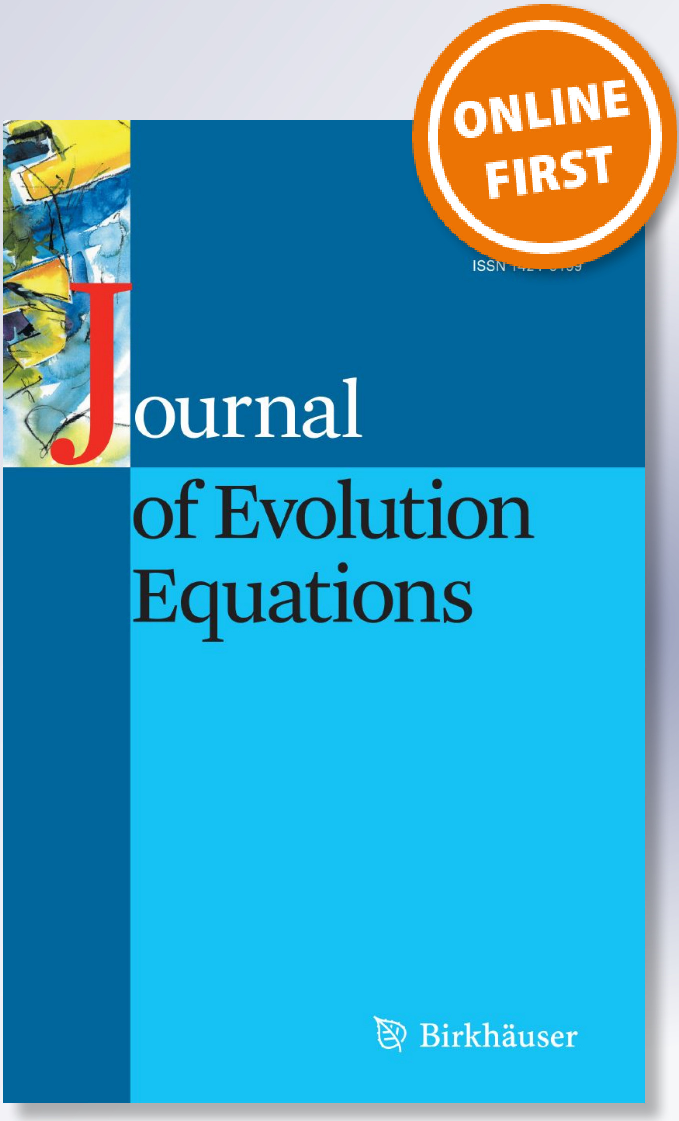
Gradient estimates for perturbed Ornstein–Uhlenbeck semigroups on infinite-dimensional convex domains

L. Angiuli, S. Ferrari & D. Pallara

Journal of Evolution Equations

ISSN 1424-3199

J. Evol. Equ.
DOI 10.1007/s00028-019-00491-y



Springer

J. Evol. Equ.

https://doi.org/10.1007/s00028-019-00491-y

Journal of Evolution Equations

Gradient estimates for perturbed Ornstein–Uhlenbeck semigroups on infinite-dimensional convex domains

L. Angiuli, S. Ferrari and D. Pallara

Abstract. Let X be a separable Hilbert space endowed with a non-degenerate centred Gaussian measure γ, and let λ_1 be the maximum eigenvalue of the covariance operator associated with γ. The associated Cameron–Martin space is denoted by H. For a sufficiently regular convex function $U : X \to \mathbb{R}$ and a convex set $\Omega \subseteq X$, we set $\nu := \mathrm{e}^{-U}\gamma$ and we consider the semigroup $(T_\Omega(t))_{t\geq 0}$ generated by the self-adjoint operator defined via the quadratic form

$$(\varphi, \psi) \mapsto \int_\Omega \langle D_H\varphi, D_H\psi\rangle_H \mathrm{d}\nu,$$

where φ, ψ belong to $D^{1,2}(\Omega, \nu)$, the Sobolev space defined as the domain of the closure in $L^2(\Omega, \nu)$ of D_H, the gradient operator along the directions of H. A suitable approximation procedure allows us to prove some pointwise gradient estimates for $(T_\Omega(t))_{t\geq 0}$. In particular, we show that

$$|D_H T_\Omega(t) f|_H^p \leq \mathrm{e}^{-p\lambda_1^{-1}t}(T_\Omega(t)|D_H f|_H^p), \quad t > 0, \ \nu\text{-a.e. in } \Omega,$$

for any $p \in [1, +\infty)$ and $f \in D^{1,p}(\Omega, \nu)$. We deduce some relevant consequences of the previous estimate, such as the logarithmic Sobolev inequality and the Poincaré inequality in Ω for the measure ν and some improving summability properties for $(T_\Omega(t))_{t\geq 0}$. In addition, we prove that if f belongs to $L^p(\Omega, \nu)$ for some $p \in (1, \infty)$, then

$$|D_H T_\Omega(t) f|_H^p \leq K_p t^{-\frac{p}{2}} T_\Omega(t)|f|^p, \quad t > 0, \ \nu\text{-a.e. in } \Omega,$$

where K_p is a positive constant depending only on p. Finally, we investigate on the asymptotic behaviour of the semigroup $(T_\Omega(t))_{t\geq 0}$ as t goes to infinity.

Introduction

This paper is a contribution to the study of infinite-dimensional elliptic and parabolic partial differential equations. The basic data are an abstract Wiener space (X, H, γ) and a quadratic form which defines a self-adjoint operator. This is a recent field of research, which finds its main motivation in stochastic analysis and its different applications to mathematical finance, statistical mechanics, hydrodynamics and quantum

Mathematics Subject Classification: 28C20, 35B40, 35D30, 35K35, 35K57, 46G12
Keywords: Asymptotic behaviour, Gradient estimates, Logarithmic Sobolev inequality, Poincaré inequality, Weighted measures, Wiener spaces.

Published online: 22 February 2019

mechanics. The simplest (still, quite challenging) case is that of a Hilbert space X endowed with a Gaussian measure γ and the Dirichlet form

$$(\varphi, \psi) \mapsto \int_X \langle D_H\varphi, D_H\psi\rangle_H \, d\gamma,$$

that defines an Ornstein–Uhlenbeck operator L which in turn generates the associated Ornstein–Uhlenbeck semigroup. Here D_H denotes the gradient along the directions of Cameron–Martin space H. Much has been done on this subject, see [11,21,26,27,33,34], relying on the available explicit Mehler's formula for the semigroup. In this case, the related stochastic differential equation is the Langevin one, i.e.

$$dX(t) = -X(t)\,dt + dW^H(t),$$

where $W^H(t)$ is a cylindrical Brownian motion. It is natural to look for generalisations of the available results, going in two directions: one is that of replacing γ with a more general measure, the other is that of considering integration on a domain $\Omega \subseteq X$. One of the main properties of Gaussian measures is that they factor according to the orthogonal decompositions of H, and this allows to get explicit formulas when integrating on the whole space X and to perform finite-dimensional approximations with increasing sequences of subspaces. Moreover, integrating on a domain requires to deal with boundary conditions (or suitable classes of test functions) that have to be assigned in order to correctly define an operator and the generated semigroup. Introducing a different measure makes the finite-dimensional approximation much more delicate and prevents to get explicit formulas even if the problem is studied in the whole space. Restricting to a domain, beside involving boundary conditions that have to be understood, makes still more difficult the infinite-dimensional approximation, and in fact, to the best of our knowledge, the only case treated in the literature is that of convex domains, see [1,6–8,12,14,18,20,31].

In this paper, we consider a log-concave weighted Gaussian measure $\nu = e^{-U}\gamma$ on a separable Hilbert space X. Here $\gamma = \mathcal{N}(0, Q_\infty)$ is the Gaussian measure with zero-mean and covariance operator $Q_\infty := -QA^{-1}$ where Q is a self-adjoint bounded non-negative and non-degenerate operator on X, $A : D(A) \subseteq X \to X$ is a self-adjoint operator such that $\langle Ax, x\rangle \le -\omega|x|^2$ ($\omega > 0$) and Q_∞ is a trace-class operator with non-negative eigenvalues $(\lambda_i)_{i\in\mathbb{N}}$. The function $U : X \to \mathbb{R}$ is convex and sufficiently regular. (Precise hypotheses are stated in Sect. 1.) We consider the quadratic form

$$\mathcal{D}_\Omega(\varphi, \psi) = \int_\Omega \langle D_H\varphi, D_H\psi\rangle_H \, d\nu, \tag{1}$$

which gives rise to the Kolmogorov operator (formally defined in a variational way through $\mathcal{D}_\Omega$)

$$L = \mathrm{Tr}(D_H^2) - \sum_{i=1}^{+\infty} \lambda_i^{-1} x_i D_i - \langle D_H U, D_H\rangle_H$$

and to the stochastic differential equation

$$\mathrm{d}X(t) = -(X(t) + DU(X(t)))\,\mathrm{d}t + Q_\infty^{1/2}\,\mathrm{d}W^H(t) + \text{boundary terms}, \tag{2}$$

[we do not enter into the details of boundary terms because we shall not come back to the stochastic side, see [6,7] for a precise formulation of Eq. (2)]. The domain we assign to the quadratic form corresponds heuristically to Neumann boundary conditions for L on $\partial\Omega$, and L generates a strongly continuous semigroup $(T_\Omega(t))_{t\geq 0}$ (simply denoted by $T_\Omega(t)$) in $L^p(\Omega, \nu)$ for $1 \leq p < \infty$. In order to study this semigroup, we proceed with a double approximation. We approximate U via Moreau–Yosida-type operators and penalise the characteristic function of Ω in order to state the problem in the whole space, eventually getting the restriction to Ω when the penalisation converges to χ_Ω. It is here that the convexity assumption on Ω is essential. Indeed, in infinite dimension there is no available procedure to mimic the standard domain decomposition and partition of unity arguments which are classical in finite dimension. Once the (approximate) problem has been formulated in the whole space, we perform a finite-dimensional approximation which provides a quite regular family of semigroups converging to $T_\Omega(t)f$ in a suitable sense and to which the results of the finite-dimensional case can be applied.

As we don't know any smoothing property of $T_\Omega(t)$ [it is not even known whether $T_\Omega(t)$ maps $C_b(\Omega)$ in $C_b(\Omega)$], we exploit the smoothing properties of the approximating semigroups. Indeed, the smoothness of the approximants is the crucial tool for many computations in this paper. Among the results that follow, one of the most relevant is the pointwise gradient estimate

$$|D_H T_\Omega(t) f|_H^p \leq \mathrm{e}^{-p\lambda_1^{-1}t}(T_\Omega(t)|D_H f|_H^p), \quad t > 0,\ \nu\text{-a.e. in } \Omega, \tag{3}$$

which holds true for any $p \in [1, +\infty)$ and f smooth enough, λ_1 being the maximum eigenvalue of the covariance operator Q_∞. Besides its own interest, estimate (3) represents the key tool in the investigation of many qualitative properties of $T_\Omega(t)$ and the related invariant measure ν. In the finite-dimensional case, gradient estimates similar to (3) are usually obtained by using the Bernstein method, which relies upon a variant of the classical maximum principle (see [29] and the reference therein) that does not have a counterpart in the infinite-dimensional case, or by using stochastic techniques, such as the Bismut–Elworthy–Li formula (see [15,21] and reference therein) and coupling methods (see, for example, [16,17,40]). On the other hand, in infinite-dimensional Wiener spaces some partial results are also available. In the case of a Gaussian measure γ and $\Omega = X$, the classical Mehler's representation formula

$$T(t)f(x) = \int_X f\left(\mathrm{e}^{-t}x + \sqrt{1-\mathrm{e}^{-2t}}\,y\right)\mathrm{d}\gamma(y),$$

gives $D_H T(t) f = \mathrm{e}^{-t}T(t)(D_H f)$, where the equality has to be meant componentwise (see [11, Proposition 1.5.6]). Again for the Gaussian measure γ on a convex subset Ω, in [12, Theorem 3.1] it is proved that $|D_H T(t) f|_H \leq \mathrm{e}^{-t}T(t)|D_H f|_H$

for any smooth function f. In this case, the idea consists in approximating the parabolic problem with a sequence of finite-dimensional parabolic problems and using the factorisation of the Gaussian measure. Clearly, this approach does not work in our case since our measure in general does not decompose as a product of measures on orthogonal subspaces. Finally, the case of a weighted Gaussian measure is also considered in [21] where a version of (3) is proved when $\Omega = X$ and the H-derivative is replaced by the Fréchet one. We point out that, in this latter case, the proof of the gradient estimate is based on purely stochastic techniques.

Hence, taking into account the existing literature, estimate (3) represents a generalisation of all the above results and the purely analytical proof we proposed, inspired by an idea due to Bakry and Émery (see [5,39]), is a novelty in the proofs of gradient estimates.

As announced, the pointwise gradient estimate (3) has several interesting consequences. First of all, it yields that the semigroup $T_\Omega(t)$ is smoothing, in the sense that it is bounded from $L^p(\Omega,\nu)$ into $D^{1,p}(\Omega,\nu)$, for any $p \in (1,\infty)$ and $t > 0$ as the estimate

$$\|D_H T_\Omega(t) f\|_{L^p(\Omega,\nu;H)} \leq C_p t^{-\frac{1}{2}} \|f\|_{L^p(\Omega,\nu)},$$

reveals. Due to the fact that the Sobolev embedding theorems fail to hold when we replace the Lebesgue measure with another general measure (as the Gaussian one), despite $T_\Omega(t)$ maps $L^p(\Omega,\nu)$ into $D^{1,p}(\Omega,\nu)$, a natural basic question is whether the semigroup $T_\Omega(t)$ is hypercontractive, i.e. if, given any $f \in L^q(\Omega,\nu)$, $q \in [1,\infty)$, the function $T_\Omega(t)f$ belongs to $L^p(\Omega,\nu)$ for some $p > q$. To give a positive answer, the starting point is the proof of a logarithmic Sobolev inequality for the measure ν which, as in the case of Gaussian measures, implies that the semigroup $T_\Omega(t)$ is hypercontractive in the L^p-spaces related to the measure ν. This last result and more improving summability properties were already known in the finite-dimensional setting for evolution operators associated with non-autonomous elliptic operator (see [3,4]). We also show a Poincaré inequality in $L^p(\Omega,\nu)$ for $p \in [2,\infty)$ that together with the hypercontractivity estimate $\|T_\Omega(t)f\|_{L^p(\Omega,\nu)} \leq c_{p,q,\Omega}\|f\|_{L^q(\Omega,\nu)}$ which holds for any $t > 0$, $f \in L^q(\Omega,\nu)$ and some $p > q$, allows us to study the asymptotic behaviour of $T_\Omega(t)f$ as $t \to +\infty$ for $f \in L^p(\Omega,\nu)$, $p > 1$, and to relate it to the behaviour of the derivative $|D_H T_\Omega(t)f|$ as $t \to +\infty$. These estimates are drawn in a more or less standard way: we have presented sketches of proofs (or even complete proofs) for the convenience of the reader.

Further consequences can be deduced, but these will be hopefully matter of other works.

Notations

For any $k \geq 0$ and $n \in \mathbb{N}$, we denote by $C^k(\mathbb{R}^n)$ the space of continuous functions with continuous derivative up to the $[k]$th order (here $[k]$ denotes the integer part of

k) such that the $[k]$-th derivative is $(k-[k])$-Hölder continuous, if $k \notin \mathbb{N}$. We use the subscript "b" to denote the space of all functions in $C^k(\mathbb{R}^n)$ which are bounded together with all their derivatives up to the $[k]$th order. $C_b^k(\mathbb{R}^n)$ is endowed with the norm

$$\|f\|_{C_b^k(\mathbb{R}^n)} := \sum_{|\alpha|\leq[k]} \|D^\alpha f\|_\infty + \sum_{|\alpha|=[k]} [D^\alpha f]_{k-[k]},$$

where $\|\cdot\|_\infty$ denotes the sup-norm and, for any $\alpha \in (0,1)$, $[\cdot]_\alpha$ is the α-Hölder seminorm. We use the subscript "loc" to denote the space of all $f \in C^{[k]}(\mathbb{R}^n)$ such that the derivatives of order $[k]$ are $(k-[k])$-Hölder continuous in any compact subset of $\mathbb{R}^n$. For any interval J and $\alpha, \beta \geq 0$, we denote by $C^{\alpha,\beta}(J \times \mathbb{R}^n)$ the usual parabolic Hölder space. The subscripts "b" and "loc" have the same meanings as above.

We also consider functions defined in infinite-dimensional spaces. X denotes a separable Hilbert space endowed with its norm $|\cdot|$ and inner product $\langle\cdot,\cdot\rangle$, while $\mathcal{L}(X)$ denotes the space of bounded linear operators from X to itself, endowed with its operator norm $\|\cdot\|_{\mathcal{L}(X)}$.

We define $C_b(X)$ to be the space of all functions $f : X \to \mathbb{R}$ which are continuous and bounded in X. For any $k \in \mathbb{N}$, we denote by $C_b^k(X)$ the space of functions $f : X \to \mathbb{R}$ which have bounded and continuous Fréchet derivatives up to the order k with norm

$$\|f\|_{C_b^k(X)} := \sum_{j=0}^{k} \|D^j f\|_\infty,$$

where D^j denotes the jth Fréchet derivative operator. Moreover, if $f : X \to \mathbb{R}$ is Lipschitz continuous, we set $[f]_{\mathrm{Lip}} = \sup_{x,y\in X,\, x\neq y} \big(|f(x)-f(y)||x-y|^{-1}\big)$.

For any $f : [0,+\infty) \times X \to \mathbb{R}$, once an orthonormal Hilbert basis $(v_i)_{i\in\mathbb{N}}$ has been fixed, we use the symbols $D_t f$, $D_i f$ to denote, respectively, the time derivative of f and the directional derivative of f in the direction of v_i. We use the same notation in $\mathbb{R}^n$ where $D_i f$ denotes the directional derivative of f along the i-th vector of the canonical basis of $\mathbb{R}^n$. Analogous meaning is given to the symbols $D_{ij} f$ and $D_{ijk} f$.

For any finite Radon measure μ on X and $1 \leq p < \infty$, the set $L^p(X,\mu)$ consists of all measurable functions $f : X \to \mathbb{R}$ such that $\|f\|^p_{L^p(X,\mu)} := \int_X |f|^p d\mu < +\infty$, while $L^\infty(X,\mu)$ is the space of all μ-essentially bounded functions with norm $\|f\|_\infty = \operatorname{ess\,sup}_{x\in X}|f(x)|$. In a similar way, we define the spaces $L^p(X,\mu;X)$ and $L^p(X,\mu;\mathcal{H}_2)$ where $\mathcal{H}_2$ is the space of Hilbert–Schmidt operators and the measurability is meant in Bochner's sense. With p' we denote the conjugate exponent of p, i.e. $1/p+1/p'=1$, with the standard convention that $1'=\infty$.

1. Assumptions and preliminary results

We start this section by listing the hypotheses we assume throughout the paper.

HYPOTHESES 1.1. *Let assume that*

(i) $Q \in \mathcal{L}(X)$ *is a self-adjoint and non-negative operator with* Ker $Q = \{0\}$;
(ii) $A : D(A) \subseteq X \to X$ *is a self-adjoint operator satisfying* $\langle Ax, x\rangle \leq -\omega|x|^2$ *for every* $x \in D(A)$ *and some positive* ω;
(iii) $Qe^{tA} = e^{tA}Q$ *for any* $t \geq 0$;
(iv) $\mathrm{Tr}(-QA^{-1}) < +\infty$.

Under Hypotheses 1.1, we can consider the Gaussian measure γ with mean zero, covariance operator $Q_\infty := -QA^{-1}$ and an orthonormal basis $(v_k)_{k\in\mathbb{N}}$ of X such that

$$Q_\infty v_k = \lambda_k v_k, \quad k \in \mathbb{N}, \tag{1.1}$$

where $(\lambda_k)_{k\in\mathbb{N}}$ is the decreasing sequence of eigenvalues of Q_∞.

