A GENERALIZATION OF GOŁĄB'S THEOREM AND APPLICATIONS TO FRACTURE MECHANICS

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Abstract. We study the lower semicontinuity for functionals of the form $K \to \int_K \varphi(x,\nu)\, d\mathcal{H}^1$ defined on compact sets in $\mathbb{R}^2$ with a finite number of connected components and finite $\mathcal{H}^1$ measure and apply the result to the study of quasi-static growth of brittle fractures in linearly elastic inhomogeneous and anisotropic bodies.

1. Introduction

In 1998, G.A. Francfort and J.-J. Marigo [10] proposed a model for the quasi-static growth of brittle fractures in elastic bodies. This model is based on Griffith's criterion of crack growth which takes into account a competition between the bulk energy given by the deformation and the surface energy given by the length of the fracture. Recently, G. Dal Maso and R. Toader [8] gave a precise mathematical formulation of the model in *dimension two* for linearly elastic homogeneous bodies under anti-planar shear.

The aim of this paper is to extend this analysis in *dimension two* to *anisotropic linearly elastic inhomogeneous bodies* subjected to *anti-planar* or *planar shear*. *Anisotropy* will be considered both in the *bulk* and in the *surface energy*.

In order to make the ideas precise, let $\Omega \subseteq \mathbb{R}^2$ be open and bounded and let $\mathcal{K}_m^f(\overline{\Omega})$ denote the family of compact subsets of $\overline{\Omega}$ with at most m connected components and finite $\mathcal{H}^1$ measure . Consider an elastic body of the form $\Omega \times \mathbb{R}$ and assume the cracks of the form $K \times \mathbb{R}$ with $K \in \mathcal{K}_m^f(\overline{\Omega})$. Assume the displacement $u : \Omega \times \mathbb{R} \to \mathbb{R}^3$ depends only on x_1, x_2. If $u(x_1,x_2) = (0,0,u_3(x_1,x_2))$, we are in the case of *anti-planar shear*, while if $u(x_1,x_2) = (u_1(x_1,x_2), u_2(x_1,x_2), 0)$, we speak of *planar shear*. In the model case, we consider the *bulk energy* (referred to a finite portion of the cylinder determined by two cross sections separated by a unit distance) of the form

$$\int_{\Omega\setminus K} \mu|Eu|^2 + \lambda|trEu|^2 \, d\mathcal{L}^2$$

where μ, λ are called *Lamé coefficients* and Eu is the symmetric part of the gradient of u.

The *surface energy* on the fracture K is given by

$$\int_K \varphi(x,\nu_x)\, d\mathcal{H}^1(x),$$

where ν_x is the *unit normal vector* at x to K and $\varphi : \overline{\Omega} \times \mathbb{R}^2 \to [0,\infty[$ is a continuous function, positively 1-homogeneous, even and convex in the second variable.

Given $\partial_D\Omega \subseteq \partial\Omega$ open in the relative topology and with a finite number of connected components, we prescribe a displacement g on $\partial_D\Omega$. The displacement $u_{g,K}$ of the elastic body relative to g and the crack K is obtained minimizing $\int_{\Omega\setminus K} \mu|Eu|^2 + \lambda|trEu|^2\, d\mathcal{L}^2$ under the condition $u = g$ on $\partial_D\Omega \setminus K$. The condition $u = g$ on $\partial_D\Omega \setminus K$ takes into account the fact that the displacement is not transmitted through a fractured region. The *total elastic* *energy* is given by

$$\mathcal{E}(g,K) := \int_{\Omega\setminus K} \mu|Eu_{g,K}|^2 + \lambda|trEu_{g,K}|^2\, d\mathcal{L}^2 + \int_K \varphi(x,\nu)\, d\mathcal{H}^1.$$

Suppose an initial crack K_0 and boundary displacements $g(t)$, $t \in [0,1]$, $g(0) = 0$, are given. By a *quasi-static growth* of the fracture, we mean an *increasing* map $t \to K(t)$ from $[0,1]$ to $\mathcal{K}_m^f(\overline{\Omega})$ with $K(0) = K_0$ and such that $K(t)$ minimizes $\mathcal{E}(g(t),K)$ among K's such that $\cup_{s<t}K(s) \subseteq K$. The constraints given by the previous cracks indicate the *irreversibility* of the growth and the *absence of healing phenomena.* We require also the *stationarity condition* $\frac{d}{ds}\mathcal{E}(g(t),K(s))|_{s=t} = 0$ and the *absolute continuity* of the total elastic energy $t \to \mathcal{E}(g(t),K(t))$ even if, as noted in [10], $t \to \int_{K(t)} \varphi(x,\nu)\, d\mathcal{H}^1$ could be discontinuous.

The quasi-static growth satisfying the stationarity condition and the absolute continuity of the total energy is obtained through a *time discretization method.* Given $\delta > 0$, we divide $[0,1]$ in N_δ intervals $[t_i^\delta, t_{i+1}^\delta]$ and we indicate by K_i^δ the solution of

$$\min\{\mathcal{E}(g(t_i^\delta),K) \,:\, K_{i-1}^\delta \subseteq K\},$$

where we consider $K_{-1}^\delta = K_0$. We make the interpolation $K^\delta(t) = K^\delta(t_i^\delta)$ if $t_i^\delta \le t < t_{i+1}^\delta$; letting $\delta \to 0$ along a suitable sequence, it turns out that $K^\delta(t) \to K(t)$ in the Hausdorff metric determining the quasi-static growth. Moreover $Eu^\delta(t) \to Eu(t)$ strongly, so that the elastic bulk energies of the approximating fractures converge to the bulk energy of the solution.

Moreover we will prove that $\int_{K^\delta(t)} \varphi(x,\nu)\, d\mathcal{H}^1$ converges to the surface energy of the solution. We conclude that the time discretization procedure gives an approximation both of the bulk and surface energy of the solution. We remark that this fact is new also in the case $\varphi \equiv 1$, that is when the surface energy depends only on the length of the fracture.

In order to deal with an *anisotropic* and *inhomogeneous* surface energy, the main step is to prove a lower semicontinuity theorem for the functional $\mathcal{F}(K) := \int_K \varphi(x,\nu_x)\, d\mathcal{H}^1$ on $\mathcal{K}_m^f(\overline{\Omega})$ with respect to Hausdorff convergence. This functional is well defined: in fact, even if K is not in general the union of m regular curves, it turns out that it is possible to define at $\mathcal{H}^1$-a.e. point $x \in K$ an *approximate unit normal vector* ν_x completely determined up to the sign. In the case K is regular, ν_x coincides with the usual normal vector. Note that for $\varphi \equiv 1$, the semicontinuity result reduces to Gołąb's theorem on the lower semicontinuity of $\mathcal{H}^1$ measure under Hausdorff convergence.

The paper is organized as follows. After some preliminaries, we prove the lower semicontinuity result in Section 4. In sections 5 and 6, we deal with the study of quasi-static growth of brittle fractures in the anti-planar and planar cases. Using shape continuity results proved in [4] and [6], we can treat inhomogeneous bulk energies: we consider quadratic forms of Eu equivalent to the standard $\int_{\Omega\setminus K} |Eu|^2\, d\mathcal{L}^2$. This cannot be done directly using the techniques of [8] where the strong convergence of the gradient of deformation is obtained through a duality argument which relies on the particular form $\int_{\Omega\setminus K} |\nabla u|^2\, d\mathcal{L}^2$ of the bulk energy.

2. NOTATIONS AND PRELIMINARIES

In what follows, $\Omega \subseteq \mathbb{R}^2$ is a bounded open set with Lipschitz boundary, $\partial_D\Omega$ is a subset of $\partial\Omega$ open in the relative topology and with a finite number of connected components.

Sets with finite perimeter. We indicate the perimeter of E in Ω by $\mathrm{P}(E,\Omega)$. Let E be a set of finite perimeter in Ω; the reduced boundary $\partial^* E$ and the approximate inner normal ν at points of $\partial^* E$ are defined such that the following identity holds:

$$\forall g \in C_c(\Omega,\mathbb{R}^2) \quad -\int_E \mathrm{div} g\, d\mathcal{L}^2 = \int_{\partial^* E} g\cdot\nu\, d\mathcal{H}^1.$$

Set $\mu_E = \nu \mathcal{H}^1 \llcorner \partial^* E$. For all $x \in \partial^* E$, indicated the map $\xi \to \frac{1}{\lambda}(\xi - x)$ by D_λ, the following *blow up property* holds: for $\lambda \to 0^+$

$$\mu_{D_\lambda(E)} \overset{*}{\rightharpoonup}_{loc} \mathcal{H}^1 \llcorner T_\nu,$$

locally weakly star in the sense of measures, where T_ν is the subspace orthogonal to ν.

We say that a sequence (E_h) of subset of Ω converges in $L^1_{\text{loc}}(\Omega)$ to E, if the corresponding characteristic functions χ_{E_h} converge in $L^1_{\text{loc}}(\Omega)$ to χ_E. If there exists $C \geq 0$ such that $\mathrm{P}(E_h, \Omega) \leq C$ for all h and $E_h \to E$ in $L^1_{\text{loc}}(\Omega)$, then E has finite perimeter in Ω and $\mu_{E_h} \overset{*}{\rightharpoonup} \mu_E$ in the weak star topology of $\mathcal{M}_b(\Omega, \mathbb{R}^2)$. For further details on sets of finite perimeter, the reader is referred to [3].

