A CHARACTERIZATION OF ENERGETIC AND BV SOLUTIONS TO ONE-DIMENSIONAL RATE-INDEPENDENT SYSTEMS

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Abstract. The notion of BV solution to a rate-independent system was introduced in [7] to describe the vanishing viscosity limit (in the dissipation term) of doubly nonlinear evolution equations. Like energetic solutions [4] in the case of *convex* energies, BV solutions provide a careful description of rate-independent evolution driven by nonconvex energies, and in particular of the energetic behavior of the system at jumps.

In this paper we study both notions in the one-dimensional setting and we obtain a full characterization of BV and energetic solutions for a broad family of energy functionals. In the case of monotone loadings we provide a simple and explicit characterization of their solutions, which allows for a direct comparison of the two concepts.

1. Introduction

Over the last decade, the analysis of rate-independent systems has received notable attention (see e.g. [4] for a thorough survey of applications). The analytical theory of rate-independent evolutions encounters some mathematical challenges, which are apparent even in the simplest example of rate-independent evolution, viz. the *doubly nonlinear* differential inclusion

$$\partial\Psi(u'(t)) + \mathrm{D}\mathcal{E}_t(u(t)) \ni 0 \quad \text{in } X^* \quad \text{a.e. in } (0,T). \tag{1.1}$$

Here X^* is the dual of a finite-dimensional linear space (and in the sequel we focus our analysis on the simplest case $X = \mathbb{R}$), $\mathrm{D}\mathcal{E}_t$ is the (space) differential of a time-dependent energy functional $\mathcal{E} \in \mathrm{C}^1(X \times [0,T];\mathbb{R})$, $\Psi : X \to [0,+\infty)$ is a convex, nondegenerate, dissipation potential: rate-independence requires that Ψ is *positively homogeneous of degree* 1.

It follows from the above conditions that the range of $\partial\Psi$ equals $K^* := \partial\Psi(0)$, which is a proper convex subset of X. Hence, if $\mathcal{E}_t(\cdot)$ is not strictly convex, one cannot expect the existence of an absolutely continuous solution to (1.1). It turns out that the natural space for candidate solutions u of (1.1) is $\mathrm{BV}([0,T];X)$, and this fact has motivated the development of various weak formulations of (1.1), which should also take into account the behavior of u at jump points.

First and foremost, we recall the notion of (global) *energetic solution*, proposed by A. Mielke and coauthors, cf. [8, 9, 4]. For the simplified rate-independent evolution (1.1), an energetic solution is a curve $u \in \mathrm{BV}([0,T];X)$ satisfying two conditions for all $t \in [0,T]$: the *global stability*

$$\mathcal{E}_t(u(t)) \le \mathcal{E}_t(z) + \Psi(z - u(t)) \qquad \text{for every } z \in X, \tag{S}$$

and the *energy balance*

$$\mathcal{E}_t(u(t)) + \mathrm{Var}_\Psi(u;[0,t]) = \mathcal{E}_0(u(0)) + \int_0^t \partial_t\mathcal{E}_s(u(s))\,\mathrm{d}s\,, \tag{E}$$

where Var_Ψ is the pointwise total variation with respect to Ψ (see (2.4) for the definition). Let us emphasize that the energetic formulation neither involves the differential $\mathrm{D}\mathcal{E}$ of the energy, nor derivatives of the function $t \mapsto u(t)$; thus, it is well suited to deal with nonsmooth energies and jumping solutions. Furthermore, as shown in [10, 4], this formulation can be considered and

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analyzed in very general ambient spaces, even with no underlying linear structure. Because of these features, the energetic concept has been exploited in several applicative contexts, see [4, 5], and the references therein.

In the case of *nonconvex* energies, the global stability condition (S) may lead the system to change instantaneously in a very drastic way, jumping into very far-apart energetic configurations. A different dynamical approach has been proposed in [7] (see also [2, 6]), by considering rate-independent evolution as limit of systems with smaller and smaller viscosity. One can thus consider the viscous approximation of (1.1), viz.

$$\partial\Psi_\varepsilon(u'(t)) + \mathrm{D}\mathcal{E}_t(u(t)) \ni 0 \quad \text{in } X^* \quad \text{a.e. in } (0,T), \tag{1.2}$$

where in the simplest case we have

$$\Psi_\varepsilon(v) := \Psi(v) + \frac{\varepsilon}{2}\Psi^2(v). \tag{1.3}$$

In fact, we focus on (1.3) just for simplicity, since much more general regularizations can be considered, see [7]. The main result of [7] is that any limit point as $\varepsilon \downarrow 0$ of the family $(u_\varepsilon)_\varepsilon$ of solutions to (1.2) is a curve $u \in \mathrm{BV}([0,T];X)$ fulfilling the *local stability condition*

$$-\mathrm{D}\mathcal{E}_t(u(t)) \in K^* \quad \text{for a.e. } t \in (0,T), \tag{S$_{\mathrm{loc}}$}$$

and the *energy balance*

$$\mathit{Var}_{\Pi,\mathcal{E}}(u;[0,t]) + \mathcal{E}_t(u(t)) = \mathcal{E}_0(u(0)) + \int_0^t \partial_t\mathcal{E}_s(u(s))\,\mathrm{d}s \quad \text{for all } t \in [0,T]. \tag{E$_{\Pi,\mathcal{E}}$}$$

Notice that (E$_{\Pi,\mathcal{E}}$) features the (pseudo)-total variation $\mathit{Var}_{\Pi,\mathcal{E}}$, suitably defined from the *vanishing viscosity contact potential*

$$\Pi(v,w) := \Psi(v)\max\big(1,\Psi_*(w)\big) \quad \text{for } (v,w) \in X \times X^* \tag{1.4}$$

with $\Psi_*(w) := \inf_{\Psi(v)\le 1}\langle w, v\rangle$. We refer to Section 2 for all details on the definition of $\mathit{Var}_{\Pi,\mathcal{E}}$ in terms of Π and $\mathcal{E}$.

Still, let us emphasize that, in general, *both* the rate-independent *and* the viscous dissipation contribute to Π, and thus to the energy balance (E$_{\Pi,\mathcal{E}}$) via the (pseudo)-total variation $\mathit{Var}_{\Pi,\mathcal{E}}$. In contrast, in the energy balance (E) for *energetic solutions* only the rate-independent dissipation is involved. In fact, (E$_{\Pi,\mathcal{E}}$) reflects the main feature of BV solutions, viz. that rate-independent and viscous effects are encompassed in the description of the solution jump trajectories, in order to provide finer (in comparison to *energetic* solutions) information on the behavior of the system. This aspect was fully explored in [7], with a thorough description of the non-jumping and jumping regimes and the related energy balances, see also Proposition 2.4 later on.

Moreover, we point out that, contrary to (S), (S$_{\mathrm{loc}}$) is a *local stability* condition. Therefore, one expects BV solutions to jump "later" and "less abruptly" than *energetic* solutions.

The latter crucial property is clearly revealed by the *characterization* of energetic and BV solutions in the one-dimensional case $X = \mathbb{R}$, which we provide in the present paper. Namely, we consider energies $\mathcal{E} : \mathbb{R} \times [0,T] \to \mathbb{R}$ and dissipation potentials $\Psi : \mathbb{R} \to [0,+\infty)$ of the form

$$\mathcal{E}_t(u) := W(u) - \ell(t)u, \quad \Psi(u) = \delta_+ u^+ + \delta_- u^- \qquad (u,t) \in \mathbb{R} \times [0,T], \tag{1.5}$$

with $W \in \mathrm{C}^1(\mathbb{R})$ a (possibly nonconvex) energy, $\ell \in \mathrm{C}^1([a,b])$ a given external loading, and $\delta_\pm > 0$. Then, (1.1) reduces to the rate-independent ODE

$$\partial\Psi(u'(t)) + W'(u(t)) \ni \ell(t) \quad \text{a.e. in } (0,T). \tag{1.6}$$

Our main Theorems 3.1 and 5.1 characterize BV and energetic solutions to (1.6), and provide the starting point for the explicit representation formulas given by Theorems 4.3 and 6.3 when the loading function ℓ is *monotone*. In the case of a strictly increasing map ℓ and of an initial datum $\bar{u}$ satisfying a slightly stronger (local/global) stability condition, we have the following simple descriptions:

- u is a BV *solution to* (1.6) if and only if it is *nondecreasing* on $[0,T]$ and solves

$$u \text{ is nondecreasing on } [0,T], \quad W'(u(t)) = \ell(t) - \delta_+ \quad \text{for all } t \in [0,T] \setminus \mathrm{J}_u, \tag{1.7}$$

where J_u is the jump set of u. In terms of the *upper monotone envelope of* W', defined by $\boldsymbol{m}^{u_0}(u) := \max_{u_0 \le v \le u} W'(v)$ (see §4.1), (1.7) can also be written as

$$\boldsymbol{m}^{u_0}(u(t)) = \ell(t) - \delta_+, \quad u_0 := u(0), \quad \ell(0) - \delta_+ \ge W'(u_0).$$

- u is an energetic solution to (1.6) if and only if

$$u \text{ is nondecreasing on } [0,T], \quad \partial W(u(t); u_0) = \ell(t) - \delta_+ \quad \text{for all } t \in [0,T], \tag{1.8}$$

where $\partial W(\cdot; u_0)$ is the (convex analysis) subdifferential (see §6.1) of the function

$$W(u; u_0) := \begin{cases} W(u) & \text{if } u \ge u_0 \\ +\infty & \text{if } u < u_0. \end{cases} \tag{1.9}$$

Introducing the convex envelope $W^{**}(\cdot; u_0)$ of $W(\cdot; u_0)$ (whose definition is given in (6.7)) and its derivative $\mathfrak{m}^{u_0}(u) := \mathrm{D}W^{**}(u; u_0)$ for $u > u_0$, (1.8) also yields the following equation for the energetic solution u:

$$\mathfrak{m}^{u_0}(u(t)) = \ell(t) - \delta_+ \quad \text{for all } t \in [0,T]. \tag{1.10}$$

Therefore, BV and energetic solutions depend monotonically on increasing loadings. Furthermore, BV solutions are governed by the upper monotone envelope of W', whereas energetic solutions involve the derivative of the convexified energy (1.9). In both cases, the initial condition provides crucial information to construct the above monotone graphs. The case of a decreasing loading ℓ can be easily recovered from the increasing one thanks to the symmetry principle recalled in Proposition 2.5. The concatenation principle allows us to extend immediately our results to the case of piecewise monotone loadings.

In Examples 4.7 and 6.5, we illustrate (1.7) and (1.8) in the simple, yet significant case of the double-well potential

$$W(u) = \frac{1}{4}(u^2 - 1)^2 \qquad u \in \mathbb{R}.$$

In this context, the input-output relation $\ell \to u$ given by (1.10) for *energetic* solutions, corresponds to the so-called *Maxwell rule*, cf. the discussion in [12, Sec. I.3]. The latter evolution mode prescribes that for all $t \in [0,T]$, the function $u(t)$ only attains *absolute minima* of the function $u \mapsto W(u) - (\ell(t) - \delta_+)u$. This corresponds to convexification of W, and causes the system to jump "early" into far-apart configurations. Instead, the evolution mode (1.7) follows the *Delay rule*, related to *hysteresis behavior.* The system accepts also *relative minima* of $u \mapsto W(u) - (\ell(t) - \delta_+)u$, and thus the function $t \mapsto u(t)$ tends to jump "as late as possible".

The *plan* of the paper is as follows: in Section 2, we recall some definitions and preliminary properties of BV functions, energetic and BV solutions in a finite-dimensional setting. Section 3 is devoted to a refined characterization of BV solutions in the one-dimensional setting. The case of monotone loadings is carefully analyzed in Section 4, after some auxiliary results on upper and lower monotone envelopes of functions. One-dimensional energetic solutions are studied in Section 5; their explicit characterization when ℓ is monotone is carried out in the last Section 6.

2. Preliminaries

In this section we recall some notation and properties related to functions in $\mathrm{BV}([0,T]; X)$, with X a *finite-dimensional* vector space, and to *energetic and* BV *solutions* of a general rate-independent system.

2.1. BV **functions.** Hereafter, we shall consider functions of bounded variation $u \in \mathrm{BV}([a,b];X)$ to be *pointwise defined at every time* $t \in [a,b]$. Notice that a function $u \in \mathrm{BV}([a,b];X)$ admits left (resp. right) limits at every $t \in (a,b]$ (resp. $t \in [a,b)$), viz. $\exists\, u_{\mathsf{l}}(t) = \lim_{s\uparrow t} u(s)$ and $\exists\, u_{\mathsf{r}}(t) = \lim_{s\downarrow t} u(s)$. We also adopt the convention $u_{\mathsf{l}}(a) := u(a)$, $u_{\mathsf{r}}(b) := u(b)$. The *pointwise* jump set J_u of u is the at most countable set defined by

$$\mathrm{J}_u := \big\{t \in [a,b] : u_{\mathsf{l}}(t) \neq u(t) \text{ or } u(t) \neq u_{\mathsf{r}}(t)\big\} \supset \text{ess-}\mathrm{J}_u := \big\{t \in (a,b) : u_{\mathsf{l}}(t) \neq u_{\mathsf{r}}(t)\big\}. \tag{2.1}$$

We denote by u' the distributional derivative of u (extended by $u(a)$ in $(-\infty,a)$ and by $u(b)$ in $(b,+\infty)$) in $\mathcal{D}'(\mathbb{R})$: it is a Radon vector measure with finite total variation $|u'|$ supported in $[a,b]$. In the one-dimensional case $X = \mathbb{R}$, u' admits the Hahn decomposition $u' = (u')^+ - (u')^-$ as the difference of two positive and mutually singular measures, such that $|u'| = (u')^+ + (u')^-$.

It is well known [1] that u' can be decomposed into the sum of its diffuse part u'_{co} and the jump part u'_{J}:

$$u' = u'_{\mathrm{co}} + u'_{\mathrm{J}}, \quad u'_{\mathrm{J}} := u' \llcorner \text{ess-}\mathrm{J}_u, \quad \text{so that } u'_{\mathrm{co}}(\{t\}) = 0 \quad \text{for every } t \in [a,b]. \tag{2.2}$$

2.2. **Energetic solutions to rate-independent systems.** We consider a general rate-independent system $(X, \mathcal{E}, \Psi)$, where the dissipation potential

$$\Psi : X \to [0,+\infty) \text{ is 1-positively homogeneous, convex, with } \Psi(v) > 0 \text{ if } v \neq 0,$$

and

$$\mathcal{E} \in \mathrm{C}^1(X \times [a,b]), \quad \mathcal{E}_t(u) := W(u) - \langle \ell(t), u\rangle,$$

for some $W \in \mathrm{C}^1(X)$ and $\ell \in \mathrm{C}^1([a,b];X^*)$. We shall also use the notation $\partial_t \mathcal{E}_t(u) := -\langle \ell'(t), u\rangle$ for the partial time derivative, and we set

$$K^* := \partial\Psi(0) = \{w \in X^* : \Psi_*(w) \leq 1\} \subset X^*, \quad \text{where} \quad \Psi_*(w) := \sup_{\Psi(v)\leq 1} \langle w, v\rangle \quad w \in X^* \tag{2.3}$$

We recall the notion of *energetic solution* to the rate-independent system $(X, \mathcal{E}, \Psi)$, cf. [8, 9, 4].

