RESTRICTING OPEN SURJECTIONS

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Abstract. We show that any continuous open surjection from a complete metric space to another metric space can be restricted to a surjection for which the domain has the same density character as the target. This improves a recent result of Aron, Jaramillo and Le Donne.

1. Introduction

In recent years, there has been an increasing interest in studying *surjectivity properties* of different classes of maps between Banach spaces or, more generally, metric spaces. For a metric space X we denote by $\text{card}(X)$ the cardinality of X and by $\text{dens}(X)$ the density character, which is defined as the smallest cardinality of a dense subset of X.

In the Banach-space setting, it was proved in [4] that every separable Banach space Y is the range of a C^1-smooth surjection $f : X \to Y$ from any infinite-dimensional separable Banach space X. In addition, some conditions are given in [4] under which f can be chosen to be C^∞-smooth. In [6], it was shown that if $X = c_0$ and $Y = \ell_2$, then f cannot be C^2-smooth. For the non-separable space $X = c_0(\omega_1)$, the results of [5] show that the existence of C^2-smooth surjections onto ℓ_2 depends on additional axioms of set theory.

The problem of surjectivity of separable restrictions has been considered in [3]. It is proved there that, for every X belonging to some class of Banach spaces which includes all C^∞-smooth spaces with density $\geq 2^{\aleph_0}$, as well as all spaces $\ell_p(\Gamma)$ with $\text{card}(\Gamma) \geq 2^{\aleph_0}$, and for every Banach space Y with dimension ≥ 2, there exists a C^∞ smooth surjection $f : X \to Y$ whose restriction to any separable subspace of X fails to be surjective. More recently, this result has been extended in [7], where it is shown to hold for every non-separable super-reflexive space X.

In the metric setting, positive results about the surjectivity of separable restrictions have been obtained in [2]. A map $f\colon X \to Y$ between metric spaces is called *density-surjective* if there is a subset $Z \subset X$ so that $\text{dens}(Z) = \text{dens}(Y)$ and $f|_Z\colon Z \to Y$ is surjective. It is shown in [2] that every *uniformly open* continuous surjection from a complete metric space to another metric space is density-surjective. This result has been refined very recently

Date: September 14, 2018.

2010 *Mathematics Subject Classification.* Primary 54E40, 54C65.

The research of Jaramillo is supported in part by MINECO grant MTM2015-65825-P (Spain). Le Donne acknowledges the support of the Academy of Finland, project no. 288501, and the European Research Council, ERC-StG grant GeoMeG. Rajala acknowledges the support of the Academy of Finland, project no. 274372.

in [8], where it is proved that the corresponding surjective restriction of a uniformly open surjection can be also chosen to be uniformly open.

In this short note we improve on the mentioned result from [2] by replacing the uniformly openess assumption by openess.

Theorem 1.1. *Let X and Y be metric spaces with X complete. Suppose that $f\colon X \to Y$ is a continuous open surjection. Then f is density-surjective.*

We will actually prove a slightly more general version of Theorem 1.1 that is recorded as Theorem 2.1.

Recall that, if $f : M \to N$ is a C^1-smooth map between Banach manifolds, a point $x \in M$ is said to be a *regular point* of f if the derivative $df(x) : T_xM \to T_{f(x)}N$ is onto. A point $y \in N$ is said to be a *regular value* of f if every $x \in f^{-1}(y)$ is a regular point; otherwise we say that y is a *critical value* of f. Also, note that every *paracompact* Banach manifold admits a complete metric (see e.g., [9, Corollary on page 2]). Now as a corollary of Theorem 2.1 and the open mapping theorem for Banach manifolds (see e.g. [1, Theorem 3.5.2]), we obtain the following consequence.

Corollary 1.2. *Let $f : M \to N$ be a C^1-smooth surjection between paracompact Banach manifolds. Suppose that the set of critical values of f is countable. Then f is density-surjective.*

2. Proof of the Theorem

We will prove the following, slightly more general version of Theorem 1.1

Theorem 2.1. *Let X and Y be metric spaces with X complete and let $Y' \subset Y$. Let $f\colon X \to Y$ be continuous such that for $X' := f^{-1}(Y')$ the map $f|_{X'}\colon X' \to Y'$ is open and surjective. Then there exists a subspace $X_0 \subset X'$ with $\mathrm{dens}(X_0) = \mathrm{dens}(Y')$ such that $f|_{X_0}\colon X_0 \to Y'$ is surjective and X_0 is relatively closed in X'.*

Let us note that without loss of generality we may consider the case where Y is infinite.

Lemma 2.2. *Let $\tilde{X}$ and $\tilde{Y}$ be metric spaces, $f\colon \tilde{X} \to \tilde{Y}$ a continuous open surjection and $r > 0$. Then there exists an open cover $\{\tilde{Y}_k\}_{k\in I}$ of $\tilde{Y}$, with $\mathrm{card}(I) \le \mathrm{dens}(\tilde{Y})$ such that for every $k \in I$ we have $\mathrm{diam}(\tilde{Y}_k) \le r$ and there exists a point $\tilde{x}_k \in f^{-1}(\tilde{Y}_k)$ such that*

$$\tilde{Y}_k \subset f(B(\tilde{x}_k, r)). \tag{2.1}$$

Consequently, we also have that for each $k \in I$ we have that the map

$$f : f^{-1}(\tilde{Y}_k) \cap B(\tilde{x}_k, r) \to \tilde{Y}_k \tag{2.2}$$

is a continuous and open surjection.