The Cameron–Martin space $(H, |\cdot|_H)$, where

$$H = \left\{ x \in X \,\middle|\, \sum_{k=1}^{+\infty} \lambda_k^{-1} \langle x, v_k\rangle^2 < +\infty \right\},$$

and $|\cdot|_H$ is the norm induced by the inner product $\langle h, k\rangle_H := \langle Q_\infty^{-1/2} h, Q_\infty^{-1/2} k\rangle$, $h, k \in H$, is a Hilbert space which is densely embedded in X. Note that, as $H = Q_\infty^{1/2} X$, the sequence $(e_k)_{k\in\mathbb{N}}$, where $e_k = \sqrt{\lambda_k} v_k$ for any $k \in \mathbb{N}$, is an orthonormal basis of H.

We need to recall the definition of Lipschitz continuous function along the Cameron–Martin space H. If Y is a Banach space with norm $\|\cdot\|_Y$, a function $F : X \to Y$ is said to be H-Lipschitz continuous if there exists a positive constant C such that

$$\|F(x+h) - F(x)\|_Y \leq C|h|_H, \tag{1.2}$$

for every $h \in H$ and γ-a.e. $x \in X$ (see [11, Section 4.5 and Section 5.11] for the basic properties of H-Lipschitz continuous functions). In particular, we point out that, by [11, Corollary 4.5.4], there exists a Borel modification of F such that (1.2) is satisfied for any $x \in X$. Henceforth we always refer to such modification. We denote with $[F]_{H\text{-Lip}}$ the best constant C appearing in (1.2).

Now, we introduce a notion of derivative weaker than the classical Fréchet one. We say that $f : X \to \mathbb{R}$ is H-differentiable at $x_0 \in X$ if there exists $\ell \in H$ such that

$$f(x_0 + h) = f(x_0) + \langle \ell, h\rangle_H + o(|h|_H), \quad \text{as } |h|_H \to 0.$$

In such a case, we set $D_H f(x_0) := \ell$ and $D_i f(x_0) := \langle D_H f(x_0), e_i\rangle_H$ for any $i \in \mathbb{N}$. The derivative $D_H f(x_0)$ is called the *Malliavin derivative* of f at x_0. In a similar way, we say that f is twice H-differentiable at x_0 if f is H-differentiable near x_0 and there exists $\mathcal{B} \in \mathcal{H}_2$ such that

$$f(x_0 + h) = f(x_0) + \langle D_H f(x_0), h\rangle_H + \frac{1}{2}\langle \mathcal{B}h, h\rangle_H + o(|h|_H^2), \quad \text{as } |h|_H \to 0.$$

In such a case, we set $D^2_H f(x_0) := \mathcal{B}$ and $D_{ij} f(x_0) := \langle D^2_H f(x_0)e_j, e_i\rangle_H$ for any $i, j \in \mathbb{N}$. We recall that if f is twice H-differentiable at x_0, then $D_{ij} f(x_0) = D_{ji} f(x_0)$ for every $i, j \in \mathbb{N}$.

REMARK 1.2. If a function $f : X \to \mathbb{R}$ is (resp. twice) Fréchet differentiable at x_0, then it is (resp. twice) H-differentiable at x_0 and it holds $D_H f(x_0) = Q_\infty Df(x_0)$, (resp. $D^2_H f(x_0) = Q^{1/2}_\infty D^2 f(x_0) Q^{1/2}_\infty$).

For any $k \in \mathbb{N}\cup\{\infty\}$, we denote by $\mathcal{F}C^k_b(X)$, the space of cylindrical C^k_b functions, i.e. the set of functions $f : X \to \mathbb{R}$ such that $f(x) = \varphi(\langle x, h_1\rangle, \dots, \langle x, h_N\rangle)$ for some $\varphi \in C^k_b(\mathbb{R}^N)$, $h_1, \dots, h_N \in H$ and $N \in \mathbb{N}$. By $\mathcal{F}C^k_b(X, H)$, we denote H-valued cylindrical C^k_b functions with finite rank.

The Sobolev spaces in the sense of Malliavin $D^{1,p}(X, \gamma)$ and $D^{2,p}(X, \gamma)$ with $p \in [1, \infty)$ are defined as the completions of the *smooth cylindrical functions* $\mathcal{F}C^\infty_b(X)$ in the norms

$$\|f\|_{D^{1,p}(X,\gamma)} := \left(\|f\|^p_{L^p(X,\gamma)} + \int_X |D_H f|^p_H d\gamma\right)^{\frac{1}{p}};$$

$$\|f\|_{D^{2,p}(X,\gamma)} := \left(\|f\|^p_{D^{1,p}(X,\gamma)} + \int_X |D^2_H f|^p_{\mathcal{H}_2} d\gamma\right)^{\frac{1}{p}}.$$

This is equivalent to consider the domain of the closure of the gradient operator, defined on smooth cylindrical functions, in $L^p(X, \gamma)$.
We define a weighted Gaussian measure considering a function $U : X \to \mathbb{R}$ that satisfies the following

HYPOTHESIS 1.3. *U is a convex function which belongs to $C^2(X) \cap D^{1,q}(X, \gamma)$ for all $q \in [1, \infty)$.*

The convexity of the function U guarantees that U is bounded from below by a linear function; therefore, it decreases at most linearly, and by Fernique's theorem (see [11, Theorem 2.8.5]) e^{-U} belongs to $L^1(X, \gamma)$. Then, we can consider the finite log-concave measure

$$\nu := \mathrm{e}^{-U}\gamma.$$

Notice that γ and ν are equivalent measures, hence saying that a statement holds γ-a.e. is the same as saying that it holds ν-a.e. Moreover, the fact that U belongs to $D^{1,q}(X, \gamma)$ for any $q \in [1, \infty)$ allows us to conclude that the operator $D_H : \mathcal{F}C^1_b(X) \to L^p(X, \nu; H)$ is closable in $L^p(X, \nu)$, $p \in (1, \infty)$ and we may define the space $D^{1,p}(X, \nu)$, $p > 1$, as the domain of its closure (still denoted by D_H). In a similar way, we can define $D^{2,p}(X, \nu)$, $p \in (1, \infty)$ (for more details, see [13,23]). The Gaussian integration by parts formula $\int_X D_i f \mathrm{d}\gamma = \frac{1}{\sqrt{\lambda_i}} \int_X \langle x, v_i\rangle f \mathrm{d}\gamma$, which holds true for any $f \in \mathcal{F}C^1_b(X)$ and $i \in \mathbb{N}$, yields that

$$\int_X \psi D_i\varphi \mathrm{d}\nu + \int_X \varphi D_i\psi \mathrm{d}\nu = \int_X \varphi\psi D_i U \mathrm{d}\nu + \frac{1}{\sqrt{\lambda_i}} \int_X \langle x, v_i\rangle \varphi\psi \mathrm{d}\nu, \qquad i \in \mathbb{N}, \tag{1.3}$$

for any $\varphi, \psi \in \mathcal{F}C_b^1(X)$, hence by density for any $\varphi, \psi \in D^{1,p}(X, \nu)$, $p \in (1, \infty)$.

In what follows, Ω denotes an open convex subset of X. In this case, the spaces $D^{1,p}(\Omega, \nu)$ and $D^{2,p}(\Omega, \nu)$, $p \in (1, \infty)$ can be defined in a similar way as in the whole space, thanks to the following result (proved in [2] in the Gaussian case).

PROPOSITION 1.4. *Let us assume that Hypotheses 1.1 and 1.3 are satisfied and let $p \in (1, \infty)$ and Ω be an open subset of X. Then the operators $D_H : \mathcal{F}C_b^\infty(\Omega) \to L^p(\Omega, \nu; H)$ and*

$$(D_H, D_H^2) : \mathcal{F}C_b^\infty(\Omega) \times \mathcal{F}C_b^\infty(\Omega) \to L^p(\Omega, \nu; H) \times L^p(\Omega, \nu; \mathcal{H}_2) \tag{1.4}$$

are closable in $L^p(\Omega, \nu)$ and $L^p(\Omega, \nu) \times L^p(\Omega, \nu)$, respectively. Here $\mathcal{F}C_b^\infty(\Omega)$ is the space of the restriction to Ω of the functions in $\mathcal{F}C_b^\infty(X)$.

Proof. We just prove that the operator $D_H : \mathcal{F}C_b^\infty(\Omega) \to L^p(\Omega, \nu; H)$ is closable in $L^p(\Omega, \nu)$, since the proof that the operator defined in (1.4) is closable in $L^p(\Omega, \nu) \times L^p(\Omega, \nu)$ is quite similar. By the linearity of the operator D_H it is enough to prove that if $(f_k)_{k\in\mathbb{N}} \subseteq \mathcal{F}C_b^\infty(\Omega)$ is such that

$$\lim_{k\to+\infty} f_k = 0 \quad \text{in } L^p(\Omega, \nu);$$
$$\lim_{k\to+\infty} D_H f_k = \Phi \quad \text{in } L^p(\Omega, \nu; H),$$

then $\Phi = 0$ ν-a.e in Ω.

By Lusin's theorem and standard arguments following from [35], the space $\mathrm{Lip}_c(\Omega)$ of the bounded Lipschitz functions u defined on X with bounded support such that $\mathrm{dist}(\mathrm{supp}\, u, \Omega^c) > 0$ is dense in $L^p(\Omega, \nu)$. So it is enough to prove that $\int_\Omega \langle \Phi, e_i\rangle_H u \mathrm{d}\nu = 0$, for every $i \in \mathbb{N}$ and $u \in \mathrm{Lip}_c(\Omega)$.

To this aim, let us fix $u \in \mathrm{Lip}_c(\Omega)$ and observe that, by the Hölder inequality, Hypothesis 1.3 and the fact that $\mathrm{e}^{-U} \in L^q(X, \gamma)$ for every $q \in [1, \infty)$, we get

$$\lim_{k\to+\infty} \int_\Omega f_k D_i u \mathrm{d}\nu \le [u]_{\mathrm{Lip}} [\nu(\Omega)]^{1/p'} \lim_{k\to+\infty} \|f_k\|_{L^p(\Omega,\nu)} = 0 \tag{1.5}$$

and

$$\lim_{k\to+\infty} \int_\Omega f_k u D_i U \mathrm{d}\nu \le \|u\|_\infty \|D_i U\|_{L^{p'q}(X,\gamma)} \|\mathrm{e}^{-U}\|^{1/p'}_{L^{q'}(X,\gamma)} \lim_{k\to+\infty} \|f_k\|_{L^p(\Omega,\nu)} = 0; \tag{1.6}$$

for every $i \in \mathbb{N}$ and $q \in (1, \infty)$. Moreover, Fernique's theorem and the quoted hypotheses imply that

$$\begin{aligned}&\lim_{k\to+\infty} \int_\Omega f_k u \frac{\langle x, v_i\rangle}{\sqrt{\lambda_i}} \mathrm{d}\nu \\ &\le \frac{\|u\|_\infty}{\sqrt{\lambda_i}} \left(\int_X |\langle x, v_i\rangle|^{p'q} \mathrm{d}\gamma \right)^{\frac{1}{p'q}} \|\mathrm{e}^{-U}\|^{1/p'}_{L^{q'}(X,\gamma)} \lim_{k\to+\infty} \|f_k\|_{L^p(\Omega,\nu)} = 0.\end{aligned} \tag{1.7}$$

Now, we claim that $\int_\Omega \langle \Phi, e_i\rangle_H u \mathrm{d}\nu = \lim_{k\to+\infty}\int_X \widetilde{u} D_i f_k \mathrm{d}\nu$, where $\widetilde{u}$ is the null extension of u out of Ω. Indeed, again by using the hypotheses listed above we get

$$\begin{aligned}&\lim_{k\to+\infty}\int_\Omega u(D_i f_k - \langle \Phi, e_i\rangle_H)\mathrm{d}\nu\\&\quad\le \|u\|_\infty[\nu(\Omega)]^{1/p'}\lim_{k\to+\infty}\left(\int_\Omega |D_i f_k - \langle \Phi, e_i\rangle_H|^p \mathrm{d}\nu\right)^{1/p} = 0.\end{aligned}$$

To conclude, let us observe that $\widetilde{u}$ is Lipschitz continuous on X, so by the integration by parts formula (1.3) and (1.5)–(1.7) we deduce

$$\begin{aligned}\int_\Omega \langle \Phi, e_i\rangle_H u\mathrm{d}\nu &= \lim_{k\to+\infty}\int_X \widetilde{u} D_i f_k \mathrm{d}\nu\\&= \lim_{k\to+\infty}\int_X f_k\left(-D_i\widetilde{u} + \widetilde{u} D_i U + \widetilde{u}\frac{\langle x, v_i\rangle}{\sqrt{\lambda_i}}\right)\mathrm{d}\nu\\&= \lim_{k\to+\infty}\int_\Omega f_k\left(-D_i u + u D_i U + u\frac{\langle x, v_i\rangle}{\sqrt{\lambda_i}}\right)\mathrm{d}\nu = 0.\end{aligned}$$

This proves the claim. □

The spaces $D^{1,p}(\Omega, \nu; H)$, $p \in (1, \infty)$, are defined in a similar way, replacing smooth cylindrical functions with H-valued smooth cylindrical functions. We recall that if $F \in D^{1,p}(\Omega, \nu; H)$, then $D_H F$ belongs to $\mathcal{H}_2$.

In the sequel, we consider boundary Cauchy problems defined in Ω and we will need some continuity properties of the *distance function along* H, $d_\Omega : X \to [0, +\infty]$, defined by

$$d_\Omega(x) := \begin{cases}\inf\{|h|_H \mid h \in H\cap(\Omega - x)\}, & H\cap(\Omega - x) \neq \emptyset;\\ +\infty, & H\cap(\Omega - x) = \emptyset,\end{cases}$$

for any $x \in X$. In the following proposition, we recall some results about the function d_Ω (see [11, Theorems 2.8.5 & 5.11.2] and [14, Section 3]).

PROPOSITION 1.5. *Let $\Omega \subseteq X$ be an open convex set. Then d_Ω^2 is H-differentiable and its Malliavin derivative is H-Lipschitz with H-Lipschitz constant less than or equal to 2, i.e.*

$$|D_H d_\Omega^2(x + h) - D_H d_\Omega^2(x)|_H \le 2|h|_H,$$

for any $h \in H$ and for ν-a.e $x \in X$. Moreover, $D_H^2 d_\Omega^2$ exists ν-a.e. in X and d_Ω^2 belongs to $D^{2,p}(X, \nu)$ for every $p \in [1, \infty)$.

In order to prove our results, we need further regularity of the second-order Malliavin derivative of the distance function along H. More precisely, we assume that

HYPOTHESIS 1.6. *Ω is an open convex subset of X such that $\nu(\partial\Omega) = 0$ and $D_H^2 d_\Omega^2$ is H-continuous γ-a.e. in X; i.e. for γ-a.e. $x \in X$*

$$\lim_{|h|_H\to 0} D_H^2 d_\Omega^2(x + h) = D_H^2 d_\Omega^2(x).$$

REMARK 1.7. We point out that there is a rather large class of subsets of X satisfying Hypothesis 1.6. For instance, by [24,28], if $\partial\Omega$ is (locally) a C^2-embedding in X of an open subset of a hyperplane in X and $\nu(\partial\Omega) = 0$, then Hypothesis 1.6 is satisfied. Easy examples are:

(i) every open ball and open ellipsoid of X;
(ii) every open hyperplane of X;
(iii) every set of the form $\Omega = \{x \in X \mid G(x) < 0\}$, where $G : X \to \mathbb{R}$ is convex, belongs to $C^2(X)$ and $D_H G$ is nonzero at every point of $\partial\Omega$ (check [28, Theorem 1(a)]).

An important tool in our analysis is the *Moreau–Yosida approximants of U along H*. We recall the main properties of this approximation, and we refer to [9, Section 12.4] for the classical theory and to [1,13,14] for the case considered here.

PROPOSITION 1.8. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper convex and lower semicontinuous function and denote by $\mathrm{dom}(f) = \{x \in X \mid f(x) < +\infty\}$. For any $\varepsilon > 0$ and $x \in X$, let us consider*

$$f_\varepsilon(x) := \inf\left\{ f(x+h) + \frac{1}{2\varepsilon}|h|_H^2 \,\middle|\, h \in H \right\}. \tag{1.8}$$

Then, the following properties hold true:

(i) *$f_\varepsilon(x) \le f(x)$ for any $\varepsilon > 0$ and $x \in X$. Moreover, $f_\varepsilon(x)$ converges monotonically to $f(x)$ for any $x \in X$, as $\varepsilon \to 0^+$;*
(ii) *f_ε is H-differentiable in X and $D_H f_\varepsilon$ is H-Lipschitz continuous in X;*
(iii) *f_ε belongs to $D^{2,p}(X,\gamma)$, whenever $f \in L^p(X,\gamma)$ for some $1 \in [1,\infty)$;*
(iv) *if $x \in \mathrm{dom}(f)$ and f belongs to $D^{1,p}(X,\gamma)$ for some $p \in [1,\infty)$, then $D_H f_\varepsilon(x)$ converges to $D_H f(x)$ as $\varepsilon \to 0^+$;*
(v) *if $f \in C^2(X) \cap D^{2,p}(X,\gamma)$ for some $p \in [1,\infty)$ and f is twice H-differentiable at every point $x \in \mathrm{dom}(f)$, then $D_H^2 f_\varepsilon(x)$ exists and converges to $D_H^2 f(x)$ as $\varepsilon \to 0^+$, for any $x \in \mathrm{dom}(f)$. Furthermore $D_H^2 f_\varepsilon$ is H-continuous in X, i.e. $\lim_{|h|_H \to 0} D_H^2 f_\varepsilon(x+h) = D_h^2 f_\varepsilon(x)$ for any $x \in X$.*

Further notation We now introduce some notations which will be largely used in the paper. For any $i, n \in \mathbb{N}$ and $x \in X$, we define $x_i := \sqrt{\lambda_i}\langle x, v_i\rangle$ and by Π_n the projection $\Pi_n : X \to \mathbb{R}^n$, $\Pi_n x := (x_1, \ldots, x_n)$. The function Σ_n denotes the embedding $\Sigma_n : \mathbb{R}^n \to H$, $\Sigma_n \xi := \sum_{k=1}^n \xi_k e_k$, for any $\xi \in \mathbb{R}^n$. Moreover, if $P_n : X \to H$ is defined by $P_n x := \sum_{i=1}^n x_i e_i$ for any $x \in X$ and $n \in \mathbb{N}$, then the conditional expectation of f, $\mathbb{E}_n f$ defined as follows

$$\mathbb{E}_n f(x) := \int_X f(P_n x + (I - P_n)y)\mathrm{d}\gamma(y), \qquad f \in L^p(X,\gamma),\ \ p \in [1,\infty),$$

enjoys some good continuity properties (see [11, Corollary 3.5.2 and Proposition 5.4.5] for a proof of the following result).

PROPOSITION 1.9. *Assume that Hypotheses 1.1 hold true and let* $1 \le p < +\infty$, $k \in \mathbb{N}$ *and* $f \in D^{k,p}(X,\gamma)$. *Then* $\mathbb{E}_n f$ *belongs to* $D^{k,p}(X,\gamma)$ *and converges to* f *in* $D^{k,p}(X,\gamma)$ *and pointwise* γ*-a.e. in* X*, as* n *tends to* $+\infty$. *Moreover* $\|\mathbb{E}_n f\|_{D^{k,p}(X,\gamma)} \le \|f\|_{D^{k,p}(X,\gamma)}$ *and*

$$D_i\mathbb{E}_n f = \begin{cases} \mathbb{E}_n D_i f & 1 \le i \le n; \\ 0 & i > n. \end{cases}$$

We conclude this section by recalling the main properties of the semigroup generated by the operator L_Ω in $L^2(\Omega,\nu)$ defined as

$$D(L_\Omega) = \left\{ u \in D^{1,2}(\Omega,\nu) \,\middle|\, \text{there exists } v_u \in L^2(\Omega,\nu) \text{ such that} \right.$$
$$\left. \int_\Omega \langle D_H u, D_H\varphi\rangle_H \mathrm{d}\nu = -\int_\Omega v_u\varphi \mathrm{d}\nu \text{ for every } \varphi \in \mathcal{F}C_b^\infty(\Omega) \right\}, \tag{1.9}$$

with $L_\Omega u := v_u$ if $u \in D(L_\Omega)$.

