A NOTE ON TOPOLOGICAL DIMENSION, HAUSDORFF MEASURE, AND RECTIFIABILITY

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Abstract. We give a sufficient condition for a general compact metric space to admit an n-rectifiable piece, as a consequence of a recent result of David Bate. Let X be a compact metric space of topological dimension n. Suppose that the n-dimensional Hausdorff measure of X, $\mathcal{H}^n(X)$, is finite. Suppose further that the lower n-density of the measure $\mathcal{H}^n$ is positive, $\mathcal{H}^n$-almost everywhere in X. Then X contains an n-rectifiable subset of positive $\mathcal{H}^n$-measure. Moreover, the assumption on the lower density is unnecessary if one uses recently announced results of Csörnyei-Jones.

1. Introduction

The purpose of this note is to record a consequence, for general metric spaces, of a recent result of Bate [2]. We prove the following fact:

Theorem 1.1. *Let X be a compact metric space of topological dimension n. Suppose that the n-dimensional Hausdorff measure of X, $\mathcal{H}^n(X)$, is finite.*

Suppose further that

$$\liminf_{r\to 0} \frac{\mathcal{H}^n(B(x,r))}{r^n} > 0 \text{ for } \mathcal{H}^n\text{-a.e. } x \in X. \tag{1}$$

Then X contains an n-rectifiable subset of positive $\mathcal{H}^n$-measure.

Moreover, assumption (1) is unnecessary if one uses recently announced results of Csörnyei-Jones.

The use in Theorem 1.1 of the results of Csörnyei-Jones arises purely through our use of the work of Bate [2] (Theorem 2.1 below), and does not directly appear in any of the proofs here. See Bate's discussion just below [2, Theorem 1.1] for details concerning the announcement of Csörnyei-Jones and the dependence of Theorem 2.1 on them.

When X is a subset of some Euclidean space, Theorem 1.1, without assuming (1) or the results of Csörnyei-Jones, appears to already be known (see [16, p. 880]), as a consequence of the Besicovitch-Federer projection

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theorem. For general metric spaces, the Besicovitch-Federer theorem is unavailable [3], but Bate's work [2] serves as our replacement. As a general rule, sufficient conditions for finding rectifiability in an abstract metric space are much rarer than for subsets of Euclidean space, where tools such as projection and density theorems are available.

When $n = 1$, Theorem 1.1 (without assuming (1) or relying on the results of Csörnyei-Jones) is a consequence of the fact that continua of finite $\mathcal{H}^1$-measure are Lipschitz images of $[0, 1]$ (see, e.g., [15, Lemma 3.7]), but this particular fact does not extend to $n > 1$.

We now recall some background: For compact metric spaces, the commonly used notions of topological dimension (Lebesgue covering dimension, large/strong inductive dimension, and small/weak inductive dimension) agree. We refer the reader to [14, Sections I.4 and II.5] for this fact and the relevant definitions. For Hausdorff measure and dimension, we refer the reader to [9, Chapter 8].

An $\mathcal{H}^n$-measurable subset E of a metric space X is called *n-rectifiable* if

$$\mathcal{H}^n(E \setminus \bigcup_{i=1}^{\infty} f_i(F_i)) = 0$$

where F_i are measurable subsets of $\mathbb{R}^n$ and $f_i : F_i \to X$ are Lipschitz maps. By a theorem of Kirchheim [12, Lemma 4], one can equivalently take f_i to be bi-Lipschitz mappings.

A subset E of a metric space X is called *purely n-unrectifiable* if it contains no n-rectifiable subsets of positive $\mathcal{H}^n$-measure.

If a compact metric space X has topological dimension n, then it is a well-known fact (see, e.g., [9, Theorem 8.15]) that $\mathcal{H}^n(X) > 0$, although certainly X may have infinite n-dimensional Hausdorff measure or even Hausdorff dimension strictly larger than n, as is the case for classical fractals. Thus, Theorem 1.1 says that in the extremal situation, one must see some Euclidean structure in the space.

Related results, in which a combination of n-dimensional topological behavior and n-dimensional measure theoretic behavior implies some type of rectifiability, can be found, for example, in [5, 7, 8, 11, 16]. These results typically employ more quantitative assumptions to obtain more quantitative conclusions than our Theorem 1.1.