Proof. For each $i \in \mathbb{N}$ define

$$\tilde{X}_i := \{\tilde{x} \in \tilde{X} \,:\, B(f(\tilde{x}), 2^{-i}r) \subset f(B(\tilde{x}, r))\}.$$

Since f is open, we have $\tilde{X} = \bigcup_{i\in\mathbb{N}} \tilde{X}_i$. It suffices to cover the set

$$\tilde{Y}^i := f(\tilde{X}_i)$$

for each $i \in \mathbb{N}$. Let $D^i = \{\tilde{y}_k^i\}_{k \in I_i}$ be a dense subset of $\tilde{Y}^i$ with $\mathrm{card}(I_i) \leq \mathrm{dens}(\tilde{Y})$. For each $k \in I_i$ select a point $\tilde{x}_k^i \in f^{-1}(\{\tilde{y}_k^i\}) \cap \tilde{X}_i$ and set

$$\tilde{Y}_k^i = B(\tilde{y}_k^i, 2^{-i} r).$$

By construction, $\tilde{Y}_k^i$ are open, $\tilde{Y}^i = \bigcup_{k \in I_i} \tilde{Y}_k^i$, $\mathrm{diam}(\tilde{Y}_k^i) \leq 2^{1-i} r \leq r$, $\tilde{x}_k^i \in f^{-1}(\tilde{Y}_k^i)$ and $\tilde{Y}_k^i \subset f(B(\tilde{x}_k^i, r))$ as was required. From (2.1), the map in (2.2) is surjective. Moreover, it is continuous and open since it is the restriction to an open set of a continuous and open map. □

Remark 2.3. Notice that in the proof of Lemma 2.2 one cannot directly use the openness of f to open neighborhoods of preimages of a dense set of points in $\tilde{Y}$ since the images of such neighborhoods need not cover the space $\tilde{Y}$. This is why the space $\tilde{X}$ is first covered by sets $\tilde{X}_i$ where the map is uniformly open on the given scale r.

Proof of Theorem 2.1. We prove the claim by repeatedly using Lemma 2.2. First, we let $r = 1$, $\tilde{X} = X'$ and $\tilde{Y} = Y'$. Lemma 2.2 gives us an open cover $\{Y_k^0\}_{k \in I_0}$ of Y' and points $\{x_k^0\}_{k \in I_0}$.

Then we continue inductively. Suppose we have defined for fixed i an open cover $\{Y_k^i\}_{k \in I_i}$ of Y' and corresponding points $\{x_k^i\}_{k \in I_i}$. We continue to cover each Y_k^i with the help of Lemma 2.2 by taking $r = 2^{-i}$, $\tilde{Y} = Y_k^i$ and $\tilde{X} = f^{-1}(Y_k^i) \cap B(x_k^i, 2^{-i})$. Lemma 2.2 now gives us a cover $\{Y_{k,j}^i\}_{j \in J_{k,i}}$ of Y_k^i by sets $Y_{k,j}^i \subset Y_k^i$. We collect all the covers of Y_k^i, for $k \in I_i$, and write the collection as

$$\{Y_k^{i+1}\}_{k \in I_{i+1}} = \bigcup_{k \in I_i} \{Y_{k,j}^i \,:\, j \in J_{k,i}\},$$

and similarly we collect all the corresponding points $\{x_k^{i+1}\}_{k \in I_{i+1}}$ for which (2.1) holds.

Now we define X_0 as the closure in X' of

$$D = \bigcup_{i=0}^{\infty} \{x_k^i \,:\, k \in I_i\}.$$

Since for each $i \in \mathbb{N}$ we have $\mathrm{card}(I_i) \leq \mathrm{dens}(Y')$, we have $\mathrm{card}(D) \leq \mathrm{dens}(Y')$. Thus $\mathrm{dens}(X_0) \leq \mathrm{dens}(Y')$.

We still need to verify that $f(X_0) = Y'$. For this, take $y \in Y'$. For each $i \in \mathbb{N}$ we select an index $k_i \in I_i$ recursively. Since $\{Y_k^0\}_{k \in I_0}$ covers Y', there exists $k_0 \in I_0$ such that $y \in Y_{k_0}^0$. Suppose now that $y \in Y_{k_i}^i$ for $i \geq 0$. Then by construction of the collection $\{Y_k^{i+1}\}_{k \in I_{i+1}}$, there exists an index $k_{i+1} \in I_{i+1}$ such that $y \in Y_{k_i}^i$ and $Y_{k_{i+1}}^{i+1} \subset Y_{k_i}^i$. Thus, by writing $y_i := f(x_{k_i}^i) \in Y_{k_i}^i$ we obtain a sequence $(y_i)_{i \in \mathbb{N}}$ such that $d(y_i, y) \leq 2^{-i}$. Moreover, by the very construction $x_{k_{i+1}}^{i+1} \in B(x_{k_i}^i, 2^{-i})$ for all $i \in \mathbb{N}$. Therefore, $(x_{k_i}^i)_{i \in \mathbb{N}}$ is a Cauchy sequence. Since X is complete, there exists $x \in X$ such that $x_{k_i}^i \to x$ as $i \to \infty$. By continuity of f we have $f(x) = y$ and thus $x \in X'$. □

As a final remark, we point out some questions concerning Theorem 1.1. First, it would be interesting to know whether the construction can be done in such a way that the restriction of f remains open. Furthermore, a natural question is whether Theorem 1.1 holds in more general settings. For example, in the case that $f : X \to Y$ is a continuous open surjection between uniform spaces, where X is complete. Also, in the case that $f : X \to Y$ is a continuous open surjection between topological spaces, where X is Čech-complete. We would like to thank the referees for suggesting these questions.

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