PROPOSITION 1.10. *Under Hypotheses 1.1, 1.3 and 1.6, the following properties hold true.*

(i) *For any* $\lambda > 0$ *and* $f \in L^2(\Omega,\nu)$*, the equation* $\lambda u - L_\Omega u = f$ *in* Ω *has a weak solution* $u \in D^{1,2}(\Omega,\nu)$*, i.e. for every* $\varphi \in D^{1,2}(\Omega,\nu)$ *it holds*

$$\lambda \int_\Omega u\varphi \mathrm{d}\nu + \int_\Omega \langle D_H u, D_H\varphi\rangle_H \mathrm{d}\nu = \int_\Omega f\varphi \mathrm{d}\nu.$$

Moreover, $u \in D^{2,2}(\Omega,\nu)$ *and the equation* $\lambda u - L_\Omega u = f$, $\lambda > 0$*, holds* ν*-a.e. in* Ω*. Denoting by* $R(\lambda, L_\Omega)$ *the resolvent operator of* L_Ω*, the following estimates hold:*

$$\|R(\lambda, L_\Omega)f\|_{L^2(\Omega,\nu)} \le \frac{\|f\|_{L^2(\Omega,\nu)}}{\lambda}, \quad \|D_H R(\lambda, L_\Omega)f\|_{L^2(\Omega,\nu;H)} \le \frac{\|f\|_{L^2(\Omega,\nu)}}{\sqrt{\lambda}}, \tag{1.10}$$

and

$$\left\| D_H^2 R(\lambda, L_\Omega) f \right\|_{L^2(\Omega,\nu;\mathcal{H}_2)} \le \sqrt{2}\|f\|_{L^2(\Omega,\nu)}, \tag{1.11}$$

Consequently, L_Ω *generates a bounded self-adjoint analytic semigroup* $T_\Omega(t)$ *in* $L^2(\Omega,\nu)$.

(ii) $T_\Omega(t)$ *can be extended to a positivity preserving contraction semigroup (still denoted by* $T_\Omega(t)$*) in* $L^p(\Omega,\nu)$ *for every* $1 \le p \le +\infty$ *and* $t \ge 0$*. In addition, it is strongly continuous in* $L^p(\Omega,\nu)$ *for any* $p \in [1,+\infty)$.

(iii) *If* $f \in C_b(\Omega)$ *has positive infimum in* Ω*, then* $T_\Omega(t)f$ *has a positive* ν*-essential infimum, for any* $t > 0$.

(iv) *For any convex function* $\varphi : \mathbb{R} \to \mathbb{R}$,

$$\varphi(T_\Omega(t)f) \le T_\Omega(t)(\varphi \circ f), \quad \nu\text{-a.e. in } \Omega, \; t > 0, \; f \in C_b(\Omega). \tag{1.12}$$

(v) *For any* $p \in (1, +\infty)$

$$T_\Omega(t)(fg) \leq (T_\Omega(t)|f|^p)^{1/p}(T_\Omega(t)|g|^{p'})^{1/p'} \quad \nu\text{-a.e. in } \Omega,\ t > 0,\ f, g \in C_b(\Omega). \tag{1.13}$$

(vi) *For any* $p \in [1, \infty)$, $f \in L^p(\Omega, \nu)$ *and* $g \in L^\infty(\Omega, \nu)$ *it holds*

$$\int_\Omega f T_\Omega(t) g \mathrm{d}\nu = \int_\Omega g T_\Omega(t) f \mathrm{d}\nu, \qquad t > 0.$$

Proof. (i) Inequalities (1.10) and (1.11) are proved in [14, Theorem 1.3], while the last statement follows from the standard theory of semigroups.

(ii) It is a consequence of [36, Theorem 2.14 and Corollary 2.18]. Indeed by these results, it is enough to prove the following two Beurling–Deny conditions:

(1) if $u \in D^{1,2}(\Omega, \nu)$, then $|u| \in D^{1,2}(\Omega, \nu)$ and $\int_\Omega |D_H|u||_H^2 \mathrm{d}\nu \leq \int_\Omega |D_H u|_H^2 \mathrm{d}\nu$.
(2) if $0 \leq u \in D^{1,2}(\Omega, \nu)$, then $u \wedge \mathbb{1} := \min\{u, 1\}$ belongs to $D^{1,2}(\Omega, \nu)$ and

$$\int_\Omega |D_H(u \wedge \mathbb{1})|_H^2 \mathrm{d}\nu \leq \int_\Omega |D_H u|_H^2 \mathrm{d}\nu. \tag{1.14}$$

Statement (1) follows from the fact that if u belongs to $D^{1,2}(\Omega, \nu)$, then there exists a sequence $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{F}C_b^1(\Omega)$ converging to u in $D^{1,2}(\Omega, \nu)$. It can be proved that the sequence $\widetilde{u}_n = \sqrt{u_n^2 + n^{-1}}$ converges to $|u|$ in $D^{1,2}(\Omega, \nu)$ as $n \to +\infty$, namely $|u|$ belongs to $D^{1,2}(\Omega, \nu)$. In addition, $D_H|u| = \mathrm{sign}(u) D_H u$ and $\int_\Omega |D_H|u||_H^2 \mathrm{d}\nu = \int_\Omega |D_H u|_H^2 \mathrm{d}\nu$ (see [19, Lemma 2.7] for further details).
To prove (2), as above we can consider a sequence $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{F}C_b^1(\Omega)$ converging to u in $D^{1,2}(\Omega, \nu)$, as n goes to infinity. Then, the sequence

$$\widetilde{u}_n = \frac{1}{2}\left(u_n + 1 - \sqrt{(u_n - 1)^2 + \frac{1}{n}}\right),$$

converges to $u \wedge \mathbb{1}$ as $n \to +\infty$, that is $u \wedge \mathbb{1} \in D^{1,2}(\Omega, \nu)$. Further,

$$D_H(u \wedge \mathbb{1}) = \frac{1}{2}(1 - \mathrm{sign}(u - 1)) D_H u$$

and (1.14) holds true (see [12, Proposition 1.1] for more details). The strong continuity follows from [36].

(iii) It follows immediately using the positivity of $T(t)$ and observing that $T(t)c = c$ for any $c \in \mathbb{R}$. Indeed $f \geq c$ implies $T(t)f \geq c$.

(iv)–(v) Due to [32, Theorem 4.3.5], there is a Markov process $(\mathcal{Y}, \mathcal{M}, (X_t)_{t\geq 0}, (P_x)_{x\in X})$ such that $T_\Omega(t)f(x) = \mathbf{E}_x(f(X_t))$ for ν-a.e $x \in X$, where $\mathbf{E}_x$ denotes the expected value with respect to the probability measure P_x. We summarise here some of the main properties of the Markov process $(\mathcal{Y}, \mathcal{M}, (X_t)_{t\geq 0}, (P_x)_{x\in X})$ for the convenience of the reader:

- $(\mathcal{Y}, \mathcal{M})$ is a measurable space;

- there exists a filtration $(\mathcal{M}_t)_{t\geq 0}$ on $(\mathcal{Y}, \mathcal{M})$ such that $(X_t)_{t\geq 0}$ is a $(\mathcal{M}_t)_{t\geq 0}$-adapted stochastic process;
- P_x, $x \in X$, are probability measures on $(\mathcal{Y}, \mathcal{M})$;
- it holds $P_x[X_{s+t} \in A|\mathcal{M}_s] = P_{X_s}[X_t \in A]$ for all Borel set $A \subseteq X$, any $s, t \geq 0$ and for P_x-a.e. $x \in X$.

We remark that in [32, Chapter 4, Section 4(b)] the authors study exactly the case we are in. The claims are easy consequences of the Jensen and Hölder inequalities.
(vi) Since $\mathcal{F}C_b^\infty(\Omega)$ is dense in $L^p(\Omega, \nu)$ for every $p \in [1, \infty)$, there exists a sequence $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{F}C_b^\infty(\Omega)$ such that $\lim_{n\to+\infty} \|f_n - f\|_{L^p(\Omega,\nu)}$. By the self-adjointness of T_Ω in $L^2(\Omega, \nu)$, we get

$$\int_\Omega f_n T_\Omega(t) g \mathrm{d}\nu = \int_\Omega g T_\Omega(t) f_n \mathrm{d}\nu, \quad t > 0.$$

By (ii) $T_\Omega(t)g \in L^\infty(\Omega, \nu)$, so letting n go to infinity we get the claim. $\square$

If $\Omega = X$, the operator in (1.9), denoted by L, acts on smooth cylindrical functions φ as follows

$$L\varphi := \mathrm{Tr}(D_H^2\varphi) - \sum_{i=1}^\infty \lambda_i^{-1}\langle x, e_i\rangle D_i\varphi - \langle D_H U, D_H\varphi\rangle_H, \quad \nu\text{-a.e in } X, \tag{1.15}$$

and it is symmetrised by the measure ν, indeed

$$\int_X \psi L\varphi \mathrm{d}\nu = -\int_X \langle D_H\varphi, D_H\psi\rangle_H \mathrm{d}\nu, \qquad \varphi, \psi \in \mathcal{F}C_b^1(X). \tag{1.16}$$

From now on, we assume that Hypotheses 1.1, 1.3 and 1.6 hold true.

2. An approximation result

The main goal of this section is Theorem 2.8 which states that for any $f \in L^2(\Omega, \nu)$ the function $T_\Omega(t)f$ can be approximated in a suitable way by smooth enough functions written in terms of semigroups depending on two parameters n and ε. These parameters take into account that the approximation procedure first reduces the problem from an infinite-dimensional setting to a finite-dimensional one, and then, by using a penalisation argument, it allows to solve the problem in the domain Ω throughout the solution of a suitable problem in the whole space.

In view of these facts, we first recall some results about parabolic and elliptic problems with unbounded coefficients in finite dimension.

2.1. Parabolic and elliptic equations in $\mathbb{R}^n$

In this subsection, we consider a convex function $\phi \in C^{2+\alpha}(\mathbb{R}^n)$ for some $\alpha \in (0, 1)$ with bounded second derivatives and a second-order differential operator $\mathcal{L}_\phi$ acting on smooth functions v as follows

$$\mathcal{L}_\phi v(\xi) = \Delta v(\xi) + \langle \mathcal{B}\xi, Dv(\xi)\rangle - \langle D\phi(\xi), Dv(\xi)\rangle, \quad \xi \in \mathbb{R}^n,$$

where $\mathcal{B}$ is a constant symmetric matrix such that $\langle \mathcal{B}\xi, \xi\rangle \le -\beta|\xi|^2$ for any $\xi \in \mathbb{R}^n$ and some $\beta > 0$.

It is known (see [29, Chapter 1] and the reference therein) that for any $\varphi \in C_b(\mathbb{R}^n)$ there exists a unique bounded classical solution v of problem

$$\begin{cases} D_t v(t,\xi) = \mathcal{L}_\phi v(t,\xi) & t > 0,\ \xi \in \mathbb{R}^n; \\ v(0,\xi) = \varphi(\xi), & \xi \in \mathbb{R}^n. \end{cases} \tag{2.1}$$

Namely v belongs to $C_b([0,+\infty) \times \mathbb{R}^n) \cap C^{1+\alpha/2,2+\alpha}_{\mathrm{loc}}((0,+\infty) \times \mathbb{R}^n)$ and solves the Cauchy problem (2.1). The uniqueness of v is a consequence of the convexity of ϕ and of the existence of a Lyapunov function, i.e. a positive function $g \in C^2(\mathbb{R}^n)$ such that $\lim_{|\xi|\to+\infty} g(\xi) = +\infty$ and

$$(\mathcal{L}_\phi g)(\xi) - \lambda g(\xi) \le 0, \qquad \xi \in \mathbb{R}^n, \tag{2.2}$$

for some $\lambda > 0$. Indeed, taking $g(\xi) = |\xi|^2$, $\xi \in \mathbb{R}^n$, we have

$$\begin{aligned}(\mathcal{L}_\phi g)(\xi) &= 2n + 2\langle \mathcal{B}\xi, \xi\rangle - 2\langle D\phi(\xi), \xi\rangle \\ &\le 2n - 2\beta|\xi|^2 - 2\langle D\phi(\xi) - D\phi(0), \xi\rangle - 2\langle D\phi(0), \xi\rangle \\ &\le 2n - 2\beta|\xi|^2 + 2|D\phi(0)||\xi|,\end{aligned}$$

where we have used that $\langle D\phi(\xi) - D\phi(0), \xi\rangle \ge 0$ for every $\xi \in \mathbb{R}^n$ so, clearly, we can find λ such that inequality (2.2) is satisfied.

In this way, we can consider the semigroup $T_\phi(t)$ associated with $\mathcal{L}_\phi$ in $C_b(\mathbb{R}^n)$ and write $v(t,\xi) = (T_\phi(t)\varphi)(\xi)$ for any $t > 0$ and $\xi \in \mathbb{R}^n$. It turns out that $T_\phi(t)$ is a positivity-preserving contractive semigroup in $C_b(\mathbb{R}^n)$.

To pass from finite to infinite dimension, we prove and exploit suitable uniform gradient estimates independent of the dimension. More precisely, we prove a dimension-free uniform estimate for the gradient of $T_\phi(t)\varphi$, $\varphi \in C^1_b(\mathbb{R}^n)$. Such kind of estimates has already been proved for semigroups associated with more general operators (see [29, Chapter 5] and the reference therein). However, since in all these estimates are not emphasised how and if the constants appearing depend on the dimension, we provide a sketch of the proofs (essentially based on the Bernstein method and the classical maximum principle) that allows us to verify that the constants are dimension-free.

PROPOSITION 2.1. *The estimate*

$$|D_\xi T_\phi(t)\varphi(\xi)| \le \frac{\|\varphi\|_\infty}{\sqrt{\beta t}} \tag{2.3}$$

holds true for any $t > 0$, $\xi \in \mathbb{R}^n$ and $\varphi \in C_b(\mathbb{R}^n)$. Here β is the positive constant which bounds from below the quadratic form associated with $-\mathcal{B}$.

Proof. It suffices to prove the claim for functions $\varphi \in C^{2+\alpha}_c(\mathbb{R}^n)$, i.e. the space of the functions in $C^{2+\alpha}(\mathbb{R}^n)$ with compact support. Indeed, if $\varphi \in C_b(\mathbb{R}^n)$ we can consider a sequence $(\varphi_m)_{m\in\mathbb{N}}$ converging to φ locally uniformly as m goes to infinity and use the fact that, up to a subsequence, $T_\phi(t)\varphi_m$ converges to $T_\phi(t)\varphi$ in $C^{1,2}_{\rm loc}((0,+\infty)\times\mathbb{R}^n)$, as m goes to infinity (see [29]). Moreover, taking advantage of the interior Schauder estimates (see [25]), we reduce ourselves to proving the claim for the solution v_R of the homogeneous Neumann–Cauchy problem associated with the equation $D_t v = \mathcal{L}_\phi v$ in $(0,T]\times B_R$, where B_R is the open ball centred at the origin with radius R large enough such that the support of φ is contained in B_R. Indeed, once (2.3) is proved for v_R, recalling that v_R converges to $T_\phi(t)\varphi$ in $C^{1,2}_{\rm loc}((0,+\infty)\times\mathbb{R}^n)$ as $R\to+\infty$, we conclude. Therefore, let $\varphi\in C^{2+\alpha}_c(\mathbb{R}^n)$ and v_R be as specified above. Then, the function

$$z_R(t,\xi) := |v_R(t,\xi)|^2 + \beta t\left|D_\xi v_R(t,\xi)\right|^2 \quad t>0,\ \xi\in B_R$$

satisfies $z_R(0,\cdot)=\varphi^2$ in B_R, $\langle D_\xi v_R,\nu\rangle\le 0$ (ν is the unit normal vector) on $(0,T]\times\partial B_R$ and solves the equation

$$\begin{aligned} D_t z_R - \mathcal{L}_\phi z_R = (\beta-1)\left|D_\xi v_R\right|^2 + \left\langle \mathcal{B} D_\xi v_R, D_\xi v_R\right\rangle \\ -\langle D^2_\xi\phi D_\xi v_R, D_\xi v_R\rangle - \beta t|D^2_\xi v_R|^2 \le 0, \end{aligned}$$

in $(0,T]\times B_R$ (in the last inequality we have used the convexity of ϕ and the assumption on the matrix $\mathcal{B}$). The classical maximum principle applied to the function $z_R-\|\varphi\|_\infty$ yields the claim in $(0,T]\times B_R$. The arbitrariness of T allows us to extend the claim in the whole $(0,+\infty)\times B_R$. $\square$

The contractivity of $T_\phi(t)$ in $C_b(\mathbb{R}^n)$ and estimate (2.3) yield some dimension-free uniform estimates for the solution (and its gradient) of the elliptic equation

$$\lambda v - \mathcal{L}_\phi v = \varphi \in C^2_b(\mathbb{R}^n), \qquad \lambda>0. \tag{2.4}$$

PROPOSITION 2.2. *For any $\lambda>0$, there exists a unique bounded classical solution v of problem* (2.4). *Moreover, v satisfies*

$$(i)\quad \|v\|_\infty \le \frac{1}{\lambda}\|\varphi\|_\infty, \qquad (ii)\quad \|Dv\|_\infty \le \sqrt{\frac{\pi}{\beta\lambda}}\|\varphi\|_\infty. \tag{2.5}$$

In addition, if $\phi\in C^3(\mathbb{R}^n)$, then v belongs to $C^3_b(\mathbb{R}^n)$.

Proof. Existence and estimates (2.5) are immediate consequences of the fact that

$$v(\xi) = \int_0^{+\infty} \mathrm{e}^{-\lambda t}(T_\phi(t)\varphi)(\xi)\mathrm{d}t \quad \xi\in\mathbb{R}^n.$$

(see [10, Propositions 3.2 & 3.4] and [37, Proposition 3.6]) and estimate (2.3).

Concerning the last statement, we prove that the third-order derivatives of v are bounded. Indeed, the classical theory of elliptic equations guarantees that v belongs to $C^3(\mathbb{R}^n)$. Moreover, [30, Theorem 1] yields that u belongs to $C^{2+\theta}_b(\mathbb{R}^n)$ for every

$0 < \theta < 1$ and $\|v\|_{C_b^{2+\theta}(\mathbb{R}^n)} \le C\|\varphi\|_{C_b^\theta(\mathbb{R}^n)}$ for some positive constant C independent of φ. Thus, we can differentiate (2.4) and obtain

$$\lambda D_j v - \mathcal{L}_\phi D_j v = D_j\varphi + (D^2\phi Dv)_j + (\mathcal{B}Dv)_j, \tag{2.6}$$

for any $j = 1, \dots, n$. Thus, taking into account that the right-hand side of (2.6) is α-Hölder continuous and bounded we can apply again [30, Theorem 1] to deduce that $D_j v \in C_b^{2+\alpha}(\mathbb{R}^n)$ for every $j = 1, \dots, n$. In particular, v belongs to $C_b^3(\mathbb{R}^n)$. □

2.2. Back to the infinite dimension

Here we apply the results of the previous subsection with $\mathcal{B} = \mathrm{diag}(-\lambda_1^{-1}, \dots, -\lambda_n^{-1})$ and $\beta = \lambda_1^{-1}$ (see (1.1) for the definition of $(\lambda_i)_{i\in\mathbb{N}}$). Moreover, we focus on the term $\langle D_H U, D_H\rangle_H$ in the operator L in (1.15). We introduce some functions that, in some sense, reduce U from infinite dimension to finite dimension and that contain a penalisation term which allows us to localise the problem in Ω. More precisely, we define $\Phi_\varepsilon : X \to \mathbb{R}$ and $\phi_{\varepsilon,n} : \mathbb{R}^n \to \mathbb{R}$, respectively, by

$$\Phi_\varepsilon(x) := U_\varepsilon(x) + \frac{1}{2\varepsilon} d_\Omega^2(x),$$
$$\phi_{\varepsilon,n}(\xi) := (\mathbb{E}_n\Phi_\varepsilon)(\Sigma_n\xi) \quad x \in X,\ \xi \in \mathbb{R}^n,\ n \in \mathbb{N},\ \varepsilon > 0,$$

where U_ε is the Moreau–Yosida approximation of U along H [see (1.8)] and $\Sigma_n : \mathbb{R}^n \to X$ is the embedding function defined in Sect. 1.

