

ENERGY CONSERVATION FOR INHOMOGENEOUS INCOMPRESSIBLE AND COMPRESSIBLE EULER EQUATIONS

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ABSTRACT. We study the conservation of energy for inhomogeneous incompressible and compressible Euler equations in a torus or a bounded domain. We provided sufficient conditions for a weak solution to conserve the energy. The spatial regularity for the density is only required to have the order of $2/3$ and when the density is constant, we recover the existing results for classical incompressible Euler equation.

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1. INTRODUCTION AND MAIN RESULTS

Let Ω be either \mathbb{T}^d or a bounded and connected domain in \mathbb{R}^d with C^2 boundary $\partial\Omega$, with $d \geq 2$. This paper studies the conservation of energy for weak solutions to *inhomogeneous incompressible*

Euler equation

$$\begin{cases} \partial_t \varrho + \nabla \cdot (\varrho \mathbf{u}) = 0, & \text{in } \Omega \times (0, T), \\ \partial_t (\varrho \mathbf{u}) + \nabla \cdot (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla P = 0, & \text{in } \Omega \times (0, T), \\ \nabla \cdot \mathbf{u} = 0, & \text{in } \Omega \times (0, T), \\ \mathbf{u}(x, t) \cdot n(x) = 0, & \text{on } \partial\Omega \times (0, T), \\ (\varrho \mathbf{u})(x, 0) = \varrho_0(x) \mathbf{u}_0(x), & \text{in } \Omega, \\ \varrho(x, 0) = \varrho_0(x), & \text{in } \Omega, \end{cases} \quad (\text{E})$$

as well as the *compressible isentropic Euler equation*

$$\begin{cases} \partial_t \varrho + \nabla \cdot (\varrho \mathbf{u}) = 0, & \text{in } \Omega \times (0, T), \\ \partial_t (\varrho \mathbf{u}) + \nabla \cdot (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla \varrho^\gamma = 0, & \text{in } \Omega \times (0, T), \\ \mathbf{u}(x, t) \cdot n(x) = 0, & \text{on } \partial\Omega \times (0, T), \\ (\varrho \mathbf{u})(x, 0) = \varrho_0(x) \mathbf{u}_0(x), & \text{in } \Omega, \\ \varrho(x, 0) = \varrho_0(x), & \text{in } \Omega, \end{cases} \quad (\text{Ec})$$

where $\gamma > 1$, $T > 0$, and $n(x)$ denotes the outward unit normal vector field to the boundary $\partial\Omega$. Note that the boundary condition $\mathbf{u} \cdot n = 0$ is neglected when $\Omega = \mathbb{T}^d$. In system (E), $\varrho : \Omega \times (0, T) \rightarrow \mathbb{R}_+$ is the scalar density of a fluid, $\mathbf{u} : \Omega \times (0, T) \rightarrow \mathbb{R}^d$ denotes its velocity and $P : \Omega \times (0, T) \rightarrow \mathbb{R}$ stands for the scalar pressure.

In his celebrated paper [Ons49] Onsager conjectured that there is dissipation of energy for homogeneous Euler equation (namely $\varrho \equiv 1$ in (E)) for weak solutions with low regularity. More precisely, if the weak solution is in C^α for $\alpha > 1/3$ then the energy is conserved while the energy is dissipated if $\alpha < 1/3$. The first landmark result concerning the loss (or gain) of energy is due to Scheffer [Sch93] in which he proved the existence of a weak solution having compact support both in time and space. This was later also recovered by Shnirelman [Shn97] for the torus \mathbb{T}^d . This direction of research has been greatly pushed forward by a series of works of De Lellis and Székelyhidi in e.g. [DS12, DS13, DS14, BDIS15]. The Onsager's conjecture has been recently settled by Isett in [Ise18a, Ise18]. The other direction, i.e. the conservation of energy, was first proved by Constant-E-Titi [CET94] for the torus \mathbb{T}^3 . The case of bounded domains is studied only recently in [BT18, BTW] in the context of Hölder spaces and in [DN18, NN18] in the context of Besov spaces.

Much less works concerning the inhomogeneous incompressible Euler equation (E) and the compressible equation (Ec) have been published in the literature and so far only the case of a bounded domain with periodic boundary condition, namely \mathbb{T}^d , has been treated (see the recent papers [LS16, FGGW17, CY]). More precisely, in [FGGW17], Feireisl *et al.* provided sufficient conditions in terms of Besov regularity both in time and space of the density ϱ , the velocity \mathbf{u} and the momentum $\mathbf{m} = \varrho \mathbf{u}$ to guarantee the conservation of the energy. Their method relies on the idea in [CET94] and requires also regularity conditions on the pressure. Recently, by using a different approach, Chen and Yu [CY] obtained the energy conservation, under a different set of regularity conditions, without any integrability assumption on the pressure. It is worth noting that the aforementioned papers only deal with the case of the torus \mathbb{T}^d .

In this paper, we provide modest sufficient conditions for a weak solution to conserve the energy for both inhomogeneous incompressible Euler equation (E) and compressible Euler equation (Ec). Our techniques are suitable to deal with both the case of a torus or a bounded domain. We need first the definition of a weak solution for (E) (a weak solution for (Ec) can be defined in the same way so we omit it here).

Definition 1.1. A triple (ϱ, \mathbf{u}, P) is called a weak solution to (E) if

(i)

$$\int_0^T \int_{\Omega} (\varrho \partial_t \varphi + \varrho \mathbf{u} \nabla \varphi) dx dt = 0$$

for every test function $\varphi \in C_0^\infty(\Omega \times (0, T))$.

(ii)

$$\int_0^T \int_{\Omega} (\varrho \mathbf{u} \cdot \partial_t \psi + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla \psi + P \nabla \cdot \psi) dx dt = 0$$

for every test vector field $\psi \in C_0^\infty(\Omega \times (0, T))^d$.

(iii) $\varrho(\cdot, t) \rightharpoonup \varrho_0$ in $\mathcal{D}'(\Omega)$ as $t \rightarrow 0$, i.e.

$$\lim_{t \rightarrow 0} \int_{\Omega} \varrho(x, t) \varphi(x) dx = \int_{\Omega} \varrho_0(x) \varphi(x) dx \quad (1)$$

for every test function $\varphi \in C_0^\infty(\Omega)$.

(iv) $(\varrho \mathbf{u})(\cdot, t) \rightharpoonup \varrho_0 \mathbf{u}_0$ in $\mathcal{D}'(\Omega)$ as $t \rightarrow 0$, i.e.

$$\lim_{t \rightarrow 0} \int_{\Omega} (\varrho \mathbf{u})(x, t) \psi(x) dx = \int_{\Omega} (\varrho_0 \mathbf{u}_0)(x) \psi(x) dx \quad (2)$$

for every test vector field $\psi \in C_0^\infty(\Omega)^d$.

To state the main results, we introduce, for $\beta > 0$, $\delta > 0$ and $p \geq 1$, in the case of the torus \mathbb{T}^d the quantity¹

$$\|f\|_{\mathcal{V}_{\delta}^{\beta, p}(\mathbb{T}^d)} := \sup_{|h| < \delta} |h|^{-\beta} \|f(\cdot + h) - f(\cdot)\|_{L^p(\mathbb{T}^d)},$$

and in the case of a bounded domain Ω the quantity

$$\|f\|_{\mathcal{V}_{\delta}^{\beta, p}(K)} := \sup_{|h| < \delta} |h|^{-\beta} \|f(\cdot + h) - f(\cdot)\|_{L^p(K)},$$

for any $K \subset\subset \Omega$ with $d(K, \partial\Omega) > 2\delta$. For $T > 0$ we denote by

$$\|f\|_{L^q(0, T; \mathcal{V}_{\delta}^{\beta, p}(\mathbb{T}^d))} = \left(\int_0^T \|f(t)\|_{\mathcal{V}_{\delta}^{\beta, p}(\mathbb{T}^d)}^q dt \right)^{1/q}, \quad \text{for } 1 \leq q < \infty,$$

$$\|f\|_{L^\infty(0, T; \mathcal{V}_{\delta}^{\beta, p}(\mathbb{T}^d))} = \operatorname{ess\,sup}_{t \in (0, T)} \|f(t)\|_{\mathcal{V}_{\delta}^{\beta, p}(\mathbb{T}^d)},$$

and similarly for $\|f\|_{L^q(0, T; \mathcal{V}_{\delta}^{\beta, p}(K))}$ with $K \subset\subset \Omega$.

Our main results read as follows.

Theorem 1.1 (Conservation of energy for (E) in a torus). *Let (ϱ, \mathbf{u}, P) be a weak solution to (E) in the case $\Omega = \mathbb{T}^d$ and assume that*

$$0 < \varrho, \varrho^{-1} \in L^\infty(\mathbb{T}^d \times (0, T)), \quad \mathbf{u} \in L^3(\mathbb{T}^d \times (0, T)), \quad P \in L^{\frac{3}{2}}(\mathbb{T}^d \times (0, T)), \quad (3)$$

$$\|\varrho\|_{L^\infty(0, T; \mathcal{V}_{\delta_0}^{\frac{2}{3}, \infty}(\mathbb{T}^d))} + \|\mathbf{u}\|_{L^3(0, T; \mathcal{V}_{\delta_0}^{\frac{1}{3}, 3}(\mathbb{T}^d))} < \infty, \quad \limsup_{\delta \rightarrow 0} \|\mathbf{u}\|_{L^3(0, T; \mathcal{V}_{\delta}^{\frac{1}{3}, 3}(\mathbb{T}^d))} = 0, \quad (4)$$

for some $\delta_0 > 0$. Then the energy for (E) conserves for all time, i.e.

$$\int_{\Omega} (\varrho |\mathbf{u}|^2)(x, t) dx = \int_{\Omega} (\varrho_0 |\mathbf{u}_0|^2)(x) dx \quad \forall t \in (0, T). \quad (5)$$

¹Clearly here is a slight abuse of notation since the definition is not a norm but only a seminorm.

Remark 1.2. Our results in Theorem 1.1 improve that of [CY] where the authors assumed in particular $\varrho \in L^p(0, T; W^{1,p}(\mathbb{T}^d))$ and $\mathbf{u} \in L^q(0, T; B_q^{\alpha,\infty}(\mathbb{T}^d))$ with $\frac{1}{p} + \frac{3}{q} \leq 1$ and $\alpha > \frac{1}{3}$. Here we are able to reduce the spatial regularity of the density ϱ to the order $\frac{2}{3}$ and keep the original order $\frac{1}{3}$ of the velocity. When $\|\mathbf{u}\|_{L^3(0,T;\mathcal{V}_\delta^{\alpha,3}(\mathbb{T}^d))} < \infty$ for some $\alpha > \frac{1}{3}$, the limit condition in (4) is automatically satisfied. When $\varrho \equiv \text{const}$ we recover (and slightly improved) the classical result for homogeneous Euler equation (see e.g. [CET94]).