Hausdorff metric on compact sets. We indicate the set of all compact subsets of $\overline{\Omega}$ by $\mathcal{K}(\overline{\Omega})$, the set of elements of $\mathcal{K}(\overline{\Omega})$ with finite $\mathcal{H}^1$ measure by $\mathcal{K}^f(\overline{\Omega})$ and, given $\lambda \geq 0$, the compact sets K with $\mathcal{H}^1(K) \leq \lambda$ by $\mathcal{K}^\lambda(\overline{\Omega})$. $\mathcal{K}(\overline{\Omega})$ can be endowed by the Hausdorff metric d_H defined by

$$d_H(K_1, K_2) := \max\left\{ \sup_{x \in K_1} \operatorname{dist}(x, K_2), \sup_{y \in K_2} \operatorname{dist}(y, K_1) \right\}$$

with the conventions $\operatorname{dist}(x, \emptyset) = \operatorname{diam}(\Omega)$ and $\sup \emptyset = 0$, so that $d_H(\emptyset, K) = 0$ if $K = \emptyset$ and $d_H(\emptyset, K) = \operatorname{diam}(\Omega)$ if $K \neq \emptyset$. It turns out that $\mathcal{K}(\overline{\Omega})$ endowed with the Hausdorff metric is a compact space (see e.g. [14]). Let $\mathcal{K}_m(\overline{\Omega})$ be the subset of $\mathcal{K}(\overline{\Omega})$ of those compact sets which have less than m connected components. Since Hausdorff convergence preserves connectedness, $\mathcal{K}_m(\overline{\Omega})$ are closed subsets of $\mathcal{K}(\overline{\Omega})$ for all m. Let $\mathcal{K}^f_m(\overline{\Omega}) := \mathcal{K}_m(\overline{\Omega}) \cap \mathcal{K}^f(\overline{\Omega})$ and given $\lambda \geq 0$, $\mathcal{K}^\lambda_m(\overline{\Omega}) := \mathcal{K}_m(\overline{\Omega}) \cap \mathcal{K}^\lambda(\overline{\Omega})$.

Hausdorff measure $\mathcal{H}^1$ is not lower semicontinuous in $\mathcal{K}(\overline{\Omega})$ with respect to Hausdorff metric. However it is lower semicontinuous if restricted to $\mathcal{K}_m(\overline{\Omega})$: for the case $m = 1$, this result is known as Gołąb's theorem (see e.g. [12]). The general case can be found in [8].

Theorem 2.1. *Let (K_n) be a sequence in $\mathcal{K}_m(\overline{\Omega})$ which converges to K in the Hausdorff metric. Then $K \in \mathcal{K}_m(\overline{\Omega})$ and for every open subset $U \subseteq \mathbb{R}^2$*

$$\mathcal{H}^1(K \cap U) \leq \liminf_n \mathcal{H}^1(K_n \cap U).$$

Structure of compact connected sets with finite $\mathcal{H}^1$ measure. It can be proved (see e.g. [9]) that if $K \in \mathcal{K}^f_1(\overline{\Omega})$, for a.e. $x \in K$ there exists an approximate unit normal vector ν_x which is characterized by

$$\mu_{D_\lambda(E)} \overset{*}{\rightharpoonup}_{loc} \mathcal{H}^1 \llcorner T_{\nu_x} \quad \text{for } \lambda \to 0^+ \tag{2.1}$$

locally weakly star in the sense of measures, where T_{ν_x} is the subspace of $\mathbb{R}^2$ orthogonal to ν_x. Moreover the map $x \to \nu_x$ is Borel measurable, so that for every continuous function $\varphi : \overline{\Omega} \times \mathbb{R}^2 \to [0, \infty[$ even in the second variable the integral

$$\int_K \varphi(x, \nu)\, d\mathcal{H}^1$$

is well defined. Clearly the functional is well defined also for $K \in \mathcal{K}^f_m(\overline{\Omega})$ with $m \geq 1$.

In section 4 we will be concerned in the problem of the lower semicontinuity of the function $K \to \int_K \varphi(x, \nu)\, d\mathcal{H}^1$ under the Hausdorff convergence.

We will use the fact that a connected set C with finite $\mathcal{H}^1$ measure is arcwise connected and moreover $\mathcal{H}^1(\overline{C}) = \mathcal{H}^1(C)$: see e.g. [8].

Reshetnyak's theorems on measures. The following theorem gives a lower semicontinuity result for functionals defined on measures; for a proof, the reader is referred to [3]. If μ is a measure, let $|\mu|$ be its total variation and let $\frac{d\mu}{d|\mu|}$ be the Radon-Nicodym derivative of μ with respect to $|\mu|$.

Theorem 2.2. *Let Ω be an open subset of $\mathbb{R}^n$ and μ, μ_k be $\mathbb{R}^m$-valued finite Radon measures in Ω; if $\mu_h \to \mu$ weakly star in $\mathcal{M}_b(\Omega, \mathbb{R}^m)$ then*

$$\int_\Omega f\left(x, \frac{d\mu}{d|\mu|}(x)\right) d|\mu|(x) \leq \liminf_{h\to\infty} \int_\Omega f\left(x, \frac{d\mu_h}{d|\mu_h|}(x)\right) d|\mu_h|(x)$$

for every lower semicontinuous function $f : \Omega \times \mathbb{R}^m \to [0, +\infty]$, positively 1-homogeneous and convex in the second variable.

We say that μ_n converges strictly to μ in $\mathcal{M}_b(\Omega, \mathbb{R}^m)$ if $\mu_n \to \mu$ weakly star and $|\mu_n|(\Omega) \to |\mu|(\Omega)$. The following theorem gives a continuity result for functional defined on measures: for a proof see [3].

Theorem 2.3. *Let Ω be an open subset of $\mathbb{R}^n$ and μ, μ_k be $\mathbb{R}^m$-valued finite Radon measures in Ω; if $\mu_h \to \mu$ strictly in $\mathcal{M}_b(\Omega, \mathbb{R}^m)$ then*

$$\lim_{h\to\infty} \int_\Omega f\left(x, \frac{d\mu_h}{d|\mu_h|}(x)\right) d|\mu_h|(x) = \int_\Omega f\left(x, \frac{d\mu}{d|\mu|}(x)\right) d|\mu|(x)$$

for every continuous and bounded function $f : \Omega \times S^{m-1} \to \mathbb{R}$.

Deny-Lions spaces. If A is an open subset of $\mathbb{R}^2$, the Deny-Lions space $L^{1,2}(A)$ is defined as

$$L^{1,2}(A) := \left\{ u \in W^{1,2}_{\text{loc}}(A) \,:\, \nabla u \in L^2(A, \mathbb{R}^2) \right\}. \tag{2.2}$$

In the case in which A is regular $L^{1,2}(A)$ coincides with the usual Sobolev space while if it is irregular, it can be strictly larger. In what follows, given $K \subseteq \overline{\Omega}$ compact and $u \in L^{1,2}(\Omega \setminus K)$, we extend ∇u to 0 on K, so that $\nabla u \in L^2(\Omega, \mathbb{R}^2)$ although ∇u is the distributional derivative of u only on $\Omega \setminus K$. The following theorem proved in [4] will be used in Section 5.

Theorem 2.4. *Let $m \geq 1$, K_n a sequence in $\mathcal{K}_m(\overline{\Omega})$ which converges to K in the Hausdorff metric and such that $\mathcal{L}^2(\Omega \setminus K_n) \to \mathcal{L}^2(\Omega \setminus K)$. Then for every $u \in L^{1,2}(\Omega \setminus K)$, there exists $u_n \in L^{1,2}(\Omega \setminus K_n)$ such that $\nabla u_n \to \nabla u$ strongly in $L^2(\Omega, \mathbb{R}^2)$.*

Consider now for A open subset of $\mathbb{R}^2$

$$LD^{1,2}(A) := \left\{ u \in W^{1,2}_{\text{loc}}(A; \mathbb{R}^2) \,:\, E(u) \in L^2(A, \mathrm{M}^{2\times 2}_{\text{sym}}) \right\}, \tag{2.3}$$

where $Eu := \frac{1}{2}(\nabla u + (\nabla u)^t)$ is the symmetric part of the gradient of u and $\mathrm{M}^{2\times 2}_{\text{sym}}$ is the space of 2×2 symmetric matrices endowed with the standard scalar product $B_1{:}B_2 := tr(B_1^t B_2)$ and the corresponding norm $|B| := (B{:}B)^{\frac{1}{2}}$.

In what follows, given $K \subseteq \overline{\Omega}$ compact and $u \in LD^{1,2}(\Omega \setminus K)$, we extend Eu to 0 on K although it coincides with the symmetric part of the distributional gradient of u only on $\Omega \setminus K$. The following result, which can be obtained combining the density result proved in [6] and Theorem 2.4, will be used in Section 6.

Theorem 2.5. *Let $m \geq 1$, K_n a sequence in $\mathcal{K}_m(\overline{\Omega})$ which converges to K in the Hausdorff metric and such that $\mathcal{L}^2(\Omega \setminus K_n) \to \mathcal{L}^2(\Omega \setminus K)$. Then for every $u \in LD^{1,2}(\Omega \setminus K)$, there exists a sequence $u_n \in LD^{1,2}(\Omega \setminus K_n)$ such that $Eu_n \to Eu$ strongly in $L^2(\Omega; \mathrm{M}^{2\times 2}_{\text{sym}})$.*

Absolutely continuous function. Given a Hilbert space X, we indicate by $AC([0,1],X)$ the space of absolutely continuous function from $[0,1]$ to X: for the main properties of this space, the reader is referred to [5]. Given $g \in AC([0,1],X)$, the time derivative, which exists a.e. in $[0,1]$, is denoted by $\dot{g}$.

3. THE MAIN RESULTS

Let $\varphi : \overline{\Omega} \times \mathbb{R}^2 \to [0,+\infty[$ a continuous function, positively 1-homogeneous, even and convex in the second variable such that for $c_1, c_2 > 0$

$$\forall (x,\nu) \in \overline{\Omega} \times \mathbb{R}^2 \ : \ c_1|\nu| \le \varphi(x,\nu) \le c_2|\nu|. \tag{3.1}$$

The main result of the paper is the following lower semicontinuity theorem.

Theorem 3.1. *The functional*

$$\begin{array}{rcl} \mathcal{F} : \mathcal{K}_m^f(\overline{\Omega}) & \longrightarrow & [0,\infty[\\ \\ K & \mapsto & \int_K \varphi(x,\nu)\, d\mathcal{H}^1 \end{array}$$

is lower semicontinuous if $\mathcal{K}_m^f(\overline{\Omega})$ is endowed with the Hausdorff metric.

The previous theorem will be used to deal with the problem of evolution of brittle fractures in linearly elastic bodies.