Definition 2.1 (Energetic solution). *A curve $u \in \mathrm{BV}([a,b];X)$ is an energetic solution of the* rate-independent system $(X, \mathcal{E}, \Psi)$ *if for all $t \in [a,b]$ it satisfies the* global stability

$$\forall\, z \in X: \qquad \mathcal{E}_t(u(t)) \leq \mathcal{E}_t(z) + \Psi(z - u(t)) \tag{S}$$

and the energy balance

$$\mathcal{E}_t(u(t)) + \mathrm{Var}_\Psi(u;[a,t]) = \mathcal{E}_a(u(a)) + \int_a^t \partial_t \mathcal{E}_s(u(s))\,\mathrm{d}s, \tag{E}$$

where

$$\mathrm{Var}_\Psi(u;[a,t]) = \sup\Big\{ \sum_{m=1}^M \Psi\big(u(t_m) - u(t_{m-1})\big) : a = t_0 < t_1 < \cdots < t_{M-1} < t_M = t\Big\} \tag{2.4}$$

denotes the pointwise Ψ-total variation of u on the interval $[a,t]$.

The following characterization of energetic solutions has been proved in [7, Prop. 2.2].

Proposition 2.2 (Differential characterization of energetic solutions). *A curve $u \in \mathrm{BV}([a,b];X)$ is an energetic solution of the rate-independent system $(X, \mathcal{E}, \Psi)$ if and only if it satisfies the* global stability *condition* (S)*, the* doubly nonlinear differential inclusion in the BV sense

$$\partial\Psi\Big(\frac{\mathrm{d}u'_{\mathrm{co}}}{\mathrm{d}\mu}(t)\Big) + \mathrm{D}W(u(t)) \ni \ell(t) \quad \textit{for } \mu\textit{-a.e. } t \in (a,b), \quad \mu := \mathscr{L}^1 + |u'_{\mathrm{co}}|, \tag{DN}$$

and the following jump conditions *at each point $t \in \mathrm{J}_u$:*

$$\begin{aligned} \mathcal{E}_t(u_{\mathsf{r}}(t)) - \mathcal{E}_t(u_{\mathsf{l}}(t)) &= -\Psi(u_{\mathsf{r}}(t) - u_{\mathsf{l}}(t)), \\ \mathcal{E}_t(u(t)) - \mathcal{E}_t(u_{\mathsf{l}}(t)) &= -\Psi(u(t) - u_{\mathsf{l}}(t)), \\ \mathcal{E}_t(u_{\mathsf{r}}(t)) - \mathcal{E}_t(u(t)) &= -\Psi(u_{\mathsf{r}}(t) - u(t)). \end{aligned} \tag{$\mathrm{J_{ener}}$}$$

The jump conditions ($\mathrm{J}_{\mathrm{ener}}$) show that, in the case of an energetic solution u the jump set J_u coincides with the essential jump set ess-J_u.

2.3. BV **solutions to rate-independent systems.** As we mentioned in the introduction, we shall restrict to BV solutions arising in the vanishing viscosity limit of (1.2), with dissipation potentials Ψ_ε of the form (1.3). In such a setting, the *vanishing viscosity contact potential* $\Pi : X \times X^* \to [0, +\infty)$ associated to the family $(\Psi_\varepsilon)_\varepsilon$ is given by

$$\Pi(v, w) := \Psi(v) \max(1, \Psi_*(w)) = \begin{cases} \Psi(v) & \text{if } w \in K^*, \\ \Psi(v)\Psi_*(w) & \text{if } w \notin K^*. \end{cases} \tag{2.5}$$

For a fixed $t \in [a, b]$, the (possibly asymmetric) Finsler cost induced by Π and (the differential of) $\mathcal{E}$ at the time t is for every $\bar{u}, u_1 \in X$ given by

$$\begin{aligned} \Delta_{\Pi,\mathcal{E}}(t; \bar{u}, u_1) := \inf \Big\{ \int_{r_0}^{r_1} \Pi(\dot{\vartheta}(r), -\mathrm{D}\mathcal{E}_t(\vartheta(r)))\,\mathrm{d}r : \\ \vartheta \in \mathrm{AC}([r_0, r_1]; X),\ \vartheta(r_0) = \bar{u},\ \vartheta(r_1) = u_1 \Big\}. \end{aligned} \tag{2.6}$$

As already observed in [7], it is not difficult to check that the infimum in (3.5) is always attained by a *Lipschitz* curve $\vartheta \in \mathrm{Lip}([r_0, r_1]; X)$ such that $\Pi(\dot{\vartheta}(r), -\mathrm{D}\mathcal{E}_t(\vartheta(r))) \equiv 1$ for a.e. $r \in (r_0, r_1)$.

For every $u \in \mathrm{BV}([a, b]; X)$ and every subinterval $[\alpha, \beta] \subset [a, b]$, the *jump variation* of u induced by $(\Pi, \mathcal{E})$ on $[\alpha, \beta]$ is

$$\begin{aligned} \mathrm{Jmp}_{\Pi,\mathcal{E}}(u; [\alpha, \beta]) := {}& \Delta_{\Pi,\mathcal{E}}(\alpha; u(\alpha), u_{\mathsf{r}}(\alpha)) + \Delta_{\Pi,\mathcal{E}}(\beta; u_{\mathsf{l}}(\beta), u(\beta)) \\ & + \sum_{t \in \mathrm{J}_u \cap (\alpha,\beta)} \big(\Delta_{\Pi,\mathcal{E}}(t; u_{\mathsf{l}}(t), u(t)) + \Delta_{\Pi,\mathcal{E}}(t; u(t), u_{\mathsf{r}}(t))\big), \end{aligned} \tag{2.7}$$

and the associated (pseudo-)total variation is

$$\mathrm{Var}_{\Pi,\mathcal{E}}(u; [\alpha, \beta]) := \int_\alpha^\beta \Psi\left(\frac{\mathrm{d}u'_{\mathrm{co}}}{\mathrm{d}\mu}\right) \mathrm{d}\mu + \mathrm{Jmp}_{\Pi,\mathcal{E}}(u; [\alpha, \beta]), \tag{2.8}$$

where μ could be *any* nonnegative and diffuse reference measure, provided that u'_{co} is absolutely continuous w.r.t. μ (e.g. $\mu = \mathscr{L}^1 + |u'_{\mathrm{co}}|$). In fact, since Ψ is 1-homogeneous, the result would be independent of μ (see e.g. [1]).

Then, we are in the position to recall the definition of BV solution given in [7].

Definition 2.3. *A curve* $u \in \mathrm{BV}([a, b]; X)$ *is a* BV *solution of the rate-independent system* $(\mathbb{R}, \mathcal{E}, \Pi)$ *if it satisfies the* local stability *condition*

$$\ell(t) - \mathrm{D}W(u(t)) \in K^* \quad \textit{for a.e. } t \in [a, b] \setminus \mathrm{J}_u, \tag{$\mathrm{S}_{\mathrm{loc}}$}$$

and the $(\Pi, \mathcal{E})$-energy balance

$$\mathrm{Var}_{\Pi,\mathcal{E}}(u; [a, t]) + \mathcal{E}_t(u(t)) = \mathcal{E}_0(u(0)) + \int_a^t \partial_t \mathcal{E}_s(u(s))\,\mathrm{d}s \quad \textit{for all } t \in [a, b]. \tag{$\mathrm{E}_{\Pi,\mathcal{E}}$}$$

In [7, Sec. 4] the following result has been proved.

Proposition 2.4 (Differential characterization of BV solutions)**.** *A curve* $u \in \mathrm{BV}([a, b]; X)$ *is a* BV *solution of the rate-independent system* $(\mathbb{R}, \mathcal{E}, \Pi)$ *if and only if it satisfies the doubly nonlinear differential inclusion* (DN)*, and it fulfills at each point* $t \in \mathrm{J}_u$ *the jump conditions:*

$$\begin{aligned} \mathcal{E}_t(u_{\mathsf{r}}(t)) - \mathcal{E}_t(u_{\mathsf{l}}(t)) &= -\Delta_{\Pi,\mathcal{E}}(t; u_{\mathsf{l}}(t), u_{\mathsf{r}}(t)), \\ \mathcal{E}_t(u(t)) - \mathcal{E}_t(u_{\mathsf{l}}(t)) &= -\Delta_{\Pi,\mathcal{E}}(t; u_{\mathsf{l}}(t), u(t)), \\ \mathcal{E}_t(u_{\mathsf{r}}(t)) - \mathcal{E}_t(u(t)) &= -\Delta_{\Pi,\mathcal{E}}(t; u(t), u_{\mathsf{r}}(t)). \end{aligned} \tag{J_{BV}}$$

As in the case of energetic solutions, (J_{BV}) yield that, for a BV solution u the jump set J_u coincides with the essential jump set ess-J_u.

2.4. **Symmetry and concatenation principles for energetic and** BV **solutions.** We state here two useful properties of energetic and BV solutions. Let us first introduce the modified energy, dissipation, and potentials

$$\tilde{\mathcal{E}}_t(u) := \mathcal{E}_t(-u), \quad \tilde{\Psi}(u) := \Psi(-u), \quad \tilde{\Pi}(u,w) := \Pi(-u,-w). \tag{2.9}$$

Proposition 2.5 (Symmetry principle)**.** *A curve $u \in \mathrm{BV}([a,b];\mathbb{R})$ is an energetic solution of the rate-independent system $(X,\mathcal{E},\Psi)$ (resp. a* BV *solution of the rate-independent system $(X,\mathcal{E},\Pi)$) if and only if the curve $\tilde{u}(t) := -u(t)$ is an energetic solution of the rate-independent system $(X,\tilde{\mathcal{E}},\tilde{\Psi})$ (resp. of the rate-independent system $(X,\tilde{\mathcal{E}},\tilde{\Pi})$).*

The *proof* follows from easy calculations, observing that

$$\tilde{\mathcal{E}}_t(\tilde{u}(t)) = \mathcal{E}_t(u(t)), \quad \tilde{u}'(t) = -u'(t), \quad \tilde{\Psi}(\tilde{u}'(t)) = \Psi(u'(t)), \quad \tilde{\Psi}_*(w) = \Psi_*(-w), \quad \tilde{K}^* = -K^*$$

Another simple property concerns the behavior of energetic and BV solutions with respect to restriction and concatenation. The proof is trivial.

Proposition 2.6 (Restriction and concatenation principle)**.**

(1) The restriction of an energetic (resp. BV*) solution in $[a,b]$ to an interval $[\alpha,\beta] \subset [a,b]$ is an energetic (resp.* BV*) solution in $[\alpha,\beta]$.*

(2) If $a = t_0 < t_1 < \cdots < t_{M-1} < T_m = b$ is a subdivision of $[a,b]$ and $u : [a,b] \to X$ is an energetic (resp. BV*) solution in each one of the intervals $[t_{j-1},t_j]$, $j = 1,\cdots,M$, then u is an energetic (resp.* BV*) solution in $[a,b]$.*

2.5. **The one-dimensional setting.** From now on we consider the particular case $X = \mathbb{R}$, which we also identify with X^*. We will denote by v^+, v^- the positive and negative part of $v \in \mathbb{R}$.

Dissipation. A dissipation potential is a function of the form

$$\Psi(v) := \delta_+ v^+ + \delta_- v^- \quad v \in \mathbb{R}, \qquad \text{for some } \delta_\pm > 0. \tag{2.10}$$

Hence, we have

$$\partial\Psi(v) = \begin{cases} \delta_+ & \text{if } v > 0, \\ [-\delta_-,\delta_+] & \text{if } v = 0, \\ -\delta_- & \text{if } v < 0 \end{cases} \qquad \text{for all } v \in \mathbb{R},$$

$$K^* = [-\delta_-,\delta_+], \qquad \Psi_*(w) = \frac{1}{\delta_+}w^+ + \frac{1}{\delta_-}w^- \quad \text{for all } w \in \mathbb{R}.$$

Energy functional. The energy is given by a function $\mathcal{E} : \mathbb{R} \times [a,b] \to \mathbb{R}$ of the form

$$\mathcal{E}_t(u) := W(u) - \ell(t)u, \tag{2.11}$$

with $\ell \in \mathrm{C}^1([a,b])$ and $W : \mathbb{R} \to \mathbb{R}$ such that

$$W \in \mathrm{C}^1(\mathbb{R}), \quad \lim_{x\to-\infty} W'(x) = -\infty, \quad \lim_{x\to+\infty} W'(x) = +\infty. \tag{2.12}$$

3. BV solutions of rate-independent systems in $\mathbb{R}$

In this section we will provide an equivalent characterization of BV solutions to the rate-independent system $(\mathbb{R},\mathcal{E},\Pi)$, in the one-dimensional setting considered in §2.5.