It's remarked that [CY] also studies the case where ϱ only belongs to $L^\infty((0, T) \times \mathbb{T}^d)$, but as a consequence they need to compensate that by assuming Besov regularity for the velocity both in time and space $\mathbf{u} \in B_p^{\beta,\infty}(0, T; B_q^{\alpha,\infty}(\mathbb{T}^d))$ with $2\alpha + \beta > 1$ and $\alpha + 2\beta > 1$.

When Ω is a bounded, connected domain with smooth boundary, we need additionally some behavior of \mathbf{u} and P near the boundary. In the sequel, $\Omega_r := \{x \in \Omega : d(x, \partial\Omega) > r\}$ for any $r \geq 0$ and $\int_E f dx := \frac{1}{\mathcal{L}^d(E)} \int_E f dx$ for any Borel set $E \subset \mathbb{R}^d$. Since Ω is a bounded, connected domain with C^2 boundary, we find $r_0 > 0$ and a unique C_b^1 -vector function $n : \Omega \setminus \Omega_{r_0} \rightarrow S^{d-1}$ such that the following holds true: for any $r \in [0, r_0)$, $x \in \Omega_r \setminus \Omega_{r_0}$ there exists a unique $x_r \in \partial\Omega_r$ such that $d(x, \partial\Omega_r) = |x - x_r|$ and $n(x)$ is the outward unit normal vector field to the boundary $\partial\Omega_r$ at x_r .

Theorem 1.3 (Conservation of energy for (E) in a bounded domain). *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with C^2 boundary $\partial\Omega$. Let (ϱ, \mathbf{u}, P) be a weak solution to (E). Assume that*

$$0 < \varrho, \varrho^{-1} \in L^\infty(\Omega \times (0, T)), \quad \mathbf{u} \in L^3(\Omega \times (0, T)), \quad P \in L^{\frac{3}{2}}(\Omega \times (0, T)), \quad (6)$$

$$\|\varrho\|_{L^\infty(0,T;\mathcal{V}_\delta^{\frac{2}{3},\infty}(\Omega_{2\delta}))} + \|\mathbf{u}\|_{L^3(0,T;\mathcal{V}_\delta^{\frac{1}{3},3}(\Omega_{2\delta}))} < \infty, \quad \limsup_{\varepsilon \rightarrow 0} \|\mathbf{u}\|_{L^3(0,T;\mathcal{V}_\varepsilon^{\frac{1}{3},3}(\Omega_\delta))} = 0 \quad \forall \delta > 0, \quad (7)$$

and

$$\left(\int_0^T \int_{\Omega \setminus \Omega_\varepsilon} |\mathbf{u}(x, t)|^3 dx dt \right)^{\frac{2}{3}} \left(\int_0^T \int_{\Omega \setminus \Omega_\varepsilon} |\mathbf{u}(x, t) \cdot n(x)|^3 dx dt \right)^{\frac{1}{3}} = o(1) \quad \text{as } \varepsilon \rightarrow 0, \quad (8)$$

$$\left(\int_0^T \int_{\Omega \setminus \Omega_\varepsilon} |P(x, t)|^{\frac{3}{2}} dx dt \right)^{2/3} \left(\int_0^T \int_{\Omega \setminus \Omega_\varepsilon} |\mathbf{u}(x, t) \cdot n(x)|^3 dx dt \right)^{\frac{1}{3}} = o(1) \quad \text{as } \varepsilon \rightarrow 0. \quad (9)$$

Then the energy for (E) conserves for all time, i.e.

$$\int_\Omega (\varrho |\mathbf{u}|^2)(x, t) dx = \int_\Omega (\varrho_0 |\mathbf{u}_0|^2)(x) dx \quad \forall t \in (0, T). \quad (10)$$

Remark 1.4.

- Note that for the case of bounded domain, we only require the density and velocity belong locally to a Besov space.
- When $\varrho \equiv \text{const}$ we recover (and slightly improved) the recent results in [NN18] with the remark that the integrability of the pressure P in (6) is only needed in the inhomogeneous incompressible case (or compressible case). See Remark 2.1.
- Conditions (8) and (9) can be replaced by the following conditions

$$\limsup_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega \setminus \Omega_\varepsilon} |\mathbf{u}(x, t)|^3 dx dt + \limsup_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega \setminus \Omega_\varepsilon} |P(x, t)|^{\frac{3}{2}} dx dt < \infty$$

and

$$\liminf_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega \setminus \Omega_\varepsilon} |\mathbf{u}(x, t) \cdot n(x)|^3 dx dt = 0.$$

- Put $\tilde{\mathbf{u}}(x, t) = \mathbf{u}(x, t) \cdot n(x)$ for any $x \in \Omega \setminus \Omega_{r_0}$. If the function

$$h : \varepsilon \mapsto h(\varepsilon) = \|\tilde{\mathbf{u}}\|_{L^3((\Omega \setminus \Omega_\varepsilon) \times (0, T))}$$

satisfies $h(\varepsilon) \leq C\varepsilon^{2/3}$ for every $\varepsilon \in (0, r_0)$ with some $C > 0$, then conditions (8) and (9) are fulfilled.

We now turn to the energy conservation for the isentropic compressible Euler equation (Ec) (with $\gamma > 1$). Put

$$\alpha := \frac{2}{3 \min\{\gamma, 2\}}.$$

Theorem 1.5 (Conservation of energy for (Ec) in a torus). *Let (ϱ, \mathbf{u}, P) be a weak solution to (Ec) in the case $\Omega = \mathbb{T}^d$. Assume that*

$$0 < \varrho, \varrho^{-1} \in L^\infty(\mathbb{T}^d \times (0, T)), \quad \mathbf{u} \in L^3(\mathbb{T}^d \times (0, T)), \quad (11)$$

$$\|\varrho\|_{L^\infty(0, T; \mathcal{V}_{\delta_0}^{\alpha, \infty}(\mathbb{T}^d))} + \|\mathbf{u}\|_{L^3(0, T; \mathcal{V}_{\delta_0}^{\frac{1}{3}, 3}(\mathbb{T}^d))} < \infty, \quad \limsup_{\delta \rightarrow 0} \|\mathbf{u}\|_{L^3(0, T; \mathcal{V}_\delta^{\frac{1}{3}, 3}(\mathbb{T}^d))} = 0, \quad (12)$$

for some $\delta_0 > 0$, and

$$\limsup_{\delta \rightarrow 0} \|\varrho\|_{L^\infty(0, T; \mathcal{V}_\delta^{\alpha, \infty}(\mathbb{T}^d))} = 0 \quad \text{if } \gamma \geq 2. \quad (13)$$

Then the energy for (Ec) conserves for all time, i.e.

$$\int_{\mathbb{T}^d} \left(\frac{1}{2}(\varrho|\mathbf{u}|^2)(x, t) + \frac{\varrho(x, t)^\gamma}{\gamma - 1} \right) dx = \int_{\mathbb{T}^d} \left(\frac{1}{2}(\varrho_0|\mathbf{u}_0|^2)(x) + \frac{\varrho_0(x)^\gamma}{\gamma - 1} \right) dx \quad \forall t > 0. \quad (14)$$

Remark 1.6. We remark that the spatial regularity of the density $\alpha \geq 1/3$ and the equality holds iff $\gamma \geq 2$, and in that case we need the asymptotic behavior (13). If $\gamma < 2$, and consequently $\alpha > 1/3$, then the condition (13) is automatically satisfied.

Theorem 1.7 (Conservation of energy for (Ec) in a bounded domain). *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with C^2 boundary $\partial\Omega$. Let (ϱ, \mathbf{u}, P) be a weak solution to (Ec). Assume that*

$$0 < \varrho, \varrho^{-1} \in L^\infty(\Omega \times (0, T)), \quad \mathbf{u} \in L^3(\Omega \times (0, T)), \quad (15)$$

$$\|\varrho\|_{L^\infty(0, T; \mathcal{V}_\delta^{\alpha, \infty}(\Omega_{2\delta}))} + \|\mathbf{u}\|_{L^3(0, T; \mathcal{V}_\delta^{\frac{1}{3}, 3}(\Omega_{2\delta}))} < \infty, \quad \limsup_{\varepsilon \rightarrow 0} \|\mathbf{u}\|_{L^3(0, T; \mathcal{V}_\varepsilon^{\frac{1}{3}, 3}(\Omega_\delta))} = 0 \quad \forall \delta > 0, \quad (16)$$

$$\limsup_{\varepsilon \rightarrow 0} \|\varrho\|_{L^\infty(0, T; \mathcal{V}_\varepsilon^{\alpha, \infty}(\Omega_{2\delta}))} = 0 \quad \forall \delta > 0, \quad \text{if } \gamma \geq 2, \quad (17)$$

$$\left(\int_0^T \int_{\Omega \setminus \Omega_\varepsilon} |\mathbf{u}(x, t)|^3 dx dt \right)^{\frac{2}{3}} \left(\int_0^T \int_{\Omega \setminus \Omega_\varepsilon} |\mathbf{u}(x, t) \cdot n(x)|^3 dx dt \right)^{\frac{1}{3}} = o(1) \quad \text{as } \varepsilon \rightarrow 0, \quad (18)$$

$$\int_0^T \int_{\Omega \setminus \Omega_\varepsilon} |\mathbf{u}(x, t) \cdot n(x)| dx dt = o(1) \quad \text{as } \varepsilon \rightarrow 0. \quad (19)$$

Then the energy for (E) conserves for all time, i.e.

$$\int_\Omega \left(\frac{1}{2}(\varrho|\mathbf{u}|^2)(x, t) + \frac{\varrho(x, t)^\gamma}{\gamma - 1} \right) dx = \int_\Omega \left(\frac{1}{2}(\varrho_0|\mathbf{u}_0|^2)(x) + \frac{\varrho_0(x)^\gamma}{\gamma - 1} \right) dx \quad \forall t \in (0, T). \quad (20)$$

The paper is organized as follows: The proofs of Theorems 1.1, 1.3, 1.5 and 1.7 will be presented in the next four Sections respectively. In the Appendix, we collect some useful estimates which will be used in the proofs.

Notation:

- For convenience, we simply write $\|f\|_{L^p}$ for $\|f\|_{L^p((0,T)\times\Omega)}$ with $\Omega = \mathbb{T}^d$ or a bounded domain and $1 \leq p \leq \infty$.
- We denote by C a generic constant, whose value can change from line to line or even the same line. Sometimes we write $C(\lambda)$ to emphasize the dependence on $\lambda > 0$.

