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WELL-POSEDNESS AND LONGTIME BEHAVIOR FOR A SINGULAR PHASE FIELD SYSTEM WITH PERTURBED PHASE DYNAMICS

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Abstract. We consider a singular phase field system located in a smooth bounded domain. In the entropy balance equation appears a logarithmic nonlinearity. The second equation of the system, deduced from a balance law for the microscopic forces that are responsible for the phase transition process, is perturbed by an additional term involving a possibly nonlocal maximal monotone operator and arising from a class of sliding mode control problems. We prove existence and uniqueness of the solution for this resulting highly nonlinear system. Moreover, under further assumptions, the longtime behavior of the solution is investigated.

1. **Introduction.** This paper is devoted to the mathematical analysis of a system of partial differential equations (PDE) arising from a thermodynamic model describing phase transitions. The system is written in terms of a rescaled balance of energy and of a balance law for the microforces that govern the phase transition. Moreover, the second equation of the system is perturbed by the presence of an additional maximal monotone nonlinearity. This paper will focus only on analytical aspects and, in particular, will investigate existence, uniqueness and longtime behavior of the solution. In order to make the presentation clear from the beginning, let us briefly introduce the main ingredients of the PDE system and give some comments on the physical meaning.

We consider a two-phase system located in a smooth bounded domain $\Omega \subseteq \mathbb{R}^3$ and let $T > 0$ denote a final time. The unknowns of the problem are the *absolute temperature* ϑ and a *phase parameter* χ which may represent the local proportion of one of the two phases. To ensure thermomechanical consistency, suitable physical constraints on χ are introduced: if it is assumed, e.g., that the two phases may coexist at each point with different proportions, it turns out to be reasonable to require that χ lies between 0 and 1, with $1 - \chi$ representing the proportion of the

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second phase. In particular, the values $\chi = 0$ and $\chi = 1$ may correspond to the pure phases, while χ is between 0 and 1 in the regions when both phases are present. Clearly, the model should provide an evolution for χ that complies with the previous physical constraint.

Now, let us state precisely the equations as well as the initial and boundary conditions. The two equations governing the evolution of ϑ and χ are recovered as balance laws. The first equation is obtained as a reduction of the energy balance equation divided by the absolute temperature ϑ (see [7, formulas (2.33)–(2.35)]). Hence, the so-called entropy balance can be written in $\Omega \times (0, T)$ as follows:

$$\partial_t(\ln \vartheta + \ell\chi) - k_0 \Delta\vartheta = F, \tag{1}$$

where $k_0 > 0$ is a thermal coefficient for the entropy flux, ℓ is a positive parameter and F stands for an external entropy source. We point out that in the previous equation one finds the entropy flux $\mathbf{Q}$, related to the heat flux vector $\mathbf{q}$ by $\mathbf{Q} = \mathbf{q}/\vartheta$, and specified by $\mathbf{Q}(t) = -k_0 \nabla\vartheta(t)$, $t \in (0, T)$. Moreover, due to the presence of the logarithm of the temperature in the entropy equation (1), the positivity of the variable representing the absolute temperature follows directly from solving the problem, i.e., from finding a solution component ϑ to which the logarithm applies. This is important, since we can avoid the use of other methods, or the setting of special assumptions, in order to guarantee the positivity of ϑ in the space-time domain.

The second equation of the system under study accounts for the phase dynamics and is deduced from a balance law for the microscopic forces that are responsible for the phase transition process. According to [21, 25], this balance reads

$$\partial_t\chi - \Delta\chi + \beta(\chi) + \pi(\chi) \ni \ell\vartheta, \tag{2}$$

where $\beta + \pi$ represents the derivative, or the subdifferential, of a double-well potential $\mathcal{W}$ defined as

$$\mathcal{W} = \tilde{\beta} + \tilde{\pi},$$

where

$$\tilde{\beta} : \mathbb{R} \longrightarrow [0, +\infty] \text{ is proper, l.s.c. and convex with } \tilde{\beta}(0) = 0, \tag{3}$$

$$\tilde{\pi} \in C^1(\mathbb{R}) \text{ and } \pi = \tilde{\pi}' \text{ is Lipschitz continuous in } \mathbb{R}. \tag{4}$$

Due to (3), the subdifferential $\beta := \partial\tilde{\beta}$ is well defined and turns out to be a maximal monotone graph. Moreover, as $\tilde{\beta}$ takes on its minimum in 0, we have that $0 \in \beta(0)$. Note that in (2) the inclusion is used in place of the equality in order to allow for the presence of a multivalued β. We recall that many different choices of $\tilde{\beta}$ and $\tilde{\pi}$ have been introduced in the literature (see, e.g., [5, 8, 20, 26]). In the case of a solid-liquid phase transition, $\mathcal{W}$ may be chosen in such a way that the full potential (cf. (2))

$$\chi \mapsto \tilde{\beta}(\chi) + \tilde{\pi}(\chi) - \ell\vartheta\chi$$

exhibits one of the two minima $\chi = 0$ and $\chi = 1$ as global minimum for equilibrium, depending on whether ϑ is below or above a critical value ϑ_c, which may represent a phase change temperature. A sample case is given by $\tilde{\pi}(\chi) = \ell\vartheta_c\chi$ and by the $\tilde{\beta}$ that coincides with the indicator function $I_{[0,1]}$ of the interval $[0, 1]$, that is,

$$\tilde{\beta}(\rho) = I_{[0,1]}(\rho) = \begin{cases} 0 & \text{if } \ 0 \le \rho \le 1 \\ +\infty & \text{elsewhere} \end{cases}$$

so that $\beta = \partial I_{[0,1]}$ is specified by

$$r \in \beta(\rho) \quad \text{if and only if} \quad r \begin{cases} \leq 0 & \text{if } \rho = 0 \\ = 0 & \text{if } 0 < \rho < 1 \\ \geq 0 & \text{if } \rho = 1 \end{cases} .$$

Of course, this yields a singular case for the potential $\mathcal{W}$, in which $\tilde{\beta}$ is not differentiable, and it is known in the literature as the double obstacle case (cf. [5,8,21])

In the present contribution, we assume that the second equation (2) of the system is perturbed by the presence of an additional maximal monotone nonlinearity, i.e.,

$$\partial_t \chi - \Delta \chi + \beta(\chi) + \pi(\chi) + \zeta \ni \ell \vartheta, \tag{5}$$

where

$$\zeta(t) \in A(\chi(t) - \chi^*) \qquad \text{for a.e. } t \in (0,T). \tag{6}$$

Here, χ^* is a positive and smooth function ($\chi^* \in H^2(\Omega)$ with null outward normal derivative on the boundary) and $A : L^2(\Omega) \to L^2(\Omega)$ is a maximal monotone operator satisfying some conditions, namely: A is the subdifferential of a proper, convex and lower semicontinuous (l.s.c.) function $\Phi : L^2(\Omega) \to \mathbb{R}$ which takes its minimum in 0, and A is linearly bounded in $L^2(\Omega)$. As widely described in [3], the role of this further nonlinearity is physically meaningful in the framework of phase transition processes.

In the last decades phase field models have attracted a number of mathematicians and applied scientists to describe many different physical phenomena. Let us just recall some results in the literature that are related to our system. Some key references are the papers [6–8]. Besides, we quote [10], where a first simplified version of the entropy system is considered, and [9, 11] for related analyses and results. Besides, let us mention the contributions [18, 19], where standard phase field systems of Caginalp type, perturbed by the presence of nonlinearities similar to (6), are considered and the existence of strong solutions, the global well-posedness of the system and the sliding mode property are proved. We also refer to [14], where the author prove the existence of solutions for a system characterized by the contemporary presence of two nonlinearities in the entropy balance equation: the resulting system is highly nonlinear and the main difficulties lie in the treatment of the doubly nonlinear equation

$$\partial_t(\ln \vartheta + \ell \chi) - k_0 \Delta \vartheta + \zeta \ni F, \quad \zeta(t) \in A(\chi(t) - \chi^*) \qquad \text{for a.e. } t \in (0,T).$$

In the first part of the present contribution we prove existence and uniqueness of the solution for the system consisting of equations (1), (5)–(6) coupled with suitable boundary and initial conditions. In particular, we prescribe a no-flux condition on the boundary for both variables:

$$\partial_\nu \vartheta = 0, \quad \partial_\nu \chi = 0 \quad \text{on } \Gamma \times (0,T), \tag{7}$$

where ∂_ν denotes the outward normal derivative on the boundary Γ of Ω. Besides, in the light of (6), initial conditions are stated for $\ln \vartheta$ and χ:

$$\ln \vartheta(0) = \ln \vartheta_0, \quad \chi(0) = \chi_0 \quad \text{in } \Omega. \tag{8}$$

The second part of the paper is concerned with the asymptotic behavior of the solution to (1), (5)–(8) as t goes to $+\infty$. Let us point out that the longtime behavior has been already investigated for related equations with memory terms

in, e.g., [4]. In our framework, we assume that $A = \rho\,\mathrm{Sign}$, where ρ is a positive coefficient, $\mathrm{Sign} :\longrightarrow 2^H$ is defined as

$$\mathrm{Sign}(v) = \begin{cases} \frac{v}{\|v\|} & \text{if } v \neq 0, \\ B_1(0) & \text{if } v = 0, \end{cases} \tag{9}$$

and $B_1(0)$ is the closed unit ball of H (it is straightforward to check that Sign satisfies the properties required for the operator A). Then, we show that the ω-limit, defined as

$$\omega := \Big\{(\vartheta_\infty, \chi_\infty) \in V \times V : \ \exists \text{ a subsequence } \ t_n \nearrow +\infty \ \text{ such that } \\ \Big(\vartheta(t_n), \chi(t_n)\Big) \rightharpoonup (\vartheta_\infty, \chi_\infty) \text{ in } V \times V\Big\}$$

is nonempty and consists only of stationary solutions. In particular, ϑ_∞ is a constant, while χ_∞ satisfies

$$-\Delta\chi_\infty + \xi_\infty + \pi(\chi_\infty) + \zeta_\infty = \ell\vartheta_\infty \quad \text{a.e. in } \Omega, \tag{10}$$

$$\xi_\infty \in \beta(\chi_\infty), \qquad \zeta_\infty \in A(\chi_\infty - \chi^*). \tag{11}$$

As far as the outline of the paper is concerned, we state precisely assumptions and main results in Section 2, then introduce the time-discrete problem (P_τ) in Section 3 and completely prove existence and uniqueness of the solution. Section 4 is devoted to the proof of several uniform estimates, independent of τ, involving the solution of (P_τ). Then, in Section 5 we pass to the limit as $\tau \searrow 0$ by means of compactness and monotonicity arguments in order to find a solution to the problem (1), (5)–(8). Finally, in Section 6 and in Section 7, respectively, we prove the uniqueness of the solution and its longtime behavior.

2. **Main results.**

2.1. **Preliminary assumptions.** We assume $\Omega \subseteq \mathbb{R}^3$ to be open, bounded, connected, of class C^1 and we write $|\Omega|$ for its Lebesgue measure. Moreover, Γ and ∂_ν still stand for the boundary of Ω and the outward normal derivative, respectively. Given a finite final time $T > 0$, for every $t \in (0, T]$ we set

$$Q_t = (0, t) \times \Omega, \quad Q = Q_T, \tag{1}$$

$$\Sigma_t = (0, t) \times \Gamma, \quad \Sigma = \Sigma_T. \tag{2}$$

We also introduce the spaces

$$H = L^2(\Omega), \qquad V = H^1(\Omega), \qquad V_0 = H^1_0(\Omega), \tag{3}$$

$$W = \{u \in H^2(\Omega) : \ \partial_\nu u = 0 \text{ on } \Gamma\}, \tag{4}$$

with usual norms $\|\cdot\|_H$, $\|\cdot\|_V$ and inner products $(\cdot,\cdot)_H$, $(\cdot,\cdot)_V$, respectively. We identify H with its dual space H', so that $W \subset V \subset H \subset V' \subset W'$ with dense and compact embeddings. Let $\langle\cdot,\cdot\rangle$ stand for the duality pairing between V' and V. The notation $\|\cdot\|_p$, $(1 \leq p \leq +\infty)$ stands for the standard norm in $L^p(\Omega)$. For short, in the notation of norms, we do not distinguish between a space (or its norm) and a power thereof.

From now on, we interpret the operator $-\Delta$ as the Laplacian operator from the space W to H, then including the Neumann homogeneous boundary condition. Moreover, we extend $-\Delta$ to an operator from V to V' by setting

$$\langle -\Delta u, v\rangle := \int_\Omega \nabla u \cdot \nabla v, \quad u, v \in V. \tag{5}$$

Throughout the paper, we account for the well-known continuous embeddings $V \subset L^q(\Omega)$ $(1 \le q \le 6)$, $W \subset C^0(\overline{\Omega})$ and for the related Sobolev inequalities:

$$\|v\|_q \le C_s \|v\|_V \qquad \text{and} \qquad \|v\|_\infty \le C_s \|v\|_W \tag{6}$$

for $v \in V$ and $v \in W$, respectively, where C_s depends on Ω only, since sharpness is not needed. We will also use a variant of the Poincaré inequality, i.e., there exists a positive constant C_p such that

$$\|v\|_V \le C_p \Big(\|v\|_{L^1(\Omega)} + \|\nabla v\|_H \Big), \qquad v \in V. \tag{7}$$

Furthermore, we make repeated use of Hölder inequality and of Young's inequalities, i.e., for every $a, b > 0$, $z \in (0,1)$ and $\delta > 0$

$$ab \le \delta a^{\frac{1}{z}} + (1-z) b^{\frac{1}{1-\delta}}, \tag{8}$$

$$ab \le \delta a^2 + \frac{1}{4\delta} b^2. \tag{9}$$

Besides, for every a, $b \in \mathbb{R}$ we have that

$$(a-b)a = \frac{1}{2}a^2 - \frac{1}{2}b^2 + \frac{1}{2}(a-b)^2. \tag{10}$$

Finally, we also recall the discrete version of the Gronwall lemma (see, e.g., [22, Prop. 2.2.1]).

Lemma 2.1. *If $(a_0, \cdots, a_N) \in [0, +\infty)^{N+1}$ and $(b_1, \cdots, b_N) \in [0, +\infty)^N$ satisfy*

$$a_m \le a_0 + \sum_{n=1}^{m-1} a_n b_n \qquad \text{for } m = 1, \cdots, N, \tag{11}$$

then

$$a_m \le a_0 \exp\left(\sum_{n=1}^{m-1} b_n \right) \qquad \text{for } m = 1, \cdots, N. \tag{12}$$

Finally, we state another useful result for the sequel.