Theorem 3.1. *A function $u \in \mathrm{BV}([a,b];\mathbb{R})$ is a* BV *solution of the rate-independent system $(\mathbb{R},\mathcal{E},\Pi)$ of §2.5 if and only if the following properties hold:*

a) u satisfies the local stability condition (equivalent to $(\mathrm{S_{loc}})$)

$$-\delta_- \le \ell(t) - W'(u(t)) \le \delta_+ \quad \textit{for every } t \in [a,b] \setminus \mathrm{J}_u. \tag{$\mathrm{S_{loc,\mathbb{R}}}$}$$

b) The function $W'(u_{\mathsf{l}})$ is continuous in $[a,b)$, the function $W'(u_{\mathsf{r}})$ is continuous in $(a,b]$ and they coincide in (a,b): we denote their common value by $W'(u_{\mathsf{lr}})$.

c) u satisfies the following precise formulation of the doubly nonlinear differential inclusion (DN)

$$W'(u_{\mathsf{lr}}(t)) = \ell(t) - \delta_+ \quad \textit{for every } t \in \operatorname{supp}\big((u')^+\big) \cap (a,b), \tag{3.1}$$

$$W'(u_{\mathsf{lr}}(t)) = \ell(t) + \delta_- \quad \textit{for every } t \in \operatorname{supp}\big((u')^-\big) \cap (a,b), \tag{3.2}$$

with obvious modifications at $t = a$, $t = b$ as in the previous point b).

d) At each jump point $t \in \mathrm{J}_u$, u satisfies the jump conditions

$$\min\big(u_{\mathsf{l}}(t), u_{\mathsf{r}}(t)\big) \le u(t) \le \max\big(u_{\mathsf{r}}(t), u_{\mathsf{l}}(t)\big), \tag{3.3}$$

and

$$\ell(t) - W'(\vartheta) \ge \delta_+ \quad \textit{if } u_{\mathsf{l}}(t) < u_{\mathsf{r}}(t), \qquad \ell(t) - W'(\vartheta) \le -\delta_- \quad \textit{if } u_{\mathsf{l}}(t) > u_{\mathsf{r}}(t), \tag{3.4}$$

for every ϑ such that $\min\big(u_{\mathsf{l}}(t), u_{\mathsf{r}}(t)\big) \le \vartheta \le \max\big(u_{\mathsf{r}}(t), u_{\mathsf{l}}(t)\big)$. *In particular* $\mathrm{J}_u = \text{ess-}\mathrm{J}_u$.

Proof. We split the argument in various steps.

Claim 1: *The local stability condition* ($\mathrm{S_{loc}}$) *is equivalent to* ($\mathrm{S_{loc,\mathbb{R}}}$).

It is sufficient to recall that $K^* = [-\delta_-, \delta_+]$.

Claim 2: *the jump conditions* ($\mathrm{J_{BV}}$) *are equivalent to* (3.3) *and* (3.4).

Taking (2.5) and (2.11) into account, the Finsler cost $\Delta_{\Pi,\mathcal{E}}(t;\cdot,\cdot)$ in fact reduces (up to a linear reparametrization) to

$$\begin{aligned}\Delta_{\Pi,\mathcal{E}}(t;\bar u,u_1) = \min\Big\{ \int_0^1 \max\Big(1, \Psi_*\big(\ell(t) - W'(\vartheta(r))\big)\Big) \Psi(\vartheta'(r))\,\mathrm{d}r :\\ \vartheta \in \mathrm{AC}([0,1];\mathbb{R}),\ \vartheta(0) = \bar u,\ \vartheta(1) = u_1\Big\},\end{aligned} \tag{3.5}$$

Let us consider, e.g., the case $\bar u \le u_1$ and notice that, if $\vartheta \in \mathrm{AC}([0,1];\mathbb{R})$ fulfils $\vartheta(0) = \bar u$ and $\vartheta(1) = u_1$, then, setting

$$r_0 := \sup\{r \in [0,1] \,:\, \vartheta(t) \le \bar u\}, \quad r_1 := \inf\{r \in [r_0,1] \,:\, \vartheta(r) \ge u_1\},$$

there holds

$$\vartheta(r_0) = \bar u, \qquad \vartheta(r_1) = u_1, \qquad \vartheta(r) \in (\bar u, u_1) \text{ for all } r \in (r_0, r_1).$$

Therefore, the value of the integral in (3.5) surely diminishes if we just consider the restriction of ϑ to the interval $[r_0, r_1]$, so that we can assume that the range of a minimizing curve in (3.5) is contained in $[\bar u, u_1]$.

We can also suppose that the competing curves ϑ in (3.5) are nondecreasing. In fact, if ϑ is absolutely continuous and connects $\bar u$ to u_1, we can consider the curve $\tilde\vartheta(r) := \max_{s\in[0,r]} \vartheta(s)$. It is easy to check that ϑ is nondecreasing and absolutely continuous, since for all $0 \le r_1 \le r_2 \le 1$

$$\tilde\vartheta(r_2) - \tilde\vartheta(r_1) \le \sup_{s_1,s_2\in[r_1,r_2]} |\vartheta(s_2) - \vartheta(s_1)| \le \int_{r_1}^{r_2} |\vartheta'(s)|\,\mathrm{d}s.$$

It follows that $\tilde\vartheta' = \vartheta'$ a.e. on the coincidence set $\{\tilde\vartheta = \vartheta\}$, whereas one can easily check that $\tilde\vartheta'(r) = 0$ where $\tilde\vartheta(r) > \vartheta(r)$ (viz., where $\tilde\vartheta(r) \ne \vartheta(r)$). From the above considerations we obtain

$$\int_0^1 \max\big(1, \Psi_*(\ell(t) - W'(\tilde\vartheta(r)))\big)\Psi(\tilde\vartheta'(r))\,\mathrm{d}r \le \int_0^1 \max\big(1, \Psi_*(\ell(t) - W'(\vartheta(r)))\big)\Psi(\vartheta'(r))\,\mathrm{d}r.$$

Therefore, it is not restrictive to assume that the curve ϑ is nondecreasing on $[0,1]$. Then, with a change of variable, from (3.5) we deduce the identity

$$\Delta_{\Pi,\mathcal{E}}(t;\bar u,u_1) = \delta_+ \int_{\bar u}^{u_1} \max\big(1, \Psi_*(\ell(t) - W'(r))\big)\,\mathrm{d}r. \tag{3.6}$$

Notice that

$$\delta_+ \max\big(1, \Psi_*(\ell(t) - W'(r))\big) \ge \max\big(\delta_+, \ell(t) - W'(r)\big) \ge \ell(t) - W'(r), \tag{3.7}$$

and

$$\delta_+ \max\big(1, \Psi_*(\ell(t)-W'(r))\big) = \ell(t)-W'(r) \quad \Leftrightarrow \quad \ell(t)-W'(r) \geq \delta_+; \tag{3.8}$$

moreover, at a jump point t with $u_{\mathsf{l}}(t) < u_{\mathsf{r}}(t)$ there holds

$$\mathcal{E}_t(u_{\mathsf{r}}(t)) - \mathcal{E}_t(u_{\mathsf{l}}(t)) = -\int_{u_{\mathsf{l}}(t)}^{u_{\mathsf{r}}(t)} \big(\ell(t)-W'(r)\big)\,\mathrm{d}r \tag{3.9}$$

Comparing ($\mathrm{J_{BV}}$) with (3.6) and (3.9) we immediately see that the first condition of ($\mathrm{J_{BV}}$) is equivalent to the first of (3.4). If (3.3) also holds, then we easily get the other two conditions of ($\mathrm{J_{BV}}$). Conversely, ($\mathrm{J_{BV}}$) yields

$$\Delta_{\Pi,\mathcal{E}}(t; u_{\mathsf{l}}(t), u_{\mathsf{r}}(t)) = \Delta_{\Pi,\mathcal{E}}(t; u_{\mathsf{l}}(t), u(t)) + \Delta_{\Pi,\mathcal{E}}(t; u(t), u_{\mathsf{r}}(t)),$$

and this identity implies (3.3) by the positivity of the integrand in (3.6). The case $u_{\mathsf{l}}(t) > u_{\mathsf{r}}(t)$ can be studied by a similar argument, see also Proposition 2.5.

Claim 3, sufficiency: *a function u satisfying a) – d) is a* BV *solution.*

In view of Prop. 2.4 and Claim 2, we simply have to check that the differential inclusion (DN) holds: it follows from (3.1), (3.2), and ($\mathrm{S_{loc,\mathbb{R}}}$) since μ is diffuse and therefore $u_{\mathsf{r}}(t) = u_{\mathsf{l}}(t) = u(t)$ for μ-a.e. $t \in [a,b]$.

Claim 4: *a) and d) imply b).*

It is immediate, since the continuity of W' and ℓ yields that the inequalities in ($\mathrm{S_{loc,\mathbb{R}}}$) hold also for u_{l} and u_{r}; (3.4) provides the opposite inequalities.

Claim 5, necessity: *a* BV *solution u satisfies a) – d).*

By the previous claims, it remains to check *c)*. The identity in (3.1) is satisfied $(u')^+$-a.e. in $[a,b]$ thanks to the differential inclusion (DN) and the jump/stability conditions (which yield (3.1) on the jump set). By Claim 4, we know that $W'(u_{\mathsf{lr}})$ is continuous, so that the identity in (3.1) holds on the support of $(u')^+$. The same argument applies to (3.2). □

The previous general result has a simple consequence: a BV solution is locally constant in a neighborhood of a point where the stability condition ($\mathrm{S_{loc,\mathbb{R}}}$) holds with a strict inequality.

Lemma 3.2. *Let $u \in \mathrm{BV}([a,b];\mathbb{R})$ be a* BV *solution of the rate-independent system $(\mathbb{R},\mathcal{E},\Pi)$ of §2.5 and let us suppose that $-\delta_- < \ell(s) - W'(u(s)) < \delta_+$ at some $s \in [a,b]$. Then, setting*

$$\alpha := \max\big\{t \in [a,s] : \Psi_*(\ell(t)-W'(u(s))) = 1\big\}, \quad \beta := \min\big\{t \in [s,b] : \Psi_*(\ell(t)-W'(u(s))) = 1\big\}$$

we have $\alpha < s < \beta$ and $u(t) \equiv u(s)$ for every $t \in [\alpha,\beta]$.

Proof. In view of (3.3) and (3.4), s is not a jump point of u, hence u is continuous at s and the set $\{t \in [a,b] : \Psi_*(\ell(t)-W'(u(t))) < 1\}$ contains a (relatively) open neighborhood I of s; since $u = u_{\mathsf{r}} = u_{\mathsf{l}}$ in I, point *c)* of Theorem 3.1 shows that $u(t) \equiv u(s)$ in I. Consider now the set $J := \big\{r \in (\alpha,\beta) : u(r) = u(s)\big\}$: we have seen that J is open, and it is easy to check that it is also closed in (α,β), so that $J = (\alpha,\beta)$ and the thesis follows. □

4. BV solutions of the rate-independent system $(\mathbb{R},\mathcal{E},\Pi)$ with monotone loadings

As mentioned in the introduction, BV solutions of rate-independent systems in $\mathbb{R}$, driven by monotone loadings, involve the notion of the *upper and lower monotone* (i.e. nondecreasing) *envelopes* of the graph of a given function (W' in our setting). In this section we first focus on a few properties of these maps and their inverses, and then we exhibit the explicit formulas characterizing BV solutions when ℓ is increasing or decreasing.

4.1. The upper monotone envelope of W'.

Definition 4.1 (Upper monotone envelope)**.** *For every $\bar u \in \mathbb{R}$, we set $\bar\ell := W'(\bar u)$, and we define the maximal monotone map $\boldsymbol{m}^{\bar u}(\cdot) : \mathbb{R} \rightrightarrows \mathbb{R}$*

$$\boldsymbol{m}^{\bar u}(u) := \max_{\bar u \le v \le u} W'(v) \quad \textit{if } u > \bar u, \qquad \boldsymbol{m}^{\bar u}(\bar u) := (-\infty, W'(\bar u)], \qquad \boldsymbol{m}^{\bar u}(u) = \emptyset \quad \textit{if } u < \bar u. \tag{4.1}$$

We call $\boldsymbol{m}^{\bar u}(\cdot)$ the upper monotone envelope *of W' in the interval $(\bar u, +\infty)$. The* contact set *is defined by*

$$C^{\bar u} := \{\bar u\} \cup \big\{u > \bar u : W'(u) = \boldsymbol{m}^{\bar u}(u)\big\}. \tag{4.2}$$

The mapping $\boldsymbol{m}^{\bar u}(\cdot)$ is monotone and surjective thanks to (2.12); it is single-valued on $(\bar u, +\infty)$ (where we identify the set $\boldsymbol{m}^{\bar u}(u)$ with its unique element with a slight abuse of notation). We can thus consider the inverse graph $\boldsymbol{p}^{\bar u}(\cdot) : \mathbb{R} \rightrightarrows \mathbb{R}$ of $\boldsymbol{m}^{\bar u}(\cdot)$: it is defined by

$$u \in \boldsymbol{p}^{\bar u}(\ell) \quad \Leftrightarrow \quad \ell \in \boldsymbol{m}^{\bar u}(u) \quad \text{for } u,\ \ell \in \mathbb{R}. \tag{4.3}$$

Clearly, $\boldsymbol{p}^{\bar u}(\cdot)$ is a maximal monotone graph in $\mathbb{R}$, and it is uniquely characterized by a left-continuous monotone function $p_{\mathsf{l}}^{\bar u}(\cdot)$ and a right-continuous monotone function $p_{\mathsf{r}}^{\bar u}(\cdot)$ such that

$$\boldsymbol{p}^{\bar u}(\ell) = [p_{\mathsf{l}}^{\bar u}(\ell), p_{\mathsf{r}}^{\bar u}(\ell)], \quad \text{i.e.} \quad \boldsymbol{m}^{\bar u}(u) \ni \ell \quad \Leftrightarrow \quad p_{\mathsf{l}}^{\bar u}(\ell) \le u \le p_{\mathsf{r}}^{\bar u}(\ell). \tag{4.4}$$

We also consider a further selection in the graph of $\boldsymbol{p}^{\bar u}(\cdot)$:

$$\boldsymbol{p}_{\mathsf{c}}^{\bar u}(\ell) := \big\{u \in \boldsymbol{p}^{\bar u}(\ell) : W'(u) = \ell\big\} = \big\{u \in C^{\bar u} : \boldsymbol{m}^{\bar u}(u) \ni \ell\big\} = \boldsymbol{p}^{\bar u}(\ell) \cap C^{\bar u}. \tag{4.5}$$

By introducing the set

$$A^{\bar u} := \big\{f : (\bar u, +\infty) \to \mathbb{R} \ :\ f \text{ is nondecreasing and fulfils } f \ge W'\big\}, \tag{4.6}$$

we have

$$\boldsymbol{m}^{\bar u}(\cdot)|_{(\bar u, +\infty)} \in A^{\bar u} \quad \text{and} \quad W'(u) \le \boldsymbol{m}^{\bar u}(u) \le f(u) \ \text{ for all } f \in A^{\bar u} \text{ and } u \in (\bar u, +\infty), \tag{4.7}$$

so that $\boldsymbol{m}^{\bar u}(\cdot)$ is the minimal nondecreasing map above the graph of W' in $(\bar u, +\infty)$. It immediately follows from (4.7) that

$$\boldsymbol{m}^{\bar u}(u) = \inf\{f(u) \ :\ f \in A^{\bar u}\} \quad \text{for all } u > \bar u. \tag{4.8}$$

The following result collects some simple properties of $p_{\mathsf{l}}^{\bar u}(\cdot)$ and $p_{\mathsf{r}}^{\bar u}(\cdot)$.