Let $a \in L^\infty(\Omega, \mathrm{M}^{2\times 2}_{\mathrm{sym}})$ such that for $\alpha_1, \alpha_2 > 0$

$$\forall x \in \Omega\,, \forall \xi \in \mathbb{R}^2 \ : \ \alpha_1|\xi|^2 \le a(x)\xi \cdot \xi \le \alpha_2 |\xi|^2. \tag{3.2}$$

Let $(\cdot,\cdot)_a$ denote the associated scalar product on $L^2(\Omega,\mathbb{R}^2)$ defined as

$$(v,w)_a = \int_\Omega \sum_{i,j=1}^2 a(x) v_i(x) w_j(x)\, d\mathcal{L}^2(x)$$

and let $||\cdot||_a$ be the relative norm.

For every $g \in H^1(\Omega)$ and $K \in \mathcal{K}_m^f(\overline{\Omega})$, we set

$$\mathcal{E}(g,K) := \min_{v \in \Gamma(g,K)} \left\{ \int_{\Omega\setminus K} a(x)\nabla v \cdot \nabla v\, d\mathcal{L}^2 + \int_K \varphi(x,\nu)\, d\mathcal{H}^1 \right\}, \tag{3.3}$$

where

$$\Gamma(g,K) := \left\{ u \in L^{1,2}(\Omega \setminus K)\,,\ u = g \,\mathrm{on}\, \partial_D\Omega \setminus K \right\}. \tag{3.4}$$

The following theorem states the existence of a quasi-static evolution for brittle fractures in linear elastic bodies under anti-planar displacement: note that both the bulk and the surface energy depend in a possibly inhomogeneous way on the anisotropy of the body.

Theorem 3.2. *Let $m \ge 1$, $g \in AC([0,1],H^1(\Omega))$, $K_0 \in \mathcal{K}_m^f(\overline{\Omega})$. Then there exists a function $K : [0,1] \to \mathcal{K}_m^f(\overline{\Omega})$ such that, letting $u(t)$ be a solution of the minimum problem (3.3) which defines $\mathcal{E}(g(t),K(t))$ for all $t \in [0,1]$,*

(a) $$K_0 \subseteq K(s) \subseteq K(t) \text{ for } 0 \le s \le t \le 1;$$

(b) $$\mathcal{E}(g(0),K(0)) \le \mathcal{E}(g(0),K) \quad \forall K \in \mathcal{K}_m^f(\overline{\Omega}),\ K_0 \subseteq K;$$

(c) $$\forall t \in]0,1] : \mathcal{E}(g(t),K(t)) \le \mathcal{E}(g(t),K) \quad \forall K \in \mathcal{K}_m^f(\overline{\Omega}),\ \cup_{s<t} K(s) \subseteq K;$$

(d) $$t \mapsto \mathcal{E}(g(t),K(t)) \text{ is absolutely continuous on } [0,1];$$

(e) $$\frac{d}{dt}\mathcal{E}(g(t),K(t)) = 2(\nabla u(t), \nabla \dot{g}(t))_a \quad \text{for a.e. } t \in [0,1],$$

$$\text{(f)} \qquad \frac{d}{ds}\mathcal{E}(g(t),K(s))|_{s=t} = 0 \quad \textit{for a.e. } t \in [0,1].$$

Let $\mathcal{L}(\mathrm{M}^{2\times 2}_{\mathrm{sym}})$ be the space of automorphism of $\mathrm{M}^{2\times 2}_{\mathrm{sym}}$ and let $A \in L^\infty(\Omega, \mathcal{L}(\mathrm{M}^{2\times 2}_{\mathrm{sym}}))$ such that there exist $\alpha_1, \alpha_2 > 0$ with

$$\forall x \in \Omega \,:\, \alpha_1 |M|^2 \leq A(x)M{:}M \leq \alpha_2 |M|^2.$$

Let us pose $(Eu, Ev)_A := \int_{\Omega\setminus K} A(x)Eu{:}Ev \, d\mathcal{L}^2$ and $||Eu||_A := (Eu, Eu)_A^{\frac{1}{2}}$.

For every $g \in H^1(\Omega;\mathbb{R}^2)$ and $K \in \mathcal{K}^f_m(\overline{\Omega})$, set

$$\mathcal{G}(g,K) = \min_{v \in \mathcal{V}(g,K)} \left\{ \int_{\Omega\setminus K} A(x)Eu{:}Eu \, d\mathcal{L}^2 + \int_K \varphi(x,\nu)\, d\mathcal{H}^1 \right\}, \tag{3.5}$$

where

$$\mathcal{V}(g,K) = \left\{ u \in LD^{1,2}(\Omega \setminus K)\,,\, u = g \text{ on } \partial_D\Omega \setminus K \right\}. \tag{3.6}$$

The following theorem states the existence of a quasi-static evolution for brittle fractures in inhomogeneous anisotropic linearly elastic bodies under planar displacement.

Theorem 3.3. *Let $m \geq 1$, $g \in AC([0,1], H^1(\Omega;\mathbb{R}^2))$, $K_0 \in \mathcal{K}^f_m(\overline{\Omega})$. Then there exists a function $K : [0,1] \to \mathcal{K}^f_m(\overline{\Omega})$ such that, letting $u(t)$ be a solution of the minimum problem (3.5) which defines $\mathcal{G}(g(t),K(t))$ for all $t \in [0,1]$,*

$$\text{(a)} \qquad K_0 \subseteq K(s) \subseteq K(t) \textit{ for } 0 \leq s \leq t \leq 1;$$

$$\text{(b)} \qquad \mathcal{G}(g(0),K(0)) \leq \mathcal{G}(g(0),K) \quad \forall K \in \mathcal{K}^f_m(\overline{\Omega}),\ K_0 \subseteq K;$$

$$\text{(c)} \qquad \forall t \in]0,1] \,:\, \mathcal{G}(g(t),K(t)) \leq \mathcal{G}(g(t),K) \quad \forall K \in \mathcal{K}^f_m(\overline{\Omega}),\ \cup_{s<t}K(s) \subseteq K;$$

$$\text{(d)} \qquad t \mapsto \mathcal{G}(g(t),K(t)) \textit{ is absolutely continuous on } [0,1];$$

$$\text{(e)} \qquad \frac{d}{dt}\mathcal{G}(g(t),K(t)) = 2(Eu(t), E\dot{g}(t))_A \quad \textit{for a.e. } t \in [0,1],$$

$$\text{(f)} \qquad \frac{d}{ds}\mathcal{G}(g(t),K(s))|_{s=t} = 0 \quad \textit{for a.e. } t \in [0,1].$$

Remark 3.4. It turns out that for every function $K : [0,1] \to \mathcal{K}^f_m(\overline{\Omega})$ which satisfies (a)-(d) of Theorem 3.2, then conditions (e) and (f) are equivalent. Similarly, for every function $K : [0,1] \to \mathcal{K}^f_m(\overline{\Omega})$ which satisfies (a)-(d) of Theorem 3.3, conditions (e) and (f) are equivalent.

We will prove theorem 3.1 in section 4 using a comparison of measures which involves a blow-up technique; theorems 3.2 and 3.3 will be proved in sections 5 and 6 respectively: a discretization in time procedure will be employed and, in the particular case in which $g(0) = 0$, we prove that this method gives an approximation of the total energy of the solution.

4. A GENERALIZATION OF GOŁĄB THEOREM

Throughout this section, let $\varphi : \overline{\Omega} \times \mathbb{R}^2 \to [0,\infty[$ be a continuous function, positively 1-homogeneous, even and convex in the second variable satisfying

$$\forall \nu \in \mathbb{R}^2 \,:\, c_1|\nu| \leq \varphi(x,\nu) \leq c_2|\nu| \tag{4.1}$$

for some $c_1, c_2 > 0$.

Let $\mathcal{C}$ be the subset of $L^1(\Omega)$ composed by characteristic functions of sets with finite perimeter in Ω.

Theorem 4.1. *Consider the functional $\mathcal{G} : \mathcal{C} \to [0, \infty[$ defined by*

$$\mathcal{G}(E) = \int_{\partial^* E} \varphi(x, \nu)\, d\mathcal{H}^1$$

where ν denotes the inner normal of E. Then $\mathcal{G}$ is lower semicontinuous with respect to the L^1 topology.

Proof. Let (E_h) be a sequence of sets with finite perimeter in Ω with $E_h \to E$ in $L^1(\Omega)$: it is sufficient to consider the case $\mathrm{P}(E_h, \Omega) \le C$ for some $C \ge 0$ independent of h. As noted in Section 2, $\mu_{E_h} \stackrel{*}{\rightharpoonup} \mu_E$ in the weak star topology of $\mathcal{M}_b(\Omega, \mathbb{R}^2)$. Since the inner normal to E_h (resp. E) is given by $\dfrac{d\mu_{E_h}}{d\mathcal{H}^1}$ (resp. by $\dfrac{d\mu_E}{d\mathcal{H}^1}$), we can use Reshetnyak lower semicontinuity theorem (see Section 2) to get the conclusion. ☐

Theorem 4.2. *Let U be an open subset of $\mathbb{R}^2$. The functional*

$$\begin{array}{rcl} \mathcal{F} : \mathcal{K}_m^f(\overline{\Omega}) & \longrightarrow & [0, \infty[\\ K & \mapsto & \int_{K \cap U} \varphi(x, \nu)\, d\mathcal{H}^1 \end{array}$$

is lower semicontinuous if $\mathcal{K}_m^f(\overline{\Omega})$ is endowed with the Hausdorff metric.

Proof. We consider preliminarily the case $m = 1$.