2. PROOF OF THEOREM 1.1

In this section we write \mathbb{T}^d instead of Ω . By smoothing (E), we obtain

$$\partial_t \varrho^\varepsilon + \nabla \cdot (\varrho \mathbf{u})^\varepsilon = 0 \quad (21)$$

and

$$\partial_t (\varrho \mathbf{u})^\varepsilon + \nabla \cdot (\varrho \mathbf{u} \otimes \mathbf{u})^\varepsilon + \nabla P^\varepsilon = 0. \quad (22)$$

for any $0 < \varepsilon < 1$.

Multiplying (22) by $(\varrho^\varepsilon)^{-1}(\varrho \mathbf{u})^\varepsilon$ then integrating on $(\tau, t) \times \mathbb{T}^d$, for $0 < \tau < t < T$, we get

$$\int_\tau^t \int_{\mathbb{T}^d} \frac{1}{\varrho^\varepsilon} (\varrho \mathbf{u})^\varepsilon \partial_t (\varrho \mathbf{u})^\varepsilon dx ds + \int_\tau^t \int_{\mathbb{T}^d} \frac{1}{\varrho^\varepsilon} (\varrho \mathbf{u})^\varepsilon \nabla \cdot (\varrho \mathbf{u} \otimes \mathbf{u})^\varepsilon dx ds + \int_\tau^t \int_{\mathbb{T}^d} \frac{1}{\varrho^\varepsilon} (\varrho \mathbf{u})^\varepsilon \nabla P^\varepsilon dx ds = 0. \quad (23)$$

Denote by (A), (B) and (C) the terms on the left-hand side of (23) respectively. We will estimate their convergence separately. Let M be a constant such that

$$\|\varrho\|_{L^\infty} + \|\varrho^{-1}\|_{L^\infty} + \|\varrho\|_{L^\infty(0,T;\mathcal{V}_{\delta_0}^{\frac{1}{3},\infty}(\mathbb{T}^d))} \leq M, \text{ for some } \delta_0 > 0.$$

2.1. Estimate of (A). We rewrite (A) as

$$\begin{aligned} (A) &= \frac{1}{2} \int_\tau^t \int_{\mathbb{T}^d} \partial_t \left(\frac{|(\varrho \mathbf{u})^\varepsilon|^2}{\varrho^\varepsilon} \right) dx ds - \frac{1}{2} \int_\tau^t \int_{\mathbb{T}^d} \frac{1}{(\varrho^\varepsilon)^2} \nabla \cdot [(\varrho \mathbf{u})^\varepsilon - \varrho^\varepsilon \mathbf{u}^\varepsilon] |(\varrho \mathbf{u})^\varepsilon|^2 dx ds \\ &\quad - \frac{1}{2} \int_\tau^t \int_{\mathbb{T}^d} \frac{1}{(\varrho^\varepsilon)^2} \nabla \cdot (\varrho^\varepsilon \mathbf{u}^\varepsilon) |(\varrho \mathbf{u})^\varepsilon|^2 dx ds \\ &=: (A1) + (A2) + (A3). \end{aligned} \quad (24)$$

The term (A3) will be cancelled with the term (B2) when estimating (B). We will study the limit of (A2) since (A1) is the desired term. By integration by parts and Hölder's inequality,

$$|(A2)| \leq \frac{1}{2} \int_0^T \left\| \nabla \cdot \left(\frac{|(\varrho \mathbf{u})^\varepsilon|^2}{(\varrho^\varepsilon)^2} \right) (s) \right\|_{L^{\frac{3}{2}}(\mathbb{T}^d)} \|((\varrho \mathbf{u})^\varepsilon - \varrho^\varepsilon \mathbf{u}^\varepsilon)(s)\|_{L^3(\mathbb{T}^d)} ds.$$

For any $\delta \in (\varepsilon, \delta_0)$, by Lemma A.2, we estimate

$$\|((\varrho \mathbf{u})^\varepsilon - \varrho^\varepsilon \mathbf{u}^\varepsilon)(s)\|_{L^3(\mathbb{T}^d)} \leq C(M) \varepsilon^{\frac{2}{3}} \|\mathbf{u}(s)\|_{\mathcal{V}_{\delta}^{\frac{1}{3},3}(\mathbb{T}^d)},$$

and

$$\left\| \nabla \cdot \left(\frac{|(\varrho \mathbf{u})^\varepsilon|^2}{(\varrho^\varepsilon)^2} \right) (s) \right\|_{L^{\frac{3}{2}}(\mathbb{T}^d)} \leq C(M) \varepsilon^{-\frac{2}{3}} \left(\|\mathbf{u}(s)\|_{L^3(\mathbb{T}^d)}^2 + \|\mathbf{u}(s)\|_{L^3(\mathbb{T}^d)} \|\mathbf{u}(s)\|_{\mathcal{V}_{\delta}^{\frac{1}{3},3}(\mathbb{T}^d)} \right).$$

Thus,

$$|(A2)| \leq C(M) \|\mathbf{u}\|_{L^3(0,T;\mathcal{V}_{\delta}^{\frac{1}{3},3}(\mathbb{T}^d))} \left(\|\mathbf{u}\|_{L^3}^2 + \|\mathbf{u}\|_{L^3} \|\mathbf{u}\|_{L^3(0,T;\mathcal{V}_{\delta}^{\frac{1}{3},3}(\mathbb{T}^d))} \right).$$

Therefore, by assumption (4),

$$\limsup_{\delta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \limsup_{\tau \rightarrow 0} |(A2)| = 0. \quad (25)$$

2.2. Estimate of (B) . By integration by parts we have

$$(B) = - \int_{\tau}^t \int_{\mathbb{T}^d} [(\varrho \mathbf{u} \otimes \mathbf{u})^{\varepsilon} - (\varrho \mathbf{u})^{\varepsilon} \otimes \mathbf{u}^{\varepsilon}] \nabla \cdot \frac{(\varrho \mathbf{u})^{\varepsilon}}{\varrho^{\varepsilon}} dx ds - \int_{\tau}^t \int_{\mathbb{T}^d} (\varrho \mathbf{u})^{\varepsilon} \otimes \mathbf{u}^{\varepsilon} \nabla \cdot \frac{(\varrho \mathbf{u})^{\varepsilon}}{\varrho^{\varepsilon}} dx ds \quad (26)$$

$$=: (B1) + (B2).$$

It can be checked by using Lemmas A.2 that

$$\| [(\varrho \mathbf{u} \otimes \mathbf{u})^{\varepsilon} - (\varrho \mathbf{u})^{\varepsilon} \otimes \mathbf{u}^{\varepsilon}] (s) \|_{L^{\frac{3}{2}}(\mathbb{T}^d)} \leq C(M) \varepsilon^{\frac{2}{3}} \left(\|\mathbf{u}(s)\|_{\mathcal{V}_{\delta}^{\frac{1}{3},3}(\mathbb{T}^d)} + \|\mathbf{u}(s)\|_{L^3(\mathbb{T}^d)} \right) \|\mathbf{u}(s)\|_{\mathcal{V}_{\delta}^{\frac{1}{3},3}(\mathbb{T}^d)}$$

and

$$\left\| \nabla \cdot \frac{(\varrho \mathbf{u})^{\varepsilon}}{\varrho^{\varepsilon}} (s) \right\|_{L^3(\mathbb{T}^d)} \leq C(M) \varepsilon^{-\frac{2}{3}} \left(\|\mathbf{u}(s)\|_{L^3(\mathbb{T}^d)} + \|\mathbf{u}(s)\|_{\mathcal{V}_{\delta}^{\frac{1}{3},3}(\mathbb{T}^d)} \right). \quad (27)$$

Thus, by Hölder's inequality,

$$\begin{aligned} |(B1)| &\leq C \int_0^T \|(\varrho \mathbf{u} \otimes \mathbf{u})^{\varepsilon} - (\varrho \mathbf{u})^{\varepsilon} \otimes \mathbf{u}^{\varepsilon}\|_{L^{\frac{3}{2}}(\mathbb{T}^d)} \left\| \nabla \cdot \frac{(\varrho \mathbf{u})^{\varepsilon}}{\varrho^{\varepsilon}} (s) \right\|_{L^3(\mathbb{T}^d)} ds \\ &\leq C(M) \left(\|\mathbf{u}\|_{L^3}^2 + \|\mathbf{u}\|_{L^3(0,T;\mathcal{V}_{\delta}^{\frac{1}{3},3}(\mathbb{T}^d))}^2 \right) \|\mathbf{u}\|_{L^3(0,T;\mathcal{V}_{\delta}^{\frac{1}{3},3}(\mathbb{T}^d))}. \end{aligned} \quad (28)$$

Therefore, by (4),

$$\limsup_{\delta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \limsup_{\tau \rightarrow 0} |(B1)| = 0. \quad (29)$$

2.3. Estimate $(A3) + (B2) = 0$. It remains to estimate the terms $(A3)$ and $(B2)$. We will prove here that they in fact cancel each other. Indeed, using integration by parts we have

$$\begin{aligned} (B2) &= \int_{\tau}^t \int_{\mathbb{T}^d} \frac{(\varrho \mathbf{u})^{\varepsilon}}{\varrho^{\varepsilon}} \nabla \cdot [(\varrho \mathbf{u})^{\varepsilon} \otimes \mathbf{u}^{\varepsilon}] dx ds = \int_{\tau}^t \int_{\mathbb{T}^d} \frac{(\varrho \mathbf{u})^{\varepsilon}}{\varrho^{\varepsilon}} [\nabla(\varrho \mathbf{u})^{\varepsilon} \mathbf{u}^{\varepsilon} + (\varrho \mathbf{u})^{\varepsilon} \nabla \mathbf{u}^{\varepsilon}] dx ds \\ &= -\frac{1}{2} \int_{\tau}^t \int_{\mathbb{T}^d} |(\varrho \mathbf{u})^{\varepsilon}|^2 \nabla \cdot \frac{\mathbf{u}^{\varepsilon}}{\varrho^{\varepsilon}} dx ds + \int_{\tau}^t \int_{\mathbb{T}^d} \frac{|(\varrho \mathbf{u})^{\varepsilon}|^2}{\varrho^{\varepsilon}} \nabla \cdot \mathbf{u}^{\varepsilon} dx ds \\ &= \int_{\tau}^t \int_{\mathbb{T}^d} |(\varrho \mathbf{u})^{\varepsilon}|^2 \left[\frac{1}{\varrho^{\varepsilon}} \nabla \cdot \mathbf{u}^{\varepsilon} - \frac{1}{2} \nabla \cdot \frac{\mathbf{u}^{\varepsilon}}{\varrho^{\varepsilon}} \right] dx ds \\ &= \frac{1}{2} \int_{\tau}^t \int_{\mathbb{T}^d} \frac{|(\varrho \mathbf{u})^{\varepsilon}|^2}{(\varrho^{\varepsilon})^2} \nabla(\varrho^{\varepsilon} \mathbf{u}^{\varepsilon}) dx ds = -(A3). \end{aligned} \quad (30)$$