It is easy to see that the assumptions of Theorem 1.1 (including (1)) do not imply n-rectifiability of the whole space X. For example, X may be the disjoint union of the unit ball in $\mathbb{R}^n$ with a metric space that is a purely n-unrectifiable Cantor set of positive n-dimensional Hausdorff measure.

More surprising is that the assumptions of Theorem 1.1 do not imply n-rectifiability even if one assumes that X is a compact n-dimensional topological manifold. In the appendix to [19], Schul and Wenger construct a compact topological n-sphere with $\mathcal{H}^n(X) < \infty$ that contains a purely n-unrectifiable subset of positive measure.

On the other hand, Theorem 1.1 implies that every open ball in a compact n-manifold with finite $\mathcal{H}^n$-measure contains an n-rectifiable subset of positive $\mathcal{H}^n$-measure. Note that there exist such manifolds with no bi-Lipschitz embedding into any Euclidean space [13, 18].

As a final remark, we point out two well-known, purely unrectifiable examples that contrast with Theorem 1.1. For one, consider the closed unit ball B in the Heisenberg group, which is a compact metric space of topological dimension 3 and Hausdorff dimension 4. This metric space B is purely 4-unrectifiable, as one can show with a standard "blowup" argument. In fact, B is also purely 2- and 3-unrectifiable, but this is more difficult to establish (see [1, Theorem 7.2]).

For a second example, consider any compact metric space (X, d) of topological dimension m and Hausdorff dimension $n \geq m$, and let $Y = (X, d^p)$ for some $p \in (0, 1)$. Then Y is a compact metric space of topological dimension m and Hausdorff dimension $n/p > m$ that is purely k-unrectifiable for each $k \in \mathbb{N}$. Indeed, if $E \subseteq \mathbb{R}^k$ is compact and $f : E \to f(E) \subset Y$ is a bi-Lipschitz map, then blowing up $f : E \to f(E)$, in the Gromov-Hausdorff sense, at a point of density of E yields a bi-Lipschitz embedding of $\mathbb{R}^k$ into a metric space of the form (Z, d^p). This is impossible, as such a space can contain no rectifiable curves. For more on such blowup arguments, we refer the reader to [6, Chapters 8-9].

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2. Proof of Theorem 1.1

Given a metric space X and $m \in \mathbb{N}$, let $\mathrm{Lip}_1(X, m)$ denote the space of bounded, 1-Lipschitz functions $f\colon X \to \mathbb{R}^m$, equipped with the supremum distance, which we denote dist. This is a complete metric space, and hence residual subsets (in the sense of Baire category) are dense.

The proof of Theorem 1.1 is based on the following recent result.

Theorem 2.1 (Bate [2, Theorem 1.1])**.** *Let X be a complete, purely n-unrectifiable metric space with $\mathcal{H}^n(X) < \infty$. Suppose further that* (1) *holds.*

Then the set of all $f \in Lip_1(X, m)$ with $\mathcal{H}^n(f(X)) = 0$ is residual.

Moreover, assumption (1) *is unnecessary if one uses recently announced results of Csörnyei-Jones.*

This has the following easy consequence.

Corollary 2.2. *Let X be a compact, purely n-unrectifiable metric space with $\mathcal{H}^n(X) < \infty$ and satisfying* (1)*.*

Let $g : X \to [0, 1]^n$ be continuous. Then there is a sequence of Lipschitz functions $f_i : X \to [0, 1]^n$ that converge to g in the supremum distance and satisfy $\mathcal{H}^n(f_i(X)) = 0$ for all $i \in \mathbb{N}$.

Moreover, assumption (1) *is unnecessary if one uses recently announced results of Csörnyei-Jones.*

Proof. Let h_i be a sequence of L_i-Lipschitz functions converging to g in the supremum distance. (The existence of such a sequence is a consequence of the Stone-Weierstrass theorem, or see [17, Lemma 2.4] for a simple direct proof.) Thus $L_i^{-1}h_i \in \mathrm{Lip}_1(X,n)$.

