problem in a ball. Let B_1 be the unit ball in \mathbb{R}^d , $B_1^+ := B_1 \cap \{x_d > 0\}$ and $B_1' = B_1 \cap \{x_d = 0\}$; we will use the notation $x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$. Let $g \in H^1(B_1^+)$ be such that $g \geq 0$ on B_1' . Let $\mathcal{H} = L^2 = L^2(B_1^+)$, $\mathcal{W} = H^2(B_1^+)$ and

$$\mathcal{K}_{\text{TH}} := \left\{ u \in H^1(B_1^+) : u \geq 0 \text{ on } B_1', u = g \text{ on } \partial B_1 \cap \{x_d > 0\} \right\}.$$

Recall that the thin-obstacle energy is given by

$$\mathcal{F}_{\text{TH}}(u) = \frac{1}{2} \int_{B_1} |\nabla u|^2 \, dx \quad \text{with} \quad \nabla \mathcal{F}_{\text{TH}}(u) = \Delta u.$$

Let φ be the unique solution of the thin-obstacle problem

$$\min_{u \in \mathcal{K}_{\text{TH}}} \mathcal{F}_{\text{TH}}(u), \quad (4.5)$$

where all functions in \mathcal{K}_{TH} are extended even to the full ball.

Proposition 4.2 (Constrained Łojasiewicz for the thin-obstacle problem). *Let \mathcal{K}_{TH} , \mathcal{F}_{TH} and φ be as above. There is a dimensional constant C_d such that*

$$(\mathcal{F}_{\text{TH}}(u) - \mathcal{F}_{\text{TH}}(\varphi))^{1/2} \leq C_d \|\nabla \mathcal{F}_{\text{TH}}(u)\|_{\mathcal{K}}, \quad (4.6)$$

for every $u \in H^2(B_1^+) \cap \mathcal{K}_{\text{TH}}$ such that

$$\frac{\partial u}{\partial x_d} \leq 0 \quad \text{on } B_1' \quad \text{and} \quad u \frac{\partial u}{\partial x_d} = 0 \quad \text{on } B_1'. \quad (4.7)$$

Proof. For simplicity of notations we drop the index TH. Let $u \in H^2(B_1^+) \cap \mathcal{K}$ be a function satisfying (4.7). By definition of $\|\cdot\|_{\mathcal{K}}$ (see (2.2)), we have the estimate

$$\begin{aligned} \|\nabla \mathcal{F}(u)\|_{\mathcal{K}} &\geq -\frac{1}{\|u - \varphi\|_{L^2}} \int_{B_1^+} (u - \varphi) \Delta u \, dx = -\frac{1}{\|u - \varphi\|_{L^2}} \int_{B_1^+} (u - \varphi) \Delta(u - \varphi) \, dx \\ &= \frac{1}{\|u - \varphi\|_{L^2}} \left(\int_{B_1^+} |\nabla(u - \varphi)|^2 \, dx - \int_{B_1'} (u - \varphi) \frac{\partial(u - \varphi)}{\partial n} \, dx' \right), \end{aligned}$$

where n is the exterior normal to B_1^+ . On the other hand, we have

$$\begin{aligned} \frac{1}{2} \int_{B_1^+} |\nabla(u - \varphi)|^2 \, dx &= \frac{1}{2} \int_{B_1^+} |\nabla u|^2 \, dx - \frac{1}{2} \int_{B_1^+} |\nabla \varphi|^2 \, dx + \int_{B_1^+} \nabla \varphi \cdot \nabla(\varphi - u) \, dx \\ &= \mathcal{F}(u) - \mathcal{F}(\varphi) + \int_{B_1'} (\varphi - u) \frac{\partial \varphi}{\partial n} \, dx'. \end{aligned} \quad (4.8)$$

Using that $\varphi \frac{\partial u}{\partial n} \geq 0$, $u \frac{\partial \varphi}{\partial n} \geq 0$ and $u \frac{\partial u}{\partial n} = \varphi \frac{\partial \varphi}{\partial n} = 0$ on B_1' , we have

$$\begin{aligned} \|\nabla \mathcal{F}(u)\|_{\mathcal{K}} &\geq \frac{1}{\|u - \varphi\|_{L^2}} \left(\frac{1}{2} \int_{B_1^+} |\nabla(u - \varphi)|^2 \, dx + \int_{B_1'} \left(\varphi \frac{\partial u}{\partial n} + u \frac{\partial \varphi}{\partial n} \right) \, dx' \right) \\ &= \frac{1}{\|u - \varphi\|_{L^2}} \left(\mathcal{F}(u) - \mathcal{F}(\varphi) + \int_{B_1'} \varphi \frac{\partial u}{\partial n} \, dx' \right) \geq \frac{\mathcal{F}(u) - \mathcal{F}(\varphi)}{\|u - \varphi\|_{L^2}}. \end{aligned}$$

On the other hand, the fact that $u - \varphi = 0$ on $\partial B_1 \cap \{x_d > 0\}$, the Poincaré inequality and (4.8) give that

$$\|u - \varphi\|_{L^2(B_1^+)}^2 \leq C_d \int_{B_1^+} |\nabla(u - \varphi)|^2 \, dx \leq 2C_d (\mathcal{F}(u) - \mathcal{F}(\varphi)),$$

which concludes the proof of (4.6). \square

4.3. Obstacle problem on a compact manifold. Let (\mathcal{M}, g) be a compact connected oriented Riemannian manifold of dimension $d \geq 2$. We denote by Δ and ∇ the Laplace-Beltrami operator and the gradient on \mathcal{M} , respectively. We denote by dV_g the volume form on \mathcal{M} , in local coordinates $dV_g = \det(g_{ij}) dx^1 \wedge \cdots \wedge dx^n$. We will denote by $L^2(\mathcal{M})$ the space of Lebesgue measurable square real integrable functions and, for $u \in L^2(\mathcal{M})$, we will use the notation $\|u\|_2 = (\int_{\mathcal{M}} u^2 \, dV_g)^{1/2}$. The associated scalar product in $L^2(\mathcal{M})$ will be denoted by

$$u \cdot v = \langle u, v \rangle = \langle u, v \rangle_{L^2(\mathcal{M})} = \int_{\mathcal{M}} uv \, dV_g, \quad \text{for } u, v \in L^2(\mathcal{M}).$$