Lemma 4.2. *Assume* (2.12)*. Then*

(1) the functions $p_{\mathsf{l}}^{\bar u}(\cdot)$ and $p_{\mathsf{r}}^{\bar u}(\cdot)$ are nondecreasing in $\mathbb{R}$ and we have

$$\bar\ell = W'(\bar u) \le \ell_1 < \ell_2 \ \Rightarrow\ p_{\mathsf{r}}^{\bar u}(\ell_1) < p_{\mathsf{l}}^{\bar u}(\ell_2). \tag{4.9}$$

(2) For every $\ell \in \mathbb{R}$ there holds:

$$\lim_{\lambda \uparrow \ell} p_{\mathsf{l}}^{\bar u}(\lambda) = p_{\mathsf{l}}^{\bar u}(\ell) \qquad \textit{and} \qquad \lim_{\lambda \downarrow \ell} p_{\mathsf{r}}^{\bar u}(\lambda) = p_{\mathsf{r}}^{\bar u}(\ell). \tag{4.10}$$

(3) For every $\ell \ge \bar\ell = W'(\bar u)$ there holds

$$W'(u) \le \ell \quad \textit{if } u \in [\bar u, p_{\mathsf{r}}^{\bar u}(\ell)]. \tag{4.11}$$

(4) For every $\ell \ge \bar\ell = W'(\bar u)$ there holds

$$W'(p_{\mathsf{l}}^{\bar u}(\ell)) = W'(p_{\mathsf{r}}^{\bar u}(\ell)) = W'(\boldsymbol{p}_{\mathsf{c}}^{\bar u}(\ell)) = \ell; \tag{4.12}$$

(5) For every $\ell \in \mathbb{R}$ there holds

$$p_{\mathsf{l}}^{\bar u}(\ell) = \min\big\{u \ge \bar u \ :\ W'(u) \ge \ell\big\}, \qquad p_{\mathsf{r}}^{\bar u}(\ell) = \inf\big\{u \ge \bar u \ :\ W'(u) > \ell\big\}. \tag{4.13}$$

Proof. First of all, (4.9) and (4.10) are general properties of the inverse of a maximal monotone map.

As for (4.11), it is an immediate consequence of the inequality $W' \le \boldsymbol{m}^{\bar u}(\cdot)$ in $[\bar u, \infty)$.

Let us check (4.12) and (4.13). The identity $W'(p_{\mathsf{l}}^{\bar u}(\ell)) = \ell$ and the first property of (4.13) ensue from (4.1): it is sufficient to notice that $W'(u) \le \boldsymbol{m}^{\bar u}(u) < \ell$ if $\bar u \le u < p_{\mathsf{l}}^{\bar u}(\ell)$ and $\boldsymbol{m}^{\bar u}(u) = \ell$ if $u = p_{\mathsf{l}}^{\bar u}(\ell)$.

The identity $W'(p_{\mathsf{r}}^{\bar u}(\ell)) = \ell$ is due to the continuity of W' and the second property of (4.13). To prove it, we observe that when $u > p_{\mathsf{r}}^{\bar u}(\ell)$ we have $\boldsymbol{m}^{\bar u}(u) > \ell$ and we know that there exists $v \in [p_{\mathsf{r}}^{\bar u}(\ell), u]$ such that $W'(v) > \ell$: since u is arbitrary we get $p_{\mathsf{r}}^{\bar u}(\ell) \ge \inf\{u \ge \bar u : W'(u) > \ell\}$. The converse inequality follows from (4.11). □

4.2. **The lower monotone envelope of W'.** In a completely similar way we can introduce the *maximal monotone map* below the graph of W' on the interval $(-\infty, \bar u]$, viz.

$$\boldsymbol{n}^{\bar u}(u) := \inf_{u \le v \le \bar u} W'(v) \quad \text{if } u < \bar u, \qquad \boldsymbol{n}^{\bar u}(\bar u) := [W'(\bar u), +\infty), \qquad \boldsymbol{n}^{\bar u}(u) = \emptyset \quad \text{if } u > \bar u, \tag{4.14}$$

which satisfies

$$\boldsymbol{n}^{\bar u}(u) = \sup\big\{f(u) \ : \ f \in B^{\bar u}\big\} \quad \text{for } u < \bar u, \tag{4.15}$$

where

$$B^{\bar u} := \big\{f : (-\infty, \bar u) \to \mathbb{R} \ : \ f \text{ is nondecreasing and fulfils } f \le W'\big\}.$$

As before, the inverse graph $\boldsymbol{q}^{\bar u}(\cdot) := (\boldsymbol{n}^{\bar u}(\cdot))^{-1} : \mathbb{R} \rightrightarrows (-\infty, \bar u]$ can be represented as $\boldsymbol{q}^{\bar u}(u) = [q_{\mathsf{l}}^{\bar u}(u), q_{\mathsf{r}}^{\bar u}(u)]$, where

$$q_{\mathsf{l}}^{\bar u}(\ell) = \sup\big\{u \le \bar u \ : \ W'(u) < \ell\big\}, \qquad q_{\mathsf{r}}^{\bar u}(\ell) = \max\big\{u \le \bar u \ : \ W'(u) \le \ell\big\}, \tag{4.16}$$

and we set

$$\boldsymbol{q}_{\mathsf{c}}^{\bar u}(\ell) := \big\{u \in \boldsymbol{q}^{\bar u}(\ell) : W'(u) = \ell\big\}. \tag{4.17}$$

If we consider the *symmetric* energy as in (2.9), viz. $\widetilde{W}(u) := W(-u)$ (so that $\widetilde{W}'(u) = -W'(-u)$), and we denote by $\tilde{\boldsymbol{m}}^{\bar u}(\cdot), \tilde p_{\mathsf{l}}^{\bar u}(\cdot)$, and $\tilde p_{\mathsf{r}}^{\bar u}(\cdot)$ the functions associated with $\widetilde{W}$ via (4.1) and (4.4), it can be easily checked that

$$\boldsymbol{n}^{\bar u}(u) = \tilde p^{(-\bar u)}(-u), \quad q_{\mathsf{r}}^{\bar u}(\ell) = -\tilde p_{\mathsf{l}}^{(-\bar u)}(-\ell), \quad q_{\mathsf{l}}^{\bar u}(\ell) = -\tilde p_{\mathsf{r}}^{(-\bar u)}(-\ell). \tag{4.18}$$

Hence the obvious analogue of Lemma 4.2 holds.

4.3. **Monotone loadings and** BV **solutions.** We apply the notions introduced in §4.1 to characterize BV solutions when ℓ is monotone.

First of all, we provide an explicit formula yielding BV solutions for an increasing loading ℓ. The case of a decreasing and of a piecewise monotone loading will easily follow by applying Propositions 2.5 and 2.6.

Theorem 4.3. *Let $\bar u \in \mathbb{R}$ and $\ell \in \mathrm{C}^1([a,b])$ be a nondecreasing loading such that*

$$\ell(a) \ge W'(\bar u) - \delta_-. \tag{4.19}$$

Any map $u : [a,b] \to \mathbb{R}$ satisfying

$$u \text{ is nondecreasing}, \quad u(t) \in \boldsymbol{p}_{\mathsf{c}}^{\bar u}(\ell(t) - \delta_+) \quad \text{for every } t \in [a,b] \setminus \mathrm{J}_u, \quad W'(u(b)) \le \ell(b) - \delta_+ \tag{4.20}$$

is a BV *solution of the rate independent system $(\mathbb{R}, \mathcal{E}, \Pi)$ of §2.5. In particular,* (4.20) *yields*

$$u(t) \in [p_{\mathsf{l}}^{\bar u}(\ell(t) - \delta_+), p_{\mathsf{r}}^{\bar u}(\ell(t) - \delta_+)] \quad \text{for every } t \in [a,b], \tag{4.21}$$

and (4.21) *is equivalent to* (4.20) *when ℓ is strictly increasing.*

Proof. We apply Theorem 3.1. It is immediate to check that u satisfies $(\mathrm{S}_{\mathrm{loc},\mathbb{R}})$.

By continuity, (4.20) yields

$$W'(u_{\mathsf{l}}(t)) + \delta_+ = W'(u_{\mathsf{r}}(t)) + \delta_+ = \ell(t) \quad \text{for every } t \in (a,b) \tag{4.22}$$

so that *b)* and *c)* are satisfied.

To check the jump conditions of *d)*, let us notice that by (4.13), (4.22), and the monotonicity of u we have

$$p_{\mathsf{l}}^{\bar u}(\ell(t) - \delta_+) \le u_{\mathsf{l}}(t) \le u(t) \le u_{\mathsf{r}}(t) \le p_{\mathsf{r}}^{\bar u}(\ell(t) - \delta_+) \quad \text{for every } t \in [a,b], \tag{4.23}$$

which yields (3.3) and (3.4) by (4.11).

The inequalities in (4.23) also show that (4.20) implies (4.21). Conversely, when ℓ is strictly increasing, it is immediate to check that any map satisfying (4.21) is nondecreasing. Since the jump set of u coincides with $\{t \in [a,b] : p_{\mathsf{l}}^{\bar u}(\ell(t) - \delta_+) < p_{\mathsf{r}}^{\bar u}(\ell(t) - \delta_+)\}$, we have $p_{\mathsf{l}}^{\bar u}(\ell(t) - \delta_+) = p_{\mathsf{r}}^{\bar u}(\ell(t) - \delta_+) = \boldsymbol{p}_{\mathsf{c}}^{\bar u}(\ell(t) - \delta_+)$ at every $t \in [a,b] \setminus \mathrm{J}_u$. Then, also the second condition of (4.20) is satisfied. $\square$

Applying Proposition 2.5 and the discussion of §4.2 we have:

Corollary 4.4. *Let $\bar u \in \mathbb{R}$ and $\ell \in \mathrm{C}^1([a,b])$ be a nonincreasing loading satisfying $\ell(a) \le W'(\bar u) + \delta_+$. Any map $u : [a,b] \to \mathbb{R}$ satisfying*

$$u \text{ is nonincreasing}, \quad u(t) \in \boldsymbol{q}_{\mathsf{c}}^{\bar u}(\ell(t) + \delta_-) \quad \text{for every } t \in [a,b] \setminus \mathrm{J}_u, \quad W'(u(b)) - \delta_- \ge \ell(b) \tag{4.24}$$

is a BV *solution of the rate independent system* $(\mathbb{R}, \mathcal{E}, \Pi)$. *In particular,* (4.24) *yields*

$$u(t) \in [q_{\mathsf{l}}^{\bar u}(\ell(t) + \delta_-), q_{\mathsf{r}}^{\bar u}(\ell(t) + \delta_-)] \quad \text{for every } t \in [a,b], \tag{4.25}$$

and (4.25) *is equivalent to* (4.24) *when ℓ is strictly decreasing.*

The next result states that, under a slightly stronger condition on the initial data, any BV solution driven by an increasing loading admits the representation (4.20).

Theorem 4.5 (Nondecreasing loadings)**.** *Let $\ell \in \mathrm{C}^1([a,b])$ be a nondecreasing loading and let $u \in \mathrm{BV}([a,b];\mathbb{R})$ be a* BV *solution of the rate-independent system* $(\mathbb{R}, \mathcal{E}, \Pi)$ *satisfying*

$$\ell(a) \ge W'(u(a)) - \delta_-, \tag{4.26}$$

$$W' < W'(u(a)) \text{ in a left neighborhood of } u(a) \text{ if } \ell(a) = W'(u(a)) - \delta_-. \tag{4.27}$$

Then u can be represented as in Theorem 4.3, i.e. it satisfies

$$u \text{ is nondecreasing}, \quad u(t) \in \boldsymbol{p}_{\mathsf{c}}^{u(a)}(\ell(t) - \delta_+) \quad \text{for every } t \in [a,b] \setminus \mathrm{J}_u, \tag{4.28}$$

and

$$u(t) \in [p_{\mathsf{l}}^{u(a)}(\ell(t) - \delta_+), p_{\mathsf{r}}^{u(a)}(\ell(t) - \delta_+)] \quad \text{for all } t \in [a,b]. \tag{4.29}$$

Proof. We apply Theorem 3.1 and we split the argument in a few steps; as usual we set $\bar u := u(a)$.
Claim 1: *There exists $\alpha \in [a,b]$ such that $\ell(t) - W'(u(t)) > -\delta_-$ for all $t \in (\alpha, b]$, and $u(t) \equiv \bar u$, $\ell(t) \equiv \ell(a)$ in $[a,\alpha]$.*

Let us consider the set

$$\Sigma := \{t \in [a,b] \,:\, \ell(t) - W'(u(t)) \le -\delta_-\} \tag{4.30}$$

and observe that by (3.3) and (3.4) we also have $\Sigma \cap (a,b) = \{t \in [a,b] \,:\, \ell(t) - W'(u_{\mathsf{lr}}(t)) = -\delta_-\}$, (with obvious modifications for $\Sigma \cap \{a,b\}$), so that Σ is closed thanks to *b)* of Theorem 3.1.

If $a \notin \Sigma$ we set $\alpha := a$ and $\Sigma_a = \emptyset$. If $a \in \Sigma$ we denote by Σ_a the connected component of Σ containing a and we set $\alpha = \max \Sigma_a$. If $\alpha > a$ then

$$\ell(t) - W'(u(t)) \le -\delta_- = \ell(t) - W'(u_{\mathsf{r}}(t)) = \ell(t) - W'(u_{\mathsf{l}}(t)) \quad \text{for every } t \in [a,\alpha], \tag{4.31}$$

so that by *c)* of Theorem 3.1 u is nonincreasing in $[a,\alpha]$. Then, (4.27), (4.31) and the jump condition (3.3) show that $a \notin \mathrm{J}_u$ (otherwise, $u_\mathsf{r}(a) < u(a)$ by (4.31) and (3.4), and (3.4) would also imply $W'(\vartheta) \geq \ell(a)+\delta_- = W'(\bar u)$ for $\vartheta \in [u_\mathsf{r}(a),\bar u]$, contrary to (4.27)), so that $u_\mathsf{r}(a) = u(a) = \bar u$. Since ℓ is nondecreasing, with a similar argument and still invoking (4.27) we conclude that $u(t) = u_\mathsf{r}(t) \equiv \bar u$ and $\ell(t) \equiv \ell(a)$ in $[a,\alpha]$. If $\alpha = b$ the claim is proved.