Lemma 2.2. *Assume that a, $b \in \mathbb{R}$ are strictly positive. Then*

$$(a-b) \le \big(\ln a^2 - \ln b^2 \big)(a+b). \tag{13}$$

Proof. We consider $a > b$ (if $b > a$ the technique of the proof is analogous) and obtain

$$(a-b) \le (\ln(a^2) - \ln(b^2))(a+b) = 2(\ln(a) - \ln(b))(a+b) = 2\ln\left(\frac{a}{b}\right)(a+b). \tag{14}$$

Then, dividing by b, we have that

$$\left(\frac{a}{b} - 1\right) \le 2\ln\left(\frac{a}{b}\right)\left(\frac{a}{b} + 1\right). \tag{15}$$

Letting $x = a/b$, we can rewrite (15) as

$$(x-1) \le 2\ln x (x+1) \qquad \text{for } x \ge 1. \tag{16}$$

Now, we observe that (13) is verified if and only if the function

$$f(x) := 2(x+1)\ln x - x + 1 \quad \text{is nonnegative for every } x \ge 1. \tag{17}$$

Since $f(1) = 0$ and $f'(x) > 0$ for every $x \ge 1$, we conclude that the proof of the lemma is complete. □

In the following, the small-case symbol c stands for different constants which depend only on Ω, on the final time T, on the shape of the nonlinearities and on the constants and the norms of the functions involved in the assumptions of our statements. On the contrary, we use different symbols to denote precise constants to which we could refer. The reader should keep in mind that the meaning of c might change from line to line and even in the same chain of inequalities.

2.2. **Statement of the problem and results.** As far as the data of our problem are concerned, let ℓ and $k_0 > 0$ be two real constants. We also consider the data F, χ^*, ϑ_0 and χ_0 such that

$$F \in H^1(0,T;H) \cap L^1(0,T;L^\infty(\Omega)), \tag{18}$$

$$\chi^* \in W, \tag{19}$$

$$\vartheta_0 \in V, \quad \vartheta_0 > 0 \text{ a.e. in } \Omega, \quad \ln \vartheta_0 \in H, \tag{20}$$

$$\chi_0 \in W. \tag{21}$$

Moreover, we introduce the functions $\tilde{\beta}$ and $\tilde{\pi}$, satisfying the conditions listed below:

$$\tilde{\beta} : \mathbb{R} \longrightarrow [0,+\infty] \text{ is lower semicontinuous and convex with } \tilde{\beta}(0) = 0, \tag{22}$$

$$\lim_{|r|\to+\infty} |r|^{-2}\tilde{\beta}(r) = +\infty, \tag{23}$$

$$\tilde{\pi} \in C^1(\mathbb{R}) \text{ and } \pi \text{ is Lipschitz continuous.} \tag{24}$$

Since $\tilde{\beta}$ is proper, lower semicontinuous and convex, the subdifferential $\beta := \partial\tilde{\beta}$ is well defined. We denote by $D(\beta)$ and $D(\tilde{\beta})$ the effective domains of β and $\tilde{\beta}$, respectively. Thanks to these assumptions, β is a maximal monotone graph. Moreover, as $\tilde{\beta}$ takes on its minimum in 0, we have that $0 \in \beta(0)$. We also assume that

$$\xi_0 \in D(\beta) \quad \text{a.e. in } \Omega \text{ and } \exists\ \chi_0 \in H \text{ such that } \xi_0 \in \beta(\chi_0) \text{ a.e. in } \Omega, \tag{25}$$

whence

$$\tilde{\beta}(\chi_0) \in L^1(\Omega). \tag{26}$$

Indeed, thanks to the definition of the subdifferential and to (22), we have that

$$0 \le \int_\Omega \tilde{\beta}(\chi_0) \le (\xi_0, \chi_0) \le \|\xi_0\|_H \|\chi_0\|_H. \tag{27}$$

In the following, the same symbol β will be used for the maximal monotone operators induced on $H \equiv L^2(\Omega)$ and $L^2(0,T;H) \equiv L^2(Q)$.

In our problem a maximal monotone operator

$$A : H \longrightarrow H \tag{28}$$

also appears. We assume that

$$\begin{aligned} &A \text{ is the subdifferential of a convex and l.s.c. function } \Phi : H \longrightarrow \mathbb{R} \\ &\text{which takes its minimum in 0 and has at most a quadratic growth.} \end{aligned} \tag{29}$$

These properties are related to our assumptions on $A = \partial\Phi$, which read

$$0 \in A(0), \quad \exists\, C_A > 0 \text{ such that } \|y\|_H \le C_A(1 + \|x\|_H) \quad \forall\, x \in H,\ \forall\, y \in Ax. \tag{30}$$

In the following, the same symbol A will be used for the maximal monotone operators induced on $L^2(0,T;H)$.

Examples of operators A. Now, we consider the operator

$$\text{sign } : \mathbb{R} \longrightarrow 2^{\mathbb{R}} \tag{31}$$

$$\operatorname{sign}(r) = \begin{cases} \frac{r}{|r|} & \text{if } r \neq 0, \\ [-1,1] & \text{if } r = 0, \end{cases} \tag{32}$$

and its nonlocal counterpart

$$\operatorname{Sign}: \qquad H \longrightarrow 2^H \tag{33}$$

$$\operatorname{Sign}(v) = \begin{cases} \frac{v}{\|v\|} & \text{if } v \neq 0, \\ B_1(0) & \text{if } v = 0, \end{cases} \tag{34}$$

where $B_1(0)$ is the closed unit ball of H. It is straightforward to check that Sign satisfies (30) and turns out to be the subdifferential of the norm function $v \mapsto \|v\|_H$. Moreover, let us recall that the subdifferential of the convex function $v \mapsto \int_\Omega |v|$ is a maximal monotone operator from $L^\infty(\Omega)$ to $2^{L^\infty(\Omega)}$ defined as in (34).

Main results. Our aim is to find a quadruplet $(\vartheta, \chi, \zeta, \xi)$ satisfying the regularity conditions

$$\vartheta \in L^2(0,T;V), \tag{35}$$

$$\vartheta > 0 \text{ a.e. in } \Omega \quad \text{and} \quad \ln\vartheta \in H^1(0,T;V') \cap L^\infty(0,T;H), \tag{36}$$

$$\chi \in H^1(0,T;H) \cap L^2(0,T;W), \tag{37}$$

$$\zeta \in L^2(0,T;H), \qquad \xi \in L^2(0,T;H), \tag{38}$$

and solving Problem (P) defined as

$$\partial_t(\ln\vartheta(t) + \ell\chi(t)) - k_0\Delta\vartheta(t) = F(t) \quad \text{in } V, \text{ for a.e. } t \in (0,T), \tag{39}$$

$$\partial_t\chi - \Delta\chi + \xi + \pi(\chi) + \zeta(t) = \ell\vartheta \quad \text{a.e. in } Q, \tag{40}$$

$$\xi \in \beta(\chi) \quad \text{a.e. in } Q, \tag{41}$$

$$\zeta(t) \in A(\chi(t) - \chi^*) \quad \text{for a.e. } t \in (0,T), \tag{42}$$

$$\partial_\nu\vartheta = 0, \qquad \partial_\nu\chi = 0 \quad \text{on } \Sigma, \tag{43}$$

$$\ln\vartheta(0) = \ln\vartheta_0, \qquad \chi(0) = \chi_0 \quad \text{in } \Omega. \tag{44}$$

In order to obtain a variational formulation of Problem (P), from (39) and (43) we infer that

$$\langle \partial_t(\ln\vartheta(t) + \ell\chi(t)), v\rangle + k_0\int_\Omega \nabla\vartheta(t)\cdot\nabla v = \int_\Omega F(t)v, \qquad v \in V. \tag{45}$$

Theorem - (Existence and uniqueness) 2.3. *Assume (18)–(30). Then problem (P) stated by (39)–(44) has a unique solution (ϑ, χ, ξ) satisfying (35)–(38) and the regularity properties*

$$\vartheta \in L^\infty(0,T;V), \qquad \zeta \in L^\infty(0,T;H), \tag{46}$$

$$\chi \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W), \qquad \xi \in L^\infty(0,T;H). \tag{47}$$

Theorem - (Longtime behavior) 2.4. *Assume (18)–(30). In addiction, if χ^* is constant, ρ is a positive parameter, $A = \rho\operatorname{Sign}$ and*

$$F_\infty = 0, \qquad F \in L^\infty(0,+\infty;L^\infty(\Omega)) \cap L^1(0,+\infty;L^1(\Omega)) \cap L^2(0,+\infty;H), \tag{48}$$

then the ω-limit, defined as

$$\omega := \Big\{(\vartheta_\infty, \chi_\infty) \in V \times V : \ \exists \text{ a subsequence } \ t_n \nearrow +\infty \ \text{ such that } \Big(\vartheta(t_n), \chi(t_n)\Big) \rightharpoonup (\vartheta_\infty, \chi_\infty) \text{ in } V \times V\Big\} \tag{49}$$

is nonempty and consists only of stationary solutions. In particular, ϑ_∞ is a constant, while χ_∞ satisfies

$$-\Delta\chi_\infty + \xi_\infty + \pi(\chi_\infty) + \zeta_\infty = \ell\vartheta_\infty \quad \text{a.e. in } \Omega, \tag{50}$$

$$\xi_\infty \in \beta(\chi_\infty), \qquad \zeta_\infty \in A(\chi_\infty - \chi^*). \tag{51}$$

3. **The approximating problem** (P_τ)**.** The following three sections are devoted to the proof of Theorem 2.3. In order to obtain this result, we introduce a backward finite differences scheme. Assume that N is a positive integer and let Z be any normed space. By setting the time step $\tau = T/N$ we introduce the interpolation maps from Z^{N+1} into either $L^\infty(0,T;Z)$ or $W^{1,\infty}(0,T;Z)$. For $(z^0, z^1, \cdots, z^N) \in Z^{N+1}$, we define the piecewise constant functions $\overline{z}_\tau$ and the piecewise linear functions $\widehat{z}_\tau$, respectively:

$$\overline{z}_\tau \in L^\infty(0,T;Z), \qquad \widehat{z}_\tau \in W^{1,\infty}(0,T;Z), \tag{1}$$

$$\overline{z}((i+s)\tau) = z^{i+1}, \quad \widehat{z}((i+s)\tau) = z^i + s(z^{i+1} - z^i), \tag{2}$$

if $0 < s < 1$ and $i = 0, \cdots, N-1$. We also define the operator

$$\delta_\tau : Z^{N+1} \to Z^N,$$

$$\delta_\tau z^i = w \quad \text{means that} \quad w^i = \frac{z^{i+1} - z^i}{\tau} \quad \text{for } i = 0, \cdots, N-2, \tag{3}$$

for $(z^0, z^1, \cdots, z^N) \in Z^{N+1}$ and $(w^0, w^1, \cdots, w^{N-1}) \in Z^N$, and we denote by

$$\partial_t \widehat{z}_\tau(t) := \delta_\tau z^{i-1} \qquad \text{for a.a. } t \in ((i-1)\tau, i\tau), \;\; i = 1, \cdots, N. \tag{4}$$

By a direct computation, it is straightforward to prove that

$$\|\overline{z}_\tau - \widehat{z}_\tau\|_{L^\infty(0,T;Z)} = \max_{i=0,\cdots,N-1} \|z_{i+1} - z_i\|_Z = \tau \|\partial_t \widehat{z}_\tau\|_{L^\infty(0,T;Z)}, \tag{5}$$

$$\|\overline{z}_\tau - \widehat{z}_\tau\|^2_{L^2(0,T;Z)} = \frac{\tau}{3} \sum_{i=0}^{N-1} \|z_{i+1} - z_i\|^2_Z = \frac{\tau^2}{3} \|\partial_t \widehat{z}_\tau\|^2_{L^2(0,T;Z)}, \tag{6}$$

$$\begin{aligned} \|\overline{z}_\tau - \widehat{z}_\tau\|^2_{L^\infty(0,T;Z)} &= \max_{i=0,\cdots,N-1} \|z_{i+1} - z_i\|^2_Z \\ &\le \sum_{i=0}^{N-1} \tau^2 \left\| \frac{z_{i+1} - z_i}{\tau} \right\|^2_Z \\ &\le \tau \|\partial_t \widehat{z}_\tau\|^2_{L^2(0,T;Z)}. \end{aligned} \tag{7}$$

Then, we consider the approximating problem (P_τ). We set

$$F^i := \frac{1}{\tau} \int_{(i-1)\tau}^{i\tau} F(s)\, ds, \qquad \text{for } i = 1, \cdots, N \tag{8}$$

and we look for two vectors $(\vartheta^0, \vartheta^1, \cdots, \vartheta^N) \in V^{N+1}$, $(\chi^0, \chi^1, \cdots, \chi^N) \in V^{N+1}$ satisfying, for $i = 1, \cdots, N$, the system

$$\vartheta^i > 0 \qquad \text{a.e. in } \Omega, \ln \vartheta^i \in H, \qquad \exists \zeta^i,\ \xi^i \in H \text{such that} \tag{9}$$

$$\tau^{1/2}\vartheta^i + \ln \vartheta^i + \ell\chi^i - \tau k_0 \Delta \vartheta^i = \tau F^i + \tau^{1/2}\vartheta^{i-1} + \ln \vartheta^{i-1} + \ell\chi^{i-1} \qquad \text{in } \Omega, \tag{10}$$

$$\chi^i - \tau\Delta\chi^i + \tau\xi^i + \tau\pi(\chi^i) + \tau\zeta^i = \chi^{i-1} + \tau\ell\vartheta^i \qquad \text{in } \Omega, \tag{11}$$

$$\zeta^i \in A(\chi^i - \chi^*) \qquad \text{in } \Omega, \tag{12}$$

$$\xi^i \in \beta(\chi^i) \qquad \text{in } \Omega, \tag{13}$$

$$\partial_\nu \vartheta^i = \partial_\nu \chi^i = 0 \qquad \text{on } \Gamma, \tag{14}$$

$$\ln \vartheta^0 = \ln \vartheta_0, \qquad \chi^0 = \chi_0 \qquad \text{in } \Omega. \tag{15}$$

Now, we rewrite the equations (10) and (11) using the piecewise constant functions $\overline{z}_\tau$ and the piecewise linear functions $\widehat{z}_\tau$ defined in (1)–(2), respectively, and obtain that

$$\tau^{1/2}\partial_t\widehat{\vartheta}_\tau + \partial_t\widehat{\ln\vartheta}_\tau + \ell\partial_t\widehat{\chi}_\tau - k_0\Delta\overline{\vartheta}_\tau = \overline{F}_\tau, \tag{16}$$

$$\partial_t\widehat{\chi}_\tau - \Delta\overline{\chi}_\tau + \overline{\xi}_\tau + \pi(\overline{\chi}_\tau) + \overline{\zeta}_\tau = \ell\overline{\vartheta}. \tag{17}$$

$$\overline{\zeta}_\tau \in A(\overline{\chi}_\tau - \chi^*) \qquad \text{in } \Omega, \tag{18}$$

$$\overline{\xi}_\tau \in \beta(\overline{\chi}_\tau) \qquad \text{in } \Omega, \tag{19}$$

$$\partial_\nu\overline{\vartheta}_\tau = \partial_\nu\overline{\chi}_\tau = 0 \qquad \text{on } \Gamma, \tag{20}$$

$$\ln\overline{\vartheta}_0 = \ln\vartheta_0, \qquad \overline{\vartheta}_0 = \chi_0 \qquad \text{in } \Omega. \tag{21}$$

In view of (18)–(21), we infer that for $i = 1$ the right-hand side of (10) is an element of H, and we have to find ϑ^1 along with ξ^1 fulfilling (9)–(10) and (12); in case we succeed, from a comparison in (10) it will turn out that $\vartheta^1 \in W$. Then, we insert ϑ^1 in the right-hand side of (11) and seek $\chi^1 \in W$ and $\xi^1 \in H$ satisfying (11) and (13). Once we recover them, we can start again our procedure, and so on. Then, it is important to show that, for a fixed i and known data F^i, ϑ^{i-1}, $\ln\vartheta^{i-1}$, χ^{i-1} we are able to find a pair (ϑ^i, χ^i) solving (9)–(14).