Let $K_n, K \in \mathcal{K}_1^f(\overline{\Omega})$ with $K_n \to K$ in the Hausdorff metric: our aim is to verify that

$$\int_{K \cap U} \varphi(x, \nu)\, d\mathcal{H}^1 \le \liminf_n \int_{K_n \cap U} \varphi(x, \nu)\, d\mathcal{H}^1.$$

Without loss of generality we may consider sequences (K_n) such that

$$\sup_n \int_{K_n \cap U} \varphi(x, \nu)\, d\mathcal{H}^1 < +\infty.$$

Let us consider the positive measures μ_n, μ in $\mathcal{M}_b(U)$

$$\mu_n(B) = \int_{K_n \cap B} \varphi(x, \nu)\, d\mathcal{H}^1,$$

$$\mu(B) = \int_{K \cap B} \varphi(x, \nu)\, d\mathcal{H}^1.$$

By (4.1), (μ_n) is bounded in $\mathcal{M}_b(U)$ and so up to a subsequence it converges in the weak-star topology of $\mathcal{M}_b(U)$ to a measure μ_0 whose support is contained in $K \cap U$. By weak convergence we have

$$\mu_0(U) \le \liminf_n \mu_n(U),$$

and so it is sufficient to prove that

$$\mu(U) \le \mu_0(U). \tag{4.2}$$

We prove instead that $\mu \le \mu_0$ using a density argument which requires a blow-up technique: we obtain (4.2) as a consequence.

Firstly consider $K_n \in \mathcal{K}_1^f(\overline{B}_1(0))$, $\mathcal{H}^1(K_n) \le C$ for some $C \ge 0$ and $K_n \to K$ in the Hausdorff metric where K is the diameter connecting the points $e_1 := (1, 0)$ and $-e_1$. Note that for every strip $S_\eta = \{x \in \mathbb{R}^2 : -\eta \le x_2 \le \eta\}$ with $\eta > 0$, $K_n \subseteq S_\eta$ and $K_n \cap \partial S_\eta = \emptyset$ for n large enough. Given $\varepsilon > 0$, let $V^\varepsilon := \{x \in \mathbb{R}^2 : -1 + \varepsilon \le x_1 \le 1 - \varepsilon\}$, $\partial^\pm V^\varepsilon$ the connected components of ∂V^ε containing the points $(1 - \varepsilon)e_1$ and $-(1 - \varepsilon)e_1$ respectively. For n large enough, since K_n is connected, there exist points $x_n^\pm \in \partial^\pm V^\varepsilon \cap K_n$ such that $x_n^\pm \to \pm(1 - \varepsilon)e_1$. Let L_n be the union of the segments connecting x_n^- to $-(1 - \varepsilon)e_1$ and

$-(1-\varepsilon)e_1$ to $-e_1$, x_n^+ to $(1-\varepsilon)e_1$ and $(1-\varepsilon)e_1$ to e_1. Note that $H_n := K_n \cup L_n$ is connected and that

$$\mathcal{H}^1(L_n) \leq 3\varepsilon$$

for n large enough.

Let E_n be the connected component of $B_1(0) \setminus H_n$ containing $\frac{1}{2}e_2$, where $e_2 := (0,1)$. As $\pm e_1 \in H_n$ and H_n converges to K in the Hausdorff metric, it is easy to see that E_n converges in L^1 to $B_1^+(0) := \{x \in B_1(0) \, : \, x_2 > 0\}$. E_n has finite perimeter because $\partial E_n \subseteq H_n$ and these sets have finite $\mathcal{H}^1$ measure (see Proposition 3.62 of [3]). By Theorem 4.1 we have

$$\begin{aligned}
\int_K \varphi(x,\nu)\, d\mathcal{H}^1 = \int_{\partial^* B_1^+(0)} \varphi(x,\nu)\, d\mathcal{H}^1 \quad &\leq \quad \liminf_n \int_{\partial^* E_n} \varphi(x,\nu)\, d\mathcal{H}^1 \leq \\
&\leq \quad \liminf_n \int_{H_n} \varphi(x,\nu)\, d\mathcal{H}^1 \leq \\
&\leq \quad \liminf_n \int_{K_n} \varphi(x,\nu)\, d\mathcal{H}^1 + c_2 \limsup_n \mathcal{H}^1(L_n) \leq \\
&\leq \quad \liminf_n \int_{K_n} \varphi(x,\nu)\, d\mathcal{H}^1 + 3c_2\varepsilon.
\end{aligned}$$

Letting $\varepsilon \to 0$, we have

$$\int_K \varphi(x,\nu)\, d\mathcal{H}^1 \leq \liminf_n \int_{K_n} \varphi(x,\nu)\, d\mathcal{H}^1. \tag{4.3}$$

To obtain the thesis, we need to prove that for $\mathcal{H}^1$-almost all points x_0 of $K \cap U$

$$\limsup_{\rho \to 0^+} \frac{\mu(B_\rho(x_0))}{2\rho} \geq \varphi(x_0, \nu_{x_0}) \tag{4.4}$$

where ν_{x_0} indicates the normal to K at x_0: this is sufficient in order to compare μ_0 and μ (see Theorem 2.56 of [3]).

Up to a rotation we may assume that $\nu_{x_0} = e_2$. Let $\rho_k \to 0^+$ and let T_k be the map defined by

$$T_k(\xi) = \frac{1}{\rho_k}(\xi - x_0)$$

which brings the ball $B_{\rho_k}(x_0)$ to the unit ball of the plane. By our choice of x_0, $\mathcal{H}^1 \llcorner T_k(K)$ converges locally weakly star in the sense of measures to $\mathcal{H}^1 \llcorner H$ where H denotes the horizontal axis of the plane.

Note that for $k \to \infty$

$$T_k(K) \cap \overline{B}_1(0) \to H \cap \overline{B}_1(0) \tag{4.5}$$

in the Hausdorff metric. In fact, up to a subsequence, $T_k(K) \cap \overline{B}_1(0) \to \widetilde{K}$ by compactness of the Hausdorff metric. Clearly $H \cap \overline{B}_1(0) \subseteq \widetilde{K}$ because if $y \in (H \cap \overline{B}_1(0)) \setminus \widetilde{K}$, there exists $\rho > 0$ such that $T_k(K) \cap B_\rho(y) = \emptyset$ definitively and so

$$\mathcal{H}^1(H \cap \overline{B}_1(0) \cap B_\rho(y)) \leq \liminf_k \mathcal{H}^1(T_k(K) \cap B_\rho(y)) = 0$$

which is absurd. Conversely, $\widetilde{K} \subseteq H \cap \overline{B}_1(0)$ because if $y \in \widetilde{K} \setminus (H \cap \overline{B}_1(0))$, there exists $\rho > 0$ such that $H \cap \overline{B}_1(0) \cap \overline{B}_\rho(y) = \emptyset$ and by the inequality

$$\limsup_k \mathcal{H}^1(\overline{B}_\rho(y) \cap T_k(K)) \leq \mathcal{H}^1(\overline{B}_\rho(y) \cap H \cap \overline{B}_1(0))$$

we deduce

$$\limsup_k \mathcal{H}^1(\overline{B}_\rho(y) \cap T_k(K)) = 0. \tag{4.6}$$

But we proved that $H \cap \overline{B}_1(0) \subseteq \widetilde{K}$ and so the points of $H \cap \overline{B}_1(0)$ are limit of points of $T_k(K)$: since every $T_k(K)$ is arcwise connected (they are connected and have finite $\mathcal{H}^1$ measure), we have that $\mathcal{H}^1(\overline{B}_\rho(y) \cap T_k(K)) \geq \rho$ definitively and this contradicts (4.6).

We may suppose that ρ_k's are chosen in such a way that

$$\mu(\partial B_{\rho_k}(x_0)) = 0 \qquad \lim_n \mu_n(\overline{B}_{\rho_k}(x_0)) = \mu(B_{\rho_k}(x_0)). \tag{4.7}$$

Since $T_k(K_n) \to T_k(K)$ in the Hausdorff metric for $n \to +\infty$ by (4.5) and (4.7) there exists a subsequence n_k such that

$$T_k(K_{n_k}) \cap \overline{B}_1(0) \to H \cap \overline{B}_1(0) \tag{4.8}$$

in the Hausdorff metric for $k \to +\infty$ and

$$\mu_{n_k}(\overline{B}_{\rho_k}(x_0)) \leq \mu(B_{\rho_k}(x_0)) + \rho_k^2.$$

We now want to use the device of the first part of the proof: we employ the notation introduced before. Let $\varepsilon > 0$, $\eta > 0$, $R_\eta^\varepsilon := S_\eta \cap V^\varepsilon$, $\partial^\pm R_\eta^\varepsilon := R_\eta^\varepsilon \cap \partial^\pm V^\varepsilon$; for k large enough $T_k(K_{n_k}) \cap V^\varepsilon \subseteq R_\eta^\varepsilon$ and if $C_k^\pm$ is the connected component of $(T_k(K_{n_k}) \cap \overline{B}_1(0)) \cup \partial^- R_\eta^\varepsilon \cup \partial^+ R_\eta^\varepsilon$ containing $\partial^\pm R_\eta^\varepsilon$, we have $(T_k(K_{n_k}) \cap \overline{B}_1(0)) \cup \partial^- R_\eta^\varepsilon \cup \partial^+ R_\eta^\varepsilon = C_k^- \cup C_k^+$. In fact if $\xi \notin C_k^- \cup C_k^+$ and C_k^ξ be the connected component of $(T_k(K_{n_k}) \cap \overline{B}_1(0)) \cup \partial^- R_\eta^\varepsilon \cup \partial^+ R_\eta^\varepsilon$ containing ξ, by (4.8), $C_k^\xi \cap \partial R_\eta^\varepsilon = \emptyset$ for k large enough and so C_k^ξ would be connected against the connectedness of $T_k(K_{n_k}) \cup \partial^- R_\eta^\varepsilon \cup \partial^+ R_\eta^\varepsilon$.

By (4.8), we deduce easily that it is possible to join a point of C_k^+ and a point of C_k^- through a line $l_k \subseteq \overline{B}_1(0)$ such that $\mathcal{H}^1(l_k) \leq \varepsilon$ for k large enough.