If $\alpha < b$ let us show that $\Sigma \setminus \Sigma_a$ is empty. Indeed, if not there exists $\sigma \in (\alpha, b] \setminus \Sigma$ and we can consider $\sigma' := \min(\Sigma \cap [\sigma, b])$; since Σ is closed, by construction $\sigma' > \sigma > \alpha$ and the definition of Σ yields $-\delta_- < \ell(t) - W'(u_\mathsf{l}(t)) < \delta_+$ in some left neighborhood (σ'', σ') of σ'. Lemma 3.2 yields that $u_\mathsf{l}(t) = u(t) \equiv u_\mathsf{l}(\sigma')$ in (σ'',σ'), and this contradicts the fact that ℓ is nondecreasing and $\ell(\sigma') - W'(u_\mathsf{l}(\sigma')) = -\delta_-$.

Claim 2: *u is nondecreasing on $[a,b]$.*

Relation (3.2) and Claim 1 imply that $(u')^-\big([a,b)\big) = 0$, so that u is nondecreasing in $[a,b)$. If b were a jump point, then by (3.4) it would be $u(b) > u_\mathsf{l}(b)$.

Claim 3: *let $B := \{t \in [\alpha,b] : W'(\bar u) + \delta_+ = \ell(t)\}$ and $\beta := \min B$ (we set $\beta := b$ if B is empty). Then $u(t) \equiv \bar u$ in $[a,\beta)$ and*

$$W'(u_\mathsf{l}(t)) = W'(u_\mathsf{r}(t)) = \ell(t) - \delta_+, \qquad W'(u(t)) \leq \ell(t) - \delta_+ \qquad \text{for all } t \in (\beta, b]. \tag{4.32}$$

In particular

$$u_\mathsf{l}(t) \geq p_\mathsf{l}^{\bar u}(\ell(t) - \delta_+) \quad \text{for every } t \in [a,b]. \tag{4.33}$$

The first statement follows from Claim 1 and Lemma 3.2.

To prove the second property in (4.32), we argue by contradiction and we suppose that a point $s \in (\beta, b]$ exists such that if $W'(u(s)) + \delta_+ > \ell(s)$. Since ℓ is nondecreasing, Claim 1 and Lemma 3.2 show that $u(t) \equiv u(s)$ for every $t \in [\alpha, s]$, so that $s \leq \beta$.

The first identity in (4.32) then follows by a continuity argument and the stability condition $(\mathrm{S}_{\mathrm{loc},\mathbb{R}})$. Inequality (4.33) ensues from (4.32) and the first characterization in (4.13).

Claim 4: *For every $t \in [a,b]$ we have $u_\mathsf{r}(t) \leq p_\mathsf{r}^{\bar u}(\ell(t) - \delta_+)$.*

If $u_\mathsf{r}(t) = \bar u$ there is nothing to prove. Otherwise, let $t \geq \beta$ and take $z \in (\bar u, u_\mathsf{r}(t))$. Since u is nondecreasing, there exists $s \in [\beta, t]$ such that $u_\mathsf{l}(s) \leq z \leq u_\mathsf{r}(s)$, so that by (4.32) and (3.4)

$$W'(z) \leq \ell(s) - \delta_+ \leq \ell(t) - \delta_+,$$

since ℓ is nondecreasing. Being $z < u_\mathsf{r}(t)$ arbitrary, the claim follows from the second of (4.13).

Conclusion:

Relation (4.29) follows from Claim 2, (4.33), and Claim 4. At every continuity point t for u we have $u_\mathsf{l}(t) = u_\mathsf{r}(t) = u(t)$, so that (4.28) is due to (4.29) and (4.32). ☐

A straightforward consequence of Proposition 2.5 and the discussion of §4.2 concerns the characterization of BV solutions in the case of a decreasing load: it can be deduced from the analysis of the increasing case.

Corollary 4.6 (Nonincreasing loadings)**.** *Let $\bar u \in \mathbb{R}$, let $\ell \in \mathrm{C}^1([a,b])$ be a nonincreasing loading satisfying $\ell(a) \leq W'(\bar u) + \delta_+$, and let us suppose that*

$$W' > W'(\bar u) \text{ in a right neighborhood of } \bar u \text{ if } \ell(a) = W'(\bar u) + \delta_+. \tag{4.34}$$

Then every BV *solution $u \in \mathrm{BV}([0,T];\mathbb{R})$ with $u(a) = \bar u$ satisfies*

$$u \text{ is nonincreasing}, \quad u(t) \in \boldsymbol{q}_\mathsf{c}^{\bar u}(\ell(t)) \quad \text{for every } t \in [a,b] \setminus \mathrm{J}_u, \tag{4.35}$$

and therefore

$$u(t) \in [q_\mathsf{l}^{\bar u}(\ell(t) + \delta_-), q_\mathsf{r}^{\bar u}(\ell(t) + \delta_-)] \quad \text{for all } t \in [a,b]. \tag{4.36}$$

We conclude this section with some examples illustrating the previous results. In particular, Example 4.8 shows that the characterization (4.29) may not hold if the loading ℓ does not comply with (4.27).

Example 4.7 (BV solutions driven by a double-well energy)**.** The double-well potential energy

$$W(u) = \frac{1}{4}(u^2-1)^2 \tag{4.37}$$

clearly fulfills condition (2.12). Note that $W'(u) = u^3 - u$, and

$$W'(u) > 0 \text{ for } u < u_1 := -\frac{1}{\sqrt{3}} \text{ and } u > u_2 := \frac{1}{\sqrt{3}}, \text{ with } W'(u) < 0 \text{ for } u_1 < u < u_2.$$

Let us set

$$\ell^* := W'(u_1) = \frac{2\sqrt{3}}{9}, \qquad \ell_* := W'(u_2) = -\frac{2\sqrt{3}}{9},$$

and, for later convenience,

$$u_* := \min\{u \in \mathbb{R} \,:\, W'(u) = \ell_*\}, \quad u^* := \max\{u \in \mathbb{R} \,:\, W'(u) = \ell^*\}. \tag{4.38}$$

We also introduce the intervals $I_1 := (-\infty, u_1]$, $I_2 := [u_1, u_2]$, $I_3 := [u_2, +\infty)$, and, for $i = 1, 2, 3$, we denote by S_i the inverse function $((W')_{|I_i})^{-1}$. Hence

$$S_1 : (-\infty, \ell^*] \to (-\infty, u_1), \; S_2 : [\ell_*, \ell^*] \to [u_1, u_2], \; S_3 : [\ell_*, +\infty) \to [u_2, +\infty). \tag{4.39}$$

Notice that the functions S_1 and S_3 are strictly increasing, whereas S_2 is strictly decreasing. Let us consider the evolution in the case of an external loading $\ell \in [0,1] \to \mathbb{R}$ such that

$$\begin{gathered}\ell \quad \text{is strictly increasing in } [0,1/2], \quad \ell \quad \text{is strictly decreasing in } [1/2,1] \\ u(0) < u_1, \quad \ell(0) := W'(u(0)) + \delta_+, \quad \ell(1/2) > \ell^* + \delta_+, \quad \ell(1) < \ell_* - \delta_-.\end{gathered}$$

There are points

$$t_1 \in (0,1/2) : \; \ell(t_1) = \ell^* + \delta_+; \qquad t_2 \in (1/2,1) : \; \ell(t_2) + \delta_- = \ell(1/2) - \delta_+; \qquad t_3 \in (t_2,1) : \; \ell(t_3) = \ell_* - \delta_-$$

All the BV solutions are then given by

$$u(t) = \begin{cases} S_1(\ell(t) - \delta_+) & \text{in } [0, t_1), \\ u' \in [u_1, u^*] & \text{if } t = t_1, \\ S_3(\ell(t) - \delta_+) & \text{in } (t_1, 1/2], \\ S_3(\ell(1/2) - \delta_+) & \text{in } [1/2, t_2], \\ S_3(\ell(t) + \delta_-) & \text{in } [t_2, t_3), \\ u'' \in [u_*, u_3] & \text{if } t = t_3, \\ S_1(\ell(t) + \delta_-) & \text{in } (t_3, 1]. \end{cases}$$

Example 4.8 (Bifurcation of BV solutions driven by critical loadings)**.** We consider the same potential energy (4.37) as in Example 4.7, but we suppose now that $\ell(1/2) = \ell^* + \delta_+$ (in this case (4.34) is not satisfied at $a = 1/2$). In addition to the solution considered before, we have another family of solutions, among which e.g.

$$u(t) = \begin{cases} S_1(\ell(t) - \delta_+) & \text{in } [0, 1/2), \\ S_2(\ell(t) - \delta_+) & \text{in } [1/2, t_*), \\ S_2(\ell(t_*) - \delta_+) & \text{in } [t_*, t_3], \\ u'' \in [u_*, u_3] & \text{if } t = t_3, \\ S_1(\ell(t) + \delta_-) & \text{in } (t_3, 1]. \end{cases}$$

where $t_* \in (1/2, t_3)$ is defined by $\ell(t_*) - \delta_- = \ell(t_3) + \delta_+$.

5. Energetic solutions of rate-independent systems in $\mathbb{R}$

In this section we provide a general characterization of energetic solutions to the rate-independent system $(\mathbb{R}, \mathcal{E}, \Psi)$ considered in §2.5.

In order to express the global stability and jump conditions for energetic solutions, let us introduce the following *one-sided global slopes*

$$W'_{\mathsf{i,r}}(u) := \inf_{z>u} \frac{W(z) - W(u)}{z-u}, \qquad W'_{\mathsf{s,l}}(u) := \sup_{z<u} \frac{W(z) - W(u)}{z-u}, \tag{5.1}$$

where the subscripts ir and sl stands for *inf-right* and *sup-left* respectively. They satisfy

$$W'_{\mathsf{i,r}}(u) \le W'(u) \le W'_{\mathsf{s,l}}(u), \quad W'_{\mathsf{s,l}}(u) = -\widetilde{W}'_{\mathsf{i,r}}(-u) \quad \text{for every } u \in \mathbb{R}, \tag{5.2}$$

and it not difficult to check that they are continuous. Indeed, it is sufficient to introduce the continuous function $V : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$$V(u,z) := \begin{cases} W'(u) & \text{if } z = u, \\ \frac{W(z)-W(u)}{z-u} & \text{if } z \neq u, \end{cases}$$

and observe, e.g. for $W'_{\mathsf{i,r}}$, that $W'_{\mathsf{i,r}}(u) = \min\{V(u,z) : z \ge u\}$ and for u in a bounded set the minimum is attained in a compact set thanks to (2.12).

Taking (5.2) into account, we observe that the global stability condition (S) can be reformulated as the system of inequalities

$$-\delta_- \le \ell(t) - W'_{\mathsf{s,l}}(u(t)) \le \ell(t) - W'(u(t)) \le \ell(t) - W'_{\mathsf{i,r}}(u(t)) \le \delta_+ \quad \text{for all } t \in [a,b]. \tag{5.3}$$

The continuity property of the one-sided slopes also yields

$$-\delta_- \le \ell(t) - W'_{\mathsf{s,l}}(u_{\mathsf{r}}(t)) \le \ell(t) - W'(u_{\mathsf{r}}(t)) \le \ell(t) - W'_{\mathsf{i,r}}(u_{\mathsf{r}}(t)) \le \delta_+ \quad \text{for all } t \in [a,b], \tag{5.4}$$

$$-\delta_- \le \ell(t) - W'_{\mathsf{s,l}}(u_{\mathsf{l}}(t)) \le \ell(t) - W'(u_{\mathsf{l}}(t)) \le \ell(t) - W'_{\mathsf{i,r}}(u_{\mathsf{l}}(t)) \le \delta_+ \quad \text{for all } t \in [a,b]. \tag{5.5}$$

We can state the main characterization theorem concerning energetic solutions, which the reader may compare with Thm. 3.1 for BV solutions.

Theorem 5.1. *A function $u \in \mathrm{BV}([a,b];\mathbb{R})$ is an energetic solution of the rate-independent system $(\mathbb{R}, \mathcal{E}, \Psi)$ of §2.5 if and only if the following properties hold:*

a) u satisfies the global stability condition (5.3) *(and therefore* (5.4) *and* (5.5)*).*
b) u satisfies the following precise formulation of the doubly nonlinear differential inclusion (DN)

$$W'(u_{\mathsf{r}}(t)) = W'_{\mathsf{i,r}}(u_{\mathsf{r}}(t)) = \ell(t) - \delta_+ \quad \text{for every } t \in \mathcal{S}_+ := \operatorname{supp}\big((u')^+\big), \tag{5.6}$$

$$W'(u_{\mathsf{r}}(t)) = W'_{\mathsf{s,l}}(u_{\mathsf{r}}(t)) = \ell(t) + \delta_- \quad \text{for every } t \in \mathcal{S}_- := \operatorname{supp}\big((u')^-\big). \tag{5.7}$$

c) At each point $t \in \mathrm{J}_u$, u fulfills the jump conditions:

$$u(t) = (1-\theta)u_{\mathsf{l}}(t) + \theta u_{\mathsf{r}}(t), \quad W(u(t)) = (1-\theta)W(u_{\mathsf{l}}(t)) + \theta W(u_{\mathsf{r}}(t)) \quad \text{for some } \theta \in [0,1], \tag{5.8}$$

and

$$W'_{\mathsf{i,r}}(z) \le W'_{\mathsf{i,r}}(u_{\mathsf{l}}(t)) = \frac{W(u_{\mathsf{r}}(t)) - W(u_{\mathsf{l}}(t))}{u_{\mathsf{r}}(t) - u_{\mathsf{l}}(t)} = \ell(t) - \delta_+ \qquad \text{if } u_{\mathsf{l}}(t) \le z < u_{\mathsf{r}}(t), \tag{5.9}$$

$$W'_{\mathsf{s,l}}(z) \ge W'_{\mathsf{s,l}}(u_{\mathsf{l}}(t)) = \frac{W(u_{\mathsf{r}}(t)) - W(u_{\mathsf{l}}(t))}{u_{\mathsf{r}}(t) - u_{\mathsf{l}}(t)} = \ell(t) + \delta_- \qquad \text{if } u_{\mathsf{l}}(t) > z \ge u_{\mathsf{r}}(t). \tag{5.10}$$

In particular, u is locally constant in the open set

$$\mathcal{I} := \Big\{ t \in [a,b] : -\delta_- < \ell(t) - W'_{\mathsf{s,l}}(u(t)) \le \ell(t) - W'_{\mathsf{i,r}}(u(t)) < \delta_+ \Big\}. \tag{5.11}$$

Since any jump point belongs either to the support of $(u')^+$, or of $(u')^-$, combining (5.8), (5.6), and (5.9) and (5.8), (5.7), and (5.10) we also get at every jump point

$$W'_{\mathsf{i,r}}(u_\mathsf{l}) = W'_{\mathsf{i,r}}(u) = W'_{\mathsf{i,r}}(u_\mathsf{r}) = W'(u_\mathsf{r}) = \frac{W(u_\mathsf{r}) - W(u_\mathsf{l})}{u_\mathsf{r} - u_\mathsf{l}} = \ell - \delta_+ \qquad \text{if } u_\mathsf{l} < u_\mathsf{r} \tag{5.12a}$$

$$W'_{\mathsf{s,l}}(u_\mathsf{l}) = W'_{\mathsf{s,l}}(u) = W'_{\mathsf{s,l}}(u_\mathsf{r}) = W'(u_\mathsf{r}) = \frac{W(u_\mathsf{r}) - W(u_\mathsf{l})}{u_\mathsf{r} - u_\mathsf{l}} = \ell + \delta_- \qquad \text{if } u_\mathsf{l} > u_\mathsf{r}. \tag{5.12b}$$

Proof. It is easy to check that, if $u \in \mathrm{BV}(a, b; \mathbb{R})$ satisfies conditions *a) – c)* then u is an energetic solution to $(\mathbb{R}, \mathcal{E}, \Psi)$: we omit the simple details.