By Theorem 2.1, we can find, for each $i \in \mathbb{N}$, a Lipschitz function $g_i \in \mathrm{Lip}_1(X,n)$ satisfying

$$\mathrm{dist}(L_i^{-1}h_i, g_i) < L_i^{-1}2^{-i} \text{ and } \mathcal{H}^n(g_i(X)) = 0.$$

Consider the 1-Lipschitz retraction $r : \mathbb{R}^n \to [0,1]^n$ given by

$$r(x_1, x_2, \dots, x_n) = (\psi(x_1), \psi(x_2), \dots, \psi(x_n)), \tag{2}$$

where

$$\psi(t) = \begin{cases} 0 & t < 0 \\ t & 0 \le t \le 1 \\ 1 & t > 1. \end{cases}$$

Lastly, set

$$f_i = r \circ (L_i g_i).$$

Since r is Lipschitz and $\mathcal{H}^n(g_i(X)) = 0$, we have $\mathcal{H}^n(f_i(X)) = 0$ for all $i \in \mathbb{N}$. Furthermore,

$$\mathrm{dist}(f_i, g) = \mathrm{dist}(r \circ (L_i g_i), r \circ g) \le \mathrm{dist}(L_i g_i, g) < 2^{-i} + \mathrm{dist}(h_i, g) \to 0.$$

□

To prove Theorem 1.1, we will also need some topological information.

Definition 2.3. Let $f\colon X \to Y$ be a continuous map between metric spaces. A point $y \in Y$ is called a *stable value* of f if there is $\epsilon > 0$ such that $y \in g(X)$ for every continuous $g\colon X \to Y$ with $\mathrm{dist}(g,f) < \epsilon$.

Some basic and well-known facts about stable values of mappings to $[0,1]^n$ are collected in the following lemma.

Lemma 2.4. *Let X be a metric space and let y be a stable value of a continuous map $f\colon X \to [0,1]^n$. Then*

(i) $y \notin \partial([0,1]^n)$,
(ii) *y is a stable value of g for each continuous $g : X \to [0,1]^n$ with $\mathrm{dist}(g,f)$ sufficiently small, and*
(iii) *$f(X)$ contains an open neighborhood of y in $[0,1]^n$.*

Proof. Part (i) is simple and explained in [10, Example VI 4]. Part (ii) is an immediate consequence of the definition of stable value.

For part (iii), recall the 1-Lipschitz retraction $r\colon \mathbb{R}^n \to [0,1]^n$ defined in (2). Note that r maps $\mathbb{R}^n \setminus [0,1]^n$ onto the boundary of $[0,1]^n$.

Let y be a stable value of $f : X \to [0,1]^n$, with parameter $\epsilon > 0$. Then, by part (i), $y \in (0,1)^n$. We claim that $f(X)$ contains $B(y,\epsilon) \cap [0,1]^n$. Consider any $y' \in B(y,\epsilon) \cap [0,1]^n$. The formula

$$h(x) = r(x + y - y')$$

defines a continuous map from $[0,1]^n$ to itself such that $h(y') = y$ and $|h(x) - x| < \epsilon$ for all $x \in [0,1]^n$.

Consider the map $g : X \to [0,1]^n$ defined by $g = h \circ f$. Then $\mathrm{dist}(f, g) < \epsilon$, so $g(x) = y$ for some $x \in X$. Therefore,

$$r(f(x) + y - y') = y.$$

Since y is not on the boundary of $[0,1]^n$, we must have $f(x) + y - y' = y$, i.e., $f(x) = y'$. □

The following theorem is the second main ingredient in the proof of Theorem 1.1.

Theorem 2.5 (Theorem III.1 of [14])**.** *Let X be a compact metric space of topological dimension n. Then there is a continuous map $g\colon X \to [0,1]^n$ with a stable value.*

The technique of using stable values to find some rectifiable structure in a metric space was used by David and Semmes [6, Section 12.3] and Bonk and Kleiner [4] in similar contexts.

Proof of Theorem 1.1. Let X be a compact metric space of topological dimension n and $\mathcal{H}^n(X) < \infty$. We claim that X contains an n-rectifiable subset of positive measure. Suppose, to the contrary, that X is purely n-unrectifiable.

Then, by Theorem 2.5, there is a continuous map $g\colon X \to [0,1]^n$ with a stable value y. By Corollary 2.2, there is a sequence f_i of Lipschitz maps from X to $[0,1]^n$ that converge to g in the supremum distance and satisfy

$$\mathcal{H}^n(f_i(X)) = 0 \tag{3}$$

for all $i \in \mathbb{N}$.

On the other hand, by Lemma 2.4(ii), when $i \in \mathbb{N}$ is sufficiently large, the map f_i must also have y as a stable value. In that case, $g_i(X)$ contains an open subset of $[0,1]^n$, by Lemma 2.4(iii). This contradicts (3). □

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