In order to apply the finite-dimensional results obtained in Sect. 2.1, we need also to regularise the function $\phi_{\varepsilon,n}$. To do this, we consider $\phi_{\varepsilon,n,\eta} : \mathbb{R}^n \to \mathbb{R}$, the convolution of $\phi_{\varepsilon,n}$ with a standard mollifier ρ_η.

First, we state some properties of the functions just introduced. In the following statement, we show that the function $\phi_{\varepsilon,n,\eta}$ belongs to $C_b^{2+\alpha}(\mathbb{R}^n)$ for any $\alpha \in (0, 1)$.

LEMMA 2.3. *For every $\varepsilon, \eta > 0$ and $n \in \mathbb{N}$, the function $\phi_{\varepsilon,n,\eta}$ belongs to $C_b^\infty(\mathbb{R}^n)$.*

Proof. Clearly, $\phi_{\varepsilon,n,\eta}$ belongs to $C^\infty(\mathbb{R}^n)$. Let us show that $D^2\phi_{\varepsilon,n,\eta}$ is bounded in $\mathbb{R}^n$. Propositions 1.5 and 1.8(ii) guarantee that Φ_ε is H-differentiable and $D_H\Phi_\varepsilon$ is H-Lipschitz continuous in X. The same holds true in $\mathbb{R}^n$ for the functions $\phi_{\varepsilon,n}$. Rademacher's theorem yields that $D\phi_{\varepsilon,n}$ is differentiable $\mathcal{L}^n$-a.e. and $D^2\phi_{\varepsilon,n}$ is $\mathcal{L}^n$-essentially bounded. This implies that $D^2\phi_{\varepsilon,n,\eta}$ are bounded in $\mathbb{R}^n$. With similar arguments, it follows that $\phi_{\varepsilon,n,\eta} \in C_b^\infty(\mathbb{R}^n)$. □

LEMMA 2.4. *Let $\varepsilon > 0$. There exists an infinitesimal sequence $(\eta_n)_{n\in\mathbb{N}}$ such that*

$$\lim_{n\to+\infty} D_H\Phi_{\varepsilon,n,\eta_n} = D_H\Phi_\varepsilon, \tag{2.7}$$
$$\lim_{n\to+\infty} D_H^2\Phi_{\varepsilon,n,\eta_n} = D_H^2\Phi_\varepsilon, \tag{2.8}$$

where $\Phi_{\varepsilon,n,\eta_n}(x) := \phi_{\varepsilon,n,\eta_n}(\Pi_n x)$ for any $x \in X$. The limits in (2.7) and (2.8) are taken in $L^2(X, \nu_\varepsilon; H)$ and $L^2(X, \nu_\varepsilon; \mathcal{H}_2)$, respectively, and ν_ε is the measure $e^{-\Phi_\varepsilon}\gamma$.

Proof. Throughout this proof, for any $n \in \mathbb{N}$ and $x, y \in X$ we set $\Gamma_n(x, y) := P_n x + (I - P_n)y$. We start by proving that (2.7) holds true for every infinitesimal sequence $(\eta_n)_{n\in\mathbb{N}}$. To this aim, let $(\eta_n)_{n\in\mathbb{N}}$ be an infinitesimal sequence. Then

$$\begin{aligned}
\int_X \Big(\left|D_H\Phi_\varepsilon - D_H\Phi_{\varepsilon,n,\eta_n}\right|_H^2 \mathrm{d}\nu_\varepsilon &\le 2\int_X |D_H\Phi_\varepsilon - D_H\mathbb{E}_n\Phi_\varepsilon|_H^2 \\
&\quad + \left|D_H\mathbb{E}_n\Phi_\varepsilon - D_H\Phi_{\varepsilon,n,\eta_n}\right|_H^2\Big)\mathrm{d}\nu_\varepsilon \\
&\le 2\left(\int_X \mathrm{e}^{-p'\Phi_\varepsilon}\mathrm{d}\gamma\right)^{\frac{1}{p'}}\left(\int_X |D_H\Phi_\varepsilon - D_H\mathbb{E}_n\Phi_\varepsilon|_H^{2p}\mathrm{d}\gamma\right)^{\frac{1}{p}} \\
&\quad + 2\int_X \left|D_H\mathbb{E}_n\Phi_\varepsilon - D_H\Phi_{\varepsilon,n,\eta_n}\right|_H^2\mathrm{d}\nu_\varepsilon. \qquad (2.9)
\end{aligned}$$

Since $D_H d_\Omega^2$ is H-Lipschitz continuous in X, the function Φ_ε belongs to $D^{1,q}(X,\gamma)$ for $q \in [1,\infty)$. Thus, Proposition 1.9 yields that the second line in (2.9) vanishes as n goes to infinity. Now

$$\begin{aligned}
&\int_X \left|D_H\mathbb{E}_n\Phi_\varepsilon - D_H\Phi_{\varepsilon,n,\eta_n}\right|_H^2\mathrm{d}\nu_\varepsilon \\
&\quad = \int_X \sum_{i=1}^n \Big(\int_X D_i\Phi_\varepsilon(\Gamma_n(x,y))\mathrm{d}\gamma(y) \\
&\qquad - \int_X\Big(\int_{\mathbb{R}^n} D_i\Phi_\varepsilon(\Gamma_n(x,y) - \eta_n(\Sigma_n\xi))\rho(\xi)\mathrm{d}\xi\Big)\mathrm{d}\gamma(y)\Big)^2\mathrm{d}\nu_\varepsilon(x) \\
&\quad \le \int_X\int_{\mathbb{R}^n}\Big(\int_X |D_H\Phi_\varepsilon(\Gamma_n(x,y)) - D_H\Phi_\varepsilon(\Gamma_n(x,y) - \eta_n(\Sigma_n(\xi))|_H^2\mathrm{d}\gamma(y)\Big) \\
&\qquad \rho(\xi)\mathrm{d}\xi\,\mathrm{d}\nu_\varepsilon(x) \\
&\quad \le [D_H\Phi_\varepsilon]_{H\text{-Lip}}^2\nu_\varepsilon(X)\eta_n\int_{\mathbb{R}^n}|\xi|^2\rho(\xi)\mathrm{d}\xi \le [D_H\Phi_\varepsilon]_{H\text{-Lip}}^2\nu_\varepsilon(X)\eta_n, \qquad (2.10)
\end{aligned}$$

and the right-hand side of (2.10) vanishes as $n \to +\infty$.

Now we prove (2.8). Propositions 1.5 and 1.8(iii) guarantee that Φ_ε belong to $D^{2,p}(X,\gamma)$ for any $p \in [1,\infty)$ and by Proposition 1.9 we immediately get that $D_H^2\mathbb{E}_n\Phi_\varepsilon$ converges to $D_H^2\Phi_\varepsilon$ in $L^2(X,\nu_\varepsilon;\mathcal{H}_2)$ as $n \to +\infty$.

In view of this fact, arguing as in (2.9), it remains to prove the existence of a vanishing sequence $(\eta_n)_{n\in\mathbb{N}}$ such that $(D_H^2\mathbb{E}_n\Phi_\varepsilon - D_H^2\Phi_{\varepsilon,n,\eta_n})_{n\in\mathbb{N}}$ is infinitesimal in $L^2(X,\nu_\varepsilon;\mathcal{H}_2)$ as n goes to infinity.

We start by showing that for any $n \in \mathbb{N}$ the function $D_H^2\Phi_{\varepsilon,n,\eta}$ converges to $D_H^2\mathbb{E}_n\Phi_\varepsilon$ in $L^2(X,\nu_\varepsilon;\mathcal{H}_2)$ as $\eta \to 0^+$. To this aim, we can argue as in (2.10) and deduce that

$$\begin{aligned}
&\int_X |D_H^2\mathbb{E}_n\Phi_\varepsilon - D_H^2\Phi_{\varepsilon,n,\eta}|_{\mathcal{H}_2}^2\mathrm{d}\nu_\varepsilon \\
&\le \int_X\int_{\mathbb{R}^n}\Big(\int_X |D_H^2\Phi_\varepsilon(\Gamma_n(x,y)) - D_H^2\Phi_\varepsilon(\Gamma_n(x,y) - \eta(\Sigma_n\xi))|_{\mathcal{H}_2}^2\mathrm{d}\gamma(y)\Big) \\
&\qquad \rho(\xi)d\xi\,\mathrm{d}\nu_\varepsilon(x).
\end{aligned}$$

By Hypotheses 1.3, 1.6 and Proposition 1.8(v), the function $D^2_H\Phi_\varepsilon$ is H-continuous. This guarantees that the integrand function vanishes as $\eta \to 0$. Moreover, as $D^2_H\Phi_\varepsilon$ is γ-essentially bounded in X, we can estimate the integrand function by a constant independent of η and apply the dominated convergence theorem to conclude.
Now, a diagonal argument yields an infinitesimal sequence satisfying (2.8) (and (2.7), too). We start by letting η_1 be such that

$$\left\| D^2_H\mathbb{E}_1\Phi_\varepsilon - D^2_H\Phi_{\varepsilon,1,\eta_1}\right\|_{L^2(X,\nu_\varepsilon,\mathcal{H}_2)} < 1.$$

Proceeding by induction, for every $n \geq 1$, we take η_{n+1} in such a way that $\eta_{n+1} < \eta_n$ and

$$\left\| D^2_H\mathbb{E}_{n+1}\Phi_\varepsilon - D^2_H\Phi_{\varepsilon,n+1,\eta_{n+1}}\right\|_{L^2(X,\nu_\varepsilon,\mathcal{H}_2)} < \frac{1}{2^n}.$$

Thus, let $\varepsilon > 0$ and $\bar{n} \in \mathbb{N}$ be such that $1 < 2^{\bar{n}-1}\varepsilon$ and $\left\| D^2_H\Phi_\varepsilon - D^2_H\mathbb{E}_n\Phi_\varepsilon\right\|_{L^2(X,\nu_\varepsilon,\mathcal{H}_2)} < \frac{\varepsilon}{2}$ for any $n \geq \bar{n}$. Then for $n \geq \bar{n}$

$$\begin{aligned}\left\| D^2_H\Phi_\varepsilon - D^2_H\Phi_{\varepsilon,n,\eta_n}\right\|_{L^2(X,\nu_\varepsilon,\mathcal{H}_2)} &\leq \left\| D^2_H\Phi_\varepsilon - D^2_H\mathbb{E}_n\Phi_\varepsilon\right\|_{L^2(X,\nu_\varepsilon,\mathcal{H}_2)} \\ &\quad + \left\| D^2_H\mathbb{E}_n\Phi_\varepsilon - D^2_H\Phi_{\varepsilon,n,\eta_n}\right\|_{L^2(X,\nu_\varepsilon,\mathcal{H}_2)} \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

So the proof is complete. □

Now, let $f \in \mathcal{F}C^\infty_b(X)$ and $\psi \in C^\infty_b(\mathbb{R}^{n_0})$ for some $n_0 \in \mathbb{N}$ be such that $f(x) = \psi(\Pi_{n_0}x)$ for any $x \in X$. Proposition 2.2 and Lemma 2.3 allow us to consider v_{ε,n,η_n}, $(n \geq n_0)$, the unique solution of (2.4) with ϕ replaced by $\phi_{\varepsilon,n,\eta_n}$ and φ replaced by ψ.

In order to come back to the infinite-dimensional setting, we define

$$\Phi_{\varepsilon,n}(x) := \phi_{\varepsilon,n,\eta_n}(\Pi_n x), \quad V_{\varepsilon,n}(x) := v_{\varepsilon,n,\eta_n}(\Pi_n x), \quad x \in X,\ \varepsilon > 0,\ n \geq n_0$$

where $(\eta_n)_{n\in\mathbb{N}}$ is the sequence of Lemma 2.4. Now we consider the operator L_ε defined as

$$\begin{aligned} D(L_\varepsilon) = \bigg\{ u \in D^{1,2}(X,\nu_\varepsilon) \,\bigg|\, &\text{there exists } v_u \in L^2(X,\nu_\varepsilon) \text{ such that} \\ &\int_X \langle D_H u, D_H\varphi\rangle_H \mathrm{d}\nu_\varepsilon = -\int_X v_u\varphi \mathrm{d}\nu_\varepsilon \text{ for every } \varphi \in \mathcal{F}C^\infty_b(\Omega)\bigg\}, \end{aligned}$$

with $L_\varepsilon u := v_u$ if $u \in D(L_\varepsilon)$. We remark that L_ε acts on smooth cylindrical functions φ as follows

$$L_\varepsilon\varphi = \mathrm{Tr}(D^2_H\varphi) - \sum_{i=1}^{+\infty}\lambda_i^{-1}\langle x, e_i\rangle D_i\varphi - \langle D_H\Phi_\varepsilon, D_H\varphi\rangle_H. \tag{2.11}$$

REMARK 2.5. Note that formulas (1.3) and (1.16) hold true also with ν, L and U replaced by ν_ε, L_ε and Φ_ε, respectively. The same arguments listed after Hypothesis 1.3 allow us to define the spaces $D^{k,p}(X,\nu_\varepsilon)$ for any $\varepsilon > 0$, $p \in (1,\infty)$ and $k = 1,2$. Moreover, if $(T_\varepsilon(t))_{t\geq 0}$ is the analytic semigroup generated by the operator L_ε in $L^2(X,\nu_\varepsilon)$, then all the properties listed in Proposition 1.10 for $T_\Omega(t)$ hold true for $T_\varepsilon(t)$, too.

PROPOSITION 2.6. *The function $V_{\varepsilon,n}$ belongs to $\mathcal{F}C_b^3(X)$ and solves*

$$\lambda V_{\varepsilon,n} - L_\varepsilon V_{\varepsilon,n} = f + \big\langle D_H\Phi_\varepsilon - D_H\Phi_{\varepsilon,n}, D_H V_{\varepsilon,n}\big\rangle_H =: f_n, \qquad \lambda > 0. \tag{2.12}$$

Moreover, f_n converges to f in $L^2(X,\nu_\varepsilon)$ and $D_H f_n$ converges to $D_H f$ in $L^1(X,\nu_\varepsilon,H)$, as n goes to infinity.

Proof. The fact that $V_{\varepsilon,n}$ belongs to $\mathcal{F}C_b^3(X)$ follows from Proposition 2.2 and Lemma 2.3. In order to obtain (2.12), we recall that $v_{\varepsilon,n,\eta_n}(\xi) = V_{\varepsilon,n}(\Sigma_n\xi)$ for any $\xi \in \mathbb{R}^n$. So we have

$$\begin{aligned}\lambda V_{\varepsilon,n}(\Sigma_n\xi) - \mathrm{Tr}(D_H^2 V_{\varepsilon,n}(\Sigma_n\xi)) + \sum_{i=1}^{+\infty}\lambda_i^{-1}\xi_i D_i V_{\varepsilon,n}(\Sigma_n\xi)&\\ + \big\langle D_H\Phi_{\varepsilon,n}(\Sigma_n\xi), D_H V_{\varepsilon,n}(\Sigma_n\xi)\big\rangle_H &= \psi(\xi).\end{aligned}$$

Now adding and subtracting $L_\varepsilon V_{\varepsilon,n}(\Sigma_n\xi)$ [see (2.11)] and letting $\xi = \Pi_n x$, we get (2.12). Observe that by Proposition 2.2 we also get the following estimate

$$\big\| D_H V_{\varepsilon,n}\big\|_\infty \leq \sqrt{\frac{\lambda_1\pi}{\lambda}}\|f\|_\infty =: K. \tag{2.13}$$

Using (2.12) and (2.13), we get

$$\begin{aligned}\int_X |f_n - f|^2 d\nu_\varepsilon &= \int_X \big|\big\langle D_H\Phi_\varepsilon - D_H\Phi_{\varepsilon,n}, D_H V_{\varepsilon,n}\big\rangle\big|_H^2 d\nu_\varepsilon\\ &\leq K^2\int_X \big|D_H\Phi_\varepsilon - D_H\Phi_{\varepsilon,n}\big|_H^2 d\nu_\varepsilon.\end{aligned}$$

and by (2.7) we obtain that f_n converges to f in $L^2(X,\nu_\varepsilon)$.

In order to prove the last part of the claim we first estimate $D_H^2 V_{\varepsilon,n}$. Differentiating (2.12) along e_j, multiplying the result by $D_j V_{\varepsilon,n}$ and then summing up from 1 to n, yield

$$\begin{aligned}\lambda\big|D_H V_{\varepsilon,n}\big|_H^2 - \sum_{j=1}^n D_j V_{\varepsilon,n} L_\varepsilon(D_j V_{\varepsilon,n}) + \sum_{i=1}^n \lambda_i^{-1}(D_i V_{\varepsilon,n})^2 + \big\langle D_H^2\Phi_\varepsilon D_H V_{\varepsilon,n}, D_H V_{\varepsilon,n}\big\rangle_H&\\ = \big\langle D_H f, D_H V_{\varepsilon,n}\big\rangle_H + \big\langle (D_H^2\Phi_\varepsilon - D_H^2\Phi_{\varepsilon,n}) D_H V_{\varepsilon,n}, D_H V_{\varepsilon,n}\big\rangle_H&\\ + \big\langle D_H^2 V_{\varepsilon,n} D_H V_{\varepsilon,n}, D_H\Phi_\varepsilon - D_H\Phi_{\varepsilon,n}\big\rangle_H.&\end{aligned}$$

Since $\lambda_i > 0$ for every $i \in \mathbb{N}$, by the convexity of Φ_ε we get

$$-\sum_{j=1}^{n} D_j V_{\varepsilon,n} L_\varepsilon(D_j V_{\varepsilon,n}) \leq \langle D_H f, D_H V_{\varepsilon,n}\rangle_H + \left\langle D_H^2 V_{\varepsilon,n} D_H V_{\varepsilon,n}, D_H\Phi_\varepsilon - D_H\Phi_{\varepsilon,n}\right\rangle_H + \left\langle (D_H^2\Phi_\varepsilon - D_H^2\Phi_{\varepsilon,n}) D_H V_{\varepsilon,n}, D_H V_{\varepsilon,n}\right\rangle_H. \qquad (2.14)$$

Thus, integrating (2.14) with respect to ν_ε and using that

$$\int_X \langle D_H u, D_H\varphi\rangle_H \mathrm{d}\nu_\varepsilon = -\int_X \varphi L_\varepsilon u \mathrm{d}\nu_\varepsilon, \qquad u \in D(L_\varepsilon),\ \varphi \in \mathcal{F}C_b^1(X),$$

we deduce

$$\int_X \left|D_H^2 V_{\varepsilon,n}\right|_{\mathcal{H}_2}^2 \mathrm{d}\nu_\varepsilon \leq K\nu_\varepsilon(X)\|D_H f\|_\infty + \sigma K \int_X \left|D_H^2 V_{\varepsilon,n}\right|_{\mathcal{H}_2}^2 \mathrm{d}\nu_\varepsilon + \frac{1}{4\sigma} K \int_X \left|D_H\Phi_\varepsilon - D_H\Phi_{\varepsilon,n}\right|_H^2 \mathrm{d}\nu_\varepsilon + K^2\left(\int_X \left|D_H^2\Phi_\varepsilon - D_H^2\Phi_{\varepsilon,n}\right|_{\mathcal{H}_2}^2 \mathrm{d}\nu_\varepsilon\right)^{1/2}.$$

for every $\sigma > 0$. Choosing $\sigma = (2K)^{-1}$, we have

$$\frac{1}{2}\int_X \left|D_H^2 V_{\varepsilon,n}\right|_{\mathcal{H}_2}^2 \mathrm{d}\nu_\varepsilon \leq K\nu_\varepsilon(X)\|D_H f\|_\infty + \frac{1}{2}K^2 \int_X \left|D\Phi_\varepsilon - D\Phi_{\varepsilon,n}\right|_H^2 \mathrm{d}\nu_\varepsilon + K^2\left(\int_X \left|D^2\Phi_\varepsilon - D^2\Phi_{\varepsilon,n}\right|_{\mathcal{H}_2}^2 \mathrm{d}\nu_\varepsilon\right)^{1/2}.$$

Thanks to (2.7) and (2.8), there is a constant $C = C(K, \varepsilon) > 0$ such that $\|D_H^2 V_{\varepsilon,n}\|_{L^2(X,\nu_\varepsilon;\mathcal{H}_2)} \leq C$ for every $n \in \mathbb{N}$. To complete the proof, we show that $D_H f_n$ converges to $D_H f$ in $L^1(X, \nu_\varepsilon; H)$. We have

$$\begin{aligned}\int_X |D_H f_n - D_H f| \mathrm{d}\nu_\varepsilon &= \int_X \left|D_H\langle D_H\Phi_\varepsilon - D_H\Phi_{\varepsilon,n}, D_H V_{\varepsilon,n}\rangle_H\right|_H \mathrm{d}\nu_\varepsilon \\ &\leq \int_X \left(\left|(D_H^2\Phi_\varepsilon - D_H^2\Phi_{\varepsilon,n}) D_H V_{\varepsilon,n}\right|_H + \left|D_H^2 V_{\varepsilon,n}(D_H\Phi_\varepsilon - D_H\Phi_{\varepsilon,n})\right|_H\right) \mathrm{d}\nu_\varepsilon \\ &\leq K\|D_H^2\Phi_\varepsilon - D_H^2\Phi_{\varepsilon,n}\|_{L^2(X,\nu_\varepsilon;\mathcal{H}_2)} \\ &\quad + \|D_H^2 V_{\varepsilon,n}\|_{L^2(X,\nu_\varepsilon;\mathcal{H}_2)}\|D_H\Phi_\varepsilon - D_H\Phi_{\varepsilon,n}\|_{L^2(X,\nu_\varepsilon;H)}.\end{aligned}$$

So, being $\|D_H^2 V_{\varepsilon,n}\|_{L^2(X,\nu_\varepsilon;\mathcal{H}_2)}$ bounded, the claim follows from (2.7) and (2.8). □

Proposition 2.6 and the Lumer–Phillips theorem yield that the resolvent set of L_ε in $L^2(X, \nu_\varepsilon)$ contains the half-line $(0, +\infty)$. In addition, from [13, Theorem 5.10], we get the following approximation result.