Theorem 3.1. *There exists some fixed value $\tau_1 \le \min\{1, T\}$, depending only on the data, such that for any time step $0 < \tau < \tau_1$ the approximating problem (P_τ) stated by (9)–(15) has a unique solution*

$$(\vartheta^0, \vartheta^1, \cdots, \vartheta^N) \in V \times W^N, \qquad (\chi^0, \chi^1, \cdots, \chi^N) \in W^{N+1}.$$

Let us now rewrite the discrete equation (10)–(10) by using the piecewise constant and piecewise linear functions defined in (1), with obvious notation, and obtain that

$$\tau^{1/2}\partial_t\widehat{\vartheta}_\tau + \partial_t\widehat{\ln\vartheta}_\tau + \ell\partial_t\widehat{\chi}_\tau - k_0\Delta\overline{\vartheta}_\tau = \overline{F}_\tau \quad \textit{a.e. in } Q, \tag{22}$$

$$\partial_t\widehat{\chi}_\tau - \Delta\overline{\chi}_\tau + \overline{\xi}_\tau + \pi(\overline{\chi}_\tau) + \overline{\zeta}_\tau = \ell\overline{\vartheta}_\tau \quad \textit{a.e. in } Q, \tag{23}$$

$$\overline{\zeta}_\tau(t) \in A(\overline{\chi}_\tau(t) - \chi^*) \quad \textit{for a.e. } t \in (0, T), \tag{24}$$

$$\overline{\xi}_\tau \in \beta(\overline{\chi}_\tau) \quad \textit{a.e. in } Q, \tag{25}$$

$$\partial_\nu\overline{\vartheta}_\tau = \partial_\nu\overline{\chi}_\tau = 0 \quad \textit{a.e. on } \Sigma, \tag{26}$$

$$\widehat{\vartheta}_\tau(0) = \vartheta_0, \quad \widehat{\chi}_\tau(0) = \chi_0 \quad \textit{a.e. in } \Omega. \tag{27}$$

3.1. **The auxiliary approximating problem** (AP_ε)**.** In this subsection we consider the auxiliary approximating problem (AP_ε) obtained by considering the approximating problem (P_τ) in each interval of range τ and replacing the operators appearing in (10)–(15) with their Yosida regularizations. About general properties of maximal monotone operators and subdifferentials of convex functiions, we refer the reader to [1, 12].

Yosida regularization of ln**.** We introduce the Yosida regularization of ln. For $\varepsilon > 0$ we set

$$\ln_\varepsilon : \mathbb{R} \longrightarrow \mathbb{R}, \qquad \ln_\varepsilon := \frac{I - (I + \varepsilon\ln)^{-1}}{\varepsilon}. \tag{28}$$

where I denotes the identity. We remark that $\ln_\varepsilon$ is monotone, Lipschitz continuous (with Lipschitz constant $1/\varepsilon$) and satisfies the following properties: denoting by $L_\varepsilon = (I + \varepsilon\ln)^{-1}$ the resolvent operator, we have that

$$\ln_\varepsilon(x) \in \ln(L_\varepsilon x) \quad \text{for all } x \in \mathbb{R},$$

$$|\ln_\varepsilon(x)| \le |\ln(x)|, \quad \lim_{\varepsilon\searrow 0}\ln_\varepsilon(x) = \ln(x) \quad \text{for all } x > 0.$$

We also introduce the nonnegative and convex functions

$$\Lambda(x) = \int_1^x \ln r \, dr, \quad \Lambda_\varepsilon(y) = \int_1^y \ln_\varepsilon r \, dr \quad \text{for all } x > 0 \text{ and } y \in \mathbb{R}. \tag{29}$$

Note that the graph $x \mapsto \ln x$ is nothing but the subdifferential of the convex function Λ extended by lower semicontinuity in 0 and with value $+\infty$ for $x < 0$. On the other hand, Λ_ε coincides with the Moreau–Yosida regularization of Λ and, in particular, we have that

$$0 \le \Lambda_\varepsilon(x) \le \Lambda(x) \quad \text{for every } x > 0. \tag{30}$$

Yosida regularization of A. We introduce the Yosida regularization of A. For $\varepsilon > 0$ we define

$$A_\varepsilon : H \longrightarrow H, \qquad A_\varepsilon = \frac{I - (I + \varepsilon A)^{-1}}{\varepsilon}. \tag{31}$$

Note that A_ε is Lipschitz-continuous (with Lipschitz constant $1/\varepsilon$) and maximal monotone in H. Moreover, A satisfies the following properties: denoting by $J_\varepsilon = (I + \varepsilon A)^{-1}$ the resolvent operator, for all $\delta > 0$ and for all $x \in H$, we have that

$$A_\varepsilon x \in A(J_\varepsilon x), \tag{32}$$

$$\|A_\varepsilon x\|_H \le \|A^0 x\|_H, \quad \lim_{\varepsilon \searrow 0} \|A_\varepsilon x - A^0 x\|_H = 0, \tag{33}$$

where $A^0 x$ is the element of the range of A having minimal norm. Let us point out a key property of A_ε, which is a consequence of (30): indeed, there holds

$$\|A_\varepsilon x\|_H \le C_A(1 + \|x\|_H) \quad \text{for all } x \in H. \tag{34}$$

Notice that $0 \in A(0)$ and $0 \in I(0)$: consequently, for every $\varepsilon > 0$ we infer that $J_\varepsilon(0) = 0$. Moreover, since A is maximal monotone, J_ε is a contraction. Then, from (30) and (32) it follows that

$$\|A_\varepsilon x\|_H \le C_A(\|J_\varepsilon x\|_H + 1) \le C_A(\|J_\varepsilon x - J_\varepsilon 0\|_H + 1) \le C_A(\|x\|_H + 1)$$

for every $x \in H$.

Yosida regularization of β. We introduce the Yosida regularization of β. For $\varepsilon > 0$ we define

$$\beta_\varepsilon : \mathbb{R} \longrightarrow \mathbb{R}, \qquad \beta_\varepsilon = \frac{I - (I + \varepsilon\beta)^{-1}}{\varepsilon}. \tag{35}$$

We remark that β_ε is Lipschitz continuous (with Lipschitz constant $1/\varepsilon$) and satisfies the following properties: denoting by $R_\varepsilon = (I + \varepsilon\beta)^{-1}$ the resolvent operator, we have that

$$\beta_\varepsilon(x) \in \beta(R_\varepsilon x) \quad \text{for all } x \in \mathbb{R},$$
$$|\beta_\varepsilon(x)| \le |\beta^0(x)|, \quad \lim_{\varepsilon \searrow 0} \beta_\varepsilon(x) = \beta^0(x) \quad \text{for all } x \in D(\beta),$$

where $\beta^0(x)$ is the element of the range of $\beta(x)$ having minimal modulus. We also introduce the Moreau–Yosida regularization of $\tilde\beta$. For $\varepsilon > 0$ and $x \in \mathbb{R}$ we define

$$\tilde\beta_\varepsilon : \mathbb{R} \longrightarrow [0, +\infty], \qquad \tilde\beta_\varepsilon(x) := \min_{y \in \mathbb{R}} \left\{ \tilde\beta(y) + \frac{1}{2\varepsilon}|x - y|^2 \right\}.$$

and we recall that

$$\tilde\beta_\varepsilon(x) \le \tilde\beta(x) \quad \text{for every } x \in \mathbb{R}. \tag{36}$$

We also observe that β_ε is the derivative of $\tilde{\beta}_\varepsilon$. Then, for every $x_1, x_2 \in \mathbb{R}$ we have that

$$\tilde{\beta}_\varepsilon(x_2) = \tilde{\beta}_\varepsilon(x_1) + \int_{x_1}^{x_2} \beta_\varepsilon(s)\, ds. \tag{37}$$

Definition of the auxiliary approximating problem (AP_ε). We fix τ and consider the auxiliary approximating problem (AP_ε) obtained by considering (10)–(15) in the interval of range τ and regularizing the operators appearing in (P_τ). We set

$$g := \tau F^i + \tau^{1/2}\vartheta^{i-1} + \ln \vartheta^{i-1} + \ell\chi^{i-1}, \qquad h := \chi^{i-1}, \tag{38}$$

and note that both g and h are prescribed elements of H (cf. (8), (18), (20), (21) and (9)). We look for a pair $(\Theta_\varepsilon, X_\varepsilon)$ such that

$$\tau^{1/2}\Theta_\varepsilon + \ln_\varepsilon \Theta_\varepsilon - \tau k_0 \Delta\Theta_\varepsilon = -\ell X_\varepsilon + g \qquad \text{in } \Omega, \tag{39}$$

$$X_\varepsilon - \tau\Delta X_\varepsilon + \tau\beta_\varepsilon(X_\varepsilon) + \tau\pi(X_\varepsilon) + \tau A_\varepsilon(X_\varepsilon - \chi^*) = \chi^{i-1} + \tau\ell\Theta_\varepsilon \qquad \text{in } \Omega, \tag{40}$$

where $\ln_\varepsilon$, A_ε and β_ε are the Yosida regularization of ln, A and β defined by (28), (31) and (35), respectively. Here, according to the extended meaning of $-\Delta$ (see (5)), we omit the specification of the boundary conditions as with (14).

Theorem 3.2. *Let g, $h \in H$. Then there exists some fixed value $\tau_2 \le \min\{1, T\}$, depending only on the data, such that for every time step $\tau \in (0, \tau_2)$ and for all $\varepsilon \in (0, 1]$ the auxiliary approximating problem (AP_ε) stated by (39)–(40) has a unique solution $(\Theta_\varepsilon, X_\varepsilon)$.*

3.2. **Existence of a solution for (AP_ε).** In order to prove the existence of the solution for the auxiliary approximating problem (AP_ε) we intend to apply [1, Corollary 1.3, p. 48]. To this aim, we point out that, for τ small enough, the two operators

$$[\tau^{1/2} I + \ln_\varepsilon - \tau k_0 \Delta] \quad \text{appearing in (39)}, \tag{41}$$

$$[I + \tau\beta_\varepsilon + \tau\pi - \tau\Delta + \tau A_\varepsilon(\cdot - \chi^*)] \quad \text{appearing in (39)}, \tag{42}$$

both with domain W and range H, are monotone and coercive. Indeed, they are the sum of a monotone, Lipschitz continuous and coercive operator:

$$\tau^{1/2} I + \ln_\varepsilon \quad \text{in (41)}, \quad \text{and} \quad I + \tau\beta_\varepsilon + \tau\pi + \tau A_\varepsilon(\,\cdot - \chi^*) \quad \text{in (42)},$$

and of a maximal monotone operator that is $-\Delta$ with a positive coefficient in front. We now check our first claim. Letting v_1, $v_2 \in H$, we have that

$$\begin{aligned} &\Big((\tau^{1/2} I + \ln_\varepsilon)(v_1) - (\tau^{1/2} I + \ln_\varepsilon)(v_2), v_1 - v_2\Big) \\ &\ge \tau^{1/2}\|v_1 - v_2\|_H^2 + \big(\ln_\varepsilon(v_1) - \ln_\varepsilon(v_2), v_1 - v_2\big). \end{aligned} \tag{43}$$

Due to the monotonicity of $\ln_\varepsilon$, the last term on the right-hand side of (43) is nonnegative. Then we infer that

$$\Big((\tau^{1/2} I + \ln_\varepsilon)(v_1) - (\tau^{1/2} I + \ln_\varepsilon)(v_2), v_1 - v_2\Big) \ge \tau^{1/2}\|v_1 - v_2\|_H^2.$$

i.e. the operator $[\tau^{1/2} I + \ln_\varepsilon]$ is strongly monotone, hence coercive in H. Next, for all v_1, $v_2 \in H$ we have that

$$\begin{aligned} &\Big((I + \tau\beta_\varepsilon + \tau\pi + \tau A_\varepsilon(\cdot - \chi^*))(v_1) - (I + \tau\beta_\varepsilon + \tau\pi + \tau A_\varepsilon(\cdot - \chi^*))(v_2), v_1 - v_2\Big) \\ &\ge \|v_1 - v_2\|_H^2 - C_\pi\tau\|v_1 - v_2\|_H^2 + \tau\Big(\beta_\varepsilon(v_1) - \beta_\varepsilon(v_2), v_1 - v_2\Big) \\ &\quad + \tau\big(A_\varepsilon(v_1 - \chi^*) - A_\varepsilon(v_2 - \chi^*), v_1 - v_2\big), \end{aligned} \tag{44}$$

where C_π denotes a Lipschitz constant for π. Since β_ε and A_ε are monotone, it turns out that

$$\tau\Big(\beta_\varepsilon(v_1)-\beta_\varepsilon(v_2), v_1-v_2\Big)\geq 0, \tag{45}$$

$$\tau(A_\varepsilon(v_1-\chi^*)-A_\varepsilon(v_2-\chi^*), v_1-v_2)\geq 0, \tag{46}$$

and, choosing $\tau_2\leq 1/2C_\pi$, from (44) we infer that

$$\Big((I+\tau\beta_\varepsilon+\tau\pi+\tau A_\varepsilon(\cdot-\chi^*))(v_1)-(I+\tau\beta_\varepsilon+\tau\pi+\tau A_\varepsilon(\cdot-\chi^*))(v_1), v_1-v_2\Big)$$
$$\geq \frac{1}{2}\|v_1-v_2\|_H^2, \tag{47}$$

whence the operator $[I+\tau\beta_\varepsilon+\tau\pi+\tau A_\varepsilon(\cdot-\chi^*)]$ is strongly monotone and coercive in H, for every $\tau\leq\tau_2$.