2.4. Estimate of (C) . Using the divergence free condition², we estimate

$$(C) = \int_{\tau}^t \int_{\mathbb{T}^d} \frac{1}{\varrho^{\varepsilon}} ((\varrho \mathbf{u})^{\varepsilon} - \varrho^{\varepsilon} \mathbf{u}^{\varepsilon}) \nabla P^{\varepsilon} dx ds. \quad (31)$$

From Hölder's inequality and Lemma A.2 we get

$$\begin{aligned} |(C)| &\leq C(\|\varrho^{-1}\|_{L^{\infty}}) \int_0^T \|(\varrho \mathbf{u})^{\varepsilon} - \varrho^{\varepsilon} \mathbf{u}^{\varepsilon}\|_{L^3(\mathbb{T}^d)} \|\nabla P^{\varepsilon}\|_{L^{\frac{3}{2}}(\mathbb{T}^d)} ds \\ &\leq C(\|\varrho^{-1}\|_{L^{\infty}}) \int_0^T \varepsilon^{\frac{2}{3}} \|\varrho\|_{\mathcal{V}_{\delta}^{\frac{2}{3},\infty}(\mathbb{T}^d)} \varepsilon^{\frac{1}{3}} \|\mathbf{u}\|_{\mathcal{V}_{\delta}^{\frac{1}{3},3}(\mathbb{T}^d)} \varepsilon^{-1} \|P\|_{L^{\frac{3}{2}}(\mathbb{T}^d)} ds \\ &\leq C(\|\varrho^{-1}\|_{L^{\infty}}) \|\varrho\|_{L^{\infty}(0,T;\mathcal{V}_{\delta}^{\frac{2}{3},\infty}(\mathbb{T}^d))} \|\mathbf{u}\|_{L^3(0,T;\mathcal{V}_{\delta}^{\frac{1}{3},3}(\mathbb{T}^d))} \|P\|_{L^{\frac{3}{2}}}. \end{aligned}$$

²We observe that the divergence free is only used here to treat the term involving the pressure. This observation will become helpful when treating compressible Euler equation in the next sections.

By the assumption (4) we have

$$\limsup_{\delta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \limsup_{\tau \rightarrow 0} |(C)| = 0.$$

Remark 2.1. We remark here that the integrability of the pressure P in (3) is only needed in the inhomogeneous or compressible equations since obviously if $\varrho \equiv \text{const}$ then $(C) \equiv 0$.

2.5. Conclusion. From the previous estimates we have,

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{\tau \rightarrow 0} \left| \int_{\tau}^t \int_{\mathbb{T}^d} \partial_t \left[\frac{1}{\varrho^\varepsilon} |(\varrho \mathbf{u})^\varepsilon|^2 \right] dx ds \right| = 0.$$

Thanks to (21) and (22), one has $\partial_t(\varrho \mathbf{u})^\varepsilon, \partial_t \varrho^\varepsilon \in L^{\frac{3}{2}}(0, T, L^\infty(\mathbb{T}^d))$. Thus,

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{\tau \rightarrow 0} \left| \int_{\mathbb{T}^d} \frac{1}{\varrho^\varepsilon} |(\varrho \mathbf{u})^\varepsilon|^2(x, t) dx - \int_{\mathbb{T}^d} \frac{1}{\varrho^\varepsilon} |(\varrho \mathbf{u})^\varepsilon|^2(x, \tau) dx \right| = 0.$$

By (1) and (2), $\varrho(\cdot, \tau) \rightharpoonup \varrho_0$ and $(\varrho \mathbf{u})(\cdot, \tau) \rightharpoonup \varrho_0 \mathbf{u}_0$ in $\mathcal{D}'(\Omega)$, so for every $x \in \mathbb{T}^d$,

$$\varrho^\varepsilon(x, \tau) \rightarrow \varrho_0^\varepsilon(x), \quad (\varrho \mathbf{u})^\varepsilon(x, \tau) \rightarrow (\varrho_0 \mathbf{u}_0)^\varepsilon(x),$$

as $\tau \rightarrow 0$. Thanks to dominated convergence theorem, it yields

$$\limsup_{\varepsilon \rightarrow 0} \left| \int_{\mathbb{T}^d} \frac{1}{\varrho^\varepsilon} |(\varrho \mathbf{u})^\varepsilon|^2(x, t) dx - \int_{\mathbb{T}^d} \frac{1}{\varrho_0^\varepsilon} |(\varrho_0 \mathbf{u}_0)^\varepsilon|^2(x) dx \right| = 0.$$

Thus, by the standard property of convolution, we derive (5). The proof is complete. \square

3. PROOF OF THEOREM 1.3

The proof of Theorem 1.3 is similar to that of Theorem 1.1, except that we have to take care of the boundary layer when taking integration by parts. Recalling the smooth version of (E) as

$$\partial_t \varrho^\varepsilon + \nabla \cdot (\varrho \mathbf{u})^\varepsilon = 0 \quad \text{in} \quad \Omega_{2\varepsilon}, \quad (32)$$

and

$$\partial_t (\varrho \mathbf{u})^\varepsilon + \nabla \cdot (\varrho \mathbf{u} \otimes \mathbf{u})^\varepsilon + \nabla P^\varepsilon = 0 \quad \text{in} \quad \Omega_{2\varepsilon}. \quad (33)$$

Take $0 < \varepsilon < \varepsilon_1/10 < \varepsilon_2/10 < \delta_0/100$. Note that if $\varepsilon_2 \rightarrow 0$ then $\varepsilon, \varepsilon_1 \rightarrow 0$. Multiplying (33) by $(\varrho^\varepsilon)^{-1}(\varrho \mathbf{u})^\varepsilon$ then integrating on $(\tau, t) \times \Omega_{\varepsilon_2}$, with $0 < \tau < t < T$, yield

$$\int_{\tau}^t \int_{\Omega_{\varepsilon_2}} \frac{1}{\varrho^\varepsilon} (\varrho \mathbf{u})^\varepsilon \partial_t (\varrho \mathbf{u})^\varepsilon dx ds + \int_{\tau}^t \int_{\Omega_{\varepsilon_2}} \frac{1}{\varrho^\varepsilon} (\varrho \mathbf{u})^\varepsilon \nabla \cdot (\varrho \mathbf{u} \otimes \mathbf{u})^\varepsilon dx ds + \int_{\tau}^t \int_{\Omega_{\varepsilon_2}} \frac{1}{\varrho^\varepsilon} (\varrho \mathbf{u})^\varepsilon \nabla P^\varepsilon dx ds = 0. \quad (34)$$

For $\varepsilon_3 > 0$ small, we integrate (34) with respect to ε_2 on $(\varepsilon_1, \varepsilon_1 + \varepsilon_3)$ to get

$$\begin{aligned} \frac{1}{\varepsilon_3} \int_{\varepsilon_1}^{\varepsilon_1 + \varepsilon_3} \int_{\tau}^t \int_{\Omega_{\varepsilon_2}} \frac{1}{\varrho^\varepsilon} (\varrho \mathbf{u})^\varepsilon \partial_t (\varrho \mathbf{u})^\varepsilon dx ds d\varepsilon_2 &+ \frac{1}{\varepsilon_3} \int_{\varepsilon_1}^{\varepsilon_1 + \varepsilon_3} \int_{\tau}^t \int_{\Omega_{\varepsilon_2}} \frac{1}{\varrho^\varepsilon} (\varrho \mathbf{u})^\varepsilon \nabla \cdot (\varrho \mathbf{u} \otimes \mathbf{u})^\varepsilon dx ds d\varepsilon_2 \\ &+ \frac{1}{\varepsilon_3} \int_{\varepsilon_1}^{\varepsilon_1 + \varepsilon_3} \int_{\tau}^t \int_{\Omega_{\varepsilon_2}} \frac{1}{\varrho^\varepsilon} (\varrho \mathbf{u})^\varepsilon \nabla P^\varepsilon dx ds d\varepsilon_2 = 0. \end{aligned} \quad (35)$$

Denote by (D), (E) and (F) the three terms on the left hand side of (35), which will be estimated separately in the following subsections. Let M_{ε_1} be a constant such that

$$\|\varrho\|_{L^\infty} + \|\varrho^{-1}\|_{L^\infty} + \|\varrho\|_{L^\infty(0, T; \mathcal{V}_{\varepsilon_1/4}^{\frac{2}{3}, \infty}(\Omega_{\varepsilon_1/2}))} \leq M_{\varepsilon_1}.$$

It's worth mentioning that in the following, we let successively $\tau \rightarrow 0$, $\varepsilon \rightarrow 0$, then $\varepsilon_1 \rightarrow 0$. Therefore, at some estimates, after letting $\varepsilon \rightarrow 0$, constants depending on M_{ε_1} (usually denoted

by $C(M_{\varepsilon_1})$ vanishes for each $\varepsilon_1 > 0$, therefore we then will be able to send $\varepsilon_1 \rightarrow 0$ without encountering any issue.