PROPOSITION 2.7. *For any $\varepsilon > 0$ and $f \in L^2(X, \nu_\varepsilon)$, there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $D^{1,2}(X, \nu_\varepsilon)$ such that $R(\lambda, L_\varepsilon) f_n$ belongs to $\mathcal{F}C_b^3(X)$ for every $n \in \mathbb{N}$ and*

$$\lim_{n\to+\infty} \|R(\lambda, L_\varepsilon) f_n - R(\lambda, L_\varepsilon) f\|_{D^{2,2}(X,\nu_\varepsilon)} = 0, \qquad \lambda > 0,$$

where $R(\lambda, L_\varepsilon)$ is the resolvent operator of L_ε [*see* (2.11)]. *In addition,*

$$\begin{aligned} \|R(\lambda, L_\varepsilon) f\|_{L^2(X,\nu_\varepsilon)} &\le \frac{1}{\lambda}\|f\|_{L^2(X,\nu_\varepsilon)}, \\ \|D_H R(\lambda, L_\varepsilon) f\|_{L^2(X,\nu_\varepsilon;H)} &\le \frac{1}{\sqrt{\lambda}}\|f\|_{L^2(X,\nu_\varepsilon)}, \end{aligned} \tag{2.15}$$

and

$$\left\| D_H^2 R(\lambda, L_\varepsilon) f \right\|_{L^2(X,\nu_\varepsilon;\mathcal{H}_2)} \le \sqrt{2}\|f\|_{L^2(X,\nu_\varepsilon)}. \tag{2.16}$$

Now, we are ready to prove the main theorem of this section.

THEOREM 2.8. *The following statements hold true.*

(i) *For any $\varepsilon > 0$ and $f \in L^2(X, \nu_\varepsilon)$, it holds that*

$$\lim_{n\to+\infty} \|T_\varepsilon(t) f_n - T_\varepsilon(t) f\|_{D^{2,2}(X,\nu_\varepsilon)} = 0, \qquad t > 0,$$

where $(f_n)_{n\in\mathbb{N}}$ is the sequence defined in Proposition 2.7. Furthermore, $T_\varepsilon(t) f_n$ belongs to $\mathcal{F}C_b^3(X)$. In addition, if $f \in D^{1,2}(X, \nu_\varepsilon)$ then $D_H f_n$ converges to $D_H f$ in $L^1(X, \nu_\varepsilon; H)$, as n goes to infinity.

(ii) *For any $f \in L^2(\Omega, \nu)$ there exists an infinitesimal sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that $T_{\varepsilon_n}(t)\widetilde{f}$ weakly converges to $T_\Omega(t) f$ in $D^{2,2}(\Omega, \nu)$, where $\widetilde{f}$ is any L^2-extension of f to X.*

Proof. The analyticity of the semigroups $T_\Omega(t)$ and $T_\varepsilon(t)$ in $L^2(\Omega, \nu)$ and $L^2(X, \nu_\varepsilon)$, respectively, and the decay estimates (1.10), (1.11), (2.15) and (2.16) (and Remark 2.5) allow us to write the following representation formulas

$$\begin{aligned} D_H^j T_\varepsilon(t) f &= \frac{1}{2\pi i}\int_\sigma e^{\lambda t} D_H^j R(\lambda, L_\varepsilon) f \,d\lambda, \qquad t > 0,\ f \in L^2(X, \nu_\varepsilon), \\ D_H^j T_\Omega(t) f &= \frac{1}{2\pi i}\int_{\sigma'} e^{\lambda t} D_H^j R(\lambda, L_\Omega) f \,d\lambda, \qquad t > 0,\ f \in L^2(\Omega, \nu), \end{aligned} \tag{2.17}$$

for any $j = 0, 1, 2$, where σ (resp. σ') is a smooth (unbounded) curve in $\mathbb{C}$ which leaves on the left a sector containing the spectrum of L_ε (resp. L_Ω).

(i) For any $j = 0, 1, 2$, we have

$$
\begin{aligned}
&\int_X |D_H^j T_\varepsilon(t) f_n - D_H^j T_\varepsilon(t) f|_j^2 \mathrm{d}\nu_\varepsilon \\
&= \frac{1}{4\pi^2} \int_X \left| \int_\sigma \mathrm{e}^{\lambda t} \Big(D_H^j R(\lambda, L_\varepsilon) f_n \mathrm{d}\lambda - D_H^j R(\lambda, L_\varepsilon) f \Big) d\lambda \right|_j^2 \mathrm{d}\nu_\varepsilon \\
&\le \frac{K(\sigma, t)}{4\pi^2} \int_\sigma \int_X \mathrm{e}^{\lambda t} |D_H^j R(\lambda, L_\varepsilon) f_n - D_H^j R(\lambda, L_\varepsilon) f|_j^2 \mathrm{d}\nu_\varepsilon \mathrm{d}\lambda,
\end{aligned}
\tag{2.18}
$$

where $|\cdot|_j$ denotes the norm in $\mathbb{R}$, H, $\mathcal{H}_2$, respectively, and $K(\sigma, t) = \int_\sigma \mathrm{e}^{\lambda t} \mathrm{d}\lambda$. We conclude observing that, by the dominated convergence theorem and the results in Proposition 2.7, the right-hand side of (2.18) vanishes as n goes to infinity. The furthermore part is consequence of Proposition 2.7 and the integral representation formula (2.17). Finally, the last assertion is an immediate consequence of Proposition 2.6.

(ii) Since $U(x) \ge \Phi_\varepsilon(x)$ for any $x \in \Omega$, by using (2.15) and (2.16) we immediately deduce that for any vanishing sequence (ε_n) and for any $f \in L^2(\Omega, \nu)$ the sequence $(R(\lambda, L_{\varepsilon_n})\widetilde{f})$ is bounded in $D^{2,2}(\Omega, \nu)$. A compactness argument yields that there exists a subsequence of (ε_n) [still denoted by (ε_n)] such that $R(\lambda, L_{\varepsilon_n})\widetilde{f}$ weakly converges to an element $u \in D^{2,2}(\Omega, \nu)$, as n goes to infinity. From [14, Theorem 5.3], it follows that $u = R(\lambda, L_\Omega) f$. Now, the proof proceeds as in (i). Indeed, for any $f, g \in L^2(\Omega, \nu)$ we have

$$
\begin{aligned}
\int_\Omega (T_{\varepsilon_n}(t)\widetilde{f}) g \mathrm{d}\nu &= \frac{1}{2\pi i} \int_\Omega \int_\sigma \mathrm{e}^{\lambda t} (R(\lambda, L_{\varepsilon_n})\widetilde{f}) g \mathrm{d}\lambda \mathrm{d}\nu \\
&= \frac{1}{2\pi i} \int_\sigma \mathrm{e}^{\lambda t} \int_\Omega (R(\lambda, L_{\varepsilon_n})\widetilde{f}) g \mathrm{d}\nu \mathrm{d}\lambda,
\end{aligned}
$$

Now, arguing as in (i), by the dominated convergence theorem we deduce

$$
\lim_{n \to +\infty} \int_\Omega (T_{\varepsilon_n}(t)\widetilde{f}) g \mathrm{d}\nu = \frac{1}{2\pi i} \int_\Omega \int_\sigma \mathrm{e}^{\lambda t} (R(\lambda, L_\Omega)) f) g \mathrm{d}\lambda \mathrm{d}\nu = \int_\Omega (T_\Omega(t) f) g \mathrm{d}\nu.
$$

In a similar fashion, it is possible to prove that $D_H T_{\varepsilon_n}(t)\widetilde{f}$ weakly converges to $D_H T_\Omega(t) f$ in $L^2(\Omega, \nu; H)$ and that $D_H^2 T_{\varepsilon_n}(t)\widetilde{f}$ weakly converges to $D_H^2 T_\Omega(t) f$ in $L^2(\Omega, \nu; \mathcal{H}_2)$. □

3. Pointwise gradient estimates

In this section, we prove some pointwise gradient estimates for $T_\Omega(t)$. As already observed in Introduction, these estimates are interesting since, firstly, they represent a generalisation to what it is known in the literature and, secondly, they allow to deduce many properties of $T_\Omega(t)$ and of the associated invariant measure ν, as the results in Sect. 4 show.

THEOREM 3.1. *For any $p \in [1,+\infty)$ and $f \in D^{1,p}(\Omega,\nu)$*

$$|D_H T_\Omega(t) f|_H^p \le e^{-p\lambda_1^{-1}t}(T_\Omega(t)|D_H f|_H^p), \quad t>0,\ \nu\text{-a.e. in } \Omega. \tag{3.1}$$

Proof. First we prove the claim with $p=1$ and $f \in \mathcal{F}C_b^\infty(\Omega)$. Next we address to the general case.
Let $f \in \mathcal{F}C_b^\infty(X) \subseteq D^{1,2}(X,\nu)(\subseteq D^{1,2}(X,\nu_\varepsilon)$, for any $\varepsilon>0)$ and g a bounded, continuous and positive function. To overcome the lack of regularity of the function $|D_H T_\Omega(t) f|_H$ at its zeros, we replace it by $\eta_\sigma(|D_H T_\Omega(t) f|_H^2)$ where $\eta_\sigma:[0,+\infty)\to[0,+\infty)$ is the concave and smooth function defined by $\eta_\sigma(\xi):=\sqrt{\sigma+\xi}-\sqrt{\sigma}$ for any $\xi\ge 0$ and $\sigma>0$. Note that η_σ is Lipschitz continuous in $[0,+\infty)$ and satisfies

$$\text{(i)}\ \eta_\sigma(\xi)\le\sqrt{\xi},\quad \text{(ii)}\ \xi\eta_\sigma'(\xi)\ge\frac{1}{2}\eta_\sigma(\xi),\quad \text{(iii)}\ \eta_\sigma'(\xi)+2\xi\eta_\sigma''(\xi)\ge 0, \tag{3.2}$$

for any $\xi\ge 0$ and $\sigma>0$.

To proceed further, we need to control the third-order spatial derivatives of $T_\Omega(t)f$. Since we are not able to do that directly on $T_\Omega(t)f$, we replace it by the double indexed approximating sequence $(T_{\varepsilon_k}(t)f_n)_{n,k\in\mathbb{N}}$ where the sequences $(\varepsilon_k)_{k\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$ are as in Theorem 2.8. More precisely, ε_k vanishes as k goes to infinity, $(T_{\varepsilon_k}(t)f_n)\subseteq\mathcal{F}C_b^3(X)$ and $D_H f_n$ converges to $D_H f$ in $L^1(X,\nu_\varepsilon;H)$ as $n\to+\infty$. Hence, for any $t>0$, $\tau,s\in[0,t]$ and $k,n\in\mathbb{N}$ we define

$$w_\tau^{\varepsilon_k,n}:=|D_H u_{\varepsilon_k,n}(\tau)|_H^2,\qquad G(s)=G_{\sigma,h}^{\varepsilon_k,n}(s):=\int_X \eta_\sigma(w_{t-s}^{\varepsilon_k,n})T_{\varepsilon_k}(s)g\,\mathrm{d}\nu_{\varepsilon_k},$$

where, to simplify the notation, we have set $u_{\varepsilon_k,n}:=T_{\varepsilon_k}(\cdot)f_n$ for any $k,n\in\mathbb{N}$. Recall that $\nu_{\varepsilon_k}=\mathrm{e}^{-\Phi_{\varepsilon_k}}\gamma$ is the invariant measure associated with $T_{\varepsilon_k}(t)$ and that by the definition of the operator L_{ε_k} we get

$$\int_X \psi_1 L_{\varepsilon_k}\psi_2\,\mathrm{d}\nu_{\varepsilon_k}=-\int_X\langle D_H\psi_1,D_H\psi_2\rangle_H\,\mathrm{d}\nu_{\varepsilon_k}, \tag{3.3}$$

with $\psi_1\in D^{1,2}(X,\nu_{\varepsilon_k})$ and $\psi_2\in D(L_{\varepsilon_k})=D^{2,2}(X,\nu_{\varepsilon_k})$ (see [13, Theorem 6.2] for the characterisation of the domain of $D(L_{\varepsilon_k})$). Theorem 2.8 guarantees that, for every $t\ge 0$, the function $u_{\varepsilon_k,n}(t,\cdot)$ belongs to $\mathcal{F}C_b^3(X)$ and, as consequence, that G is differentiable in $(0,t)$. Thus, taking into account that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\eta_\sigma(w_{t-s}^{\varepsilon_k,n})&=\eta_\sigma'(w_{t-s}^{\varepsilon_k,n})\frac{\mathrm{d}}{\mathrm{d}s}|D_H u_{\varepsilon_k,n}(t-s)|_H^2\\&=-2\eta_\sigma'(w_{t-s}^{\varepsilon_k,n})\langle D_H u_{\varepsilon_k,n}(t-s),D_H(L_{\varepsilon_k}u_{\varepsilon_k,n}(t-s))\rangle_H\end{aligned}$$

and using (3.3) twice, we deduce

$$\begin{aligned}G'(s)=&-2\int_X\eta_\sigma'(w_{t-s}^{\varepsilon_k,n})\langle D_H u_{\varepsilon_k,n}(t-s),D_H(L_{\varepsilon_k}u_{\varepsilon_k,n}(t-s))\rangle_H T_{\varepsilon_k}(s)g\,\mathrm{d}\nu_{\varepsilon_k}\\&-\int_X\eta_\sigma'(w_{t-s}^{\varepsilon_k,n})\langle D_H T_{\varepsilon_k}(s)g,D_H w_{t-s}^{\varepsilon_k,n}\rangle_H\,\mathrm{d}\nu_{\varepsilon_k}\end{aligned}$$

$$
\begin{aligned}
&= -2\int_X \eta_\sigma'(w_{t-s}^{\varepsilon_k,n})\langle D_H u_{\varepsilon_k,n}(t-s), D_H(L_{\varepsilon_k}u_{\varepsilon_k,n}(t-s))\rangle_H T_{\varepsilon_k}(s)g\mathrm{d}\nu_{\varepsilon_k}\\
&\quad -\int_X \langle D_H(\eta_\sigma'(w_{t-s}^{\varepsilon_k,n})T_{\varepsilon_k}(s)g), D_H w_{t-s}^{\varepsilon_k,n}\rangle_H \mathrm{d}\nu_{\varepsilon_k}\\
&\quad +\int_X \eta_\sigma''(w_{t-s}^{\varepsilon_k,n})T_{\varepsilon_k}(s)g|D_H w_{t-s}^{\varepsilon_k,n}|_H^2\mathrm{d}\nu_{\varepsilon_k}\\
&= -2\int_X \eta_\sigma'(w_{t-s}^{\varepsilon_k,n})\langle D_H u_{\varepsilon_k,n}(t-s), D_H(L_{\varepsilon_k}u_{\varepsilon_k,n}(t-s))\rangle_H T_{\varepsilon_k}(s)g\mathrm{d}\nu_{\varepsilon_k}\\
&\quad +\int_X \eta_\sigma'(w_{t-s}^{\varepsilon_k,n})T_{\varepsilon_k}(s)gL_{\varepsilon_k}(w_{t-s}^{\varepsilon_k,n})\mathrm{d}\nu_{\varepsilon_k}+\int_X \eta_\sigma''(w_{t-s}^{\varepsilon_k,n})T_{\varepsilon_k}(s)g|D_H w_{t-s}^{\varepsilon_k,n}|_H^2\mathrm{d}\nu_{\varepsilon_k}\\
&=2\int_X \eta_\sigma'(w_{t-s}^{\varepsilon_k,n})T_{\varepsilon_k}(s)g\\
&\quad\times\left(\frac{1}{2}L_{\varepsilon_k}(w_{t-s}^{\varepsilon_k,n})-\langle D_H u_{\varepsilon_k,n}(t-s), D_H(L_{\varepsilon_k}u_{\varepsilon_k,n}(t-s))\rangle_H\right)\mathrm{d}\nu_{\varepsilon_k}\\
&\quad +\int_X \eta_\sigma''(w_{t-s}^{\varepsilon_k,n})T_{\varepsilon_k}(s)g|D_H w_{t-s}^{\varepsilon_k,n}|_H^2\mathrm{d}\nu_{\varepsilon_k}.
\end{aligned}\tag{3.4}
$$

Now, a straightforward computation and Hypothesis 1.1 yield that

$$
\begin{aligned}
&\frac{1}{2}L_{\varepsilon_k}(w_\cdot^{\varepsilon_k,n})-\langle D_H u_{\varepsilon_k,n}, D_H(L_{\varepsilon_k}u_{\varepsilon_k,n})\rangle_H\\
&\quad= |D_H^2 u_{\varepsilon_k,n}|_{\mathcal{H}_2}^2+\sum_{i=1}^{+\infty}\lambda_i^{-1}(D_i u_{\varepsilon_k,n})^2+\langle D_H^2\Phi_{\varepsilon_k}D_H u_{\varepsilon_k,n}, D_H u_{\varepsilon_k,n}\rangle_H\\
&\quad\geq |D_H^2 u_{\varepsilon_k,n}|_{\mathcal{H}_2}^2+\lambda_1^{-1}|D_H u_{\varepsilon_k,n}|_H^2+\langle D_H^2\Phi_{\varepsilon_k}D_H u_{\varepsilon_k,n}, D_H u_{\varepsilon_k,n}\rangle_H.
\end{aligned}
$$

In addition, it is easy to prove that

$$
|D_H w_\cdot^{\varepsilon_k,n}|_{\mathcal{H}_2}^2 = 4|D_H^2 u_{\varepsilon_k,n}D_H u_{\varepsilon_k,n}|_{\mathcal{H}_2}^2 \leq 4|D_H^2 u_{\varepsilon_k,n}|_{\mathcal{H}_2}^2 w_\cdot^{\varepsilon_k,n}. \tag{3.5}
$$