Considering $H_k := (T_k(K_{n_k}) \cap \overline{B}_1(0)) \cup L_k \cup l_k$, H_k is connected in $\overline{B}_1(0)$ and converges to $H \cap \overline{B}_1(0)$ in the Hausdorff metric. Applying (4.3) with $\varphi = \varphi(x_0, \cdot)$, and since $\sup\{|\varphi(x_0 + \rho_k(\cdot), \nu) - \varphi(x_0, \nu)|\} \to 0$ in $\overline{B}_1(0) \times S^1$ uniformly by the continuity of φ, we get

$$2\varphi(x_0, e_2) \leq \liminf_k \int_{H_k} \varphi(x_0, \nu)\, d\mathcal{H}^1 \leq \liminf_k \int_{T_k(K_{n_k}) \cap \overline{B}_1(0)} \varphi(x_0 + \rho_k x, \nu)\, d\mathcal{H}^1 + 3c_2\varepsilon.$$

Letting $\varepsilon \to 0$, we obtain

$$2\varphi(x_0, e_2) \leq \liminf_k \int_{T_k(K_{n_k}) \cap \overline{B}_1(0)} \varphi(x_0 + \rho_k x, \nu)\, d\mathcal{H}^1. \tag{4.9}$$

Now we are ready to conclude: in fact

$$\begin{aligned}
\limsup_{\rho \to 0} \frac{\mu(B_\rho(x_0))}{2\rho} &\geq \limsup_k \frac{\mu(B_{\rho_k}(x_0))}{2\rho_k} \geq \\
&\geq \liminf_k \frac{\mu_{n_k}(\overline{B}_{\rho_k}(x_0))}{2\rho_k} = \\
&= \frac{1}{2} \liminf_k \int_{T_k(K_{n_k}) \cap \overline{B}_1(0)} \varphi(x_0 + \rho_k x, \nu)\, d\mathcal{H}^1 \geq \varphi(x_0, e_2),
\end{aligned}$$

the last inequality coming from (4.9).

Let's now turn to the case $m \geq 2$. Let $(K_n) \in \mathcal{K}_m^f(\overline{\Omega})$ converges to K; up to a subsequence, we may suppose that there exists $m' \leq m$ such that each K_n has exactly m' connected components $\widehat{K}_n^1, \cdots, \widehat{K}_n^{m'}$. We may suppose moreover that for all i, $\widehat{K}_n^i \to \widehat{K}^i$ in the Hausdorff metric: it is readily seen that $K = \cup_{i=1}^{m'} \widehat{K}^i$ so that, using the lower semicontinuity for the case $m = 1$, we obtain

$$\begin{aligned}\int_{K\cap U} \varphi(x,\nu)\,d\mathcal{H}^1 &\le \sum_{i=1}^{m'} \int_{\widehat{K}^i\cap U} \varphi(x,\nu)\,d\mathcal{H}^1 \le \\ &\le \liminf_n \sum_{i=1}^{m'} \int_{\widehat{K}^i_n\cap U} \varphi(x,\nu)\,d\mathcal{H}^1 = \\ &= \liminf_n \int_{K_n\cap U} \varphi(x,\nu)\,d\mathcal{H}^1.\end{aligned}$$

□

Theorem 3.1 is now proved: it is sufficient to apply Theorem 4.2 with $U = B_R(0)$, $\overline{\Omega} \subseteq B_R(0)$.

Corollary 4.3. *Let (H_n) be a sequence in $\mathcal{K}(\overline{\Omega})$ which converges to H in the Hausdorff metric. Let $m \ge 1$ and let (K_n) be a sequence in $\mathcal{K}^f_m(\overline{\Omega})$ which converges to K in the Hausdorff metric. Then*

$$\int_{K\setminus H} \varphi(x,\nu)\,d\mathcal{H}^1 \le \liminf_n \int_{K_n\setminus H_n} \varphi(x,\nu)\,d\mathcal{H}^1.$$

Proof. Let $\varepsilon > 0$ and let $H^\varepsilon = \{x \in \overline{\Omega} : \mathrm{dist}(x,H) \le \varepsilon\}$. Definitively $H_n \subseteq H^\varepsilon$ so that $K_n \setminus H^\varepsilon \subseteq K_n \setminus H_n$. Applying Theorem 4.2 with $U = \mathbb{R}^2 \setminus H^\varepsilon$, we have

$$\int_{K\setminus H^\varepsilon} \varphi(x,\nu)\,d\mathcal{H}^1 \le \liminf_n \int_{K_n\setminus H^\varepsilon} \varphi(x,\nu)\,d\mathcal{H}^1 \le \liminf_n \int_{K_n\setminus H_n} \varphi(x,\nu)\,d\mathcal{H}^1.$$

Letting ε go to zero, we obtain the thesis. □

The following result will be useful in sections 5 and 6.

Theorem 4.4. *Given $m \ge 1$, let (H_n) be a sequence in $\mathcal{K}^f_m(\overline{\Omega})$ which converges to H in the Hausdorff metric, and let $K \in \mathcal{K}^f_m(\overline{\Omega})$ with $H \subseteq K$. Then there exists a sequence (K_n) in $\mathcal{K}^f_m(\overline{\Omega})$ which converges to K in the Hausdorff metric and such that $H_n \subseteq K_n$ and*

$$\int_{K\setminus H} \varphi(x,\nu)\,d\mathcal{H}^1 = \lim_n \int_{K_n\setminus H_n} \varphi(x,\nu)\,d\mathcal{H}^1. \tag{4.10}$$

Proof. Following Lemma 3.8 of [8], the connected components C_i of $K \setminus H$ are at least countable and satisfy $\mathcal{H}^1(\overline{C_i}) = \mathcal{H}^1(C_i)$. Since $H_n \to H$ in the Hausdorff metric and Ω has Lipschitz boundary, we can find arcs Z^i_n in $\overline{\Omega}$ joining H_n and C_i such that $\mathcal{H}^1(Z^i_n) \to 0$ as $n \to \infty$. Given h, consider $K^h_n := \cup_{i=1}^h Z^i_n \cup \cup_{i=1}^h \overline{C}_i$; we have $K^h_n \in \mathcal{K}^f_h(\overline{\Omega})$, $K^h_n \to K^h := \cup_{i=1}^h \overline{C}_i$ in the Hausdorff metric. Note that $\nu\mathcal{H}^1 \llcorner K^h_n \to \nu\mathcal{H}^1 \llcorner K^h$ strictly for $n \to \infty$. By Theorem 2.3, since $\mathcal{H}^1(\overline{C_i}) = \mathcal{H}^1(C_i)$, we have

$$\begin{aligned}\lim_n \int_{K^h_n} \varphi(x,\nu)\,d\mathcal{H}^1 &= \int_{K^h} \varphi(x,\nu)\,d\mathcal{H}^1 = \\ &= \int_{\cup_{i=1}^h C_i} \varphi(x,\nu)\,d\mathcal{H}^1 \le \int_{K\setminus H} \varphi(x,\nu)\,d\mathcal{H}^1.\end{aligned}$$

Choose $h_n \to +\infty$ such that

$$\limsup_n \int_{K^{h_n}_n} \varphi(x,\nu)\,d\mathcal{H}^1 \le \int_{K\setminus H} \varphi(x,\nu)\,d\mathcal{H}^1,$$

so that

$$\lim_n \sum_{i=1}^{h_n} \mathcal{H}^1(Z^i_n) = 0.$$

If we pose $K_n := H_n \cup K_n^{h_n}$, we have $K_n \in \mathcal{K}_m^f(\overline{\Omega})$, $K_n \to K$ in the Hausdorff metric and

$$\limsup_n \int_{K_n \setminus H_n} \varphi(x,\nu)\, d\mathcal{H}^1 \leq \limsup_n \int_{K_n^{h_n}} \varphi(x,\nu)\, d\mathcal{H}^1 \leq \int_{K\setminus H} \varphi(x,\nu)\, d\mathcal{H}^1.$$

The converse inequality comes from Corollary 4.3. □

5. THE ANTI-PLANAR ANISOTROPIC CASE

In this section we deal with quasi-static growth of brittle fractures in inhomogeneous anisotropic linearly elastic bodies under anti-planar displacements. We employ the notation of Section 3.

We begin with the following lemma which extends Theorem 2.4 considering boundary data. The idea is due to A. Chambolle.

Lemma 5.1. *Let $m \geq 1$, K_n a sequence in $\mathcal{K}_m(\overline{\Omega})$ which converges to K in the Hausdorff metric and such that $\mathcal{L}^2(\Omega \setminus K_n) \to \mathcal{L}^2(\Omega \setminus K)$. Let $g_n \to g$ strongly in $H^1(\Omega)$ and let $\Gamma(g_n, K_n)$ and $\Gamma(g, K)$ be the sets introduced in (3.4). Then for every $u \in \Gamma(g, K)$, there exists $u_n \in \Gamma(g_n, K_n)$ such that $\nabla u_n \to \nabla u$ strongly in $L^2(\Omega, \mathbb{R}^2)$.*

Proof. Consider Ω' a regular open set containing $\overline{\Omega}$ and pose $\partial_N\Omega := \partial\Omega \setminus \partial_D\Omega$. Since $\partial\Omega$ is regular, we may extend g_n and g to $H^1(\Omega')$ and suppose $g_n \to g$ strongly in $H^1(\Omega')$. Note that if $H_n := K_n \cup \overline{\partial_N\Omega}$ and $H = K \cup \overline{\partial_N\Omega}$, $H_n, H \in \mathcal{K}_{m'}(\overline{\Omega'})$, $H_n \to H$ in the Hausdorff metric and $\mathcal{L}^2(\Omega' \setminus H_n) \to \mathcal{L}^2(\Omega' \setminus H)$. Consider

$$v := \begin{cases} u & \text{in } \Omega \\ g & \text{in } \Omega' \setminus \Omega \end{cases}$$

Clearly $v \in L^{1,2}(\Omega' \setminus H)$; we may apply Theorem 2.4 and deduce that there exists $v_n \in L^{1,2}(\Omega' \setminus H_n)$ such that $\nabla v_n \to \nabla v$ strongly in $L^2(\Omega', \mathbb{R}^2)$. Note that we may assume $(v_n - v)$ has null average on $\Omega' \setminus \overline{\Omega}$, because we are allowed to add constants to v_n; since $\Omega' \setminus \overline{\Omega}$ is regular, by Poincaré inequality we obtain $v_n \to v$ strongly in $H^1(\Omega' \setminus \overline{\Omega})$. Let E_Ω be a linear extension operator from $H^1(\Omega' \setminus \overline{\Omega})$ to $H^1(\Omega')$. If $w_n := (v_n - v)|_{\Omega' \setminus \overline{\Omega}}$, we can choose

$$u_n := v_n - E_\Omega w_n + (g_n - g)$$

restricted to Ω. It is readily seen that $u_n \in \Gamma(g_n, K_n)$ and $\nabla u_n \to \nabla u$ strongly in $L^2(\Omega, \mathbb{R}^2)$. □

By standard arguments, it can be proved that the minimum of problem (3.3) is attained. Moreover, it can be shown that, since $\Omega \setminus K$ is not guaranteed to be regular, this minimum is in general not attained in $H^1(\Omega \setminus K)$ when the boundary data g is not bounded: the reader is referred to [11]. The following proposition deals with the behavior of minima when the compact set K varies.