3.1. Estimate of (D) . The term (D) is rewritten as

$$\begin{aligned}
 (D) &= \frac{1}{2} \frac{1}{\varepsilon_3} \int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3} \int_{\tau}^t \int_{\Omega_{\varepsilon_2}} \partial_t \left(\frac{|(\varrho \mathbf{u})^\varepsilon|^2}{\varrho^\varepsilon} \right) dx ds d\varepsilon_2 \\
 &\quad - \frac{1}{2} \frac{1}{\varepsilon_3} \int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3} \int_{\tau}^t \int_{\Omega_{\varepsilon_2}} \frac{1}{(\varrho^\varepsilon)^2} \nabla \cdot [(\varrho \mathbf{u})^\varepsilon - \varrho^\varepsilon \mathbf{u}^\varepsilon] |(\varrho \mathbf{u})^\varepsilon|^2 dx ds d\varepsilon_2 \\
 &\quad - \frac{1}{2} \frac{1}{\varepsilon_3} \int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3} \int_{\tau}^t \int_{\Omega_{\varepsilon_2}} \frac{1}{(\varrho^\varepsilon)^2} \nabla \cdot (\varrho^\varepsilon \mathbf{u}^\varepsilon) |(\varrho \mathbf{u})^\varepsilon|^2 dx ds d\varepsilon_2 \\
 &=: (D1) + (D2) + (D3).
 \end{aligned} \tag{36}$$

We will only estimate $(D2)$ since $(D3)$ will be estimated together with $(E3)$ later and $(D1)$ is a desired term. Using integration by parts we have

$$\begin{aligned}
 |(D2)| &\leq \frac{1}{2} \left| \frac{1}{\varepsilon_3} \int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3} \int_{\tau}^t \int_{\partial\Omega_{\varepsilon_2}} \frac{1}{(\varrho^\varepsilon)^2} |(\varrho \mathbf{u})^\varepsilon|^2 [(\varrho \mathbf{u})^\varepsilon - \varrho^\varepsilon \mathbf{u}^\varepsilon] \cdot n(\theta) d\mathcal{H}^{d-1}(\theta) ds d\varepsilon_2 \right| \\
 &\quad + \frac{1}{2} \left| \frac{1}{\varepsilon_3} \int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3} \int_{\tau}^t \int_{\Omega_{\varepsilon_2}} [(\varrho \mathbf{u})^\varepsilon - \varrho^\varepsilon \mathbf{u}^\varepsilon] \nabla \left(\frac{|(\varrho \mathbf{u})^\varepsilon|^2}{(\varrho^\varepsilon)^2} \right) dx ds d\varepsilon_2 \right| \\
 &=: (D21) + (D22).
 \end{aligned}$$

The term $(D22)$ can be estimated similarly to $(A2)$ as

$$\begin{aligned}
 (D22) &\leq \frac{1}{2} \int_{\tau}^t \int_{\Omega_{\varepsilon_1}} \left| [(\varrho \mathbf{u})^\varepsilon - \varrho^\varepsilon \mathbf{u}^\varepsilon] \nabla \left(\frac{|(\varrho \mathbf{u})^\varepsilon|^2}{(\varrho^\varepsilon)^2} \right) \right| dx ds \\
 &\leq C(M_{\varepsilon_1}) \|\mathbf{u}\|_{L^3(0,T;\mathcal{V}_{\varepsilon}^{\frac{1}{3},3}(\Omega_{\frac{\varepsilon_1}{2}}))} \left[\|\mathbf{u}\|_{L^3}^2 + \|\mathbf{u}\|_{L^3} \|\mathbf{u}\|_{L^3(0,T;\mathcal{V}_{\varepsilon}^{\frac{1}{3},3}(\Omega_{\frac{\varepsilon_1}{2}}))} \right].
 \end{aligned} \tag{37}$$

Therefore, by using (7),

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{\tau \rightarrow 0} (D22) = 0. \tag{38}$$

For $(D21)$ we use the coarea formula: for any $0 < r_1 < r_2 < 1$, and $g \in L^1(\Omega_{r_1} \setminus \Omega_{r_2})$,

$$\int_{\Omega_{r_1} \setminus \Omega_{r_2}} g(x) dx = \int_{r_1}^{r_2} \int_{\partial\Omega_\nu} g(\theta) d\mathcal{H}^{d-1}(\theta) d\nu, \tag{39}$$

and the fact that $\varepsilon_3 \approx \mathcal{L}^d(\Omega_{\varepsilon_1} \setminus \Omega_{\varepsilon_1+\varepsilon_3})$ to get

$$(D21) \leq C \left| \int_{\tau}^t \int_{\Omega_{\varepsilon_1} \setminus \Omega_{\varepsilon_1+\varepsilon_3}} \frac{1}{(\varrho^\varepsilon)^2} |(\varrho \mathbf{u})^\varepsilon|^2 [(\varrho \mathbf{u})^\varepsilon - \varrho^\varepsilon \mathbf{u}^\varepsilon] \cdot n(x) dx ds \right|.$$

Since $\varrho, \varrho^{-1} \in L^\infty(\Omega \times (0, T))$, $\mathbf{u} \in L^3(\Omega \times (0, T))$, we have

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{\tau \rightarrow 0} (D21) = 0. \tag{40}$$

3.2. **Estimate of (E) .** Using integration by parts we have

$$\begin{aligned}
(E) &= \frac{1}{\varepsilon_3} \int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3} \int_{\tau}^t \int_{\partial\Omega_{\varepsilon_2}} \frac{1}{\varrho^\varepsilon} (\varrho \mathbf{u})^\varepsilon (\varrho \mathbf{u} \otimes \mathbf{u})^\varepsilon n(\theta) d\mathcal{H}^{d-1}(\theta) ds d\varepsilon_2 \\
&\quad - \frac{1}{\varepsilon_3} \int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3} \int_{\tau}^t \int_{\Omega_{\varepsilon_2}} (\varrho \mathbf{u} \otimes \mathbf{u})^\varepsilon \nabla \cdot \frac{(\varrho \mathbf{u})^\varepsilon}{\varrho^\varepsilon} dx ds d\varepsilon_2 \\
&=: (E1) - \frac{1}{\varepsilon_3} \int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3} \int_{\tau}^t \int_{\Omega_{\varepsilon_2}} [(\varrho \mathbf{u} \otimes \mathbf{u})^\varepsilon - (\varrho \mathbf{u})^\varepsilon \otimes \mathbf{u}^\varepsilon] \nabla \cdot \frac{(\varrho \mathbf{u})^\varepsilon}{\varrho^\varepsilon} dx ds d\varepsilon_2 \\
&\quad - \frac{1}{\varepsilon_3} \int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3} \int_{\tau}^t \int_{\Omega_{\varepsilon_2}} (\varrho \mathbf{u})^\varepsilon \otimes \mathbf{u}^\varepsilon \nabla \cdot \frac{(\varrho \mathbf{u})^\varepsilon}{\varrho^\varepsilon} dx ds d\varepsilon_2 \\
&=: (E1) + (E2) + (E3).
\end{aligned} \tag{41}$$

By the coarea formula (39), we have

$$(E1) = \frac{1}{\varepsilon_3} \int_{\tau}^t \int_{\Omega_{\varepsilon_1} \setminus \Omega_{\varepsilon_1+\varepsilon_3}} \frac{1}{\varrho^\varepsilon} (\varrho \mathbf{u})^\varepsilon (\varrho \mathbf{u} \otimes \mathbf{u})^\varepsilon n(x) dx ds.$$

Therefore, letting successively $\tau \rightarrow 0$, $\varepsilon \rightarrow 0$ and $\varepsilon_1 \rightarrow 0$ and then using the fact $\varepsilon_3 \approx \mathcal{L}^d(\Omega \setminus \Omega_{\varepsilon_3})$ and Hölder's inequality, we obtain

$$\begin{aligned}
\limsup_{\varepsilon_1, \varepsilon \rightarrow 0} \limsup_{\tau \rightarrow 0} (E1) &\leq \frac{C(\|\varrho\|_{L^\infty}, \|\varrho^{-1}\|_{L^\infty})}{\varepsilon_3} \int_0^T \int_{\Omega \setminus \Omega_{\varepsilon_3}} |\mathbf{u}|^2 |\mathbf{u} \cdot n| dx ds \\
&\leq C(\|\varrho\|_{L^\infty}, \|\varrho^{-1}\|_{L^\infty}) \left(\int_0^T \int_{\Omega \setminus \Omega_{\varepsilon_3}} |\mathbf{u}|^3 dx ds \right)^{\frac{2}{3}} \left(\int_0^T \int_{\Omega \setminus \Omega_{\varepsilon_3}} |\mathbf{u} \cdot n|^3 dx ds \right)^{\frac{1}{3}}.
\end{aligned}$$

Thus, by assumption (8), we derive

$$\limsup_{\varepsilon_3 \rightarrow 0} \limsup_{\varepsilon_1, \varepsilon \rightarrow 0} \limsup_{\tau \rightarrow 0} |(E1)| = 0.$$

By proceeding as in estimating (B1) we derive

$$\begin{aligned}
|(E2)| &\leq C \int_{\tau}^t \int_{\Omega_{\varepsilon_1}} |(\varrho \mathbf{u} \otimes \mathbf{u})^\varepsilon - (\varrho \mathbf{u})^\varepsilon \otimes \mathbf{u}^\varepsilon| \left| \nabla \cdot \frac{(\varrho \mathbf{u})^\varepsilon}{\varrho^\varepsilon} \right| dx ds \\
&\leq C \int_0^T \|(\varrho \mathbf{u} \otimes \mathbf{u})^\varepsilon - (\varrho \mathbf{u})^\varepsilon \otimes \mathbf{u}^\varepsilon\|_{L^{\frac{3}{2}}(\Omega_{\varepsilon_1})} \left\| \nabla \cdot \frac{(\varrho \mathbf{u})^\varepsilon}{\varrho^\varepsilon} \right\|_{L^3(\Omega_{\varepsilon_1})} ds \\
&\leq C(M_{\varepsilon_1}) \left(\|\mathbf{u}\|_{L^3}^2 + \|\mathbf{u}\|_{L^3(0,T;\mathcal{V}_{\varepsilon}^{\frac{1}{3},3}(\Omega_{\frac{\varepsilon_1}{2}}))}^2 \right) \|\mathbf{u}\|_{L^3(0,T;\mathcal{V}_{\varepsilon}^{\frac{1}{3},3}(\Omega_{\frac{\varepsilon_1}{2}}))}.
\end{aligned} \tag{42}$$

Therefore, by assumption (4), it follows that

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{\tau \rightarrow 0} |(E2)| = 0. \tag{43}$$

3.3. **Estimate of (F) .** By using the divergence free condition³ we can split (F) as

$$\begin{aligned}
 (F) &= \frac{1}{\varepsilon_3} \int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3} \int_{\tau}^t \int_{\Omega_{\varepsilon_2}} \frac{1}{\varrho^\varepsilon} ((\varrho \mathbf{u})^\varepsilon - \varrho^\varepsilon \mathbf{u}^\varepsilon) \nabla P^\varepsilon dx ds d\varepsilon_2 + \frac{1}{\varepsilon_3} \int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3} \int_{\tau}^t \int_{\Omega_{\varepsilon_2}} \mathbf{u}^\varepsilon \nabla P^\varepsilon dx ds d\varepsilon_2 \\
 &=: (F1) + \frac{1}{\varepsilon_3} \int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3} \int_{\tau}^t \int_{\partial\Omega_{\varepsilon_2}} P^\varepsilon \mathbf{u}^\varepsilon \cdot n(\theta) d\mathcal{H}^{d-1}(\theta) ds d\varepsilon_2 \\
 &=: (F1) + (F2).
 \end{aligned} \tag{44}$$