The Sobolev space $H^1(\mathcal{M})$ on (\mathcal{M}, g) is defined as the closure of the smooth functions on \mathcal{M} with respect to the norm

$$\|u\|_{H^1} = \int_{\mathcal{M}} (g(\nabla u, \nabla u) + u^2) \, dV_g.$$

Moreover, we will often use the notations $\nabla u \cdot \nabla v := g(\nabla u, \nabla v)$, for the scalar product with respect to the metric g , and $|\nabla u|^2 := g(\nabla u, \nabla u)$, for the induced norm. The higher order Sobolev spaces $H^k(\mathcal{M})$ are defined analogously. The Laplace-Beltrami operator Δ is defined on $H^2(\mathcal{M})$

with values in $L^2(\mathcal{M})$ and also by duality, as an operator $\Delta : H^1(\mathcal{M}) \rightarrow H^{-1}(\mathcal{M})$; in both cases we will use the notation

$$\int_{\mathcal{M}} (\Delta u) v dV_g := - \int_{\mathcal{M}} \nabla u \cdot \nabla v dV_g \quad \text{for every } u, v \in H^1(\mathcal{M}).$$

It is well known that the spectrum of the Laplace-Beltrami operator Δ is discrete and can be written as an increasing sequence of real positive eigenvalues $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_k \leq \dots$ counted with their multiplicity. The corresponding eigenfunctions $\phi_j \in H^1(\mathcal{M})$, $j \in \mathbb{N}$, are smooth on \mathcal{M} and form an infinite orthonormal basis of $L^2(\mathcal{M})$, precisely,

$$\int_{\mathcal{M}} uv dV_g = \delta_{ij} \quad \text{and} \quad \int_{\mathcal{M}} \nabla u \cdot \nabla v dV_g = \lambda_i \delta_{ij}.$$

Let $\lambda \in \mathbb{R}$ be given and $\mathcal{F}_{\text{OB}}^\lambda$ be the functional

$$\mathcal{F}_{\text{OB}}^\lambda(u) = \frac{1}{2} \int_{\mathcal{M}} (|\nabla u|^2 - \lambda u^2) dV_g + \int_{\mathcal{M}} u dV_g. \quad (4.9)$$

For $u, v \in H^1(\mathcal{M})$, we will use the notation

$$v \cdot \nabla \mathcal{F}_{\text{OB}}^\lambda(u) = \nabla \mathcal{F}_{\text{OB}}^\lambda(u)[v] = \delta \mathcal{F}_{\text{OB}}^\lambda(u)[v] = \lim_{t \rightarrow 0} \frac{1}{t} \left(\mathcal{F}_{\text{OB}}^\lambda(u + tv) - \mathcal{F}_{\text{OB}}^\lambda(u) \right),$$

and we notice that

$$\nabla \mathcal{F}_{\text{OB}}^\lambda(u) = -\Delta u - \lambda u + 1 \quad \text{for every } u \in H^2(\mathcal{M})$$

Let $\mathcal{K}_{\text{OB}}^\mathcal{M} := \{u \in L^2(\mathcal{M}) : u \geq 0\}$. For $u \in \mathcal{K}_{\text{OB}}^\mathcal{M} \cap H^2(\mathcal{M})$, we define $\|\nabla \mathcal{F}_{\text{OB}}^\lambda(u)\|_{\mathcal{K}_{\text{OB}}^\mathcal{M}}$ as in (2.2).

Let λ be an eigenvalue of the Laplace-Beltrami operator Δ on \mathcal{M} . Then all the critical points of $\mathcal{F}_{\text{OB}}^\lambda$ are of the form $\lambda^{-1} + \phi$, for some λ -eigenfunction ϕ . We denote by \mathcal{S}_λ the set of all non-negative critical points of the functional $\mathcal{F}_{\text{OB}}^\lambda$. Notice that $\mathcal{F}_{\text{OB}}^\lambda$ is constant on \mathcal{S}_λ , that is,

$$(2\lambda)^{-1} \text{vol}(\mathcal{M}) = \mathcal{F}_{\text{OB}}^\lambda(\mathcal{S}_\lambda) = \mathcal{F}_{\text{OB}}^\lambda(\varphi), \quad \forall \varphi \in \mathcal{S}_\lambda.$$

Remark 4.3. The set \mathcal{S}_λ is bounded both in $L^2(\mathcal{M})$ and $H^1(\mathcal{M})$. Indeed, suppose that this is not the case. Then there is a sequence of λ -eigenfunctions ϕ_n such that $\|\phi_n\|_2 = 1$ and a sequence $C_n \rightarrow \infty$ such that $\lambda^{-1} + C_n \phi_n \in \mathcal{S}_\lambda$. But then we have also that $\psi_n := C_n^{-1} \lambda^{-1} + \phi_n \in \mathcal{S}_\lambda$. Now, since ϕ_n is bounded in $H^1(\mathcal{M})$, up to a subsequence, ϕ_n and ψ_n converge strongly in $L^2(\mathcal{M})$ to a function $\phi \in H^1(\mathcal{M})$, such that $\phi \geq 0$ on \mathcal{M} , $\int \phi^2 dV_g = 1$ and ϕ is a λ -eigenfunction, which is a contradiction.