Now, in order to prove Theorem 3.2, we divide the proof into two steps. In the first step, we fix $\overline{\Theta}_\varepsilon\in H$ on the right hand side of (40) and find a solution X_ε for (40). In the second step, we insert on the right hand side of (39) the solution X_ε obtained in the first step and find a solution Θ_ε to (39). Now, let $\overline{\Theta}_{1,\varepsilon}$ and $\overline{\Theta}_{2,\varepsilon}$ be two different initial data. We denote by $X_{1,\varepsilon}$, $X_{2,\varepsilon}$ the corresponding solutions for (40) obtained in the first step and by $\Theta_{1,\varepsilon}$, $\Theta_{2,\varepsilon}$ the related solution of (39) founded in the second step.

Hence, taking the difference between the two equations (39) written for $\overline{\Theta}_{1,\varepsilon}$ and $\overline{\Theta}_{2,\varepsilon}$ and testing the result by $(X_{1,\varepsilon}-X_{2,\varepsilon})$, we have that

$$\Big((I+\tau\beta_\varepsilon+\tau\pi+\tau A_\varepsilon(\cdot-\chi^*))(X_{1,\varepsilon})-(I+\tau\beta_\varepsilon+\tau\pi+\tau A_\varepsilon(\cdot-\chi^*))(X_{2,\varepsilon}), X_{1,\varepsilon}-X_{2,\varepsilon}\Big)$$
$$+\tau\int_\Omega|\nabla(X_{1,\varepsilon}-X_{2,\varepsilon})|^2\leq\tau\ell\Big(\overline{\Theta}_{1,\varepsilon}-\overline{\Theta}_{2,\varepsilon}, X_{1,\varepsilon}-X_{2,\varepsilon}\Big). \tag{48}$$

Then, applying (47) and (9), to the first term on the left hand side of (48) and to the right hand side of (48), respectively, we infer that

$$\frac{1}{2}\|X_{1,\varepsilon}-X_{2,\varepsilon}\|_H^2+\tau\int_\Omega|\nabla(X_{1,\varepsilon}-X_{2,\varepsilon})|^2\leq\frac{1}{4}\|X_{1,\varepsilon}-X_{2,\varepsilon}\|_H^2+\tau^2\ell^2\|\overline{\Theta}_{1,\varepsilon}-\overline{\Theta}_{2,\varepsilon}\|_H^2,$$

whence

$$\|X_{1,\varepsilon}-X_{2,\varepsilon}\|_H^2\leq 4\tau^2\ell^2\|\overline{\Theta}_{1,\varepsilon}-\overline{\Theta}_{2,\varepsilon}\|_H^2. \tag{49}$$

Now, we take the difference between the corresponding equations (39) written for the solutions $X_{1,\varepsilon}$, $X_{2,\varepsilon}$ obtained in the first step and test by $(\Theta_{1,\varepsilon}-\Theta_{2,\varepsilon})$. We obtain that

$$\tau^{1/2}\|\Theta_{1,\varepsilon}-\Theta_{2,\varepsilon}\|_H^2+(\ln_\varepsilon\Theta_{1,\varepsilon}-\ln_\varepsilon\Theta_{2,\varepsilon},\Theta_{1,\varepsilon}-\Theta_{2,\varepsilon})+\tau k_0\int_\Omega|\nabla(\Theta_{1,\varepsilon}-\Theta_{2,\varepsilon})|^2$$
$$\leq\frac{\ell^2}{2\tau^{1/2}}\|X_{1,\varepsilon}-X_{2,\varepsilon}\|_H^2+\frac{\tau^{1/2}}{2}\|\Theta_{1,\varepsilon}-\Theta_{2,\varepsilon}\|_H^2. \tag{50}$$

Since $\ln_\varepsilon$ is monotone, the second term on the left hand side of (50) is nonnegative. Moreover, recalling (43) and using it in the left-hand side of (50), we infer that

$$\tau^{1/2}\|\Theta_{1,\varepsilon}-\Theta_{2,\varepsilon}\|_H^2\leq\frac{\ell^2}{\tau^{1/2}}\|X_{1,\varepsilon}-X_{2,\varepsilon}\|_H^2$$

Then, by combining this inequality with (49), we deduce that

$$\|\Theta_{1,\varepsilon}-\Theta_{2,\varepsilon}\|_H^2\leq 4\tau\ell^4\|\overline{\Theta}_{1,\varepsilon}-\overline{\Theta}_{2,\varepsilon}\|_H^2, \tag{51}$$

whence we obtain a contraction mapping for every $\tau \leq \tau_2$, provided that $\tau_2 \leq 1/(8\ell^4)$. Finally, by applying the Banach fixed point theorem, we conclude that there exists a unique solution $(\Theta_\varepsilon, X_\varepsilon)$ to the auxiliary problem (AP_ε).

3.3. **A priori estimates on AP_ε.** In this subsection we derive a series of a priori estimates, independent of ε, inferred from the equations (39)–(40) of the auxiliary approximating problem (AP_ε).

First a priori estimate. We test (39) by $\tau(\Theta_\varepsilon - \vartheta^*)$ and (40) by X_ε, then we sum up. By exploiting the cancellation of the suitable corresponding terms and recalling the definition (29) of Λ_ε, we obtain that

$$\tau^{3/2}\|\Theta_\varepsilon\|_H^2 + \tau\Lambda_\varepsilon(\Theta_\varepsilon) + \tau^2 k_0 \int_\Omega |\nabla\Theta_\varepsilon|^2$$
$$+ \|X_\varepsilon\|_H^2 + \tau\int_\Omega |\nabla X_\varepsilon|^2 + \tau\big(\beta_\varepsilon(X_\varepsilon), X_\varepsilon\big) + \tau\Big(A_\varepsilon(X_\varepsilon - \chi^*), X_\varepsilon - \chi^*\Big)$$
$$\leq -\tau\Big(\pi(X_\varepsilon), X_\varepsilon\Big) - \tau\Big(A_\varepsilon(X_\varepsilon - \chi^*), \chi^*\Big) + \tau(g, \Theta_\varepsilon) + \tau(h, X_\varepsilon). \tag{52}$$

Let us note that all the terms on the left-hand side are nonnegative. Due to (19) and the continuity of the positive function ϑ^*, (30) helps us in estimating the second term on the right-hand side of (52):

$$\tau\Lambda_\varepsilon(\vartheta^*) \leq \tau\Lambda(\vartheta^*) \leq c\,\tau. \tag{53}$$

Due to the sub-linear growth of A_ε and the Lipschitz continuity of π the first two terms on the right hand side of (52) can be estimated as

$$-\tau\big(\pi(X_\varepsilon), X_\varepsilon\big) \leq \tau(C_\pi|X_\varepsilon|^2 + c), \tag{54}$$

$$-\tau\Big(A_\varepsilon(X_\varepsilon - \chi^*), \chi^*\Big) \leq \tau|X_\varepsilon|^2 + c. \tag{55}$$

Since g, $h \in H$ and (38) holds, by applying the Young inequality (9) to the other therms on the right hand side of (52), we find that

$$\tau(g, \Theta_\varepsilon) \leq \frac{\tau^{3/2}}{4}\|\Theta_\varepsilon\|_H^2 + c, \qquad \tau(h, X_\varepsilon) \leq \frac{\tau^{3/2}}{4}\|X_\varepsilon\|_H^2 + c. \tag{56}$$

Then, due to (54)–(56), from (52) we infer that

$$\tau^{3/4}\|\Theta_\varepsilon\|_H + \tau\|\nabla\Theta_\varepsilon\|_H + \|X_\varepsilon\|_H + \tau^{1/2}\|\nabla X_\varepsilon\|_H^2 \tag{57}$$

taing into account tat, e.g., $\tau \leq 1$.

Second a priori estimate. We test (40) by $\beta_\varepsilon(X_\varepsilon)$ and obtain that

$$\tau\int_\Omega \beta_\varepsilon'(X_\varepsilon)|\nabla X_\varepsilon|^2 + \tau\int_\Omega |\beta_\varepsilon(X_\varepsilon)|^2$$
$$\leq -\tau\int_\Omega \pi(X_\varepsilon)\beta_\varepsilon(X_\varepsilon) - \tau\int_\Omega A_\varepsilon(X_\varepsilon - \chi^*)\beta_\varepsilon(X_\varepsilon) + \tau\ell\int_\Omega \Theta_\varepsilon\beta_\varepsilon(X_\varepsilon) + \int_\Omega h\beta_\varepsilon(X_\varepsilon). \tag{58}$$

Thanks to the monotonicity of β_ε and to the condition $\beta_\varepsilon(0) = 0$, the terms on the left-hand side are nonnegative. As π is Lipschitz continuous and A_ε has a linear growth, applying the Young inequality (9) to every term on the right hand side of (58) and using (57), for $0 < \tau \leq 1$ we obtain that

$$-\tau\int_\Omega \pi(X_\varepsilon)\beta_\varepsilon(X_\varepsilon) \leq \frac{\tau}{5}\int_\Omega |\beta_\varepsilon(X_\varepsilon)|^2 + c, \tag{59}$$

$$-\tau\int_\Omega A_\varepsilon(X_\varepsilon-\chi^*)\beta_\varepsilon(X_\varepsilon)\le\frac{\tau}{5}\int_\Omega|\beta_\varepsilon(X_\varepsilon)|^2+c, \tag{60}$$

$$\tau\ell\int_\Omega\Theta_\varepsilon\beta_\varepsilon(X_\varepsilon)\le\frac{\tau}{5}\int_\Omega|\beta_\varepsilon(X_\varepsilon)|^2+\frac{c}{\tau^{1/2}}, \tag{61}$$

$$\int_\Omega h\beta_\varepsilon(X_\varepsilon)\le\frac{\tau}{5}\int_\Omega|\beta_\varepsilon(X_\varepsilon)|^2+\frac{c}{\tau}. \tag{62}$$

Then, owing to (59)–(62), from (58) it follows that

$$\tau\|\beta_\varepsilon(X_\varepsilon)\|_H^2\le c(1+\tau^{-1}),\quad\text{so that}\quad\tau\|\beta_\varepsilon(X_\varepsilon)\|_H\le c. \tag{63}$$

Hence, by comparison in (40), we conclude that $\tau\|\Delta X_\varepsilon\|_H\le c$ and, from (57) and standard elliptic regularity results,

$$\tau\|X_\varepsilon\|_W\le c. \tag{64}$$

Finally, recalling (34), (19) and (57), we immediately deduce that

$$\tau\|A_\varepsilon(X_\varepsilon-\chi^*)\|_H\le\tau\,C_A(1+\|X_\varepsilon\|_H+\|\chi^*\|_H)\le c. \tag{65}$$

Third a priori estimate. Next, (39) by $\ln_\varepsilon\Theta_\varepsilon$ and obtain that

$$\begin{aligned}&\|\ln_\varepsilon\Theta_\varepsilon\|_H^2+\tau k_0\int_\Omega\ln_\varepsilon'(\Theta_\varepsilon)|\nabla\Theta_\varepsilon|^2\\&\le-\tau^{1/2}(\Theta_\varepsilon,\ln_\varepsilon\Theta_\varepsilon)-\ell(X_\varepsilon,\ln_\varepsilon\Theta_\varepsilon)+(g,\ln_\varepsilon\Theta_\varepsilon).\end{aligned} \tag{66}$$

Then, by applying the Cauchy–Schwarz inequality to every term on the right-hand side and using (9) and (57), we infer that

$$\|\ln_\varepsilon\Theta_\varepsilon\|_H\le\tau^{1/2}\|\Theta_\varepsilon\|_H\le c(\tau^{-1/4}+1), \tag{67}$$

whence

$$\tau^{1/4}\|\ln_\varepsilon\Theta_\varepsilon\|_H\le c. \tag{68}$$

Moreover, due to (68), by comparison in (39) it is straightforward to see that $\tau^{5/4}\|\Delta\Theta_\varepsilon\|_H\le c$ and consequently

$$\tau^{5/4}\|\Theta_\varepsilon\|_W\le c. \tag{69}$$

3.4. **Passage to the limit as $\varepsilon\searrow 0$.** In this subsection we pass to the limit as $\varepsilon\searrow 0$ and prove that the limit of subsequences of solutions $(\Theta_\varepsilon,X_\varepsilon)$ for (AP_ε) (see (39)–(40)) yields a solution (ϑ^i,χ^i) to (10)–(15); then we can conclude that the problem (P_τ) has a solution.

Since the constants appearing in (57), (63)–(64) and (68)–(69) do not depend on ε, we infer that, at least for a subsequence, there exist some limit functions $(\vartheta^i,\chi^i,L^i,Z^i,B^i)$ such that

$$\Theta_\varepsilon\rightharpoonup\vartheta^i\quad\text{in}\quad W, \tag{70}$$

$$X_\varepsilon\rightharpoonup\chi^i\quad\text{in}\quad W, \tag{71}$$

$$\beta_\varepsilon(X_\varepsilon)\rightharpoonup B^i\quad\text{in}\quad H, \tag{72}$$

$$\ln_\varepsilon(\Theta_\varepsilon)\rightharpoonup L^i\quad\text{in}\quad H, \tag{73}$$

$$A_\varepsilon(X_\varepsilon-\chi^*)\rightharpoonup Z\quad\text{in}\quad H, \tag{74}$$

as $\varepsilon\searrow 0$. Thanks to the well known Sobolev imbedding, from (70) and (71) we infer that

$$\Theta_\varepsilon\to\vartheta^i\quad\text{in}\quad V, \tag{75}$$

$$X_\varepsilon\to\chi^i\quad\text{in}\quad V. \tag{76}$$

Besides, as π is a Lipschitz continuous function, we have that

$$|\pi(X_\varepsilon) - \pi(\chi^i)| \le C_\pi |X_\varepsilon - \chi^i|, \tag{77}$$

whence, thanks to (76), we obtain that

$$\pi(X_\varepsilon) \to \pi(\chi^i) \qquad \text{in } H, \tag{78}$$

as $\varepsilon \searrow 0$. Now, we pass to the limit on $\ln_\varepsilon(\Theta_\varepsilon)$, $A_\varepsilon(\Theta_\varepsilon - \vartheta^*)$ and $\beta_\varepsilon(X_\varepsilon)$. In view of a general convergence result involving maximal monotone operators (see, e.g., [1, Proposition 1.1, p. 42]), thanks to the strong convergences in H ensured by (75)–(76) and to the weak convergences in (73)–(74), we conclude that

$$B^i \in \beta(\chi^i), \qquad L^i \in \ln(\chi^i), \qquad Z^i \in A(\chi^i - \chi^*). \tag{79}$$

In conclusion, using (75)–(76) and (78)–(79), we can pass to the limit as $\varepsilon \searrow 0$ in (39)–(40) obtaining (10)–(15) for the limiting functions ϑ^i and χ^i.