Proposition 5.2. *Let $m \geq 1$, $\lambda \geq 0$, (K_n) a sequence in $\mathcal{K}_m^\lambda(\overline{\Omega})$ which converges to K in the Hausdorff metric, (g_n) a sequence in $H^1(\Omega)$ which converges to g strongly in $H^1(\Omega)$. Let u_n be a solution of the minimum problem*

$$\min_{v \in \Gamma(g_n, K_n)} ||\nabla u||_a^2 \tag{5.1}$$

and let u be a solution of the minimum problem

$$\min_{v \in \Gamma(g, K)} ||\nabla u||_a^2, \tag{5.2}$$

where $\Gamma(g_n, K_n)$ and $\Gamma(g, K)$ are defined as in (3.4).

Then $\nabla u_n \to \nabla u$ strongly in $L^2(\Omega, \mathbb{R}^2)$.

Proof. Using g_n as test function, we obtain

$$||\nabla u_n||_a \leq ||\nabla g_n||_a \leq c < +\infty.$$

By (3.2), there exists $\psi \in L^2(\Omega, \mathbb{R}^2)$ such that, up to a subsequence, $\nabla u_n \rightharpoonup \psi$ weakly in $L^2(\Omega, \mathbb{R}^2)$. It is not difficult to prove that there exists $u \in L^2_{loc}(\Omega)$ such that $\nabla u = \psi$ in $\Omega \setminus K$. Moreover by means of Poincaré inequality, we deduce that $u = g$ on $\partial_D \Omega \setminus K$. According to Lemma 5.1, let $v_n \in \Gamma(g_n, K_n)$ with $\nabla v_n \to \nabla u$ strongly in $L^2(\Omega, \mathbb{R}^2)$; since $||u_n||_a \leq ||v_n||_a$ by minimality of u_n, we obtain $\limsup_n ||u_n||_a \leq ||u||_a$. This proves $\nabla u_n \to \nabla u$ strongly in $L^2(\Omega, \mathbb{R}^2)$. □

We now turn to the proof of Theorem 3.2. We use a discretization in time. Given $\delta > 0$, let N_δ be the largest integer such that $\delta N_\delta \leq 1$; for $i \geq 0$ we pose $t_i^\delta = i\delta$ and for $0 \leq i \leq N_\delta$ we pose $g_i^\delta = g(t_i^\delta)$. Define K_i^δ as a solution of the minimum problem

$$\min_K \left\{ \mathcal{E}(g_i^\delta, K) \,:\, K \in \mathcal{K}_m^f(\overline{\Omega}),\, K_{i-1}^\delta \subseteq K \right\}, \tag{5.3}$$

where $K_{-1}^\delta = K_0$.

Lemma 5.3. *The minimum problem (5.3) admits a solution.*

Proof. We proceed by induction. Suppose K_{i-1}^δ is constructed and that $\lambda > \mathcal{E}(g_i^\delta, K_{i-1}^\delta)$. Let (K_n) be a minimizing sequence of problem (5.3) and let u_n be a solution of the minimum problem (3.3) which defines $\mathcal{E}(g_i^\delta, K_n)$. Up to a subsequence, $K_n \to K$ in the Hausdorff metric and $K_{i-1}^\delta \subseteq K$. Since

$$||\nabla u_n||_a^2 + \int_{K_n} \varphi(x, \nu)\, d\mathcal{H}^1 \leq \lambda$$

for n large, we have that

$$\int_{K_n} \varphi(x, \nu)\, d\mathcal{H}^1 \leq \lambda.$$

We have $K_n \in \mathcal{K}_m^{\alpha_1^{-1}\lambda}(\overline{\Omega})$ and applying Proposition 5.2, we have $||u_n||_a \to ||u||_a$ where u is a solution of problem (3.3) which defines $\mathcal{E}(g_i^\delta, K)$; moreover by Theorem 4.2, we get

$$\int_K \varphi(x, \nu)\, d\mathcal{H}^1 \leq \liminf_n \int_{K_n} \varphi(x, \nu)\, d\mathcal{H}^1 \leq \lambda.$$

Thus $K \in \mathcal{K}_m^f(\overline{\Omega})$ and $\mathcal{E}(g_i^\delta, K) \leq \liminf_n \mathcal{E}(g_i^\delta, K_n)$. We conclude that K is a solution of the minimum problem (5.3). □

Now, consider the following piecewise constant interpolation: put $g^\delta(t) = g_i^\delta$, $K^\delta(t) = K_i^\delta$, $u^\delta(t) = u_i^\delta$ for $t_i^\delta \leq t < t_{i+1}^\delta$, where u_i^δ is a solution of problem (3.3) which defines $\mathcal{E}(g_i^\delta, K_i^\delta)$.

Lemma 5.4. *There exists a positive function $\rho(\delta)$, converging to zero as $\delta \to 0$, such that for all $s < t$ in* $[0, 1]$,

$$\begin{aligned} ||\nabla u^\delta(t)||_a^2 + \int_{K^\delta(t)} \varphi(x, \nu)\, d\mathcal{H}^1 \quad &\leq \quad ||\nabla u^\delta(s)||_a^2 + \int_{K^\delta(s)} \varphi(x, \nu)\, d\mathcal{H}^1 + \\ &\quad +2 \int_{t_i^\delta}^{t_j^\delta} (\nabla u^\delta(t), \nabla \dot{g}(t))_a\, dt + \rho(\delta) \end{aligned} \tag{5.4}$$

where $t_i^\delta \leq s < t_{i+1}^\delta$ and $t_j^\delta \leq t < t_{j+1}^\delta$.

Proof. Inequality (5.4) is precisely

$$||\nabla u_j^\delta||_a^2 + \int_{K_j^\delta} \varphi(x, \nu)\, d\mathcal{H}^1 \leq ||\nabla u_i^\delta||_a^2 + \int_{K_i^\delta} \varphi(x, \nu)\, d\mathcal{H}^1 + 2 \int_{t_i^\delta}^{t_j^\delta} (\nabla u^\delta(t), \nabla \dot{g}(t))_a\, dt + \rho(\delta).$$

To obtain this one, it is sufficient to adapt the proof of Lemma 7.3 in [8]. □

Lemma 5.5. *There exists a constant C, depending only on g and K_0, such that*

$$||\nabla u^\delta(t)||_a \leq C \qquad \int_{K^\delta(t)} \varphi(x,\nu)\, d\mathcal{H}^1 \leq C$$

for every $\delta > 0$ and $t \in [0,1]$. In particular, there exists $\lambda > 0$ such that for all $t \in [0,1]$, $K^\delta(t) \in \mathcal{K}_m^\lambda(\overline{\Omega})$.

Proof. Put $\eta = \max_t\{||\nabla g(t)||_a, ||\nabla \dot{g}(t)||_a\}$. Clearly $||\nabla u^\delta(t)||_a \leq ||\nabla g^\delta(t)||_a \leq \eta$ since $g^\delta(t)$ is an admissible displacement for $K^\delta(t)$. Clearly from inequality (5.4) with $s = 0$, we obtain

$$\begin{aligned} ||\nabla u^\delta(t)||_a^2 + \int_{K^\delta(t)} \varphi(x,\nu)\, d\mathcal{H}^1 &\leq ||\nabla u^\delta(0)||_a^2 + \int_{K^\delta(0)} \varphi(x,\nu)\, d\mathcal{H}^1 + \\ &\quad +2\int_0^{t_j^\delta} (\nabla u^\delta(t), \nabla \dot{g}(t))_a\, dt + \rho(\delta) \leq \\ &\leq ||\nabla u_0^\delta||_a^2 + \int_{K_0} \varphi(x,\nu)\, d\mathcal{H}^1 + 2\eta^2 + \rho(\delta). \end{aligned}$$

The last term depends only on g and K_0 and so we obtain the first part of the thesis. The second one comes from (3.1). □

Lemma 5.6. *Let C be the constant of Lemma 5.5. There exists an increasing function $K : [0,1] \to \mathcal{K}_m^f(\overline{\Omega})$ (that is $K(s) \subseteq K(t)$ for every $0 \leq s \leq t \leq 1$), such that, for every $t \in [0,1]$, $K^\delta(t)$ converges to $K(t)$ in the Hausdorff metric as $\delta \to 0$ along a suitable sequence independent of t. Moreover if $u(t)$ is a solution of the minimum problem (3.3) which defines $\mathcal{E}(g(t), K(t))$, for every $t \in [0,1]$ we have $\nabla u^\delta(t) \to \nabla u(t)$ strongly in $L^2(\Omega, \mathbb{R}^2)$.*

Proof. The first part is a variant of Helly's theorem for monotone function: for a proof see Lemma 7.5 of [8]; the second part comes directly from Lemma 5.5 and Proposition 5.2. □

Fix now the sequence (δ_n) and the increasing map $t \to K(t)$ given by Lemma 5.6. We indicate $K^{\delta_n}(t)$ by $K_n(t)$ and $u^{\delta_n}(t)$ by $u_n(t)$.

The following property of the pair $(g(t), K(t))$ is important for subsequent results.