We next prove a Łojasiewicz inequality in a neighborhood of the family of critical points \mathcal{S}_λ . For any $u \in L^2(\mathcal{M})$, we set

$$\text{dist}_2(u, \mathcal{S}_\lambda) := \inf \{ \|u - \varphi\|_2 : \varphi \in \mathcal{S}_\lambda \},$$

and, for any $\gamma \in (0, 1/2]$, we define the function $f_\gamma : \mathbb{R} \rightarrow \mathbb{R}^+$ as

$$f_\gamma(t) := \begin{cases} t^{1/2}, & \text{if } t \geq 1, \\ t^{1-\gamma}, & \text{if } 0 \leq t \leq 1, \\ 0, & \text{if } t \leq 0, \end{cases} \quad (4.10)$$

Proposition 4.4 (Constraint Łojasiewicz inequality for the obstacle on a manifold). *Let $\lambda > 0$ be an eigenfunction of Δ on \mathcal{M} , and let $\mathcal{F}_{\text{OB}}^\lambda$, \mathcal{S}_λ be as above. Then, there are constants $C, \delta > 0$ (depending on (\mathcal{M}, g) and λ) and $\gamma \in (0, 1/2)$ (depending only on the dimension $d = \dim \mathcal{M}$) such that:*

$$f_\gamma(\mathcal{F}_{\text{OB}}^\lambda(u) - \mathcal{F}_{\text{OB}}^\lambda(\mathcal{S}_\lambda)) \leq C \|\nabla \mathcal{F}_{\text{OB}}^\lambda(u)\|_{\mathcal{K}}, \quad (4.11)$$

for every $u \in H^2(\mathcal{M}) \cap \mathcal{K}_{\text{OB}}^\mathcal{M}$ such that $\text{dist}_2(u, \mathcal{S}_\lambda) \leq \delta$. In particular, for every $E \geq 1$ and every $u \in H^2(\mathcal{M}) \cap \mathcal{K}_{\text{OB}}^\mathcal{M}$ satisfying $\text{dist}_2(u, \mathcal{S}_\lambda) \leq \delta$ and $\mathcal{F}_{\text{OB}}^\lambda(u) - \mathcal{F}_{\text{OB}}^\lambda(\mathcal{S}_\lambda) \leq E$, we have

$$(\mathcal{F}_{\text{OB}}^\lambda(u) - \mathcal{F}_{\text{OB}}^\lambda(\mathcal{S}_\lambda))_+^{1-\gamma} \leq CE^{\frac{1}{2}-\gamma} \|\nabla \mathcal{F}_{\text{OB}}^\lambda(u)\|_{\mathcal{K}}. \quad (4.12)$$

Proof. For the sake of simplicity we set $\mathcal{F} = \mathcal{F}_{\text{OB}}^\lambda$, $\mathcal{K} = \mathcal{K}_{\text{OB}}^M$ and $\mathcal{S} = \mathcal{S}_\lambda$. Let $\varphi \in \mathcal{S}$ be such that $\|u - \varphi\|_2 \leq 2\delta$. Notice that $u - \varphi$ can be uniquely decomposed in Fourier series as $u - \varphi = Q_- + Q_0 + \eta$, where Q_- contains only lower eigenmodes (corresponding to eigenvalues $< \lambda$), Q_0 is a λ -eigenfunction and

$$\eta(x) = \sum_{\{j:\lambda_j>\lambda\}} c_j \phi_j(x),$$

which contains only higher eigenfunctions (corresponding to eigenvalues $> \lambda$). Thus, $u = Q_- + Q_0 + \varphi + \eta$ and $\|Q_-\|_2, \|Q_0\|_2, \|\eta\|_2 \leq 2\delta$. We now consider $M := \max_{x \in \mathcal{M}} \{-Q_-(x) - Q_0(x) - \varphi(x)\}$ and suppose that the maximum is realized in a point $x_M \in \mathcal{M}$. Notice that since $Q_- + Q_0$ is a finite sum of (smooth) eigenfunctions, there is a constant $C > 0$ (depending on \mathcal{M} and λ) such that $\|Q_- + Q_0\|_{L^\infty} \leq C\delta$. Thus, if $M > 0$, then $x_M \in \{\varphi < C\delta\}$ and $M \leq C\delta$. We now choose δ such that $10C\delta < c_\lambda := \lambda^{-1}$ and we claim that the function

$$\tilde{u} = Q_- + Q_0 + \varphi + \frac{2M}{c_\lambda}(c_\lambda - \varphi)$$

is non-negative. Indeed, it is sufficient to consider the following two cases:

- on the set $\{\varphi \geq 2C\delta\}$, we have that

$$\tilde{u} = \left(Q_- + Q_0 + \frac{1}{2}\varphi\right) + 2M + \varphi \left(\frac{1}{2} - \frac{2M}{c_\lambda}\right) \geq 0,$$

since each of the three terms is non-negative;

- on the set $\{\varphi \leq 2C\delta\}$, we have that

$$\tilde{u} \geq Q_- + Q_0 + \varphi + \frac{2M}{c_\lambda}(c_\lambda - 2C\delta) \geq Q_- + Q_0 + \varphi + M \geq 0.$$

Next, we can compute

$$\begin{aligned} -(\tilde{u} - u) \cdot \nabla \mathcal{F}(u) &= \int_{\mathcal{M}} (-\Delta u - \lambda u + 1) \left(\eta - \frac{2M}{c_\lambda}(c_\lambda - \varphi)\right) dV_g \\ &= \int_{\mathcal{M}} (-\Delta u - \lambda u + 1) \eta dV_g \\ &= \int_{\mathcal{M}} (-\Delta(Q_- + Q_0 + \eta) - \lambda(Q_- + Q_0 + \eta)) \eta dV_g \\ &= \int_{\mathcal{M}} (|\nabla \eta|^2 - \lambda \eta^2) dV_g = \sum_{j:\lambda_j>\lambda} c_j^2 (\lambda_j - \lambda) = 2\mathcal{F}(\eta), \end{aligned}$$

where in the second equality we used the fact that

$$\int_{\mathcal{M}} (-\Delta u - \lambda u + 1) (c_\lambda - \varphi) dV_g = 0,$$

since $-\Delta u - \lambda u$ is orthogonal to every λ -eigenfunction and $c_\lambda - \varphi$ is a λ -eigenfunction, by Remark 4.3, and so its integral on \mathcal{M} vanishes, due to the fact that $\lambda > 0$.