3.5. **Uniqueness of the solution of (P_τ).** In this section we prove that the approximating problem (P_τ) stated by (10)–(15) has a unique solution. Then, the proof of Theorem 3.1 will be complete.

We write problem (P_τ) for two solutions $(\vartheta_1^i, \chi_1^i)$, $(\vartheta_2^i, \chi_2^i)$. and set $\vartheta^i := \vartheta_1^i - \vartheta_2^i$ and $\chi^i := \chi_1^i - \chi_2^i$. Then, we multiply by $\tau\vartheta^i$ the difference between the corresponding equations (10) and by χ^i the difference between the corresponding equations (11). Adding the resultant equations, we obtain that

$$\tau^{3/2}\|\vartheta^i\|_H^2 + \tau\big(\ln\vartheta_1^i - \ln\vartheta_2^i, \vartheta_1^i - \vartheta_2^i\big) + \tau\big(\zeta_1^i - \zeta_2^i, \chi_1^i - \chi^* - (\chi_2^i - \chi^*)\big) + \tau^2\int_\Omega |\nabla\vartheta^i|^2$$
$$+ \|\chi^i\|_H^2 + \tau\int_\Omega |\nabla\chi^i|^2 + \tau(\xi_1^i - \xi_2^i, \chi_1^i - \chi_2^i) = -\tau\big(\pi(\chi_1^i) - \pi(\chi_2^i), \chi_1^i - \chi_2^i\big). \tag{80}$$

Since ln, A and β are monotone, the second, the third and the seventh term on the left hand side of (80) are nonnegative. Besides, if $\tau \le 1/(2C_\pi)$, thanks to the Lipschitz continuity of π, the right hand side of (80) can be estimated as

$$\tau\big(\pi(\chi_1^i) - \pi(\chi_2^i), \chi_1^i - \chi_2^i\big) \le \frac{1}{2}\|\chi^i\|_H^2. \tag{81}$$

Then, due to (81), from (80) we conclude that

$$\tau^{3/2}\|\vartheta^i\|_H^2 + \tau^2\int_\Omega |\nabla\vartheta^i|^2 + \frac{1}{2}\|\chi^i\|_H^2 + \tau\int_\Omega |\nabla\chi^i|^2 \le 0, \tag{82}$$

whence we easily conclude that $\vartheta^i = \chi^i = 0$, i.e., $\vartheta_1^i = \vartheta_2^i$ and $\chi_1^i = \chi_2^i$.

4. **Uniform estimates on (AP_τ).** In this section we deduce some uniform estimates, independent of τ and inferred from the equations (10)–(15) of the approximating problem (P_τ).

First uniform estimate. We add (10) and (11) tested by ϑ^i and $(\chi^i - \chi^{i-1})/\tau$, respectively. Adding $(\chi^i, \chi^i - \chi^{i-1})$ to both side of the resultant equation and exploiting the cancellation of the suitable corresponding terms, we obtain that

$$\tau^{1/2}(\vartheta^i - \vartheta^{i-1}, \vartheta^i) + \big(\ln\vartheta^i - \ln\vartheta^{i-1}, \vartheta^i\big) + \tau k_0\int_\Omega |\nabla\vartheta^i|^2$$
$$+ \tau\left\|\frac{\chi^i - \chi^{i-1}}{\tau}\right\|_H^2 + (\chi^i, \chi^i - \chi^{i-1}) + (\nabla\chi^i, \nabla\chi^i - \nabla\chi^{i-1}) + (\xi^i, \chi^i - \chi^{i-1})$$
$$= \tau(F^i, \vartheta^i) - \big(\pi(\chi^i) - \chi^i, \chi^i - \chi^{i-1}\big) - \tau\left(\zeta^i, \frac{\chi^i - \chi^{i-1}}{\tau}\right). \tag{1}$$

Due to (10), we can rewrite the first, the fifth and the sixth term on the left hand side of (1) as

$$\tau^{1/2}(\vartheta^i-\vartheta^{i-1},\vartheta^i)=\frac{\tau^{1/2}}{2}\|\vartheta^i\|_H^2-\frac{\tau^{1/2}}{2}\|\vartheta^{i-1}\|_H^2+\frac{\tau^{1/2}}{2}\|\vartheta^i-\vartheta^{i-1}\|_H^2, \tag{2}$$

$$(\chi^i,\chi^i-\chi^{i-1})+(\nabla\chi^i,\nabla\chi^i-\nabla\chi^{i-1})=\frac{1}{2}\|\chi^i\|_V^2-\frac{1}{2}\|\chi^{i-1}\|_V^2+\frac{1}{2}\|\chi^i-\chi^{i-1}\|_V^2. \tag{3}$$

Moreover, since the function $u\longmapsto e^u$ is convex and e^u turns out to be its subdifferential, by setting $u^i=\ln\vartheta^i$ we obtain that

$$\begin{aligned}\big(\ln\vartheta^i-\ln\vartheta^{i-1},\vartheta^i\big)=\big(u^i-u^{i-1},e^{u^i}\big)&\geq\int_\Omega e^{u^i}-\int_\Omega e^{u^{i-1}}\\&=\|\vartheta^i\|_{L^1(\Omega)}-\|\vartheta^{i-1}\|_{L^1(\Omega)}.\end{aligned} \tag{4}$$

Since β is the subdifferential of $\tilde\beta$, it follows that

$$(\xi^i,\chi^i-\chi^{i-1})\geq\int_\Omega\tilde\beta(\chi^i)-\int_\Omega\tilde\beta(\chi^{i-1}), \tag{5}$$

while, due (9) and the sub-linear growth of A stated by (30), we obtain that

$$-\tau\left(\zeta^i,\frac{\chi^i-\chi^{i-1}}{\tau}\right)\leq C_1\tau(1+\|\chi^i\|_H^2)+\frac{\tau}{4}\left\|\frac{\chi^i-\chi^{i-1}}{\tau}\right\|_H^2,$$

where the constant C_1 depends on C_A, $\|\vartheta^*\|_H$ and C_p. Due to the the boundedness of F^i in $L^\infty(\Omega)$ and the Lipschitz continuity of π, we also infer that

$$\tau(F^i,\vartheta^i)\leq\tau\|F^i\|_{L^\infty(\Omega)}\|\vartheta^i\|_{L^1(\Omega)}, \tag{6}$$

$$\begin{aligned}-\big(\pi(\chi^i)-\chi^i,\chi^i-\chi^{i-1}\big)&\leq c\tau(1+\|\chi^i\|_H)\left\|\frac{\chi^i-\chi^{i-1}}{\tau}\right\|_H\\&\leq\frac{\tau}{4}\left\|\frac{\chi^i-\chi^{i-1}}{\tau}\right\|_H^2+\tau C_2(1+\|\chi^i\|_H^2),\end{aligned} \tag{7}$$

where C_2 depends on C_π and $|\pi(0)|$. Now, we apply the estimates (2)–(7) to the corresponding terms of (1) and sum up for $i=1,\cdots,n$, as $n\leq N$. We obtain that

$$\begin{aligned}&\frac{\tau^{1/2}}{2}\|\vartheta^n\|_H^2+\sum_{i=1}^n\frac{\tau^{1/2}}{2}\|\vartheta^i-\vartheta^{i-1}\|_H^2+\|\vartheta^n\|_{L^1(\Omega)}+\frac{k_0}{2}\sum_{i=1}^n\tau\|\nabla\vartheta^i\|_H^2\\&+\frac{1}{2}\sum_{i=1}^n\tau\left\|\frac{\chi^i-\chi^{i-1}}{\tau}\right\|_H^2+\frac{1}{2}\|\chi^n\|_V^2+\frac{1}{2}\sum_{i=1}^n\|\chi^i-\chi^{i-1}\|_V^2+\int_\Omega\tilde\beta(\chi^n)\\&\leq\frac{\tau^{1/2}}{2}\|\vartheta_0\|_H^2+\|\vartheta_0\|_{L^1(\Omega)}+\frac{1}{2}\|\chi_0\|_V^2+\int_\Omega\tilde\beta(\chi_0)+\tau\sum_{i=1}^n\|F^i\|_{L^\infty(\Omega)}\|\vartheta^i\|_{L^1(\Omega)}\\&\qquad+C_1\sum_{i=1}^n\tau\|\vartheta^i\|_{L^1(\Omega)}+C_2\sum_{i=1}^n\tau\|\chi^i\|_H^2+c.\end{aligned} \tag{8}$$

On account of (20)–(21) and (26), the first four terms on the right hand side of (8) are bounded. Now, recalling the definition of F^i (see (8)), we have that

$$\tau\sum_{i=1}^n\|F^i\|_{L^\infty(\Omega)}\|\vartheta^i\|_{L^1(\Omega)}$$

$$= \|\vartheta^n\|_{L^1(\Omega)} \int_{(n-1)\tau}^{n\tau} \|F(s)\|_{L^\infty(\Omega)}\, ds + \sum_{i=1}^{n-1} \|F^i\|_{L^\infty(\Omega)} \|\vartheta^i\|_{L^1(\Omega)}.$$

Thanks to the absolute continuity of the integral, if τ is small enough (independent of n) we have that

$$\int_{(n-1)\tau}^{n\tau} \|F(s)\|_{L^\infty(\Omega)}\, ds \le \frac{1}{4}, \qquad C_1\tau \le \frac{1}{4}, \qquad C_2\tau \le \frac{1}{4}. \tag{9}$$

Then, on the basis of (9), from (8) we infer that

$$\begin{aligned}
&\frac{\tau^{1/2}}{2}\|\vartheta^n\|_H^2 + \sum_{i=1}^n \frac{\tau^{1/2}}{2}\|\vartheta^i - \vartheta^{i-1}\|_H^2 + \frac{1}{2}\|\vartheta^n\|_{L^1(\Omega)} + \frac{k_0}{2}\sum_{i=1}^n \tau\|\nabla\vartheta^i\|_H^2 \\
&+ \frac{1}{2}\sum_{i=1}^n \tau\left\|\frac{\chi^i - \chi^{i-1}}{\tau}\right\|_H^2 + \frac{1}{4}\|\chi^n\|_V^2 + \frac{1}{2}\sum_{i=1}^n \|\chi^i - \chi^{i-1}\|_V^2 + \int_\Omega \tilde\beta(\chi^n) \\
&\le c + \tau\sum_{i=1}^{n-1}\Big(C_1\|\vartheta^i\|_{L^1(\Omega)}\|F^i\|_{L^\infty(\Omega)}\|\vartheta^i\|_{L^1(\Omega)} + C_2\|\chi^i\|_H\Big).
\end{aligned} \tag{10}$$

Now, we easily deduce that

$$\sum_{i=1}^{n-1}\tau C_1 \le \sum_{i=1}^{N}\tau C_1 = C_1 T, \qquad \sum_{i=1}^{n-1}\tau C_2 \le \sum_{i=1}^{N}\tau C_2 = C_2 T. \tag{11}$$

Beside, according to (18), we have that

$$\sum_{i=1}^{n-1}\tau\|F^i\|_{L^\infty(\Omega)} \le \sum_{i=1}^{N}\int_{(i-1)\tau}^{i\tau}\|F^i(s)\|_{L^\infty(\Omega)}\, ds = \int_0^T \|F^i(s)\|_{L^\infty(\Omega)}\, ds \le c. \tag{12}$$

Then, we can apply (12) and, recalling the notation (1), we conclude that

$$\begin{aligned}
&\tau^{1/2}\|\overline{\vartheta}_\tau\|^2_{L^\infty(0,T;H)} + \tau^{3/2}\|\partial_t\widehat{\vartheta}_\tau\|^2_{L^2(0,T;H)} + \|\overline{\vartheta}_\tau\|_{L^\infty(0,T;L^1(\Omega))} + \|\nabla\overline{\vartheta}_\tau\|_{L^2(0,T;H)} \\
&+ \|\partial_t\widehat{\chi}_\tau\|^2_{L^2(0,T;H)} + \|\overline{\chi}_\tau\|^2_{L^\infty(0,T;V)} + \tau\|\partial_t\widehat{\chi}_\tau\|^2_{L^2(0,T;V)} + \|\tilde\beta(\overline{\chi}_\tau)\|^2_{L^\infty(0,T;L^1(\Omega))} \le c,
\end{aligned} \tag{13}$$

Besides, in view of (12) and due to the sub-linear growth of A stated by (30) and to (19), we deduce that

$$\|\overline{\zeta}_\tau\|_{L^2(0,T;H)} \le c, \tag{14}$$

Since the third and the fourth term of the left hand side of (13) are bounded, using (7), we also infer that

$$\|\overline{\vartheta}_\tau\|_{L^2(0,T;V)} \le c. \tag{15}$$

Finally, by comparison in (10), we conclude that

$$\|\ln\overline{\vartheta}_\tau\|_{H^1(0,T;V')} \le c. \tag{16}$$

Second uniform estimate. We formally test (11) by ξ^i and obtain

$$(\chi^i - \chi^{i-1}, \xi^i) + \tau\|\xi^i\|_H^2 \le \tau\big(\pi(\chi^i) + \ell\vartheta^i, \xi^i\big) - \tau(\zeta^i, \xi^i). \tag{17}$$

We point out that the previous estimate (17) can be rigorously derived by testing (40) by $\beta_\varepsilon(X_\varepsilon)$ and then passing to the limit as $\varepsilon \searrow 0$. Since β is the subdifferential of $\tilde\beta$, we have that

$$(\chi^i - \chi^{i-1}, \xi^i) \ge \int_\Omega \tilde\beta(\chi^i) - \int_\Omega \tilde\beta(\chi^{i-1}). \tag{18}$$

Due to the Lipschitz continuity of π, applying the Young inequality (9) to the first term on the right hand side of (17), we deduce that

$$\tau\big(\pi(\chi^i)+\ell\vartheta^i,\xi^i\big)\le\frac{1}{4}\tau\|\xi^i\|_H^2+c\tau\big(1+\|\chi^i\|_H^2+\|\vartheta^i\|_H^2\big). \tag{19}$$

Moreover, due to the sub-linear growth of A stated by (30), using (13), we have that

$$-\tau(\zeta^i,\xi^i)\le c\tau(1+\|\chi^i\|_H^2)+\frac{\tau}{4}\|\xi^i\|_H^2. \tag{20}$$

Now, combining (17)–(20) and summing up for $i=1,\cdots,n$, $n\le N$, we infer that

$$\int_\Omega\tilde\beta(\chi^n)+\frac{1}{2}\sum_{i=1}^n\tau\|\xi^i\|_H^2\le\int_\Omega\tilde\beta(\chi_0)+c\sum_{i=1}^n\tau\big(1+\|\chi^i\|_H^2+\|\vartheta^i\|_H^2\big), \tag{21}$$

whence, due to (13)–(16), we obtain that

$$\|\overline{\xi}_\tau\|_{L^2(0,T;H)}\le c. \tag{22}$$

Finally, by comparison in (17), we conclude that $\|\Delta\overline{\chi}_\tau\|_{L^2(0,T;H)}\le c$. Then, thanks to (13) and elliptic regularity, we find out that

$$\|\overline{\chi}_\tau\|_{L^2(0,T;W)}\le c. \tag{23}$$

Third uniform estimate. We introduce the function $\psi_n:\mathbb{R}\longmapsto\mathbb{R}$ obtained by truncating the logarithmic function in the following way:

$$\psi_n(u)=\begin{cases}\ln(u) & \text{if } u\ge 1/n,\\ -\ln(n) & \text{if } u<1/n.\end{cases}$$