Lemma 5.7. *For every $t \in [0,1]$ we have*

$$\mathcal{E}(g(t), K(t)) \leq \mathcal{E}(g(t), K) \quad \forall K \in \mathcal{K}_m^f(\overline{\Omega}),\ K(t) \subseteq K. \tag{5.5}$$

Moreover

$$\mathcal{E}(g(0), K(0)) \leq \mathcal{E}(g(0), K) \quad \forall K \in \mathcal{K}_m^f(\overline{\Omega}),\ K_0 \subseteq K. \tag{5.6}$$

Proof. Let $t \in [0,1]$ and $K \in \mathcal{K}_m^f(\overline{\Omega})$ with $K(t) \subseteq K$. Since $K_n(t) \to K(t)$ in the Hausdorff metric as $\delta_n \to 0$, by Theorem 4.4 there exists a sequence (K_n) in $\mathcal{K}_m^f(\overline{\Omega})$ converging to K in the Hausdorff metric, such that $K_n(t) \subseteq K_n$ and

$$\int_{K_n \setminus K_n(t)} \varphi(x,\nu)\, d\mathcal{H}^1 \to \int_{K \setminus K(t)} \varphi(x,\nu)\, d\mathcal{H}^1. \tag{5.7}$$

By Lemma 5.5, there exists $\lambda > 0$ such that $K_n(t) \in \mathcal{K}_m^\lambda(\overline{\Omega})$ for all n. By (5.7), we deduce that there exists $\lambda' > \lambda$ with $K_n \in \mathcal{K}_m^{\lambda'}(\overline{\Omega})$ for all n.

Let v_n and v solutions of problems (3.3) which define $\mathcal{E}(g_n(t), K_n)$ and $\mathcal{E}(g(t), K)$. By minimality of $K_n(t)$ we have $\mathcal{E}(g_n(t), K_n(t)) \leq \mathcal{E}(g_n(t), K_n)$ and so

$$||\nabla u_n(t)||_a^2 \leq ||\nabla v_n||_a^2 + \int_{K_n \setminus K_n(t)} \varphi(x,\nu)\, d\mathcal{H}^1; \tag{5.8}$$

as $\delta_n \to 0$, $\nabla u_n(t) \to \nabla u(t)$ and $\nabla v_n \to \nabla v$ strongly in $L^2(\Omega, \mathbb{R}^2)$ by Proposition 5.2: passing to the limit in (5.8) and adding to both sides $\int_{K(t)} \varphi(x,\nu)\, d\mathcal{H}^1$, by (5.7) we have the thesis.

A similar proof holds for (5.6). □

Lemma 5.8. *The function $t \to \mathcal{E}(g(t), K(t))$ is absolutely continuous and*

$$\frac{d}{dt}\mathcal{E}(g(t), K(t)) = 2(\nabla u(t), \nabla \dot{g}(t))_a \quad \textit{for a.e } t \in [0,1]$$

where $u(t)$ is a solution of the minimum problem (3.3) which defines $\mathcal{E}(g(t), K(t))$.

Proof. We rewrite (5.4) in the following form

$$||\nabla u_n(t)||_a^2 + \int_{K_n(t)\setminus K_n(s)} \varphi(x,\nu)\, d\mathcal{H}^1 \leq ||\nabla u_n(s)||_a^2 + 2\int_{t_i^{\delta_n}}^{t_j^{\delta_n}} (\nabla u_n(t), \nabla \dot{g}(t))_a\, dt + \rho(\delta_n)$$

for $s \leq t$ and $t_i^{\delta_n} \leq s < t_{i+1}^{\delta_n}$ and $t_j^{\delta_n} \leq t < t_{j+1}^{\delta_n}$. Passing to the limit for $\delta_n \to 0$, using Corollary 4.3 we obtain

$$||\nabla u(t)||_a^2 + \int_{K(t)\setminus K(s)} \varphi(x,\nu)\, d\mathcal{H}^1 \leq ||\nabla u(s)||_a^2 + 2\int_s^t (\nabla u(\tau), \nabla \dot{g}(\tau))_a\, d\tau,$$

so that

$$\begin{aligned} ||\nabla u(t)||_a^2 + \int_{K(t)} \varphi(x,\nu)\, d\mathcal{H}^1 \quad &\leq \quad ||\nabla u(s)||_a^2 + \int_{K(s)} \varphi(x,\nu)\, d\mathcal{H}^1 + \\ &\quad +2\int_s^t (\nabla u(\tau), \nabla \dot{g}(\tau))_a\, d\tau. \end{aligned}$$

Following Lemma 6.5 of [8], we can prove that the function $F(g) := \mathcal{E}(g, K(t))$ is differentiable on $H^1(\Omega)$ and its differential is given by $dF(g)h = 2(\nabla u(t), \nabla h)_a$ where $u(t)$ is a solution of problem (3.3) which defines $\mathcal{E}(g, K(t))$. By Lemma 5.7, we obtain

$$\mathcal{E}(g(t), K(t)) - \mathcal{E}(g(s), K(s)) \geq \mathcal{E}(g(t), K(t)) - \mathcal{E}(g(s), K(t)) = 2\int_s^t (\nabla u(\tau,t), \nabla \dot{g}(\tau))_a d\tau$$

where $u(\tau, t)$ is a solution of the minimum problem (3.3) which defines $\mathcal{E}(g(\tau), K(t))$. We can conclude that $t \to \mathcal{E}(g(t), K(t))$ is absolutely continuous since $||\nabla u(t)||_a$ and $||\nabla u(\tau,t)||_a$ are bounded by Lemma 5.5. Moreover, dividing the previous inequalities by $t-s$ and letting $s \to t$, since $\nabla u(\tau, t) \to \nabla u(t)$ strongly in $L^2(\Omega, \mathbb{R}^2)$ for $\tau \to t$, we obtain

$$\frac{d}{dt}\mathcal{E}(g(t), K(t)) = 2(\nabla u(t), \nabla \dot{g}(t))_a \quad \textit{for a.e } t \in [0,1].$$

□

We now turn to the proof of Theorem 3.2. Points (a) and (b) are proved in lemmas 5.6 and 5.7 while points (d) and (e) are proved in Lemma 5.8. Point (f) and its equivalence to point (e) stated in Remark 3.4 are proved adapting Lemma 6.4 of [8]. To prove point (c), we need the following lemma.

Lemma 5.9. *Let $K : [0,1] \to \mathcal{K}_m^f(\overline{\Omega})$ be a map which satisfies lemmas 5.7 and 5.8. Then for every $t \in]0,1]$,*

$$\mathcal{E}(g(t), K(t)) \leq \mathcal{E}(g(t), K) \quad \forall K \in \mathcal{K}_m^f(\overline{\Omega}) : \cup_{s<t} K(s) \subseteq K.$$

Proof. Consider $t \in]0,1]$ and $K \in \mathcal{K}_m^f(\overline{\Omega})$ such that $\cup_{s<t}K(s) \subseteq K$. For $0 \leq s < t$ we have $K(s) \subseteq K$ and so by Lemma 5.7, $\mathcal{E}(g(s), K(s)) \leq \mathcal{E}(g(s), K)$. By Lemma 5.8, these expressions continuously depend on s and so passing to the limit for $s \to t$, we obtain the thesis. □

Consider now the particular case in which $g(0) = 0$: there exists a solution $K(t)$ to the problem of evolution such that $K(0) = K_0$ because in the time discretization method employed, we can choose $K^\delta(0) = K_0$. Under this assumption, we prove that this method gives an approximation of the energy of the solution.

We pose

$$\mathcal{E}_n(t) = ||\nabla u_n(t)||_a^2 + \int_{K_n(t)} \varphi(x,\nu)\, d\mathcal{H}^1$$

and

$$\mathcal{E}(t) = \mathcal{E}(g(t), K(t)) = ||\nabla u(t)||_a^2 + \int_{K(t)} \varphi(x,\nu)\, d\mathcal{H}^1.$$

The following convergence result holds.

Proposition 5.10. *For all $t \in [0,1]$ the following facts hold:*

(a) $K_n(t) \to K(t)$ *in the Hausdorff metric;*
(b) $\nabla u_n(t) \to \nabla u(t)$ *strongly in* $L^2(\Omega, \mathbb{R}^2)$*;*
(c) $\int_{K_n(t)} \varphi(x,\nu)\, d\mathcal{H}^1 \to \int_{K(t)} \varphi(x,\nu)\, d\mathcal{H}^1$.

In particular $\mathcal{E}_n(t) \to \mathcal{E}(t)$ *for all* $t \in [0,1]$.

Proof. We have already proved points (a) and (b) in Lemma 5.6. Since the functions $t \to \int_{K_n(t)} \varphi(x,\nu)\, d\mathcal{H}^1$ are increasing and bounded, we may suppose that, by Helly's theorem, they converge pointwise to a bounded increasing function $h : [0,1] \to [0,\infty[$ i.e. for all $t \in [0,1]$

$$\lim_n \int_{K_n(t)} \varphi(x,\nu)\, d\mathcal{H}^1 = h(t).$$

Moreover by Theorem 4.2 we have that $\int_{K(t)} \varphi(x,\nu)\, d\mathcal{H}^1 \le h(t)$ for all $t \in [0,1]$ and by construction $\lambda(0) = \int_{K(0)} \varphi(x,\nu)\, d\mathcal{H}^1$; in particular we have for all $t \in [0,1]$

$$\mathcal{E}(t) \le ||\nabla u(t)||_a^2 + h(t)$$

and $\mathcal{E}(0) = ||\nabla u(0)||_a^2 + h(0)$. Passing to the limit in (5.4), by (b) we obtain

$$||\nabla u(t)||_a^2 + h(t) \le ||\nabla u(s)||_a^2 + h(s) + 2\int_s^t (\nabla u(t), \nabla \dot{g}(t))_a\, dt. \tag{5.9}$$

Since by condition (e) of Theorem 3.2

$$\mathcal{E}(t) - \mathcal{E}(0) = 2\int_0^t (\nabla u(\tau), \nabla \dot{g}(\tau))_a\, d\tau,$$

we have

$$\begin{aligned} ||\nabla u(t)||_a^2 + h(t) \quad &- \quad \mathcal{E}(t) = \\ &\le \quad 2\int_0^t (\nabla u(\tau), \nabla \dot{g}(\tau))_a\, d\tau - 2\int_0^t (\nabla u(\tau), \nabla \dot{g}(\tau))_a\, d\tau = 0. \end{aligned}$$

We conclude that $h(t) = \int_{K(t)} \varphi(x,\nu)\, d\mathcal{H}^1$ for all $t \in [0,1]$. This proves point (c) and the thesis is obtained. □

6. THE PLANAR ANISOTROPIC CASE

In this section we breafily sketch the modifications of the arguments used in the previous section in order to deal with the evolution of fractures in inhomogeneous anisotropic linearly elastic bodies under planar displacements. We employ the notation of Section 3.