Notice that since the set of eigenvalues is discrete, there is a (spectral gap) constant $G(\lambda) > 0$ such that $\lambda_j - \lambda \geq G(\lambda)$, whenever $\lambda_j - \lambda > 0$. In particular, we have the inequality

$$2\mathcal{F}(\eta) = \sum_{j:\lambda_j>\lambda} c_j^2 (\lambda_j - \lambda) \geq G(\lambda) \sum_{j:\lambda_j>\lambda} c_j^2 = G(\lambda) \|\eta\|_2^2.$$

Thus, we get

$$\begin{aligned} \|\nabla \mathcal{F}(u)\|_{\mathcal{K}} &\geq \frac{-(\tilde{u} - u) \cdot \nabla \mathcal{F}(u)}{\|u - \tilde{u}\|_2} \geq \frac{2\mathcal{F}(\eta)}{\frac{2M}{c_\lambda} \|c_\lambda - \varphi\|_2 + \|\eta\|_2} \\ &\geq \frac{2\mathcal{F}(\eta)}{M \frac{2}{c_\lambda} (\|c_\lambda\|_2 + \|\varphi\|_2) + (G(\lambda)\mathcal{F}(\eta))^{1/2}} \geq C \frac{\mathcal{F}(\eta)}{M + \mathcal{F}(\eta)^{1/2}}, \end{aligned} \tag{4.13}$$

where C is a constant depending only on λ and \mathcal{M} . On the other hand, we have

$$2(\mathcal{F}(u) - \mathcal{F}(\varphi)) = \int_{\mathcal{M}} (|\nabla(u - \varphi)|^2 - \lambda(u - \varphi)^2) dV_g \leq \int_{\mathcal{M}} (|\nabla\eta|^2 - \lambda\eta^2) dV_g = 2\mathcal{F}(\eta). \quad (4.14)$$

Now, in order to get (4.11), it only remains to estimate M and put together (4.13) and (4.14). We notice that:

- $Q_- + Q_0 + \varphi$ is a (finite) linear combination of (orthonormal and smooth) eigenfunctions corresponding to eigenvalues $\leq \lambda$;
- the L^2 norm of $Q_- + Q_0 + \varphi$ is bounded by a universal constant.

As a consequence, there is a universal (Lipschitz) constant L , depending only on λ and \mathcal{M} , such that $\|\nabla(Q_- + Q_0 + \varphi)\|_{L^\infty(\mathcal{M})} \leq L$. Thus, since the negative part $\psi := -\inf\{(Q_- + Q_0 + \varphi), 0\}$ is such that $\sup\psi = M$ is small enough (bounded by a constant depending on \mathcal{M} and λ , as already mentioned above), we get that there is a constant C (depending on \mathcal{M}) such that

$$\|\psi\|_{L^2(\mathcal{M})}^2 \geq CL^{-d}M^{d+2} = CL^{-d}\|\psi\|_{L^\infty(\mathcal{M})}^{d+2}.$$

Since $u \geq 0$ on \mathcal{M} , we have that $\psi \leq \eta$ and so,

$$M^{d+2} \leq C^{-1}L^d\|\eta\|_2^2 \leq \frac{2L^d}{CG(\lambda)}\mathcal{F}(\eta),$$

which, together with (4.13) and (4.14), we get (4.11) with $\gamma = \frac{1}{d+2}$. \square

4.4. The thin-obstacle problem on the sphere. Let $\mathcal{K}_{\text{TH}}^{\mathbb{S}}$ be the set of functions on the sphere which are non-negative on the equator $\{x_d = 0\} \cap \partial B_1$. Let $m \in \mathbb{N}$, $\lambda := \lambda(2m) = 2m(2m + d - 2)$ and $\mathcal{F}_{\text{TH}}^\lambda$ be the functional

$$\mathcal{F}_{\text{TH}}^\lambda(u) = \frac{1}{2} \int_{\partial B_1} (|\nabla u|^2 - \lambda u^2) d\mathcal{H}^{d-1}.$$

Thus, $\nabla \mathcal{F}_{\text{TH}}^\lambda(u) = -\Delta u - \lambda u$, where Δ is the Laplace-Beltrami operator on the sphere. Notice that the operator $\nabla \mathcal{F}_{\text{TH}}^\lambda : H^2(\partial B_1) \rightarrow L^2(\partial B_1)$ is symmetric, that is, for any $u, v \in H^2(\partial B_1)$ we have

$$v \cdot \nabla \mathcal{F}_{\text{TH}}^\lambda(u) = \int_{\partial B_1} v(-\Delta u - \lambda u) d\mathcal{H}^{d-1} = \int_{\partial B_1} u(-\Delta v - \lambda v) d\mathcal{H}^{d-1} = u \cdot \nabla \mathcal{F}_{\text{TH}}^\lambda(v).$$

The set of critical points of $\mathcal{F}_{\text{TH}}^\lambda$ is precisely the set of the eigenfunctions of the spherical Laplacian, corresponding to the eigenvalue λ . Let \mathcal{S}_λ be the set of critical points of $\mathcal{F}_{\text{TH}}^\lambda$, which are positive on $\{x_d = 0\} \cap \partial B_1$. Notice that $\mathcal{F}_{\text{TH}}^\lambda$ vanishes on \mathcal{S}_λ , that is, $\mathcal{F}_{\text{TH}}^\lambda(\mathcal{S}_\lambda) \equiv 0$.