It is easy to see that ψ_n is an increasing and Lipschitz continuous function. Then, defining

$$j_n(u)=\int_1^u\psi_n(s)\,ds,\qquad u\in\mathbb{R}\qquad\text{and}\qquad j(u)=\int_1^u\ln s\,ds,\qquad u>0 \tag{24}$$

and testing (10) by $\psi_n(\vartheta^i)$, we obtain that

$$\begin{aligned}\tau^{1/2}\big(\vartheta^i-\vartheta^{i-1},\psi_n(\vartheta^i)\big)+\big(\ln\vartheta^i-\ln\vartheta^{i-1},\psi_n(\vartheta^i)\big)+\tau k_0\int_{\Omega\cap\{\vartheta^i\ge 1/n\}}\frac{|\nabla\vartheta^i|^2}{\vartheta^i}\\ =-\ell\big(\chi^i-\chi^{i-1},\psi_n(\vartheta^i)\big)+\tau\big(F^i,\psi_n(\vartheta^i)\big).\end{aligned} \tag{25}$$

Recalling that j_n is a convex function with derivative ψ_n, we have that

$$\tau^{1/2}\big(\vartheta^i-\vartheta^{i-1},\psi_n(\vartheta^i)\big)\ge\tau^{1/2}\int_\Omega j_n(\vartheta^i)-\tau^{1/2}\int_\Omega j_n(\vartheta^{i-1}), \tag{26}$$

whence, from (25) we infer that

$$\begin{aligned}\tau k_0\int_{\Omega\cap\{\vartheta^i\ge 1/n\}}\frac{|\nabla\vartheta^i|^2}{\vartheta^i}\le\tau^{1/2}\int_\Omega j_n(\vartheta^{i-1})-\tau^{1/2}\int_\Omega j_n(\vartheta^i)\\ -\int_\Omega(\ln\vartheta^i-\ln\vartheta^{i-1})\psi_n(\vartheta^i)-\int_\Omega\big(\ell(\chi^i-\chi^{i-1})-\tau F^i\big)\psi_n(\vartheta^i).\end{aligned} \tag{27}$$

Due to the properties of the subdifferential we have that

$$0\le j(\vartheta^k)\le j(1)+(\ln\vartheta^k,\vartheta^k-1)\qquad\text{for } k=0,1,\dots,N. \tag{28}$$

Since $\ln\vartheta^k \in H$, $\vartheta^k > 0$ a.e. in Ω and $\vartheta^k \in H$, from (28) we infer that $j(\vartheta^k) \in L^1(\Omega)$; consequently, passing to the limit as $n \to +\infty$, we obtain that

$$\begin{aligned} \psi_n(\vartheta^k) &\to \ln\vartheta^k && \text{in } H \text{ and a.e. in } \Omega, \\ j_n(\vartheta^k) &\to j(\vartheta^k) && \text{in } L^1(\Omega) \text{ and a.e. in } \Omega, \end{aligned}$$

for $k = 0, 1, \dots, N$. Then, taking the $\liminf$ in (27) as $n \to +\infty$ and applying the Fatou Lemma and (10), we have that

$$\tau k_0 \int_\Omega \frac{|\nabla\vartheta^i|^2}{\vartheta^i} \le \tau^{1/2}\int_\Omega j(\vartheta^{i-1}) - \tau^{1/2}\int_\Omega j(\vartheta^i) - \frac{1}{2}\int_\Omega |\ln\vartheta^i|^2$$
$$- \frac{1}{2}\int_\Omega |\ln\vartheta^i - \ln\vartheta^{i-1}|^2 + \frac{1}{2}\int_\Omega |\ln\vartheta^{i-1}|^2 - \int_\Omega \big(\ell(\chi^i - \chi^{i-1}) - \tau F^i\big)\ln\vartheta^i. \quad (29)$$

Now, sum up (29) for $i = 1, \cdots, n$, with $k \le N$ and obtain that

$$\tau^{1/2}\int_\Omega j(\vartheta^k) + \frac{1}{2}\|\ln\vartheta^k\|_H^2 + \frac{1}{2}\sum_{i=1}^k \tau^2 \left\| \frac{\ln\vartheta^i - \ln\vartheta^{i-1}}{\tau}\right\|_H^2 + k_0\sum_{i=1}^k \tau\int_\Omega \frac{|\nabla\vartheta^i|^2}{\vartheta^i}$$
$$\le \tau^{1/2}\int_\Omega j(\vartheta_0) + \frac{1}{2}\|\ln\vartheta_0\|_H^2 + \frac{1}{4}\sum_{i=1}^k \tau\|\ln\vartheta^i\|_H^2$$
$$+ c\sum_{i=1}^k \tau\left\|\frac{\chi^i - \chi^{i-1}}{\tau}\right\|_H^2 + c\sum_{i=1}^k \tau\|F^i\|_H^2. \quad (30)$$

We observe that, if $\tau \le 1$, then

$$\begin{aligned} \frac{1}{4}\sum_{i=1}^k \tau\|\ln\vartheta^i\|_H^2 &= \frac{1}{4}\sum_{i=1}^{k-1} \tau\|\ln\vartheta^i\|_H^2 + \frac{1}{4}\tau\|\ln\vartheta^k\|_H^2 \\ &\le \frac{1}{4}\sum_{i=1}^{k-1} \tau\|\ln\vartheta^i\|_H^2 + \frac{1}{4}\|\ln\vartheta^k\|_H^2 \end{aligned} \quad (31)$$

We also notice that the fourth and the fifth term on the right hand side of (30) are bounded by a positive constant c, due to (13) and (14), respectively. Moreover, thanks to (18) and to the definition (8) of F^i, using the Hölder inequality, the last term on the right hand side of (30) can be estimated as follows:

$$\begin{aligned} c\sum_{i=1}^k \tau\|F^i\|_H^2 &\le c\sum_{i=1}^k \tau\left\|\frac{1}{\tau}\int_{(i-1)\tau}^{i\tau} F(s)\,ds\right\|_H^2 \\ &\le c\sum_{i=1}^k \int_{(i-1)\tau}^{i\tau} \|F(s)\|_H^2\,ds \le c\|F\|_{L^2(0,T;H)}^2. \end{aligned} \quad (32)$$

Then, combining (30) with (31)–(32) (see also (28) and (20)), we infer that

$$\tau^{1/2}\int_\Omega j(\vartheta^k) + \frac{1}{4}\|\ln\vartheta^k\|_H^2 + \frac{1}{2}\sum_{i=1}^k \tau^2 \left\| \frac{\ln\vartheta^i - \ln\vartheta^{i-1}}{\tau}\right\|_H^2 + k_0\sum_{i=1}^k \tau\int_\Omega \frac{|\nabla\vartheta^i|^2}{\vartheta^i}$$
$$\le c + \frac{1}{4}\sum_{i=1}^{k-1} \tau\|\ln\vartheta^i\|_H^2, \quad (33)$$

whence, by applying (12), we conclude that

$$\tau^{1/2}\|j(\overline{\vartheta}_\tau)\|_{L^\infty(0,T;L^1(\Omega))} + \|\ln\overline{\vartheta}_\tau\|_{L^\infty(0,T;H)} + \|\nabla\overline{\vartheta}_\tau^{1/2}\|_{L^2(0,T;H)} \le c. \quad (34)$$

Moreover, due to (13), we also infer that

$$\|\overline{\vartheta}_\tau^{1/2}\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \leq c. \tag{35}$$

Summary of the uniform estimates. Let us collect the previous a priori estimates. From (13)–(14), (22)–(23) and (34)–(35) we conclude that there exists a constant $c > 0$, independent of τ, such that

$$\begin{aligned}
&\|\overline{\vartheta}_\tau\|_{L^\infty(0,T;V)} + \|\widehat{\vartheta}_\tau\|_{L^\infty(0,T;V)} + \tau^{1/4}\|\widehat{\vartheta}_\tau\|_{L^\infty(0,T;V)\cap L^2(0,T;H)} \\
&+ \|\overline{\ln\vartheta_\tau}\|_{H^1(0,T;V')\cap L^\infty(0,T;H)} + \|\overline{\vartheta}_\tau^{1/2}\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|\partial_t\widehat{\vartheta^{1/2}}_\tau\|_{L^2(0,T;V)} \\
&+ \|\overline{\chi}_\tau\|_{L^\infty(0,T;W)} + \|\partial_t\widehat{\chi}_\tau\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \\
&+ \|\overline{\zeta}_\tau\|_{L^\infty(0,T;H)} + \|\overline{\xi}_\tau\|_{L^\infty(0,T;H)} \leq c.
\end{aligned} \tag{36}$$

5. **Passage to the limit as $\tau \searrow 0$.** Thanks to (36) and the well-known weak or weak* compactness results, we deduce that, at least for a subsequence, there exists eight limit functions ϑ, $\widehat{\vartheta}$, λ, $\widehat{\lambda}$, w, $\widehat{w}$, χ, $\widehat{\chi}$, ξ and ζ such that

$$\overline{\vartheta}_\tau \rightharpoonup^* \vartheta \quad \text{in} \quad L^\infty(0,T;V), \tag{1}$$
$$\widehat{\vartheta}_\tau \rightharpoonup^* \widehat{\vartheta} \quad \text{in} \quad L^\infty(0,T;V), \tag{2}$$
$$\tau^{1/4}\widehat{\vartheta}_\tau \rightharpoonup 0 \quad \text{in} \quad H^1(0,T;H), \tag{3}$$
$$\overline{\ln\vartheta}_\tau \rightharpoonup^* \lambda \quad \text{in} \quad H^1(0,T;V')\cap L^\infty(0,T;H), \tag{4}$$
$$\overline{\vartheta}_\tau^{1/2} \rightharpoonup^* w \quad \text{in} \quad L^\infty(0,T;H)\cap L^2(0,T;V), \tag{5}$$
$$\widehat{\vartheta^{1/2}}_\tau \rightharpoonup \widehat{w} \quad \text{in} \quad H^1(0,T;H)\cap L^2(0,T;V), \tag{6}$$
$$\overline{\chi}_\tau \rightharpoonup^* \chi \quad \text{in} \quad L^\infty(0,T;W), \tag{7}$$
$$\widehat{\chi}_\tau \rightharpoonup \widehat{\chi} \quad \text{in} \quad W^{1,\infty}(0,T;H)\cap H^1(0,T;V)\cap L^\infty(0,T;W), \tag{8}$$
$$\overline{\xi}_\tau \rightharpoonup^* \xi \quad \text{in} \quad L^\infty(0,T;H), \tag{9}$$
$$\overline{\zeta}_\tau \rightharpoonup^* \zeta \quad \text{in} \quad L^\infty(0,T;H), \tag{10}$$

as $\tau \searrow 0$. Let us stress that, thanks to (4) and [27, Lemma 8, p. 84]), we have that

$$\overline{\ln\vartheta}_\tau \to \lambda \quad \text{in} \quad C^0([0,T];V'), \tag{11}$$
$$\overline{\vartheta}_\tau^{1/2} \to w \quad \text{in} \quad L^2(0,T;H), \tag{12}$$
$$\overline{\chi}_\tau \to \chi \quad \text{in} \quad C^0([0,T];H^{2-\delta}), \tag{13}$$

for every $0 < \delta < 2$. Now, we observe that $\vartheta = \widehat{\vartheta}$. Indeed, thanks to (6), we have that

$$\|\overline{\vartheta}_\tau - \widehat{\vartheta}_\tau\|_{L^2(0,T;H)} \leq \frac{\tau}{\sqrt{3}}\|\partial_t\widehat{\vartheta}_\tau\|_{L^2(0,T;H)} \leq c\tau^{3/4}. \tag{14}$$

From (14) we conclude that

$$\lim_{\tau\searrow 0}\|\overline{\vartheta}_\tau - \widehat{\vartheta}_\tau\|_{L^2(0,T;H)} = 0 \tag{15}$$

and the previous convergence still holds a.e. in Q, at least for a subsequence. Then, $\vartheta = \widehat{\vartheta}$. Moreover, we notice that $w = \widehat{w}$. Indeed, due to (5)–(7), we obtain that

$$\|\overline{\vartheta}_\tau^{1/2} - \widehat{\vartheta^{1/2}}_\tau\|_{L^2(0,T;H)} \leq \frac{\tau}{\sqrt{3}}\|\partial_t\widehat{\vartheta^{1/2}}_\tau\|_{L^2(0,T;H)} \leq c\tau. \tag{16}$$

From (16) we infer that

$$\lim_{\tau\searrow 0}\|\overline{\vartheta}_\tau^{1/2} - \widehat{\vartheta^{1/2}}_\tau\|_{L^2(0,T;H)} = 0 \tag{17}$$

and the previous convergence still holds a.e. in Q, at least for a subsequence. Then, $w = \widehat{w}$ and

$$\lim_{\tau \searrow 0} \overline{\vartheta}_\tau^{1/2} = w \qquad \text{a.e. in } Q, \tag{18}$$

whence

$$\overline{\vartheta}_\tau \to w^2 \equiv \vartheta \qquad \text{a.e. in } Q \text{ and strongly in } L^2(0,T;H). \tag{19}$$

Finally, we infer that $\chi = \widehat{\chi}$. Indeed, due to (5)–(7), we have that

$$\|\overline{\chi}_\tau - \widehat{\chi}_\tau\|_{L^\infty(0,T;V)} \le \frac{\tau}{\sqrt{3}} \|\partial_t \widehat{\chi}_\tau\|_{L^\infty(0,T;V)} \le c\tau. \tag{20}$$

Consequently, thanks to (20) and [27, Lemma 8, p. 84], we conclude that

$$\lim_{\tau \searrow 0} \|\overline{\chi}_\tau - \widehat{\chi}_\tau\|_{C^0([0,T];V)} = 0, \tag{21}$$

as well as

$$\lim_{\tau \searrow 0} \|\overline{\chi}_\tau - \widehat{\chi}_\tau\|_{L^\infty(0,T;V)} = 0, \tag{22}$$

and the previous convergence still holds a.e. in Q, at least for a subsequence. Then, $\chi = \widehat{\chi}$.