The following lemma can be obtained with arguments similar to those of Lemma 5.1.

Lemma 6.1. *Let $m \ge 1$, K_n a sequence in $\mathcal{K}_m(\overline{\Omega})$ which converges to K in the Hausdorff metric and such that $\mathcal{L}^2(\Omega \setminus K_n) \to \mathcal{L}^2(\Omega \setminus K)$. Let $g_n \to g$ strongly in $H^1(\Omega, \mathbb{R}^2)$ and let $\mathcal{V}(g_n, K_n)$ and $\mathcal{V}(g, K)$ be the sets introduced in (3.6). Then for every $u \in \mathcal{V}(g, K)$, there exists $u_n \in \mathcal{V}(g_n, K_n)$ such that $Eu_n \to Eu$ strongly in $L^2(\Omega, \mathrm{M}^{2\times 2}_{\mathrm{sym}})$.*

By standard techniques, it can be proved that the minimum in problem (3.5) is attained. The following result is similar to Proposition 5.2 and deals with the behavior of these minima as K varies.

Proposition 6.2. *Let $m \geq 1$ and $\lambda \geq 0$, let K_n be a sequence in $\mathcal{K}_m^\lambda(\overline{\Omega})$ which converges to K in the Hausdorff metric, and let g_n be a sequence in $H^1(\Omega)$ which converges to g strongly in $H^1(\Omega)$. Let u_n be a solution of the minimum problem*

$$\min_{v \in \mathcal{V}(g_n, K_n)} ||Ev||_A^2, \tag{6.1}$$

and let u be a solution of the minimum problem

$$\min_{v \in \mathcal{V}(g, K)} ||Ev||_A^2 \tag{6.2}$$

where $\mathcal{V}(g_n, K_n)$ and $\mathcal{V}(g, K)$ are defined as in (3.6).

Then $Eu_n \to Eu$ strongly in $L^2(\Omega, \mathrm{M}^{2\times 2}_{\mathrm{sym}})$.

Proof. Using g_n as test function we obtain $||Eu_n||_A \leq ||Eg_n||_A \leq c < +\infty$. By assumption on A, there exists $\sigma \in L^2(\Omega, \mathrm{M}^{2\times 2}_{\mathrm{sym}})$ such that up to a subsequence $Eu_n \rightharpoonup \sigma$ weakly in $L^2(\Omega, \mathrm{M}^{2\times 2}_{\mathrm{sym}})$. It is not difficult to prove that there exists $u \in L^2_{loc}(\Omega, \mathbb{R}^2)$ such that $Eu = \sigma$ in $\Omega \setminus K$. Moreover by means of Korn-Poincaré inequality, we deduce that $u = g$ on $\partial_D \Omega \setminus K$. According to Lemma 6.1, let $v_n \in \mathcal{V}(g_n, K_n)$ with $Ev_n \to Eu$ strongly in $L^2(\Omega, \mathrm{M}^{2\times 2}_{\mathrm{sym}})$; since $||Eu_n||_A \leq ||Ev_n||_A$ by minimality of u_n, we obtain

$$\limsup_n ||Eu_n||_A \leq \limsup_n ||Ev_n||_A = ||Eu||_A.$$

This proves $Eu_n \to Eu$ strongly in $L^2(\Omega, \mathrm{M}^{2\times 2}_{\mathrm{sym}})$. □

We employ again a time discretization process. As before given $\delta > 0$, let N_δ be the largest integer such that $\delta N_\delta \leq 1$; for $i \geq 0$ we pose $t_i^\delta = i\delta$ and for $0 \leq i \leq N_\delta$ we pose $g_i^\delta = g(t_i^\delta)$. Define K_i^δ as a solution of the minimum problem

$$\min_K \left\{ \mathcal{G}(g_i^\delta, K) \,:\, K \in \mathcal{K}_m^f(\overline{\Omega}),\, K_{i-1}^\delta \subseteq K \right\}, \tag{6.3}$$

where $K_{-1}^\delta = K_0$.

Lemma 6.3. *The minimum problem (6.3) admits a solution.*

Proof. We proceed by induction. Suppose K_{i-1}^δ is constructed and that $\lambda > \mathcal{G}(g_i^\delta, K_{i-1}^\delta)$. Let (K_n) be a minimizing sequence of problem (6.3) and let u_n be a solution of the minimum problem (3.5) which defines $\mathcal{G}(g_i^\delta, K_n)$. Up to a subsequence $K_n \to K$ in the Hausdorff metric and $K_{i-1}^\delta \subseteq K$. Since

$$||Eu_n||_A^2 + \int_{K_n} \varphi(x, \nu)\, d\mathcal{H}^1 \leq \lambda$$

for n large enough, we have that

$$\int_{K_n} \varphi(x, \nu)\, d\mathcal{H}^1 \leq \lambda;$$

We have $K_n \in \mathcal{K}_m^{\alpha_1^{-1}\lambda}(\overline{\Omega})$ and applying Proposition 6.2, we have $||Eu_n||_A \to ||Eu||_A$ where u is a solution of problem (3.5) which defines $\mathcal{E}(g_i^\delta, K)$; by Theorem 4.2, we get

$$\int_K \varphi(x, \nu)\, d\mathcal{H}^1 \leq \liminf_n \int_{K_n} \varphi(x, \nu)\, d\mathcal{H}^1 \leq \lambda.$$

Thus $K \in \mathcal{K}_m^f(\overline{\Omega})$ and $\mathcal{G}(g_i^\delta, K) \leq \liminf_n \mathcal{G}(g_i^\delta, K_n)$. We conclude that K is a solution of the minimum problem (6.3). □

Consider as before the piecewise constant interpolation obtained putting $g^\delta(t) = g_i^\delta$, $K^\delta(t) = K_i^\delta$, $u^\delta(t) = u_i^\delta$ for $t_i^\delta \le t < t_{i+1}^\delta$, where u_i^δ is a solution of problem (3.5) which defines $\mathcal{G}(g_i^\delta, K_i^\delta)$.

Lemma 6.4. *There exists a positive function $\rho(\delta)$, converging to zero as $\delta \to 0$, such that for all $s < t$ in $[0,1]$*

$$\begin{aligned} ||Eu^\delta(t)||_A^2 + \int_{K^\delta(t)} \varphi(x,\nu)\, d\mathcal{H}^1 \quad &\le \quad ||Eu^\delta(s)||_A^2 + \int_{K^\delta(s)} \varphi(x,\nu)\, d\mathcal{H}^1 + \\ &+2\int_{t_i^\delta}^{t_j^\delta} (Eu^\delta(t), E\dot{g}(t))_A\, dt + \rho(\delta) \end{aligned}$$

where $t_i^\delta \le s < t_{i+1}^\delta$ and $t_j^\delta \le t < t_{j+1}^\delta$. In particular there exists $C > 0$ depending only on g and K_0 such that for all $t \in [0,1]$

$$||Eu^\delta(t)||_A \le C \qquad \int_{K^\delta(t)} \varphi(x,\nu)\, d\mathcal{H}^1 \le C.$$

Proof. It is sufficient to adapt lemmas 5.4 and 5.5. □

Using Proposition 6.2 and the previous lemma we obtain

Lemma 6.5. *There exists an increasing function $K : [0,1] \to \mathcal{K}_m^f(\overline{\Omega})$ (that is $K(s) \subseteq K(t)$ for every $0 \le s \le t \le 1$), such that, for every $t \in [0,1]$, $K^\delta(t)$ converges to $K(t)$ in the Hausdorff metric as $\delta \to 0$ along a suitable sequence independent of t. Moreover if $u(t)$ is a solution of the minimum problem (3.5) which defines $\mathcal{G}(g(t), K(t))$, for every $t \in [0,1]$ we have $Eu^\delta(t) \to Eu(t)$ strongly in $L^2(\Omega, \mathrm{M}^{2\times 2}_{\mathrm{sym}})$.*

The proof of Theorem 3.3 can now be obtained using arguments similar to those of lemmas 5.7, 5.8 and 5.9 of Section 5.

Consider now the particular case in which $g(0) = 0$: there exists a solution $K(t)$ to the problem of evolution such that $K(0) = K_0$ because in the time discretization method employed we can choose $K^\delta(0) = K_0$. Under this assumption, as in the anti-planar case, the discretization method gives an approximation of the energy of the solution.

In fact, if we pose $K_n(t) := K^{\delta_n}(t)$ and

$$\mathcal{G}_n(t) := ||Eu_n(t)||_A^2 + \int_{K_n(t)} \varphi(x,\nu)\, d\mathcal{H}^1,$$

$$\mathcal{G}(t) := \mathcal{G}(g(t), K(t)) = ||Eu(t)||_A^2 + \int_{K(t)} \varphi(x,\nu)\, d\mathcal{H}^1,$$

the following approximation result holds.

Proposition 6.6. *As $\delta_n \to 0$ for all $t \in [0,1]$ the following facts hold:*

(a) *$K_n(t) \to K(t)$ in the Hausdorff metric;*
(b) *$Eu_n(t) \to Eu(t)$ strongly in $L^2(\Omega, \mathrm{M}^{2\times 2}_{\mathrm{sym}})$;*
(c) *$\int_{K_n(t)} \varphi(x,\nu)\, d\mathcal{H}^1 \to \int_{K(t)} \varphi(x,\nu)\, d\mathcal{H}^1$.*

In particular $\mathcal{G}_n(t) \to \mathcal{G}(t)$ for all $t \in [0,1]$.

Proof. It is sufficient to adapt Proposition 5.10. □

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