The term $(F1)$ is estimated similarly to the term (C) in the case $\Omega = \mathbb{T}^d$, thus by Lemma A.1 2), we obtain

$$\begin{aligned}
 |(F1)| &\leq \int_0^T \int_{\Omega_{\varepsilon_1}} \left| \frac{1}{\varrho^\varepsilon} \right| |(\varrho \mathbf{u})^\varepsilon - \varrho^\varepsilon \mathbf{u}^\varepsilon| |\nabla P^\varepsilon| dx ds \\
 &\leq C(M_{\varepsilon_1}) \|\varrho\|_{L^\infty(0,T;\mathcal{V}_{\frac{3}{4}}^{\frac{3}{2},\infty}(\Omega_{\frac{\varepsilon_1}{2}}))} \|\mathbf{u}\|_{L^3(0,T;\mathcal{V}_{\varepsilon}^{\frac{1}{3},3}(\Omega_{\frac{\varepsilon_1}{2}}))} \|P\|_{L^{\frac{3}{2}}}.
 \end{aligned} \tag{45}$$

Hence, by assumption (4),

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{\tau \rightarrow 0} |(F1)| = 0. \tag{46}$$

For the term $(F2)$ we use the coarea formula (39) to obtain

$$(F2) = \frac{1}{\varepsilon_3} \int_{\tau}^t \int_{\Omega_{\varepsilon_1} \setminus \Omega_{\varepsilon_1+\varepsilon_3}} P^\varepsilon \mathbf{u}^\varepsilon \cdot n(x) dx ds.$$

Letting successively $\tau \rightarrow 0$, $\varepsilon \rightarrow 0$ and $\varepsilon_1 \rightarrow 0$ and then using the fact $\varepsilon_3 \approx \mathcal{L}^d(\Omega \setminus \Omega_{\varepsilon_3})$ and Hölder inequality, we obtain

$$\begin{aligned}
 \limsup_{\varepsilon_1, \varepsilon \rightarrow 0} \limsup_{\tau \rightarrow 0} |(F2)| &= \left| \frac{1}{\varepsilon_3} \int_0^t \int_{\Omega \setminus \Omega_{\varepsilon_3}} P(x, s) \mathbf{u}(x, s) \cdot n(x) dx ds \right| \\
 &\leq C \left(\int_0^T \int_{\Omega \setminus \Omega_{\varepsilon_3}} |P(x, s)|^{\frac{3}{2}} dx ds \right)^{\frac{2}{3}} \left(\int_0^T \int_{\Omega \setminus \Omega_{\varepsilon_3}} |\mathbf{u}(x, s) \cdot n(x)|^3 dx ds \right)^{\frac{1}{3}}.
 \end{aligned}$$

By the assumption (9) we obtain

$$\limsup_{\varepsilon_3 \rightarrow 0} \limsup_{\varepsilon_1, \varepsilon \rightarrow 0} \limsup_{\tau \rightarrow 0} |(F2)| = 0. \tag{47}$$

³Again, the divergence free condition is only used here to deal with the pressure. This allows us to reuse other estimates in the case of compressible Euler equation.

3.4. Estimate of $(D3) + (E3) = o(1)$. Using integration by parts we compute (similarly to $(A3) + (B2) = 0$)

$$\begin{aligned}
(E3) &= -\frac{1}{\varepsilon_3} \int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3} \int_{\tau}^t \int_{\partial\Omega_{\varepsilon_2}} (\varrho \mathbf{u})^\varepsilon \otimes \mathbf{u}^\varepsilon \frac{(\varrho \mathbf{u})^\varepsilon}{\varrho^\varepsilon} \cdot n(\theta) d\mathcal{H}^{d-1}(\theta) ds d\varepsilon_2 \\
&\quad + \frac{1}{\varepsilon_3} \int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3} \int_{\tau}^t \int_{\Omega_{\varepsilon_2}} \frac{(\varrho \mathbf{u})^\varepsilon}{\varrho^\varepsilon} \nabla \cdot [(\varrho \mathbf{u})^\varepsilon \otimes \mathbf{u}^\varepsilon] dx ds d\varepsilon_2 \\
&=: (E31) + \frac{1}{\varepsilon_3} \int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3} \int_{\tau}^t \int_{\Omega_{\varepsilon_2}} \frac{(\varrho \mathbf{u})^\varepsilon}{\varrho^\varepsilon} [\nabla \cdot (\varrho \mathbf{u})^\varepsilon \mathbf{u}^\varepsilon + (\varrho \mathbf{u})^\varepsilon \nabla \cdot \mathbf{u}^\varepsilon] dx ds d\varepsilon_2 \\
&= (E31) + \frac{1}{2} \frac{1}{\varepsilon_3} \int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3} \int_{\tau}^t \int_{\partial\Omega_{\varepsilon_2}} |(\varrho \mathbf{u})^\varepsilon|^2 \frac{\mathbf{u}^\varepsilon}{\varrho^\varepsilon} \cdot n(\theta) d\mathcal{H}^{d-1}(\theta) ds d\varepsilon_2 \\
&\quad - \frac{1}{2} \frac{1}{\varepsilon_3} \int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3} \int_{\tau}^t \int_{\Omega_{\varepsilon_2}} |(\varrho \mathbf{u})^\varepsilon|^2 \nabla \cdot \frac{\mathbf{u}^\varepsilon}{\varrho^\varepsilon} dx ds d\varepsilon_2 \\
&\quad + \frac{1}{\varepsilon_3} \int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3} \int_0^t \int_{\Omega_{\varepsilon_2}} \frac{|(\varrho \mathbf{u})^\varepsilon|^2}{\varrho^\varepsilon} \nabla \cdot \mathbf{u}^\varepsilon dx ds d\varepsilon_2 \\
&=: (E31) + (E32) + \frac{1}{2} \frac{1}{\varepsilon_3} \int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3} \int_{\tau}^t \int_{\partial\Omega_{\varepsilon_2}} |(\varrho \mathbf{u})^\varepsilon|^2 \frac{\nabla \cdot (\varrho^\varepsilon \mathbf{u}^\varepsilon)}{(\varrho^\varepsilon)^2} dx ds d\varepsilon_2 \\
&= (E31) + (E32) - (D3).
\end{aligned} \tag{48}$$

Therefore, it remains only to estimate $(E31)$ and $(E32)$. By using a similar argument as in estimating $(E1)$, together with the coarea formula, Hölder's inequality and the fact $\varepsilon_3 \approx \mathcal{L}^d(\Omega \setminus \Omega_{\varepsilon_3})$, we obtain

$$\begin{aligned}
&\limsup_{\varepsilon_1, \varepsilon \rightarrow 0} \limsup_{\tau \rightarrow 0} (|(E31)| + |(E32)|) \\
&\leq C(\|\varrho\|_{L^\infty}, \|\varrho^{-1}\|_{L^\infty}) \left(\int_0^T \int_{\Omega \setminus \Omega_{\varepsilon_3}} |\mathbf{u}|^3 dx ds \right)^{\frac{2}{3}} \left(\int_0^T \int_{\Omega \setminus \Omega_{\varepsilon_3}} |\mathbf{u} \cdot n|^3 dx ds \right)^{\frac{1}{3}}.
\end{aligned}$$

By assumption (8), we deduce

$$\limsup_{\varepsilon_3 \rightarrow 0} \limsup_{\varepsilon_1, \varepsilon \rightarrow 0} \limsup_{\tau \rightarrow 0} (|(E31)| + |(E32)|) = 0. \tag{49}$$

Combining (48) and (49) leads to

$$\limsup_{\varepsilon_3 \rightarrow 0} \limsup_{\varepsilon_1, \varepsilon \rightarrow 0} \limsup_{\tau \rightarrow 0} |(D3) + (E3)| = 0.$$

Finally, by collecting the above estimates, we get

$$\limsup_{\varepsilon_3 \rightarrow 0} \limsup_{\varepsilon_1, \varepsilon \rightarrow 0} \limsup_{\tau \rightarrow 0} |(D1)| = 0.$$

3.5. Conclusion. From the estimate of (D) , (E) and (F) we have

$$\limsup_{\varepsilon_3 \rightarrow 0} \limsup_{\varepsilon, \varepsilon_1 \rightarrow 0} \limsup_{\tau \rightarrow 0} \left| \frac{1}{\varepsilon_3} \int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3} \int_0^t \int_{\Omega_{\varepsilon_2}} \partial_t \left(\frac{|(\varrho \mathbf{u})^\varepsilon|^2}{\varrho^\varepsilon} \right) dx ds d\varepsilon_2 \right| = 0.$$

Arguing similarly to the case of torus $\Omega = \mathbb{T}^d$ we obtain finally the results of Theorem 1.3,

$$\int_{\Omega} (\varrho |\mathbf{u}|^2)(x, t) dx = \int_{\Omega} (\varrho_0 |\mathbf{u}_0|^2)(x) dx \quad \forall t \in (0, T).$$

4. PROOF OF THEOREM 1.5

As proof of Theorem 1.1, we have for any $0 < \tau < t < T$,

$$\int_{\tau}^t \int_{\mathbb{T}^d} \frac{1}{\varrho^{\varepsilon}} (\varrho \mathbf{u})^{\varepsilon} \partial_t (\varrho \mathbf{u})^{\varepsilon} dx ds + \int_{\tau}^t \int_{\mathbb{T}^d} \frac{1}{\varrho^{\varepsilon}} (\varrho \mathbf{u})^{\varepsilon} \nabla \cdot (\varrho \mathbf{u} \otimes \mathbf{u})^{\varepsilon} dx ds + \int_{\tau}^t \int_{\mathbb{T}^d} \frac{1}{\varrho^{\varepsilon}} (\varrho \mathbf{u})^{\varepsilon} \nabla (\varrho^{\gamma})^{\varepsilon} dx ds = 0. \quad (50)$$

Let M be a constant such that

$$\|\varrho\|_{L^{\infty}} + \|\varrho^{-1}\|_{L^{\infty}} + \|\varrho\|_{L^{\infty}(0,T;\mathcal{V}_{\delta_0}^{\frac{1}{3},\infty}(\mathbb{T}^d))} + \|\varrho\|_{L^{\infty}(0,T;\mathcal{V}_{\delta_0}^{\alpha,\infty}(\mathbb{T}^d))} \leq M.$$

In view of the proof of Theorem 1.1 (recalling that the free divergence condition in the incompressible case is used only to deal with the pressure term, which will be estimated separately here), we have

$$\limsup_{\delta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \limsup_{\tau \rightarrow 0} \left| \frac{1}{2} \int_{\tau}^t \int_{\mathbb{T}^d} \partial_t \left(\frac{|(\varrho \mathbf{u})^{\varepsilon}|^2}{\varrho^{\varepsilon}} \right) dx dt + \int_{\tau}^t \int_{\mathbb{T}^d} \frac{1}{\varrho^{\varepsilon}} (\varrho \mathbf{u})^{\varepsilon} \nabla (\varrho^{\gamma})^{\varepsilon} dx ds \right| = 0 \quad (51)$$