We discuss here the converse implication. Hence, let u be an energetic solution of $(\mathbb{R}, \mathcal{E}, \Psi)$. We have already shown point *a)*; let us first consider point *c)*. The first property of (5.8) easily follows by summing the identities of the jump conditions $(\mathrm{J}_{\mathrm{ener}})$, thus obtaining

$$\Psi(u_\mathsf{r}(t) - u_\mathsf{l}(t)) = \Psi(u_\mathsf{r}(t) - u(t)) + \Psi(u(t) - u_\mathsf{l}(t)). \tag{5.13}$$

In particular, this implies that $\min(u_\mathsf{l}(t), u_\mathsf{r}(t)) \le u(t) \le \max\big(u_\mathsf{r}(t), u_\mathsf{l}(t)\big)$, so that $u(t)$ is a convex combination of $u_\mathsf{l}(t)$ and $u_\mathsf{r}(t)$ with a uniquely determined coefficient $\theta \in [0, 1]$. Conditions $(\mathrm{J}_{\mathrm{ener}})$ then yield the corresponding property for $W(u(t))$.

Let us now consider, e.g., the case $u_\mathsf{l}(t) < u_\mathsf{r}(t)$ and prove (5.9). From the first of the jump conditions $(\mathrm{J}_{\mathrm{ener}})$ and the definition of the right global slope $W'_{\mathsf{i,r}}(\cdot)$, we find

$$\ell(t) - W'_{\mathsf{i,r}}(u_\mathsf{l}(t)) \ge \ell(t) - \frac{W(u_\mathsf{r}(t)) - W(u_\mathsf{l}(t))}{u_\mathsf{r}(t) - u_\mathsf{l}(t)} = \delta_+.$$

Combining this inequality with (5.5) we conclude that the identities in (5.9) hold. If now $z \in [u_\mathsf{l}(t), u_\mathsf{r}(t))$ we obtain

$$\begin{aligned} W(z) - W(u_\mathsf{r}(t)) &= W(z) - W(u_\mathsf{l}(t)) - W'_{\mathsf{i,r}}(u_\mathsf{l}(t))(u_\mathsf{r}(t) - u_\mathsf{l}(t)) \\ &\ge W'_{\mathsf{i,r}}(u_\mathsf{l}(t))(z - u_\mathsf{l}(t)) - W'_{\mathsf{i,r}}(u_\mathsf{l}(t))(u_\mathsf{r}(t) - u_\mathsf{l}(t)) = W'_{\mathsf{i,r}}(u_\mathsf{l}(t))(z - u_\mathsf{r}(t)), \end{aligned} \tag{5.14}$$

where the second inequality ensues from the definition (5.1) of $W'_{\mathsf{i,r}}$. Dividing by $z - u_\mathsf{r}(t)$ we then have that $W'_{\mathsf{i,r}}(z) \le W'_{\mathsf{i,r}}(u_\mathsf{l}(t))$. The proof of (5.10) is completely analogous.

Concerning *b)*, we notice that (DN) yields

$$W'(u(t)) = \ell(t) - \delta_+ \quad \text{for } (u'_{\mathrm{co}})^+\text{-a.e. } t \in [a, b]$$

so that (5.6) holds by continuity and by (5.4) in $\operatorname{supp}\big(u'\big)^+ \setminus \mathrm{J}_u$. On the other hand, for every $t \in \mathrm{J}_u \cap \operatorname{supp}\big(u'\big)^+$ we have $u_\mathsf{l}(t) < u_\mathsf{r}(t)$ so that, dividing inequality (5.14) by $z - u_\mathsf{r}(t)$ and passing to the limit as $z \uparrow u_\mathsf{r}(t)$ we obtain

$$W'(u_\mathsf{r}(t)) \le W'_{\mathsf{i,r}}(u_\mathsf{l}(t)) = \ell(t) - \delta_+ \tag{5.15}$$

and applying (5.4) we conclude the proof of (5.6). The identities in (5.7) follow by the same argument.

It remains to check (5.11): by (5.9) and (5.10) any $t \in \mathcal{I}$ is a continuity point for u; the continuity properties of $W'_{\mathsf{i,r}}(\cdot)$ and $W'_{\mathsf{s,l}}(\cdot)$ then show that a neighborhood of t is also contained in $\mathcal{I}$, so that $\mathcal{I}$ is open and disjoint from J_u. Relations (5.6) and (5.7) then yield that $u' = 0$ in the sense of distributions in $\mathcal{I}$, so that u is locally constant. □

6. Energetic solutions of the rate-independent system $(\mathbb{R}, \mathcal{E}, \Psi)$ with monotone loadings

6.1. **Convex envelopes and their subdifferentials.** This section is devoted to some preliminary convex analysis results, which turn out to be useful for the characterization of energetic solutions driven by monotone loadings.

Let $\mathcal{W} : \mathbb{R} \to (-\infty, +\infty]$ be a function with proper non-empty domain $D(\mathcal{W}) := \{u \in \mathbb{R} : \mathcal{W}(u) < \infty\}$. For our purposes, we will assume that

$$D(\mathcal{W}) \text{ is a closed interval, } \mathcal{W} \text{ is of class } C^1 \text{ in } D(\mathcal{W}) \text{ and it is bounded from below.} \tag{6.1}$$

The (convex analysis) subdifferential of $\mathcal{W}$ is the multivalued map $\partial\mathcal{W} : \mathbb{R} \rightrightarrows \mathbb{R}$ defined by

$$\xi \in \partial\mathcal{W}(u) \quad \Leftrightarrow \quad W(u) + \xi(z-u) \le W(z) \quad \text{for every } z \in \mathbb{R}. \tag{6.2}$$

Clearly, $\partial\mathcal{W}(u)$ is empty if $u \notin D(\mathcal{W})$. In the present one-dimensional setting, we have a simple characterization of the subdifferential in terms of the one-sided slopes defined in (5.1):

$$\xi \in \partial\mathcal{W}(u) \quad \Leftrightarrow \quad \mathcal{W}'_{\mathsf{s,l}}(u) \le \xi \le \mathcal{W}'_{\mathsf{i,r}}(u). \tag{6.3}$$

If $u \in \mathrm{int}\big(D(\mathcal{W})\big)$, then $\partial\mathcal{W}(u)$ coincides with $\{\mathcal{W}'(u)\}$ or it is empty. By (6.3), the former case can also be characterized by

$$\partial\mathcal{W}(u) \neq \emptyset \quad \Leftrightarrow \quad \mathcal{W}'_{\mathsf{s,l}}(u) = \mathcal{W}'(u) = \mathcal{W}'_{\mathsf{i,r}}(u) \quad \text{for every } u \in \mathrm{int}\big(D(\mathcal{W})\big); \tag{6.4}$$

if this is the case, the common value on the right-hand side of (6.4) is the unique element of $\partial\mathcal{W}(u)$.

The *Fenchel-Moreau conjugate* of $\mathcal{W}$ is the function

$$\mathcal{W}^*(\ell) := \sup_{u \in \mathbb{R}} (\ell\, u - \mathcal{W}(u)), \tag{6.5}$$

and a further iteration of the conjugation yields

$$\mathcal{W}^{**}(u) := \sup_{\ell \in \mathbb{R}} \big(u\, \ell - \mathcal{W}^*(\ell)\big). \tag{6.6}$$

Indeed, $\mathcal{W}^{**}$ coincides with the *l.s.c. convex envelope of* $\mathcal{W}$, i.e. the maximal convex and l.s.c. function less than $\mathcal{W}$. It can also be defined by

$$\mathcal{W}^{**}(u) = \sup\{au + b \ : \ ay + b \le \mathcal{W}(y) \quad \text{for all } y \in \mathbb{R}.\} \tag{6.7}$$

We introduce the contact set

$$\mathfrak{C}(\mathcal{W}) := \big\{u \in \mathbb{R} : \mathcal{W}(u) = \mathcal{W}^{**}(u)\big\} \tag{6.8}$$

and its complement

$$\mathfrak{B}(\mathcal{W}) := \{u \in \mathbb{R} : \ \mathcal{W}(u) - \mathcal{W}^{**}(u) > 0\} = D(\mathcal{W}) \setminus \mathfrak{C}(\mathcal{W}); \tag{6.9}$$

It can be shown (see e.g. [11, Lemma 3.3]) that $\mathfrak{B}(\mathcal{W}) \subset D(\mathcal{W})$ is open in $\mathbb{R}$, hence it is the disjoint union of a (at most) countable collection of open intervals. Further, for every connected component (α, β) of $\mathfrak{B}(\mathcal{W})$ there holds

$$\mathcal{W}^{**}(\alpha) = \mathcal{W}(\alpha), \quad \mathcal{W}^{**}(\beta) = \mathcal{W}(\beta), \quad \mathcal{W}^{**}((1-\theta)\alpha + \theta\beta) = (1-\theta)\mathcal{W}(\alpha) + \theta\mathcal{W}(\beta) \tag{6.10}$$

for all $\theta \in [0,1]$. Using (6.10), it can be checked that $\mathcal{W}^{**} \in \mathrm{C}^1(D(\mathcal{W}))$. It is well known (see e.g. [3, Ch. 1, §5.1]) that

$$\partial\mathcal{W}(u) \neq \emptyset \quad \Leftrightarrow \quad \mathcal{W}(u) = \mathcal{W}^{**}(u), \quad \text{and in this case } \partial\mathcal{W}(u) = \partial\mathcal{W}^{**}(u). \tag{6.11}$$

Moreover, for every $\ell \in \mathbb{R}$

$$\partial\mathcal{W}^{-1}(\ell) = \operatorname*{Argmin}_{u \in \mathbb{R}} \big(\mathcal{W}(u) - \ell u\big) = \mathfrak{C}(\mathcal{W}) \cap \partial\mathcal{W}^*(\ell) \subset \partial\mathcal{W}^*(\ell) \tag{6.12}$$

and

$$\partial\mathcal{W}^*(\ell) = (\partial\mathcal{W}^{**})^{-1}(\ell) = \mathrm{co}\big((\partial\mathcal{W})^{-1}(\ell)\big), \tag{6.13}$$

where the latter set is the convex hull of $(\partial\mathcal{W})^{-1}(\ell)$.

We apply the above notions to the functions $\mathcal{W}(\cdot) := W(\cdot; \bar{u})$, $\bar{u} \in \mathbb{R}$, defined by

$$W(u; \bar{u}) := W(u) + \mathrm{I}_{[\bar{u}, \infty)}(u) = \begin{cases} +\infty & \text{if } u < \bar{u}, \\ W(u) & \text{if } u \ge \bar{u} \end{cases} \quad \text{for } u \in \mathbb{R}. \tag{6.14}$$

We consider their conjugate $W^*(\cdot;\bar u)$, and the subdifferentials

$$\mathfrak{p}^{\bar u}(\ell) := \partial W^*(\ell;\bar u) = (\partial W^{**}(\cdot;\bar u))^{-1}(\ell), \quad \mathfrak{p}^{\bar u}_{\mathsf{c}}(\ell) := \big(\partial W(\cdot;\bar u)\big)^{-1}(\ell) = \partial W^*(\ell;\bar u) \cap \mathfrak{C}^{\bar u}, \tag{6.15}$$

where $\mathfrak{C}^{\bar u} := \mathfrak{C}(W(\cdot;\bar u))$.

We observe that (cf. (4.4))

$$\mathfrak{p}^{\bar u}(\ell) = [\mathfrak{p}^{\bar u}_{\mathsf{l}}(\ell), \mathfrak{p}^{\bar u}_{\mathsf{r}}(\ell)] \quad \text{with} \quad \begin{cases} \mathfrak{p}^{\bar u}_{\mathsf{l}}(\ell) = \min\{u \geq \bar u \ : \ \partial W^{**}(u;\bar u) \ni \ell\}, \\ \mathfrak{p}^{\bar u}_{\mathsf{r}}(\ell) = \max\{u \geq \bar u \ : \ \partial W^{**}(u;\bar u) \ni \ell\}. \end{cases} \tag{6.16}$$

Furthermore, the functions $\mathfrak{p}^{\bar u}_{\mathsf{l}}$ and $\mathfrak{p}^{\bar u}_{\mathsf{r}}$ are nondecreasing and satisfy the obvious continuity properties

$$\begin{aligned} (\ell_n \downarrow \ell \text{ as } n \to \infty\ ) \ &\Rightarrow \ \lim_{n\to\infty} \mathfrak{p}^{\bar u}_{\mathsf{r}}(\ell_n) = \mathfrak{p}^{\bar u}_{\mathsf{r}}(\ell), \\ (\ell_n \uparrow \ell \text{ as } n \to \infty\ ) \ &\Rightarrow \ \lim_{n\to\infty} \mathfrak{p}^{\bar u}_{\mathsf{l}}(\ell_n) = \mathfrak{p}^{\bar u}_{\mathsf{l}}(\ell); \end{aligned} \tag{6.17}$$

There is a last interesting property which relates the one-sided slopes $W'_{\mathsf{s,l}}(\cdot), W'_{\mathsf{i,r}}(\cdot)$, their upper monotone envelopes, and the convex envelope of $\mathcal{W}(\cdot) = W(\cdot;\bar u)$.