Proposition 4.5 (Constrained Łojasiewicz inequality for the thin-obstacle on the sphere). *Let $\mathcal{K}_{\text{TH}}^{\mathbb{S}}$, $\mathcal{F}_{\text{TH}}^\lambda$, \mathcal{S}_λ and $\lambda = \lambda(2m)$ be as above. Then, for every $A > 0$, there are a constant C , depending on A , the dimension d and the homogeneity $2m$, and a constant $\delta > 0$, depending only on d and m , such that*

$$f_\gamma(\mathcal{F}_{\text{TH}}^\lambda(u)) \leq C \|\nabla \mathcal{F}_{\text{TH}}^\lambda(u)\|_{\mathcal{K}_{\text{TH}}^{\mathbb{S}}} \quad \text{for every } u \in \mathcal{K}_{\text{TH}}^{\mathbb{S}} \cap H^2(\partial B_1) \quad (4.15)$$

such that $\text{dist}_2(u, \mathcal{S}_\lambda) \leq \delta$ and $\|u\|_2 \leq A$,

where $\gamma = 1/d$, f_γ is the function defined in (4.10) and $\text{dist}_2(u, \mathcal{S}_\lambda)$ is the $L^2(\partial B_1)$ distance from u to the set \mathcal{S}_λ . In particular, there is a constant $C_{d,m}$, depending only on m and d , such that, for every $E \geq 1$ and $A \geq 1$, we have

$$(\mathcal{F}_{\text{TH}}^\lambda(u))_+^{1-\gamma} \leq C_{d,m} E^{1/2-1/d} A^{1-2/d} \|\nabla \mathcal{F}_{\text{TH}}^\lambda(u)\|_{\mathcal{K}}, \quad (4.16)$$

for every $u \in H^2(\partial B_1) \cap \mathcal{K}_{\text{TH}}^{\mathbb{S}}$ such that $\text{dist}_2(u, \mathcal{S}_\lambda) \leq \delta$, $\|u\|_2 \leq A$ and $\mathcal{F}_{\text{TH}}^\lambda(u) \leq E$.

Proof. For the sake of simplicity, we set $\mathcal{F} = \mathcal{F}_{\text{TH}}^\lambda$, $\mathcal{K} = \mathcal{K}_{\text{TH}}^{\mathbb{S}}$ and $\mathcal{S} = \mathcal{S}_\lambda$. Let $u \in \mathcal{K}$ be such that $\mathcal{F}(u) > \mathcal{F}(\mathcal{S}) = 0$. By the definition of $\|\nabla \mathcal{F}\|_{\mathcal{K}}$, it is sufficient to prove that there is a function $\tilde{u} \in \mathcal{K}$ such that

$$\frac{-(\tilde{u} - u) \cdot \nabla \mathcal{F}(u)}{\|\tilde{u} - u\|_2} \geq C f_\gamma(\mathcal{F}(u)). \quad (4.17)$$

We notice that u can be decomposed in Fourier series on ∂B_1 , using the eigenfunctions of the spherical Laplacian. We write u as $u = v_- + v_0 + v_+$ and we set $\tilde{u} = v_- + \tilde{v}_0$, where:

- v_- is the projection of u on the space of eigenfunctions corresponding to eigenvalues $< \lambda(2m)$;
- v_+ is the projection of u on the space of eigenfunctions corresponding to eigenvalues $> \lambda(2m)$;
- v_0 and \tilde{v}_0 are critical points for \mathcal{F} (eigenfunctions for $\lambda(2m)$); \tilde{v}_0 will be chosen later.

We first calculate the left-hand side of (4.17). For the L^2 norm we have

$$\|\tilde{u} - u\|_2 = (\|\tilde{v}_0 - v_0\|_2^2 + \|v_+\|_2^2)^{1/2}.$$

For the scalar product, we integrate by parts and use the orthogonality of the eigenfunctions:

$$\begin{aligned} -(\tilde{u} - u) \cdot \nabla \mathcal{F}(u) &= (v_0 - \tilde{v}_0 + v_+) \cdot \nabla \mathcal{F}(u) = u \cdot \nabla \mathcal{F}(v_0 - \tilde{v}_0 + v_+) \\ &= u \cdot \nabla \mathcal{F}(v_+) = v_+ \cdot \nabla \mathcal{F}(u) = v_+ \cdot \nabla \mathcal{F}(v_+) = \mathcal{F}(v_+). \end{aligned}$$

Thus, we obtain

$$-\frac{\tilde{u} - u}{\|\tilde{u} - u\|_2} \cdot \nabla \mathcal{F}(u) = \frac{\mathcal{F}(v_+)}{(\|\tilde{v}_0 - v_0\|_2^2 + \|v_+\|_2^2)^{1/2}}. \quad (4.18)$$

We now aim to estimate $\|\tilde{v}_0 - v_0\|_2^2 + \|v_+\|_2^2$ by $\mathcal{F}(v_+)$. First, we notice that

$$\mathcal{F}(v_+) = \frac{1}{2} \sum_{j=2m+1}^{\infty} c_j^2 (\lambda(j) - \lambda(2m)) \geq \frac{\lambda(2m+1) - \lambda(2m)}{2} \sum_{j=2m+1}^{\infty} c_j^2 = \frac{4m+d-1}{2} \|v_+\|_2^2, \quad (4.19)$$

which estimates the second term. In order to give a bound for $\|\tilde{v}_0 - v_0\|_2$ we will need to choose \tilde{v}_0 carefully. Let ψ be an eigenfunction of the spherical Laplacian, corresponding to the eigenvalue $\lambda(2m)$, and such that $\psi = 1$ on $\{x_d = 0\} \cap \partial B_1$. We choose

$$\tilde{v}_0 = M\psi + v_0 \quad \text{where} \quad M = - \inf_{\{x_d=0\} \cap \partial B_1} (v_- + v_0).$$