Thus, using (3.4) and (3.5), taking into account the convexity of Ω and U and the fact that $\eta_\sigma''\leq 0$ in $(0,+\infty)$ we deduce that

$$
\begin{aligned}
G'(s) &\geq 2\int_X [\eta_\sigma'(w_{t-s}^{\varepsilon_k,n})+2\eta_\sigma''(w_{t-s}^{\varepsilon_k,n})w_{t-s}^{\varepsilon_k,n}]T_{\varepsilon_k}(s)g|D_H^2 u_{\varepsilon_k,n}(t-s)|_{\mathcal{H}_2}^2\mathrm{d}\nu_{\varepsilon_k}\\
&\quad +2\lambda_1^{-1}\int_X \eta_\sigma'(w_{t-s}^{\varepsilon_k,n})w_{t-s}^{\varepsilon_k,n}T_{\varepsilon_k}(s)g\mathrm{d}\nu_{\varepsilon_k}\\
&\geq \lambda_1^{-1}\int_X \eta_\sigma(w_{t-s}^{\varepsilon_k,n})T_{\varepsilon_k}(s)g\mathrm{d}\nu_{\varepsilon_k}=\lambda_1^{-1}G(s),
\end{aligned}
$$

where in the last inequality we have used also (3.2)(ii)–(iii). Integrating the previous estimate with respect to s in $(0,t)$, we get $G(0)\leq \mathrm{e}^{-\lambda_1^{-1}t}G(t)$, and letting $\sigma\to 0$ we deduce

$$
\int_X |D_H u_{\varepsilon_k,n}(t)|_H g\mathrm{d}\nu_{\varepsilon_k} \leq \mathrm{e}^{-\lambda_1^{-1}t}\int_X |D_H f_n|_H T_{\varepsilon_k}(t)g\mathrm{d}\nu_{\varepsilon_k}. \tag{3.6}
$$

Proposition 1.10(vi), Remark 2.5 and formula (3.6) imply

$$\int_X |D_H u_{\varepsilon_k,n}(t)|_H g \mathrm{d}\nu_{\varepsilon_k} \le \mathrm{e}^{-\lambda_1^{-1}t} \int_X (T_{\varepsilon_k}(t)|D_H f_n|_H) g \mathrm{d}\nu_{\varepsilon_k}. \tag{3.7}$$

Since formula (3.7) holds true for every positive, bounded and continuous function g and the measures ν_ε and ν are equivalent, we get $|D_H u_{\varepsilon_k,n}(t)|_H \le \mathrm{e}^{-\lambda_1^{-1}t} T_{\varepsilon_k}(t)|D_H f_n|_H$, ν-a.e. in X for every $k, n \in \mathbb{N}$ and $t \ge 0$. From Theorem 2.8, up to subsequences, we get that $|D_H u_{\varepsilon_k,n}(t)|_H$ and $T_{\varepsilon_k}(t)|D_H f_n|_H$ pointwise converge ν-a.e. in Ω to $|D_H T_\Omega(t) f|_H$ and $T_\Omega(t)|D_H f|_H$, respectively, as $k, n \to +\infty$. This yields (3.1) with $p = 1$ and $f \in \mathfrak{F}C_b^\infty(\Omega)$. Formula (1.12) allows to extend the previous estimate to any $p \in (1, \infty)$.
Finally, let $f \in D^{1,p}(\Omega, \nu)$ and let $(g_n)_{n\in\mathbb{N}} \subseteq \mathfrak{F}C_b^\infty(\Omega)$ be a sequence converging to f in $D^{1,p}(\Omega, \nu)$ and pointwise ν-a.e. in Ω. Formula (3.1) with f replaced by $g_n - g_m$ and the invariance of ν with respect to $T_\Omega(t)$ give that the sequence $(D_H T_\Omega(t) g_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\Omega, \nu; H)$. Since $T_\Omega(t) g_n$ converges to $T_\Omega(t) f$ in $L^p(\Omega, \nu)$ and the operator D_H is closable in $L^p(\Omega, \nu)$, we obtain that $D_H T_\Omega(t) g_n$ converges to $D_H T_\Omega(t) f$ in $L^p(\Omega, \nu; H)$. Writing (3.1) with f replaced by g_n and letting $n \to +\infty$ yield the claim in the general case. □

COROLLARY 3.2. *For any $p \in (1, +\infty)$ and $f \in D^{1,p}(X, \nu)$, it holds that*

$$\lim_{t\to 0^+} \|D_H T_\Omega(t) f\|_{L^p(\Omega,\nu;H)} = \|D_H f\|_{L^p(\Omega,\nu;H)}.$$

Proof. By the strong continuity of $T_\Omega(t)$ and the lower semicontinuity of the L^p-norm of the gradient, we have

$$\|D_H f\|_{L^p(\Omega,\nu;H)} \le \liminf_{t\to 0^+} \|D_H T_\Omega(t) f\|_{L^p(\Omega,\nu;H)}.$$

Hence, by (3.1)

$$\begin{aligned}\int_\Omega |D_H f|_H^p \mathrm{d}\nu &\le \liminf_{t\to 0^+} \int_\Omega |D_H T_\Omega(t) f|_H^p \mathrm{d}\nu \le \limsup_{t\to 0^+} \int_\Omega |D_H T_\Omega(t) f|_H^p \mathrm{d}\nu \\ &\le \lim_{t\to 0^+} \mathrm{e}^{-p\lambda_1^{-1}t} \int_\Omega T_\Omega(t)|D_H f|_H^p \mathrm{d}\nu = \int_\Omega |D_H f|_H^p \mathrm{d}\nu\end{aligned}$$

and the proof is complete. □

Now we prove a pointwise gradient-function estimate for $T_\Omega(t) f$ whenever $f \in L^p(\Omega, \nu)$ and $p \in (1, \infty)$. The proof is similar to [29, Theorem 6.2.2]; however, it cannot be directly adapted to $T_\Omega(t) f$ in view of the possible lack of regularity of its derivatives. To overcome this difficulty and the additional complications due to the infinite-dimensional setting, we use again the approximants in Theorem 2.8.

THEOREM 3.3. *For $p \in (1, +\infty)$, $f \in L^p(\Omega, \nu)$ and $t > 0$ there exists a positive constant K_p, depending only on p, such that*

$$|D_H T_\Omega(t) f|_H^p \le K_p t^{-\frac{p}{2}} T_\Omega(t)|f|^p, \quad \nu\text{-a.e. in } \Omega. \tag{3.8}$$

As a consequence, we get

$$\|D_H T_\Omega(t)f\|_{L^p(\Omega,\nu;H)} \leq K_p^{\frac{1}{p}} t^{-\frac{1}{2}} \|f\|_{L^p(\Omega,\nu)}. \tag{3.9}$$

Proof. We remark that (3.9) is an easy consequence of (3.8), so it is enough to prove (3.8). We divide the proof in two steps. In the first step, we prove that if $f \in \mathcal{F}C_b^\infty(X)$, then for every $\varepsilon, s > 0$ and $p \in (1,2]$ there exists $K_p > 0$, depending only on p, such that

$$|D_H T_\varepsilon(s) f_n|_H^p \leq K_p s^{-\frac{p}{2}} T_\varepsilon(s)|f_n|^p, \quad \nu_\varepsilon\text{-a.e. in } X, \tag{3.10}$$

(see Theorem 2.8). In the second step, we prove (3.8) for any $p \in (1,\infty)$ and $f \in L^p(\Omega,\nu)$.

Step 1 Let us differentiate the function

$$G_{\delta,n}(t) = T_\varepsilon(s-t)\left(\left(|T_\varepsilon(t)f_n|^2+\delta\right)^{p/2} - \delta^{p/2}\right), \quad 0 < t < s,$$

where $\varepsilon, \delta > 0$ and $p \in (1,2]$. Setting $\phi_{\varepsilon,\delta,n}(t) := |T_\varepsilon(t)f_n|^2 + \delta$, we have

$$\begin{aligned} G'_{\delta,n}(t) =& - L_\varepsilon T_\varepsilon(s-t)\left(\left(\phi_{\varepsilon,\delta,n}(t)\right)^{p/2} - \delta^{p/2}\right) \\ &+ T_\varepsilon(s-t)\left(p\left(\phi_{\varepsilon,\delta,n}(t)\right)^{(p-2)/2}(T_\varepsilon(t)f_n)(L_\varepsilon T_\varepsilon(t)f_n)\right) \\ =& T_\varepsilon(s-t)\Big[-L_\varepsilon\left(\left(\phi_{\varepsilon,\delta,n}(t)\right)^{p/2} - \delta^{p/2}\right) \\ &+ p(T_\varepsilon(t)f_n)(L_\varepsilon T_\varepsilon(t)f_n)\left(\phi_{\varepsilon,\delta,n}(t)\right)^{(p-2)/2}\Big]. \end{aligned} \tag{3.11}$$

By Theorem 2.8, the function $\left(\phi_{\varepsilon,\delta,n}(t)\right)^{p/2} - \delta^{p/2}$ belongs to $\mathcal{F}C_b^3(X)$, hence from the definition of L_ε (see (2.11)) we get

$$\begin{aligned} L_\varepsilon\left(\left(\phi_{\varepsilon,\delta,n}(t)\right)^{p/2} - \delta^{p/2}\right) =& p\left(\phi_{\varepsilon,\delta,n}(t)\right)^{(p-2)/2}(T_\varepsilon(t)f_n)(L_\varepsilon T_\varepsilon(t)f_n) \\ &+ p\left(\phi_{\varepsilon,\delta,n}(t)\right)^{(p-2)/2}|D_H T_\varepsilon(t)f_n|_H^2 \\ &+ p(p-2)\left(\phi_{\varepsilon,\delta,n}(t)\right)^{(p-4)/2}(T_\varepsilon(t)f_n)^2|D_H T_\varepsilon(t)f_n|_H^2. \end{aligned} \tag{3.12}$$

Combining (3.11) and (3.12), we get

$$\begin{aligned} G'_{\delta,n}(t) =& - pT_\varepsilon(s-t)\left(\left(\phi_{\varepsilon,\delta,n}(t)\right)^{(p-2)/2}|D_H T_\varepsilon(t)f_n|_H^2\right) \\ &+ p(2-p)T_\varepsilon(s-t)\left(\left(\phi_{\varepsilon,\delta,n}(t)\right)^{(p-4)/2}(T_\varepsilon(t)f_n)^2|D_H T_\varepsilon(t)f_n|_H^2\right). \end{aligned}$$

Since the semigroup $(T_\varepsilon(t))_{t\geq 0}$ is positivity preserving (see Proposition 1.10(ii) and Remark 2.5), we get

$$G'_{\delta,n}(t) \leq p(1-p)T_\varepsilon(s-t)\left(\left(\phi_{\varepsilon,\delta,n}(t)\right)^{(p-2)/2}|D_H T_\varepsilon(t)f_n|_H^2\right). \tag{3.13}$$

Now integrating (3.13) from 0 to s with respect to t, we get

$$\begin{aligned}
&T_\varepsilon(s)\Big(\big(|f_n|^2+\delta\big)^{p/2}-\delta^{p/2}\Big)-\Big(|T_\varepsilon(s)f_n|^2+\delta\Big)^{p/2}+\delta^{p/2}\\
&\quad\le p(1-p)\int_0^s T_\varepsilon(s-t)\Big(\big(|T_\varepsilon(t)f_n|^2+\delta\big)^{(p-2)/2}|D_HT_\varepsilon(t)f_n|_H^2\Big)\mathrm{d}t.
\end{aligned}$$

Using again that $(T_\varepsilon(t))_{t\ge0}$ is positivity preserving, from the previous inequality we get

$$p(p-1)\int_0^s T_\varepsilon(s-t)\Big(\big(|T_\varepsilon(t)f_n|^2+\delta\big)^{(p-2)/2}|D_HT_\varepsilon(t)f_n|_H^2\Big)\mathrm{d}t\le\Big(|T_\varepsilon(s)f_n|^2+\delta\Big)^{p/2}. \tag{3.14}$$

By the semigroup property, (3.1), (1.12), (1.13), Remark 2.5 and the Young inequality, we get for every $\eta>0$

$$\begin{aligned}
|D_HT_\varepsilon(s)f_n|_H^p&=|D_HT_\varepsilon(s-t)T_\varepsilon(t)f_n|_H^p\\
&\le \mathrm{e}^{-p\lambda_1^{-1}(s-t)}T_\varepsilon(s-t)|D_HT_\varepsilon(t)f_n|_H^p\\
&\le \mathrm{e}^{-p\lambda_1^{-1}(s-t)}T_\varepsilon(s-t)\Big(\big(\phi_{\varepsilon,\delta,n}(t)\big)^{-\frac{p(2-p)}{4}}|D_HT_\varepsilon(t)f_n|_H^p\big(\phi_{\varepsilon,\delta,n}(t)\big)^{\frac{p(2-p)}{4}}\Big)\\
&\le \mathrm{e}^{-p\lambda_1^{-1}(s-t)}\Big(T_\varepsilon(s-t)\Big(\big(\phi_{\varepsilon,\delta,n}(t)\big)^{\frac{p}{2}-1}|D_HT_\varepsilon(t)f_n|_H^2\Big)\Big)^{p/2}\\
&\qquad\cdot\Big(T_\varepsilon(s-t)\big(\phi_{\varepsilon,\delta,n}(t)\big)^{\frac{p}{2}}\Big)^{1-\frac{p}{2}}\\
&\le \mathrm{e}^{-p\lambda_1^{-1}(s-t)}\frac{p}{2}\eta^{2/p}T_\varepsilon(s-t)\Big(\big(\phi_{\varepsilon,\delta,n}(t)\big)^{\frac{p}{2}-1}|D_HT_\varepsilon(t)f_n|_H^2\Big)\\
&\quad+\mathrm{e}^{-p\lambda_1^{-1}(s-t)}\Big(1-\frac{p}{2}\Big)\eta^{2/(p-2)}T_\varepsilon(s-t)\Big(|T_\varepsilon(t)f_n|^p+\delta^{p/2}\Big)\\
&\le \mathrm{e}^{-p\lambda_1^{-1}(s-t)}\frac{p}{2}\eta^{2/p}T_\varepsilon(s-t)\Big(\big(\phi_{\varepsilon,\delta,n}(t)\big)^{\frac{p}{2}-1}|D_HT_\varepsilon(t)f_n|_H^2\Big)\\
&\quad+\mathrm{e}^{-p\lambda_1^{-1}(s-t)}\Big(1-\frac{p}{2}\Big)\eta^{2/(p-2)}T_\varepsilon(s-t)\Big(T_\varepsilon(t)|f_n|^p+\delta^{p/2}\Big)
\end{aligned}\tag{3.15}$$

Multiplying (3.15) by $\mathrm{e}^{p\lambda_1^{-1}(s-t)}$, integrating from 0 to s with respect to t, and recalling (3.14) we get

$$\begin{aligned}
\frac{\mathrm{e}^{p\lambda_1^{-1}(s-t)}-1}{p\lambda_1^{-1}}|D_HT_\varepsilon(s)f_n|_H^p&\le\frac{\eta^{2/p}}{2(p-1)}\Big(|T_\varepsilon(s)f_n|^2+\delta\Big)^{p/2}\\
&\quad+\Big(1-\frac{p}{2}\Big)\eta^{2/(p-2)}s\Big(T_\varepsilon(s)|f_n|^p+\delta^{p/2}\Big).
\end{aligned}$$

Letting $\delta\to0^+$ and applying (1.12), we obtain

$$\frac{\mathrm{e}^{p\lambda_1^{-1}(s-t)}-1}{p\lambda_1^{-1}}|D_HT_\varepsilon(s)f_n|_H^p\le\Big(\frac{\eta^{2/p}}{2(p-1)}+\Big(1-\frac{p}{2}\Big)\eta^{2/(p-2)}s\Big)T_\varepsilon(s)|f_n|^p$$

whence

$$\frac{e^{p\lambda_1^{-1}(s-t)}-1}{p\lambda_1^{-1}}|D_H T_\varepsilon(s) f_n|_H^p \le \min_{\eta>0}\left\{\frac{\eta^{2/p}}{2(p-1)}+\left(1-\frac{p}{2}\right)\eta^{2/(p-2)}s\right\}T_\varepsilon(s)|f_n|^p$$
$$=: c_p s^{1-\frac{p}{2}}T_\varepsilon(s)|f_n|^p.$$

for some positive constant c_p depending only on p. Setting $t=0$, and recalling that the function $s/(e^{p\lambda_1^{-1}s}-1)$ is bounded from above, we get (3.10).

Step 2 If $p\in(2,\infty)$ it suffices to write $|D_H T_\varepsilon(s) f_n|_H^p=(|D_H T_\varepsilon(s) f_n|_H^2)^{p/2}$ and to apply (3.10) with $p=2$. Then, using (1.13) together with Remark 2.5, we get (3.10) for every $p\in(1,\infty)$. Due to the properties listed in Theorem 2.8, letting $n\to+\infty$ and $\varepsilon\to 0$, up to a subsequence we get (3.8) for every $f\in\mathcal{F}C_b^\infty(X)$. Moreover, integrating it on Ω and using that ν is the invariant measure associated with $T_\Omega(t)$, we get

$$\int_\Omega |D_H T_\Omega(s) f|_H^p d\nu \le K_p s^{-\frac{p}{2}}\int_\Omega |f|^p d\nu. \tag{3.16}$$

for any $f\in\mathcal{F}C_b^\infty(\Omega)$ and $p\in(1,\infty)$. Finally, we extend estimate (3.16) to any $f\in L^p(\Omega,\nu)$ arguing by approximation as in the last part of the proof of Theorem 3.1. To this aim, let $f\in L^p(\Omega,\nu)$ and let $(g_n)_{n\in\mathbb{N}}$ be a sequence of functions in $\mathcal{F}C_b^\infty(\Omega)$ converging to f in $L^p(\Omega,\nu)$. Then, for every $n,k\in\mathbb{N}$

$$\int_\Omega |D_H T_\Omega(s) g_n - D_H T_\Omega(s) g_k|_H^p d\nu \le K_p s^{-\frac{p}{2}}\int_\Omega |g_n-g_k|^p d\nu.$$

So the sequence $(D_H T_\Omega(s) g_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\Omega,\nu;H)$. The closability of the operator $D_H:\mathcal{F}C_b^\infty(\Omega)\to L^p(\Omega,\nu)$ in $L^p(\Omega,\nu)$ and the fact that for any $s>0$ the sequence $(T_\Omega(s)g_n)_{n\in\mathbb{N}}$ converges to $T_\Omega(s)f$ we get that $\lim_{n\to+\infty} D_H T_\Omega(s) g_n = D_H T_\Omega(s) f$ in $L^p(\Omega,\nu;H)$. Hence, writing (3.16) with f replaced by g_n and letting $n\to+\infty$, we conclude. □

The pointwise gradient estimate (3.1) implies that $\|D_H T_\Omega(t) f\|_{L^p(\Omega,\nu;H)}$ vanishes as $t\to+\infty$ and $f\in D^{1,p}(\Omega,\nu)$. Actually using (3.8), we get the same result when f belongs to $L^p(\Omega,\nu)$.

COROLLARY 3.4. *Let $p\in(1,\infty)$ and $t\ge 1$. For every $f\in L^p(\Omega,\nu)$*

$$\|D_H T_\Omega(t) f\|_{L^p(\Omega,\nu;H)} \le C_p e^{-\lambda_1^{-1}t}\|f\|_{L^p(\Omega,\nu)},$$

where $C_p=K_p^{1/p}e^{\lambda_1^{-1}}$ and K_p is the positive constant in Theorem 3.3.