Lemma 6.1. *For $\bar u \in \mathbb{R}$ let us consider the upper monotone envelope*

$$\mathfrak{r}^{\bar u}(u) := \max_{\bar u \leq v \leq u} W'_{\mathsf{i,r}}(v) \quad \textit{if } u > \bar u, \qquad \mathfrak{r}^{\bar u}(\bar u) := (-\infty, W'_{\mathsf{i,r}}(\bar u)], \qquad \mathfrak{r}^{\bar u}(u) := \emptyset \quad \textit{if } u < \bar u. \tag{6.18}$$

Then, there holds

$$\partial W^{**}(u;\bar u) = \mathfrak{r}^{\bar u}(u) \quad \textit{for every } u \in \mathbb{R}. \tag{6.19}$$

In particular,

$$\begin{aligned} \ell \in \mathfrak{r}^{\bar u}(u) \quad &\Leftrightarrow \quad u \in \mathfrak{p}^{\bar u}(\ell) = [\mathfrak{p}^{\bar u}_{\mathsf{l}}(\ell), \mathfrak{p}^{\bar u}_{\mathsf{r}}(\ell)], \\ \ell = W'_{\mathsf{i,r}}(u) \in \mathfrak{r}^{\bar u}(u) \quad &\Leftrightarrow \quad u \in \mathfrak{p}^{\bar u}_{\mathsf{c}}(\ell), \end{aligned} \tag{6.20}$$

and

$$\mathfrak{p}^{\bar u}_{\mathsf{l}}(\ell) = \min\{u \geq \bar u : W'_{\mathsf{i,r}}(u) \geq \ell\}, \qquad \mathfrak{p}^{\bar u}_{\mathsf{r}}(\ell) = \inf\{u \geq \bar u : W'_{\mathsf{i,r}}(u) > \ell\}. \tag{6.21}$$

Proof. Let us start by proving (6.19): it is sufficient to consider the case $u > \bar u$. Let $\mathcal{W}(u) = W(u;\bar u)$. If $u \in \mathfrak{C}^{\bar u} \cap (\bar u, \infty)$ then, by (6.4) and (6.11) we have $W'_{\mathsf{i,r}}(u) = W'(u) = \frac{\mathrm{d}}{\mathrm{d}u} W^{**}(u;\bar u) \leq \mathfrak{r}^{\bar u}(u)$. Let us consider the case when $u \notin \mathfrak{C}^{\bar u}$ and let (a,b) be the connected component of $\mathfrak{B}(\mathcal{W})$ containing u. Since $(W^{**})'$ is constant in (a,b), then $(\mathcal{W}^{**})'(u) = (\mathcal{W}^{**})'(a) \leq \mathfrak{r}^{\bar u}(a) \leq \mathfrak{r}^{\bar u}(u)$. All in all, we have $(\mathcal{W}^{**})'(u) \leq \mathfrak{r}^{\bar u}(u)$ for every $u > \bar u$. Since for every connected component (a,b) of $\mathfrak{B}(\mathcal{W})$ we have

$$\mathcal{W}'_{\mathsf{i,r}}(v) \leq \frac{\mathcal{W}(b) - \mathcal{W}(v)}{b - v} \leq \frac{\mathcal{W}^{**}(b) - \mathcal{W}^{**}(v)}{b - v} = (\mathcal{W}^{**})'(v) \qquad \text{for every } v \in (a,b), \tag{6.22}$$

we also deduce that $(\mathcal{W}^{**})' \geq \mathcal{W}'_{\mathsf{i,r}}(\cdot)$. Combining the fact that $\mathfrak{r}^{\bar u}(u)$ is the minimal monotone map above $\mathcal{W}'_{\mathsf{i,r}}(\cdot)$ in $(\bar u, \infty)$, and the monotonicity of $(\mathcal{W}^{**})'$, we conclude.

The first equivalence in (6.20) is a consequence of (6.15) and (6.16), and (6.21) corresponds to (4.13). Concerning the second equivalence in (6.20), if $\bar u < u \in \mathfrak{p}^{\bar u}_{\mathsf{c}}(\ell)$ then (6.4) yields $\ell = \mathcal{W}'_{\mathsf{i,r}}(u)$. Conversely, if $u \in [\mathfrak{p}^{\bar u}_{\mathsf{l}}(\ell), \mathfrak{p}^{\bar u}_{\mathsf{r}}(\ell)]$ and $W'_{\mathsf{i,r}}(u) = \ell$ then we get $u \in \mathfrak{C}^{\bar u}$. In fact, this property obviously holds if $u \in \{\mathfrak{p}^{\bar u}_{\mathsf{l}}(\ell), \mathfrak{p}^{\bar u}_{\mathsf{r}}(\ell)\}$; if $u \in (\mathfrak{p}^{\bar u}_{\mathsf{l}}(\ell), \mathfrak{p}^{\bar u}_{\mathsf{r}}(\ell))$ then, combining $W'_{\mathsf{i,r}}(u) = \ell = (\mathcal{W}^{**})'(u)$ with (6.22), we conclude that all inequalities in (6.22) hold as equalities. Hence, we find

$$\frac{\mathcal{W}(u) - \mathcal{W}(\mathfrak{p}^{\bar u}_{\mathsf{r}}(\ell))}{u - \mathfrak{p}^{\bar u}_{\mathsf{r}}(\ell)} = \frac{W^{**}(\mathfrak{p}^{\bar u}_{\mathsf{l}}(u)) - \mathcal{W}(\mathfrak{p}^{\bar u}_{\mathsf{r}}(\ell))}{u - \mathfrak{p}^{\bar u}_{\mathsf{r}}(\ell)},$$

whence $\mathcal{W}(u) = \mathcal{W}^{**}(u)$. □

6.2. **Energetic solutions of** $(\mathbb{R}, \mathcal{E}, \Psi)$ **with monotone loadings.**

Theorem 6.2. *Let* $\bar{u} \in \mathbb{R}$ *and* $\ell \in C^1([a,b])$ *be a nondecreasing loading such that*

$$W'_{\mathsf{s,l}}(\bar{u}) - \delta_- \le \ell(a) \le W'_{\mathsf{i,r}}(\bar{u}) + \delta_+. \tag{6.23}$$

Any map $u : [a,b] \to \mathbb{R}$ *satisfying*

$$u(a) = \bar{u}, \quad u \text{ is nondecreasing}, \quad u(t) \in \mathfrak{p}_{\mathsf{c}}^{\bar{u}}(\ell(t) - \delta_+) \quad \text{for every } t \in (a,b] \tag{6.24}$$

is an energetic solution of $(\mathbb{R}, \mathcal{E}, \Psi)$*. In particular,* (6.24) *yields*

$$u(t) \in [\mathfrak{p}_{\mathsf{l}}^{\bar{u}}(\ell(t) - \delta_+), \mathfrak{p}_{\mathsf{r}}^{\bar{u}}(\ell(t) - \delta_+)]. \tag{6.25}$$

Notice that, if ℓ is strictly increasing, then any selection $u(t)$ of $\mathfrak{p}_{\mathsf{c}}^{\bar{u}}(\ell(t) - \delta_+)$ is also strictly increasing, so that the second condition of (6.24) is automatically satisfied.

Proof. Notice that (6.23) and (6.24) yield

$$\ell(t) - \delta_+ \in \partial W(u(t); \bar{u}) \quad \text{for every } t \in [a,b]; \tag{6.26}$$

Indeed, (6.26) holds also at $t = a$ in view of (6.3) applied to $\mathcal{W}(u) := W(u; \bar{u})$, since in this case $\mathcal{W}'_{\mathsf{s,l}}(\bar{u}) = -\infty$ and $\mathcal{W}'_{\mathsf{i,r}}(\bar{u}) = W'_{\mathsf{i,r}}(\bar{u})$, cf. (6.14). A further application of (6.3) yields the global stability condition (5.3). Since the subdifferential has a closed graph, we also have

$$\ell(t) - \delta_+ \in \partial W(u_{\mathsf{r}}(t); \bar{u}), \quad \ell(t) - \delta_+ \in \partial W(u_{\mathsf{l}}(t); \bar{u}) \quad \text{for every } t \in [a,b]. \tag{6.27}$$

Let us set $\alpha := \inf\{t > a : u(t) > u(a)\}$. Since $u(a)$ satisfies the global stability condition by (6.23), the function u is clearly a constant energetic solution in $[a, \alpha]$. Thus, Proposition 2.6 shows that it is not restrictive to assume that $\alpha = a$.

In this case, $u_{\mathsf{r}}(t) > u(a)$ for every $t > a$, so that $u_{\mathsf{r}}(t)$ belongs to the interior of the domain of $W(\cdot; \bar{u})$; it follows from (6.24) that $W'(u_{\mathsf{r}}(t)) = W'_{\mathsf{i,r}}(u_{\mathsf{r}}(t)) = \ell(t) - \delta_+$ in $(a,b]$, and also at $t = a$ by passing to the limit in the equation as $t \downarrow a$. The formulation (5.6) of (DN) is thus satisfied in $[a,b]$.

Let us check now the point *c)* of Theorem 5.1 concerning the jump conditions. If $t \in \mathrm{J}_u$, then in view of (6.27) $\mathfrak{p}_{\mathsf{c}}^{\bar{u}}(\ell(t) - \delta_+)$ contains two distinct points $u_{\mathsf{l}}(t) < u_{\mathsf{r}}(t)$, so that the graph of $W^{**}(\cdot; \bar{u})$ is linear on $[u_{\mathsf{l}}(t), u_{\mathsf{r}}(t)]$. Since $u_{\mathsf{l}}(t), u(t), u_{\mathsf{r}}(t)$ belong to the contact set $\mathfrak{C}^{\bar{u}}$, in view of (6.10) the jump conditions (5.8) and (5.9) are also satisfied, and therefore u is an energetic solution. □

We are now in the position to state the analogue of Theorem 4.5 for energetic solutions. We also show a direct link with the notion of BV solution. To do this, we introduce the modified energy

$$\bar{W}(u) := W(\bar{u}) + \int_{\bar{u}}^{u} W'_{\mathsf{i,r}}(r)\,\mathrm{d}r, \quad \bar{\mathcal{E}}_t(u) := \bar{W}(u) - \ell(t)u. \tag{6.28}$$

Theorem 6.3. *Let* $\ell \in C^1([a,b])$ *be a nondecreasing loading and let* $u \in \mathrm{BV}([a,b]; \mathbb{R})$ *be an energetic solution of* $(\mathbb{R}, \mathcal{E}, \Psi)$ *satisfying*

$$\frac{W(z) - W(u(a))}{z - u(a)} > W'_{\mathsf{s,l}}(u(a)) \quad \text{for every } z < u(a) \quad \text{if} \quad W'_{\mathsf{s,l}}(u(a)) = \ell(a) + \delta_-, \tag{6.29a}$$

$$W'(z) < W'(u(a)) \text{ in a left neighborhood of } u(a) \quad \text{if} \quad W'(u(a)) = \ell(a) + \delta_-. \tag{6.29b}$$

Then

a) u *is also a* BV *solution of the rate-independent system* $(\mathbb{R}, \bar{\mathcal{E}}, \Pi)$*.*
b) u *can be represented as in Theorem 6.2, i.e.*

$$u \text{ is nondecreasing}, \quad u(t) \in \mathfrak{p}_{\mathsf{c}}^{u(a)}(\ell(t) - \delta_+) \quad \text{for every } t \in (a,b]. \tag{6.30}$$

Proof. Claim 1: *Let us suppose that ℓ and u are constant in the interval (ρ,σ) and $\sigma \in \mathrm{J}_u$ with $u_\mathsf{r}(\sigma) < u_\mathsf{l}(\sigma)$. If $\rho \in \mathrm{J}_u$ then also $u_\mathsf{r}(\rho) < u_\mathsf{l}(\rho)$.*

Let us denote by $\bar\ell$ and $\bar u$ the constant value of ℓ and u on (ρ,σ); by continuity $\ell(\sigma) = \ell(\rho) = \bar\ell$ and $u_\mathsf{l}(\sigma) = u_\mathsf{r}(\rho) = \bar u$. We argue by contradiction and we suppose that $\bar u = u_\mathsf{r}(\rho) > u_\mathsf{l}(\rho)$. The jump conditions (5.9) and (5.10) yield

$$W(\bar u) - W(u_\mathsf{r}(\sigma)) = (\bar\ell + \delta_-)(\bar u - u_\mathsf{r}(\sigma)), \quad W(\bar u) - W(u_\mathsf{l}(\rho)) = (\bar\ell - \delta_+)(\bar u - u_\mathsf{l}(\rho)). \tag{6.31}$$

Taking the difference of the two identities we get

$$\begin{aligned} W(u_\mathsf{l}(\rho)) - W(u_\mathsf{r}(\sigma)) &= (\delta_- + \delta_+)(\bar u - u_\mathsf{l}(\rho)) + (\bar\ell + \delta_-)(u_\mathsf{l}(\rho) - u_\mathsf{r}(\sigma)) &(6.32)\\ &= (\delta_- + \delta_+)(\bar u - u_\mathsf{r}(\sigma)) + (\bar\ell - \delta_+)(u_\mathsf{l}(\rho) - u_\mathsf{r}(\sigma)) &(6.33)\end{aligned}$$

Clearly, the case $u_\mathsf{l}(\rho) = u_\mathsf{r}(\sigma)$ is impossible; if $u_\mathsf{l}(\rho) > u_\mathsf{r}(\sigma)$, upon dividing by $u_\mathsf{l}(\rho) - u_\mathsf{r}(\sigma)$ (6.32) yields

$$W'_{\mathsf{s},\mathsf{l}}(u_\mathsf{l}(\rho)) > \bar\ell + \delta_-$$

which contradicts the first inequality of (5.5); if $u_\mathsf{l}(\rho) < u_\mathsf{r}(\sigma)$, upon dividing by $u_\mathsf{l}(\rho) - u_\mathsf{r}(\sigma)$ (6.33) yields

$$W'_{\mathsf{i},\mathsf{r}}(u_\mathsf{l}(\rho)) < \bar\ell - \delta_+$$

which contradicts the last inequality of (5.5).