By integration by parts

$$\begin{aligned} (G) &:= \int_{\tau}^t \int_{\mathbb{T}^d} \frac{1}{\varrho^{\varepsilon}} (\varrho \mathbf{u})^{\varepsilon} \nabla (\varrho^{\varepsilon})^{\gamma} dx ds + \int_{\tau}^t \int_{\mathbb{T}^d} \frac{1}{\varrho^{\varepsilon}} (\varrho \mathbf{u})^{\varepsilon} \nabla [(\varrho^{\gamma})^{\varepsilon} - (\varrho^{\varepsilon})^{\gamma}] dx ds \\ &= -\frac{\gamma}{\gamma-1} \int_{\tau}^t \int_{\mathbb{T}^d} \nabla \cdot (\varrho \mathbf{u})^{\varepsilon} (\varrho^{\varepsilon})^{\gamma-1} dx ds - \int_{\tau}^t \int_{\mathbb{T}^d} \nabla \cdot \left[\frac{1}{\varrho^{\varepsilon}} (\varrho \mathbf{u})^{\varepsilon} \right] [(\varrho^{\gamma})^{\varepsilon} - (\varrho^{\varepsilon})^{\gamma}] dx ds \\ &=: (G1) + (G2). \end{aligned}$$

Since $\partial_t \varrho^{\varepsilon} + \nabla \cdot (\varrho \mathbf{u})^{\varepsilon} = 0$,

$$(G1) = \frac{\gamma}{\gamma-1} \int_{\tau}^t \int_{\mathbb{T}^d} \partial_t \varrho^{\varepsilon} (\varrho^{\varepsilon})^{\gamma-1} dx ds = \frac{1}{\gamma-1} \int_{\tau}^t \int_{\mathbb{T}^d} \partial_t (\varrho^{\varepsilon})^{\gamma} dx ds.$$

To deal with (G2), we use the following estimate: for any $a > 0, b > -a$

$$|(a+b)^{\gamma} - (a^{\gamma} + \gamma a^{\gamma-1} b)| \leq C|b|^{\gamma} + C(a+b)^{\gamma-2}|b|^2. \quad (52)$$

We write

$$(\varrho^{\gamma})^{\varepsilon}(x, s) - (\varrho^{\varepsilon})^{\gamma}(x, s) = \int_{\mathbb{T}^d} \varrho(x-y, s)^{\gamma} \omega_{\varepsilon}(y) dy - \left(\int_{\mathbb{T}^d} \varrho(x-y, s) \omega_{\varepsilon}(y) dy \right)^{\gamma}.$$

By applying (52) with $a = \varrho(x, s)$, $b = \varrho(x-y, s) - \varrho(x, s)$ and $a = \varrho(x, s)$, $b = \int_{\mathbb{T}^d} (\varrho(x-y, s) - \varrho(x, s)) \omega_{\varepsilon}(y) dy$ and Hölder's inequality we get

$$\begin{aligned} |(\varrho^{\gamma})^{\varepsilon}(x, s) - (\varrho^{\varepsilon})^{\gamma}(x, s)| &\leq C \int_{\mathbb{T}^d} |\varrho(x-y, s) - \varrho(x, s)|^{\gamma} \omega_{\varepsilon}(y) dy \\ &\quad + C \int_{\mathbb{T}^d} \varrho(x-y, s)^{\gamma-2} |\varrho(x-y, s) - \varrho(x, s)|^2 \omega_{\varepsilon}(y) dy \\ &\quad + C \left(\int_{\mathbb{T}^d} \varrho(x-y, s) \omega_{\varepsilon}(y) dy \right)^{\gamma-2} \int_{\mathbb{T}^d} |\varrho(x-y, s) - \varrho(x, s)|^2 \omega_{\varepsilon}(y) dy. \end{aligned} \quad (53)$$

Note that, for any $p \geq 1$ and any $\delta \in (\varepsilon, \delta_0)$,

$$\int_{\mathbb{T}^d} |\varrho(x-y, s) - \varrho(x, s)|^p \omega_{\varepsilon}(y) dy \leq \varepsilon^{\alpha p} \|\varrho\|_{L^{\infty}(0,T;\mathcal{V}_{\delta}^{\alpha,\infty}(\mathbb{T}^d))}^p \quad \text{for a.e. } s \in (0, T). \quad (54)$$

By applying (54) to the terms on the right-hand side of (53) with $p = \gamma$ and $p = 2$ successively, we obtain

$$\|(\varrho^\gamma)^\varepsilon - (\varrho^\varepsilon)^\gamma\|_{L^\infty(\mathbb{T}^d \times (0, T))} \leq C(M)\varepsilon^{\alpha \min\{\gamma, 2\}} = C(M)\varepsilon^{\frac{2}{3}}.$$

Hence,

$$\begin{aligned} |(G2)| &\leq C(M)\varepsilon^{\frac{2}{3}} \left\| \nabla \cdot \frac{1}{\varrho^\varepsilon} (\varrho \mathbf{u})^\varepsilon \right\|_{L^1(\mathbb{T}^d \times (0, T))} \\ &\leq C(M)\varepsilon^{\frac{2}{3}} \left(\varepsilon^{-1+\alpha} \|\mathbf{u}\|_{L^1} \|\varrho\|_{L^\infty(0, T; \mathcal{V}_\delta^{\alpha, \infty}(\mathbb{T}^d))} + \varepsilon^{-\frac{2}{3}} \|\mathbf{u}\|_{L^3(0, T; \mathcal{V}_\delta^{\frac{1}{3}, 3}(\mathbb{T}^d))} \right). \end{aligned} \quad (55)$$

Therefore, by assumption (12), (13)

$$\limsup_{\delta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} |(G2)| = 0. \quad (56)$$

4.1. Conclusion. From the previous estimates we have

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{\tau \rightarrow 0} \left| \int_\tau^t \int_{\mathbb{T}^d} \partial_t \left[\frac{1}{2} \frac{1}{\varrho^\varepsilon} |(\varrho \mathbf{u})^\varepsilon|^2 + \frac{(\varrho^\varepsilon)^\gamma}{\gamma - 1} \right] dx ds \right| = 0.$$

By proceeding as in Section 4.1 with additional argument concerning the term ϱ^γ , we derive the energy conservation (14). \square

5. PROOF OF THEOREM 1.7

The proof of Theorem 1.7 is similar to that of Theorem 1.5, except that we have to take care of the boundary layer when taking integration by parts. Take $0 < \varepsilon < \varepsilon_1/10 < \varepsilon_3/10 < \delta_0/100$, as proof of Theorem 1.3, we have

$$\begin{aligned} &\frac{1}{\varepsilon_3} \int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3} \int_\tau^t \int_{\Omega_{\varepsilon_2}} \frac{1}{\varrho^\varepsilon} (\varrho \mathbf{u})^\varepsilon \partial_t (\varrho \mathbf{u})^\varepsilon dx ds d\varepsilon_2 + \frac{1}{\varepsilon_3} \int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3} \int_\tau^t \int_{\Omega_{\varepsilon_2}} \frac{1}{\varrho^\varepsilon} (\varrho \mathbf{u})^\varepsilon \nabla \cdot (\varrho \mathbf{u} \otimes \mathbf{u})^\varepsilon dx ds d\varepsilon_2 \\ &\quad + \frac{1}{\varepsilon_3} \int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3} \int_\tau^t \int_{\Omega_{\varepsilon_2}} \frac{1}{\varrho^\varepsilon} (\varrho \mathbf{u})^\varepsilon \nabla (\varrho^\gamma)^\varepsilon dx ds d\varepsilon_2 = 0. \end{aligned} \quad (57)$$

Let M_{ε_1} be a constant such that

$$\|\varrho\|_{L^\infty} + \|\varrho^{-1}\|_{L^\infty} + \|\varrho\|_{L^\infty(0, T; \mathcal{V}_{\frac{\varepsilon_1}{4}}^{\frac{1}{3}, \infty}(\Omega_{\varepsilon_1/2}))} + \|\varrho\|_{L^\infty(0, T; \mathcal{V}_{\frac{\varepsilon_1}{4}}^{\alpha, \infty}(\Omega_{\varepsilon_1/2}))} \leq M_{\varepsilon_1}.$$

In view of the proof of Theorem 1.3, we have

$$\begin{aligned} &\limsup_{\varepsilon_3 \rightarrow 0} \limsup_{\varepsilon_1, \varepsilon \rightarrow 0} \limsup_{\tau \rightarrow 0} \left| \frac{1}{2} \frac{1}{\varepsilon_3} \int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3} \int_\tau^t \int_{\Omega_{\varepsilon_2}} \partial_t \left(\frac{|(\varrho \mathbf{u})^\varepsilon|^2}{\varrho^\varepsilon} \right) dx ds d\varepsilon_2 \right. \\ &\quad \left. + \frac{1}{\varepsilon_3} \int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3} \int_\tau^t \int_{\Omega_{\varepsilon_2}} \frac{1}{\varrho^\varepsilon} (\varrho \mathbf{u})^\varepsilon \nabla (\varrho^\gamma)^\varepsilon dx ds d\varepsilon_2 \right| = 0. \end{aligned}$$

By integration by parts and the coarea formula (39) we have

$$\begin{aligned} (H) &:= \frac{1}{\varepsilon_3} \int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3} \int_\tau^t \int_{\Omega_{\varepsilon_2}} \frac{1}{\varrho^\varepsilon} (\varrho \mathbf{u})^\varepsilon \nabla (\varrho^\varepsilon)^\gamma dx ds d\varepsilon_2 \\ &\quad - \frac{1}{\varepsilon_3} \int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3} \int_\tau^t \int_{\Omega_{\varepsilon_2}} \nabla \cdot \frac{1}{\varrho^\varepsilon} (\varrho \mathbf{u})^\varepsilon ((\varrho^\gamma)^\varepsilon - (\varrho^\varepsilon)^\gamma) dx ds d\varepsilon_2 \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\varepsilon_3} \int_{\tau}^t \int_{\Omega_{\varepsilon_1} \setminus \Omega_{\varepsilon_1 + \varepsilon_3}} \frac{1}{\varrho^{\varepsilon}} (\varrho \mathbf{u})^{\varepsilon} ((\varrho^{\gamma})^{\varepsilon} - (\varrho^{\varepsilon})^{\gamma}) n(x) dx ds \\
& =: (H1) + (H2) + (H3).
\end{aligned}$$