Passage to the limit on the initial values. Since $\overline{\chi}_\tau \rightharpoonup^* \chi$ in $L^\infty(0,T;W)$ and $\widehat{\chi}_\tau \rightharpoonup \widehat{\chi}$ in $L^\infty(0,T;W)$ (cf. (7)–(8)), we infer that

$$\partial_\nu \vartheta = 0 \qquad \text{on } \Sigma, \tag{23}$$

whence (43) is verified. Moreover, due to the strong convergences of of $\ln(\overline{\vartheta}_\tau)$ and $\overline{\vartheta}_\tau$ stated by (11) and (12), respectively, we obtain that

$$\ln \vartheta(0) = \ln \vartheta_0, \qquad \chi(0) = \chi_0 \quad \text{in } \Omega, \tag{24}$$

whence (44) is deduced.

Passage to the limit on the logarithmic nonlinearity. Due to the weak convergence of $\overline{\vartheta}_\tau$ ensured by (1) and to the strong convergence of $\ln(\overline{\vartheta}_\tau)$ stated by (11), we have that

$$\lim_{\tau \searrow 0} \int_0^T \int_\Omega \left(\overline{\ln \vartheta_\tau}\right) \overline{\vartheta}_\tau = \lim_{\tau \searrow 0} \int_0^T \langle \overline{\ln} \vartheta_\tau, \overline{\vartheta}_\tau \rangle = \int_0^T \langle \lambda, \vartheta \rangle = \int_0^T \int_\Omega \lambda \vartheta, \tag{25}$$

whence $\lambda = \ln \vartheta$ and the equation (45) is also achieved.

Passage to the limit on the other nonlinearities. In this paragraph we check that $\xi \in \beta(\varphi)$ a.e. in Q and that $\zeta(t) \in A(\vartheta(t) - \chi^*)$ for a.e. $t \in [0,T]$. Denoting with the same symbol β the operator induced by β on $L^2(0,T;H)$, we recall that

$$\overline{\chi}_\tau \to \chi \quad \text{in} \quad L^2(0,T;H) \equiv L^2(Q), \tag{26}$$

$$\overline{\xi}_\tau \rightharpoonup \xi \quad \text{in} \quad L^2(0,T;H). \tag{27}$$

Consequently, due to [2, Proposition 2.2, p. 38], we conclude that

$$\xi \in \beta(\varphi) \qquad \text{in } L^2(0,T;H), \tag{28}$$

and this is equivalent to saying that

$$\xi \in \beta(\varphi) \qquad \text{a.e. in } Q. \tag{29}$$

Now, denoting by the same symbol A the operator induced by A on $L^2(0,T;H)$, we recall that

$$\overline{\chi}_\tau \to \chi \quad \text{in} \quad L^2(0,T;H), \tag{30}$$

$$\overline{\zeta}_\tau \rightharpoonup \zeta \quad \text{in} \quad L^2(0,T;H), \tag{31}$$

as $\tau \searrow 0$, whence, setting

$$\eta_\tau := \overline{\chi}_\tau - \chi^*, \qquad \eta := \chi - \chi^*,$$

we have that

$$\eta_\tau \longrightarrow \eta \quad \text{in } L^2(0,T;H), \tag{32}$$

as $\tau \searrow 0$. Due to [2, Proposition 2.2, p. 38], (30) and (32), we conclude that

$$\zeta \in \mathcal{A}(\eta) \quad \text{in } L^2(0,T;H), \tag{33}$$

and this is equivalent to saying that

$$\zeta(t) \in A(\chi(t) - \chi^*) \qquad \text{for a.e. } t \in [0,T]. \tag{34}$$

6. **Uniqueness.** In this section we prove the uniqueness of the solution of Problem (P) (see (39)–(44) and Theorem 2.3).

We integrate (39) over $(0,t)$ and we obtain

$$\ln\vartheta(t) + \ell\chi(t) - k_0\int_0^t \Delta\vartheta(s)\,ds = \ln\vartheta_0 + \ell\chi_0 + \int_0^t F(s)\,ds. \tag{1}$$

Then, we couple (1) with (40)–(44). We assume that F, χ^*, ϑ_0 and χ_0 are given as in (18)–(21) and (ϑ_i,χ_i), $i=1,2$, are the corresponding solutions of problem (P) (see (39)–(44)). Then we write both (1) and (40) for such solutions and multiply the difference of the first equations by $\vartheta := \vartheta_1 - \vartheta_2$ and the difference of the second ones by $\chi := \chi_1 - \chi_2$. Finally, we sum the equalities that we have obtained to each other and integrate over Q_t. We have that

$$\begin{aligned}
&\int_{Q_t}\Big(\ln(\vartheta_1) - \ln(\vartheta_2)\Big)(\vartheta_1-\vartheta_2) + \frac{k_0}{2}\int_\Omega\left|\nabla\int_0^t\vartheta\right|^2\\
&+\int_{Q_t}\Big(A(\chi_1-\chi^*) - A(\chi_2-\chi^*)\Big)(\chi_1-\chi_2) + \frac{1}{2}\int_\Omega|\chi(t)|^2 + \int_{Q_t}|\nabla\chi|^2\\
&+\int_{Q_t}\Big(\beta(\chi_1)-\beta(\chi_2)\Big)(\chi_1-\chi_2) = -\int_{Q_t}\Big(\pi(\chi_1)-\pi(\chi_2)\Big)(\chi_1-\chi_2).
\end{aligned} \tag{2}$$

Due to the Lipschitz continuity of π, the right hand side of (2) can be estimated as follows

$$-\int_{Q_t}\Big(\pi(\chi_1)-\pi(\chi_2)\Big)(\chi_1-\chi_2) \le C_\pi\int_{Q_t}|\chi|^2, \tag{3}$$

while the third and the last term on the left hand side of (2), due to the monotonicity of A and β, respectively, can be treated in this way:

$$\begin{aligned}
&\int_{Q_t}\Big(A(\chi_1-\chi^*) - A(\chi_2-\chi^*)\Big)(\chi_1-\chi_2)\\
&= \int_{Q_t}\Big(A(\chi_1-\chi^*) - A(\chi_2-\chi^*)\Big)\Big((\chi_1-\chi^*) - (\chi_2-\chi^*)\Big) \ge 0,
\end{aligned} \tag{4}$$

$$\int_{Q_t}\Big(\beta(\chi_1)-\beta(\chi_2)\Big)(\chi_1-\chi_2) \ge 0. \tag{5}$$

Finally, using (3)–(5) and applying the Gronwall Lemma to (2), we infer that

$$\int_{Q_t}\Big(\ln(\vartheta_1)-\ln(\vartheta_2)\Big)(\vartheta_1-\vartheta_2) + \frac{k_0}{2}\int_\Omega\left|\nabla\int_0^t\vartheta\right|^2 + \frac{1}{2}\int_\Omega|\chi(t)|^2 + \int_{Q_t}|\nabla\chi|^2 \le 0. \tag{6}$$

Consequently, since ln is strictly monotone, we conclude that

$$\vartheta_1 = \vartheta_2, \qquad \chi_1 = \chi_2. \tag{7}$$

7. **Long time behavior.** In this section we prove Theorem 2.4. Our procedure is the following. First we perform a number of a priori estimates that provide some compactness and ensure, in particular, that the ω-limit is nonempty and fulfills the basic properties stated in Theorem 2.4. Then, we pick any element $(\vartheta_\infty, \chi_\infty) \in \omega$ and prove its relationship with the limit problem

$$-k_0 \Delta \vartheta_\infty = 0 \quad \text{for a.e. in } \Omega, \tag{1}$$

$$-\Delta \chi_\infty + \xi_\infty + \pi(\chi_\infty) + \zeta_\infty = \ell \vartheta_\infty \quad \text{a.e. in } \Omega, \tag{2}$$

$$\xi_\infty \in \beta(\chi_\infty), \qquad \zeta_\infty \in A(\chi_\infty - \chi^*) \quad \text{a.e. in } \Omega. \tag{3}$$

Our argument is the following. We choose a sequence t_n such that $t_n \nearrow +\infty$, as $\to +\infty$, according to definition (49) and introduce the auxiliary functions

$$\vartheta_n(t) := \vartheta(t + t_n) \qquad \text{and} \qquad \chi_n(t) := \chi(t + tn), \qquad t \in [0, +\infty) \tag{4}$$

which solve problems close to (39)–(44). We show that the a priori estimates derived in the previous steps yield a number of estimates for such functions which allow us to take a weak limit point $(\vartheta^\infty, \chi^\infty)$ of the sequence $\{(\vartheta_n, \chi_n)\}$. We infer that $(\vartheta^\infty, \chi^\infty)$ solves a system close to (1)–(3), and the last step of the proof is to show that $(\vartheta^\infty, \chi^\infty)$ does not depend on time and coincides with the original pair $(\vartheta_\infty, \chi_\infty)$ of the ω-limit.

Our proof relies on a number a priori estimates. However, the regularity of the solution is not sufficient to completely justify the calculation we would like to perform. Therefore, we should come back to the procedure used in [6], where an analogous problem has been solved by passing to the limit as $\varepsilon \searrow 0$ in an approximating problem depending on the positive parameter ε, and prove a priori estimates which are uniform with respect to ε. However, in order not to make the exposition too heavy, we prefer to proceed formally on the solution of problem (39)–(44). Of course, we think of a more regular structure and of smoother initial data for a while, but it is understood that we cannot use constants related to such a further regularity. We remark that the source term F and the boundary datum are smooth enough by assumption and that no new property of the initial data ϑ_0 and χ_0 is needed (i.e., more regularity is assumed just for the approximating initial data) since we use weighted test functions, if necessary. Now, we recall the main feature of the approximating problem, which has the following form

$$\partial_t(\varepsilon \vartheta_\varepsilon + \ln_\varepsilon \vartheta_\varepsilon + \ell \chi_\varepsilon) - k_0 \Delta \vartheta_\varepsilon = F \quad \text{a.e. in } Q, \tag{5}$$

$$\partial_t \chi_\varepsilon - \Delta \chi_\varepsilon + \xi_\varepsilon + \pi(\chi_\varepsilon) + \zeta_\varepsilon = \ell \vartheta_\varepsilon \quad \text{a.e. in } Q, \tag{6}$$

$$\xi_\varepsilon \in \beta_\varepsilon(\chi_\varepsilon) \quad \text{a.e. in } Q, \tag{7}$$

$$\zeta_\varepsilon(t) \in A_\varepsilon(\chi_\varepsilon(t) - \chi^*) \quad \text{for a.e. } t \in (0, T), \tag{8}$$

$$\partial_\nu \vartheta_\varepsilon = 0, \qquad \partial_\nu \chi_\varepsilon = 0 \quad \text{on } \Sigma, \tag{9}$$

$$\ln_\varepsilon \vartheta_\varepsilon(0) = \ln_\varepsilon \vartheta_0, \qquad \chi_\varepsilon(0) = \chi_0 \quad \text{in } \Omega, \tag{10}$$

where $\ln_\varepsilon$, A_ε and β_ε are the Yosida regularization of ln, A and β defined by (28), (31) and (35), respectively. It has been proved that problem (5)–(10) has a solution $(\vartheta_\varepsilon, \chi_\varepsilon, \xi_\varepsilon)$ and that such a solution tends to (ϑ, χ, ξ) in some appropriate topology as $\varepsilon \searrow 0$, at least for a subsequence.

7.1. **A priori estimates.** Under the assumptions of Theorem 2.4 the solution of problem (39)–(44) will satisfy new a priori estimates, provided that corresponding uniform estimates are fulfilled by the solution of problem (5)–(10) and that just norms related either to reflexive Banach spaces or to dual spaces of separable Banach spaces are involved. Hence, everything would work if we were dealing with the approximating problem. In order to clarify this point, we write a remark after each formal estimate.

First a priori estimate We take the difference between (39) and (1) and test it by ϑ. Then, we multiply (40) by $\partial_t\chi$, sum the obtained equality to each other and integrate over $(0,t)$. We obtain that

$$\int_\Omega \vartheta(t) + k_0\int_{Q_t}|\nabla\vartheta|^2 + \int_{Q_t}|\partial_t\chi|^2 + \frac{1}{2}\int_\Omega|\nabla\chi(t)|^2 + \int_\Omega\tilde\beta(\chi(t))$$
$$+\rho\int_0^t(\mathrm{Sign}(\chi(s)-\chi^*),\partial_t\chi(s))_H\,ds = \int_\Omega\vartheta_0 + \int_{Q_t}F\vartheta + \frac{1}{2}\int_\Omega|\nabla\chi_0|^2$$
$$+\int_\Omega\tilde\beta(\chi_0) - \int_\Omega\tilde\pi(\chi(t)) + \int_\Omega\tilde\pi(\chi_0) \tag{11}$$

and treat each term that need some manipulation, separately. Since Sign is the subdifferential of the map $\|\cdot\|: H\to\mathbb{R}$, we have that

$$\rho\int_{Q_t}\mathrm{Sign}(\chi-\chi^*)\partial_t\chi = \rho\|\chi(t)-\chi^*\|_H - \rho\|\chi_0-\chi^*\|_H. \tag{12}$$

Now, we deal with the π term. It is easy to see that (22)–(24) imply that

$$r^2 \le \delta\tilde\beta(r) + c_\delta \qquad \text{and} \qquad |\pi(r)| \le c(r^2+1) \qquad \text{for every } r \text{ in } \mathbb{R}, \tag{13}$$

where δ denotes an arbitrary positive parameter, whose value is chosen whenever it is convenient to do it. Hence, we deduce that

$$-\int_\Omega\tilde\pi(\chi(t)) \le \frac{1}{2}\int_\Omega\tilde\beta(\chi(t)) + c. \tag{14}$$

Moreover, we have that

$$\int_{Q_t}F\vartheta = \int_0^t\|F(s)\|_\infty\|\vartheta(s)\|_{L^1(\Omega)}\,ds. \tag{15}$$

Due to the (20)–(21), (26), (48) and the previous estimates, we infer that

$$\int_\Omega\vartheta(t) + k_0\int_{Q_t}|\nabla\vartheta|^2 + \int_{Q_t}|\partial_t\chi|^2 + \frac{1}{2}\int_\Omega|\nabla\chi(t)|^2 + \frac{1}{2}\int_\Omega\tilde\beta(\chi(t))$$
$$+\rho\int_\Omega\|\chi(t)-\chi^*\|_H \le c\left(1+\int_0^t\|F(s)\|_\infty\|\vartheta(s)\|_{L^1(\Omega)}\,ds\right). \tag{16}$$

Applying the Gronwall lemma and using (48), from (16) we conclude that

$$\|\vartheta\|_{L^2(0,+\infty;V)\cap L^\infty(0,+\infty;L^1(\Omega))} + \|\partial_t\chi\|_{L^2(0,+\infty;H)} + \|\chi\|_{L^\infty(0,+\infty;V)}$$
$$+\|\tilde\beta(\chi)\|_{L^\infty(0,+\infty;L^1(\Omega))} + \rho\|\chi\|_{L^\infty(0,+\infty;H)} \le c. \tag{17}$$

Moreover, due to (48), by comparison in (39), we have that

$$\|\partial_t(\ln\vartheta)\|_{L^2(0,+\infty;V')} \le c. \tag{18}$$

Remark As said before, the above estimates (17)–(18) should be performed on the approximating problems. Doing that, we would obtain a uniform bound for both ϑ_ε and $\ln_\varepsilon(\vartheta_\varepsilon)$ in the space $L^\infty(0,+\infty;L^1(\Omega))$. Moreover, we note that the main

trouble in our formal procedure relies on the fact that the time derivative $\partial_t \ln \vartheta$ belongs just to $L^2(0,T;V')$. On the contrary, as the graph of the logarithm is replaced by a bi-Lipschitz relation (see (5)), the corresponding term of the approximating problem is a function.