We now claim that there is a constant $C_{d,m} > 0$, depending on d and m , such that

$$M^d \leq C_{d,m} (\max\{1, A\})^{d-2} \mathcal{F}(v_+). \quad (4.20)$$

First of all, we notice that there is a constant L_m , depending only on d and m , such that all the L^2 -normalized eigenfunctions corresponding to eigenvalues $\leq \lambda(2m)$ are globally L_m -Lipschitz continuous on $\partial B_1 = \mathbb{S}^{d-1}$, that is,

$$\|\nabla \phi\|_{L^\infty(\partial B_1)} \leq L_m \quad \text{for every} \quad \phi : \partial B_1 \rightarrow \mathbb{R} \quad \text{such that}$$

$$-\Delta \phi = \lambda \phi \quad \text{on} \quad \partial B_1 \quad \text{and} \quad \int_{\partial B_1} \phi^2 d\mathcal{H}^{d-1} = 1, \quad \text{where} \quad 0 \leq \lambda \leq \lambda(2m).$$

On the other hand, by construction, $v_- + v_0$ is a linear combination of eigenfunctions ϕ_j , each one normalized in $L^2(\partial B_1)$ and corresponding to a different eigenvalue $\lambda(j) \leq \lambda(2m)$, that is, we have

$$v_- + v_0 = \sum_{j=0}^{2m} c_j \phi_j \quad \text{and} \quad \|v_- + v_0\|_2^2 = \sum_{j=0}^{2m} c_j^2.$$

Thus, we get the following Lipschitz bound for $v_- + v_0$,

$$\|\nabla(v_- + v_0)\|_{L^\infty(\partial B_1)} \leq \sum_{j=0}^{2m} |c_j| \|\nabla \phi_j\|_{L^\infty(\partial B_1)} \leq L_m \sum_{j=0}^{2m} |c_j| \leq (2m)^{1/2} L_m \|v_- + v_0\|_2 \leq L,$$

where we set

$$L = (2m)^{1/2} L_m \max\{A, 1\} \geq (2m)^{1/2} L_m \|u\|_2 \geq (2m)^{1/2} L_m \|v_- + v_0\|_2.$$

Analogously, we get that M is bounded by a universal constant depending only on δ , and not on A . Indeed, if Q is the $L^2(\mathbb{S}^{d-1})$ -projection of u on the set of (nonnegative) functions \mathcal{S}_λ , we get that, at every point on the sphere $\partial B_1 = \mathbb{S}^{d-1}$,

$$|\min\{0, v_- + v_0\}| \leq |\min\{0, v_- + v_0 - Q\}| \quad \text{and} \quad |\nabla(v_- + v_0 - Q)| \leq (2m)^{1/2} L_m \delta.$$

Which proves that, for $\delta > 0$ small enough,

$$M = \max_{\{x_d=0\} \cap \mathbb{S}^{d-1}} |\min\{0, v_- + v_0\}| \leq \max_{\{x_d=0\} \cap \mathbb{S}^{d-1}} |\min\{0, v_- + v_0 - Q\}| \leq C_{d,m} \delta.$$

Next, we compute

$$\int_{\mathbb{S}^{d-2}} v_+^2 d\mathcal{H}^{d-2} \geq \int_{\mathbb{S}^{d-2}} (\min\{0, v_- + v_0\})^2 d\mathcal{H}^{d-2} \geq C_d M^2 \left(\frac{M}{L}\right)^{d-2} = \frac{C_{d,m} M^d}{(\max\{A, 1\})^{d-2}},$$

for some constant $C_{d,m}$ depending only on d and on m ; the first inequality follows from the fact that $v_- + v_0 + v_+$ is non-negative on \mathbb{S}^{d-2} , while the second one is the consequence of the facts that M is small and that $\min\{0, v_- + v_0\}$ is L -Lipschitz. Now, by the trace inequality on the sphere ∂B_1 and the fact that the expansion of v_+ contains only eigenfunctions corresponding to frequencies higher than $\lambda(2m)$, we get

$$\int_{\mathbb{S}^{d-2}} v_+^2 d\mathcal{H}^{d-2} \leq C_d \int_{\partial B_1} (|\nabla_\theta v_+|^2 + v_+^2) d\mathcal{H}^{d-1} \leq C_{d,m} \mathcal{F}(v_+),$$

which concludes the proof of (4.20). Finally, (4.20) and (4.19) give

$$\|\nabla \mathcal{F}(u)\|_{\mathcal{K}} \geq \frac{\mathcal{F}(v_+)}{(M^2 \|\psi\|_2^2 + \|v_+\|_2^2)^{1/2}} \geq \frac{C_{d,m} \mathcal{F}(v_+)}{\left((\max\{1, A\})^{2(d-2)/d} \mathcal{F}(v_+)^{2/d} + \mathcal{F}(v_+)\right)^{1/2}}.$$