By using (9), $\varrho, \varrho^{-1} \in L^{\infty}(\Omega \times (0, T))$ and $\varepsilon_3 \approx \mathcal{L}^d(\Omega \setminus \Omega_{\varepsilon_3})$ we obtain easily

$$\limsup_{\varepsilon_3 \rightarrow 0} \limsup_{\varepsilon_1, \varepsilon \rightarrow 0} \limsup_{\tau \rightarrow 0} |(H3)| = 0.$$

Using the integration by parts, together with the coarea formula (see (39)) and the fact that $\partial_t \varrho^{\varepsilon} + \nabla \cdot (\varrho \mathbf{u})^{\varepsilon} = 0$, we get

$$\begin{aligned}
(H1) & = -\frac{\gamma}{\gamma-1} \frac{1}{\varepsilon_3} \int_{\varepsilon_1}^{\varepsilon_1 + \varepsilon_3} \int_{\tau}^t \int_{\Omega_{\varepsilon_2}} \nabla \cdot (\varrho \mathbf{u})^{\varepsilon} (\varrho^{\varepsilon})^{\gamma-1} dx ds d\varepsilon_2 \\
& + \frac{\gamma}{\gamma-1} \frac{1}{\varepsilon_3} \int_{\tau}^t \int_{\Omega_{\varepsilon_1} \setminus \Omega_{\varepsilon_1 + \varepsilon_3}} (\varrho \mathbf{u})^{\varepsilon} \cdot n (\varrho^{\varepsilon})^{\gamma-1} dx ds d\varepsilon_2 \\
& = \frac{1}{\gamma-1} \frac{1}{\varepsilon_3} \int_{\varepsilon_1}^{\varepsilon_1 + \varepsilon_3} \int_{\tau}^t \int_{\Omega_{\varepsilon_2}} \partial_t (\varrho^{\varepsilon})^{\gamma} dx ds + (H11).
\end{aligned}$$

By assumption (19) and $\varrho \in L^{\infty}(\Omega \times (0, T))$ and the fact that $\varepsilon_3 \approx \mathcal{L}^d(\Omega \setminus \Omega_{\varepsilon_3})$, we assert

$$\limsup_{\varepsilon_3 \rightarrow 0} \limsup_{\varepsilon_1, \varepsilon \rightarrow 0} \limsup_{\tau \rightarrow 0} |(H11)| = 0.$$

Using an argument similar to the one leading to (55), we can show that

$$|(H2)| \leq C(M_{\varepsilon_1}) \varepsilon^{\frac{2}{3}} \left(\varepsilon^{-1+\alpha} \|\mathbf{u}\|_{L^1} \|\varrho\|_{L^{\infty}(0, T; \mathcal{V}_{\varepsilon}^{\alpha, \infty}(\Omega_{\frac{\varepsilon_1}{2}}))} + \varepsilon^{-\frac{2}{3}} \|\mathbf{u}\|_{L^3(0, T; \mathcal{V}_{\varepsilon}^{\frac{1}{3}, 3}(\Omega_{\frac{\varepsilon_1}{2}}))} \right). \quad (58)$$

This, combined with the assumptions (16) and (17), yields

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{\tau \rightarrow 0} |(H2)| = 0.$$

Therefore, we conclude

$$\limsup_{\varepsilon_3 \rightarrow 0} \limsup_{\varepsilon_1, \varepsilon \rightarrow 0} \limsup_{\tau \rightarrow 0} \left| \frac{1}{\gamma-1} \frac{1}{\varepsilon_3} \int_{\varepsilon_1}^{\varepsilon_1 + \varepsilon_3} \int_{\tau}^t \int_{\Omega_{\varepsilon_2}} \partial_t \left[\frac{1}{2} \frac{1}{\varrho^{\varepsilon}} |(\varrho \mathbf{u})^{\varepsilon}|^2 + \frac{(\varrho^{\varepsilon})^{\gamma}}{\gamma-1} \right] dx ds \right| = 0.$$

This gives the energy conservation (20). The proof is complete. \square

APPENDIX A. APPENDIX

In this section, we collect some lemmata and estimates which are used for the proofs of the main results.

For a function $f : \Omega \rightarrow \mathbb{R}$ we denote by $f^{\varepsilon} = f \star \omega_{\varepsilon}$ the smoothing version of f , where $\omega_{\varepsilon}(x) = (1/\varepsilon^d) \omega(x/\varepsilon)$ is a standard mollifier in \mathbb{R}^d .

Lemma A.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with C^2 boundary $\partial\Omega$.*

1) *Let $\beta \in (0, 1)$ and $1 \leq p \leq \infty$. Then for any function $f : \Omega \rightarrow \mathbb{R}$ and small $0 < \varepsilon \leq \frac{\delta}{2}$, there holds*

$$\|\nabla f^{\varepsilon}\|_{L^p(\Omega_{\delta})} \leq C \|\nabla \omega\|_{L^{\frac{p}{p-1}}} \varepsilon^{-1} \|f\|_{L^p(\Omega_{\frac{\delta}{2}})}, \quad (59)$$

$$\|\nabla f^{\varepsilon}\|_{L^p(\Omega_{\delta})} \leq C \|\nabla \omega\|_{L^{\frac{p}{p-1}}} \varepsilon^{-1+\beta} \|f\|_{\mathcal{V}_{\delta}^{\beta, p}(\Omega)}. \quad (60)$$

2) Let $\beta_1, \beta_2 \in (0, 1)$, $p \geq 1, p_1, p_2 \geq 1$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\delta_1 > 0, \delta_2 > 0, \delta \geq \max\{\delta_1, \delta_2\}$. Then for any functions $g_1, g_2 : \Omega \rightarrow \mathbb{R}$ and small $0 < \varepsilon \leq \frac{\min\{\delta_1, \delta_2\}}{2}$, there holds

$$\|(g_1 g_2)^\varepsilon - g_1^\varepsilon g_2^\varepsilon\|_{L^p(\Omega_\delta)} \leq C \varepsilon^{\beta_1 + \beta_2} \|\omega\|_{L^{\frac{p}{p-1}}} \|g_1\|_{V_{\delta_1}^{\beta_1, p_1}(\Omega)} \|g_2\|_{V_{\delta_2}^{\beta_2, p_2}(\Omega)}. \quad (61)$$

3) Let $\beta \in (0, 1)$ and $p \geq 1$. Then for any functions $g_1, g_2 : \Omega \rightarrow \mathbb{R}$ and small $\delta \in (0, 1)$, there holds

$$\|g_1 g_2\|_{V_\varepsilon^{\beta, p}(\Omega_\delta)} \leq C \left(\|g_1\|_{L^\infty(\Omega_{\frac{\delta}{2}})} \|g_2\|_{V_\varepsilon^{\beta, p}(\Omega_{\frac{\delta}{2}})} + \|g_1\|_{V_\delta^{\beta, \infty}(\Omega_{\frac{\delta}{2}})} \|g_2\|_{L^p(\Omega_{\frac{\delta}{2}})} \right), \quad (62)$$

for any $0 < \varepsilon < \delta/4$.

Proof. 1) Since $\int_{\mathbb{R}^d} \nabla \omega_\varepsilon(y) dy = 0$, it follows that, for a.e. $x \in \Omega_\delta$,

$$|\nabla f^\varepsilon(x)| = \left| \int_{\mathbb{R}^d} [f(x-y) - f(x)] \nabla \omega_\varepsilon(y) dy \right| \leq \varepsilon^{-1-\frac{d}{p}} \left(\int_{|y| < \varepsilon} |f(x-y) - f(x)|^p dy \right)^{\frac{1}{p}} \|\nabla \omega\|_{L^{\frac{p}{p-1}}}.$$

This yields (59) and (60).

2) It is not hard to see that

$$\begin{aligned} |(g_1 g_2)^\varepsilon(x) - g_1^\varepsilon(x) g_2^\varepsilon(x)| &\leq \int_{\mathbb{R}^d} |g_1(x-y) - g_1(x)| |g_2(x-y) - g_2(x)| \omega_\varepsilon(y) dy \\ &\quad + |g_1^\varepsilon(x) - g_1(x)| |g_2^\varepsilon(x) - g_2(x)|. \end{aligned}$$

Thus, using Holder's inequality, we get (61).

3) For every $x \in \Omega_\delta$ and $h \in \mathbb{R}^d$ such that $|h| < \delta/2$, we have

$$|(fg)(x+h) - (fg)(x)| \leq |f(x+h)| |g(x+h) - g(x)| + |f(x+h) - f(x)| |g(x)|.$$

Clearly, this gives (62). □

Lemma A.2.

1) Let $\beta \in (0, 1)$ and $1 \leq p \leq \infty$. Then for any function $f : \mathbb{T}^d \rightarrow \mathbb{R}$ and $\varepsilon > 0$, there holds

$$\begin{aligned} \|\nabla f^\varepsilon\|_{L^p(\mathbb{T}^d)} &\leq C \|\nabla \omega\|_{L^{\frac{p}{p-1}}} \varepsilon^{-1} \|f\|_{L^p(\mathbb{T}^d)}, \\ \|\nabla f^\varepsilon\|_{L^p(\mathbb{T}^d)} &\leq C \|\nabla \omega\|_{L^{\frac{p}{p-1}}} \varepsilon^{-1+\beta} \|f\|_{V_\delta^{\beta, p}(\mathbb{T}^d)}. \end{aligned}$$

2) Let $\beta_1, \beta_2 \in (0, 1)$ and $p \geq 1, p_1, p_2 \geq 1$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then for any functions $g_1, g_2 : \mathbb{T}^d \rightarrow \mathbb{R}$ and $\varepsilon > 0$, there holds

$$\|(g_1 g_2)^\varepsilon - g_1^\varepsilon g_2^\varepsilon\|_{L^p(\mathbb{T}^d)} \leq C \varepsilon^{\beta_1 + \beta_2} \|\omega\|_{L^{\frac{p}{p-1}}} \|g_1\|_{V_\delta^{\beta_1, p_1}(\mathbb{T}^d)} \|g_2\|_{V_\delta^{\beta_2, p_2}(\mathbb{T}^d)}.$$

3) Let $\beta \in (0, 1)$ and $p \geq 1$. Then for any functions $g_1, g_2 : \mathbb{T}^d \rightarrow \mathbb{R}$ and $\varepsilon > 0$, there holds

$$\|g_1 g_2\|_{V_\varepsilon^{\beta, p}(\mathbb{T}^d)} \leq C \left(\|g_1\|_{L^\infty(\mathbb{T}^d)} \|g_2\|_{V_\varepsilon^{\beta, p}(\mathbb{T}^d)} + \|g_1\|_{V_\varepsilon^{\beta, \infty}(\mathbb{T}^d)} \|g_2\|_{L^p(\mathbb{T}^d)} \right).$$

Proof. The proof is similar to that of Lemma A.1 and we omit it. □

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