Proof. By (3.1), (3.9), the semigroup property and the fact that ν is invariant with respect to $T_\Omega(t)$ we get

$$\begin{aligned}\int_\Omega |D_H T_\Omega(t) f|_H^p \mathrm{d}\nu &= \int_\Omega |D_H T_\Omega(t-1) T_\Omega(1) f|_H^p \mathrm{d}\nu \\ &\leq \mathrm{e}^{-p\lambda_1^{-1}(t-1)} \int_\Omega T_\Omega(t-1)|D_H T_\Omega(1) f|_H^p \mathrm{d}\nu \\ &\leq \mathrm{e}^{-p\lambda_1^{-1}(t-1)} \int_\Omega |D_H T_\Omega(1) f|_H^p \mathrm{d}\nu \\ &\leq K_p \mathrm{e}^{-p\lambda_1^{-1}(t-1)} \int_\Omega T_\Omega(1)|f|^p \mathrm{d}\nu \\ &\leq K_p \mathrm{e}^{-p\lambda_1^{-1}(t-1)} \int_\Omega |f|^p \mathrm{d}\nu,\end{aligned}$$

for any $t \geq 1,\ f \in L^p(\Omega, \nu)$. This concludes the proof. □

4. Logarithmic Sobolev inequality and other consequences

Logarithmic Sobolev inequalities are important tools in the study of Gaussian Sobolev spaces since they represent the counterpart of the Sobolev embeddings which in general fail to hold when the Lebesgue measure is replaced by other measures, as for example the Gaussian one. In infinite dimension, such inequalities are known for the Gaussian measure on the whole space (see [11, Theorem 5.5.1]) and on convex domains (see [12, Proposition 3.5]). In the weighted Gaussian case, the inequality is known in the whole space (see [21, Proposition 11.2.19]), for Fréchet differentiable functions. In this section, we use the pointwise gradient estimates (3.1) and (3.8) to prove logarithmic Sobolev inequalities for weighted Gaussian measures on convex domains generalising all the above results. We also collect some consequences of the logarithmic Sobolev inequality (4.4). To simplify the notation we set, if $f \in L^1(X, \nu_\varepsilon)$ and $g \in L^1(X, \nu)$

$$m_\varepsilon(f) := \frac{1}{\nu_\varepsilon(X)} \int_X f \mathrm{d}\nu_\varepsilon, \quad m_\Omega(g) := \frac{1}{\nu(\Omega)} \int_\Omega g \mathrm{d}\nu. \tag{4.1}$$

First of all, we study the asymptotic behaviour of the semigroup $(T_\varepsilon(t))_{t\geq 0}$.

LEMMA 4.1. *For any $\varepsilon > 0$ and $f \in \mathcal{F}C_b^1(X)$*

$$\lim_{t\to+\infty} T_\varepsilon(t) f(x) = m_\varepsilon(f), \quad \nu_\varepsilon\text{-a.e. } x \in X. \tag{4.2}$$

In addition, if $f \leq 1$ and has a positive infimum, then

$$\begin{aligned}\lim_{t\to+\infty} \int_X (T_\varepsilon(t) f) \log(T_\varepsilon(t) f) d\nu_\varepsilon &= \left(\int_X f d\nu_\varepsilon\right) \log\left(m_\varepsilon(f)\right) \\ &= \nu_\varepsilon(X) m_\varepsilon(f) \log\left(m_\varepsilon(f)\right). \end{aligned} \tag{4.3}$$

Proof. First of all note that since the function $(0,1] \ni x \mapsto x\,|\log x|$ has a maximum, formula (4.3) can be obtained by (4.2) and the dominated convergence theorem. The proof of (4.2) is divided in three steps.

Step 1 Let us show that there exists a sequence $(t_k)_{k\in\mathbb{N}} \subseteq [0,+\infty)$, such that $t_k \to +\infty$ as $k \to +\infty$ and $T_\varepsilon(t_k)f \to g_\varepsilon$ weakly in $L^2(X,\nu_\varepsilon)$ for some $g_\varepsilon \in L^2(X,\nu_\varepsilon)$, as k goes to infinity. To do this, it is sufficient to consider a sequence $(t_n)_{n\in\mathbb{N}}$ tending to $+\infty$ as $n \to +\infty$ and to recall that $T_\varepsilon(t_n)$ is a contraction in $L^2(X,\nu_\varepsilon)$.

Step 2 Here we claim that g_ε is H-invariant, i.e. $g_\varepsilon(x+h) = g_\varepsilon(x)$ for γ-a.e. $x \in X$ and for every $h \in H$. For any $\varphi \in C_b(X)$, we have

$$
\begin{aligned}
&\left|\int_X \big[g_\varepsilon(x+h) - g_\varepsilon(x)\big]\varphi(x)\mathrm{d}\nu_\varepsilon(x)\right| \\
&\quad\le \left|\int_X \big[g_\varepsilon(x+h) - (T_\varepsilon(t_k)f)(x+h)\big]\varphi(x)\mathrm{d}\nu_\varepsilon(x)\right| && (I_1)\\
&\qquad+ \left|\int_X \big[(T_\varepsilon(t_k)f)(x+h) - (T_\varepsilon(t_k)f_n)(x+h)\big]\varphi(x)\mathrm{d}\nu_\varepsilon(x)\right| && (I_2)\\
&\qquad+ \left|\int_X \big[(T_\varepsilon(t_k)f_n)(x+h) - (T_\varepsilon(t_k)f_n)(x)\big]\varphi(x)\mathrm{d}\nu_\varepsilon(x)\right| && (I_3)\\
&\qquad+ \left|\int_X \big[(T_\varepsilon(t_k)f_n)(x) - (T_\varepsilon(t_k)f)(x)\big]\varphi(x)\mathrm{d}\nu_\varepsilon(x)\right| && (I_4)\\
&\qquad+ \left|\int_X \big[(T_\varepsilon(t_k)f)(x) - g_\varepsilon(x)\big]\varphi(x)\mathrm{d}\nu_\varepsilon(x)\right| && (I_5)
\end{aligned}
$$

where $(f_n)_{n\in\mathbb{N}}$ is the sequence in Theorem 2.8. The regularity of $T_\varepsilon(t_k)f_n$ and (3.7) allow us to estimate (I_3) as follows

$$
\begin{aligned}
(I_3) &= \left|\int_X \left(\int_0^1 \langle D_H T_\varepsilon(t_k) f_n(x+sh), h\rangle_H \mathrm{d}s\right)\varphi(x)\mathrm{d}\nu_\varepsilon(x)\right| \\
&\le \mathrm{e}^{-\lambda_1^{-1}t_k}|h|_H\|\varphi\|_\infty \int_0^1\int_X (T_\varepsilon(t_k)|D_H f_n|_H)(x+sh)\mathrm{d}\nu_\varepsilon(x)\mathrm{d}s \\
&\le \mathrm{e}^{-\lambda_1^{-1}t_k}|h|_H\|\varphi\|_\infty \int_0^1\int_X |D_H f_n(x+sh)|_H\mathrm{d}\nu_\varepsilon(x)\mathrm{d}s \\
&\le \mathrm{e}^{-\lambda_1^{-1}t_k}|h|_H\|\varphi\|_\infty \left(\int_0^1\int_X |D_H f(x+sh)|_H\mathrm{d}\nu_\varepsilon(x)\mathrm{d}s + M\right) \\
&\le \mathrm{e}^{-\lambda_1^{-1}t_k}|h|_H\|\varphi\|_\infty (\nu_\varepsilon(X)\|D_H f\|_\infty + M),
\end{aligned}
$$

for some positive M, where in the second to last line we took into account that $\|D_H f_n\|_{L^1(X,\nu_\varepsilon;H)}$ converges to $\|D_H f\|_{L^1(X,\nu_\varepsilon;H)}$ as $n \to +\infty$. Now, for every $\eta > 0$ we can choose k large enough such that $(I_1)+(I_3)+(I_5) \le \eta/2$ and n such that $(I_2)+(I_4) \le \eta/2$. This proves the claim.

Step 3 In this step, we complete the proof. By [11, Theorem 2.5.2], a H-invariant function coincides γ-a.e. in X (hence ν-a.e. in X as well) with a constant function, i.e. there exists $c \in \mathbb{R}$ such that $g_\varepsilon(x) = c$ for γ-a.e. $x \in X$. We get

$$c = \frac{1}{\nu_\varepsilon(X)} \int_X c \mathrm{d}\nu_\varepsilon = \frac{1}{\nu_\varepsilon(X)} \lim_{k \to +\infty} \int_X T_\varepsilon(t_k) f \mathrm{d}\nu_\varepsilon = m_\varepsilon(f)$$

where in the last equality we used the invariance of ν_ε with respect to $T_\varepsilon(t)$. Since our arguments are independent of the sequence $(t_k)_{k\in\mathbb{N}}$, we get (4.2). □

REMARK 4.2. In view of the method used in the proof, the results in Lemma 4.1 cannot be easily extended to the semigroup $T_\Omega(t)$. However, as we prove in Proposition 4.7, the asymptotic behaviour of $T_\Omega(t)$ as $t \to +\infty$ can be obtained also with a precise decay estimate.

Now we are ready to prove that the measure ν satisfies a logarithmic Sobolev inequality in Ω. The idea of the proof is to apply the Deuschel and Stroock method (see [22]) to the measure ν_ε and then taking the limit as $\varepsilon \to 0$.

PROPOSITION 4.3. *For $p \in [1, \infty)$ and $f \in \mathcal{F}C_b^1(\Omega)$, the following inequality holds:*

$$\int_\Omega |f|^p \log |f|^p d\nu \le \nu(\Omega) m_\Omega(|f|^p) \log \left(m_\Omega(|f|^p)\right) + \frac{p^2\lambda_1}{2} \int_\Omega |f|^{p-2} |D_H f|_H^2 \chi_{\{f\neq 0\}} d\nu. \tag{4.4}$$

Proof. We split the proof in two parts. In the first part, we prove the claim when f satisfies some additional hypotheses, and in the second part we show (4.4) in its full generality.

Step 1 Here we prove (4.4) with ν and Ω replaced by ν_ε and X, and f in $\mathcal{F}C_b^1(X)$ such that there exists a positive constant c with $c \le f \le 1$. To this aim, we consider the function

$$F_\varepsilon(t) = \int_X (T_\varepsilon(t) f^p) \log(T_\varepsilon(t) f^p) \mathrm{d}\nu_\varepsilon, \qquad t \ge 0.$$

which is well defined thanks to Proposition 1.10(ii)–(iii) and Remark 2.5.

Our aim is to find a bound from below for the derivative of F_ε. Indeed, we show that $F_\varepsilon'(t) \ge c_1 \mathrm{e}^{-c_2 t} \int_X f^{p-2} |D_H f|_H^2 \mathrm{d}\nu_\varepsilon$, for some positive constants c_1 and c_2. We start by observing that

$$\begin{aligned}
F_\varepsilon'(t) &= \int_X (L_\varepsilon T_\varepsilon(t) f^p) \log(T_\varepsilon(t) f^p) \mathrm{d}\nu_\varepsilon + \int_X L_\varepsilon T_\varepsilon(t) f^p \mathrm{d}\nu_\varepsilon \\
&= - \int_X \left\langle D_H T_\varepsilon(t) f^p, D_H \log(T_\varepsilon(t) f^p) \right\rangle_H \mathrm{d}\nu_\varepsilon \\
&= - \int_X \frac{|D_H T_\varepsilon(t) f^p|_H^2}{T_\varepsilon(t) f^p} \mathrm{d}\nu_\varepsilon
\end{aligned}$$

where we used that $\int_X L_\varepsilon\varphi \mathrm{d}\nu_\varepsilon = 0$ for any $\varphi \in D(L_\varepsilon)$, the definition of L_ε and the integration by parts formula. By (1.13) and Remark 2.5, we have $T_\varepsilon(t)|D_H f^p|_H \le \left(T_\varepsilon(t)\frac{|D_H f^p|_H^2}{f^p}\right)^{1/2}(T_\varepsilon(t)f^p)^{1/2}$. Hence, by using (3.1) we deduce

$$F_\varepsilon'(t) \ge -\mathrm{e}^{-2\lambda_1^{-1}t}\int_X \frac{(T_\varepsilon(t)|D_H f^p|_H)^2}{T_\varepsilon(t)f^p}\mathrm{d}\nu_\varepsilon \ge -\mathrm{e}^{-2\lambda_1^{-1}t}\int_X T_\varepsilon(t)\left(\frac{|D_H f^p|_H^2}{f^p}\right)\mathrm{d}\nu_\varepsilon$$
$$= -\mathrm{e}^{-2\lambda_1^{-1}t}p^2\int_X f^{p-2}|D_H f|_H^2\mathrm{d}\nu_\varepsilon.$$

Integrating from 0 to $+\infty$ and using (4.3), we get

$$\int_X f^p\log f^p \mathrm{d}\nu_\varepsilon \le \left(\int_X f^p\mathrm{d}\nu_\varepsilon\right)\log\left(m_\varepsilon(f^p)\right) + \frac{p^2\lambda_1}{2}\int_X f^{p-2}|D_H f|_H^2\mathrm{d}\nu_\varepsilon.$$

Finally letting $\varepsilon \to 0$ and recalling that ν_ε weakly* converges to $\chi_\Omega\nu$, we get the claim.

Step 2 Now, for any $f \in \mathcal{F}C_b^1(\Omega)$ and $n \in \mathbb{N}$, let consider the sequence $(f_n)_{n\in\mathbb{N}}$ defined by $f_n = (1+\|f\|_\infty)^{-1}\sqrt{f^2+n^{-1}}$. Step 1 yields that

$$\int_\Omega f_n^p\log(f_n^p)\mathrm{d}\nu \le \left(\int_\Omega f_n^p\mathrm{d}\nu\right)\log\left(m_\Omega(f_n^p)\right) + \frac{p^2\lambda_1}{2}\int_\Omega f_n^{p-2}|D_H f_n|_H^2\mathrm{d}\nu. \tag{4.5}$$

Observing that there exists a positive constant $c_{n,p}$ such that $c_{n,p} \le f_n^p \le 1$ for any $n \in \mathbb{N}$ and using the fact that the function $x \mapsto x|\log x|$ is bounded in $(0, 1]$, by the dominated convergence theorem the left-hand side of (4.5) converges to

$$(1+\|f\|_\infty)^{-p}\int_\Omega |f|^p\log\left[(1+\|f\|_\infty)^{-p}|f|^p\right]\mathrm{d}\nu,$$

and the first term in the right-hand side of (4.5) converges to

$$\left((1+\|f\|_\infty)^{-p}\int_\Omega |f|^p\mathrm{d}\nu\right)\log\left(\frac{m_\Omega(|f|^p)}{(1+\|f\|_\infty)^p}\right).$$

Since $|D_H f_n|_H \le (1+\|f\|_\infty)^{-1}|D_H f|_H$ for every $n \in \mathbb{N}$, by the monotone convergence theorem if $p \in [1, 2)$, and by Lebesgue's dominated convergence theorem otherwise, we obtain

$$\lim_{n\to+\infty}\int_\Omega {f_n}^{p-2}|D_H f_n|_H^2\mathrm{d}\nu = (1+\|f\|_\infty)^{-p}\int_\Omega |f|^{p-2}|D_H f|_H^2\chi_{\{f\neq 0\}}\mathrm{d}\nu.$$

So the statement follows letting n to infinity in (4.5). $\square$

As it is well known the logarithmic Sobolev inequality has several interesting consequences. Among them, we point out the following, related to our setting: once a log-Sobolev inequality with respect to the measure ν has been proved, a summability improving property of $T_\Omega(t)$ follows. Indeed, we are able to show that $T_\Omega(t)$ maps $L^q(\Omega,\nu)$ into $L^p(\Omega,\nu)$ for some $p>q$. The technique used to prove this property is quite standard. However, for the sake of completeness, we provide a proof of it.

PROPOSITION 4.4. *Let $t>0$ and $p,q\in(1,+\infty)$ be such that $p\le(q-1)e^{2\lambda_1^{-1}t}+1$. Then, the operator $T_\Omega(t)$ maps $L^q(\Omega,\nu)$ in $L^p(\Omega,\nu)$ and*

$$\|T_\Omega(t)f\|_{L^p(\Omega,\nu)}\le[\nu(\Omega)]^{\frac{1}{p}-\frac{1}{q}}\|f\|_{L^q(\Omega,\nu)},\quad t>0,\ f\in L^q(\Omega,\nu). \tag{4.6}$$

Proof. Let $f\in\mathcal{F}C_b^1(\Omega)$, with a positive global infimum, and let $p(t):=(q-1)\mathrm{e}^{2\lambda_1^{-1}t}+1$. For $s\ge0$, we set

$$G(s):=\left(\frac{1}{\nu(\Omega)}\int_\Omega(T_\Omega(s)f)^{p(s)}d\nu\right)^{1/p(s)}=:\left(\frac{1}{\nu(\Omega)}F(s)\right)^{1/p(s)}$$

and we prove that G is a non-increasing function in $(0,+\infty)$. Before starting we want to recall that $T_\Omega(s)$ maps $\mathcal{F}C_b^1(\Omega)$ into $D^{1,2}(\Omega,\nu)\cap L^\infty(\Omega,\nu)$, due to the definition of the operator $T_\Omega(s)$ and Proposition 1.10(ii). This guarantees that all the integrals we are going to write are well defined and finite. So, using (1.9), we get

$$\begin{aligned}F'(s)=p'(s)\int_\Omega(T_\Omega(s)f)^{p(s)}\log(T_\Omega(s)f)\mathrm{d}\nu-p(s)(p(s)-1)\\ \times\int_\Omega(T_\Omega(s)f)^{p(s)-2}|D_HT_\Omega(s)f|_H^2\mathrm{d}\nu.\end{aligned} \tag{4.7}$$

Now we set $u(s):=T_\Omega(s)f$, and we differentiate the function G. Taking into account (4.7), we get

$$\begin{aligned}G'&=G\Bigg(-\frac{p'}{p^2}\log(m_\Omega(u^p))+\frac{1}{p\int_\Omega u^p\mathrm{d}\nu}\\&\qquad\times\left(p'\int_\Omega u^p\log u\mathrm{d}\nu-p(p-1)\int_\Omega u^{p-2}|D_Hu|_H^2\mathrm{d}\nu\right)\Bigg)\\&=G\frac{p'}{p^2\int_\Omega u^p\mathrm{d}\nu}\left(-\left(\int_\Omega u^p\mathrm{d}\nu\right)\log\left(m_\Omega(u^p)\right)+\int_\Omega u^p\log u^p\mathrm{d}\nu\right)\\&\qquad-\frac{G(p-1)}{\int_\Omega u^p\mathrm{d}\nu}\int_\Omega u^{p-2}|D_Hu|_H^2\mathrm{d}\nu.\end{aligned}$$

Since $p'(s)=2\lambda_1^{-1}(q-1)\mathrm{e}^{2\lambda_1^{-1}s}\ge0$, we can apply (4.4) to get

$$G'(s)\le(G(s))^{1-p(s)}\left(\frac{p'(s)\lambda_1}{2}-(p(s)-1)\right)\int_\Omega(T_\Omega(s)f)^{p(s)-2}|D_HT_\Omega(s)f|_H^2\mathrm{d}\nu=0.$$

This proves that G is a decreasing function, which means that $G(0)\ge G(t)$ for every $t>0$, i.e.

$$\|T_\Omega(t)f\|_{L^{p(t)}(\Omega,\nu)}\le[\nu(\Omega)]^{\frac{1}{p(t)}-\frac{1}{q}}\|f\|_{L^q(\Omega,\nu)}.$$

So we get (4.6) for a function $f \in \mathcal{F}C_b^1(\Omega)$ with positive global infimum. Indeed, if $p < p(t)$

$$\begin{aligned}\|T_\Omega(t)f\|_{L^p(\Omega,\nu)} &\le [\nu(\Omega)]^{\frac{p(t)-p}{p(t)p}} \|T_\Omega(t)f\|_{L^{p(t)}(\Omega,\nu)} \\ &\le [\nu(\Omega)]^{\frac{p(t)-p}{p(t)p}} [\nu(\Omega)]^{\frac{1}{p(t)}-\frac{1}{q}} \|f\|_{L^q(\Omega,\nu)} = [\nu(\Omega)]^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q(\Omega,\nu)}.\end{aligned}$$

Arguing as in the second step of the proof of Proposition 4.3, we obtain (4.6) for a general $f \in \mathcal{F}C_b^1(\Omega)$. The density of the space $\mathcal{F}C_b^1(\Omega)$ in $L^q(\Omega,\nu)$ allows us to conclude the proof. □

From the logarithmic Sobolev inequality follows the asymptotic behaviour of $T_\Omega(t)f$ as t goes to infinity, whenever f belongs to $L^2(\Omega,\nu)$. This can be done thanks to the Poincaré inequality.