Second a priori estimate We set for convenience $\alpha(t) = \tanh(t)$ for $t \geq 0$ and note that bot α and α' are bounded by 1. Now, we take the difference between (39) and (1). and test it by $\alpha\partial_t\vartheta = \alpha\partial_t(\vartheta - \vartheta_\infty)$. Next, we differentiate (40) with respect to time and test it by $\alpha\partial_t\chi$. Finally, we add the equalities we get to each other, integrate over $(0,t)$ and obtain that

$$\int_{Q_t} \alpha\frac{|\partial_t\vartheta|^2}{\vartheta} + k_0\int_{Q_t} \alpha\nabla(\vartheta-\vartheta_\infty)\partial_t\Big(\nabla(\vartheta-\vartheta_\infty)\Big) + \int_{Q_t}\alpha\partial_{tt}\chi\partial_t\chi + \int_{Q_t}\alpha|\nabla\partial_t\chi|^2$$
$$+\int_{Q_t}\alpha\beta'(\chi)|\partial_t\chi|^2 + \rho\int_0^t \alpha(s)(\partial_t \operatorname{Sign}(\chi(s)-\chi^*), \partial_t\chi(s))_H\, ds$$
$$= \int_{Q_t}\alpha F\vartheta - \int_{Q_t}\alpha\pi'(\chi)|\partial_t\chi|^2. \tag{19}$$

We treat each term that need some manipulation, separately. First, integrating by parts the second term on the left hand side of (19) and using (17), we obtain

$$k_0\int_{Q_t}\alpha\nabla(\vartheta-\vartheta_\infty)\partial_t\Big(\nabla(\vartheta-\vartheta_\infty)\Big)$$
$$= k_0\left[\frac{\alpha(t)}{2}\int_\Omega|\nabla(\vartheta(t)-\vartheta_\infty)|^2 - \frac{1}{2}\int_{Q_t}\alpha'|\nabla(\vartheta-\vartheta_\infty)|^2\right]$$
$$\geq k_0\frac{\alpha(t)}{2}\int_\Omega|\nabla(\vartheta(t)-\vartheta_\infty)|^2 - c.$$

Now, we deal with the third term on the left hand side of (19). With an analogous strategy, we infer that

$$\int_{Q_t}\alpha\partial_{tt}\chi\partial_t\chi \geq \frac{\alpha(t)}{2}\int_\Omega|\partial_t\vartheta(t)|^2 - c. \tag{20}$$

Since β and Sign are monotone, the last two terms on the left hand side of (19) are nonnegative. Besides, due to (48) and (17) the first term on the right hand side of (19) can be treated as follows:

$$\int_{Q_t}\alpha F\vartheta = \alpha\int_\Omega F(t)(\vartheta-\vartheta_\infty) - \int_{Q_t}\alpha' F(\vartheta-\vartheta_\infty) - \int_{Q_t}\alpha'(\vartheta-\vartheta_\infty)\partial_t F \leq c. \tag{21}$$

Indeed, from (48) we easily deduce that

$$\|\partial_t F\|_{L^1(0,+\infty;L^\infty(\Omega))} \leq c. \tag{22}$$

Finally, due to (17) and the Lipschitz continuity of π, we have that

$$-\int_{Q_t}\alpha\pi'(\chi)|\partial_t\chi|^2 \leq \int_{Q_t}|\alpha||\pi'(\chi)||\partial_t\chi|^2 \leq C_\pi\int_{Q_t}|\partial_t\chi|^2 \leq c. \tag{23}$$

Combining (19) with the previous estimates, we infer that

$$\int_{Q_t}\alpha\frac{|\partial_t\vartheta|^2}{\vartheta} + k_0\frac{\alpha(t)}{2}\int_\Omega|\nabla(\vartheta(t)-\vartheta_\infty)|^2 + \frac{\alpha(t)}{2}\int_\Omega|\partial_t\vartheta(t)|^2$$
$$+\int_{Q_t}\alpha|\nabla\partial_t\chi|^2 + \int_{Q_t}\alpha\beta'(\chi)|\partial_t\chi|^2 \leq c, \tag{24}$$

whence we conclude that

$$\|\partial_t \vartheta^{1/2}\|_{L^2(1,+\infty;H)} + \|\vartheta\|_{L^\infty(1,+\infty;V)} + \|\partial_t \chi\|_{L^\infty(1,+\infty;H)\cap L^2(1,+\infty;V)} \le c. \tag{25}$$

Moreover, due to (17), (25), the Lipschitz continuity of π and the Sobolev imbedding, we infer that

$$\|\pi(\chi)\|_{L^\infty(1,+\infty;L^6(\Omega))} \le c, \tag{26}$$

and, thanks to sub linear growth of Sign, by comparison in (40) we conclude that

$$\|\chi\|_{L^\infty(1,+\infty;W)} + \|\xi\|_{L^\infty(1,+\infty;H)} + \rho\|\operatorname{Sign}(\chi-\chi^*)\|_{L^\infty(1,+\infty;H)} \le c(1+\rho). \tag{27}$$

Remark In the above argument, we have differentiated (40). Such a procedure would be correct when dealing with the approximating problem (5)–(10), provided that its solution is smooth enough. Now, one could go through the proofs of [6] and see that the approximating solution is smoother provided that the data, the functions ln, and the graphs β and A are approximated with some more care. On the other hand, the passage to the limit as $\epsilon \searrow 0$ uses just very general properties and does not rely on a precise approximation. For instance, as far as β is concerned, one can see that the monotonicity of β_ε and the Mosco convergence of its primitive $\tilde{\beta}_\varepsilon$ to $\tilde{\beta}$ are sufficient to handle the ξ term.

Third a priori estimate Now, we prove that

$$\|\partial_t \vartheta\|_{L^2(1,+\infty;L^{12/7}(\Omega))} \le c. \tag{28}$$

Indeed, due to (25) and the Sobolev imbedding, we have that

$$\begin{aligned}
\|\partial_t \vartheta\|_{L^2(1,+\infty;L^{12/7}(\Omega))} &\le 2\|\vartheta^{1/2}\|_{L^\infty(1,+\infty;L^{12}(\Omega))}\|\partial_t \vartheta^{1/2}\|_{L^2(1,+\infty;H)} \\
&\le 2\|\vartheta\|^{1/2}_{L^\infty(1,+\infty;L^6(\Omega))}\|\partial_t \vartheta^{1/2}\|_{L^2(1,+\infty;H)} \\
&\le 2c\|\vartheta\|^{1/2}_{L^\infty(1,+\infty;V)}\|\partial_t \vartheta^{1/2}\|_{L^2(1,+\infty;H)} \le c.
\end{aligned} \tag{29}$$

7.2. **Study of the ω-limit.** First of all, we observe that ϑ is an $L^{12/7}(\Omega)$ valued continuous function on $[1,+\infty)$ and χ is a V valued continuous function on the same interval, thanks to (28) and (25), respectively. Accounting for the estimates of $\|\vartheta\|_{L^\infty(1,+\infty;V)}$ and $\|\chi\|_{L^\infty(1,+\infty;W)}$ given by (25) and (26), we deduce that (the continuous representatives of) ϑ and χ are continuous also with respect to the weak topologies of V and W, respectively. Hence, we have

$$\|\vartheta(t)\|_V + \|\chi(t)\|_W \le c \qquad \text{for every } t \ge 1 \tag{30}$$

and the ω-limit given by (49) is nonempty and contained in $V \times W$. Next, using the compact embeddings $W \subset V \subset H$, we immediately see that ω is relatively compact in $H \times V$. Moreover, general results (see, e.g., [24]) imply that it is compact and connected with respect to the strong topology of $H \times V$. Hence, to prove Theorem 2.4, it remains to show that for every element $(\vartheta_\infty, \chi_\infty) \in \omega$ the pair $(\vartheta_\infty, \chi_\infty)$ yields a solution to problem (1)–(3). Therefore, we pick $(\vartheta_\infty, \chi_\infty) \in \omega$ and a sequence $\{t_n\} \nearrow +\infty$ such that

$$(\vartheta(t_n), \chi(t_n)) \to (\vartheta_\infty, \chi_\infty) \qquad \text{strongly in } H \times V. \tag{31}$$

We can assume $t_n \ge 1$ for every n. Moreover, as any subsequence of $\{t_n\}$ enjoys the same properties of the original sequence, we do not change the notation when passing to a subsequence. We introduce the functions ϑ_n and χ_n given by (4) and the functions ξ_n, and F_n defined similarly, i.e.,

$$\xi_n(t) := \xi(t+t_n), \quad \zeta_n(t) := \zeta(t+t_n), \quad F_n(t) := F(t+tn), \quad t \in [0,+\infty). \tag{32}$$

Note that $(\vartheta_n, \chi_n, \xi_n, \zeta_n)$ solves the system

$$\partial_t(\ln \vartheta_n(t) + \ell\chi_n(t)) - k_0\Delta\vartheta_n(t) = F_n(t) \quad \text{in } V', \text{ for a.e. } t \in (0,T), \tag{33}$$

$$\partial_t\chi_n(t) - \Delta\chi_n(t) + \xi_n(t) + \pi(\chi_n(t)) + \zeta_n(t) = \ell\vartheta_n(t) \quad \text{in } H, \text{ for a.e. } t \in (0,T), \tag{34}$$

$$\xi_n \in \beta(\chi_n) \quad \text{a.e. in } Q, \tag{35}$$

$$\zeta_n(t) \in A(\chi_n(t) - \chi^*) \quad \text{for a.e. } t \in (0,T), \tag{36}$$

$$\partial_\nu\vartheta_n = 0, \qquad \partial_\nu\chi_n = 0 \quad \text{on } \Sigma, \tag{37}$$

$$\ln\vartheta_n(0) = \ln\vartheta(t_n), \qquad \chi_n(0) = \chi(t_n) \quad \text{in } \Omega. \tag{38}$$

Now, we would pass to the limit in such a system as $n \nearrow +\infty$. More precisely, we look at all functions involved in (33)–(38) and take their weak limits. Then, we identify such limits in term of the element $(\vartheta_\infty, \chi_\infty)$ of ω we have fixed in (31).

7.3. **Conclusion.** Collecting all the estimates we have obtained in the previous steps, we deduce that there exist functions ϑ^∞, χ^∞, ξ^∞, ζ^∞ on $(0,+\infty)$ such that the following weak star convergences hold (for a subsequence) for every $T \in (0,+\infty)$

$$\vartheta_n \to \vartheta^\infty \quad \text{in} \quad L^\infty(0,T;V) \cap H^1(0,T;L^{12/7}(\Omega)), \tag{39}$$

$$\chi_n \to \chi^\infty \quad \text{in} \quad L^\infty(0,T;W) \cap H^1(0,T;V) \cap W^{1,\infty}(0,T;H), \tag{40}$$

$$\xi_n \to \xi^\infty \quad \text{in} \quad L^\infty(0,T;H), \tag{41}$$

$$\zeta_n \to \zeta^\infty \quad \text{in} \quad L^\infty(0,T;H). \tag{42}$$

Using [27, Cor. 4, Sec. 8], we deduce the strong convergences

$$\vartheta_n \to \vartheta^\infty \quad \text{in} \quad C^0([0,T];H), \tag{43}$$

$$\chi_n \to \chi^\infty \quad \text{in} \quad C^0([0,T];V). \tag{44}$$

Moreover, the convergence we have obtained are sufficient to identify the limits of the nonlinear terms. For instance, we have $\xi^\infty \in \beta(\chi^\infty)$ and $\zeta^\infty \in A(\chi^\infty - \chi^*)$ since we can apply, e.g., [2, Proposition 2.2, p. 38]. On the other hand, just by direct computation, one sees that a bound of the form $\|w\|_{L^p(1,+\infty;X)} \le c$, where $p < +\infty$ and X a Banach space, implies $w_n \to 0$ strongly in $L^p(0,+\infty;X)$, where $w_n(t) := w(t+t_n)$. Hence, the a priori estimates (17), (25)–(27) and (28) yield the strong convergences

$$\partial_t(\ln\vartheta_n + \ell\chi_n) \to 0 \quad \text{in} \quad L^2(0,+\infty;V_0'), \tag{45}$$

$$\partial_t(\chi_n) \to 0 \quad \text{in} \quad L^2(0,+\infty;H), \tag{46}$$

$$\partial_t(\vartheta_n) \to 0 \quad \text{in} \quad L^2(0,+\infty;L^{12/7}(\Omega)), \tag{47}$$

respectively. In particular, we can take $n \nearrow +\infty$ in (33)–(38) and get

$$-k_0\Delta\vartheta^\infty = 0 \quad \text{in } V_0', \text{ a.e. in } (0,+\infty), \tag{48}$$

$$-\Delta\chi^\infty + \xi^\infty + \pi(\chi^\infty) + \zeta^\infty = \ell\vartheta^\infty \quad \text{a.e. in } Q_\infty, \tag{49}$$

$$\xi^\infty \in \beta(\chi^\infty), \qquad \zeta^\infty \in A(\chi^\infty - \chi^*) \quad \text{a.e. in } Q_\infty. \tag{50}$$

Moreover, the boundary conditions are fulfilled as well, since ϑ^∞ and χ^∞ take values in V_0 and in W, respectively. The proof of Theorem 2.4 is complete whenever we show that

$$\vartheta^\infty(t) = \vartheta_\infty \qquad \text{and} \qquad \chi^\infty(t) = \chi_\infty \qquad \text{for every } t \in [0,+\infty). \tag{51}$$

For every t we have

$$\vartheta^\infty(t) = \vartheta^\infty(0) + \int_0^t \partial_t\vartheta^\infty(s)\,ds = \lim_{n\to+\infty}\vartheta_n(0) + \lim_{n\to+\infty}\int_0^t \partial_t\vartheta_n(s)\,ds = \vartheta_\infty, \tag{52}$$

the limits being understood in H, by (31) and (47). The same argument used for ϑ^∞, applied to (46), leads to $\chi^\infty = \chi_\infty$ and this completes the proof.

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