Claim 2: *Let $a < \rho \le \sigma \in \mathrm{J}_u$ with $u_\mathsf{r}(\sigma) < u_\mathsf{l}(\sigma)$, and let us assume that $\rho \notin \mathrm{J}_u$ and ℓ and u are constant in the interval (ρ,σ) if $\rho < \sigma$. If $u_\mathsf{r}(t) \le u_\mathsf{r}(\rho)$ in a left neighborhood of ρ, then there exists $\varepsilon > 0$ such that $\ell(t) \equiv \ell(\rho) = \ell(\sigma)$ and $u(t) \equiv u(\rho) = u_\mathsf{l}(\sigma)$ for every $t \in [\rho - \varepsilon, \rho]$*

As in the previous claim, let us denote by $\bar\ell = \ell(\rho)$ and $\bar u = u_\mathsf{l}(\sigma)$ the constant value of ℓ and u on (ρ,σ). We argue by contradiction and we assume that there exists a sequence $t_n < \rho$ converging to ρ such that $u_n := u_\mathsf{r}(t_n) \uparrow \bar u$, $\ell_n := \ell(t_n) \uparrow \bar\ell$ and $u_n + \ell_n < \bar u + \bar\ell$. The global stability and the jump condition (5.12b) yield

$$\begin{aligned}(\bar\ell + \delta_-)(\bar u - u_\mathsf{r}(\sigma)) &= W(\bar u) - W(u_\mathsf{r}(\sigma)) = W(\bar u) - W(u_n) + W(u_n) - W(u_\mathsf{r}(\sigma))\\ &\le W'(\bar u)(\bar u - u_n) + o(\bar u - u_n) + (\ell_n + \delta_-)(u_n - u_\mathsf{r}(\sigma)) \quad \text{as } n \to \infty.\end{aligned} \tag{6.34}$$

If for some $\bar n \in \mathbb{N}$ $u_n \equiv \bar u$ for every $n \ge \bar n$ then we get

$$(\bar\ell + \delta_-)(\bar u - u_\mathsf{r}(\sigma)) \le (\ell_n + \delta_-)(\bar u - u_\mathsf{r}(\sigma)),$$

which contradicts the assumption $\ell_n < \bar\ell$. Up to extracting a suitable subsequence we can then assume that $u_n < \bar u$ for every $n \in \mathbb{N}$, so that $\bar u \in \mathrm{supp}(u')^+$ and $W'(\bar u) = \bar\ell - \delta_+$ by (5.6). From (6.34) we then get

$$(\bar\ell + \delta_-)(\bar u - u_n) + (\bar\ell + \delta_-)(u_n - u_\mathsf{r}(\sigma)) \le (\bar\ell - \delta_+)(\bar u - u_n) + (\ell_n + \delta_-)(u_n - u_\mathsf{r}(\sigma)) + o(\bar u - u_n)$$

as $n \to \infty$. so that

$$(\delta_- + \delta_+)(\bar u - u_n) + (\bar\ell - \ell_n)(u_n - u_\mathsf{r}(\sigma)) \le o(\bar u - u_n) \quad \text{as } n \to \infty,$$

which can be satisfied only if for n sufficiently big $u_n \equiv \bar u$ and $\ell_n \equiv \bar\ell$. Hence, we have a contradiction.

Claim 3: *If for some $t \in \mathrm{J}_u$ $u_\mathsf{l}(t) = u(a)$ and $\ell(t) = \ell(a)$, then $u_\mathsf{l}(t) < u_\mathsf{r}(t)$.*

It is sufficient to notice that (6.29a) is not compatible with (5.10).

Claim 4:

Let $a < \sigma' < \sigma \le b$ such that

$$\ell(t) - W'(u_\mathsf{r}(t)) > -\delta_- = \ell(\sigma) - W'(u_\mathsf{r}(\sigma)) \quad \textit{for every } t \in [\sigma',\sigma). \tag{6.35}$$

Then $\sigma \notin \mathrm{J}_u$.

We argue by contradiction and assume that $\sigma \in \mathrm{J}_u$. In view of (5.12a), necessarily

$$u_\mathsf{l}(\sigma) > u_\mathsf{r}(\sigma), \tag{6.36}$$

and (5.7) shows that u_{r} is nondecreasing in $[\sigma',\sigma)$.

Let $\mathcal{R} := \{\rho \in [a,\sigma] : u_{\mathsf{r}}(t) \equiv u_{\mathsf{l}}(\sigma),\ \ell(t) \equiv \ell(\sigma) \ \text{ for all } t \in [\rho,\sigma]\}$. $\mathcal{R}$ is clearly closed and contains σ. Let us show that $\mathcal{R}$ is open in $[a,\sigma]$: it is sufficient to prove that for every $\rho \in \mathcal{R}\cap(a,\sigma]$, the set $\mathcal{R}$ contains a left neighborhood of ρ ($\mathcal{R}$ obviously contains also a right neighborhood of ρ if $\rho < \sigma$).

When $\rho > \sigma'$ we can easily check that we can apply Claim 2. Indeed, $u_{\mathsf{r}}(t) \le u_{\mathsf{r}}(\rho)$ in a left neighborhood of ρ since u_{r} is nondecreasing in $[\sigma',\sigma)$. Further, ρ cannot be a jump point for u thanks to Claim 1, which prevents a jump point with $u_{\mathsf{l}}(\rho) < u_{\mathsf{r}}(\rho)$, and the fact that $\ell(\rho) - W'(u_{\mathsf{r}}(\rho)) > -\delta_-$, which prevents a jump point with $u_{\mathsf{l}}(\rho) > u_{\mathsf{r}}(\rho)$. By the way, this shows that $\mathcal{R} \supset [\sigma',\sigma]$.

If $\rho \le \sigma'$ we can still apply Claim 2, since we have that $\ell(\rho) = \ell(\sigma')$ and $u_{\mathsf{r}}(\rho) = u_{\mathsf{r}}(\sigma')$, so that $\ell(\rho) - W'(u_{\mathsf{r}}(\rho)) > -\delta_-$, u is nondecreasing in a left neighborhood of ρ by (5.7) and $\rho \notin \mathrm{J}_u$ by the same arguments above.

Since $\mathcal{R}$ is both open and closed, we conclude that $\mathcal{R} = [a,\sigma]$. Claim 1 and Claim 3 yield that $a \notin \mathrm{J}_u$ so that $u_{\mathsf{l}}(\sigma) = u(a)$ and $\ell(\sigma) = \ell(a)$. A further application of Claim 3 gives the contradiction with (6.36).

Claim 5: *There exists $\alpha \in [a,b]$ such that $\ell(t) - W'(u_{\mathsf{r}}(t)) > -\delta_-$ for all $t \in (\alpha,b]$ and $u(t) \equiv u(a),\ \ell(t) \equiv \ell(a)$ in $[a,\alpha]$.*

Let us consider the set

$$\Sigma := \{t \in [a,b] \ :\ W'(u_{\mathsf{r}}(t)) = \ell(t) + \delta_-\}, \tag{6.37}$$

and observe that

$$t_n \in \Sigma, \quad t_n \downarrow t \quad \Rightarrow \quad t \in \Sigma. \tag{6.38}$$

If $a \in \Sigma$ we denote by Σ_a the connected component of Σ containing a, and we set $\alpha = \sup \Sigma_a$. If $\alpha > a$, then

$$W'(u_{\mathsf{r}}(t)) = \ell(t) + \delta_- \quad \text{for every } t \in [a,\alpha], \tag{6.39}$$

so that by (5.6) u is nonincreasing in $[a,\alpha]$. Assumption (6.29a) and the jump conditions (5.12b) imply that $a \notin \mathrm{J}_u$ and $u_{\mathsf{r}}(a) = u(a)$. Since also ℓ is nondecreasing we conclude by (6.29b) and (5.7) that $u(t) \equiv u(a)$ and $\ell(t) \equiv \ell(a)$ in $[a,\alpha]$; moreover, by the same argument, $\alpha \notin \mathrm{J}_u$ so that $\alpha \in \Sigma$. When $a \notin \Sigma$ we simply set $\alpha := a$ and $\Sigma_a = \emptyset$.

The Claim then follows if we show that $\Sigma \setminus \Sigma_a$ is empty. This is trivial if $\alpha = b$. If $\alpha < b$ we suppose $\Sigma \setminus \Sigma_a \neq \emptyset$ and we argue by contradiction.

We can find points $\alpha_2 > \alpha_1 > \alpha$ such that $\alpha_1 \notin \Sigma$, and $\alpha_2 \in \Sigma$. By (6.38) we can consider $\sigma := \min(\Sigma \cap [\alpha_1, b]) > \alpha_1 > \alpha$; Claim 4 (with $\sigma' := \alpha_1$) yields that $\sigma \notin \mathrm{J}_u$, so that we can find $\varepsilon > 0$ such that

$$-\delta_- < \ell(t) - W'(u_{\mathsf{r}}(t)) < \delta_+ \quad \text{for every } t \in (\sigma - \varepsilon, \sigma). \tag{6.40}$$

Point *b)* of Theorem 5.1 implies that $u_{\mathsf{r}}(t) = u(t) \equiv u_{\mathsf{l}}(\sigma) = u_{\mathsf{r}}(\sigma)$ is constant in $(\sigma-\varepsilon,\sigma)$. Hence, $W'(u_{\mathsf{r}}(t)) \equiv W'(u_{\mathsf{r}}(\sigma)) = \ell(\sigma) + \delta_- \ge \ell(t) + \delta_-$ for every $t \in (\sigma - \varepsilon, \sigma)$, since ℓ is nondecreasing and $\sigma \in \Sigma$. This contradicts (6.40).

Claim 6: *u is nondecreasing on $[a,b]$.*

It follows immediately from (5.7) and Claim 5.

Claim 7: *Let $B := \{t \in [\alpha,b] : W'_{\mathsf{i,r}}(u(a)) + \delta_+ = \ell(t)\}$ and let $\beta := \min B$ (with the convention $\beta := b$ if B is empty). Then $u(t) \equiv u(a)$ in $[a,\beta)$ and*

$$W'_{\mathsf{i,r}}(u_{\mathsf{l}}(t)) = W'_{\mathsf{i,r}}(u_{\mathsf{r}}(t)) = \ell(t) - \delta_+ \quad \textit{for all } t \in (\beta, b]. \tag{6.41}$$

In particular, $u_{\mathsf{l}}(t) \ge \mathfrak{p}_{\mathsf{l}}^{\bar u}(\ell(t) - \delta_+)$ for all $t \in (\beta, b]$.

The first statement follows from the previous Claim and (5.6).

To prove the second identity in (6.41) for $u_{\mathsf{r}}(t)$, we argue by contradiction and we suppose that a point $s \in (\beta, b]$ exists such that if $W'_{\mathsf{i},\mathsf{r}}(u_{\mathsf{r}}(s)) + \delta_+ > \ell(s)$. Then, by (5.12a) s is not a jump point, and therefore in view of point b) of Thm. 5.1, u is locally constant around s. Since ℓ is nondecreasing, because of (5.6) we conclude that $u(t) \equiv u(s)$ for every $t \in [\alpha, s]$, so that $s \le \beta$, a contradiction.

The first identity in (6.41) then follows by continuity. The last statement of Claim 7 is a consequence of the first of (6.21).

Claim 8: *for all $t \in [a,b]$ we have $u_{\mathsf{r}}(t) \le \mathfrak{p}_{\mathsf{r}}^{\bar u}(\ell(t) - \delta_+)$.*

The statement is obvious if $u_{\mathsf{r}}(t) = \bar u = u(a)$. If $u_{\mathsf{r}}(t) > \bar u$, let $z \in (\bar u, u_{\mathsf{r}}(t))$: since u is nondecreasing we find $s \in [a,t]$ such that $z \in [u_{\mathsf{l}}(s), u_{\mathsf{r}}(s)]$, so that (5.9) (in the case $s \in \mathrm{J}_u$) or (5.6) (in the case $u_{\mathsf{l}}(s) = u_{\mathsf{r}}(s)$) yield

$$W'_{\mathsf{i},\mathsf{r}}(z) \le \ell(s) - \delta_+ \le \ell(t) - \delta_+.$$

Since z is arbitrary, we conclude by the second of (6.21).

Claim 9: *conclusion.*

In order to check point *a)*, we apply Theorem 4.3. In fact, due to the previous claims, u fulfills (4.20) with the energy $\bar W$ (6.28).

Recalling Lemma 6.1, it is easy to check that also *b)* holds. In fact Claims 7 and 8 yield

$$u(t) \in [\mathfrak{p}_{\mathsf{l}}^{\bar u}(\ell(t) - \delta_+), \mathfrak{p}_{\mathsf{r}}^{\bar u}(\ell(t) - \delta_+)] = \mathfrak{p}^{\bar u}(\ell(t) - \delta_+) \quad \text{for every } t \in (a,b]. \tag{6.42}$$

Identities (6.41) and (6.20) show that $u_{\mathsf{r}}(t), u_{\mathsf{l}}(t) \in \mathfrak{C}^{\bar u}$ for $t > a$. The jump condition (5.8) also yields that $u(t) \in \mathfrak{C}^{\bar u}$, and we conclude by definition (6.15). □

Remark 6.4. In view of Proposition 2.5, a characterization of energetic solutions analogous to the one provided in Thm. 6.3 could be given in the case of a decreasing loading as well, cf. also Corollary 4.6.

Finally, we conclude with an example, to be compared with Ex. 4.7, which illustrates energetic evolution in the case of a *nonconvex* double-well potential energy.

Example 6.5 (Energetic solutions driven by a double-well energy)**.** We consider the very same setting as in Example 4.7, viz. the double-well energy

$$W(x) = \frac{1}{4}(x^2-1)^2 \quad \text{for all } x \in \mathbb{R},$$

and a strictly increasing loading ℓ on $[0,T]$. We start from an initial datum $\bar u \in (u_*, -1)$ with u_* as in (4.38). In order to illustrate the input-output relation on the interval $[0, t_1]$, we consider the convexification of W on $[\bar u, +\infty)$, viz.

$$W^{**}(u;\bar u) = \begin{cases} \frac14 (u^2-1)^2 & \text{if } u \le -1, \\ 0 & \text{if } -1 \le u \le 1, \\ \frac14 (u^2-1)^2 & \text{if } u \ge 1. \end{cases} \qquad \text{with} \qquad \frac{\mathrm{d}}{\mathrm{d}u} W^{**}(u;\bar u) = \begin{cases} u^3 - u & \text{if } u \le -1, \\ 0 & \text{if } -1 \le u \le 1, \\ u^3 - u & \text{if } u \ge 1. \end{cases}$$

Therefore, taking into account the explicit form of $\frac{\mathrm{d}}{\mathrm{d}u} W^{**}(\cdot;\bar u)^{-1}$ in terms of the functions S_i, $i = 1,3$, in (4.39), we find the energetic solution

$$u(t) \begin{cases} = S_1(\ell(t) - \delta_+) & \text{if } \ell(t) < \delta_+, \\ \in [-1,1] & \text{if } \ell(t) = \delta_+, \\ = S_3(\ell(t) - \delta_+) & \text{if } \ell(t) > \delta_+ \end{cases} \quad \text{for all } t \in [0,T].$$

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