We now notice that, for δ small enough, we have

$$\mathcal{F}(v_+) = \mathcal{F}(u) - \mathcal{F}(v_-) \leq \mathcal{F}(u) + \lambda(2m) \|v_-\|_2^2 \leq \mathcal{F}(u) + \lambda(2m) \delta^2 \leq E + 1,$$

where we used that $\mathcal{F}(u) \leq E$. Thus, up to changing the constant $C_{d,m}$, we get

$$\mathcal{F}(u)^{1-1/d} \leq \mathcal{F}(v_+)^{1-1/d} \leq C_{d,m} E^{1/2-1/d} (\max\{1, A\})^{(d-2)/d} \|\nabla \mathcal{F}(u)\|_{\mathcal{K}},$$

which concludes the proof. \square

5. PROOF OF THE MAIN RESULTS

5.1. Proof of Theorem 1.7. We first consider the case of the parabolic obstacle problem (1.14). By Proposition 4.1, we have that the functional \mathcal{F}_{OB} satisfies the constrained Łojasiewicz inequality with $\gamma = 1/2$, while Lemma 2.7 implies that the flow of \mathcal{F}_{OB} is continuous with respect to the initial datum. Thus, by Proposition 2.10 and Corollary 2.12, we have that the solution $u : [0, +\infty[\mapsto L^2(B_1)$ of (1.14) converges in $L^2(B_1)$ to the unique stationary solution φ , which is also the unique solution of (1.12). In order to get the convergence in $H^1(B_1)$ and its rate, we estimate

$$\begin{aligned} \int_{B_1} |\nabla(u(t) - \varphi)|^2 dx &= \int_{B_1} |\nabla u(t)|^2 dx - \int_{B_1} |\nabla \varphi|^2 dx - 2 \int_{B_1} \nabla \varphi \cdot \nabla(u(t) - \varphi) dx \\ &= \int_{B_1} |\nabla u(t)|^2 dx - \int_{B_1} |\nabla \varphi|^2 dx + 2 \int_{B_1} (u(t) - \varphi) \mathbb{1}_{\{\varphi > 0\}} dx \\ &= 2(\mathcal{F}_{\text{OB}}(u(t)) - \mathcal{F}_{\text{OB}}(\varphi)) + 2|B_1|^{1/2} \|u(t) - \varphi\|_{L^2(B_1)}. \end{aligned}$$

Thus, (1.22) follows by the exponential estimates in Corollary 2.12.

In the case of the thin-obstacle problem (1.20), we first apply Proposition 4.2 obtaining that the constrained Łojasiewicz inequality (with $\gamma = 1/2$) holds for the functional \mathcal{F}_{TH} , along the solution $u(t)$ of (1.20). On the other hand, Lemma 2.7 implies that the flow of \mathcal{F}_{TH} is continuous

with respect to the initial datum. As a consequence, by Corollary 2.12 (and Remark 2.11), we have that the solution $u(t)$ converges in $L^2(B_1)$ to the unique solution φ of (1.18). Using the notations $H = \{x_d > 0\}$ and $B_1^+ = H \cap B_1$, we calculate

$$\begin{aligned} \int_{B_1^+} |\nabla(u(t) - \varphi)|^2 dx &= \int_{B_1^+} |\nabla u(t)|^2 dx - \int_{B_1^+} |\nabla \varphi|^2 dx + 2 \int_{\partial H \cap B_1} \frac{\partial \varphi}{\partial x_d} (u(t) - \varphi) dx' \\ &\leq \int_{B_1^+} |\nabla u(t)|^2 dx - \int_{B_1^+} |\nabla \varphi|^2 dx = \mathcal{F}_{\text{TH}}(u(t)) - \mathcal{F}_{\text{TH}}(\varphi). \end{aligned}$$

Thus, the conclusion follows by the estimate (2.21) of Corollary 2.12. \square

5.2. Proof of Theorem 1.8. Let us first treat the case of the parabolic obstacle problem (1.16) on the sphere. Due to Lemma 2.7 and Proposition 4.4, the hypotheses of Proposition 2.10 are satisfied. Thus, $u(t)$ converges to a function u_∞ , which is a critical point of $\mathcal{F}_{\text{OB}}^\lambda$ in $\mathcal{K}_{\text{OB}}^\mathbb{S}$ (in the sense of (2.3)). Now, since the critical points of $\mathcal{F}_{\text{OB}}^\lambda$ in $\mathcal{K}_{\text{OB}}^\mathbb{S}$ (which are in a neighborhood of \mathcal{S}_λ) are classified, we get that $u_\infty \in \mathcal{S}_\lambda$. Moreover, Proposition 2.10 implies that

$$\|u(t) - u_\infty\|_{L^2(\mathbb{S})} \leq Ct^{-\frac{\gamma}{1-2\gamma}} \quad \text{and} \quad \mathcal{F}_{\text{OB}}^\lambda(u(t)) - \mathcal{F}_{\text{OB}}^\lambda(u_\infty) \leq Ct^{-\frac{1}{1-2\gamma}}.$$

Now, reasoning as in the proof of Theorem 1.7, we obtain

$$\begin{aligned} \int_{\mathbb{S}} |\nabla(u(t) - u_\infty)|^2 &= \int_{\mathbb{S}} |\nabla u(t)|^2 - \int_{\mathbb{S}} |\nabla u_\infty|^2 - 2 \int_{\mathbb{S}} \nabla u_\infty \cdot \nabla(u(t) - u_\infty) \\ &= \int_{B_1} |\nabla u(t)|^2 dx - \int_{B_1} |\nabla u_\infty|^2 dx + 2 \int_{B_1} (u(t) - u_\infty)(1 - \lambda u_\infty) dx \\ &= 2(\mathcal{F}_{\text{OB}}^\lambda(u(t)) - \mathcal{F}_{\text{OB}}^\lambda(u_\infty)) + \lambda \int_{\mathbb{S}} |u(t) - u_\infty|^2, \end{aligned}$$

which finally gives (1.23).