PROPOSITION 4.5. *Let $p \in [2,\infty)$ and $f \in D^{1,p}(\Omega,\nu)$. Then*

$$\|f - m_\Omega(f)\|_{L^p(\Omega,\nu)} \le K\|D_H f\|_{L^p(\Omega,\nu;H)}, \tag{4.8}$$

where K is a positive constant depending only on p, λ_1 and $\nu(\Omega)$. Furthermore, if $p = 2$, then $K = \lambda_1^{1/2}$.

Proof. We divide the proof in two steps. In the first step, we prove (4.8) for $p = 2$, while in the second step we prove the claim for $p \in (2,\infty)$.

Step 1 We use an idea of [38] (see also [4, Theorem 5.2]). Let $f \in \mathcal{F}C_b^1(\Omega)$, $\eta > 0$ and consider the function $f_\eta = 1 + \eta(f - m_\Omega(f))$. Recalling that $(1+\xi)^2 \log(1+\xi)^2 = 2\xi + 3\xi^2 + o(\xi^2)$ as $\xi \to 0$, we get

$$\int_\Omega f_\eta^2 \log f_\eta^2 \mathrm{d}\nu - \left(\int_\Omega f_\eta^2 \mathrm{d}\nu\right) \log\left(m_\Omega(f_\eta^2)\right) = 2\eta^2 \int_\Omega (f - m_\Omega(f))^2 \mathrm{d}\nu + o(\eta^2).$$

By (4.4), with $p = 2$ and f replaced by f_η, we get

$$2\eta^2 \int_\Omega (f - m_\Omega(f))^2 \mathrm{d}\nu + o(\eta^2) \le 2\lambda_1 \int_\Omega |D_H f_\eta|_H^2 \mathrm{d}\nu = 2\lambda_1\eta^2 \int_\Omega |D_H f|_H^2 \mathrm{d}\nu.$$

Letting $\eta \to 0^+$, we get (4.8) for a function f belonging to $\mathcal{F}C_b^1(\Omega)$. Then by the density of $\mathcal{F}C_b^1(\Omega)$ in $D^{1,2}(\Omega,\nu)$, we get

$$\int_\Omega (f - m_\Omega(f))^2 \mathrm{d}\nu \le \lambda_1 \int_\Omega |D_H f|_H^2 \mathrm{d}\nu, \quad f \in D^{1,2}(\Omega,\nu). \tag{4.9}$$

Step 2 Now let assume that $p \in (2,\infty)$. If $g \in D^{1,p}(\Omega,\nu)$, then $|g|^{p/2} \in D^{1,2}(\Omega,\nu)$. This can be seen by approximating g by a sequence of functions in $\mathcal{F}C_b^1(\Omega)$, which is dense in $D^{1,p}(\Omega,\nu)$. Applying (4.9), with f replaced by $|g|^{p/2}$, we get

$$\int_\Omega |g|^p \mathrm{d}\nu - \frac{1}{\nu(\Omega)}\left(\int_\Omega |g|^{p/2} \mathrm{d}\nu\right)^2 \le \frac{\lambda_1 p^2}{4} \int_\Omega |g|^{p-2} |D_H g|_H^2 \mathrm{d}\nu \tag{4.10}$$

Applying the Young inequality to the right-hand side of (4.10), for every $\eta > 0$ we have

$$\int_\Omega |g|^p \mathrm{d}\nu \le \frac{\lambda_1 p(p-2)\eta^{p/(p-2)}}{4}\int_\Omega |g|^p \mathrm{d}\nu + \frac{\lambda_1 p}{2\eta^{p/2}}\int_\Omega |D_H g|_H^p \mathrm{d}\nu + \frac{1}{\nu(\Omega)}\left(\int_\Omega |g|^{p/2}\mathrm{d}\nu\right)^2.$$

Choosing $\eta > 0$ such that $\eta^{p/(p-2)} \le 4/(\lambda_1 p(p-2))$ and $K(p,\eta) := 1-(\lambda_1 p(p-2)\eta^{p/(p-2)})/4$, we deduce

$$K(p,\eta)\int_\Omega |g|^p \mathrm{d}\nu \le \frac{\lambda_1 p}{2\eta^{p/2}}\int_\Omega |D_H g|_H^p \mathrm{d}\nu + \frac{1}{\nu(\Omega)}\left(\int_\Omega |g|^{p/2}\mathrm{d}\nu\right)^2. \tag{4.11}$$

Now we proceed by induction. If $p \in (2,4)$, then

$$\int_\Omega |g|^{p/2}\mathrm{d}\nu \le \left(\int_\Omega |g|^2 \mathrm{d}\nu\right)^{p/4}[\nu(\Omega)]^{(4-p)/4}$$

and so by (4.11), for every $p \in (2,4]$

$$K(p,\eta)\int_\Omega |g|^p \mathrm{d}\nu \le \frac{\lambda_1 p}{2\eta^{p/2}}\int_\Omega |D_H g|_H^p \mathrm{d}\nu + \frac{1}{[\nu(\Omega)]^{(2-p)/2}}\left(\int_\Omega |g|^2\mathrm{d}\nu\right)^{p/2}.$$

If we let $g = f - m_\Omega(f)$ for a function $f \in D^{1,p}(\Omega,\nu)$, we get

$$K(p,\eta)\int_\Omega |f-m_\Omega(f)|^p \mathrm{d}\nu \le \frac{\lambda_1 p}{2\eta^{p/2}}\int_\Omega |D_H f|_H^p \mathrm{d}\nu + \frac{1}{[\nu(\Omega)]^{(2-p)/2}}\left(\int_\Omega |f-m_\Omega(f)|^2\mathrm{d}\nu\right)^{p/2}.$$

By (4.9), we get

$$K(p,\eta)\int_\Omega |f-m_\Omega(f)|^p \mathrm{d}\nu \le \frac{\lambda_1 p}{2\eta^{p/2}}\int_\Omega |D_H f|_H^p \mathrm{d}\nu + \frac{\lambda_1^{p/2}}{[\nu(\Omega)]^{(p^2-4)/(2p)}}\int_\Omega |D_H f|^p\mathrm{d}\nu, \tag{4.12}$$

which proves the statement when $p \in (2,4]$. Now let $p \in (4,8]$. For any $f \in D^{1,p}(\Omega,\nu)$ we apply (4.11) to the function $g = f - m_\Omega(f)$, and since $p/2 \in (2,4]$, we can use (4.12) with $p/2$ instead of p, to get the thesis for $p \in (4,8]$. Iterating the above procedure, we conclude the proof. □

A standard consequence of the Poincaré inequality is the convergence of $T_\Omega(t)f$ to $m_\Omega(f)$ (see (4.1)) in $L^2(\Omega,\nu)$, as the following exponential decay estimate shows.

COROLLARY 4.6. *If $f \in L^2(\Omega,\nu)$, then*

$$\|T_\Omega(t)f - m_\Omega(f)\|_{L^2(\Omega,\nu)} \le e^{-\lambda_1^{-1}t}\|f\|_{L^2(\Omega,\nu)}. \tag{4.13}$$

As a consequence for every $f \in L^2(\Omega,\nu)$, it holds

$$\lim_{t\to+\infty} T_\Omega(t)f = m_\Omega(f), \qquad \nu\text{-a.e. in } \Omega.$$

Proof. Let $G(s) = \int_\Omega (T_\Omega(s)f - m_\Omega(f))^2 \mathrm{d}\nu$. Using (1.9) and (4.8), we get

$$\begin{aligned} G'(s) &= \frac{\mathrm{d}}{\mathrm{d}s}\int_\Omega (T_\Omega(s)f - m_\Omega(f))^2 \mathrm{d}\nu = 2\int_\Omega (T_\Omega(s)f)(L_\Omega T_\Omega(s)f)\mathrm{d}\nu \\ &= -2\int_\Omega |D_H T_\Omega(s)f|_H^2 \mathrm{d}\nu \leq -\frac{2}{\lambda_1}\int_\Omega (T_\Omega(s)f - m_\Omega(T_\Omega(s)f))^2 \mathrm{d}\nu \\ &= -\frac{2}{\lambda_1}\int_\Omega (T_\Omega(s)f - m_\Omega(f))^2 \mathrm{d}\nu = -\frac{2}{\lambda_1}G(s). \end{aligned}$$

Thus, $G(t) \leq \mathrm{e}^{-2\lambda_1^{-1}t}G(0)$, which means

$$\begin{aligned} &\int_\Omega (T_\Omega(t)f - m_\Omega(f))^2 d\nu \\ &\quad\leq \mathrm{e}^{-2\lambda_1^{-1}t}\int_\Omega (f - m_\Omega(f))^2 d\nu \\ &\quad= \mathrm{e}^{-2\lambda_1^{-1}t}\left[\int_\Omega f^2 d\nu - 2\frac{1}{\nu(\Omega)}\left(\int_\Omega f d\nu\right)^2 + \frac{1}{\nu(\Omega)}\left(\int_\Omega f d\nu\right)^2\right] \\ &\quad\leq \mathrm{e}^{-2\lambda_1^{-1}t}\int_\Omega f^2 d\nu. \end{aligned}$$

This concludes the proof. □

Once the Poincaré inequality, with $p = 2$, the gradient estimate (3.9) and a hypercontractivity type estimate like (4.6) are available, we can establish a relationship between the asymptotic behaviour of $T_\Omega(t)f$ and that of $|D_H T_\Omega(t)f|_H$ as $t \to +\infty$, whenever $f \in L^p(\Omega, \nu)$, $p \in (1, \infty)$. More precisely, arguing as in [4, Theorem 5.3] we can prove the following result, that extends the decay estimate (4.13) to any $p \in (1, \infty)$. We skip the proof due to its length and the fact that it does not present any substantial difference with the one contained in [4, Theorem 5.3]

PROPOSITION 4.7. *For any $p \in (1, \infty)$, consider the sets*

$$\begin{aligned} \mathcal{A}_p = \Big\{\omega \in \mathbb{R} \,\Big|\, &\exists M_{p,\omega} > 0 \text{ s.t. } \|T_\Omega(t)f - m_\Omega(f)\|_{L^p(\Omega,\nu)} \leq M_{p,\omega}e^{\omega t}\|f\|_{L^p(\Omega,\nu)}, \\ &t > 0,\ f \in L^p(\Omega, \nu)\Big\}; \\ \mathcal{B}_p = \Big\{\omega \in \mathbb{R} \,\Big|\, &\exists N_{p,\omega} > 0 \text{ s.t. } \|D_H T_\Omega(t)f\|_{L^p(\Omega,\nu;H)} \leq N_{p,\omega}e^{\omega t}\|f\|_{L^p(\Omega,\nu)}, \\ &t > 1,\ f \in L^p(\Omega, \nu)\Big\}. \end{aligned}$$

Then the sets $\mathcal{A}_p$ and $\mathcal{B}_p$ are independent of p and they coincide. In particular, by Corollary 3.4, for any $p \in (1, \infty)$ there exists a positive constant K_{p,λ_1}, depending only on p and λ_1, such that for every $t > 0$ and $f \in L^p(\Omega, \nu)$, the inequality

$$\|T_\Omega(t)f - m_\Omega(f)\|_{L^p(\Omega,\nu)} \leq K_{p,\lambda_1}e^{-\lambda_1^{-1}t}\|f\|_{L^p(\Omega,\nu)}$$

holds. As a consequence, for every $p \in (1, \infty)$ and $f \in L^p(\Omega, \nu)$

$$\lim_{t\to+\infty} T_\Omega(t)f = m_\Omega(f), \quad \nu\text{-a.e. in } \Omega.$$

Acknowledgements

The authors are members of GNAMPA of the Italian Istituto Nazionale di Alta Matematica (INdAM). L. A. and S. F. have been partially supported by the INdAM-GNAMPA Project "Equazioni e sistemi di equazioni di Kolmogorov in dimensione finita e non" (2017). S. F. has been partially supported by the INdAM-GNAMPA Project "Equazioni e sistemi di equazioni ellittiche e paraboliche a coefficienti illimitati" (2018). D. P. has been partially supported by the INFN and by the INdAM-GNAMPA Projects "Regolarità massimale per alcuni operatori lineari ellittici degeneri" (2017) and "Teoria geometrica della misura in spazi metrici con misura" (2018). The authors have been also partially supported by the PRIN project "Deterministic and stochastic evolution equations" of the Italian Ministry of Education, MIUR.

REFERENCES

[1] D. Addona, G. Cappa, S. Ferrari, *On the domain of elliptic operators defined in subsets of Wiener spaces*, eprint arXiv:1706.05260 (2017).

[2] D. Addona, G. Menegatti, M. Miranda Jr., *On integration by parts formula on open convex sets in Wiener spaces*, eprint arXiv:1808.06825, (2018).

[3] L. Angiuli, L. Lorenzi, *On improvement of summability properties in nonautonomous Kolmogorov equations*, Comm. Pure Appl. Anal. **13** (2014), 1237–1265.

[4] L. Angiuli, L. Lorenzi, A. Lunardi, *Hypercontractivity and asymptotic behavior in nonautonomous Kolmogorov equations*, Comm. Partial Differential Equations **38** (2013), 2049–2080.

[5] D. Bakry, M. Émery, *Diffusions hypercontractives* in "Séminaire de probabilités, XIX, 1983/84", Lecture Notes in Math., vol 1123 (1985), Springer, Berlin, 177–206.

[6] V. Barbu, G. Da Prato, L. Tubaro, *Kolmogorov equation associated to the stochastic reflection problem on a smooth convex set of a Hilbert space*, Ann. Probab. **37** (2009), 1427–1458.

[7] V. Barbu, G. Da Prato, L. Tubaro, *Kolmogorov equation associated to the stochastic reflection problem on a smooth convex set of a Hilbert space II*, Ann. Inst. Henri Poincaré Probab. Stat. **47** (2011), 699–724.

[8] V. Barbu, G. Da Prato, L. Tubaro, *The stochastic reflection problem in Hilbert spaces*, Comm. Partial Differential Equations **37** (2012), 352–367

[9] H.H. Bauschke, P.L. Combettes, Convex analysis and monotone operator theory in Hilbert spaces, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, Springer, New York, 2011.

[10] M. Bertoldi, S. Fornaro *Gradient estimates in parabolic problems with unbounded coefficients*, Studia Math. **165** (2004), 221–254.

[11] V. I. Bogachev, Gaussian measures, Mathematical Surveys and Monographs, vol. 62, American Mathematical Society, Providence, RI, 1998.

[12] G. Cappa, *On the Ornstein-Uhlenbeck operator in convex sets of Banach spaces*, eprint (2015), https://doi.org/10.4064/sm8229-3-2018.

[13] G. Cappa, S. Ferrari, *Maximal Sobolev regularity for solutions of elliptic equations in infinite dimensional Banach spaces endowed with a weighted Gaussian measure*, J. Differential Equations **261** (2016), 7099–7131.

[14] G. Cappa, S. Ferrari, *Maximal Sobolev regularity for solutions of elliptic equations in Banach spaces endowed with a weighted Gaussian measure: the convex subset case*, J. Math. Anal. Appl. **458** (2018), 300–331.

[15] S. Cerrai, Second order PDE's in finite and infinite dimension, Lecture Notes in Mathematics, vol. 1762, Springer-Verlag, Berlin, 2001.

[16] M. Cranston, *Gradient estimates on manifolds using coupling*, J. Funct. Anal. **99** (1991), 110–124.
[17] M. Cranston, *A probabilistic approach to gradient estimates*, Canad. Math. Bull. **35** (1992), 46–55.
[18] G. Da Prato, A. Lunardi, *On the Dirichlet semigroup for Ornstein-Uhlenbeck operators in subsets of Hilbert spaces*, J. Funct. Anal. **259** (2010), 2642–2672.
[19] G. Da Prato, A. Lunardi, *Sobolev regularity for a class of second order elliptic PDE's in infinite dimension*, Ann. Probab. **42** (2014), 2113–2160.
[20] G. Da Prato, A. Lunardi, *Maximal Sobolev regularity in Neumann problems for gradient systems in infinite dimensional domains*, Ann. Inst. Henri Poincaré Probab. Stat. **51** (2015), 1102–1123.
[21] G. Da Prato, J. Zabczyk, Second order partial differential equations in Hilbert spaces, London Mathematical Society Lecture Note Series, vol. 293, Cambridge University Press, Cambridge, 2002.
[22] J. Deuschel, D. W. Stroock, *Hypercontractivity and spectral gap of symmetric diffusions with applications to the stochastic Ising models*, J. Funct. Anal. **92** (1990), 30–48.
[23] S. Ferrari, *Sobolev spaces with respect to a weighted Gaussian measures in infinite dimensions*, eprint arXiv:1510.08283 (2015).
[24] S. Fitzpatrick, R. R. Phelps, *Differentiability of the metric projection in Hilbert space*, Trans. Amer. Math. Soc. **270** (1982), 483–501.
[25] A. Friedman, Partial differential equations of parabolic type, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964.
[26] M. Hino, *On Dirichlet spaces over convex sets in infinite dimensions* in "Finite and infinite dimensional analysis in honor of Leonard Gross (New Orleans, LA, 2001)", Contemp. Math., vol. 317 (2003), Amer. Math. Soc., Providence, RI, 143–156.
[27] M. Hino, *Dirichlet spaces on H-convex sets in Wiener space*, Bull. Sci. Math. **135** (2011), 667–683.
[28] R. B. Holmes, *Smoothness of certain metric projections on Hilbert space*, Trans. Amer. Math. Soc. **184** (1973), 87–100.
[29] L. Lorenzi, Analytical methods for Kolmogorov equations, Second Edition, Monographs and Research Notes in Mathematics, CRC Press, 2017.
[30] A. Lunardi, *Schauder theorems for linear elliptic and parabolic problems with unbounded coefficients in* $\mathbb{R}^n$, Studia Math. **128** (1998), 171–198.
[31] A. Lunardi, M. Miranda Jr., D. Pallara, *BV functions on convex domains in Wiener spaces*, Potential Anal. **43** (2015), 23-48.
[32] Z. M. Ma, M. Röckner, Introduction to the theory of (nonsymmetric) Dirichlet forms, Universitext, Springer-Verlag, Berlin, 1992.
[33] J. Maas, J. van Neerven, *On analytic Ornstein-Uhlenbeck semigroups in infinite dimensions*, Arch. Math. (Basel) **89** (2007), 226–236.
[34] J. Maas, J. van Neerven, *Gradient estimates and domain identification for analytic Ornstein-Uhlenbeck operators* in "Parabolic problems", Progr. Nonlinear Differential Equations Appl., vol. 80 (2011), Birkhäuser/Springer Basel AG, Basel, 463–477.
[35] M. Mandelkern, *On the uniform continuity of Tietze extensions*, Arch. Math. (Basel) **55** (1990), 387–388.
[36] E. M. Ouhabaz, Analysis of heat equations on domains, London Mathematical Society Monographs Series, vol. 31, Princeton University Press, Princeton, NJ, 2005.
[37] E. Priola, *On a class of Markov type semigroups in spaces of uniformly continuous and bounded functions*, Studia Math. **136** (1999), 271–295.
[38] O. S. Rothaus, *Logarithmic Sobolev inequalities and the spectrum of Schrödinger operators*, J. Funct. Anal. **42** (1981), 110–120.
[39] G. Savaré, *Self-improvement of the Bakry-Émery condition and Wasserstein contraction of the heat flow in* $\mathrm{RCD}(K,\infty)$ *metric measure spaces*, Discrete Contin. Dyn. Syst. **34** (2014), 1641–1661.
[40] F.-Y. Wang, *On estimation of the logarithmic Sobolev constant and gradient estimates of heat semigroups*, Probab. Theory Related Fields **108** (1997), 87–101.

L. Angiuli, S. Ferrari and D. Pallara
Dipartimento di Matematica e Fisica "Ennio de Giorgi"
Università del Salento
Via Per Arnesano snc
73100 Lecce
Italy
E-mail: luciana.angiuli@unisalento.it

S. Ferrari
E-mail: simone.ferrari@unisalento.it

D. Pallara
E-mail: diego.pallara@unisalento.it