The case of the parabolic thin-obstacle problem is analogous. We first notice that the L^2 -norm of $u(t)$ is decreasing in t as far as $\mathcal{F}_{\text{TH}}^\lambda(u(t))$ remains positive. Indeed, since $u(t)$ (multiplied by any positive constant) is always an admissible test function in the differential inequality (1.21), we have that

$$(u'(t) + \nabla \mathcal{F}_{\text{TH}}^\lambda(u(t))) \cdot u(t) = 0.$$

Thus, taking the derivative of $\|u(t)\|^2 = \|u(t)\|_{L^2(\partial B_1)}^2$, we get

$$\frac{d}{dt} \|u(t)\|^2 = 2u'(t) \cdot u(t) = -2u(t) \cdot \nabla \mathcal{F}_{\text{TH}}^\lambda(u(t)) = -2\mathcal{F}_{\text{TH}}^\lambda(u(t)) \leq 0,$$

where the last equality is just an integration by parts. This implies that $\|u(t)\| \leq \|u_0\|$ for every $t \geq 0$. In particular, the Łojasiewicz inequality (4.16) from Proposition 4.5 holds all along the flow $u(t)$ with a uniform constant, depending only on the L^2 -norm and the energy of the initial datum u_0 . Thus, using Lemma 2.7 and Proposition 4.4 with $\gamma \in]0, 1/2[$, we get by Proposition 2.10 that $u(t)$ converges to a function u_∞ , which is a critical point of $\mathcal{F}_{\text{TH}}^\lambda$ in $\mathcal{K}_{\text{TH}}^\mathbb{S}$. Since the critical points of $\mathcal{F}_{\text{TH}}^\lambda$ in $\mathcal{K}_{\text{TH}}^\mathbb{S}$ are classified, we get that $u_\infty \in \mathcal{S}_\lambda$. We denote by \mathbb{S}^+ the upper half-sphere $\{x_d > 0\} \cap \partial B_1$ and we calculate

$$\begin{aligned} \int_{\mathbb{S}^+} |\nabla(u(t) - u_\infty)|^2 &= \int_{\mathbb{S}^+} |\nabla u(t)|^2 - \int_{\mathbb{S}^+} |\nabla u_\infty|^2 - 2\lambda \int_{\mathbb{S}^+} u_\infty(u(t) - u_\infty) \\ &= \mathcal{F}_{\text{TH}}(u(t)) - \mathcal{F}_{\text{TH}}(u_\infty) + \lambda \int_{\mathbb{S}^+} |u(t) - u_\infty|^2. \end{aligned}$$

Now, using Proposition (2.10), we get (1.23). \square

5.3. Proof of Theorem 1.10. Both the claims (OB) and (TH) follow by Proposition 3.1. Let us check that the conditions (SL), (FL) and (LS) of Proposition 3.1 are satisfied.

(SL) In order, to prove the slicing condition (SL), we set

$$\mathcal{G}(u) := \int_{B_1} |\nabla u|^2 dx - k \int_{\partial B_1} u^2 d\mathcal{H}^{d-1}, \quad (5.1)$$

for some $k > 0$ and $u = u(r, \theta)$. Setting $\theta \in \partial B_1$, $d\theta = d\mathcal{H}^{d-1}$ and we calculate

$$\begin{aligned} \mathcal{G}(r^k u) &= \int_0^1 \int_{\partial B_1} \left(|k r^{k-1} u + r^k \partial_r u|^2 + r^{2k-2} |\nabla_\theta u|^2 \right) d\theta r^{d-1} dr - k \int_{\partial B_1} u^2 d\theta \\ &= \int_0^1 \int_{\partial B_1} \left(k^2 r^{2k-2} u^2 + r^{2k} |\partial_r u|^2 + k r^{2k-1} \partial_r (u^2) + r^{2k-2} |\nabla_\theta u|^2 \right) d\theta r^{d-1} dr - k \int_{\partial B_1} u^2 d\theta \\ &= \int_0^1 \int_{\partial B_1} \left(k^2 r^{2k-2} u^2 + r^{2k} |\partial_r u|^2 - k(2k + d - 2) r^{2k-2} u^2 + r^{2k-2} |\nabla_\theta u|^2 \right) d\theta r^{d-1} dr \\ &= \int_0^1 r^{2k+d-3} \int_{\partial B_1} \left(|\nabla_\theta u|^2 - k(k + d - 2) u^2 \right) d\theta dr + \int_0^1 r^{2k+d-1} \int_{\partial B_1} |\partial_r u|^2 d\theta dr \end{aligned}$$

that is, if we set

$$\mathcal{F}(\phi) := \int_{\partial B_1} \left(|\nabla_\theta \phi|^2 - \lambda(k) \phi^2 \right) d\mathcal{H}^{d-1}, \quad \text{where } \lambda(k) = k(k + d - 2), \quad (5.2)$$

then we have the slicing equality

$$\mathcal{G}(r^k u) = \int_0^1 \mathcal{F}(u(r, \cdot)) r^{2k+d-3} dr + \int_0^1 r^{2k+d-1} \int_{\partial B_1} |\partial_r u|^2 d\mathcal{H}^{d-1} dr. \quad (5.3)$$

Now, setting $k = 2m$, $\lambda = \lambda(2m)$, $\mathcal{G} = \mathcal{G}_{\text{TH}}$ and $\mathcal{F} = \mathcal{F}_{\text{TH}}^\lambda$, we get the slicing inequality (SL) for the thin-obstacle problem (at the singular points, where the homogeneity of the blow-up is $2m$). Using the identity (5.3) with $k = 1$ and $\lambda = 2d$, together with a simple change in polar coordinates, we get (SL) for the obstacle problem, with $\mathcal{F} = \mathcal{F}_{\text{OB}}^\lambda$ and $\mathcal{G} = \mathcal{G}_{\text{OB}}$.

(FL) The existence of the flow (FL) follows from the general existence result [4].

(LS) For what concerns the constrained Łojasiewicz inequality (LS), in the case of the obstacle problem it is enough to apply Proposition 4.4 with $\mathcal{M} = \mathbb{S}^{d-1}$ and $\lambda = 2d$, while in the case of the thin-obstacle it follows from Proposition 4.5. \square

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